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On Yau's problem of evolving one curve to another: Convex case

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Abstract

A revised Yau's Curvature Difference Flow is considered to deform one convex curve X_0 to another one $\widetilde{X}$. It is proved that this flow exists globally on time interval $[0, +\infty)$ and the evolving curve, preserving its convexity and bounded area A, converges to a fixed limiting curve X_∞ (congruent to $\sqrt{A/\widetilde{A}}\ \widetilde{X}$) as time tends to infinity, where $\widetilde{A}$ is the area bounded by the target curve $\widetilde{X}$.

MSC: 53A04; 35K15

Keywords: Convex curve; Yau's Curvature Difference Flow; C^∞ convergence; Steiner point

1. Introduction

In 2007, S.T. Yau proposed an interesting problem that **whether one can use a parabolic curvature flow method to evolve one curve X_0 to another $\widetilde{X}$ either in finite time or in infinite time**. In the same year, Y.-C. Lin and D.-H. Tsai constructed a linear parabolic model [21] to evolve a convex curve to another one, where a curve is called convex if it is closed, embedded and has positive curvature everywhere. They showed that if the two curves have same length and the curvature is bounded above during the evolution, then X_0 can be deformed into $\widetilde{X}$.

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To settle Yau's problem, Tsai suggested an evolution model, which is called as "Yau's Curvature Difference Flow (YCDF)":

$$\begin{cases} \frac{\partial X}{\partial t}(\varphi,t)=[\kappa(\varphi,t)-\widetilde{\kappa}(\varphi)]N(\varphi,t) \ \text{ in } S^1\times(0,\omega), \\ X(\varphi,0)=X_0(\varphi) \ \text{ on } S^1, \end{cases} \tag{1.1}$$

where κ and $\widetilde{\kappa}$ are the curvature of the evolving curve X and the target curve $\widetilde{X}$ separately and N is the unit inner normal of X.

The significance of YCDF is that it stops if X_0 converges to $\widetilde{X}$. But in general, this flow may not deform X_0 into $\widetilde{X}$. For example, let X_0 and $\widetilde{X}$ be two convex curves with curvature denoted by κ_0 and $\widetilde{\kappa}$, respectively. If $\kappa_0>\widetilde{\kappa}$ everywhere then the results in Chapter 3 of the book by Chou and Zhu [9] or their paper [8] tell us that the evolving curve will shrink to a self-similar solution of the famous curve shortening flow. In the embedded case, the limiting curve is a circle after a proper rescaling [1,3].

Early in the year 1993, Gage and Li [13,15] studied a flow for convex curves

$$\begin{cases} \frac{\partial X}{\partial t}=(\widetilde{p}/\widetilde{\kappa})\kappa N \ \text{ in } S^1\times(0,\omega), \\ X(\varphi,0)=X_0(\varphi) \ \text{ on } S^1, \end{cases} \tag{1.2}$$

where $\widetilde{p}:=-\langle\widetilde{X},\widetilde{N}\rangle$ is the support function of the target curve $\widetilde{X}$. They showed that the evolving curve converges to the shape of the target curve as $X(\cdot,t)$ shrinks to a point, provided that the convex body bounded by $\widetilde{X}$ is symmetric. Their result implies that Yau's problem is solved for convex initial X_0 and convex, symmetric target $\widetilde{X}$. Motivated by the work of Gage and Li and the models for phase transitions [4,5,17], Chou–Zhu in 1999 considered an anisotropic flow [7]

$$\begin{cases} \frac{\partial X}{\partial t}=(g(\theta)\kappa+F)N \ \text{ in } S^1\times(0,\omega), \\ X(\theta,0)=X_0(\theta) \ \text{ on } S^1, \end{cases} \tag{1.3}$$

where θ denotes the tangent angle, g is of the form $g=d^2f/d\theta^2+f$ and F is a constant. They showed that the evolving curve X is uniformly convex on the maximal time interval $[0,t_{\max})$ and the behavior depends on the value of F when $t\to t_{\max}$. This result was generalized to a more widely used model by Chou–Zhu [8]. As a spacial case, they proved that there exists a constant F^* such that the flow

$$\begin{cases} \frac{\partial X}{\partial t}=(\kappa-F^*\widetilde{\kappa})N \ \text{ in } S^1\times(0,\omega), \\ X(\varphi,0)=X_0(\varphi) \ \text{ on } S^1, \end{cases} \tag{1.4}$$

exists globally and X converges to a stationary solution, where $\widetilde{\kappa}$ is a positive, periodic function. This flow (1.4) is an alternative model to evolve X_0 to $\widetilde{X}$ if one chooses $\widetilde{\kappa}$ as the curvature of the target curve. Because the critical number F^* is obtained via a contradiction argument, one may not easily find its value for a concrete X_0. Lin–Tsai [22] did some work to estimate F^* in the case that the target curve $\widetilde{X}$ is a unit circle.

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Inspired by Yau's problem and the work by Chou–Zhu [7], Chou–Zhu [8] and Lin–Tsai [21], we studied a revised YCDF of the form:

$$\begin{cases} \frac{\partial X}{\partial t}(\varphi,t) = [\kappa(\varphi,t) - \lambda(t)\widetilde{\kappa}(\varphi)]N(\varphi,t) \ \text{ in } S^1 \times (0,\omega), \\ X(\varphi,0) = X_0(\varphi) \ \text{ on } S^1, \end{cases} \tag{1.5}$$

where $\lambda(t) = 2\pi / \int_{S^1} \widetilde{\kappa} g d\varphi$ and g is the metric of X. The nonlocal term $\lambda(t)$ is used to keep the bounded area A of $X(\cdot,t)$ a constant. Comparing to F^* in Equation (1.4), the coefficient $\lambda(t)$ is dependent on time and explicitly expressed.

If the target curve $\widetilde{X}$ is a circle ($\widetilde{\kappa}$ is a positive constant) then this flow is Gage's area-preserving flow [11]. If the initial curve is homothetic to $\widetilde{X}$ then the evolving curve is stationary under the flow. The main result of this paper is as follows.

Main Theorem. *Let X_0 and $\widetilde{X}$ be two smooth, closed and embedded curves with positive curvature everywhere. The flow (1.5) with initial X_0 and referential $\widetilde{X}$ exists globally on time interval $[0,+\infty)$, preserves the bounded area A and the convexity of every evolving curve $X(\cdot,t)$ and deforms X_0 into a fixed curve X_∞ (congruent to $\sqrt{A/\widetilde{A}}\ \widetilde{X}$) as time tends to infinity, where $\widetilde{A}$ is the area bounded by the curve $\widetilde{X}$.*

The proof of Main Theorem is composed of the following three aspects:

First of all, one needs to bound the curvature κ of the evolving curve X to obtain the global existence of the flow (1.5). The positivity of κ is a direct application of maximum principle of parabolic equations. The uniform upper bound of κ can be obtained via a careful analysis of an auxiliary function Q which introduced firstly by Chou [6].

Then, since whether the evolution equation of κ is degenerate or not is unknown, it's quite hard to estimate the gradient of κ directly. The L^2-norm of κ_θ can be bounded by the uniform upper bound of κ and the evolution of entropy $\int_0^{2\pi} \ln \kappa d\theta$. Then the convergence of the curvature follows from Ascoli–Arzela Theorem and the monotony of the Minkowski length $\mathfrak{L}$. By the convergence of κ and its positivity, it has a positive and lower bound. So the evolution equation of κ is not degenerate.

Finally, one has to prove that the evolving curve X converges as $t \to \infty$. In the previous study of non-local flows for convex curves, the speed of the flow can be proved exponentially decaying. However, due to the complexity of the evolution equation of κ, the exponential decay of $|\kappa - \lambda\widetilde{\kappa}|$ seems quite difficult to be obtained. One can follow the method in [7] or Chapter 3 of [9] to prove the convergence of the Steiner point and the support function, leading to the convergence of the evolving curve to a fixed limit.

In Section 2, it is proved that the flow (1.5) exists in time interval $[0,+\infty)$ and preserves convexity and the bounded area of the evolving curve. In Section 3, it is shown that the curvature of the evolving curve converges to a limit $\sqrt{\widetilde{A}/A}\widetilde{\kappa}$ in C^∞ sense, where $\widetilde{\kappa}$ is the curvature of the target curve. In Section 4, Main Theorem is proved via showing that the evolving curve $X(\cdot,t)$ converges to a fixed curve X_∞ as time goes to infinity.

2. Existence

In this section, some basic properties of the flow (1.5) will be explored, such as the short time existence, the convexity of every curve $X(\cdot,t)$. To extend the flow on time interval $[0,+\infty)$, it is proved that the curvature κ of the evolving curve is bounded for all $t \geq 0$.

The estimate of the curvature is a key step towards to the global existence of the flow. A very useful method constructed by Kai-Seng Chou [6] can give an upper bound of κ independent of time. The idea is first to define some an auxiliary function, denoted by Q. Then applying the comparison principle of parabolic equations can give us an upper bound of Q, yielding a uniform bound of the curvature.

One can also follow Gage–Hamilton's method [14] to show that the flow (2.1) exists globally via proving the similar geometric estimate, integral estimate and pointwise estimate of κ. The advantage of the method in this paper is that it provides an uniformly upper bound of the curvature. The uniform bound of the curvature is very helpful to the study of the convergence of the flow.

2.1. Short time existence

Let $\theta = \theta(\varphi,t)$ be the tangent angle of $X(\cdot,t)$. This parameter plays an important role in the study of curvature flows for convex curves, because it can simplify some important quantities, such as the Frenet frame $T = (\cos\theta, \sin\theta)$, $N = (-\sin\theta, \cos\theta)$. In order to make θ independent of time, one can add a proper tangent component to Equation (1.5) to obtain

$$\begin{cases} \frac{\partial X}{\partial t} = (\kappa - \lambda\widetilde{\kappa})N + \alpha T \ \text{ in } S^1 \times (0,\omega), \\ X(\varphi,0) = X_0(\varphi) \ \text{ on } S^1. \end{cases} \tag{2.1}$$

As is well known that the flow (2.1) differs from that of (1.5) by a reparametrization. From now on, let s be the arc length parameter and set $\beta := \kappa - \lambda\widetilde{\kappa}$. Under the flow (2.1) the metric g and the Frenet frame evolve according to

$$\frac{\partial g}{\partial t} = \left(\frac{\partial \alpha}{\partial s} - \beta\kappa\right) g,$$
$$\frac{\partial T}{\partial t} = \left(\alpha\kappa + \frac{\partial \beta}{\partial s}\right) N, \qquad \frac{\partial N}{\partial t} = -\left(\alpha\kappa + \frac{\partial \beta}{\partial s}\right) T$$

and the relation for the operators $\partial/\partial s$ and $\partial/\partial t$ is

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = -\left(\frac{\partial \alpha}{\partial s} - \beta\kappa\right)\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}.$$

Noticing that

$$\frac{\partial T}{\partial t} = (-\sin\theta, \cos\theta)\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial t} N,$$

one obtains the evolution of the tangent angle

$$\frac{\partial \theta}{\partial t} = \alpha\kappa + \frac{\partial \beta}{\partial s}.$$

If $X(\cdot, t)$ is a family of convex curves then $\kappa(\cdot, t) > 0$ and one can use θ to parametrize every curve. Now one can choose

$$\alpha = -\frac{1}{\kappa}\frac{\partial \beta}{\partial s} \tag{2.2}$$

to make sure $\frac{\partial \theta}{\partial t} \equiv 0$. Because $\frac{\partial \theta}{\partial s} = \kappa$ (by the definition of the curvature), the metric of the evolving curve is $g = \frac{1}{\kappa}$. The evolution equation of κ is as follows

$$\begin{aligned}\frac{\partial \kappa}{\partial t} &= \frac{\partial^2 \theta}{\partial t \partial s} = -\left(\frac{\partial \alpha}{\partial s} - \beta\kappa\right)\kappa + \frac{\partial}{\partial s}\frac{\partial \theta}{\partial t} \\ &= \frac{\partial}{\partial s}\frac{\partial \theta}{\partial t} - \frac{\partial \alpha}{\partial s}\kappa + \beta\kappa^2.\end{aligned}$$

Thus under the flow (2.1), the curvature κ satisfies a Cauchy problem

$$\begin{cases} \frac{\partial \kappa}{\partial t} = \kappa^2\left(\frac{\partial^2 \kappa}{\partial \theta^2} - \lambda\frac{\partial^2 \widetilde{\kappa}}{\partial \theta^2} + \kappa - \lambda\widetilde{\kappa}\right) \text{ in } [0, 2\pi] \times (0, \omega), \\ \kappa(\theta, 0) = \kappa_0(\theta) \text{ on } [0, 2\pi], \end{cases} \tag{2.3}$$

where κ_0 is the curvature of the initial curve X_0. (2.3) is a quasilinear parabolic equation with a nonlocal term

$$\lambda = \frac{2\pi}{\int_0^{2\pi} \widetilde{\kappa}/\kappa d\theta}$$

and a positive initial value which satisfies the closing condition

$$\int_0^{2\pi} \frac{e^{i\theta}}{\kappa_0} d\theta = 0.$$

Once this Cauchy problem has a positive smooth solution, the above closing condition holds for κ and there exists a family of convex curves drawn by κ, satisfying Equation (2.1). So one can pay attention to Equation (2.3).

The linearization of the evolution equation of the curvature κ at κ_0 is

$$\begin{aligned}\frac{\partial \kappa}{\partial t} = \kappa_0^2\frac{\partial^2 \kappa}{\partial \theta^2} + \left(2\kappa_0\frac{\partial^2 \kappa_0}{\partial \theta^2} - 2\lambda(0)\kappa_0\frac{\partial^2 \widetilde{\kappa}}{\partial \theta^2} + 3\kappa_0^2 - 2\lambda(0)\widetilde{\kappa}\kappa_0\right)\kappa \\ -\left(\kappa_0^2\frac{\partial^2 \widetilde{\kappa}}{\partial \theta^2} + \widetilde{\kappa}\kappa_0^2\right)\frac{\lambda(0)^2}{2\pi}\int_0^{2\pi}\frac{\widetilde{\kappa}}{\kappa_0^2}\kappa d\theta.\end{aligned}$$

It is a uniformly parabolic equation with smooth coefficients. So Equation (2.3) has a unique smooth solution on $[0, 2\pi] \times [0, t_0)$, where t_0 is small (see Fact 3 in Chapter 1 of the book [9]). One can also follow [14] or [19] using Leray–Schauder fixed point theorem to show the short time existence of the Cauchy problem (2.3).

Theorem 2.1. *The flow (2.1) has a unique smooth solution on $S^1 \times [0, t_0)$ for some $t_0 > 0$.*

2.2. Convexity of the evolving curve

The continuity of κ implies that there exists $t_1 \in (0, t_0)$ such that the evolving curve $X(\cdot, t)$ is convex for all $t \in [0, t_1)$. So one can use θ to parametrize the curve in a short time interval. In fact, the evolving curve is convex whenever the flow exists. It is an application of maximum principle for parabolic equations. In order to use maximum principle, one first needs to show that the nonlocal term λ in the evolution equation of κ is bounded.

Lemma 2.2. *If the flow (2.1) exists in time interval $[0, \omega)$ and the evolving curve is convex for $t \in [0, t_0)(t_0 \le \omega)$, then the area bounded by $X(\cdot, t)$ is fixed and the length of the evolving curve and the nonlocal term $\lambda(t)$ can be bounded by*

$$\sqrt{4\pi A} \le L(t) \le \sqrt{L^2(0) + \frac{2A}{\widetilde{m}} \int_0^{2\pi} \widetilde{\kappa} d\theta} := C, \tag{2.4}$$

$$\frac{2\pi}{\widetilde{M} C} \le \lambda(t) \le \frac{2\pi}{\widetilde{m}\sqrt{4\pi A}} := \Lambda, \tag{2.5}$$

where $\widetilde{M} := \max\{\widetilde{\kappa}(\theta) | \theta \in [0, 2\pi]\}$, $\widetilde{m} := \min\{\widetilde{\kappa}(\theta) | \theta \in [0, 2\pi]\}$.

Proof. Under the flow (2.1), we have set $\beta = \kappa - \lambda\widetilde{\kappa}$. Compute that

$$\frac{dA}{dt} = -\int_0^{2\pi} \beta/\kappa d\theta = -\int_0^{2\pi} \left(1 - \lambda\frac{\widetilde{\kappa}}{\kappa}\right) d\theta = 0.$$

So this flow preserves the bounded area of the evolving curve. The length of X satisfies that

$$\frac{dL}{dt} = -\int_0^{2\pi} (\kappa - \lambda\widetilde{\kappa}) d\theta = -\int_0^{2\pi} \kappa d\theta + \lambda \int_0^{2\pi} \widetilde{\kappa} d\theta.$$

Noticing that Gage Inequality [10] says

$$\int_0^{2\pi} \kappa d\theta \ge \frac{\pi L}{A}$$

and λ has upper bound

$$\lambda = \frac{2\pi}{\int_0^{2\pi} \widetilde{\kappa}/\kappa d\theta} \le \frac{2\pi}{\widetilde{m}L},$$

one obtains

$$\frac{dL}{dt} \le -\frac{\pi L}{A} + \frac{2\pi}{\widetilde{m}L}\int_0^{2\pi} \widetilde{\kappa} d\theta.$$

So $L^2(t) \le L^2(0)\exp(-\frac{\pi}{A}t) + \frac{2A}{\widetilde{m}}\int_0^{2\pi} \widetilde{\kappa} d\theta (1 - \exp(-\frac{\pi}{A}t))$, which implies the right hand side of Inequality (2.4). The left hand side is a corollary of the classical isoperimetric inequality. Inequality (2.5) is a direct corollary of (2.4). □

Since $\lambda(t)$ is bounded by constants independent of time, one can use maximum principle to show that the flow preserves the convexity of the evolving curve.

Lemma 2.3. *The evolving curve under the flow (2.1) is convex.*

Proof. Suppose the flow exists in time interval $[0,\omega)$ and there is a smallest $t_1 \in (0,\omega)$ such that the minimum of $\kappa(\cdot, t_1)$ with respect to θ is 0. Let θ_* be the point such that $\kappa(\cdot, t)$ attains its minimum value

$$\kappa(\theta_*, t) = \min\{\kappa(\theta, t) | \theta \in [0, 2\pi]\} := \kappa_{\min}(t),$$

for $t \in (0, t_1]$. Set the constants $\widetilde{M}_i := \max\left\{\left|\frac{d^i\widetilde{\kappa}}{d\theta^i}\right| \middle| \theta \in [0, 2\pi]\right\}, i = 1, 2, 3, \cdots$. At the point (θ_*, t), one has

$$\begin{aligned}\frac{\partial\kappa}{\partial t} &\ge \kappa^2\left(\kappa - \lambda\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2} - \lambda\widetilde{\kappa}\right)\\ &\ge -\kappa^2(\Lambda\widetilde{M}_2 + \Lambda\widetilde{M}).\end{aligned}$$

Using Hamilton's technique of maximum principle [18], the differential inequality

$$\frac{d\kappa_{\min}}{dt} \ge -\kappa_{\min}^2(\Lambda\widetilde{M}_2 + \Lambda\widetilde{M})$$

holds in Lipschitz sense and implies that

$$\kappa_{\min}(t) \ge \left(\frac{1}{\kappa_{\min}(0)} + (\Lambda\widetilde{M}_2 + \Lambda\widetilde{M})t\right)^{-1} > 0, \tag{2.6}$$

for all $t \in [0, t_1]$. It contradicts the hypothesis of t_1. □

Once the flow (2.1) exists in time interval $[0,\omega)$, the evolving curve is always convex. The parameter θ can be used to formulate evolution equations from now on.

2.3. Bound of the curvature and global existence

In order to extend the flow, one needs to bound the curvature. Bonnesen Inequality [23] for convex curves in the plane says that

$$-\pi r^2 + rL - A \geq 0, \quad r_{in} \leq r \leq r_{out},$$

where r_{in} and r_{out} are the inradius and outradius of the domain bounded by X, respectively. So

$$1 \leq \frac{r_{out}}{r_{in}} \leq \frac{L + \sqrt{L^2 - 4\pi A}}{L - \sqrt{L^2 - 4\pi A}} = (\sqrt{I_r} + \sqrt{I_r - 1})^2,$$

where $I_r(t) = L^2/(4\pi A)$ is the isoperimetric ratio of X.

Let us turn to the flow (2.1). By the bound of $L(t)$, the isoperimetric ratio satisfies that

$$1 \leq I_r(t) \leq \frac{C^2}{4\pi A},$$

for all $t \in [0, \omega)$. Under the flow, the ratio of outradius and inradius is bounded by

$$\frac{r_{out}(t)}{r_{in}(t)} \leq (\sqrt{\frac{C^2}{4\pi A}} + \sqrt{\frac{C^2}{4\pi A} - 1})^2 := \sigma, \quad t \in [0, \omega).$$

Hence $r_{out}(t) \leq \sigma r_{in}(t)$. Because $\pi r_{out}(t)^2 \geq A$, one obtains that

$$r_{in}(t) \geq \sigma^{-1}\sqrt{A/\pi}, \tag{2.7}$$

for all $t \in [0, \omega)$. The inradius has a time-independent positive lower bound. Now one can show that the curvature of the evolving curve has an upper bound independent of time.

Lemma 2.4. *There exists a positive constant M independent of time such that*

$$\kappa(\theta, t) \leq M, \quad (\theta, t) \in [0, 2\pi] \times [0, +\infty). \tag{2.8}$$

Proof. Let $E(0)$ be a circle enclosed by X_0 with radius $r(0) = r_{in}(0)$. Let us shrink $E(0)$ and X_0 by the famous curve shortening flow to obtain two family of curves, denoted by $E(t)$ and $Y(t)$ respectively. By the maximum principle, $E(t)$ is bounded by $Y(t)$ for every t. Because the velocity of the evolving curve $X(\cdot, t)$ differs from that of $Y(t)$ by $-\lambda(t)\widetilde{\kappa}N$, $X(\cdot, t)$ contains $Y(t)$ as time goes. So $E(t)$ is enclosed by $X(\cdot, t)$ for all $t \in [0, \omega)$. The radius of $E(t)$ is given by $r(t) = \sqrt{r^2(0) - 2t}$, $t \in [0, r^2(0)/2]$.

Define $p := -\langle X, N\rangle$ the support function of the evolving curve with respect to the center of $E(0)$ as the origin O. The support function $p(\theta, t)$ satisfies that

$$p(\theta, t) \geq \sqrt{r^2(0) - 2t}, \quad (\theta, t) \in [0, 2\pi] \times [0, \min\{r^2(0)/2, \omega\}).$$

Let us choose $t_1 = 3r^2(0)/8$ so that $p(\theta, t) \geq r(0)/2$. Since $p(\theta, t)$ is the distance from the point O to the tangent line of X,

$$p(\theta, t) \leq |X(\theta, t)| < \frac{L(t)}{2} \leq C.$$

Define an auxiliary function

$$Q = \frac{\kappa}{p - \delta} > 0, \quad \delta = \frac{r(0)}{4},$$

for all $t \in [0, t_1]$. One has $0 < 2\delta \leq p \leq C$, where both δ and C are constants independent of time. Noticing that

$$\frac{\partial p}{\partial t} = \lambda\widetilde{\kappa} - \kappa, \quad \frac{\partial Q}{\partial \theta} = \frac{1}{p-\delta}\frac{\partial \kappa}{\partial \theta} - \frac{\kappa}{(p-\delta)^2}\frac{\partial p}{\partial \theta},$$

$$\frac{\partial^2 Q}{\partial \theta^2} = \frac{1}{p-\delta}\frac{\partial^2 \kappa}{\partial \theta^2} - \frac{2}{(p-\delta)^2}\frac{\partial p}{\partial \theta}\frac{\partial \kappa}{\partial \theta} - \frac{\kappa}{(p-\delta)^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{2\kappa}{(p-\delta)^3}\left(\frac{\partial p}{\partial \theta}\right)^2,$$

one has the evolution equation of Q:

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \kappa^2\frac{\partial^2 Q}{\partial \theta^2} + \frac{2\kappa^2}{p-\delta}\frac{\partial p}{\partial \theta}\frac{\partial Q}{\partial \theta} - \delta(p-\delta)Q^3 - \lambda(p-\delta)\frac{\partial^2\widetilde{\kappa}}{\partial \theta^2}Q^2 - \lambda(p-\delta)\widetilde{\kappa}Q^2 \\ &\quad + 2Q^2 - \frac{\lambda\widetilde{\kappa}}{p-\delta}Q.\end{aligned}$$

Whenever $Q > \frac{1}{\delta^2}(\Lambda\widetilde{M}_2(C-\delta)+2)$, one obtains that

$$\begin{aligned}&-\delta(p-\delta)Q^3 - \lambda(p-\delta)\frac{\partial^2\widetilde{\kappa}}{\partial \theta^2}Q^2 - \lambda(p-\delta)\widetilde{\kappa}Q^2 + 2Q^2 - \frac{\lambda\widetilde{\kappa}}{p-\delta}Q \\ &\leq Q^2[-\delta^2 Q + \Lambda\widetilde{M}_2(C-\delta)+2)] < 0\end{aligned}$$

and the function $Q_{\max}(t) = \max\{Q(\theta, t)|\theta \in [0, 2\pi]\}$ decreases. By the maximum principle of parabolic equations, Q has an upper bound

$$Q(\theta, t) \leq \max\left\{Q_{\max}(0), \frac{1}{\delta^2}[\Lambda\widetilde{M}_2(C-\delta)+2]\right\}, \tag{2.9}$$

for all $t \in [0, t_1]$. Repeating the same process as above, one can show that inequality (2.9) holds for $t \in [nt_1, (n+1)t_1]$, $n = 1, 2, \cdots$. Thus there exists a positive constant M independent of time such that (2.8) holds. □

If the curvature κ is bounded by $0 < \kappa \leq M$ for all time, all the derivatives $\frac{\partial^i \kappa}{\partial \theta^i}$ grow at most exponentially by the maximum principle. So the flow (2.1) exists globally.

Theorem 2.5. *The flow (2.1) exists in time interval* $[0, +\infty)$.

The upper bound of κ in Lemma 2.4 is independent of time. An advantage of this fact is that one can use the uniform bound for κ to obtain its convergence.

3. Convergence of the curvature

In this section, we shall prove the C^∞-convergence of the flow (1.5). The flow has C^2-convergence if the curvature of the evolving curve possesses a limit as time tends to infinity. If all the derivatives of the curvature converges then we say the flow converges in the C^∞ sense.

In order to prove the convergence of the curvature, some estimate of the derivative κ_θ is needed. Usually, the gradient estimate of parabolic equations can be obtained by applying maximum principle to some auxiliary functions. However, this strategy works only if the equation is uniformly parabolic, i.e., the curvature κ should have a positive lower bound for all $t \geq 0$ in our case. Up to now, it is not known whether the equation of κ is uniformly parabolic for all time. The method in the classical theory of parabolic equations is not quite usable.

We shall use the entropy $\int_0^{2\pi} \ln \kappa d\theta$ and the uniform upper bound of κ to estimate L^2-norm of κ_θ, yielding the convergence of the curvature. The C^∞-convergence of the flow follows from a classical method of parabolic equations.

3.1. C^2-convergence

In the previous study of non-local flows which deform convex curves into circles, the C^2-convergence of the flow relies on the Hausdorff convergence of the evolving curve to a finite circle, i.e., the isoperimetric difference $L(t)^2 - 4\pi A(t)$ tends to 0 as time tends to infinity. For two given convex curves X and $\widetilde{X}$ in the plane, one also has a similar inequality of Wulff isoperimetric difference (see [16])

$$\mathfrak{L}^2 - 4A\widetilde{A} \geq 0,$$

where $\mathfrak{L} := \int_0^{2\pi} \widetilde{p}/\kappa d\theta$ is called Minkowski length of X with respect to $\widetilde{X}$ and the equality holds if and only if X is homothetic to $\widetilde{X}$.

Under the flow (2.1), the Minkowski length $\mathfrak{L}$ satisfies that

$$\begin{aligned}
\frac{d\mathfrak{L}}{dt} &= \int_0^{2\pi} -\frac{\widetilde{p}}{\kappa^2}\kappa^2 \left(\frac{\partial^2 \kappa}{\partial\theta^2} - \lambda\frac{\partial^2 \widetilde{\kappa}}{\partial\theta^2} + \kappa - \lambda\widetilde{\kappa}\right) d\theta \\
&= -\int_0^{2\pi} \left(\frac{\partial^2 \widetilde{p}}{\partial\theta^2} + \widetilde{p}\right)(\kappa - \lambda\widetilde{\kappa}) d\theta \\
&= -\int_0^{2\pi} \frac{1}{\widetilde{\kappa}}(\kappa - \lambda\widetilde{\kappa}) d\theta \\
&= -\int_0^{2\pi} \frac{\kappa}{\widetilde{\kappa}} d\theta + \frac{(2\pi)^2}{\int_0^{2\pi} \frac{\widetilde{\kappa}}{\kappa} d\theta} \leq 0,
\end{aligned}$$

where Cauchy–Schwarz Inequality is used. So Wulff isoperimetric difference is decreasing with respect to time. In the case of $\widetilde{\kappa} \equiv$ const., Gage [11] showed that the isoperimetric difference satisfies

$$\frac{d}{dt}(L^2 - 4\pi A) \le -\frac{2\pi}{A}(L^2 - 4\pi A),$$

by using his famous inequality (Gage Inequality, see [10]) for convex curves. So the isoperimetric difference converges to 0 under Gage's area preserving flow. However, the so called Wulff–Gage Inequality (see [16])

$$\int_0^{2\pi} \frac{\kappa}{\widetilde{\kappa}^2} \widetilde{p} d\theta \ge \frac{\widetilde{A}\mathfrak{L}}{A}$$

holds under a condition that the target $\widetilde{X}$ is centrosymmetric. It seems not easy to prove $\lim_{t\to\infty}(\mathfrak{L}^2 - 4A\widetilde{A}) = 0$ directly. So it is not appropriate to prove the C^2-convergence of the flow via the method of using Hausdorff convergence.

To guarantee the existence of convergent subsequence of the curvature, we plan to bound an L^2-estimate of κ_θ by using the entropy estimate.

Lemma 3.1. *The quantity $\int_0^{2\pi} \left(\frac{\partial\kappa}{\partial\theta}\right)^2 d\theta$ is uniformly bounded under the flow (2.1).*

Proof. Using the evolution equation of κ, one can compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}\left(\frac{\partial\kappa}{\partial\theta}\right)^2 d\theta &= \int_0^{2\pi} \frac{\partial\kappa}{\partial\theta}\frac{\partial}{\partial\theta}\left[\kappa^2\left(\frac{\partial^2\kappa}{\partial\theta^2} - \lambda\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2} + \kappa - \lambda\widetilde{\kappa}\right)\right] d\theta \\
&= -\int_0^{2\pi} \frac{\partial^2\kappa}{\partial\theta^2}\kappa^2\left(\frac{\partial^2\kappa}{\partial\theta^2} - \lambda\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2} + \kappa - \lambda\widetilde{\kappa}\right) d\theta \\
&= -\int_0^{2\pi} \kappa^2\left(\frac{\partial^2\kappa}{\partial\theta^2}\right)^2 d\theta + \lambda\int_0^{2\pi}\kappa^2\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2}\frac{\partial^2\kappa}{\partial\theta^2}d\theta - \int_0^{2\pi}\kappa^3\frac{\partial^2\kappa}{\partial\theta^2}d\theta \\
&\quad + \lambda\int_0^{2\pi}\widetilde{\kappa}\kappa^2\frac{\partial^2\kappa}{\partial\theta^2}d\theta.
\end{aligned}$$

Using integration by parts, one obtains

$$\lambda\int_0^{2\pi}\kappa^2\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2}\frac{\partial^2\kappa}{\partial\theta^2}d\theta = -\lambda\int_0^{2\pi}\frac{\partial\kappa}{\partial\theta}\left(2\kappa\frac{\partial\kappa}{\partial\theta}\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2} + \kappa^2\frac{\partial^3\widetilde{\kappa}}{\partial\theta^3}\right)d\theta$$

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$$= -\lambda \int_0^{2\pi} 2\kappa \left(\frac{\partial \kappa}{\partial \theta}\right)^2 \frac{\partial^2 \widetilde{\kappa}}{\partial \theta^2} d\theta - \lambda \int_0^{2\pi} \kappa^2 \frac{\partial \kappa}{\partial \theta} \frac{\partial^3 \widetilde{\kappa}}{\partial \theta^3} d\theta$$

$$\leq 2\Lambda M \widetilde{M}_2 \int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + \frac{\Lambda}{2} \left[\int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + 2\pi M^4 (\widetilde{M}_3)^2 \right],$$

$$-\int_0^{2\pi} \kappa^3 \frac{\partial^2 \kappa}{\partial \theta^2} d\theta = 3 \int_0^{2\pi} \kappa^2 \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta \leq 3M^2 \int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta,$$

$$\lambda \int_0^{2\pi} \widetilde{\kappa} \kappa^2 \frac{\partial^2 \kappa}{\partial \theta^2} d\theta = -\lambda \int_0^{2\pi} \frac{\partial \kappa}{\partial \theta} \left(\frac{\partial \widetilde{\kappa}}{\partial \theta} \kappa^2 + 2\kappa \widetilde{\kappa} \frac{\partial \kappa}{\partial \theta} \right) d\theta$$

$$= -2\lambda \int_0^{2\pi} \kappa \widetilde{\kappa} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta - \lambda \int_0^{2\pi} \kappa^2 \frac{\partial \widetilde{\kappa}}{\partial \theta} \frac{\partial \kappa}{\partial \theta} d\theta$$

$$\leq 2\Lambda M \widetilde{M} \int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + \frac{\Lambda}{2} \left[\int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + 2\pi M^4 \widetilde{M}^2 \right],$$

where the constant $\widetilde{M}_i$ is the bound of $|\frac{\partial^i \widetilde{\kappa}}{\partial \theta^i}|$ used in the proof of Lemma 2.3. There exist two constants C_1 and C_2 independent of time such that

$$\frac{1}{2} \frac{d}{dt} \int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta \leq C_1 \int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + C_2.$$

Noticing that

$$\frac{d}{dt} \int_0^{2\pi} \ln \kappa \, d\theta = \int_0^{2\pi} \kappa \left(\frac{\partial^2 \kappa}{\partial \theta^2} - \lambda \frac{\partial^2 \widetilde{\kappa}}{\partial \theta^2} + \kappa - \lambda \widetilde{\kappa} \right) d\theta$$

$$\leq -\int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + \Lambda M \widetilde{M}_2 \cdot 2\pi + M^2 \cdot 2\pi,$$

one obtains

$$\frac{d}{dt} \left[\int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + (2C_1 + 1) \int_0^{2\pi} \ln \kappa \, d\theta \right]$$

$$
\begin{aligned}
&\le -\int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + 2C_2 + 2\pi(2C_1+1)(\Lambda M \widetilde{M}_2 + M^2) \\
&\le -\left[\int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + (2C_1+1)\int_0^{2\pi} \ln\kappa\, d\theta\right] \\
&\quad + (2C_1+1)\cdot 2\pi \ln M + 2C_2 + 2\pi(2C_1+1)(\Lambda M \widetilde{M}_2 + M^2).
\end{aligned}
$$

Integrating the above inequality can give us that

$$
\int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta + (2C_1+1)\int_0^{2\pi} \ln\kappa\, d\theta \le C_3,
$$

for some constant C_3 independent of time. So

$$
\begin{aligned}
\int_0^{2\pi} \left(\frac{\partial \kappa}{\partial \theta}\right)^2 d\theta &\le C_3 - (2C_1+1)\int_0^{2\pi} \ln\kappa\, d\theta \\
&= C_3 + (2C_1+1)\int_0^{2\pi} \ln\rho\, d\theta \\
&\le C_3 + (2C_1+1)\int_0^{2\pi} \rho\, d\theta \\
&= C_3 + (2C_1+1)L(t) \\
&\le C_3 + (2C_1+1)C := C_4,
\end{aligned}
$$

where ρ is the radius of curvature and C, given by Inequality (2.4), is a constant dependent on the initial curve X_0. $\square$

The estimate of $\int_0^{2\pi} \left(\frac{\partial\kappa}{\partial\theta}\right)^2 d\theta$ is a key step to the proof of convergence for the flow (2.1). Let θ_1, θ_2 be two arbitrary points in $[0, 2\pi]$. It follows from Lemma 3.1,

$$
|\kappa(\theta_1,t) - \kappa(\theta_2,t)| = \left|\int_{\theta_1}^{\theta_2} \frac{\partial\kappa}{\partial\theta} d\theta\right| \le \sqrt{|\theta_1-\theta_2|}\sqrt{\int_0^{2\pi}\left(\frac{\partial\kappa}{\partial\theta}\right)^2 d\theta} \le \sqrt{C_4}|\theta_1-\theta_2|^{\frac{1}{2}},
$$

where $C_4 = C_3 + (2C_1+1)C$ is a constant independent of time. So $\kappa(\cdot,t)$ is equicontinuous. Because the curvature is also uniformly bounded by Lemma 2.2 ($\kappa > 0$) and Lemma 2.4 ($\kappa \le M$). Ascoli–Arzelà Theorem tells us that there is a convergent subsequence of $\kappa(\cdot,t)$. Furthermore one has the following convergence.

Theorem 3.2. *Under the flow (2.1), the curvature of the evolving curve has a limit*

$$\lim_{t\to\infty}\kappa(\theta,t)=\sqrt{\frac{\widetilde{A}}{A}}\widetilde{\kappa}(\theta). \tag{3.1}$$

Proof. Let $\kappa(\theta,t_i)$ be a convergent subsequence of the curvature, where $t_i\to\infty$ as $i\to\infty$. Denote by $\kappa_\infty(\theta)$ the limit of $\kappa(\theta,t_i)$. Compute that

$$\begin{aligned}
\frac{d^2\mathfrak{L}}{dt^2}&=\frac{d}{dt}\left(-\int_0^{2\pi}\frac{\kappa}{\widetilde{\kappa}}d\theta+\frac{2\pi}{\int_0^{2\pi}\frac{\widetilde{\kappa}}{\kappa}d\theta}\right)\\
&=-\int_0^{2\pi}\frac{\kappa^2}{\widetilde{\kappa}}\left(\frac{\partial^2\kappa}{\partial\theta^2}-\lambda\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2}+\kappa-\lambda\widetilde{\kappa}\right)d\theta-\frac{\lambda^2}{2\pi}\int_0^{2\pi}-\widetilde{\kappa}\left(\frac{\partial^2\kappa}{\partial\theta^2}-\lambda\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2}+\kappa-\lambda\widetilde{\kappa}\right)d\theta\\
&=2\int_0^{2\pi}\frac{\kappa}{\widetilde{\kappa}}\left(\frac{\partial\kappa}{\partial\theta}\right)^2d\theta-\int_0^{2\pi}\frac{\kappa^2}{\widetilde{\kappa}^2}\frac{\partial\kappa}{\partial\theta}\frac{\partial\widetilde{\kappa}}{\partial\theta}d\theta+\lambda\int_0^{2\pi}\frac{\kappa^2}{\widetilde{\kappa}}\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2}d\theta-\int_0^{2\pi}\frac{\kappa^3}{\widetilde{\kappa}}d\theta\\
&\quad+\lambda\int_0^{2\pi}\kappa^2d\theta+\frac{\lambda^2}{2\pi}\int_0^{2\pi}\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2}\kappa d\theta-\frac{\lambda^3}{2\pi}\int_0^{2\pi}\widetilde{\kappa}\frac{\partial^2\widetilde{\kappa}}{\partial\theta^2}d\theta\\
&\quad+\frac{\lambda^2}{2\pi}\int_0^{2\pi}\widetilde{\kappa}\kappa d\theta-\frac{\lambda^3}{2\pi}\int_0^{2\pi}\widetilde{\kappa}^2d\theta.
\end{aligned}$$

Since

$$2\int_0^{2\pi}\frac{\kappa}{\widetilde{\kappa}}\left(\frac{\partial\kappa}{\partial\theta}\right)^2d\theta\le 2\frac{M}{\widetilde{m}}C_4,$$

$$-\int_0^{2\pi}\frac{\kappa^2}{\widetilde{\kappa}^2}\frac{\partial\kappa}{\partial\theta}\frac{\partial\widetilde{\kappa}}{\partial\theta}d\theta\le\frac{M^2\widetilde{M}_1}{\widetilde{m}^2}\sqrt{2\pi}\sqrt{C_4},$$

and all the other terms of $\frac{d^2\mathfrak{L}}{dt^2}$ have bounds independent of time, there is a constant C_5 depending on the initial curve such that

$$\left|\frac{d^2\mathfrak{L}}{dt^2}\right|\le C_5.$$

Noticing that $\frac{d\mathfrak{L}}{dt}$ is nonpositive, one can integrate of it to obtain

$$\int_0^\infty \frac{d\mathfrak{L}}{dt}dt = \int_0^{2\pi} \frac{\widetilde{p}}{\kappa_\infty}d\theta - \mathfrak{L}(0) > -\mathfrak{L}(0).$$

It follows from (i) $\frac{d\mathfrak{L}}{dt} \le 0$, (ii) the boundedness of $\left|\frac{d^2\mathfrak{L}}{dt^2}\right|$ and (iii) the lower bound of $\int_0^\infty \frac{d\mathfrak{L}}{dt}dt$, there exists a limit immediately

$$\lim_{t\to\infty} \frac{d\mathfrak{L}}{dt} = 0.$$

So the limit of the subsequence $\kappa(\theta, t_i)$ satisfies that

$$-\int_0^{2\pi} \frac{\kappa_\infty}{\widetilde{\kappa}}d\theta + \frac{2\pi}{\int_0^{2\pi} \frac{\widetilde{\kappa}}{\kappa_\infty}d\theta} = 0.$$

The equality of Cauchy–Schwarz Inequality tells us that $\kappa_\infty/\widetilde{\kappa}$ is a constant. Because the ratio of the area bounded by the limiting curve and that by the target curve is $A/\widetilde{A}$, one has the limit (3.1) for subsequence $\{\kappa(\theta, t_i)\}$.

It is proved that every convergent subsequence of $\kappa(\cdot, t)$ tends to the same limit. So the curvature itself converges and Equation (3.1) holds. □

3.2. C^∞-convergence

In the following, we shall show that all the derivatives of the curvature are uniformly bounded. Combining the convergence of κ (3.1), one obtains that the curvature converges in the C^∞ sense.

Because $\kappa(\theta, t)$ converges as $t \to \infty$, there exists a positive constant m independent of time such that

$$\kappa(\theta, t) \ge m > 0 \tag{3.2}$$

for all $\theta \in [0, 2\pi]$ and $t \in [0, \infty)$. So the evolution equation of the curvature (2.4) is always uniformly parabolic. This is an important condition to show the C^∞-convergence of κ. The method in Lin–Tsai's paper [20] can be applied here. From now on, subindex stands for partial derivatives, such as $f_t = \frac{\partial f}{\partial t}$, $f_\theta = \frac{\partial f}{\partial \theta}$, $f_{\theta\theta} = \frac{\partial^2 f}{\partial \theta^2}$, $\cdots$.

Lemma 3.3. *There is a positive constant M_1 dependent on X_0 and $\widetilde{X}$ such that*

$$|\kappa_\theta| \le M_1. \tag{3.3}$$

Proof. Define $\varphi = \kappa_\theta + \frac{\mu}{2}\kappa^2$. Then $\varphi_\theta = \kappa_{\theta\theta} + \mu\kappa\kappa_\theta$ and $\varphi_{\theta\theta} = \kappa_{\theta\theta\theta} + \mu(\kappa_\theta)^2 + \mu\kappa\kappa_{\theta\theta}$. Using the evolution equation of κ, one can compute that

$$\begin{aligned}\varphi_t = {} & \kappa^2\varphi_{\theta\theta} + 2\kappa\kappa_\theta\varphi_\theta - 3\mu\kappa^2\varphi^2 + 3\mu^2\kappa^4\varphi + 3\kappa^2\varphi - 2\lambda\widetilde{\kappa}_{\theta\theta}\kappa\varphi - 2\lambda\widetilde{\kappa}\kappa\varphi \\ & - \frac{3}{4}\mu^3\kappa^6 - \frac{1}{2}\mu\kappa^4 - \lambda\widetilde{\kappa}_{\theta\theta\theta}\kappa^2 - \lambda\widetilde{\kappa}_\theta\kappa^2.\end{aligned}$$

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Set $\mu = 1$. Since $\varphi(\theta_*, t) > 0$ at the point θ_* where $\varphi(\cdot, t)$ attains its maximum value with respect to θ, one has estimate at this point (θ_*, t)

$$\varphi_t \le \kappa^2 \varphi_{\theta\theta} + 2\kappa\kappa_\theta\varphi_\theta - 3m^2\varphi^2 + (3M^4 + M^2 + 2\widetilde{M}_2\Lambda M + 2\Lambda\widetilde{M}M)\varphi$$
$$+ \frac{3}{2}M^4 + \Lambda\widetilde{M}_3M^2 + \Lambda\widetilde{M}_1M^2.$$

Once $\varphi \ge \frac{-C_1+\sqrt{C_1^2+12m^2C_2}}{-6m^2}$, $\varphi_{\max}(t) := \max\{\varphi(\theta, t)|\theta \in [0, 2\pi]\}$ is decreasing as time goes, where $C_1 = 3M^4 + M^2 + 2\Lambda\widetilde{M}_2M + 2\Lambda\widetilde{M}M$ and $C_2 = \frac{3}{2}M^4 + \Lambda\widetilde{M}_3M^2 + \Lambda\widetilde{M}_1M^2$. The maximum principle tells us that there exists a constant

$$\Phi_1 = \max\left\{\varphi_{\max}(0), \frac{-C_1 + \sqrt{C_1^2 + 12m^2C_2}}{-6m^2}\right\}$$

such that $\varphi \le \Phi_1$ for all $(\theta, t) \in [0, 2\pi] \times [0, +\infty)$.

Set $\mu = -1$. If $\varphi(\theta_*, t) \le 0$ at the point θ_* where $\varphi(\cdot, t)$ attains its minimum value with respect to θ, then

$$\varphi_t \ge \kappa^2 \varphi_{\theta\theta} + 2\kappa\kappa_\theta\varphi_\theta + 3m^2\varphi^2 - (3M^4 + M^2 + 2\Lambda M\widetilde{M}_2 + 2\Lambda\widetilde{M}M)\varphi$$
$$-\frac{3}{2}M^4 - \Lambda M^2\widetilde{M}_3 - \Lambda M^2\widetilde{M}_1.$$

A similar argument implies that there exists a constant

$$\Phi_2 = \min\left\{0,\ \varphi_{\min}(0),\ \frac{-C_1 - \sqrt{C_1^2 + 12m^2C_2}}{6m^2}\right\}$$

such that $\varphi \ge \Phi_2$ for all $(\theta, t) \in [0, 2\pi] \times [0, +\infty)$.

Combining the above two cases, one obtains a uniform bound of κ_θ in (3.3). □

Lemma 3.4. *There is a positive constant M_2 dependent on X_0 and $\widetilde{X}$ such that*

$$|\kappa_{\theta\theta}| \le M_2. \tag{3.4}$$

Proof. The proof of this lemma is similar to that of Lemma 3.3. Considering the function $\varphi = \kappa_{\theta\theta} + \frac{\mu}{2}(\kappa_\theta)^2$, one has the evolution equation

$$\varphi_t = \kappa^2\varphi_{\theta\theta} + 4\kappa\kappa_\theta\varphi_\theta - (\mu\kappa - 2)\kappa\varphi^2 + \text{lower order terms of } \varphi.$$

Choosing $\mu = \frac{2M+1}{m^2}$, one has $-(\mu\kappa - 2)\kappa \le -1$. So the above equation implies that

$$\varphi_t \le \kappa^2\varphi_{\theta\theta} + 4\kappa\kappa_\theta\varphi_\theta - \varphi^2 + f_1(\lambda, \kappa, \kappa_\theta)\varphi + f_2(\lambda, \kappa, \kappa_\theta),$$

where f_1 and f_2 are polynomials with bounded and smooth coefficients. Once φ is large enough, $\varphi_{\max}(t)$ is decreasing as time goes. So $\kappa_{\theta\theta}$ must stay bounded above on $[0, 2\pi] \times [0, \infty)$. The proof of the lower bound is similar. $\square$

Since $|\kappa_\theta|$ and $|\kappa_{\theta\theta}|$ are uniformly bounded, there is a convergent subsequence $\kappa_\theta(\theta, t_i)$ as t_i tends to infinity. Theorem 3.2 tells us that $\kappa(\theta, t)$ converges to $\sqrt{\widetilde{A}/A}\widetilde{\kappa}(\theta)$ when $t \to \infty$. So every convergent subsequence of κ_θ tends to $\sqrt{\widetilde{A}/A}\widetilde{\kappa}_\theta$. One obtains that:

Corollary 3.5. *The derivative κ_θ converges to $\sqrt{\widetilde{A}/A}\widetilde{\kappa}_\theta$ as time tends to infinity.*

Lemma 3.6. *There is a positive constant M_i dependent on X_0 and $\widetilde{X}$ such that the ith derivative of κ with respect to θ satisfies*

$$\left|\kappa^{(i)}\right| \le M_i, \quad i = 3, 4, \cdots . \tag{3.5}$$

Proof. The induction method will be used. Suppose $\kappa_\theta, \kappa_{\theta\theta}, \cdots, \kappa^{(n-1)}$ are all bounded on $[0, 2\pi] \times [0, \infty)$ for $n \ge 3$. Choosing a similar function $\varphi = \kappa^{(n)} + \frac{\mu}{2}(\kappa^{(n-1)})^2$, one has its evolution equation

$$\begin{aligned}\varphi_t = \kappa^2\varphi_{\theta\theta} + (2n\kappa\kappa_\theta - \mu\kappa^2\kappa^{(n-1)})\varphi_\theta - \mu\kappa^2\varphi^2 \\ + P_1(\mu, \kappa, \kappa_\theta, \kappa_{\theta\theta}, \cdots, \kappa^{(n-1)})\varphi + P_2(\mu, \kappa, \kappa_\theta, \kappa_{\theta\theta}, \cdots, \kappa^{(n-1)}),\end{aligned}$$

where P_1 and P_2 are polynomials with of κ, $\widetilde{\kappa}$ and their derivatives. Noticing that $0 < m \le \kappa \le M$, one can choose proper μ and follow the same argument in the proof of Lemma 3.3 or Lemma 3.4 to show that φ and $\kappa^{(n)}$ are also uniformly bounded on $[0, 2\pi] \times [0, \infty)$. $\square$

Using the evolution equation of the curvature, one obtains that all the derivatives of the κ are uniformly bounded. Since κ converges as $t \to \infty$, one has the C^∞ convergence of κ:

Corollary 3.7. *Under the flow (2.1), one has the convergence of the derivatives*

$$\lim_{t\to\infty} \frac{\partial^n \kappa}{\partial\theta^n} = \sqrt{\frac{\widetilde{A}}{A}} \frac{\partial^n \widetilde{\kappa}}{\partial\theta^n}, \quad n = 2, 3, 4, \cdots .$$

4. Convergence of the evolving curve

As is well known, the curvature uniquely determines a curve up to transformations using the Euclidean group. Although the curvature of the evolving curve under the flow (2.1) is proved to converge a limit (Equation (3.1)) as time goes to infinity, the study of this flow is not quite complete. To show that the flow effectively deforms X_0 into the target curve, one needs to prove that the evolving curve $X(\cdot, t)$ can not escape to infinity or oscillate indefinitely. To fulfill the proof of Main Theorem, we shall show that the evolving curve converges to a fixed limiting curve X_∞ (congruent to $\sqrt{A/\widetilde{A}}\,\widetilde{X}$) as time tends to infinity. The ingredient is to prove the convergence of the support function by considering the movement of the Steiner point (see Definition 4.1).

In order to better understand the convergence of the flow (2.1), we first review a special case, i.e. Gage's area-preserving flow. Under his flow, one has the evolution equation of the curvature

$$\kappa_t = \kappa^2(\kappa_{\theta\theta} + \kappa - \frac{2\pi}{L}).$$

Furthermore, one obtains that

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}(\kappa_\theta)^2 d\theta = -\int_0^{2\pi}\kappa^2(\kappa_{\theta\theta})^2 d\theta + 2\int_0^{2\pi}\kappa(\kappa_\theta)^2(\kappa - \frac{2\pi}{L})d\theta + \int_0^{2\pi}\kappa^2(\kappa_\theta)^2 d\theta.$$

Since $\lim_{t\to\infty}\kappa = \sqrt{\frac{\pi}{A}}$ (a constant), one has an inequality $\int_0^{2\pi}(\kappa_{\theta\theta})^2 d\theta \geq (4-\varepsilon)\int_0^{2\pi}(\kappa_\theta)^2 d\theta$ (Lemma 5.7.9 in [14]) for $\varepsilon > 0$ is small and t is large enough. Thus there exists T_0 such that

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}(\kappa_\theta)^2 d\theta \leq \Big[-(\frac{\pi}{A} - \varepsilon)(4-\varepsilon) + 3\Big(\frac{\pi}{A} + \varepsilon\Big)\Big]\int_0^{2\pi}(\kappa_\theta)^2 d\theta,$$

when $t > T_0$. Choosing ε small enough, one has the exponential decay of $\int_0^{2\pi}(\kappa_\theta)^2 d\theta$. Because

$$\begin{aligned}
&\int_0^{2\pi}(\kappa - \frac{2\pi}{L})^2 d\theta \\
&= \int_0^{L}(\kappa - \frac{2\pi}{L})^2\kappa ds \leq M\int_0^{L}(\kappa - \frac{2\pi}{L})^2 ds \\
&\overset{\xi=(2\pi s)/L}{=\!=\!=\!=} M\int_0^{2\pi}(\kappa - \frac{2\pi}{L})^2\frac{L}{2\pi}d\xi \leq \frac{L}{2\pi}M\int_0^{2\pi}(\kappa_\xi)^2 d\xi \\
&= \frac{L}{2\pi}M\int_0^{L}(\kappa_s)^2\frac{L^2}{4\pi^2}\frac{2\pi}{L}ds \leq \frac{L^2}{4\pi^2}M^2\int_0^{2\pi}(\kappa_\theta)^2 d\theta \\
&\leq \frac{L_0^2}{4\pi^2}M^2\int_0^{2\pi}(\kappa_\theta)^2 d\theta,
\end{aligned}$$

Sobolev Inequality tells us that the speed of the flow $|\kappa - \frac{2\pi}{L}|$ also exponentially decays. So the evolving curve converges to a fixed limiting curve.

Because the curvature under the flow (2.1) does not always converge to a constant, the above technique is not applicable to show that the speed of the flow $|\kappa - \lambda\widetilde{\kappa}|$ exponentially decays. Thanks to the work by Chou–Zhu [7] (or see Chapter 3 of [9]), one can solve this problem by considering the evolution of Steiner point and the support function for $X(\cdot, t)$.

Definition 4.1 *(see [12]).* Let $X(\theta)$ be a convex curve in the plane parameterized by its tangent angle θ. The Steiner point of this curve is defined by

$$S = \frac{1}{2\pi}\int_0^{2\pi} X(\theta)d\theta.$$

Denote by $p := -\langle X, N\rangle$ the support function with respect to the origin. The Steiner point can be expressed as

$$S = \frac{1}{2\pi}\int_0^{2\pi}(p_\theta T - pN)d\theta = -\frac{1}{\pi}\int_0^{2\pi} pNd\theta$$
$$= \frac{1}{\pi}\left(\int_0^{2\pi} p\sin\theta d\theta,\ -\int_0^{2\pi} p\cos\theta d\theta\right).$$

Remark 4.2. Since the Steiner point is mean value of X, it lies in the domain bounded by this convex curve. One can choose another base point to define a new support function of X, but the Steiner point is irrelevant to the choice of base point. Inequality 2.4 tells us that the length of the evolving curve is bounded by a constant independent of time. The width ($< \frac{L(t)}{2}$) of the region bounded by $X(\cdot, t)$ also has an upper bound independent of time. So the evolving curve under the flow (2.1) can not escape to infinity if the Steiner point $S(t)$ converges to a fixed point.

Now we shift the support function to $\widehat{p} = p + \langle S(t), N\rangle$ and consider a function $I(t)$ of $\widehat{p}$:

$$I(t) = \int_0^{2\pi} \frac{\widehat{p}}{\widetilde{\kappa}}d\theta.$$

$I(t)$ has a lower bound since $I(t) \geq \frac{1}{\widetilde{M}}\int_0^{2\pi}\widehat{p}d\theta = \frac{L(t)}{\widetilde{M}} \geq \frac{\sqrt{4\pi A}}{\widetilde{M}}$. One can compute the evolution of $I(t)$ to obtain

$$\frac{dI}{dt} = \int_0^{2\pi}\frac{p_t}{\widetilde{\kappa}}d\theta + \int_0^{2\pi}\frac{1}{\widetilde{\kappa}}\left\langle\frac{dS}{dt}, N\right\rangle d\theta$$
$$= \int_0^{2\pi}\frac{\lambda\widetilde{\kappa} - \kappa}{\widetilde{\kappa}}d\theta + \left\langle\frac{dS}{dt}, \int_0^{2\pi}\frac{1}{\widetilde{\kappa}}Nd\theta\right\rangle$$
$$= \int_0^{L}\frac{(\lambda\widetilde{\kappa} - \kappa)\kappa}{\widetilde{\kappa}}ds + \left\langle\frac{dS}{dt}, 0\right\rangle$$

$$
\begin{aligned}
&= -\int_0^L \frac{(\lambda\widetilde{\kappa}-\kappa)^2}{\widetilde{\kappa}} ds + \lambda \int_0^L (\lambda\widetilde{\kappa}-\kappa) ds \\
&= -\int_0^{2\pi} \frac{(\lambda\widetilde{\kappa}-\kappa)^2}{\widetilde{\kappa}\kappa} d\theta + 0 = -\int_0^{2\pi} \frac{(p_t)^2}{\widetilde{\kappa}\kappa} d\theta,
\end{aligned}
$$

where we have used the closing condition of convex curves $\int_0^{2\pi} \frac{1}{\kappa} N d\theta = 0$ and the area-preserving property of the flow. $I(t)$ is decreasing and has a lower bound, so it converges as time tends to infinity. From the evolution of I, one obtains

$$
\int_0^{2\pi} (p_t)^2 d\theta \leq -\widetilde{M} M \frac{dI}{dt}. \tag{4.1}
$$

Lemma 4.3. *Define* $\widetilde{S} = \int_0^{2\pi} \frac{p}{\widetilde{\kappa}^2} N d\theta$. $\widetilde{S}(t)$ *converges to a fixed point under the flow (2.1).*

Proof. Because the limit (3.1) implies that $\lim_{t\to\infty} p_t = -\lim_{t\to\infty} (\kappa - \lambda\widetilde{\kappa}) = 0$, there exists $T_0 > 0$ such that $|p_t/\lambda\widetilde{\kappa}| < 1$ and $1 - p_t/\lambda\widetilde{\kappa} \geq \frac{1}{2}$ if $t > T_0$. It follows from the closing condition,

$$
\begin{aligned}
0 &= \int_0^{2\pi} \frac{e^{i\theta}}{\kappa} d\theta = \int_0^{2\pi} \frac{e^{i\theta}}{\lambda\widetilde{\kappa} - p_t} d\theta \\
&= \int_0^{2\pi} \frac{e^{i\theta}}{\lambda\widetilde{\kappa}} \left[1 + \frac{p_t}{\lambda\widetilde{\kappa}} + \left(\frac{p_t}{\lambda\widetilde{\kappa}}\right)^2 + \left(\frac{p_t}{\lambda\widetilde{\kappa}}\right)^3 + \cdots \right] d\theta \\
&= 0 + \int_0^{2\pi} \frac{p_t e^{i\theta}}{\lambda^2\widetilde{\kappa}^2} d\theta + \int_0^{2\pi} \frac{(p_t)^2 e^{i\theta}}{\lambda^3\widetilde{\kappa}^3} \left(1 + \frac{p_t}{\lambda\widetilde{\kappa}} + \cdots\right) d\theta,
\end{aligned}
$$

i.e.,

$$
\int_0^{2\pi} \frac{p_t e^{i\theta}}{\widetilde{\kappa}^2} d\theta = -\frac{1}{\lambda} \int_0^{2\pi} \frac{(p_t)^2 e^{i\theta}}{\widetilde{\kappa}^3} \frac{1}{1 - p_t/\lambda\widetilde{\kappa}} d\theta.
$$

Computing the module of both sides can give us that

$$
\left[\left(\int_0^{2\pi} \frac{p_t \cos\theta}{\widetilde{\kappa}^2} d\theta \right)^2 + \left(\int_0^{2\pi} \frac{p_t \sin\theta}{\widetilde{\kappa}^2} d\theta \right)^2 \right]^{\frac{1}{2}}
$$

$$\leq \frac{2}{\lambda}\left[\left(\int_0^{2\pi}\frac{(p_t)^2|\cos\theta|}{\widetilde{\kappa}^3}d\theta\right)^2+\left(\int_0^{2\pi}\frac{(p_t)^2|\sin\theta|}{\widetilde{\kappa}^3}d\theta\right)^2\right]^{\frac{1}{2}}$$

$$\leq \frac{2\sqrt{2}}{\lambda\widetilde{m}^3}\int_0^{2\pi}(p_t)^2 d\theta.$$

It follows from the above estimate and Inequality (4.1),

$$\left|\frac{d\widetilde{S}}{dt}\right| = \left|\int_0^{2\pi}\frac{p_t}{\widetilde{\kappa}^2}N d\theta\right|$$

$$= \left[\left(\int_0^{2\pi}\frac{p_t}{\widetilde{\kappa}^2}\sin\theta d\theta\right)^2+\left(\int_0^{2\pi}\frac{p_t}{\widetilde{\kappa}^2}\cos\theta d\theta\right)^2\right]^{\frac{1}{2}}$$

$$\leq \frac{2\sqrt{2}}{\lambda\widetilde{m}^3}\int_0^{2\pi}(p_t)^2 d\theta \leq -\frac{2\sqrt{2}\widetilde{M}M}{\lambda\widetilde{m}^3}\frac{dI}{dt}.$$

Let t' and t'' be large enough positive time. Since

$$|\widetilde{S}(t')-\widetilde{S}(t'')| = \left|\int_{t'}^{t''}\frac{d\widetilde{S}}{dt}dt\right| \leq \frac{2\sqrt{2}\widetilde{M}M}{\lambda\widetilde{m}^3}\left|\int_{t'}^{t''}\frac{dI}{dt}dt\right| = \frac{2\sqrt{2}\widetilde{M}M}{\lambda\widetilde{m}^3}|I(t')-I(t'')|$$

and $I(t)$ converges as $t\to\infty$, $\widetilde{S}(t)$ converges as well. □

By its definition, $\widehat{p} = -\langle X - S(t), N\rangle$ means the distance from Steiner point to the tangent of X at θ. So

$$0 < \widehat{p} < \frac{L}{2} \leq \frac{C}{2}. \tag{4.2}$$

Rotate the xy-coordinate system such that the Steiner point can be expressed as $S(t) = (S_1(t), 0)$, where $S_1(t) \geq 0$. Noticing that

$$\widetilde{S} = \int_0^{2\pi}\frac{\widehat{p}}{\widetilde{\kappa}^2}N d\theta - \int_0^{2\pi}\frac{\langle S, N\rangle}{\widetilde{\kappa}^2}N d\theta$$

$$= \int_0^{2\pi}\frac{\widehat{p}}{\widetilde{\kappa}^2}N d\theta - \int_0^{2\pi}\frac{(S_1\sin^2\theta, -S_1\sin\theta\cos\theta)}{\widetilde{\kappa}^2}d\theta,$$

one obtains that S_1 (i.e., $|S|$) has a bound independent of time. So $|X|$ has a bound

$$|X(\cdot,t)| \le |X(\cdot,t) - S(t)| + |S(t)| < \frac{L(t)}{2} + |S(t)|.$$

Thus the evolving curve can not escape to infinity in the plane. To prevent the case of indefinite oscillation, one needs to show that the support function p of the evolving curve also converges.

Lemma 4.4. *The support function of the evolving curve converges as time tends to infinity. So the evolving curve itself converges.*

Proof. Because $|p| \le |X|$ for all θ and all t, the support function is also bounded by a constant independent of time. Therefore, $p_{\theta\theta} = 1/\kappa - p$ has a bound independent of time and so is $p_\theta = \int_0^\theta p_{\theta\theta}(\phi,t)d\phi + p_{\theta\theta}(0,t)$. There is a convergent subsequence of $p(\cdot,t)$.

Every convergent subsequence of the support function determines a family of convergent evolving curve. Suppose $X(\theta,t')$ and $X(\theta,t'')$ converge to different $X_1(\theta)$ and $X_2(\theta)$ respectively. Denote by p_i the support function of X_i, $i=1,2$. Since $(\kappa_i(\theta,t) - \lambda(t)\widetilde{\kappa}(\theta)) \to 0$ as $t \to \infty$ for every tangent angle θ, X_1 and X_2 differs by a translation. So

$$p_1 - p_2 = l_1\cos\theta + l_2\sin\theta$$

for some l_1 and l_2. Since $\frac{\sin\theta}{\widetilde{\kappa}}$ and $\frac{\cos\theta}{\widetilde{\kappa}}$ are linearly independent, the Cauchy–Schwarz inequality implies

$$\left[\int_0^{2\pi} \frac{\cos^2\theta}{\widetilde{\kappa}^2}d\theta \int_0^{2\pi} \frac{\sin^2\theta}{\widetilde{\kappa}^2}d\theta\right]^{1/2} > \int_0^{2\pi} \frac{\sin\theta\cos\theta}{\widetilde{\kappa}^2}d\theta.$$

Thus, from the following identity

$$\begin{aligned}&\int_0^{2\pi} (l_1\cos\theta + l_2\sin\theta)\frac{N}{\widetilde{\kappa}^2}d\theta \\ &= \int_0^{2\pi} p_1\frac{N}{\widetilde{\kappa}^2}d\theta - \int_0^{2\pi} p_2\frac{N}{\widetilde{\kappa}^2}d\theta = \lim_{t'\to\infty}\widetilde{S}(t') - \lim_{t''\to\infty}\widetilde{S}(t'') = 0,\end{aligned}$$

one can conclude $l_1 = l_2 = 0$, i.e. $p_1 \equiv p_2$. Since the support function uniquely determines the curve, it is shown that the evolving curve converges. □

The proof of the Main Theorem is a combination of Lemma 2.4, Theorem 2.5, Theorem 3.2, Corollary 3.7, Lemma 4.3 and Lemma 4.4.

In 2001, Andrews [2] introduced Minkowski differential geometry and considered a volume-preserving anisotropic mean curvature flow for convex surfaces. Andrews' model (1) can deform a convex surface into its Wulff shape. This work may be a good reference to construct a flow which deforms a convex surface to another one.

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It is similar to Gage's area-preserving flow for convex curves [11], the flow (1.5) can not evolve generic embedded curves to convex curves. We end this paper by asking a general case of Yau's problem: Whether one can define a parabolic curvature flow to evolve one closed and embedded curve X_0 to another one?

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