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Blow-up phenomena of semilinear wave equations and their weakly coupled systems

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Abstract

In this paper we consider the wave equations with power type nonlinearities including time-derivatives of unknown functions and their weakly coupled systems. We propose a framework of test function methods and give a simple proof of the derivation of sharp upper bounds for lifespan of solutions to nonlinear wave equations and their systems. We point out that for respective critical cases, we use a family of self-similar solutions to the linear wave equation including Gauss's hypergeometric functions, which are originally introduced by Zhou [59]. We emphasize that our framework does not require the pointwise positivity of the initial data even in the high dimensional case $N \geq 4$. Moreover, we find a new (p, q)-curve for the system $\partial_t^2 u - \Delta u = |v|^q$, $\partial_t^2 v - \Delta v = |\partial_t u|^p$ with lifespan estimates for small solutions in a new region.

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Contents

1. Introduction

In this paper we consider the semilinear wave equations with power type nonlinearities including derivatives of unknown functions and their weakly coupled systems

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = G\big(u(x,t), \partial_t u(x,t)\big), & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = \varepsilon f(x), & x \in \mathbb{R}^N, \\ \partial_t u(x,0) = \varepsilon g(x), & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = G_1\big(v(x,t), \partial_t v(x,t)\big), & (x,t) \in \mathbb{R}^N \times (0,T), \\ \partial_t^2 v(x,t) - \Delta v(x,t) = G_2\big(u(x,t), \partial_t u(x,t)\big), & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = \varepsilon f_1(x), \quad v(x,0) = \varepsilon f_2(x), & x \in \mathbb{R}^N, \\ \partial_t u(x,0) = \varepsilon g_1(x), \quad \partial_t v(x,0) = \varepsilon g_2(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $\partial_t = \partial/\partial t$, $\Delta = \sum_{j=1}^N \partial^2/\partial x_j^2$ and $T > 0$. The nonlinear terms G, G_1 and G_2 are nonnegative and smooth (specified later), and u and v are unknown functions. Throughout this paper the initial values (f, g), (f_1, g_1) and (f_2, g_2) satisfy the following condition

$$(f, g) \in C_c^\infty(\mathbb{R}^N), \quad I[g] := \int_{\mathbb{R}^N} g(x)\, dx > 0. \tag{1.3}$$

Finally, the parameter $\varepsilon > 0$ describes the smallness of the corresponding initial value. The aim of the present paper is to give a simple way to derive the corresponding sharp lifespan estimates of blowup solutions to (1.1) and (1.2) via a test function method.

The problem of blowup phenomena of (1.1) has a long history. The study of this kind problem with $G(u) = |u|^p$ has been started by John [26]. He proved the following fact when $N = 3$.

- If $1 < p < 1 + \sqrt{2}$, then the solution of (1.1) blows up in finite time for non-zero initial value.
- If $p > 1 + \sqrt{2}$, then there exists a global solution with a small initial value.

After that Strauss [51] conjectured that the threshold for dividing blowup phenomena in finite time for arbitrary "positive" small initial value and global existence of small solutions is given by

$$p_S(N) = \sup\{p > 1\,;\ \gamma_S(N, p) > 0\}, \quad \gamma_S(N, p) = 2 + (N + 1)p - (N - 1)p^2,$$

which provides $p_S(3) = 1 + \sqrt{2}$ as proved in John [26]. In the case $N = 1$, Kato [29] proved blowup phenomena in finite time for arbitrary "positive" small initial value with $1 < p < p_S(1) = \infty$. There are many subsequent papers dealing with blowup phenomena. Then by the contributions until Yordanov–Zhang [58] and Zhou [63], the complete picture of blowup phenomena and existence of global solutions are clarified including the critical situation $p = p_S(N)$ (see also Glassey [13,14], Sideris [50], Schaeffer [47], Rammaha [46], Georgiev–Lindblad–Sogge [11] and Lai–Zhou [36]).

The lifespan of blowup solutions to (1.1) has been intensively considered. Here we refer Lindblad [40], Zhou [59–61], Lindblad–Sogge [41], Di Pomponio–Georgiev [8], Takamura–Wakasa [54]. In view of the previous works listed above, the precise behavior of lifespan of small solutions with respect to the parameter $\varepsilon > 0$ sufficiently small:

$$\mathrm{LifeSpan}(u) \approx \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma_S(N,p)}} & \text{if } 1 < p < p_S(N), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_S(N). \end{cases}$$

An alternative proof of lifespan estimate with critical case $p = p_S(N)$ with the use of Gauss's hypergeometric function can be found in Zhou [61] and Zhou–Han [65].

Similar problem for (1.1) with $G = |\partial_t u|^p$ can be found (see e.g., John [27], Sideris [49], Masuda [43], Schaeffer [48] Rammaha [45], Agemi [1], Hidano–Tsutaya [17] Tzvetkov [55], Zhou [62] and Hidano–Wang–Yokoyama [18]). The complete picture of the blowup phenomena for small solutions can be summarized as follows:

$$\mathrm{LifeSpan}(u) \approx \begin{cases} C\varepsilon^{-(\frac{1}{p-1} - \frac{N-1}{2})^{-1}} & \text{if } 1 < p < \frac{N+1}{N-1}, \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = \frac{N+1}{N-1}, \\ \infty & \text{if } \frac{N+1}{N-1} < p < \frac{N}{N-2}. \end{cases}$$

We should remark that the global existence of small solutions to (1.1) with $G = |\partial_t u|^p$ is only proved under the initial value with radially symmetric in high spatial dimension.

The problem (1.1) with the combined type $G = |u|^q + |\partial_t u|^p$ has been recently discussed by Zhou–Han [64] and Hidano–Wang–Yokoyama [19]. In [19], the borderline of the position of (p, q) for blowup phenomena of small solutions is determined. Surprisingly, in the threshold case they proved the global existence of small solutions. Therefore the situation is completely different from both the cases $G(u) = |u|^p$ and $G = |\partial_t u|^p$.

In this connection, a similar interesting structure has been analyzed for the weakly coupled problem (1.2). In this case the interaction of each unknown functions u and v plays an important role. In particular, the situation depends heavily on the structure of the nonlinear terms G_1 and G_2. This means that even in the special case $G_1 = |v|^p$ and $G_2 = |u|^q$, the position of (p, q) (with describes the effect of nonlinearity) is quite important to discuss the behavior of solutions to (1.2). From this viewpoint, many mathematicians tried to find blowup phenomena and global solutions of small solutions. Here we refer Del Santo–Georgiev–Mitidieri [4], Deng [6], Del Santo–Mitidieri [5], Deng [7], Kubo–Ohta [31], Agemi–Kurokawa–Takamura [2], Kurokawa–Takamura [34], Kurokawa [33], Georgiev–Takamura–Zhou [12], Kurokawa–Takamura–Wakasa [35] for the case $G_1 = |v|^p$ and $G_2 = |u|^q$, Deng [7], Xu [57], Kubo–Kubota–Sunagawa [32] for the case $G_1 = |\partial_t v|^p$ and $G_2 = |\partial_t u|^q$, and Hidano–Yokoyama [20] for the case $G_1 = |v|^q$ and $G_2 = |\partial_t u|^p$.

Recently in Ikeda–Sobajima [24], an alternative test function method for nonlinear heat, Schrödinger, and damped wave equations has been introduced. This argument provides sharp upper bounds of lifespan for respective equations. Of course each equation has a huge amount of previous works (see e.g., Fujita [10], Hayakawa [16], Sugitani [52], Kobayashi–Sirao–Tanaka [28], Li–Nee [38] for the heat equations, Ikeda–Wakasugi [25], Fujiwara–Ozawa [15] for the Schrödinger equations and Li–Zhou [39], Lin–Nishihara–Zhai [42], Ikeda–Ogawa [21] Lai–Zhou [37] for the damped wave equations the references therein). Despite of this, the technique in [24] gives us a short proof of sharp upper bound of lifespan of small solutions to respective equations and in some cases, in particular, this technique enables us to find new estimates for the Schrödinger equation. Moreover, it worth noticing that the initial value is not required the positivity in the point-wise sense even in high dimensional cases $N \geq 4$. Therefore we expect that by introducing the technique in [24] into the analysis of wave equations, one can give an alternative proof of sharp (for many cases) upper bound of lifespan of small solutions and the assumption on the initial value can be weakened.

The first purpose of the present paper is to propose a framework of test function methods for nonlinear wave equations due to [24] and give precise lifespan estimates for problems (1.1) and (1.2) without the assumption of the positivity of initial value in the point-wise sense. The second is to find a new blowup region for the case (1.2) with $G_1 = |v|^q$ and $G_2 = |\partial_t u|^p$ with lifespan estimates for respective cases. Since in the present paper we focus our attention to the framework of test function methods, we do not enter a discussion for existence of solutions to the respective problems. At this point, we refer Sideris [50], Kapitanskii [30], Hidano–Wang–Yokoyama [18], Georgiev–Takamura–Zhou [12], Kubo–Kubota–Sunagawa [32], Hidano–Yokoyama [20] and their references therein.

The present paper is organized as follows: In Section 2, to explain our argument, we demonstrate the short derivation of the upper bound of lifespan for the special case $\partial_t^2 u - \Delta u = |u|^p$ with $1 < p < p_S(N)$. Section 3 is devoted to describe the properties of super-solutions to the wave equations and self-similar solutions to the linear wave equation including Gauss's hypergeometric functions, which is introduced in Zhou [59]. Some useful lemmas indicating our test function method are stated and proved also in Section 3. The main results are stated at the beginning of each Sections 4, 5, 6, 7, 8 and 9. More precisely, in Section 4, we discuss

$$\partial_t^2 u - \Delta u = |u|^p \quad \text{in } \mathbb{R}^N \times (0, T)$$

for the critical case $p = p_S(N)$. Although the sharp lifespan estimate has been proved by Takamura–Wakasa [54] and an alternative proof was given by Zhou–Han [65], we will give a (much) simpler proof. Then we discuss the equation

$$\partial_t^2 u - \Delta u = |\partial_t u|^p \quad \text{in } \mathbb{R}^N \times (0,T)$$

in Section 5, which is related to the Glassey conjecture. The equation with a combined type nonlinearity

$$\partial_t^2 u - \Delta u = |u|^q + |\partial_t u|^p \quad \text{in } \mathbb{R}^N \times (0,T)$$

will be dealt with in Section 6. After that the weakly coupled systems

$$\begin{cases} \partial_t^2 u - \Delta u = a_{11}|v|^{p_{11}} + a_{12}|\partial_t v|^{p_{12}}, & \text{in } \mathbb{R}^N \times (0,T), \\ \partial_t^2 v - \Delta v = a_{21}|u|^{p_{21}} + a_{22}|\partial_t u|^{p_{22}}, & \text{in } \mathbb{R}^N \times (0,T), \end{cases}$$

are considered when $a_{12} = a_{22} = 0$ in Section 7, when $a_{11} = a_{21} = 0$ in Section 8 and when $a_{21} = a_{12} = 0$ in Section 9, respectively. We point out that in Section 9, a new blowup position of (p,q) is found and lifespan estimates including critical situations (on the critical curve) are derived.

2. Alternative proof of blowup of $\partial_t^2 u - \Delta u = |u|^p$ for $1 < p < p_S(N)$

To begin with, we consider the following problem

$$\begin{cases} \partial_t^2 u_\varepsilon - \Delta u_\varepsilon = |u_\varepsilon|^p & \text{in } \mathbb{R}^N \times (0,T), \\ u_\varepsilon(0) = \varepsilon f & \text{in } \mathbb{R}^N, \\ \partial_t u_\varepsilon(0) = \varepsilon g & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

where we assume that f and g satisfies (1.3). In this section we use

$$\gamma_S(N,p) = 2 + (N+1)p - (N-1)p^2, \quad p_S(N) = \sup\{p > 1\ ;\ \gamma_S(N,p) > 0\}.$$

Definition 2.1. Let $f, g \in C_c^\infty(\mathbb{R}^N)$ and $p > 1$. The function

$$u \in C([0,T); H^1(\mathbb{R}^N)) \cap C^1([0,T); L^2(\mathbb{R}^N)) \cap L^p(0,T; L^p(\mathbb{R}^N))$$

is called a weak solution of (2.1) in $(0,T)$ if $u(0) = \varepsilon f$, $\partial_t u(0) = \varepsilon g$ and for every $\Psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$,

$$\varepsilon \int_{\mathbb{R}^N} g(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} |u(x,t)|^p \Psi(x,t)\,dx\,dt$$
$$= \int_0^T \int_{\mathbb{R}^N} \Big(-\partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t) \cdot \nabla \Psi(x,t) \Big)\,dx\,dt.$$

Proposition 2.1. *Let f, g satisfy (1.3) and let u_ε be a weak solution to (2.1) satisfying* $\operatorname{supp} u_\varepsilon \subset \{(x,t) \in \mathbb{R}^N \times [0,T]\,;\ |x| \le r_0 + t\}$ *for* $r_0 = \sup\{|x|\,;\ x \in \operatorname{supp}(f,g)\}$. *Set T_ε as a lifespan of u_ε given by*

$$T_\varepsilon = \sup\{T > 0\,;\ \textit{there exists a solution to (2.1) in } (0,T)\}.$$

If $1 < p < p_S(N)$ (that is, $\gamma_S(N,p) > 0$), then $T_\varepsilon < \infty$. Moreover, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$,

$$T_\varepsilon \le \begin{cases} C\varepsilon^{-\frac{p-1}{2}} & \textit{if } N = 1,\ 1 < p < \infty, \\ C\varepsilon^{-\frac{p-1}{3-p}} & \textit{if } N = 2,\ 1 < p \le 2, \\ C\varepsilon^{-2p(p-1)/\gamma_S(2,p)} & \textit{if } N = 2,\ 2 < p < p_S(2), \\ C\varepsilon^{-2p(p-1)/\gamma_S(N,p)} & \textit{if } N \ge 3,\ 1 < p < p_S(N). \end{cases}$$

Proof. If $T_\varepsilon \le 1$, then the assertion is trivial by choosing ε_0 sufficiently small. Suppose that $T_\varepsilon > 1$ and take $T \in (1, T_\varepsilon)$. Put $\eta \in C^\infty([0,\infty))$ satisfying

$$\eta(s) = \begin{cases} 1 & s < 1/2 \\ \text{decreasing} & 1/2 < s < 1, \\ 0 & s > 1, \end{cases} \qquad \eta_T(s) = \eta(s/T).$$

By the definition of weak solution of (2.1) in $(0,T)$, we see from $\Psi = \eta_T(t)^{2p'}$ (multiplying compactly supported smooth function χ on $\mathbb{R}^N$ satisfying $\chi = 1$ if $x \in B(0, r_0 + T)$)

$$\begin{aligned} & I[g]\varepsilon + \int_0^T \eta_T^{2p'} \int_{\mathbb{R}^N} |u_\varepsilon|^p \, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}^N} \Big(-\partial_t u_\varepsilon \partial_t (\eta_T^{2p'}) + \nabla u_\varepsilon \cdot \nabla(\eta_T^{2p'}) \Big)\, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}^N} u_\varepsilon \partial_t^2 (\eta_T^{2p'})\, dx\, dt \\ &= \frac{2p'}{T^2} \int_0^T \eta_T^{2p'-2} \int_{\mathbb{R}^N} u_\varepsilon \Big(\eta(t/T)\eta''(t/T) + (2p'-1)(\eta'(t/T))^2 \Big)\, dx\, dt \\ &\le \frac{2p'(\|\eta''\|_{L^\infty} + (2p'-1)\|\eta'\|_{L^\infty}^2)}{T^2} \int_0^T \eta_T^{2p'/p} \int_{\mathbb{R}^N} |u_\varepsilon|\, dx\, dt \end{aligned}$$

$$\leq \frac{[2p'(\|\eta''\|_{L^\infty} + (2p'-1)\|\eta'\|_{L^\infty}^2)]^{p'}}{p'T^{2p'}} \int_0^T \int_{B(r_0+t)} dx\,dt + \frac{1}{p}\int_0^T \eta_T^{2p'} \int_{\mathbb{R}^N} |u_\varepsilon|^p\,dx\,dt,$$

where we have used the finite propagation property. This yields

$$p' I[g]\varepsilon + \int_0^T \eta_T^{2p'} \int_{\mathbb{R}^N} |u_\varepsilon|^p\,dx\,dt \leq C_1 T^{N-1-\frac{2}{p-1}}, \tag{2.2}$$

with

$$C_1 := \frac{[2p'(\|\eta''\|_{L^\infty} + (2p'-1)\|\eta'\|_{L^\infty}^2)]^{p'}(1+r_0)^{N+1}|S^{N-1}|}{N+1}$$

and the volume of N-dimensional unit sphere $|S^{N-1}|$. Since the choice of $T \in (1, T_\varepsilon)$ is arbitrary, the above inequality implies the first and second estimates for T_ε.

To obtain the third and fourth estimates for T_ε, we introduce a special solution to the linear wave equation as follows:

$$w_\lambda(x,t) = \lambda^{N-1}\Big((\lambda+t)^2 - |x|^2\Big)^{-\frac{N-1}{2}}, \quad \lambda > r_0.$$

Noting that $w_\lambda(x,0) \to 1$ and $\partial_t w_\lambda(x,0) \to 0$ as $\lambda \to \infty$ uniformly on $\mathrm{supp}(f,g)$, we see from the dominated convergence theorem that there exists $\lambda_0 > r_0$ such that

$$\int_{\mathbb{R}^N} g(x) w_{\lambda_0}(x,0) - f(x)\partial_t w_{\lambda_0}(x,0)\,dx \geq \frac{1}{2}\int_{\mathbb{R}^N} g(x)\,dx = \frac{1}{2}I[g] > 0.$$

Taking $\Psi = w_{\lambda_0}\eta_T^{2p'}$ (multiplying compactly supported smooth function χ on $\mathbb{R}^N$ satisfying $\chi = 1$ if $x \in B(0, r_0+T)$) in the definition of weak solutions, we have

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^N} g(x) w_{\lambda_0}(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} |u_\varepsilon|^p \Psi\,dx\,dt \\
&= \int_0^T \int_{\mathbb{R}^N} \Big(-\partial_t u_\varepsilon \partial_t \Psi + \nabla u_\varepsilon \cdot \nabla \Psi\Big)\,dx\,dt \\
&= \varepsilon \int_{\mathbb{R}^N} f(x)\partial_t w_{\lambda_0}(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} u_\varepsilon\Big(\partial_t^2 \Psi - \Delta \Psi\Big)\,dx\,dt.
\end{aligned}$$

Neglecting the second term in the left-hand side and using Hölder's inequality and the definition of w_{λ_0}, we deduce

$$
\begin{aligned}
I[g]\varepsilon &\le 2\int_0^T\int_{\mathbb{R}^N} u_\varepsilon\Big(\partial_t^2\Psi-\Delta\Psi\Big)\,dx\,dt\\
&= 4p'\int_0^T \eta_T^{2p'-2}\int_{\mathbb{R}^N} u_\varepsilon\bigg(2\partial_t w_{\lambda_0}\frac{\eta'(t/T)\eta(t/T)}{T}\\
&\quad + w_{\lambda_0}\frac{\eta''(t/T)\eta(t/T)+(2p'-1)(\eta'(t/T))^2}{T^2}\bigg)\,dx\,dt\\
&\le C_2T^{-N-1}\int_{T/2}^T \eta_T^{2p'/p}\int_{\mathbb{R}^N}|u_\varepsilon|\left(1-\frac{|x|^2}{(\lambda_0+t)^2}\right)^{-\frac{N+1}{2}}\,dx\,dt\\
&\le C_2T^{-N-1}\left(\int_{T/2}^T \eta_T^{2p'}\int_{\mathbb{R}^N}|u_\varepsilon|^p\,dx\,dt\right)^{\frac{1}{p}}\left(\int_{T/2}^T\int_{B(0,r_0+t)}\left(1-\frac{|x|}{\lambda_0+t}\right)^{-\frac{N+1}{2}p'}\,dx\,dt\right)^{\frac{1}{p'}}\\
&\le C_2'\left(T^{-N+\frac{N-1}{2}p}\int_0^T \eta_T^{2p'}\int_{\mathbb{R}^N}|u_\varepsilon|^p\,dx\,dt\right)^{\frac{1}{p}},
\end{aligned}
$$

for some $C_2>0$ and $C_2'>0$. Therefore we have

$$
\Big(I[g]\varepsilon\Big)^pT^{N-\frac{N-1}{2}p}\le C_2'\int_0^T\eta_T^{2p'}\int_{\mathbb{R}^N}|u_\varepsilon|^p\,dx\,dt. \tag{2.3}
$$

Combining (2.2) and (2.3), we obtain

$$
\Big(I[g]\varepsilon\Big)^pT^{N-\frac{N-1}{2}p}\le C_1C_2'T^{N-1-\frac{2}{p-1}}
$$

which implies the third and fourth estimates for T_ε. □

Remark 2.1. The upper bound of T_ε is not sharp in the case $(N,p)=(2,2)$. Indeed, Lindblad [40] and Takamura [53] proved the estimate $T_\varepsilon\le Ca(\varepsilon)$ with $a^2\varepsilon^2\log(1+a)=1$ by using a refined concentration estimate (similar to (3.5)) which is deduced from pointwise estimates for solutions to the linear wave equation.

Remark 2.2. In Yordanov–Zhang [58] and the subsequent papers, to prove a lower bound for $\int_{\mathbb{R}^N}|u|^p\,dx$, the positive radially symmetric solution $e^{-t}\phi(x)$ of $\partial_t^2u-\Delta u=0$ with the function ϕ satisfying $\phi-\Delta\phi=0$ was used. However, their treatment requires the positivity of initial value in the point-wise sense, in particular for high spatial dimensional cases. In contrast, the proof of Proposition 2.1 only needs the positivity of $I[g]$ by virtue of a new choice of the solution w_λ.

Remark 2.3. In the proof of Proposition 2.1, we do not use neither an auxiliary result for second order ordinary inequalities nor an iteration argument. The view-point from the proof of Proposition 2.1 may give us an easier understanding about blowup phenomena for sub-critical case.

3. Preliminaries for general cases

To analyze more general equations and systems, we introduce the super-solutions to wave equations and self-similar solutions. In this section, we state the fundamental properties of the super-solutions to wave equations and self-similar solutions.

We note that even if some notations overlap with ones in the previous section, we state them again for the reader's convenience.

3.1. Super-solutions of the linear wave equation and their properties

First we introduce super-solutions of wave equations.

Definition 3.1. Let (f, g) satisfy (1.3). The function $u \in H^1(0, T; L^2(\mathbb{R}^N)) \cap L^2(0, T; H^1(\mathbb{R}^N))$ is called a super-solution of $\partial_t^2 u - \Delta u = H$ with $u(0) = \varepsilon f$ and $\partial_t u(0) = \varepsilon g$ and $H \in L^1(0, T; L^1(\mathbb{R}^N))$ if $u(0) = \varepsilon f$ and

$$\varepsilon \int_{\mathbb{R}^N} g(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} H\Psi\,dx\,dt \le \int_0^T \int_{\mathbb{R}^N} (-\partial_t u \partial_t \Psi + \nabla u \cdot \nabla \Psi)\,dx\,dt$$

for every nonnegative function $\Psi \in C_c^1(\mathbb{R}^N \times [0, T))$.

Then we will use two kinds of families of cut-off functions with respect to time variables; $\eta \in C^\infty([0, \infty))$ satisfying

$$\eta(s) = \begin{cases} 1 & s < 1/2, \\ \text{decreasing} & 1/2 < s < 1, \\ 0 & s > 1 \end{cases}$$

and

$$\eta^*(s) = \begin{cases} 0 & s < 1/2, \\ \eta(s) & s \ge 1/2 \end{cases}$$

and for $k \ge 2$, $R > 0$,

$$\eta_R(t) = \eta\left(\frac{t}{R}\right), \quad \eta_R^*(t) = \eta^*\left(\frac{t}{R}\right), \quad \psi_R(t) = [\eta_R(t)]^k, \quad \psi_R^*(t) = [\eta_R^*(t)]^k.$$

The functions η_R^* and ψ_R^* are used only to justify the following estimates.

Lemma 3.1. *Let $k \geq 2$ and $R \geq 1$. For every $t \geq 0$,*

$$|\partial_t \psi_R(t)| \leq \frac{k\|\eta'\|_{L^\infty}}{R}[\psi_R^*(t)]^{1-\frac{1}{k}}, \quad |\partial_t^2 \psi_R(t)| \leq \frac{k\Big((k-1)\|\eta'\|_{L^\infty}^2 + \|\eta''\eta\|_{L^\infty}\Big)}{R^2}[\psi_R^*(t)]^{1-\frac{2}{k}}.$$

Proof. Noting that

$$\partial_t \psi_R(t) = k\eta_R'(t)[\eta_R(t)]^{k-1},$$
$$\partial_t^2 \psi_R(t) = k\left((k-1)(\eta_R'(t))^2 + \eta_R(t)\eta_R''(t)\right)[\eta_R(t)]^{k-2},$$

we easily obtain the desired inequalities. □

By using ψ_R as a test function for super-solutions, we obtain the following lemma.

Lemma 3.2. *Let $1 < p < \infty$ and $k \geq 2p'$, and let (f, g) satisfy (1.3) and let u be a super-solution of $\partial_t^2 u - \Delta u = H$ with $u(0) = \varepsilon f$, $\partial_t u(0) = \varepsilon g$, $H \in L^2(0, T; L^2(\mathbb{R}^N))$ and $\operatorname{supp} u \subset \{(x,t) \in \mathbb{R}^N \times [0, T]\,;\ |x| \leq r_0 + t\}$ for $r_0 = \sup\{|x|\,;\ x \in \operatorname{supp}(f, g)\}$. Then the following inequalities hold:*

(i) *For every $1 \leq R < T$*

$$I[g]\varepsilon + \int_0^T \int_{\mathbb{R}^N} H\psi_R\,dx\,dt \leq C_1 R^{-2} \int_0^T \int_{\mathbb{R}^N} |u|[\psi_R^*]^{\frac{1}{p}}\,dx\,dt. \tag{3.1}$$

(ii) *For every $1 \leq R < T$*

$$I[g]\varepsilon + \int_0^T \int_{\mathbb{R}^N} H\psi_R\,dx\,dt \leq C_2 R^{-1} \int_0^T \int_{\mathbb{R}^N} |\partial_t u|[\psi_R^*]^{\frac{1}{p}}\,dx\,dt. \tag{3.2}$$

Proof. By the definition of super-solution of $\partial_t^2 u - \Delta u = H \geq 0$, choosing $\Psi = \psi_R$ (with multiplying compactly supported smooth function ζ satisfying $\zeta \equiv 1$ on supp u), we have

$$I[g]\varepsilon + \int_0^T \int_{\mathbb{R}^N} H\psi_R\,dx\,dt \leq -\int_0^T \int_{\mathbb{R}^N} \partial_t u \partial_t \psi_R\,dx\,dt.$$

Then we can obtain **(ii)** by using Lemma 3.1 with $k \geq p'$. On the other hand, noting that

$$\int_{\mathbb{R}^N} \partial_t u \partial_t \psi_R\,dx = \frac{d}{dt}\int_{\mathbb{R}^N} u\partial_t \psi_R\,dx - \int_{\mathbb{R}^N} u\partial_t^2 \psi_R\,dx,$$

we obtain **(i)** from Definition 3.1 with $k \geq 2p'$. □

For a general (smooth) test function Ψ, we can find the following relation with respect to $\partial_t^2\Psi - \Delta\Psi$.

Lemma 3.3. *Let u be a super-solution of $\partial_t^2 u - \Delta u = H$ with $u(0) = \varepsilon f$, $\partial_t u(0) = \varepsilon g$ and $H \ge 0$. Then the following inequalities hold:*

(i) *For $\Psi \in C^2(\mathbb{R}^N \times [0, T))$ satisfying $\Psi \ge 0$,*

$$\varepsilon \int_{\mathbb{R}^N} (g\Psi(\cdot, 0) - f\partial_t\Psi(\cdot, 0))\,dx + \int_0^T\int_{\mathbb{R}^N} H\Psi\,dx\,dt \le \int_0^T\int_{\mathbb{R}^N} u(\partial_t^2\Psi - \Delta\Psi)\,dx\,dt. \tag{3.3}$$

(ii) *For $\widetilde{\Psi} \in C^3(\mathbb{R}^N \times [0, T))$ satisfying $\partial_t\widetilde{\Psi} \ge 0$,*

$$\varepsilon \int_{\mathbb{R}^N} (g\partial_t\widetilde{\Psi}(\cdot, 0) - f\Delta\widetilde{\Psi}(\cdot, 0))\,dx + \int_0^T\int_{\mathbb{R}^N} H\partial_t\widetilde{\Psi}\,dx\,dt \le \int_0^T\int_{\mathbb{R}^N} \partial_t u(\partial_t^2\widetilde{\Psi} - \Delta\widetilde{\Psi})\,dx\,dt. \tag{3.4}$$

Proof. **(i)** Integration by parts yields that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N} (-\partial_t u\partial_t\Psi + \nabla u\cdot\nabla\Psi)\,dx\,dt &= \frac{d}{dt}\left[-\int_{\mathbb{R}^N} u\partial_t\Psi\,dx\right] + \int_0^T\int_{\mathbb{R}^N} (u\partial_t^2\Psi - \Delta u\Psi)\,dx\,dt\\ &= \varepsilon\int_{\mathbb{R}^N} f\partial_t\Psi(x, 0)\,dx + \int_0^T\int_{\mathbb{R}^N} u(\partial_t^2\Psi - \Delta\Psi)\,dx\,dt.\end{aligned}$$

Connecting the definition of super-solution, we deduce (3.3).

(ii) Applying **(i)** $\Psi = \partial_t\widetilde{\Psi} \in C_c^2(\mathbb{R}^N \times [0, T))$, we have

$$\varepsilon \int_{\mathbb{R}^N} (g\partial_t\widetilde{\Psi}(\cdot, 0) - f\partial_t^2\widetilde{\Psi}(\cdot, 0))\,dx + \int_0^T\int_{\mathbb{R}^N} H\widetilde{\Psi}\,dx\,dt \le \int_0^T\int_{\mathbb{R}^N} u\partial_t(\partial_t^2\widetilde{\Psi} - \Delta\widetilde{\Psi})\,dx\,dt.$$

Noting that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N} u\partial_t(\partial_t^2\widetilde{\Psi} - \Delta\widetilde{\Psi})\,dx\,dt\\ &= \int_0^T \frac{d}{dt}\left[\int_{\mathbb{R}^N} u(\partial_t^2\widetilde{\Psi} - \Delta\widetilde{\Psi})\,dx\right]dt - \int_0^T\int_{\mathbb{R}^N} \partial_t u(\partial_t^2\widetilde{\Psi} - \Delta\widetilde{\Psi})\,dx\,dt\end{aligned}$$

$$= -\varepsilon \int_{\mathbb{R}^N} f(\partial_t^2 \widetilde{\Psi}(\cdot,0) - \Delta \widetilde{\Psi}(\cdot,0))\,dx\,dt - \int_0^T \int_{\mathbb{R}^N} \partial_t u(\partial_t^2 \widetilde{\Psi} - \Delta \widetilde{\Psi})\,dx\,dt,$$

we have (3.4). □

The following two lemmas describe the concentration phenomena of the wave near the light cone $\partial B(0, 1+t)$, which is essentially the same as an estimate given in Yordanov–Zhang [58] (but under a weaker assumption). In their proofs, we use a special solution of the linear wave equation given by

$$V(x,t) = t(t^2 - |x|^2)^{-\frac{N+1}{2}} = t^{-N}\left(1 - \frac{|x|^2}{t^2}\right)^{-\frac{N+1}{2}}$$

in $\mathcal{Q} = \{(x,t) \in \mathbb{R}^N \times [0,\infty)\ ;\ |x| < t\}$. By the notation in next subsection, we see $V(x,t) = \Phi_\beta(x,t)$ with $\beta = N$ (see Definition 3.2 below).

Lemma 3.4. *Let f, g satisfy (1.3) and let u be a super-solution of $\partial_t^2 u - \Delta u = 0$ with $u(0) = \varepsilon f$, $\partial_t u(0) = \varepsilon g$ and $\operatorname{supp} u \subset \{(x,t) \in \mathbb{R}^N \times [0,T]\ ;\ |x| \le r_0 + t\}$ for $r_0 = \sup\{|x|\ ;\ x \in \operatorname{supp}(f,g)\}$. Then for every $p > 1$ and $k \ge 2p'$, there exists a constant $\delta_1 = \delta_1(N,p,k,f,g) > 0$ (independent of ε) such that for every $1 \le R < T$*

$$\delta_1\Big(I[g]\varepsilon\Big)^p R^{N-\frac{N-1}{2}p} \le \int_0^T \int_{\mathbb{R}^N} |u|^p \psi_R^*\,dx\,dt. \tag{3.5}$$

Proof. Put

$$v_\lambda(x,t) = \lambda^N V(x, \lambda + t), \quad (x,t) \in \mathcal{Q}_\lambda = \{(x,t) \in \mathbb{R}^N \times [0,\infty)\ ;\ (x, \lambda + t) \in \mathcal{Q}\},$$

for $\lambda > r_0$ and then $\operatorname{supp} u \subset \mathcal{Q}_\lambda$. Noting that

$$\partial_t V(x,t) = -t^{-N-1}\left(N + \frac{|x|^2}{t^2}\right)\left(1 - \frac{|x|^2}{t^2}\right)^{-\frac{N+3}{2}},$$

we see that

$$\int_{\mathbb{R}^N} g(x) v_\lambda(x,0) - f(x)\partial_t v_\lambda(x,0)\,dx$$

$$= \int_{\mathbb{R}^N} g(x)\left(1 - \frac{|x|^2}{\lambda^2}\right)^{-\frac{N+1}{2}} dx + \frac{1}{\lambda}\int_{\mathbb{R}^N} f(x)\left(N + \frac{|x|^2}{\lambda^2}\right)\left(1 - \frac{|x|^2}{\lambda^2}\right)^{-\frac{N+3}{2}} dx.$$

Since the pair (f,g) satisfies (1.3), the dominated convergence theorem implies that there exists $\lambda_0 > r_0$ such that

$$\int_{\mathbb{R}^N} g(x)v_{\lambda_0}(x,0) - f(x)\partial_t v_{\lambda_0}(x,0)\,dx \geq \frac{1}{2}\int_{\mathbb{R}^N} g(x)\,dx > 0. \tag{3.6}$$

On the other hand, since u is a super-solution of $\partial_t^2 u - \Delta u = 0$, choosing $\Psi = v_{\lambda_0}\psi_R$ in Lemma 3.3 **(i)**, we see from the fact $\partial_t^2 v_{\lambda_0} - \Delta v_{\lambda_0} = 0$ and Lemma 3.1 that

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^N} g(x)v_{\lambda_0}(x,0) - f(x)\partial_t v_{\lambda_0}(x,0)\,dx \\
&\leq \int_0^T \int_{\mathbb{R}^N} u(\partial_t^2(v_{\lambda_0}\psi_R) - \Delta(v_{\lambda_0}\psi_R))\,dx\,dt \\
&\leq C\int_0^T \int_{\mathbb{R}^N} |u|\left(\frac{|\partial_t v_{\lambda_0}|}{R} + \frac{v_{\lambda_0}}{R^2}\right)[\psi_R^*]^{\frac{1}{p}}\,dx\,dt \\
&\leq C\left(\int_0^T \int_{\mathbb{R}^N} |u|^p \psi_R^*\,dx\,dt\right)^{\frac{1}{p}} \left(\int_{\frac{R}{2}}^R \int_{B(0,1+t)} \left(\frac{|\partial_t v_{\lambda_0}|}{R} + \frac{v_{\lambda_0}}{R^2}\right)^{p'} dx\,dt\right)^{\frac{1}{p'}}.
\end{aligned} \tag{3.7}$$

Since $|x| \leq 1+t$ and $R/2 \leq t \leq R$ yield

$$\frac{|\partial_t v_{\lambda_0}|}{R} + \frac{v_{\lambda_0}}{R^2} \leq CR^{-N-2}\left(1 - \frac{|x|^2}{(\lambda_0+t)^2}\right)^{-\frac{N+3}{2}} \leq CR^{-N-2}\left(1 - \frac{|x|}{\lambda_0+t}\right)^{-\frac{N+3}{2}},$$

a direct calculation implies

$$\int_{\frac{R}{2}}^R \int_{B(0,1+t)} \left(\frac{|\partial_t v_{\lambda_0}|}{R} + \frac{v_{\lambda_0}}{R^2}\right)^{p'} dx\,dt \leq CR^{N-(\frac{N+1}{2})p'}. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we obtain (3.5). □

Lemma 3.5. *Let f, g satisfy* (1.3) *and let u be a super-solution of $\partial_t^2 u - \Delta u = 0$ with $u(0) = \varepsilon f$, $\partial_t u(0) = \varepsilon g$ and $\operatorname{supp} u \subset \{(x,t) \in \mathbb{R}^N \times [0,T]\,;\ |x| \leq r_0 + t\}$ for $r_0 = \sup\{|x|\,;\ x \in \operatorname{supp}(f,g)\}$. Then for every $q > 1$ and $k \geq 2q'$, there exists a constant $\delta_1' = \delta_1'(N,q,k,f,g) > 0$ such that for every $1 \leq R < T$,*

$$\delta_1'\Big(I[g]\varepsilon\Big)^q R^{N-\frac{N-1}{2}q} \leq \int_0^T \int_{\mathbb{R}^N} |\partial_t u|^q \psi_R^*\,dx\,dt. \tag{3.9}$$

Proof. Set $w_\lambda(x,t) = -\lambda^{N+1}V_\lambda(x,t) = -\lambda v_\lambda(x,t)$ for $\lambda > 1$ and then

$$\partial_t(w_\lambda\psi_R) = \lambda(-\partial_t v_\lambda\psi_R - v_\lambda\partial_t\psi_R) \geq 0.$$

Noting that $\partial_t^2 w_\lambda - \Delta w_\lambda = 0$, as in the proof of Lemma 3.4, we can verify that there exists $\lambda_0' > 1$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x)\partial_t w_{\lambda_0'}(x,0) - f(x)\Delta w_{\lambda_0'}(x,0)\,dx \\
&\quad = -\lambda\int_{\mathbb{R}^N} g(x)\partial_t v_{\lambda_0'}(x,0)\,dx + \lambda\int_{\mathbb{R}^N} f(x)\partial_t^2 v_{\lambda_0'}(x,0)\,dx \\
&\quad \geq \frac{N}{2}I[g].
\end{aligned}$$

Applying Lemma 3.3 **(ii)** with $\widetilde{\Psi} = w_{\lambda_0'}\psi_R$, we have

$$\begin{aligned}
\varepsilon\int_{\mathbb{R}^N} g(x)\partial_t v_{\lambda_0'}(x,0) - f(x)\Delta v_{\lambda_0'}(x,0)\,dx &\leq -\int_0^T\int_{\mathbb{R}^N} \partial_t u(\partial_t^2(w_{\lambda_0'}\psi_R) - \Delta(w_{\lambda_0'}\psi_R))\,dx\,dt \\
&\leq C\int_0^T\int_{\mathbb{R}^N} |\partial_t u|\left(\frac{|\partial_t v_{\lambda_0'}|}{R} + \frac{v_{\lambda_0'}}{R^2}\right)[\psi_R^*]^{\frac{1}{p}}\,dx\,dt \\
&\leq CR^{\frac{N-1}{2}-\frac{N}{q}}\left(\int_0^T\int_{\mathbb{R}^N} |\partial_t u|^q\psi_R^*\,dx\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

The above inequalities imply (3.9). □

3.2. *Self-similar solutions including Gauss's hypergeometric functions*

In the respective critical case of blowup phenomena for wave equations, we need precise information about the behavior of solutions to the linear wave equation. Therefore, next we introduce a family of self-similar solutions to $\partial_t^2 u - \Delta u = 0$ including Gauss's hypergeometric functions, which also can be found in Zhou [59–61], Zhou–Han [65] and also Ikeda–Sobajima [22,23].

Definition 3.2. Let $N \geq 2$. For $0 < \beta < \infty$, define

$$\Phi_\beta(x,t) = (t+|x|)^{-\beta}F\left(\beta, \frac{N-1}{2}, N-1; \frac{2|x|}{t+|x|}\right) \quad (x,t) \in \mathcal{Q},$$

where $F(a,b,c;z)$ is the Gauss hypergeometric function with a parameter (a,b,c) given by

$$F(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\quad z\in[0,1)$$

with the Pochhammer symbol $(d)_0=1$ and $(d)_n=\prod_{k=1}^{n}(d+k-1)$ for $n\in\mathbb{N}$. Also we set

$$\Phi_{\beta,\lambda}(x,t)=\lambda^{\beta}\Phi_\beta(x,\lambda+t),\quad (x,t)\in\mathcal{Q}_\lambda=\{(x,t)\in\mathbb{R}^N\times[0,\infty)\,;\,(x,\lambda+t)\in\mathcal{Q}\}.$$

Remark 3.1. In particular, the following formula for hypergeometric functions is known:

$$F\left(a,a+\frac{1}{2},c;z\right)=(1+\sqrt{z})^{-2a}F\left(2a,c-\frac{1}{2},2c-1;\frac{2\sqrt{z}}{1+\sqrt{z}}\right)$$

(see Beals–Wong [3, Section 8.9.6]). Putting $a=\frac{\beta}{2}$, $c=\frac{N}{2}$, we have

$$\Phi_\beta(x,t)=(t+|x|)^{-\beta}F\left(\beta,\frac{N-1}{2},N-1;\frac{2|x|}{t+|x|}\right)=t^{-\beta}F\left(\frac{\beta}{2},\frac{\beta+1}{2},\frac{N}{2};\frac{|x|^2}{t^2}\right).$$

This formula implies $\Phi_\beta\in C^\infty(\mathcal{Q})$. In one-dimensional case, the critical exponent for respective case does not appear.

Then the family $\{\Phi_\beta\}_{\beta>0}$ satisfies the following properties. For detail, see [65] and also [22, 23].

Lemma 3.6. *The following assertions hold:*

- **(i)** $\partial_t^2\Phi_\beta-\Delta\Phi_\beta=0$ *on* $\mathcal{Q}$.
- **(ii)** $\partial_t\Phi_\beta=-\beta\Phi_{\beta+1}$ *on* $\mathcal{Q}$.
- **(iii)** *If* $0\le\beta<\frac{N-1}{2}$, *then* $t^{-\beta}\le\Phi_\beta(x,t)\le K_\beta t^{-\beta}$ *on* $\mathcal{Q}$.
- **(iv)** *If* $\beta>\frac{N-1}{2}$, *then*

$$k_\beta t^{-\beta}\left(1-\frac{|x|^2}{t^2}\right)^{\frac{N-1}{2}-\beta}\le\Phi_\beta(x,t)\le K_\beta t^{-\beta}\left(1-\frac{|x|^2}{t^2}\right)^{\frac{N-1}{2}-\beta}.$$

In view of the properties of $\{\Phi_\beta\}_{\beta>0}$, we will take $\Psi=\Phi_{\beta,\lambda}\psi_R$. Then we have the following lemma.

Lemma 3.7. *Let* (f,g) *satisfy* (1.3) *and let* u *be a super-solution of* $\partial_t^2u-\Delta u=H$ *with* $u(0)=\varepsilon f$, $\partial_t u(0)=\varepsilon g$, $H\in L^2(0,T;L^2(\mathbb{R}^N))$ *and* $\operatorname{supp}u\subset\{(x,t)\in\mathbb{R}^N\times[0,T]\,;\,|x|\le r_0+t\}$ *for* $r_0=\sup\{|x|\,;\,x\in\operatorname{supp}(f,g)\}$. *Then for every* $\beta>0$, *there exist constants* $\lambda_\beta>0$, $C_{\beta,1}>0$ *and* $C_{\beta,2}>0$ *such that the followings hold:*

- **(i)** *If* $k\ge 2p'$ *and* $\beta>0$, *then for every* $\lambda\ge\lambda_\beta$ *and* $1\le R<T$

$$\frac{1}{2}I[g]\varepsilon+\int_0^T\int_{\mathbb{R}^N}H\Phi_{\beta,\lambda}\psi_R\,dx\,dt\le C_{\beta,1}R^{-1}\int_0^T\int_{\mathbb{R}^N}|u|\Phi_{\beta+1,\lambda}[\psi_R^*]^{\frac{1}{p}}\,dx\,dt \tag{3.10}$$

(ii) *If $k \ge 2p'$ and $\beta > 1$, then for every $\lambda \ge \lambda_\beta$ and $1 \le R < T$*

$$\frac{1}{2}I[g]\varepsilon + \int_0^T \int_{\mathbb{R}^N} H\Phi_{\beta,\lambda}\psi_R \, dx\, dt \le C_{\beta,2}R^{-1}\int_0^T \int_{\mathbb{R}^N} |\partial_t u|\Phi_{\beta,\lambda}[\psi_R^*]^{\frac{1}{p}} \, dx\, dt \tag{3.11}$$

Proof. Put

$$c_{\beta,\lambda}(f,g) = \int_{\mathbb{R}^N} g\Phi_{\beta,\lambda} + \beta f \Phi_{\beta+1,\lambda} \, dx.$$

Then we easily see that $c_{\beta,\lambda}(f,g) \to \int_{\mathbb{R}^N} g\, dx$ as $\lambda \to \infty$. Therefore there exists $\lambda_\beta > 0$ such that for every $\lambda \ge \lambda_\beta$,

$$c_{\beta,\lambda}(f,g) \ge \frac{1}{2}I[g].$$

Next, observe that for $R \ge 1$ and $(x,t) \in \mathcal{Q}_\lambda \cap \operatorname{supp}\psi_R^*$,

$$\begin{aligned}
\Phi_{\beta,\lambda}(x,t) &\le 2\lambda^\beta(\lambda+t)(\lambda+t+|x|)^{-\beta-1}F\left(\beta, \frac{N-1}{2}, N-1, \frac{2|x|}{\lambda+t+|x|}\right) \\
&\le 2\lambda^\beta(\lambda+t)(\lambda+t+|x|)^{-\beta-1}F\left(\beta+1, \frac{N-1}{2}, N-1, \frac{2|x|}{\lambda+t+|x|}\right) \\
&\le 2\lambda^{\beta+1}(1+t)(\lambda+t+|x|)^{-\beta-1}F\left(\beta+1, \frac{N-1}{2}, N-1, \frac{2|x|}{\lambda+t+|x|}\right) \\
&\le 4R\Phi_{\beta+1,\lambda}(x,t).
\end{aligned}$$

Therefore by Lemma 3.1 and Lemma 3.6 **(i)**, **(ii)**, we see that

$$\begin{aligned}
|\partial_t^2(\Phi_{\beta,\lambda}\psi_R) - \Delta(\Phi_{\beta,\lambda}\psi_R)| &\le 2|\partial_t\Phi_{\beta,\lambda}\partial_t\psi_R| + \Phi_{\beta,\lambda}|\partial_t^2\psi_R| \\
&\le CR^{-1}\Phi_{\beta,\lambda+1}[\psi_R^*]^{1-\frac{2}{k}}.
\end{aligned}$$

Choosing $\Psi = \Phi_{\beta,\lambda}\psi_R$ in Lemma 3.3, we have **(i)**. For **(ii)**, we choose $\Psi = -(\beta-1)^{-1} \times \lambda\Phi_{\beta-1,\lambda}\psi_R$ for $\beta > 1$ and $\lambda > \lambda_\beta$ (the same choice as the case **(i)**). Noting that

$$\partial_t\Psi = \Phi_{\beta,\lambda}\psi_R - \frac{\lambda}{\beta-1}\Phi_{\beta,\lambda}\partial_t\psi_R \ge \Phi_{\beta,\lambda}\psi_R,$$

we can deduce **(ii)**. □

Throughout the present paper we often use the following lemma.

Lemma 3.8. *For every* $R \ge 1$,

$$\int_{\frac{R}{2}}^{R} \int_{B(0,1+t)} \Phi_{\beta,\lambda}^{p'}\,dx\,dt \le \begin{cases} C_\lambda R^{N+1-\beta p'} & \text{if } \beta \in [0, \frac{N+1}{2} - \frac{1}{p}), \\ C_\lambda R^{N-(\frac{N-1}{2})p'} \log R & \text{if } \beta = \frac{N+1}{2} - \frac{1}{p}, \\ C_\lambda R^{N-(\frac{N-1}{2})p'} & \text{if } \beta \in (\frac{N+1}{2} - \frac{1}{p}, \infty). \end{cases}$$

Proof. All of the assertions are verified by using Lemma 3.6 **(iii)** and **(iv)**. □

3.3. Lemmas for lifespan estimates for respective critical cases

To provide an upper bound of lifespan of solutions to respective critical cases, we need to adopt the framework proposed in [24]. For the proof we refer the one of [24, Proposition 2.1].

Definition 3.3. For nonnegative function $w \in L^1_{\rm loc}([0,T); L^1(\mathbb{R}^N))$, set

$$Y[w](R) = \int_0^R \left(\int_0^T \int_{\mathbb{R}^N} w(x,t)\psi_\sigma^*(t)\,dx\,dt \right) \sigma^{-1} d\sigma, \quad R \in (0,T).$$

Then $Y[w]$ has the following properties.

Lemma 3.9. *For* $w \in L^1_{\rm loc}([0,T); L^1(\mathbb{R}^N))$, $Y[w](\cdot) \in C^1((0,T))$ *and for every* $R \in (0,T)$

$$\frac{d}{dR} Y[w](R) = R^{-1} \int_0^T \int_{\mathbb{R}^N} w(x,t)\psi_R^*(t)\,dx\,dt,$$

$$Y[w](R) \le \int_0^T \int_{\mathbb{R}^N} w(x,t)\psi_R(t)\,dx\,dt.$$

It worth noticing that in critical cases the behavior of $Y[|u|^p \Phi_\beta]$ is crucial to obtain not only the blowup phenomena but also the upper bound of lifespan for respective problems. The following lemma provides the sharp upper bound of solutions for respective problems.

Lemma 3.10. *Let* $2 < t_0 < T$, $0 \le \phi \in C^1([t_0,T))$. *Assume that*

$$\begin{cases} \delta \le K_1 t\phi'(t), & t \in (t_0,T), \\ \phi(t)^{p_1} \le K_2 t(\log t)^{p_2-1}\phi'(t), & t \in (t_0,T) \end{cases} \tag{3.12}$$

with $\delta, K_1, K_2 > 0$ *and* $p_1, p_2 > 1$. *If* $p_2 < p_1 + 1$, *then there exist positive constants* δ_0 *and* K_3 *(independent of* δ*) such that*

$$T \le \exp(K_3 \delta^{-\frac{p_1-1}{p_1-p_2+1}})$$

when $0 < \delta < \delta_0$.

Proof. If $T \le t_0^4$, then we can choose $\delta_0' = \left(4K_0^{-1}\log t_0\right)^{-\frac{p_1-p_2+1}{p_1-1}}$. Therefore we assume $t_0^4 < T$. By the first inequality in (3.12), we have for every $t \in (t_0^2, T)$,

$$\phi(t) = \phi(t^{1/2}) + \int_{t^{1/2}}^{t} \phi'(s)\,ds \ge \frac{\delta}{K_1}(\log t - \log t^{1/2}) = \frac{\delta}{2K_1}\log t.$$

On the other hand, let $t_1 \in (t_0^4, T)$ be arbitrary fixed. The second inequality in (3.12) implies

$$\frac{d}{dt}[\phi(t)]^{1-p_1} \le -\frac{p_1-1}{K_2} t^{-1}(\log t)^{1-p_2}, \quad t \in (t_0^2, T)$$

and therefore integrating it over $[t_1^{1/2}, t_1]$, we deduce

$$\begin{aligned}
[\phi(t_1)]^{1-p_1} &\le [\phi(t_1^{1/2})]^{1-p_1} - \frac{p_1-1}{K_2}\int_{t_1^{1/2}}^{t_1} s^{-1}(\log s)^{1-p_2}\,ds \\
&\le \left[\frac{\delta}{4K_1}\log t_1\right]^{1-p_1} - \frac{p_1-1}{K_2}\int_{1/2}^{1} \sigma^{1-p_2}\,d\sigma\ (\log t_1)^{2-p_2} \\
&\le \left(\left[\frac{\delta}{4K_1}\right]^{1-p_1} - \frac{p_1-1}{K_2}\int_{1/2}^{1} \sigma^{1-p_2}\,d\sigma(\log t_1)^{1+p_1-p_2}\right)(\log t_1)^{1-p_1}.
\end{aligned}$$

This yields that

$$(\log t_1)^{1+p_1-p_2} \le \frac{(4K_1)^{p_1-1}K_2}{p_1-1}\left(\int_{1/2}^{1} \sigma^{1-p_2}\,d\sigma\right)^{-1} \delta^{-(p_1-1)}.$$

Since the choice of $t_1 \in (t_0^4, T)$ is arbitrary, we obtain the desired upper bound of T. $\square$

4. The case $\partial_t^2 u - \Delta u = G(u)$

The first problem is the following classical Cauchy problem of the following semilinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = G(u), & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(0) = \varepsilon f, & x \in \mathbb{R}^N, \\ \partial_t u(0) = \varepsilon g, & x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

where the nonlinearity $G \in C^1(\mathbb{R})$ satisfies

$$G(0) = 0, \quad G(s) \geq a|s|^p, \quad s \in \mathbb{R}$$

for some $a > 0$ and $p > 1$. In this case the definition of weak solutions is the following:

Definition 4.1. Let $f, g \in C_c^\infty(\mathbb{R}^N)$ and $p > 1$. The function

$$u \in C([0,T); H^1(\mathbb{R}^N)) \cap C^1([0,T); L^2(\mathbb{R}^N)), \quad G(u) \in L^1(0,T; L^1(\mathbb{R}^N))$$

is called a weak solution of (4.1) in $(0,T)$ if $u(0) = \varepsilon f$, $\partial_t u(0) = \varepsilon g$ and for every $\Psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$,

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^N} g(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} G(u(x,t))\Psi(x,t)\,dx\,dt \\
&= \int_0^T \int_{\mathbb{R}^N} \Big(-\partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t) \cdot \nabla \Psi(x,t) \Big)\,dx\,dt.
\end{aligned}$$

In order to give a unified viewpoint with weakly coupled systems, we introduce

$$\Gamma_S(N,p) = \left(1 + \frac{1}{p}\right)(p-1)^{-1} - \frac{N-1}{2} = \frac{\gamma_S(N,p)}{2p(p-1)}.$$

The case of $G(s) = |s|^p$ with $1 < p < p_S(N)$ was already shown in Section 2. The essence of the proof for $1 < p < p_S(N)$ is the same as in Section 2. Therefore we would state all assertions but prove only for the case $p = p_S(N)$. The assertion is formulated by the upper bound of maximal existence time of solutions to (4.1).

Proposition 4.1. *Let (f,g) satisfy (1.3) and let u be a solution to (4.1) in $(0,T)$ satisfying $\operatorname{supp} u \subset \{(x,t) \in \mathbb{R}^N \times [0,T]\,;\ |x| \leq r_0 + t\}$ for $r_0 = \sup\{|x|\,;\ x \in \operatorname{supp}(f,g)\}$. If*

$$\Gamma_S(N,p) \geq 0$$

(that is, $1 < p \leq p_S(N)$), then T has the following upper bound

$$T \leq \begin{cases} C\varepsilon^{-\frac{p-1}{2}} & \text{if } N = 1,\ 1 < p < \infty, \\ C\varepsilon^{-\frac{p-1}{3-p}} & \text{if } N = 2,\ 1 < p \leq 2, \\ C\varepsilon^{-\Gamma_S(2,p)^{-1}} & \text{if } N = 2,\ 2 < p < p_S(2), \\ C\varepsilon^{-\Gamma_S(N,p)^{-1}} & \text{if } N \geq 3,\ 1 < p < p_S(N), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } N \geq 2,\ p = p_S(N) \end{cases}$$

for every $\varepsilon \in (0, \varepsilon_0]$, where ε_0 and C are positive constants independent of ε.

Proof. We only show the estimate for T for the case $N \geq 2$, $p = p_S(N)$. We will deduce differential inequalities for $Y = Y[|u|^p \Phi_{\beta,\lambda}]$ defined in Lemma 3.9 with $\beta = \beta_p = \frac{N-1}{2} - \frac{1}{p}$ and $\lambda = \lambda_{\beta_p}$ (given in Lemma 3.7). By virtue of Lemma 3.9, we see from the inequality in Lemma 3.4 that

$$
\begin{aligned}
Y'(R)R &= \int_0^T \int_{\mathbb{R}^N} |u|^p \Phi_{\beta,\lambda} \psi_R^* \, dx \, dt \\
&\geq \left(\frac{\lambda}{\lambda+1}\right)^{\beta} R^{-\beta} \int_0^T \int_{\mathbb{R}^N} |u|^p \psi_R^* \, dx \, dt \\
&\geq \delta_1 \left(\frac{\lambda}{\lambda+1}\right)^{\beta} \left(\varepsilon I[g]\right)^p,
\end{aligned}
$$

where we used $\beta_p = N - \frac{N-1}{2} p$ by the assumption $\gamma_S(N, p) = 0$. Moreover, using Lemma 3.7 **(i)** with $\beta = \beta_p$ and Lemma 3.8, we have

$$
\begin{aligned}
[Y(R)]^p \leq \left(\int_0^T \int_{\mathbb{R}^N} |u|^p \Phi_{\beta,\lambda} \psi_R \, dx \, dt\right)^p &\leq C R^{\frac{N-1}{2} p - N} (\log R)^{p-1} \int_0^T \int_{\mathbb{R}^N} |u|^p \psi_R^* \, dx \, dt \\
&\leq C (\log R)^{p-1} \int_0^T \int_{\mathbb{R}^N} |u|^p \Phi_{\beta,\lambda} \psi_R^* \, dx \, dt \\
&= C R (\log R)^{p-1} Y'(R).
\end{aligned}
$$

Applying Lemma 3.10 with $p_1 = p_2 = p$, we obtain $T \leq \exp(C\varepsilon^{-p(p-1)})$. The proof is complete. □

5. The case $\partial_t^2 u - \Delta u = G(\partial_t u)$

In this section we consider the following semilinear wave equation with the nonlinearity governed by derivatives of the unknown function:

$$
\begin{cases}
\partial_t^2 u - \Delta u = G(\partial_t u) & (x,t) \in \mathbb{R}^N \times (0,T), \\
u(0) = \varepsilon f & x \in \mathbb{R}^N, \\
\partial_t u(0) = \varepsilon g & x \in \mathbb{R}^N,
\end{cases}
\tag{5.1}
$$

where the nonlinearity $G \in C^1(\mathbb{R}^N)$ satisfies

$$
G(0) = 0, \quad G(\sigma) \geq b|\sigma|^p, \quad \sigma \in \mathbb{R}
$$

for some $b > 0$ and $p > 1$. In this case the definition of weak solutions is the following:

Definition 5.1. Let $f, g \in C_c^\infty(\mathbb{R}^N)$ and $p > 1$. The function

$$u \in C([0,T); H^1(\mathbb{R}^N)) \cap C^1([0,T); L^2(\mathbb{R}^N)), \quad G(\partial_t u) \in L^1(0,T; L^1(\mathbb{R}^N))$$

is called a weak solution of (5.1) in $(0,T)$ if $u(0) = \varepsilon f$, $\partial_t u(0) = \varepsilon g$ and for every $\Psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$,

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^N} g(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} G(\partial_t u(x,t))\Psi(x,t)\,dx\,dt \\
&= \int_0^T \int_{\mathbb{R}^N} \Big(-\partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t)\cdot\nabla\Psi(x,t)\Big)\,dx\,dt.
\end{aligned}$$

For the problem (5.1), we set

$$\Gamma_G(N,p) = \frac{1}{p-1} - \frac{n-1}{2}.$$

The exponent $p_G(N) = \frac{N+1}{N-1}$ ($\Gamma_G(N, p_G(N)) = 0$) is so-called Glassey exponent. For the convenience, we put $p_G(1) = \infty$.

Proposition 5.1. *Let (f,g) satisfy (1.3) and let u be a solution to (5.1) in $(0,T)$ satisfying $\operatorname{supp} u \subset \{(x,t) \in \mathbb{R}^N \times [0,T]\,;\ |x| \le r_0 + t\}$ for $r_0 = \sup\{|x|\,;\ x \in \operatorname{supp}(f,g)\}$. If*

$$\Gamma_G(N,p) \ge 0$$

then T has the following upper bound

$$T \le \begin{cases} C\varepsilon^{-\Gamma_G(N,p)^{-1}} & \text{if } 1 < p < p_G(N), \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } N \ge 2,\ p = p_G(N) \end{cases}$$

for every $\varepsilon \in (0, \varepsilon_0]$, where ε_0 and C are positive constants independent of ε.

Proof. Note that u is a super-solution of $\partial_t^2 u - \Delta u = b|\partial_t u|^p$. Choosing $\beta > \frac{N-1}{2} + 1$ and $\lambda = \lambda_\beta$ in Lemma 3.7 **(ii)** and Lemma 3.8, we have

$$\begin{aligned}
&\frac{\varepsilon}{2} I[g] + b\int_0^T \int_{\mathbb{R}^N} |\partial_t u|^p \Phi_{\beta,\lambda}\psi_R\,dx\,dt \\
&\le CR^{-1}\int_0^T \int_{\mathbb{R}^N} |\partial_t u|\Phi_{\beta,\lambda}[\psi_R^*]^{\frac{1}{p}}\,dx\,dt
\end{aligned}$$

$$\le CR^{-1}\left(\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{R}{2}}^R\int_{B(0,r_0+t)}\Phi_{\beta,\lambda}\,dx\,dt\right)^{\frac{1}{p'}}$$

$$\le CR^{-(\frac{1}{p-1}-\frac{N-1}{2})\frac{1}{p'}}\left(\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt\right)^{\frac{1}{p}}. \tag{5.2}$$

Setting $Y = Y[|\partial_t u|^p\Phi_{\beta,\lambda}]$ defined in Lemma 3.9, we obtain

$$\left(\frac{\varepsilon}{2}I[g] + bY(R)\right)^p \le CR^{-\Gamma_G(N,p)(p-1)+1}Y'(R).$$

Solving the above differential inequality, we can deduce

$$R \le \begin{cases} C\varepsilon^{-(\frac{1}{p-1}-\frac{N-1}{2})^{-1}} & \text{if } \Gamma_G(N,p) > 0,\\ \exp(C\varepsilon^{-(p-1)}) & \text{if } \Gamma_G(N,p) = 0.\end{cases}$$

Since the choice of $R \in (1, T)$ is arbitrary, we could derive the desired upper bound of T. □

Remark 5.1. We can also see from $\psi_R^* \le \psi_R$ that if $\Gamma_G(N, p) > 0$, then by Young's inequality we have

$$\frac{\varepsilon}{2}I[g] \le CR^{-(\frac{1}{p-1}-\frac{N-1}{2})}$$

and therefore we can easily get the desired lifespan estimate for $1 < p < \frac{N+1}{N-1}$. However, this argument does not work in the critical situation $p = \frac{N+1}{N-1}$.

6. The case of a combined type $\partial_t^2 u - \Delta u = G(u, \partial_t u)$

In this section we discuss the semilinear wave equation with a nonlinearity of a combined type

$$\begin{cases} \partial_t^2 u - \Delta u = G(u, \partial_t u) & (x,t) \in \mathbb{R}^N \times (0,T),\\ u(0) = \varepsilon f & x \in \mathbb{R}^N,\\ \partial_t u(0) = \varepsilon g & x \in \mathbb{R}^N,\end{cases} \tag{6.1}$$

where the nonlinearity $G \in C^1(\mathbb{R}^2)$ satisfies

$$G(0,0) = 0, \quad G(s,\sigma) \ge a|s|^q + b|\sigma|^p \quad (s,\sigma) \in \mathbb{R}^2$$

for some $a, b > 0$ and $p, q > 1$. This problem has been considered by Zhou–Han [64] Hidano–Wang–Yokoyama [19] and Wang–Zhou [56].

In this case the definition of weak solutions is the following:

Definition 6.1. Let $f, g \in C_c^\infty(\mathbb{R}^N)$ and $p > 1$. The function

$$u \in C([0,T); H^1(\mathbb{R}^N)) \cap C^1([0,T); L^2(\mathbb{R}^N)), \quad G(u, \partial_t u) \in L^1(0,T; L^1(\mathbb{R}^N))$$

is called a weak solution of (6.1) in $(0,T)$ if $u(0) = \varepsilon f$, $\partial_t u(0) = \varepsilon g$ and for every $\Psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$,

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^N} g(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} G\big(u(x,t), \partial_t u(x,t)\big)\Psi(x,t)\,dx\,dt \\
&= \int_0^T \int_{\mathbb{R}^N} \Big(-\partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t) \cdot \nabla \Psi(x,t)\Big)\,dx\,dt.
\end{aligned}$$

For the problem (6.1), we set

$$\Gamma_{\mathrm{comb}}(N,p,q) = \frac{q+1}{p(q-1)} - \frac{N-1}{2}.$$

The following assertion is already given by Hidano–Wang–Yokoyama [19].

Proposition 6.1. *Let (f,g) satisfy (1.3) and let u be a solution to (6.1) in $(0,T)$ satisfying* $\operatorname{supp} u \subset \{(x,t) \in \mathbb{R}^N \times [0,T)\ ;\ |x| \le r_0 + t\}$ *for* $r_0 = \sup\{|x|\ ;\ x \in \operatorname{supp}(f,g)\}$. *If*

$$\max\{\Gamma_S(N,q), \Gamma_G(N,p)\} \ge 0 \text{ or } \Gamma_{\mathrm{comb}}(N,p,q) > 0,$$

then T has the following upper bound

$$T \le \begin{cases}
\exp(C\varepsilon^{-(p-1)}) & \text{if } p = \frac{N+1}{N-1},\ q > 1 + \frac{4}{N-1}, \\
C\varepsilon^{-\Gamma_G(N,q)^{-1}} & \text{if } p < \frac{N+1}{N-1},\ q > 2p-1, \\
C\varepsilon^{-\Gamma_{\mathrm{comb}}(N,p,q)^{-1}} & \text{if } p \le q \le 2p-1,\ \Gamma_{\mathrm{comb}}(N,p,q) > 0, \\
C\varepsilon^{-\Gamma_S(N,p)^{-1}} & \text{if } p > q, \quad q < p_S(N), \\
\exp(C\varepsilon^{q(q-1)}) & \text{if } p \ge q = p_S(N)
\end{cases}$$

for every $\varepsilon \in (0, \varepsilon_0]$, where ε_0 and C are positive constants independent of ε.

Remark 6.1. In the case $\Gamma_S(N,q) < 0$, $\Gamma_G(N,p) < 0$ and $\Gamma_{\mathrm{comb}}(N,p,q) \le 0$, Hidano–Wang–Yokoyama [19] proved global existence of small solutions to (6.1) when $N = 2, 3$. Therefore although it is open but one can expect that the same conclusion can be proved for all dimensions.

Remark 6.2. In Wang–Zhou [56], the lower estimate for lifespan of solutions to (6.1) with $N = 4$ and $p \in \{2\} \cup [3, \infty)$ is given. Therefore in these cases, the upper bound for T in Proposition 6.1 is sharp.

Proof. We have already proved the first, second, fourth and fifth cases in Propositions 4.1 and 5.1 because G satisfies both $G \geq a|s|^q$ and $G \geq b|\sigma|^q$. Therefore we only consider the third case. Observe that u is a super-solution of $\partial_t^2 u - \Delta u = 0$. By virtue of Lemma 3.5, we already have

$$\delta_1'\Big(I[g]\varepsilon\Big)^p R^{N-\frac{N-1}{2}p} \leq \int_0^T \int_{\mathbb{R}^N} |\partial_t u|^p \psi_R^* \, dx\, dt.$$

On the other hand, since u is a super-solution of $\partial_t^2 u - \Delta u \geq a|u|^q + b|\partial_t u|^p$, Lemma 3.2 with Young's inequality implies

$$\begin{aligned} I[g]\varepsilon + \int_0^T \int_{\mathbb{R}^N} (a|u|^q + b|\partial_t u|^p)\psi_R^* \, dx\, dt &\leq C R^{-2} \int_0^T \int_{\mathbb{R}^N} |u|[\psi_R^*]^{\frac{1}{q}} \, dx\, dt \\ &\leq \frac{a^{-\frac{1}{q-1}} C^{q'}}{q'} R^{N-\frac{q+1}{q-1}} + \frac{a}{q} \int_0^T \int_{\mathbb{R}^N} |u|^q \psi_R^* \, dx\, dt. \end{aligned}$$

Combining the above inequalities, we deduce

$$b\delta_1'\Big(I[g]\varepsilon\Big)^p R^{N-\frac{N-1}{2}p} \leq \frac{a^{-\frac{1}{q-1}} C^{q'}}{q'} R^{N-\frac{q+1}{q-1}}.$$

Since the choice of $R \in (1, T)$ is arbitrary, this gives the third estimate for T. $\square$

7. The case of the system $\partial_t^2 u - \Delta u = G_1(v)$ and $\partial_t^2 v - \Delta v = G_2(u)$

The problem in this section is the following weakly coupled semilinear wave equations

$$\begin{cases} \partial_t^2 u - \Delta u = G_1(v), & (x,t) \in \mathbb{R}^N \times (0,T), \\ \partial_t^2 v - \Delta v = G_2(u) & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(0) = \varepsilon f_1 & x \in \mathbb{R}^N, \\ \partial_t u(0) = \varepsilon g_1 & x \in \mathbb{R}^N, \\ v(0) = \varepsilon f_2 & x \in \mathbb{R}^N, \\ \partial_t v(0) = \varepsilon g_2 & x \in \mathbb{R}^N, \end{cases} \tag{7.1}$$

where the nonlinearities $G_1 \in C^1(\mathbb{R})$ and $G_2 \in C^1(\mathbb{R})$ satisfy

$$G_1(0) = 0, \quad G_2(0) = 0, \quad G_1(s) \geq a|s|^p, \quad G_2(s) \geq b|s|^q \quad s \in \mathbb{R}$$

for some $a, b > 0$ and $p, q > 1$. The problem (7.1) with $G_1(s) = |s|^p$ and $G_2(s) = |s|^q$ is studied by Deng [7], Kubo–Ohta [31], Agemi–Kurokawa–Takamura [2], Kurokawa–Takamura–Wakasa

[35]. The aim of this section is to find the same result about upper bound of lifespan of solutions to (7.1) by using a test function method similar to the one in Section 4.

In this case the definition of weak solutions is the following:

Definition 7.1. Let $f_1, f_2, g_1, g_2 \in C_c^\infty(\mathbb{R}^N)$. The pair of functions

$$(u,v) \in C([0,T);(H^1(\mathbb{R}^N))^2) \cap C^1([0,T);(L^2(\mathbb{R}^N))^2),$$
$$G_2(u) \in L^1(0,T;L^1(\mathbb{R}^N)), \quad G_1(v) \in L^1(0,T;L^1(\mathbb{R}^N))$$

is called a weak solution of (7.1) in $(0,T)$ if $(u,v)(0) = (\varepsilon f_1, \varepsilon f_2)$, $(\partial_t u, \partial_t v)(0) = (\varepsilon g_1, \varepsilon g_2)$ and for every $\Psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$,

$$\varepsilon \int_{\mathbb{R}^N} g_1(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} G_1\big(v(x,t)\big)\Psi(x,t)\,dx\,dt$$
$$= \int_0^T \int_{\mathbb{R}^N} \Big(-\partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t)\cdot \nabla\Psi(x,t)\Big)\,dx\,dt,$$
$$\varepsilon \int_{\mathbb{R}^N} g_2(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} G_2\big(u(x,t)\big)\Psi(x,t)\,dx\,dt$$
$$= \int_0^T \int_{\mathbb{R}^N} \Big(-\partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t)\cdot \nabla\Psi(x,t)\Big)\,dx\,dt.$$

As in the previous works listed above, we introduce

$$F_{SS}(N,p,q) = \left(p + 2 + \frac{1}{q}\right)(pq-1)^{-1} - \frac{N-1}{2}.$$

The assertion for the estimates for T is the following. The result has been given until Kurokawa–Takamura–Wakasa [35].

Proposition 7.1. *Let (f_1, g_1) and (f_2, g_2) satisfy (1.3) and let (u,v) be a weak solution of the system (7.1) satisfying* $\operatorname{supp}(u,v) \subset \{(x,t) \in \mathbb{R}^N \times [0,T)\,;\ |x| \le r_0 + t\}$ *for* $r_0 = \sup\{|x|\,;\ x \in \operatorname{supp}(f_1, f_2, g_1, g_2)\}$. *If*

$$\Gamma_{SS}(N,p,q) = \max\{F_{SS}(N,p,q), F_{SS}(N,q,p)\} \ge 0,$$

then T has the following upper bound

$$T \le \begin{cases} C\varepsilon^{-\Gamma_{SS}(N,p,q)^{-1}} & \textit{if } \Gamma_{SS}(N,p,q) > 0, \\ \exp(C\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}) & \textit{if } \Gamma_{SS}(N,p,q) = 0,\ p \neq q, \\ \exp(C\varepsilon^{-p(p-1)}) & \textit{if } \Gamma_{SS}(N,p,q) = 0,\ p = q \end{cases}$$

for every $\varepsilon \in (0, \varepsilon_0]$, *where* ε_0 *and* C *are positive constants independent of* ε.

Proof. We assume $F_{SS}(N,p,q) \ge F_{SS}(N,q,p)$, otherwise, we can interchange u and v. Moreover, we already have the following estimates by Lemma 3.4:

$$\delta_1\Big(I[g_1]\varepsilon\Big)^q R^{N-\frac{N-1}{2}q} \le \int_0^T \int_{\mathbb{R}^N} |u|^q \psi_R^*\, dx\, dt. \tag{7.2}$$

Now we consider the case $F_{SS}(N,p,q) > 0$. By Lemma 3.2, we have

$$\begin{aligned} \left(\int_0^T \int_{\mathbb{R}^N} |v|^p \psi_R\, dx\, dt\right)^q &\le CR^{-2+(N-1)(q-1)} \int_0^T \int_{\mathbb{R}^N} |u|^q \psi_R^*\, dx\, dt, \\ \left(\int_0^T \int_{\mathbb{R}^N} |u|^q \psi_R\, dx\, dt\right)^p &\le CR^{-2+(N-1)(p-1)} \int_0^T \int_{\mathbb{R}^N} |v|^p \psi_R^*\, dx\, dt. \end{aligned} \tag{7.3}$$

These imply

$$\begin{aligned} \left(\int_0^T \int_{\mathbb{R}^N} |u|^q \psi_R\, dx\, dt\right)^{pq} &\le C\left(R^{-2+(N-1)(p-1)} \int_{\mathbb{R}^N} |v|^p \psi_R^*\, dx\, dt\right)^q \\ &\le CR^{[-2+(N-1)(p-1)]q-2+(N-1)(q-1)} \int_0^T \int_{\mathbb{R}^N} |u|^q \psi_R^*\, dx\, dt \\ &\le CR^{(N-1)(pq-1)-2(q+1)} \int_0^T \int_{\mathbb{R}^N} |u|^q \psi_R^*\, dx\, dt, \end{aligned}$$

and hence

$$\int_0^T \int_{\mathbb{R}^N} |u|^q \psi_R\, dx\, dt \le CR^{N-1-\frac{2(q+1)}{pq-1}} = CR^{N-\frac{pq+2q+1}{pq-1}}.$$

Combining (7.2), we deduce

$$\Big(I[g_1]\varepsilon\Big)^q \le CR^{\frac{N-1}{2}q-\frac{pq+2q+1}{pq-1}} = CR^{-qF_{SS}(N,p,q)}.$$

Since $R \in (1,T)$ is arbitrary, we have the upper bound for T.

Next we consider the critical case $F_{SS}(N,p,q)=0$. If $F_{SS}(N,p,q)=F_{SS}(N,q,p)$, then we have $p=q=p_S(N)$. In this case, we consider the following differential inequality

$$\partial_t^2(u+v)-\Delta(u+v)=b|u|^p+a|v|^p \ge 2^{-p}\min\{a,b\}(|u|+|v|)^p \ge 2^{-p}\min\{a,b\}|u+v|^p.$$

Applying Proposition 4.1 with a replaced with $2^{-p}\min\{a,b\}$, we can obtain $T \le \exp(C\varepsilon^{-p(p-1)})$. Here we assume $0=F_{SS}(N,p,q)>F_{SS}(N,q,p)$. Then combining (7.2) and (7.3), we have

$$\int_0^T\int_{\mathbb{R}^N}|v|^p\psi_R^*\,dx\,dt \ge \delta_1'\Big(I[g_1]\varepsilon\Big)^{pq}R^{(N-\frac{N-1}{2}q)p+2-(N-1)(p-1)} = \delta_1'\Big(I[g_1]\varepsilon\Big)^{pq}R^{\frac{N-1}{2}-\frac{1}{q}}$$

where we have used $F_{SS}(N,p,q)=0$. We see from Lemma 3.6 **(iii)** that

$$\int_0^T\int_{\mathbb{R}^N}|v|^p\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt \ge \delta_1'\Big(I[g_1]\varepsilon\Big)^{pq}$$

with $\beta=\beta_q=\frac{N-1}{2}-\frac{1}{q}$ and $\lambda=\lambda_{\beta_q}$. By using Lemma 3.7 **(i)** with $\beta=\beta_q$, Lemma 3.8, (7.3) and the condition $F_{SS}(N,p,q)=0$, we have

$$\begin{aligned}
\left(\int_0^T\int_{\mathbb{R}^N}|v|^p\Phi_{\beta,\lambda}\psi_R\,dx\,dt\right)^{pq} &\le CR^{-(N-\frac{N-1}{2}q)p}(\log R)^{p(q-1)}\left(\int_0^T\int_{\mathbb{R}^N}|u|^q\psi_R^*\,dx\,dt\right)^p\\
&\le CR^{-(N-\frac{N-1}{2}q)p-2+(N-1)(p-1)}(\log R)^{p(q-1)}\int_0^T\int_{\mathbb{R}^N}|v|^p\psi_R^*\,dx\,dt\\
&\le C(\log R)^{p(q-1)}\int_0^T\int_{\mathbb{R}^N}|v|^p\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt.
\end{aligned}$$

In view of Lemma 3.9, taking $Y=Y[|v|^p\Phi_{\beta,\lambda}]$, we deduce

$$\begin{cases}\varepsilon^{pq}\le CRY'(R),\\ [Y(R)]^{pq}\le CR(\log R)^{p(q-1)}Y'(R).\end{cases}$$

Applying Lemma 3.10 with $\delta=\varepsilon^{pq}$, $p_1=pq$ and $p_2=p(q-1)+1$, we obtain

$$T\le \exp(C\varepsilon^{-q(pq-1)}).$$

The proof is complete. □

8. The case of the system $\partial_t^2 u - \Delta u = G_1(\partial_t v)$ and $\partial_t^2 v - \Delta v = G_2(\partial_t u)$

We consider the following weakly coupled system of semilinear wave equations with nonlinearities including derivatives

$$\begin{cases} \partial_t^2 u - \Delta u = G_1(\partial_t v), & (x,t) \in \mathbb{R}^N \times (0,T), \\ \partial_t^2 v - \Delta v = G_2(\partial_t u) & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(0) = \varepsilon f_1 & x \in \mathbb{R}^N, \\ \partial_t u(0) = \varepsilon g_1 & x \in \mathbb{R}^N, \\ v(0) = \varepsilon f_2 & x \in \mathbb{R}^N, \\ \partial_t v(0) = \varepsilon g_2 & x \in \mathbb{R}^N, \end{cases} \tag{8.1}$$

where the nonlinearities $G_1 \in C^1(\mathbb{R})$ and $G_2 \in C^1(\mathbb{R})$ satisfy

$$G_1(0) = 0, \quad G_2(0) = 0, \quad G_1(\sigma) \ge a|\sigma|^p, \quad G_2(\sigma) \ge b|\sigma|^q, \quad \sigma \in \mathbb{R}$$

for some $a, b > 0$ and $p, q > 1$. The blowup phenomena of the system (8.1) is studied in Deng [7]. It seems that the upper bound of lifespan of solutions to (8.1) has not been obtained so far. In the present paper we obtain an upper bound of lifespan by our technique similar to Section 5.

In this case the definition of weak solutions is the following:

Definition 8.1. Let $f_1, f_2, g_1, g_2 \in C_c^\infty(\mathbb{R}^N)$. The pair of functions

$$(u, v) \in C([0,T); (H^1(\mathbb{R}^N))^2) \cap C^1([0,T); (L^2(\mathbb{R}^N))^2),$$
$$G_2(\partial_t u) \in L^1(0,T; L^1(\mathbb{R}^N)), \quad G_1(\partial_t v) \in L^1(0,T; L^1(\mathbb{R}^N))$$

is called a weak solution of (8.1) in $(0,T)$ if $(u,v)(0) = (\varepsilon f_1, \varepsilon f_2)$, $(\partial_t u, \partial_t v)(0) = (\varepsilon g_1, \varepsilon g_2)$ and for every $\Psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$,

$$\begin{aligned} &\varepsilon \int_{\mathbb{R}^N} g_1(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} G_1\big(\partial_t v(x,t)\big)\Psi(x,t)\,dx\,dt \\ &= \int_0^T \int_{\mathbb{R}^N} \Big(- \partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t) \cdot \nabla \Psi(x,t)\Big)\,dx\,dt, \\ &\varepsilon \int_{\mathbb{R}^N} g_2(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} G_2\big(\partial_t u(x,t)\big)\Psi(x,t)\,dx\,dt \\ &= \int_0^T \int_{\mathbb{R}^N} \Big(- \partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t) \cdot \nabla \Psi(x,t)\Big)\,dx\,dt. \end{aligned}$$

In this case, set

$$F_{GG}(N,p,q)=\frac{p+1}{pq-1}-\frac{N-1}{2}.$$

The exponent F_{GG} seems to play the same rule (with the shift of dimension N to $N-1$) as the one for weakly coupled heat equations in Escobedo–Herrero [9] (see also Nishihara–Wakasugi [44] for weakly coupled system of damped wave equations).

Proposition 8.1. *Let* (f_1,g_1) *and* (f_2,g_2) *satisfy* (1.3) *and let* (u,v) *be a weak solution of the system* (8.1) *satisfying* $\operatorname{supp}(u,v)\subset\{(x,t)\in\mathbb{R}^N\times[0,T)\ ;\ |x|\le r_0+t\}$ *for* $r_0=\sup\{|x|\ ;\ x\in \operatorname{supp}(f_1,f_2,g_1,g_2)\}$. *If*

$$\Gamma_{GG}(N,p,q)=\max\{F_{GG}(N,p,q),F_{GG}(N,q,p)\}\ge 0,$$

then T *has the following upper bound*

$$T\le\begin{cases}C\varepsilon^{-\Gamma_{GG}(N,p,q)^{-1}} & \text{if } \Gamma_{GG}(N,p,q)>0,\\ \exp(C\varepsilon^{-(pq-1)}) & \text{if } \Gamma_{GG}(N,p,q)=0,\quad p\neq q,\\ \exp(C\varepsilon^{-(p-1)}) & \text{if } \Gamma_{GG}(N,p,q)=0,\quad p=q\end{cases}$$

for every $\varepsilon\in(0,\varepsilon_0]$, *where* ε_0 *and* C *are positive constants independent of* ε.

Proof. As in the proof of Proposition 7.1, we only consider the case $F_{GG}(N,p,q)\ge F_{GG}(N,q,p)$ (that is, $p\ge q$). Lemma 3.7 **(ii)** with $\beta>\frac{N-1}{2}+1$ and $\lambda=\lambda_\beta$ and Lemma 3.8 imply

$$\begin{aligned}\frac{\varepsilon}{2}I[g_1]+a\int_0^T\int_{\mathbb{R}^N}|\partial_t v|^p\Phi_{\beta,\lambda}\psi_R\,dx\,dt &\le CR^{-1}\int_0^T\int_{\mathbb{R}^N}|\partial_t u|\Phi_{\beta,\lambda}[\psi_R^*]^{\frac{1}{q}}\,dx\,dt\\ &\le CR^{-(\frac{1}{q-1}-\frac{N-1}{2})\frac{1}{q'}}\left(\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^q\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt\right)^{\frac{1}{q}},\end{aligned}$$

and similarly,

$$\begin{aligned}\frac{\varepsilon}{2}I[g_2]+b\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^q\Phi_{\beta,\lambda}\psi_R\,dx\,dt &\le CR^{-1}\int_0^T\int_{\mathbb{R}^N}|\partial_t v|\Phi_{\beta,\lambda}[\psi_R^*]^{\frac{1}{p}}\,dx\,dt\\ &\le CR^{-(\frac{1}{p-1}-\frac{N-1}{2})\frac{1}{p'}}\left(\int_0^T\int_{\mathbb{R}^N}|\partial_t v|^p\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt\right)^{\frac{1}{p}}.\end{aligned}$$

Combining these inequalities, we deduce

$$\left(\frac{\varepsilon}{2}I[g_1]+a\int_0^T\int_{\mathbb{R}^N}|\partial_t v|^p\Phi_{\beta,\lambda}\psi_R\,dx\,dt\right)^{pq}\le CR^{(\frac{N-1}{2})(pq-1)-p-1}\int_0^T\int_{\mathbb{R}^N}|\partial_t v|^p\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt.$$

Using $Y=Y[|\partial_t v|^p\Phi_{\beta,\lambda}]$ in Lemma 3.9, we can verify

$$T\le\begin{cases}C\varepsilon^{-F_{GG}(N,p,q)^{-1}} & \text{if } F_{GG}(N,p,q)>0,\\ \exp(C\varepsilon^{-(pq-1)}) & \text{if } F_{GG}(N,p,q)=0.\end{cases}$$

Note that in the case $F_{GG}(N,p,q)=F_{GG}(N,q,p)=0$, we have $p=q$ and then $\Gamma_G(N,p)=0$. Applying Proposition 5.1 to the inequality

$$\partial_t^2(u+v)-\Delta(u+v)\ge 2^{-p}\min\{a,b\}|\partial_t(u+v)|^p,$$

we have $T\le\exp(C\varepsilon^{-(p-1)})$. □

9. The case of system $\partial_t^2u-\Delta u=G_1(v)$ and $\partial_t^2v-\Delta v=G_2(\partial_t u)$

To close the paper, in the last section we consider the weakly coupled system of semilinear wave equations of the form

$$\begin{cases}\partial_t^2u-\Delta u=G_1(v), & (x,t)\in\mathbb{R}^N\times(0,T),\\ \partial_t^2v-\Delta v=G_2(\partial_t u) & (x,t)\in\mathbb{R}^N\times(0,T),\\ u(0)=\varepsilon f_1 & x\in\mathbb{R}^N,\\ \partial_t u(0)=\varepsilon g_1 & x\in\mathbb{R}^N,\\ v(0)=\varepsilon f_2 & x\in\mathbb{R}^N,\\ \partial_t v(0)=\varepsilon g_2 & x\in\mathbb{R}^N,\end{cases}\tag{9.1}$$

where the nonlinearities $G_1\in C^1(\mathbb{R})$ and $G_2\in C^1(\mathbb{R})$ satisfy

$$G_1(0)=0,\quad G_2(0)=0,\quad G_1(s)\ge a|s|^q,\quad G_2(\sigma)\ge b|\sigma|^p,\quad s,\sigma\in\mathbb{R}$$

for some $a,b>0$ and $p,q>1$. In this case the definition of weak solutions is the following:

Definition 9.1. Let $f_1,f_2,g_1,g_2\in C_c^\infty(\mathbb{R}^N)$. The pair of functions (u,v) satisfying

$$(u,v)\in C([0,T);(H^1(\mathbb{R}^N))^2)\cap C^1([0,T);(L^2(\mathbb{R}^N))^2),$$
$$G_2(\partial_t u)\in L^1(0,T;L^1(\mathbb{R}^N)),\quad G_1(v)\in L^1(0,T;L^1(\mathbb{R}^N))$$

is called a weak solution of (9.1) in $(0,T)$ if $(u,v)(0)=(\varepsilon f_1,\varepsilon f_2)$, $(\partial_t u,\partial_t v)(0)=(\varepsilon g_1,\varepsilon g_2)$ and for every $\Psi\in C_c^\infty(\mathbb{R}^N\times[0,T))$,

$$\varepsilon \int_{\mathbb{R}^N} g_1(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} G_1\big(v(x,t)\big)\Psi(x,t)\,dx\,dt$$
$$= \int_0^T \int_{\mathbb{R}^N} \Big(- \partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t)\cdot \nabla \Psi(x,t)\Big)\,dx\,dt,$$
$$\varepsilon \int_{\mathbb{R}^N} g_2(x)\Psi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} G_2\big(\partial_t u(x,t)\big)\Psi(x,t)\,dx\,dt$$
$$= \int_0^T \int_{\mathbb{R}^N} \Big(- \partial_t u(x,t)\partial_t \Psi(x,t) + \nabla u(x,t)\cdot \nabla \Psi(x,t)\Big)\,dx\,dt.$$

Here we introduce two kinds of exponent for the problem (9.1).

$$F_{SG,1}(N,p,q) = \left(\frac{1}{p} + 1 + q\right)(pq-1)^{-1} - \frac{N-1}{2},$$
$$F_{SG,2}(N,p,q) = \left(2 + \frac{1}{q}\right)(pq-1)^{-1} - \frac{N-1}{2}.$$

The problem (9.1) is recently discussed in Hidano–Yokoyama [20] and the blowup phenomena for small solutions are shown in the case $F_{SG,1}(N,p,q) > 0$. The other condition $F_{SG,2}(N,p,q) \geq 0$ is now carried out by our test function method. Furthermore, we can also find the lifespan estimate for (p,q) on the borderline case.

Proposition 9.1. *Let (f_1, g_1) and (f_2, g_2) satisfy (1.3) and let (u, v) be a weak solution of the system (9.1) satisfying $\mathrm{supp}(u,v) \subset \{(x,t) \in \mathbb{R}^N \times [0,T)\ ;\ |x| \leq r_0 + t\}$ for $r_0 = \sup\{|x|\ ;\ x \in \mathrm{supp}\,(f_1, f_2, g_1, g_2)\}$. If*

$$\Gamma_{SG}(N,p,q) = \max\{F_{SG,1}(N,p,q), F_{SG,2}(N,p,q)\} \geq 0,$$

then T has the following upper bound:

$$T \leq \begin{cases} C\varepsilon^{-\Gamma_{SG}(N,p,q)^{-1}} & \text{if } \Gamma_{SG}(N,p,q) > 0, \\ \exp(C\varepsilon^{-q(pq-1)}) & \text{if } \Gamma_{SG,1}(N,p,q) = 0 > F_{SG,2}(N,p,q), \\ \exp(C\varepsilon^{-p(pq-1)}) & \text{if } \Gamma_{SG,1}(N,p,q) < 0 = F_{SG,2}(N,p,q), \\ \exp(C\varepsilon^{-(pq-1)}) & \text{if } \Gamma_{SG,1}(N,p,q) = 0 = F_{SG,2}(N,p,q) \end{cases}$$

for every $\varepsilon \in (0, \varepsilon_0]$, where ε_0 and C are positive constants independent of ε.

Remark 9.1. On the critical curve, we could find lifespan estimates including exponential functions. At the intersection point of two critical curves $\{\Gamma_{SG,1} = 0\}$ and $\{\Gamma_{SG,2} = 0\}$, some discontinuity in the sense of lifespan estimates appears.

Proof. (The case $\Gamma_{SG}(N,p,q)>0$.) By Lemma 3.4 for v and Lemma 3.5 for $\partial_t u$, we have

$$\delta_1\Big(I[g_1]\varepsilon\Big)^p R^{N-\frac{N-1}{2}p} \le \int_0^T\int_{\mathbb{R}^N} |\partial_t u|^p \psi_R^*\,dx\,dt, \tag{9.2}$$

$$\delta_1'\Big(I[g_2]\varepsilon\Big)^q R^{N-\frac{N-1}{2}q} \le \int_0^T\int_{\mathbb{R}^N} |v|^q \psi_R^*\,dx\,dt. \tag{9.3}$$

On the other hand, since u is a super-solution of $\partial_t^2 u-\Delta u=G=|v|^q$ and v is a super-solution of $\partial_t^2 v-\Delta v=\widetilde{G}=|\partial_t u|^p$ using Lemma 3.2 **(ii)** with u, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N} |v|^q\psi_R\,dx\,dt &\le CR^{-1}\int_0^T\int_{\mathbb{R}^N}|\partial_t u|[\psi_R^*]^{\frac{1}{p}}\,dx\,dt\\ &\le CR^{\frac{-1+N(p-1)}{p}}\left(\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R^*\,dx\,dt\right)^{\frac{1}{p}}\end{aligned} \tag{9.4}$$

and using Lemma 3.2 **(i)** for v, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N} |\partial_t u|^p\psi_R\,dx\,dt &\le CR^{-2}\int_0^T\int_{\mathbb{R}^N}|v|[\psi_R^*]^{\frac{1}{q}}\,dx\,dt\\ &\le CR^{\frac{-2+(N-1)(q-1)}{q}}\left(\int_0^T\int_{\mathbb{R}^N}|v|^q\psi_R^*\,dx\,dt\right)^{\frac{1}{q}}.\end{aligned} \tag{9.5}$$

Combining the above inequalities, we deduce

$$\begin{aligned}\left(\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R\,dx\,dt\right)^{pq} &\le CR^{[-2+(N-1)(q-1)]p-1+N(p-1)}\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R^*\,dx\,dt\\ &\le CR^{-pq-p-1+N(pq-1)}\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R^*\,dx\,dt\end{aligned}$$

and

$$\left(\int_0^T\int_{\mathbb{R}^N}|v|^q\psi_R\,dx\,dt\right)^{pq}\le CR^{[-1+N(p-1)]q-2+(N-1)(q-1)}\int_0^T\int_{\mathbb{R}^N}|v|^q\psi_R^*\,dx\,dt$$
$$\le CR^{-2q-1+N(pq-1)}\int_0^T\int_{\mathbb{R}^N}|v|^q\psi_R^*\,dx\,dt.$$

These yield that

$$\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R\,dx\,dt\le CR^{N-\frac{pq+p+1}{pq-1}}$$
$$\int_0^T\int_{\mathbb{R}^N}|v|^q\psi_R\,dx\,dt\le CR^{N-\frac{2q+1}{pq-1}},$$

and therefore combining (9.2) and (9.3), we obtain the desired estimates for T for $\Gamma_{SG}(N, p,q)>0$.

(The case $F_{SG,1}(N,p,q)=0>F_{SG,2}(N,p,q)$.) Observe that the condition $F_{SG,1}(N, p,q)=0$ yields

$$\left(N-\frac{N-1}{2}p-\beta_q\right)q+\left(N-\frac{N-1}{2}q-\beta_p-1\right)=-F_{SG,1}(N,p,q)=0. \tag{9.6}$$

We see by (9.2) and (9.5) that

$$\int_0^T\int_{\mathbb{R}^N}|v|^q\psi_R^*\,dx\,dt\ge C^{-q}R^{2-(N-1)(q-1)}\left(\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R\,dx\,dt\right)^q$$
$$\ge C^{-q}\delta_1^q\Big(I[g_1]\varepsilon\Big)^{pq}R^{(N-\frac{N-1}{2}p)q+2-(N-1)(q-1)}$$
$$\ge C^{-q}\delta_1^q\Big(I[g_1]\varepsilon\Big)^{pq}R^{(N-\frac{N-1}{2}p-\beta_q)q+N-\frac{N-1}{2}q}$$
$$=C^{-q}\delta_1^q\Big(I[g_1]\varepsilon\Big)^{pq}R^{\beta_p+1}$$

and therefore

$$\int_0^T\int_{\mathbb{R}^N}|v|^q\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt\ge C^{-q}\delta_1^q\Big(\varepsilon I[g_1]\Big)^{pq}$$

with $\beta=\beta_p+1$ and $\lambda=\lambda_{\beta_p+1}$. On the other hand, using Lemma 3.7 **(ii)** with $\beta=\beta_p+1$, Lemma 3.8, (9.5) and the condition $F_{SG,1}(N,p,q)=0$ again, we deduce

$$
\begin{aligned}
\left(\int_0^T\int_{\mathbb{R}^N} |v|^q \Phi_{\beta,\lambda}\psi_R\,dx\,dt\right)^{pq} &\le \left(\frac{C}{R}\int_0^T\int_{\mathbb{R}^N} |\partial_t u|\Phi_{\beta,\lambda}[\psi_R]^{\frac{1}{p}}\,dx\,dt\right)^{pq}\\
&\le CR^{-(N-\frac{N-1}{2}p)q}(\log R)^{q(p-1)}\left(\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R^*\,dx\,dt\right)^q\\
&\le CR^{-(\beta_p+1)}(\log R)^{q(p-1)}\int_0^T\int_{\mathbb{R}^N}|v|^q\psi_R^*\,dx\,dt\\
&\le C(\log R)^{q(p-1)}\int_0^T\int_{\mathbb{R}^N}|v|^q\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt.
\end{aligned}
$$

Lemmas 3.9 and 3.10 with $\delta=\varepsilon^{pq}$, $p_1=pq$ and $p_2=q(p-1)+1$ imply $T\le \exp(C\varepsilon^{-p(pq-1)})$.

(The case $F_{SG,1}(N,p,q)<0=F_{SG,2}(N,p,q)$**.)** Observe that the condition $F_{SG,2}(N,p,q)=0$ yields

$$
\left(N-\frac{N-1}{2}p-\beta_q\right)+\left(N-\frac{N-1}{2}q-\beta_p-1\right)p=-F_{SG,2}(N,p,q)=0. \tag{9.7}
$$

We see by (9.3) and (9.4) that

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R^*\,dx\,dt &\ge C^{-p}R^{1-N(p-1)}\left(\int_0^T\int_{\mathbb{R}^N}|v|^q\psi_R\,dx\,dt\right)^p\\
&\ge C^{-p}\delta_2^p\Big(I[g_2]\varepsilon\Big)^{pq}R^{(N-\frac{N-1}{2}q)p+1-N(p-1)}\\
&= C^{-p}\delta_2^p\Big(I[g_2]\varepsilon\Big)^{pq}R^{\beta_q}
\end{aligned}
$$

and therefore

$$
\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt\ge C^{-p}\delta_2^p\Big(I[g_2]\varepsilon\Big)^{pq}
$$

with $\beta=\beta_q$ and $\lambda=\lambda_{\beta_q}$. On the other hand, using Lemma 3.7 **(i)** with $\beta=\beta_q$, Lemma 3.8, (9.5) and the condition $F_{SG,2}(N,p,q)=0$, we have

$$
\begin{aligned}
\left(\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\Phi_{\beta,\lambda}\psi_R\,dx\,dt\right)^{pq} &\le \left(\frac{C}{R}\int_0^T\int_{\mathbb{R}^N}|v|\Phi_{\beta+1,\lambda}[\psi_R^*]^{\frac{1}{q}}\,dx\,dt\right)^{pq}\\
&\le CR^{-(N-\frac{N-1}{2}q)p}(\log R)^{p(q-1)}\left(\int_0^T\int_{\mathbb{R}^N}|v|^q\psi_R^*\,dx\,dt\right)^{p}\\
&\le CR^{-\beta_q}(\log R)^{p(q-1)}\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R^*\,dx\,dt\\
&\le C(\log R)^{p(q-1)}\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt. \qquad (9.8)
\end{aligned}
$$

Lemmas 3.9 and 3.10 with $\delta=\varepsilon^{pq}$, $p_1=pq$ and $p_2=p(q-1)+1$ imply $T\le\exp(C\varepsilon^{-q(pq-1)})$.

(The case $F_{SG,1}(N,p,q)=F_{SG,2}(N,p,q)=0$.) In this case we see from (9.6) and (9.7) that

$$
N-\frac{N-1}{2}p=\beta_q,\quad N-\frac{N-1}{2}q=\beta_p+1.
$$

This implies that (9.2) can be rewritten as

$$
\begin{aligned}
\delta_1\Big(I[g_1]\varepsilon\Big)^p &\le R^{-N+\frac{N-1}{2}p}\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\psi_R^*\,dx\,dt\\
&\le C\int_0^T\int_{\mathbb{R}^N}|\partial_t u|^p\Phi_{\beta,\lambda}\psi_R^*\,dx\,dt
\end{aligned}
$$

with $\beta=\beta_q$ and $\lambda=\lambda_{\beta_q}$. In view of the above estimate and (9.8), Lemma 3.9 with $w=|\partial_t u|^p\Phi_{\beta,\lambda}$ and Lemma 3.10 with $\delta=\varepsilon^p$, $p_1=pq$ and $p_2=p(q-1)+1$ imply $T\le\exp(C\varepsilon^{-(pq-1)})$. □

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