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The interplay of critical regularity of nonlinearities in a weakly coupled system of semi-linear damped wave equations

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Dedicated to Professor Daniele Del Santo from University of Trieste in honor of his 60$^{\text{th}}$ birthday

Abstract

We would like to study a weakly coupled system of semi-linear classical damped wave equations with moduli of continuity in nonlinearities whose powers belong to the critical curve in the $p - q$ plane. The main goal of this paper is to find out sharp conditions of these moduli of continuity which classify between global (in time) existence of small data solutions and finite time blow-up of (even) small data solutions.

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1. Introduction

Recently, there exist numerous contributions to non-linear damped wave equations with local or nonlocal nonlinearity as well. One of these models is the following semi-linear Cauchy problem for the classical damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & x \in \mathbb{R}^n,\ t \geq 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $p > 1$. The authors in [9] proved the global (in time) existence of small data energy solutions for

$$p > p_{\mathrm{Fuj}}(n) = 1 + \frac{2}{n},$$

the so-called Fujita exponent, and for $p \leq n/(n-2)$ if $n \geq 3$. Besides, they also indicated a blow-up result in the inverse case $1 < p < p_{\mathrm{Fuj}}(n)$ which was improved to $1 < p \leq p_{\mathrm{Fuj}}(n)$ in the paper [10] by using the well-known test function method so far. For this reason, we can say that the Fujita exponent distinguishes the admissible range of powers p in (1) into those possessing global (in time) existence of small data solutions (stability of the zero solution) and those producing a blow-up behavior (even for small data). However, to determine the critical nonlinearity, it seems too rough to restrict (1) to the scale of power nonlinearities $\{|u|^p\}_{p>1}$. Quite recently, the second author and collaborators have discussed this issue for the following Cauchy problem in the paper [1]:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^{p_{\mathrm{Fuj}}(n)} \mu(|u|), & x \in \mathbb{R}^n,\ t \geq 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $\mu = \mu(|u|)$ stands for a modulus of continuity, a well-known notation to describe the regularity of a function with respect to desired variables, here with respect to u. This means that it provides an additional regularity of the non-linear term in comparison with the power nonlinearity $|u|^{p_{\mathrm{Fuj}}(n)}$. More precisely, the authors in the cited paper have found out sharp conditions for the critical regularity of the non-linear term of (2), namely,

$$\int_0^c \frac{\mu(s)}{s} ds < \infty \quad \text{and} \quad \int_0^c \frac{\mu(s)}{s} ds = \infty,$$

where c is a sufficiently small positive constant. Both conditions separate the global (in time) existence of small data Sobolev solutions and the blow-up behavior of Sobolev solutions, respectively.

During the last decades, the study of Cauchy problems for weakly coupled systems of equations in place of exploiting single equations only has been achieving a great attention from many mathematicians because of their wide applications in various disciplines. One of the most typical problems is the following weakly coupled system of semi-linear classical damped wave equations (see, for example, [5–8] and references therein):

$$\begin{cases} u_{tt} - \Delta u + u_t = |v|^p, & x \in \mathbb{R}^n,\ t \geq 0, \\ v_{tt} - \Delta v + v_t = |u|^q, & x \in \mathbb{R}^n,\ t \geq 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

with $p, q > 1$. Particularly, the authors in the papers [5,7] investigated the global (in time) existence of small data Sobolev solutions to (3) in low space dimensions $n = 1, 2, 3$, which was extended for any space dimensions $n \geq 1$ in the paper [6] afterwards by using weighted energy estimates. Here the following condition for a pair of exponents (p, q) comes into play:

$$\frac{1 + \max\{p, q\}}{pq - 1} < \frac{n}{2}.$$

When this condition is no longer true, non-existence results of global (in time) Sobolev solutions to (3) were proved in [5–7]. For this reason, one may claim that the critical curve of the exponents (p, q) in (3) in the $p - q$ plane is described by the condition

$$\frac{1 + \max\{p, q\}}{pq - 1} = \frac{n}{2}. \tag{4}$$

The main interest of this paper is strongly inspired by the recent paper of the second author [1] for a possible connection between (2) and (3). A natural question arises that whether it is sharp or not to obtain the critical curve (4) in the scale of pairs of power nonlinearities $\{|v|^p, |u|^q\}_{p,q>1}$. Hence, the key motivation for this article is to give an answer to this question. Namely, let us consider the Cauchy problem for the following weakly coupled system of semi-linear classical damped wave equations with moduli of continuity terms in power nonlinearities:

$$\begin{cases} u_{tt} - \Delta u + u_t = |v|^{p^*} \mu_1(|v|), & x \in \mathbb{R}^n,\ t \geq 0, \\ v_{tt} - \Delta v + v_t = |u|^{q^*} \mu_2(|u|), & x \in \mathbb{R}^n,\ t \geq 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{5}$$

where the functions $\mu_1 = \mu_1(|v|)$ and $\mu_2 = \mu_2(|u|)$ are some suitable moduli of continuity. We assume that the pair of exponents (p^*, q^*) with $p^* > 1$ and $q^* > 1$ belongs to the critical curve described by (4) in the $p - q$ plane. Our main purpose of this paper is that we would like to understand the effect of interaction of additional regularities in power nonlinearities $|v|^{p^*}$ and $|u|^{q^*}$, which are given by these moduli of continuity, not only on the global (in time) existence of small data Sobolev solutions but also on finite time blow-up of Sobolev solutions. Especially, we are interested in looking for a threshold by exploring the following optimal conditions for μ_1 and μ_2:

$$\int_0^c \frac{1}{s}\big(\mu_1(s)\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(s)\big)^{\frac{1}{q^*+1}}\,ds < \infty \quad \text{or} \quad \int_0^c \frac{1}{s}\big(\mu_1(s)\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(s)\big)^{\frac{1}{q^*+1}}\,ds = \infty,$$

which leads either to global (in time) existence results or to non-existence results of global (in time) solutions individually (see later, Theorems 1.1 and 1.2). In the proofs we develop some ideas coming from the paper [1] to the weakly coupled system of type (5). Through this work, one should recognize that our results are not a simple generalization of those in [1]. Concretely, there are two points worthy to be mentioned. The first point as we can see is that allowing loss of decay appropriately, which has never appeared in [1], comes into play to find out these conditions for μ_1 and μ_2 in guaranteeing the existence of global (in time) Sobolev solutions. In other words, we can feel more explicitly how the required assumptions of additional regularities of nonlinearities follow essentially from using some suitable loss of decay. This gives a new interplay in comparison with the previous research papers in terms of the study of weakly coupled systems (see, for instance, [5–7]). Moreover, the other point worth to be noticed is that the technical choice of a test function with a parameter depending on p^*, q^* brings some remarkable benefits in the proof of the blow-up result.

Notations

- We denote $[s] := \max\big\{k \in \mathbb{Z} \,:\, k \le s\big\}$ as the integer part of $s \in \mathbb{R}$.
- For later convenience, hereafter C denotes a suitable positive constant and may have different value from line to line.
- For two given nonnegative functions f and g, we write $f \lesssim g$ when $f \le Cg$. We write $f \approx g$ when $g \lesssim f \lesssim g$.
- As usual, H^m and $\dot{H}^m$, with $m \in \mathbb{N}$, denote Sobolev spaces based on the L^2 space.

Main results

Without loss of generality, if we assume $p^* \le q^*$, then the critical curve in the $p-q$ plane for (5) becomes

$$\frac{1+q^*}{p^*q^*-1} = \frac{n}{2}. \tag{6}$$

Our main results concerned with the case $p^* \le q^*$ read as follows.

Theorem 1.1 ***(Global existence)***. *Let $n = 1, 2$. Assume that the following assumptions of moduli of continuity hold:*

$$s\mu_j'(s) = O\big(\mu_j(s)\big) \quad \text{as } s \to +0 \text{ with } j = 1, 2. \tag{7}$$

Moreover, we suppose that one of the following conditions is satisfied:

$$\text{i)} \int_0^c \frac{\mu_1(s)}{s} ds < \infty \quad \text{and} \quad \int_0^c \frac{\mu_2(s)}{s} ds < \infty. \tag{8}$$

$$\text{ii)} \text{ If } \int_0^c \frac{\mu_1(s)}{s} ds = \infty \ \text{ or } \ \int_0^c \frac{\mu_2(s)}{s} ds = \infty, \ \text{ then } \int_0^c \frac{1}{s}\big(\mu_1(s)\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(s)\big)^{\frac{1}{q^*+1}} ds < \infty. \tag{9}$$

Especially, when $q^* = p^*$, a further assumption required is that $s \in (0, c] \to \dfrac{\mu_1(s)}{\mu_2(s)}$ is a decreasing function. Here $c > 0$ is a suitable small constant. Then, there exists a constant $\epsilon > 0$ such that for any small data

$$\big((u_0, u_1),\ (v_0, v_1)\big) \in \mathcal{A} := \Big(\big(L^1 \cap H^{1+[n/2]}\big) \times \big(L^1 \cap H^{[n/2]}\big)\Big)^2$$

satisfying the assumption

$$\|(u_0, u_1), (v_0, v_1)\|_{\mathcal{A}} := \|u_0\|_{H^{1+[n/2]}} + \|u_0\|_{L^1} + \|u_1\|_{H^{[n/2]}} + \|u_1\|_{L^1} \le \epsilon,$$

we have a uniquely determined global (in time) small data Sobolev solution

$$(u, v) \in \Big(\mathcal{C}\big([0, \infty), H^1 \cap L^\infty\big)\Big)^2$$

to (5). The following estimates hold for $k = 0, 1$:

$$\begin{aligned}
\big\|\nabla^k u(t, \cdot)\big\|_{L^2} &\lesssim (1 + t)^{-\frac{n}{4} - \frac{k}{2} + \sigma(p^*, q^*)} \ell(t) \|(u_0, u_1), (v_0, v_1)\|_{\mathcal{A}},\\
\|u(t, \cdot)\|_{L^\infty} &\lesssim (1 + t)^{-\frac{n}{2} + \sigma(p^*, q^*)} \ell(t) \|(u_0, u_1), (v_0, v_1)\|_{\mathcal{A}},\\
\big\|\nabla^k v(t, \cdot)\big\|_{L^2} &\lesssim (1 + t)^{-\frac{n}{4} - \frac{k}{2}} \|(u_0, u_1), (v_0, v_1)\|_{\mathcal{A}},\\
\|v(t, \cdot)\|_{L^\infty} &\lesssim (1 + t)^{-\frac{n}{2}} \|(u_0, u_1), (v_0, v_1)\|_{\mathcal{A}},
\end{aligned}$$

where

$$\sigma(p^*, q^*) := \frac{q^* - p^*}{p^* q^* - 1} \tag{10}$$

and the weight function $\ell = \ell(t)$ is defined by

$$\ell(t) := \begin{cases} 1 & \text{if (8) holds},\\ \left(\dfrac{\mu_1\big(c(1+t)^{-\varepsilon}\big)}{\mu_2\big(c(1+t)^{-\varepsilon}\big)}\right)^{\frac{1}{q^*+1}} & \text{if (9) holds}, \end{cases} \tag{11}$$

with a sufficiently small constant $\varepsilon > 0$.

Remark 1.1. We want to point out that the constant $\sigma(p^*, q^*)$ and the weight function $\ell = \ell(t)$ appearing in the estimates for solutions in Theorem 1.1 represent some loss of decay in comparison with the corresponding estimates of Sobolev solutions to the corresponding linear Cauchy problem with vanishing right-hand side.

Example 1.1. We give some examples of moduli of continuity μ_1 and μ_2 satisfying the assumptions (8) and (9) in Theorem 1.1.

- The assumption (8) is fulfilled if we choose μ_1 and μ_2 by one of the following moduli of continuity:

1. $\mu(s) = s^\alpha$ with $\alpha \in (0, 1]$;
2. $\mu(s) = \big(\log(1+s)\big)^\alpha$ with $\alpha \in (0, 1]$;
3. $\mu(0) = 0$ and $\mu(s) = \Big(\log\frac{1}{s}\Big)^{-\alpha}$ with $\alpha > 1$;
4. $\mu(0) = 0$ and $\mu(s) = \Big(\log\frac{1}{s}\Big)^{-1}\Big(\log\log\frac{1}{s}\Big)^{-1}\cdots\Big(\underbrace{\log\cdots\log}_{m \text{ times } \log}\frac{1}{s}\Big)^{-\alpha}$ with $m \in \mathbb{N}$, $\alpha > 1$.

- The assumption (9) is fulfilled if we choose μ_1 and μ_2 by one of the following pairs of moduli of continuity:
  1. $\mu_1(0) = 0$ and $\mu_1(s) = \Big(\log\frac{1}{s}\Big)^{-\alpha_1}$ with $0 < \alpha_1 \le 1$,
     $\mu_2(0) = 0$ and $\mu_2(s) = \Big(\log\frac{1}{s}\Big)^{-\alpha_2}$ with $\alpha_2 > 1$,
     or $\mu_1(0) = 0$ and $\mu_1(s) = \Big(\log\frac{1}{s}\Big)^{-\alpha_1}$ with $\alpha_1 > 1$,
     $\mu_2(0) = 0$ and $\mu_2(s) = \Big(\log\frac{1}{s}\Big)^{-\alpha_2}$ with $0 < \alpha_2 \le 1$,
     provided that
     $$\frac{q^*}{q^*+1}\alpha_1 + \frac{1}{q^*+1}\alpha_2 > 1;$$
  2. $\mu_1(0) = 0$ and $\mu_1(s) = \Big(\log\frac{1}{s}\Big)^{-1}\Big(\log\log\frac{1}{s}\Big)^{-1}\cdots\Big(\underbrace{\log\cdots\log}_{m \text{ times } \log}\frac{1}{s}\Big)^{-\alpha_1}$ with $m \in \mathbb{N}$, $0 < \alpha_1 \le 1$,
     $\mu_2(0) = 0$ and $\mu_2(s) = \Big(\log\frac{1}{s}\Big)^{-1}\Big(\log\log\frac{1}{s}\Big)^{-1}\cdots\Big(\underbrace{\log\cdots\log}_{m \text{ times } \log}\frac{1}{s}\Big)^{-\alpha_2}$ with $m \in \mathbb{N}$, $\alpha_2 > 1$,
     or $\mu_1(0) = 0$ and $\mu_1(s) = \Big(\log\frac{1}{s}\Big)^{-1}\Big(\log\log\frac{1}{s}\Big)^{-1}\cdots\Big(\underbrace{\log\cdots\log}_{m \text{ times } \log}\frac{1}{s}\Big)^{-\alpha_1}$ with $m \in \mathbb{N}$, $\alpha_1 > 1$,
     $\mu_2(0) = 0$ and $\mu_2(s) = \Big(\log\frac{1}{s}\Big)^{-1}\Big(\log\log\frac{1}{s}\Big)^{-1}\cdots\Big(\underbrace{\log\cdots\log}_{m \text{ times } \log}\frac{1}{s}\Big)^{-\alpha_2}$ with $m \in \mathbb{N}$, $0 < \alpha_2 \le 1$,
     provided that
     $$\frac{q^*}{q^*+1}\alpha_1 + \frac{1}{q^*+1}\alpha_2 > 1.$$

Intuitively, from the two latter examples one can think of the modulus of continuity

$$\mu_{1,2} := \mu_{1,2}(s) = \big(\mu_1(s)\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(s)\big)^{\frac{1}{q^*+1}}$$

as a "middle" modulus of continuity between μ_1 and μ_2 being subject to the following conditions:

$$\int_0^c \frac{\mu_1(s)}{s} ds = \infty \quad \text{and} \quad \int_0^c \frac{\mu_2(s)}{s} ds < \infty$$

or

$$\int_0^c \frac{\mu_1(s)}{s} ds < \infty \quad \text{and} \quad \int_0^c \frac{\mu_2(s)}{s} ds = \infty.$$

Then, we may take, among other things, suitable choices of μ_1 as well as μ_2 to claim that $\mu_{1,2}$ satisfies

$$\int_0^c \frac{\mu_{1,2}(s)}{s} ds < \infty.$$

Theorem 1.2 *(**Blow-up**). Assume that the initial data $u_0 = v_0 = 0$ and $u_1, v_1 \in L^1$ satisfy the following relations:*

$$\int_{\mathbb{R}^n} u_1(x)\,dx > 0 \quad \textit{and} \quad \int_{\mathbb{R}^n} v_1(x)\,dx > 0. \tag{12}$$

Moreover, we suppose the following assumptions of moduli of continuity:

$$s^k \mu_j^{(k)}(s) = o\big(\mu_j(s)\big) \quad \textit{as } s \to +0 \textit{ with } j, k = 1, 2, \tag{13}$$

and

$$\int_0^c \frac{1}{s}\big(\mu_1(s)\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(s)\big)^{\frac{1}{q^*+1}}\,ds = \infty, \tag{14}$$

where $c > 0$ is a suitable small constant. Then, there is no global (in time) Sobolev solution to (5).

Example 1.2. We give some examples of moduli of continuity μ_1 and μ_2 fulfilling the assumption (14) in Theorem 1.2. One may choose μ_1 and μ_2 as follows:

1. $\mu_1(0) = 0$ and $\mu_1(s) = \Big(\log \frac{1}{s}\Big)^{-\alpha_1}$ with $\alpha_1 > 0$,
   $\mu_2(0) = 0$ and $\mu_2(s) = \Big(\log \frac{1}{s}\Big)^{-\alpha_2}$ with $\alpha_2 > 0$,
   provided that

$$\frac{q^*}{q^*+1}\alpha_1 + \frac{1}{q^*+1}\alpha_2 \le 1;$$

2. $\mu_1(0)=0$ and $\mu_1(s)=\Big(\log\frac{1}{s}\Big)^{-1}\Big(\log\log\frac{1}{s}\Big)^{-1}\cdots\Big(\underbrace{\log\cdots\log}_{m \text{ times } \log}\frac{1}{s}\Big)^{-\alpha_1}$ with $m\in\mathbb{N}$, $\alpha_1>0$,

   $\mu_2(0)=0$ and $\mu_2(s)=\Big(\log\frac{1}{s}\Big)^{-1}\Big(\log\log\frac{1}{s}\Big)^{-1}\cdots\Big(\underbrace{\log\cdots\log}_{m \text{ times } \log}\frac{1}{s}\Big)^{-\alpha_2}$ with $m\in\mathbb{N}$, $\alpha_2>0$,

   provided that

$$\frac{q^*}{q^*+1}\alpha_1+\frac{1}{q^*+1}\alpha_2\le 1.$$

2. Proofs of main results

2.1. Global (in time) existence of small data solutions

In order to prove the global (in time) existence of small data Sobolev solutions, the following preliminary lemmas come into play.

Lemma 2.1 *(Lemma 1 in [4]). The Sobolev solutions to the corresponding linear Cauchy problem to (5) with vanishing right-hand side satisfy the following estimates:*

$$\big\|\nabla^k w(t,\cdot)\big\|_{L^2}\lesssim(1+t)^{-\frac{n}{4}-\frac{k}{2}}\big(\|w_0\|_{L^1}+\|w_0\|_{H^k}+\|w_1\|_{L^1}+\|w_1\|_{H^{k-1}}\big),$$

with $k=0,\,1,\,1+[n/2]$ *and*

$$\|w(t,\cdot)\|_{L^\infty}\lesssim(1+t)^{-\frac{n}{2}}\big(\|w_0\|_{L^1}+\|w_0\|_{H^{1+[n/2]}}+\|w_1\|_{L^1}+\|w_1\|_{H^{[n/2]}}\big),$$

where w *stands for* u *or* v.

Lemma 2.2. *Let* $\mu_1=\mu_1(s)$ *and* $\mu_2=\mu_2(s)$ *be moduli of continuity. Then, the following estimates hold:*

(a) $$\int_0^t(1+t-\tau)^{-\alpha}(1+\tau)^{-1}\big(\mu_1\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau$$
$$\lesssim(1+t)^{-\alpha}\int_0^t(1+\tau)^{-1}\big(\mu_1\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau$$

   for any $\alpha\le 1$ *and for all* $\beta_1,\beta_2,\gamma\ge 0$,

(b) $$\int_0^\infty(1+\tau)^{-1}\big(\mu_1\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau=C_0\int_0^C\frac{1}{s}\big(\mu_1(s)\big)^{\beta_1}\big(\mu_2(s)\big)^{\beta_2}ds$$

   for any $\beta_1,\beta_2\ge 0$ *and for all* $\gamma>0$, *where* $C_0=C_0(C,\gamma)$ *is a suitable positive constant.*

Proof. To prove the first estimate, we divide the integral on the left-hand side into two parts as follows:

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-\alpha}(1+\tau)^{-1}\big(\mu_1\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau\\
&\quad=\int_0^{t/2} (1+t-\tau)^{-\alpha}(1+\tau)^{-1}\big(\mu_1\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau\\
&\qquad+\int_{t/2}^{t} (1+t-\tau)^{-\alpha}(1+\tau)^{-1}\big(\mu_1\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau\\
&\quad=: I_1(t)+I_2(t).
\end{aligned}$$

Using the relation $1+t-\tau\approx 1+t$ for any $\tau\in[0,t/2]$ one derives

$$\begin{aligned}
I_1(t)&\lesssim (1+t)^{-\alpha}\int_0^{t/2}(1+\tau)^{-1}\big(\mu_1\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau\\
&\lesssim (1+t)^{-\alpha}\int_0^{t}(1+\tau)^{-1}\big(\mu_1\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau. \qquad (15)
\end{aligned}$$

In addition, we notice the relation $1+\tau\approx 1+t$ for any $\tau\in[t/2,t]$ to deal with I_2 in the following way:

$$I_2(t)\lesssim (1+t)^{-\alpha}\int_{t/2}^{t}(1+t-\tau)^{-\alpha}(1+\tau)^{\alpha-1}\big(\mu_1\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau.$$

Due to the hypothesis $\alpha\le 1$ and $\gamma\ge 0$, it holds

$$\begin{cases}(1+\tau)^{\alpha-1}\le (1+t-\tau)^{\alpha-1}\\ (1+\tau)^{-\gamma}\le (1+t-\tau)^{-\gamma}\end{cases}\qquad \text{for any } \tau\in[t/2,t].$$

As a result, this leads to

$$\begin{aligned}
I_2(t)&\lesssim (1+t)^{-\alpha}\int_{t/2}^{t}(1+t-\tau)^{-1}\big(\mu_1\big(C(1+t-\tau)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+t-\tau)^{-\gamma}\big)\big)^{\beta_2}d\tau\\
&\lesssim (1+t)^{-\alpha}\int_0^{t/2}(1+\rho)^{-1}\big(\mu_1\big(C(1+\rho)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\rho)^{-\gamma}\big)\big)^{\beta_2}d\rho\\
&\lesssim (1+t)^{-\alpha}\int_0^{t}(1+\rho)^{-1}\big(\mu_1\big(C(1+\rho)^{-\gamma}\big)\big)^{\beta_1}\big(\mu_2\big(C(1+\rho)^{-\gamma}\big)\big)^{\beta_2}d\rho, \qquad (16)
\end{aligned}$$

where we have used the increasing property of the functions μ_1, μ_2 as well as the change of variables $\rho = t - \tau$. Combining (15) and (16) we may conclude the estimate (a). Finally, a standard change of variables implies immediately the estimate (b). $\square$

Lemma 2.3 *(Gagliardo-Nirenberg inequality, see [2,3]). Let $j, m \in \mathbb{N}$ with $j < m$ and $w \in H^m(\mathbb{R}^n)$. Let us assume $\frac{j}{m} \leq \theta \leq 1$ and $1 \leq r, r_1, r_2 \leq \infty$ such that*

$$j - \frac{n}{r} = \Big(m - \frac{n}{r_1}\Big)\theta - \frac{n}{r_2}(1-\theta).$$

Then, it holds

$$\big\|\nabla^j w\big\|_{L^r} \lesssim \big\|\nabla^m w\big\|_{L^{r_1}}^{\theta}\, \|w\|_{L^{r_2}}^{1-\theta},$$

provided that $(m - \frac{n}{r_1}) - j \notin \mathbb{N}$, that is, $\frac{n}{r_1} > m - j$ or $\frac{n}{r_1} \notin \mathbb{N}$.

If $(m - \frac{n}{r_1}) - j \in \mathbb{N}$, then this inequality holds provided that $\frac{j}{m} \leq \theta < 1$.

Proof of Theorem 1.1. We introduce the solution space

$$X(t) := \Big(\mathcal{C}\big([0,t], H^1 \cap L^\infty\big)\Big)^2$$

with the norm

$$\begin{aligned}
\|(u,v)\|_{X(t)} := \sup_{0\leq\tau\leq t} \Big(& (1+\tau)^{\frac{n}{4}-\sigma(p^*,q^*)}\big(\ell(\tau)\big)^{-1}\|u(\tau,\cdot)\|_{L^2} \\
& + (1+\tau)^{\frac{n}{4}+\frac{1}{2}-\sigma(p^*,q^*)}\big(\ell(\tau)\big)^{-1}\big\|\nabla u(\tau,\cdot)\big\|_{L^2} \\
& + (1+\tau)^{\frac{n}{2}-\sigma(p^*,q^*)}\big(\ell(\tau)\big)^{-1}\|u(\tau,\cdot)\|_{L^\infty} \\
& + (1+\tau)^{\frac{n}{4}}\|v(\tau,\cdot)\|_{L^2} + (1+\tau)^{\frac{n}{4}+\frac{1}{2}}\big\|\nabla v(\tau,\cdot)\big\|_{L^2} + (1+\tau)^{\frac{n}{2}}\|v(\tau,\cdot)\|_{L^\infty}\Big),
\end{aligned}$$

where the parameter $\sigma(p^*,q^*)$ and the weight function $\ell = \ell(\tau)$ are determined as in (10) and (11), respectively. We denote by $\mathcal{K}_0 = \mathcal{K}_0(t,x)$ and $\mathcal{K}_1 = \mathcal{K}_1(t,x)$ the fundamental solutions to the corresponding linear Cauchy problems for (5). Then, Sobolev solutions to (5) with vanishing right-hand sides are defined by

$$\begin{cases} u^{\ln}(t,x) = \mathcal{K}_0(t,x) \ast_x u_0(x) + \mathcal{K}_1(t,x) \ast_x u_1(x), \\ v^{\ln}(t,x) = \mathcal{K}_0(t,x) \ast_x v_0(x) + \mathcal{K}_1(t,x) \ast_x v_1(x). \end{cases}$$

Thanks to Duhamel's principle, Sobolev solutions to (5) are interpreted as solutions to the following system of non-linear integral equations:

$$\begin{cases} u(t,x) = u^{\ln}(t,x) + \int\limits_0^t \mathcal{K}_1(t-\tau,x) *_x \big(|v(\tau,x)|^{p^*}\mu_1(|v(\tau,x)|)\big) d\tau =: u^{\ln}(t,x) + u^{\mathrm{nl}}(t,x), \\ v(t,x) = v^{\ln}(t,x) + \int\limits_0^t \mathcal{K}_1(t-\tau,x) *_x \big(|u(\tau,x)|^{q^*}\mu_2(|u(\tau,x)|)\big) d\tau =: v^{\ln}(t,x) + v^{\mathrm{nl}}(t,x). \end{cases}$$

For all $t > 0$, we define the operator

$$\Psi: \quad (u,v) \in X(t) \mapsto \Psi(u,v)(t,x) = \big(u^{\ln}(t,x) + u^{\mathrm{nl}}(t,x), v^{\ln}(t,x) + v^{\mathrm{nl}}(t,x)\big).$$

Our aim is to apply Banach's fixed point theorem to arrive at global (in time) existence of small data Sobolev solutions to (5). To establish this, we need to indicate that the operator Ψ satisfies the following two inequalities:

$$\|\Psi(u,v)\|_{X(t)} \lesssim \|(u_0,u_1),(v_0,v_1)\|_{\mathcal{A}} + \|(u,v)\|_{X(t)}^{p^*} + \|(u,v)\|_{X(t)}^{q^*}, \tag{17}$$

$$\begin{aligned} \|\Psi(u,v) - \Psi(\bar{u},\bar{v})\|_{X(t)} \lesssim \|(u,v) - (\bar{u},\bar{v})\|_{X(t)} \Big(& \|(u,v)\|_{X(t)}^{p^*-1} + \|(\bar{u},\bar{v})\|_{X(t)}^{p^*-1} \\ & + \|(u,v)\|_{X(t)}^{q^*-1} + \|(\bar{u},\bar{v})\|_{X(t)}^{q^*-1} \Big). \end{aligned} \tag{18}$$

At first, we conclude the estimate

$$\big\|(u^{\ln}, v^{\ln})\big\|_{X(t)} \lesssim \|(u_0,u_1),(v_0,v_1)\|_{\mathcal{A}}$$

by Lemma 2.1. Hence, it suffices to prove the following inequality instead of (17):

$$\big\|(u^{\mathrm{nl}}, v^{\mathrm{nl}})\big\|_{X(t)} \lesssim \|(u,v)\|_{X(t)}^{p^*} + \|(u,v)\|_{X(t)}^{q^*}. \tag{19}$$

Before verifying the inequality (19), we take account of the following auxiliary estimates for $\tau \in [0,t]$:

$$\begin{aligned} &\big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} \\ &\quad \lesssim (1+\tau)^{-1+\frac{q^*-p^*}{p^*q^*-1}}\, \mu_1\Big(c(1+\tau)^{-\frac{1+q^*}{p^*q^*-1}}\Big)\|(u,v)\|_{X(t)}^{p^*}, \end{aligned} \tag{20}$$

$$\begin{aligned} &\big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} \\ &\quad \lesssim (1+\tau)^{-1}\big(\ell(\tau)\big)^{q^*}\, \mu_2\Big(c(1+\tau)^{-\frac{1+p^*}{p^*q^*-1}}\ell(\tau)\Big)\|(u,v)\|_{X(t)}^{q^*}, \end{aligned} \tag{21}$$

$$\begin{aligned} &\big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^1\cap H^1} \\ &\quad \lesssim (1+\tau)^{-1+\frac{q^*-p^*}{p^*q^*-1}}\, \mu_1\Big(c(1+\tau)^{-\frac{1+q^*}{p^*q^*-1}}\Big)\|(u,v)\|_{X(t)}^{p^*}, \end{aligned} \tag{22}$$

$$\begin{aligned} &\big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^1\cap H^1} \\ &\quad \lesssim (1+\tau)^{-1}\big(\ell(\tau)\big)^{q^*}\, \mu_2\Big(c(1+\tau)^{-\frac{1+p^*}{p^*q^*-1}}\ell(\tau)\Big)\|(u,v)\|_{X(t)}^{q^*}, \end{aligned} \tag{23}$$

where $c > 0$ is a suitable small constant. Indeed, we may re-write

$$\big\||v(\tau,\cdot)|^{p^*}\big\|_{L^1\cap L^2} = \|v(\tau,\cdot)\|^{p^*}_{L^{p^*}} + \|v(\tau,\cdot)\|^{p^*}_{L^{2p^*}},$$
$$\big\||u(\tau,\cdot)|^{q^*}\big\|_{L^1\cap L^2} = \|u(\tau,\cdot)\|^{q^*}_{L^{q^*}} + \|u(\tau,\cdot)\|^{q^*}_{L^{2q^*}}.$$

The application of Gagliardo-Nirenberg inequality from Lemma 2.3 to control the norms

$$\|v(\tau,\cdot)\|_{L^{p^*}}, \quad \|v(\tau,\cdot)\|_{L^{2p^*}}, \quad \|u(\tau,\cdot)\|_{L^{q^*}} \quad \text{and} \quad \|u(\tau,\cdot)\|^{q^*}_{L^{2q^*}}$$

yields immediately

$$\big\||v(\tau,\cdot)|^{p^*}\big\|_{L^1\cap L^2} \lesssim (1+\tau)^{-\frac{n}{2}(p^*-1)}\|(u,v)\|^{p^*}_{X(t)},$$
$$\big\||u(\tau,\cdot)|^{q^*}\big\|_{L^1\cap L^2} \lesssim (1+\tau)^{-\frac{n}{2}(q^*-1)+\sigma(p^*,q^*)q^*}\big(\ell(\tau)\big)^{q^*}\|(u,v)\|^{q^*}_{X(t)}.$$

Since μ_1, μ_2 are increasing functions, from the definition of the norm in $X(t)$ one obtains

$$\begin{aligned}\big\|\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^\infty} &\le \mu_1\big(\|v(\tau,\cdot)\|_{L^\infty}\big) \le \mu_1\Big(C(1+\tau)^{-\frac{n}{2}}\|(u,v)\|_{X(t)}\Big)\\ &\le \mu_1\Big(c(1+\tau)^{-\frac{n}{2}}\Big)\end{aligned}$$

and

$$\begin{aligned}\big\|\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^\infty} &\le \mu_2\big(\|u(\tau,\cdot)\|_{L^\infty}\big) \le \mu_2\Big(C(1+\tau)^{-\frac{n}{2}+\sigma(p^*,q^*)}\ell(\tau)\|(u,v)\|_{X(t)}\Big)\\ &\le \mu_2\Big(c(1+\tau)^{-\frac{n}{2}+\sigma(p^*,q^*)}\ell(\tau)\Big)\end{aligned}$$

with $c := C\varepsilon_0$, where ε_0 is a sufficiently small constant such that $\|(u,v)\|_{X(t)} \le \varepsilon_0$. For this reason, we may arrive for $\tau \in [0,t]$ at

$$\begin{aligned}\big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} &\lesssim \big\||v(\tau,\cdot)|^{p^*}\big\|_{L^1\cap L^2}\,\big\|\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^\infty}\\ &\lesssim (1+\tau)^{-\frac{n}{2}(p^*-1)}\,\mu_1\Big(c(1+\tau)^{-\frac{n}{2}}\Big)\|(u,v)\|^{p^*}_{X(t)}\\ &\lesssim (1+\tau)^{-1+\frac{q^*-p^*}{p^*q^*-1}}\,\mu_1\Big(c(1+\tau)^{-\frac{1+q^*}{p^*q^*-1}}\Big)\|(u,v)\|^{p^*}_{X(t)}\end{aligned}$$

and

$$\begin{aligned}&\big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^1\cap L^2}\\ &\quad\lesssim \big\||u(\tau,\cdot)|^{q^*}\big\|_{L^1\cap L^2}\,\big\|\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^\infty}\\ &\quad\lesssim (1+\tau)^{-\frac{n}{2}(q^*-1)+\sigma(p^*,q^*)q^*}\big(\ell(\tau)\big)^{q^*}\,\mu_2\Big(c(1+\tau)^{-\frac{n}{2}+\sigma(p^*,q^*)}\ell(\tau)\Big)\|(u,v)\|^{q^*}_{X(t)}\\ &\quad\lesssim (1+\tau)^{-1}\big(\ell(\tau)\big)^{q^*}\,\mu_2\Big(c(1+\tau)^{-\frac{1+p^*}{p^*q^*-1}}\ell(\tau)\Big)\|(u,v)\|^{q^*}_{X(t)},\end{aligned}$$

where we notice that the relations

$$-\frac{n}{2}(p^*-1)=-1+\frac{q^*-p^*}{p^*q^*-1},\quad -\frac{n}{2}(q^*-1)+\sigma(p^*,q^*)q^*=-1,$$
$$-\frac{n}{2}+\sigma(p^*,q^*)=-\frac{1+p^*}{p^*q^*-1}$$

are valid due to (6) and (10). This completes the proof of (20) and (21). In order to show the two remaining estimates (22) and (23), we take into consideration

$$\big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^1\cap H^1}=\big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^1\cap L^2}+\big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{\dot{H}^1}, \tag{24}$$

$$\big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^1\cap H^1}=\big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^1\cap L^2}+\big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{\dot{H}^1}. \tag{25}$$

Therefore, it is reasonable to control the two additional norms only

$$\big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{\dot{H}^1}\quad\text{and}\quad\big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{\dot{H}^1}.$$

Observing the assumption (7) one derives the relation

$$\Big|\nabla\big(|v(\tau,x)|^{p^*}\mu_1(|v(\tau,x)|)\big)\Big|\lesssim|v(\tau,x)|^{p^*-1}\mu_1\big(|v(\tau,x)|\big)\,|\nabla v(\tau,x)|.$$

Thus, it follows that

$$\begin{aligned}\big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{\dot{H}^1}&\lesssim\|v(\tau,\cdot)\|_{L^\infty}^{p^*-1}\big\|\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^\infty}\|\nabla v(\tau,\cdot)\|_{L^2}\\&\lesssim(1+\tau)^{-\frac{n}{2}(p^*-1)-\frac{n}{4}-\frac{1}{2}}\mu_1\Big(c(1+\tau)^{-\frac{n}{2}}\Big)\|(u,v)\|_{X(t)}^{p^*}.\end{aligned} \tag{26}$$

In the same way we obtain

$$\begin{aligned}&\big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{\dot{H}^1}\\&\quad\lesssim(1+\tau)^{-\frac{n}{2}(q^*-1)-\frac{n}{4}-\frac{1}{2}+\sigma(p^*,q^*)q^*}\big(\ell(\tau)\big)^{q^*}\ \mu_2\Big(c(1+\tau)^{-\frac{n}{2}+\sigma(p^*,q^*)}\ell(\tau)\Big)\|(u,v)\|_{X(t)}^{q^*}.\end{aligned} \tag{27}$$

Collecting (20), (24), (26) and (21), (25), (27) we may conclude (22) and (23), respectively.

Let us come back to show the inequality (19). Our strategy is to use the estimates from Lemma 2.1 and the derived estimates from (20) to (23) to achieve the following estimates for $k=0,1$:

$$\big\|\nabla^k u^{\mathrm{nl}}(t,\cdot)\big\|_{L^2}\lesssim\int_0^t(1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}}\big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^1\cap L^2}d\tau$$

$$\lesssim \|(u,v)\|_{X(t)}^{p^*} \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}} (1+\tau)^{-1+\frac{q^*-p^*}{p^*q^*-1}}\, \mu_1\Big(c(1+\tau)^{-\frac{1+q^*}{p^*q^*-1}}\Big) d\tau,$$

$$\big\|u^{\rm nl}(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} \displaystyle\int_0^t (1+t-\tau)^{-\frac{n}{2}} \big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} d\tau & \text{if } n=1 \\ \displaystyle\int_0^t (1+t-\tau)^{-\frac{n}{2}} \big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big)\big\|_{L^1\cap H^1} d\tau & \text{if } n=2 \end{cases}$$

$$\lesssim \|(u,v)\|_{X(t)}^{p^*} \int_0^t (1+t-\tau)^{-\frac{n}{2}} (1+\tau)^{-1+\frac{q^*-p^*}{p^*q^*-1}}\, \mu_1\Big(c(1+\tau)^{-\frac{1+q^*}{p^*q^*-1}}\Big) d\tau,$$

and

$$\big\|\nabla^k v^{\rm nl}(t,\cdot)\big\|_{L^2} \lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}} \big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} d\tau$$

$$\lesssim \|(u,v)\|_{X(t)}^{q^*} \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}} (1+\tau)^{-1} \big(\ell(\tau)\big)^{q^*} \mu_2\Big(c(1+\tau)^{-\frac{1+p^*}{p^*q^*-1}}\ell(\tau)\Big) d\tau,$$

$$\big\|v^{\rm nl}(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} \displaystyle\int_0^t (1+t-\tau)^{-\frac{n}{2}} \big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} d\tau & \text{if } n=1 \\ \displaystyle\int_0^t (1+t-\tau)^{-\frac{n}{2}} \big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big)\big\|_{L^1\cap H^1} d\tau & \text{if } n=2 \end{cases}$$

$$\lesssim \|(u,v)\|_{X(t)}^{q^*} \int_0^t (1+t-\tau)^{-\frac{n}{2}} (1+\tau)^{-1} \big(\ell(\tau)\big)^{q^*} \mu_2\Big(c(1+\tau)^{-\frac{1+p^*}{p^*q^*-1}}\ell(\tau)\Big) d\tau.$$

According to (11) let us divide our considerations into the following two cases:

- **Case 1:** Let us assume (8). So, we take $\ell(\tau)\equiv 1$. For this reason, we can proceed as follows:

$$\begin{aligned}\big\|\nabla^k u^{\rm nl}(t,\cdot)\big\|_{L^2} &\lesssim (1+t)^{\frac{q^*-p^*}{p^*q^*-1}} \|(u,v)\|_{X(t)}^{p^*} \\ &\quad\times \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}} (1+\tau)^{-1}\, \mu_1\Big(c(1+\tau)^{-\frac{1+q^*}{p^*q^*-1}}\Big) d\tau \\ &\lesssim (1+t)^{-\frac{n}{4}-\frac{k}{2}+\frac{q^*-p^*}{p^*q^*-1}} \|(u,v)\|_{X(t)}^{p^*} \int_0^t (1+\tau)^{-1}\, \mu_1\Big(c(1+\tau)^{-\frac{1+q^*}{p^*q^*-1}}\Big) d\tau\end{aligned}$$

$$
\begin{aligned}
&\leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}+\frac{q^*-p^*}{p^*q^*-1}} \|(u,v)\|^{p^*}_{X(t)} \int_0^\infty (1+\tau)^{-1}\, \mu_1\Big(c(1+\tau)^{-\frac{1+q^*}{p^*q^*-1}}\Big) d\tau \\
&= C(1+t)^{-\frac{n}{4}-\frac{k}{2}+\frac{q^*-p^*}{p^*q^*-1}} \|(u,v)\|^{p^*}_{X(t)} \int_0^c \frac{\mu_1(s)}{s} ds \\
&\lesssim (1+t)^{-\frac{n}{4}-\frac{k}{2}+\frac{q^*-p^*}{p^*q^*-1}} \|(u,v)\|^{p^*}_{X(t)},
\end{aligned}
$$

where we applied Lemma 2.2 after choosing

$$
\alpha = \frac{n}{4} + \frac{k}{2}, \quad \beta_1 = 1, \quad \beta_2 = 0, \quad \gamma = \frac{1+q^*}{p^*q^*-1}
$$

and used the assumption (8) as well. Moreover, one may estimate

$$
\begin{aligned}
\big\|\nabla^k v^{\mathrm{nl}}(t,\cdot)\big\|_{L^2} &\lesssim \|(u,v)\|^{q^*}_{X(t)} \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}} (1+\tau)^{-1}\, \mu_2\Big(c(1+\tau)^{-\frac{1+p^*}{p^*q^*-1}}\Big) d\tau \\
&\lesssim (1+t)^{-\frac{n}{4}-\frac{k}{2}} \|(u,v)\|^{q^*}_{X(t)} \int_0^t (1+\tau)^{-1}\, \mu_2\Big(c(1+\tau)^{-\frac{1+p^*}{p^*q^*-1}}\Big) d\tau \\
&\leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|(u,v)\|^{q^*}_{X(t)} \int_0^\infty (1+\tau)^{-1}\, \mu_2\Big(c(1+\tau)^{-\frac{1+p^*}{p^*q^*-1}}\Big) d\tau \\
&= C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|(u,v)\|^{q^*}_{X(t)} \int_0^c \frac{\mu_2(s)}{s} ds \\
&\lesssim (1+t)^{-\frac{n}{4}-\frac{k}{2}} \|(u,v)\|^{q^*}_{X(t)},
\end{aligned}
$$

where we have applied Lemma 2.2 by choosing

$$
\alpha = \frac{n}{4} + \frac{k}{2}, \quad \beta_1 = 0, \quad \beta_2 = 1, \quad \gamma = \frac{1+p^*}{p^*q^*-1}
$$

and used the assumption (8) as well. Analogously, we also obtain the following estimates:

$$
\begin{aligned}
\big\|u^{\mathrm{nl}}(t,\cdot)\big\|_{L^\infty} &\lesssim (1+t)^{-\frac{n}{2}+\frac{q^*-p^*}{p^*q^*-1}} \|(u,v)\|^{p^*}_{X(t)}, \\
\big\|v^{\mathrm{nl}}(t,\cdot)\big\|_{L^\infty} &\lesssim (1+t)^{-\frac{n}{2}} \|(u,v)\|^{q^*}_{X(t)},
\end{aligned}
$$

by the choice of

$$
\alpha = \frac{n}{2}, \quad \beta_1 = 1, \quad \beta_2 = 0, \quad \gamma = \frac{1+q^*}{p^*q^*-1}
$$

or

$$\alpha = \frac{n}{2}, \quad \beta_1 = 0, \quad \beta_2 = 1, \quad \gamma = \frac{1+p^*}{p^*q^*-1}.$$

From the definition of the norm in $X(t)$, collecting all the above derived estimates completes the proof of inequality (19).

- **Case 2:** Let us assume (9). So, we take

$$\ell(\tau) = \left(\frac{\mu_1\big(c(1+\tau)^{-\varepsilon}\big)}{\mu_2\big(c(1+\tau)^{-\varepsilon}\big)}\right)^{\frac{1}{q^*+1}}.$$

Following similar arguments as we did in the treatment of Case 1 we may estimate

$$\begin{aligned}
&\big\|\nabla^k u^{\mathrm{nl}}(t,\cdot)\big\|_{L^2} \\
&\lesssim \|(u,v)\|_{X(t)}^{p^*}\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}}(1+\tau)^{-1+\frac{q^*-p^*}{p^*q^*-1}}\ell(\tau)\big(\ell(\tau)\big)^{-1}\mu_1\Big(c(1+\tau)^{-\varepsilon}\Big)d\tau \\
&\qquad\qquad\qquad\qquad \big(\text{since } \mu_1 \text{ is an increasing function}\big) \\
&\lesssim (1+t)^{\frac{q^*-p^*}{p^*q^*-1}}\ell(t)\|(u,v)\|_{X(t)}^{p^*} \\
&\times \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}}(1+\tau)^{-1}\Big(\mu_1\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{q^*}{q^*+1}}\Big(\mu_2\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{1}{q^*+1}}d\tau \\
&\qquad\qquad\qquad\qquad \big(\text{by (28) in Remark 2.1}\big) \\
&\lesssim (1+t)^{-\frac{n}{4}-\frac{k}{2}+\frac{q^*-p^*}{p^*q^*-1}}\ell(t)\|(u,v)\|_{X(t)}^{p^*} \\
&\times \int_0^t (1+\tau)^{-1}\Big(\mu_1\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{q^*}{q^*+1}}\Big(\mu_2\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{1}{q^*+1}}d\tau \\
&\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}+\frac{q^*-p^*}{p^*q^*-1}}\ell(t)\|(u,v)\|_{X(t)}^{p^*} \\
&\times \int_0^\infty (1+\tau)^{-1}\Big(\mu_1\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{q^*}{q^*+1}}\Big(\mu_2\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{1}{q^*+1}}d\tau \\
&= C(1+t)^{-\frac{n}{4}-\frac{k}{2}+\frac{q^*-p^*}{p^*q^*-1}}\ell(t)\|(u,v)\|_{X(t)}^{p^*}\int_0^c \frac{1}{s}\big(\mu_1(s)\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(s)\big)^{\frac{1}{q^*+1}}ds \\
&\lesssim (1+t)^{-\frac{n}{4}-\frac{k}{2}+\frac{q^*-p^*}{p^*q^*-1}}\ell(t)\|(u,v)\|_{X(t)}^{p^*},
\end{aligned}$$

where we have applied Lemma 2.2 after choosing

$$\alpha = \frac{n}{4} + \frac{k}{2}, \quad \beta_1 = \frac{q^*}{q^*+1}, \quad \beta_2 = \frac{1}{q^*+1}, \quad \gamma = \varepsilon$$

as well as used the assumption (9). In such a way one also has

$$\begin{aligned}
\big\|\nabla^k v^{\mathrm{nl}}(t,\cdot)\big\|_{L^2} &\lesssim \|(u,v)\|_{X(t)}^{q^*} \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}}(1+\tau)^{-1}\big(\ell(\tau)\big)^{q^*}\ \mu_2\big(c(1+\tau)^{-\varepsilon}\big)d\tau \\
&\qquad\qquad\qquad\qquad\big(\text{by (29) in Remark 2.1}\big) \\
&= \|(u,v)\|_{X(t)}^{q^*} \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}}(1+\tau)^{-1} \\
&\qquad \times \Big(\mu_1\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{q^*}{q^*+1}} \Big(\mu_2\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{1}{q^*+1}} d\tau \\
&\lesssim (1+t)^{-\frac{n}{4}-\frac{k}{2}} \|(u,v)\|_{X(t)}^{q^*} \\
&\qquad \times \int_0^t (1+\tau)^{-1} \Big(\mu_1\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{q^*}{q^*+1}} \Big(\mu_2\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{1}{q^*+1}} d\tau \\
&\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|(u,v)\|_{X(t)}^{q^*} \\
&\qquad \times \int_0^\infty (1+\tau)^{-1} \Big(\mu_1\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{q^*}{q^*+1}} \Big(\mu_2\big(c(1+\tau)^{-\varepsilon}\big)\Big)^{\frac{1}{q^*+1}} d\tau \\
&= C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|(u,v)\|_{X(t)}^{q^*} \int_0^c \frac{1}{s}\big(\mu_1(s)\big)^{\frac{q^*}{q^*+1}} \big(\mu_2(s)\big)^{\frac{1}{q^*+1}} ds \\
&\lesssim (1+t)^{-\frac{n}{4}-\frac{k}{2}} \|(u,v)\|_{X(t)}^{q^*},
\end{aligned}$$

where we have employed Lemma 2.2 after choosing

$$\alpha = \frac{n}{4} + \frac{k}{2}, \quad \beta_1 = \frac{q^*}{q^*+1}, \quad \beta_2 = \frac{1}{q^*+1}, \quad \gamma = \varepsilon$$

and used the assumption (9) as well. Similarly, we may derive the following estimates:

$$\begin{aligned}
\big\|u^{\mathrm{nl}}(t,\cdot)\big\|_{L^\infty} &\lesssim (1+t)^{-\frac{n}{2}+\frac{q^*-p^*}{p^*q^*-1}}\ell(t)\|(u,v)\|_{X(t)}^{p^*}, \\
\big\|v^{\mathrm{nl}}(t,\cdot)\big\|_{L^\infty} &\lesssim (1+t)^{-\frac{n}{2}}\|(u,v)\|_{X(t)}^{q^*},
\end{aligned}$$

where we applied Lemma 2.2 after choosing

$$\alpha = \frac{n}{2}, \quad \beta_1 = \frac{q^*}{q^*+1}, \quad \beta_2 = \frac{1}{q^*+1}, \quad \gamma = \varepsilon.$$

From the definition of the norm in $X(t)$, we combine all the above derived estimates to complete the proof of inequality (19) in both cases.

Next, let us prove the inequality (18). For two elements (u, v) and $(\bar{u}, \bar{v})$ from $X(t)$, it is obvious that

$$\Psi(u,v)(t,x) - \Psi(\bar{u},\bar{v})(t,x) = \big(u^{\mathrm{nl}}(t,x) - \bar{u}^{\mathrm{nl}}(t,x), v^{\mathrm{nl}}(t,x) - \bar{v}^{\mathrm{nl}}(t,x)\big).$$

Then, we use the same strategies as in the proof of the inequality (19) to gain the following estimates with $k = 0, 1$:

$$\begin{aligned}
&\big\|\nabla^k\big(u^{\mathrm{nl}} - \bar{u}^{\mathrm{nl}}\big)(t,\cdot)\big\|_{L^2} \\
&\qquad \lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}} \big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big) - |\bar{v}(\tau,\cdot)|^{p^*}\mu_1\big(|\bar{v}(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} d\tau, \\
&\big\|\big(u^{\mathrm{nl}} - \bar{u}^{\mathrm{nl}}\big)(t,\cdot)\big\|_{L^\infty} \\
&\quad \lesssim \begin{cases} \displaystyle\int_0^t (1+t-\tau)^{-\frac{n}{2}} \big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big) - |\bar{v}(\tau,\cdot)|^{p^*}\mu_1\big(|\bar{v}(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} d\tau & \text{if } n=1, \\ \displaystyle\int_0^t (1+t-\tau)^{-\frac{n}{2}} \big\||v(\tau,\cdot)|^{p^*}\mu_1\big(|v(\tau,\cdot)|\big) - |\bar{v}(\tau,\cdot)|^{p^*}\mu_1\big(|\bar{v}(\tau,\cdot)|\big)\big\|_{L^1\cap H^1} d\tau & \text{if } n=2, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\nabla^k\big(v^{\mathrm{nl}} - \bar{v}^{\mathrm{nl}}\big)(t,\cdot)\big\|_{L^2} \\
&\quad \lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}} \big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big) - |\bar{u}(\tau,\cdot)|^{q^*}\mu_2\big(|\bar{u}(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} d\tau, \\
&\big\|\big(v^{\mathrm{nl}} - \bar{v}^{\mathrm{nl}}\big)(t,\cdot)\big\|_{L^\infty} \\
&\quad \lesssim \begin{cases} \displaystyle\int_0^t (1+t-\tau)^{-\frac{n}{2}} \big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big) - |\bar{u}(\tau,\cdot)|^{q^*}\mu_2\big(|\bar{u}(\tau,\cdot)|\big)\big\|_{L^1\cap L^2} d\tau & \text{if } n=1, \\ \displaystyle\int_0^t (1+t-\tau)^{-\frac{n}{2}} \big\||u(\tau,\cdot)|^{q^*}\mu_2\big(|u(\tau,\cdot)|\big) - |\bar{u}(\tau,\cdot)|^{q^*}\mu_2\big(|\bar{u}(\tau,\cdot)|\big)\big\|_{L^1\cap H^1} d\tau & \text{if } n=2. \end{cases}
\end{aligned}$$

Applying the mean value theorem gives the following integral representation:

$$|v(\tau,x)|^{p^*}\mu_1\big(|v(\tau,x)|\big) - |\bar{v}(\tau,x)|^{p^*}\mu_1\big(|\bar{v}(\tau,x)|\big)$$

$$= \big(v(\tau,x)-\bar{v}(\tau,x)\big)\int_0^1 d_{|v|}G\big(\omega v(\tau,x)+(1-\omega)\bar{v}(\tau,x)\big)\,d\omega,$$

where $G(v)=|v|^{p^*}\mu_1(|v|)$. Since the condition (7) of moduli of continuity holds, one gets

$$d_{|v|}G(v)=p^*|v|^{p^*-1}\mu_1(|v|)+|v|^{p^*}d_{|v|}\mu_1(|v|)\lesssim |v|^{p^*-1}\mu_1(|v|).$$

Thus, it follows that

$$\begin{aligned}&\Big||v(\tau,x)|^{p^*}\mu_1\big(|v(\tau,x)|\big)-|\bar{v}(\tau,x)|^{p^*}\mu_1\big(|\bar{v}(\tau,x)|\big)\Big|\\ &\quad\lesssim \big|v(\tau,x)-\bar{v}(\tau,x)\big|\int_0^1 \big|\omega v(\tau,x)+(1-\omega)\bar{v}(\tau,x)\big|^{p^*-1}\mu_1\Big(\big|\omega v(\tau,x)+(1-\omega)\bar{v}(\tau,x)\big|\Big)\,d\omega.\end{aligned}$$

Similarly, we also obtain

$$\begin{aligned}&\Big||u(\tau,x)|^{q^*}\mu_2\big(|u(\tau,x)|\big)-|\bar{u}(\tau,x)|^{q^*}\mu_2\big(|\bar{u}(\tau,x)|\big)\Big|\\ &\quad\lesssim \big|u(\tau,x)-\bar{u}(\tau,x)\big|\int_0^1 \big|\omega u(\tau,x)+(1-\omega)\bar{u}(\tau,x)\big|^{q^*-1}\mu_2\Big(\big|\omega u(\tau,x)+(1-\omega)\bar{u}(\tau,x)\big|\Big)\,d\omega.\end{aligned}$$

By the aid of Hölder's inequality and applying the same tools as in the proof of inequality (19), we arrive at the inequality (18). Summarizing, the proof of Theorem 1.1 is completed. □

Remark 2.1. Here we want to underline that in the proof of Theorem 1.1 we have used the following auxiliary properties of the weight function $\ell=\ell(\tau)$ in Case 2:

$$\text{i) } (1+\tau)^{\frac{q^*-p^*}{p^*q^*-1}}\ell(\tau) \quad \text{is increasing for } \tau>0; \tag{28}$$

$$\text{ii) } (1+\tau)^{-\frac{1+p^*}{p^*q^*-1}}\ell(\tau)\le (1+\tau)^{-\varepsilon} \tag{29}$$

for a sufficiently small and positive ε. Indeed, by change of variables $s=c(1+\tau)^{-\varepsilon}$ we may re-write

$$\begin{aligned}f(\tau):=(1+\tau)^{\frac{q^*-p^*}{p^*q^*-1}}\ell(\tau)&=(1+\tau)^{\frac{q^*-p^*}{p^*q^*-1}}\left(\frac{\mu_1\big(c(1+\tau)^{-\varepsilon}\big)}{\mu_2\big(c(1+\tau)^{-\varepsilon}\big)}\right)^{\frac{1}{q^*+1}}\\ &=C\,s^{-\frac{q^*-p^*}{\varepsilon(p^*q^*-1)}}\left(\frac{\mu_1(s)}{\mu_2(s)}\right)^{\frac{1}{q^*+1}}=C\left(s^{-\frac{(q^*-p^*)(q^*+1)}{\varepsilon(p^*q^*-1)}}\,\frac{\mu_1(s)}{\mu_2(s)}\right)^{\frac{1}{q^*+1}}.\end{aligned}$$

For this reason, in order to prove that $f=f(\tau)$ is an increasing function, it suffices to verify that

$$h_1(s):=s^{-\frac{(q^*-p^*)(q^*+1)}{\varepsilon(p^*q^*-1)}}\,\mu_1(s)$$

is a decreasing function in the case $q^* > p^*$ due to the increasing property of the function μ_2. Here we take into consideration that in the case $q^* = p^*$ the assumption

$$s \in (0, c] \to \frac{\mu_1(s)}{\mu_2(s)} \text{ is a decreasing function}$$

implies immediately that $f = f(\tau)$ is an increasing function. To complete the case $q^* > p^*$, we have

$$\begin{aligned}
&h_1'(s) \\
&= -\frac{(q^*-p^*)(q^*+1)}{\varepsilon(p^*q^*-1)} s^{-\frac{(q^*-p^*)(q^*+1)}{\varepsilon(p^*q^*-1)}-1}\mu_1(s) + s^{-\frac{(q^*-p^*)(q^*+1)}{\varepsilon(p^*q^*-1)}}\mu_1'(s) \\
&\le s^{-\frac{(q^*-p^*)(q^*+1)}{\varepsilon(p^*q^*-1)}-1}\mu_1(s)\Big(-\frac{(q^*-p^*)(q^*+1)}{\varepsilon(p^*q^*-1)} + C\Big) \quad \big(\text{since } s\mu_1'(s) \le C\mu_1(s) \text{ from (7)}\big) \\
&\le 0,
\end{aligned}$$

after the choice of a sufficiently small constant $\varepsilon > 0$. This provides the first statement (28). In an analogous way, we may conclude the second one (29).

2.2. Blow-up result

To prove our result, the following generalized Jensen's inequality comes into play.

Lemma 2.4 *(Lemma 8 in [1]). Let $\eta = \eta(x)$ be a nonnegative function almost everywhere on Ω, provided that η is positive on a set of positive measure. Then, for each convex function h on $\mathbb{R}$ the following inequality holds:*

$$h\left(\frac{\int\limits_\Omega f(x)\eta(x)\,dx}{\int\limits_\Omega \eta(x)\,dx}\right) \le \frac{\int\limits_\Omega h\big(f(x)\big)\eta(x)\,dx}{\int\limits_\Omega \eta(x)\,dx},$$

where f is any nonnegative function such that all the above integrals are meaningful.

Proof of Theorem 1.2. Our proof relies on ideas from the recent paper [1] of the second author and collaborators, where the paper is devoted to the study of the single semi-linear damped wave equation (2). First of all, we introduce a test function $\varphi = \varphi(\rho)$ fulfilling

$$\varphi \in \mathcal{C}_0^\infty\big([0,\infty)\big) \text{ and } \varphi(\rho) = \begin{cases} 1 & \text{if } \quad 0 \le \rho \le 1/2, \\ \text{decreasing} & \text{if } \quad 1/2 \le \rho \le 1, \\ 0 & \text{if } \quad \rho \ge 1. \end{cases}$$

Also, we introduce the function $\varphi^* = \varphi^*(\rho)$ as follows:

$$\varphi^*(\rho) = \begin{cases} 0 & \text{if} \quad 0 \le \rho < 1/2, \\ \varphi(\rho) & \text{if} \quad 1/2 \le \rho < \infty. \end{cases}$$

Let R be a large parameter in $[0, \infty)$. We introduce two functions

$$\phi_R = \phi_R(t,x) = \Big(\varphi\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu+2} \quad \text{and} \quad \phi_R^* = \phi_R^*(t,x) = \Big(\varphi^*\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu+2},$$

where the parameter $\nu > 0$ will be fixed later. Then, we may observe that

$$\begin{aligned} &\operatorname{supp}\phi_R \subset Q_R := \big\{(t,x)\ :\ t^2+|x|^4 \le R^4\big\}, \\ &\operatorname{supp}\phi_R^* \subset Q_R^* := Q_R \setminus \big\{(t,x)\ :\ t^2+|x|^4 < R^4/2\big\}. \end{aligned}$$

Now we define the following two functionals:

$$I_R := \int_0^\infty \int_{\mathbb{R}^n} |v(t,x)|^{p^*} \mu_1\big(|v(t,x)|\big)\phi_R(t,x)\,dxdt = \int_{Q_R} |v(t,x)|^{p^*} \mu_1\big(|v(t,x)|\big)\phi_R(t,x)\,d(x,t),$$

$$J_R := \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^{q^*} \mu_2\big(|u(t,x)|\big)\phi_R(t,x)\,dxdt = \int_{Q_R} |u(t,x)|^{q^*} \mu_2\big(|u(t,x)|\big)\phi_R(t,x)\,d(x,t).$$

Let us assume that $(u,v) = (u(t,x), v(t,x))$ is a global (in time) Sobolev solution to (5) for data satisfying the assumptions of the theorem. We multiply the left-hand sides of (5) by $\phi_R = \phi_R(t,x)$ and integrate by parts to achieve

$$\begin{aligned} 0 \le I_R &= -\int_{\mathbb{R}^n} u_1(x)\phi_R(0,x)\,dx + \int_{Q_R} u(t,x)\big(\partial_t^2\phi_R(t,x) - \Delta\phi_R(t,x) - \partial_t\phi_R(t,x)\big)\,d(x,t) \\ &=: -\int_{\mathbb{R}^n} u_1(x)\phi_R(0,x)\,dx + I_R^*, \end{aligned} \tag{30}$$

and

$$\begin{aligned} 0 \le J_R &= -\int_{\mathbb{R}^n} v_1(x)\phi_R(0,x)\,dx + \int_{Q_R} v(t,x)\big(\partial_t^2\phi_R(t,x) - \Delta\phi_R(t,x) - \partial_t\phi_R(t,x)\big)\,d(x,t) \\ &=: -\int_{\mathbb{R}^n} v_1(x)\phi_R(0,x)\,dx + J_R^*. \end{aligned} \tag{31}$$

To estimate I_R^* and J_R^*, a straightforward calculation gives the following estimates:

$$\big|\partial_t\phi_R(t,x)\big| \lesssim \frac{1}{R^2}\Big(\varphi^*\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu+1},$$

$$\big|\partial_t^2\phi_R(t,x)\big| \lesssim \frac{1}{R^4}\Big(\varphi^*\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu},$$

$$\big|\Delta\phi_R(t,x)\big| \lesssim \frac{1}{R^2}\Big(\varphi^*\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu},$$

where we have used the support conditions of ϕ_R and ϕ_R^*. As a consequence, we arrive at

$$|I_R^*| \lesssim \frac{1}{R^2}\int\limits_{Q_R} |u(t,x)|\,\Big(\varphi^*\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu} d(x,t) = \frac{1}{R^2}\int\limits_{Q_R^*} |u(t,x)|\,\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}} d(x,t), \quad (32)$$

and

$$|J_R^*| \lesssim \frac{1}{R^2}\int\limits_{Q_R} |v(t,x)|\,\Big(\varphi^*\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu} d(x,t) = \frac{1}{R^2}\int\limits_{Q_R^*} |v(t,x)|\,\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}} d(x,t). \quad (33)$$

Let us now turn to estimate the above integrals. For this purpose, we define two functions

$$\Phi_p = \Phi_p(s) = s^{p^*}\mu_1(s) \quad \text{and} \quad \Phi_q = \Phi_q(s) = s^{q^*}\mu_2(s).$$

We have

$$\begin{aligned}
&\Phi_q\Big(|u(t,x)|\,\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}}\Big)\\
&\quad = |u(t,x)|^{q^*}\,\big(\phi_R^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}}\mu_2\Big(|u(t,x)|\,\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}}\Big)\\
&\quad \le |u(t,x)|^{q^*}\,\big(\phi_R^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}}\mu_2\big(|u(t,x)|\big) = \Phi_q\big(|u(t,x)|\big)\,\big(\phi_R^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}} \qquad (34)
\end{aligned}$$

since $\mu = \mu(s)$ is an increasing function and it holds

$$0 \le \big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}} \le 1$$

for any $\nu > 0$. It is obvious from the assumption (13) that

$$\Phi_q''(s) = s^{q^*-2}\Big(q^*(q^*-1)\mu_2(s) + 2q^*s\mu_2'(s) + s^2\mu_2''(s)\Big) \ge 0,$$

that is, Φ_q is a convex function on a small interval $(0, c_0]$ with a sufficiently small constant $c_0 > 0$. Additionally, we can choose a convex continuation of Φ_q outside this interval to guarantee that Φ_q is convex on $[0,\infty)$. The application of the generalized Jensen's inequality from Lemma 2.4 with $h(s) = \Phi_q(s)$, $f(t,x) = |u(t,x)|\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}}$, $\eta \equiv 1$ and $\Omega \equiv Q_R^*$ leads to the following estimate:

$$\Phi_q\left(\frac{\int\limits_{Q_R^*} |u(t,x)|\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}}\,d(x,t)}{\int\limits_{Q_R^*} 1\,d(x,t)}\right) \leq \frac{\int\limits_{Q_R^*} \Phi_q\Big(|u(t,x)|\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}}\Big)\,d(x,t)}{\int\limits_{Q_R^*} 1\,d(x,t)}.$$

Taking account of

$$\int\limits_{Q_R^*} 1\,d(x,t) \approx R^{n+2}$$

it follows

$$\Phi_q\left(\frac{\int\limits_{Q_R^*} |u(t,x)|\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}}\,d(x,t)}{R^{n+2}}\right) \leq \frac{\int\limits_{Q_R^*} \Phi_q\Big(|u(t,x)|\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}}\Big)\,d(x,t)}{R^{n+2}}. \tag{35}$$

From the estimates (34) and (35) one derives

$$\Phi_q\left(\frac{\int\limits_{Q_R^*} |u(t,x)|\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}}\,d(x,t)}{R^{n+2}}\right) \leq \frac{\int\limits_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\,\big(\phi_R^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}}\,d(x,t)}{R^{n+2}}. \tag{36}$$

Due to the fact that $\mu = \mu(s)$ is a strictly increasing function, $\Phi_q = \Phi_q(s)$ is also a strictly increasing function on $[0,\infty)$. As a result, it implies from (36) that

$$\int\limits_{Q_R^*} |u(t,x)|\big(\phi_R^*(t,x)\big)^{\frac{\nu}{\nu+2}}\,d(x,t) \leq R^{n+2}\,\Phi_q^{-1}\left(\frac{\int\limits_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\,\big(\phi_R^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}}\,d(x,t)}{R^{n+2}}\right). \tag{37}$$

Collecting the estimates (30), (32) and (37) we obtain

$$I_R + \int\limits_{\mathbb{R}^n} u_1(x)\phi_R(0,x)\,dx \lesssim R^n\,\Phi_q^{-1}\left(\frac{\int\limits_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\,\big(\phi_R^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}}\,d(x,t)}{R^{n+2}}\right).$$

In the same manner using (31) and (33) one also gets

$$J_R + \int_{\mathbb{R}^n} v_1(x)\phi_R(0,x)\,dx \lesssim R^n\, \Phi_p^{-1}\left(\frac{\int_{Q_R^*} \Phi_p\big(|v(t,x)|\big)\,\big(\phi_R^*(t,x)\big)^{\frac{\nu p^*}{\nu+2}}\,d(x,t)}{R^{n+2}}\right).$$

Thanks to the assumption (12) we have

$$\int_{\mathbb{R}^n} u_1(x)\phi_R(0,x)\,dx > 0 \quad \text{and} \quad \int_{\mathbb{R}^n} v_1(x)\phi_R(0,x)\,dx > 0$$

for all $R \geq R_0$, where R_0 is a sufficiently large, positive constant. Thus, it holds

$$I_R \lesssim R^n\, \Phi_q^{-1}\left(\frac{\int_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\,\big(\phi_R^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}}\,d(x,t)}{R^{n+2}}\right), \tag{38}$$

$$J_R \lesssim R^n\, \Phi_p^{-1}\left(\frac{\int_{Q_R^*} \Phi_p\big(|v(t,x)|\big)\,\big(\phi_R^*(t,x)\big)^{\frac{\nu p^*}{\nu+2}}\,d(x,t)}{R^{n+2}}\right), \tag{39}$$

for all $R \geq R_0$. Next, for $\lambda > 0$ and $s > 0$ we define the following auxiliary functions:

$$g_q = g_q(\lambda) = \int_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\,\big(\phi_\lambda^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}}\,d(x,t) \quad \text{and} \quad G_q = G_q(s) = \int_0^s g_q(\lambda)\lambda^{-1}\,d\lambda,$$

$$g_p = g_p(\lambda) = \int_{Q_R^*} \Phi_p\big(|v(t,x)|\big)\,\big(\phi_\lambda^*(t,x)\big)^{\frac{\nu p^*}{\nu+2}}\,d(x,t) \quad \text{and} \quad G_p = G_p(s) = \int_0^s g_p(\lambda)\lambda^{-1}\,d\lambda.$$

Therefore, we can express

$$G_q(R) = \int_0^R \Big(\int_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\,\big(\phi_\lambda^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}}\,d(x,t)\Big)\lambda^{-1}\,d\lambda$$

$$= \int\limits_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\Big(\int\limits_0^R \Big(\varphi^*\Big(\frac{t^2+|x|^4}{\lambda^4}\Big)\Big)^{\nu q^*} \lambda^{-1}\, d\lambda\Big)\, d(x,t).$$

By performing the change of variables $\bar{\lambda} = \dfrac{t^2+|x|^4}{\lambda^4}$, we take into account that to given $(x,t) \in Q_R^*$ we have

$$\bar{\lambda} = \frac{t^2+|x|^4}{\lambda^4} \in \Big[\frac{1}{2}, 1\Big] \text{ on the support of } \varphi^* \text{ for } \lambda \in (0, R).$$

Moreover, the function φ^* is decreasing on $\big[\frac{1}{2}, \infty\big)$. Hence, we may conclude from

$$\frac{t^2+|x|^4}{R^4} \le \frac{t^2+|x|^4}{\lambda^4} \quad \text{for} \quad \lambda \in (0, R)$$

that

$$\varphi^*\Big(\frac{t^2+|x|^4}{\lambda^4}\Big) \le \varphi^*\Big(\frac{t^2+|x|^4}{R^4}\Big)$$

as far as

$$\frac{t^2+|x|^4}{R^4} \ge \frac{1}{2}.$$

But this is clear due to $(x,t) \in Q_R^*$. Summarizing we arrive at the following chain of inequalities:

$$\begin{aligned}
G_q(R) &= \frac{1}{4}\int\limits_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\left(\int\limits_{\frac{t^2+|x|^4}{R^4}}^{\infty} \big(\varphi^*(\bar{\lambda})\big)^{\nu q^*}\, \bar{\lambda}^{-1}\, d\bar{\lambda}\right) d(x,t) \\
&\le \frac{1}{4}\int\limits_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\Big(\varphi^*\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu q^*}\left(\int\limits_{\frac{1}{2}}^{1} \bar{\lambda}^{-1}\, d\bar{\lambda}\right) d(x,t) \qquad (40) \\
&\qquad\qquad \big(\text{since } \operatorname{supp}\varphi^* \subset [1/2, 1]\big) \\
&\le \frac{\log 2}{4}\int\limits_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\Big(\varphi\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu q^*} d(x,t) \\
&\qquad\qquad \big(\text{since } \varphi^* \equiv \varphi \text{ in } [1/2, 1]\big) \\
&\le C\int\limits_{Q_R} \Phi_q\big(|u(t,x)|\big)\Big(\varphi\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu q^*} d(x,t),
\end{aligned}$$

since φ is decreasing. An analogous argument implies

$$G_p(R) \le C \int\limits_{Q_R} \Phi_p\big(|v(t,x)|\big)\Big(\varphi\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu p^*} d(x,t).$$

Let us now choose $\nu \ge \max\Big\{\dfrac{2}{p^*-1}, \dfrac{2}{q^*-1}\Big\} = \dfrac{2}{p^*-1}$. Thus, it follows that

$$G_q(R) \le C \int\limits_{Q_R} |u(t,x)|^{q^*} \mu_2\big(|u(t,x)|\big)\Big(\varphi\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu+2} d(x,t) = C J_R, \tag{41}$$

$$G_p(R) \le C \int\limits_{Q_R} |v(t,x)|^{p^*} \mu_1\big(|v(t,x)|\big)\Big(\varphi\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu+2} d(x,t) = C I_R. \tag{42}$$

Furthermore, the following relations hold:

$$\frac{\mathrm{d}G_q}{\mathrm{d}s}(s)s = g_q(s), \quad \text{in particular,} \quad \Big(\frac{\mathrm{d}G_q}{\mathrm{d}s}\Big)(s=R)R = g_q(R),$$
$$\frac{\mathrm{d}G_p}{\mathrm{d}s}(s)s = g_p(s), \quad \text{in particular,} \quad \Big(\frac{\mathrm{d}G_p}{\mathrm{d}s}\Big)(s=R)R = g_p(R),$$

which imply

$$G_q'(R)R = g_q(R) = \int\limits_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\,\big(\phi_R^*(t,x)\big)^{\frac{\nu q^*}{\nu+2}} d(x,t), \tag{43}$$

$$G_p'(R)R = g_p(R) = \int\limits_{Q_R^*} \Phi_p\big(|v(t,x)|\big)\,\big(\phi_R^*(t,x)\big)^{\frac{\nu p^*}{\nu+2}} d(x,t). \tag{44}$$

Combining the estimates (38), (39) and from (41) to (44) gives

$$\frac{G_q(R)}{C} \le J_R \le C R^n\, \Phi_p^{-1}\Big(\frac{G_p'(R)}{R^{n+1}}\Big),$$
$$\frac{G_p(R)}{C} \le I_R \le C R^n\, \Phi_q^{-1}\Big(\frac{G_q'(R)}{R^{n+1}}\Big).$$

Consequently, we may conclude

$$\Phi_p\Big(\frac{G_q(R)}{C R^n}\Big) \le \frac{G_p'(R)}{R^{n+1}},$$
$$\Phi_q\Big(\frac{G_p(R)}{C R^n}\Big) \le \frac{G_q'(R)}{R^{n+1}},$$

for all $R \geq R_0$. Then, recalling the definition of the functions Φ_p and Φ_q we derive

$$\Big(\frac{G_q(R)}{CR^n}\Big)^{p^*}\mu_1\Big(\frac{G_q(R)}{CR^n}\Big) \leq \frac{G'_p(R)}{R^{n+1}},$$
$$\Big(\frac{G_p(R)}{CR^n}\Big)^{q^*}\mu_2\Big(\frac{G_p(R)}{CR^n}\Big) \leq \frac{G'_q(R)}{R^{n+1}},$$

for all $R \geq R_0$. These estimates imply

$$\frac{1}{CR^{n(p^*-1)-1}}\mu_1\Big(\frac{G_q(R)}{CR^n}\Big)\big(G_q(R)\big)^{p^*} \leq G'_p(R),$$
$$\frac{1}{CR^{n(q^*-1)-1}}\mu_2\Big(\frac{G_p(R)}{CR^n}\Big)\big(G_p(R)\big)^{q^*} \leq G'_q(R),$$

for all $R \geq R_0$. Due to the increasing property of the functions $\mu_1 = \mu_1(s)$, $\mu_2 = \mu_2(s)$, $G_p = G_p(R)$ and $G_q = G_q(R)$, the following inequalities hold:

$$\frac{1}{CR^{n(p^*-1)-1}}\mu_1\Big(\frac{G_q(R_0)}{CR^n}\Big)\big(G_q(R)\big)^{p^*} \leq G'_p(R),$$
$$\frac{1}{CR^{n(q^*-1)-1}}\mu_2\Big(\frac{G_p(R_0)}{CR^n}\Big)\big(G_p(R)\big)^{q^*} \leq G'_q(R).$$

Hence,

$$\frac{C}{R^{n(p^*-1)-1}}\mu_1\big(C_0R^{-n}\big)\big(G_q(R)\big)^{p^*} \leq G'_p(R),$$
$$\frac{C}{R^{n(q^*-1)-1}}\mu_2\big(C_0R^{-n}\big)\big(G_p(R)\big)^{q^*} \leq G'_q(R),$$

for all $R \geq R_0$, where $C_0 = C_0(C, R_0) := \dfrac{1}{C}\min\big\{G_p(R_0), G_q(R_0)\big\}$. For simplicity, putting $\tau := R$ and denoting

$$\theta_1(\tau) := \frac{1}{\tau^{n(p^*-1)-1}}\mu_1\big(C_0\tau^{-n}\big), \quad \theta_2(\tau) := \frac{1}{\tau^{n(q^*-1)-1}}\mu_2\big(C_0\tau^{-n}\big),$$

we obtain the following system of ordinary differential inequalities for $\tau \geq R_0$:

$$G'_p(\tau) \geq C\,\theta_1(\tau)\big(G_q(\tau)\big)^{p^*}, \tag{45}$$
$$G'_q(\tau) \geq C\,\theta_2(\tau)\big(G_p(\tau)\big)^{q^*}. \tag{46}$$

For any $r > R_0$, after multiplying (45) by $G'_q(\tau)$ and integrating by parts over $[R_0, r]$ we arrive at

$$G_p(r)G_q'(r) - G_p(R_0)G_q'(R_0) - \int_{R_0}^{r} G_p(\tau)G_q''(\tau)\,d\tau$$
$$\geq \frac{C}{p^*+1}\theta_1(r)\big(G_q(r)\big)^{p^*+1} - \frac{C}{p^*+1}\theta_1(R_0)\big(G_q(R_0)\big)^{p^*+1}$$
$$- \frac{C}{p^*+1}\int_{R_0}^{r} \theta_1'(\tau)\big(G_q(\tau)\big)^{p^*+1}\,d\tau.$$

This relation is equivalent to

$$G_p(r)G_q'(r) + \int_{R_0}^{r} \frac{G_p(\tau)G_q'(\tau)}{\tau}\,d\tau - \int_{R_0}^{r} \frac{G_p(\tau)g_q'(\tau)}{\tau}\,d\tau$$
$$\geq \frac{C}{p^*+1}\theta_1(r)\big(G_q(r)\big)^{p^*+1} + \Big(G_p(R_0)G_q'(R_0) - \frac{C}{p^*+1}\theta_1(R_0)\big(G_q(R_0)\big)^{p^*+1}\Big)$$
$$- \frac{C}{p^*+1}\int_{R_0}^{r} \theta_1'(\tau)\big(G_q(\tau)\big)^{p^*+1}\,d\tau, \tag{47}$$

where we have used the equality

$$G_q''(\tau) = \frac{g_q'(\tau) - G_q'(\tau)}{\tau}.$$

To control the right-hand side (RHS) of (47), at first let us choose a sufficiently small constant $C = C(R_0) > 0$ to verify the inequality

$$G_p(R_0)G_q'(R_0) - \frac{C}{p^*+1}\theta_1(R_0)\big(G_q(R_0)\big)^{p^*+1} \geq 0.$$

Hence, we can see that

$$\text{RHS of (47)} \geq \frac{C}{p^*+1}\theta_1(r)\big(G_q(r)\big)^{p^*+1} - \frac{C}{p^*+1}\int_{R_0}^{r} \theta_1'(\tau)\big(G_q(\tau)\big)^{p^*+1}\,d\tau. \tag{48}$$

A direct calculation gives the following equality:

$$\theta_1'(\tau) = -\Big(n(p^*-1) - 1 + n\frac{C_0\tau^{-n}\mu_1'\big(C_0\tau^{-n}\big)}{\mu_1\big(C_0\tau^{-n}\big)}\Big)\frac{\theta_1(\tau)}{\tau}.$$

Now we distinguish our considerations into the following two cases:

- If $p^* \geq 1 + \frac{1}{n}$, then $n(p^*-1)-1 \geq 0$. Thanks to the assumption (13), it is clear that $\theta_1'(\tau) \leq 0$. From (48) this implies immediately

$$\text{RHS of (47)} \geq \frac{C}{p^*+1}\theta_1(r)\big(G_q(r)\big)^{p^*+1}. \tag{49}$$

- If $1 < p^* < 1 + \frac{1}{n}$, then $n(p^*-1)-1 < 0$. Thanks to the assumption (13) again, one has $\theta_1'(\tau) \geq 0$ for large R_0. From (48) we deduce

$$\text{RHS of (47)} \geq \frac{C}{p^*+1}\theta_1(r)\big(G_q(r)\big)^{p^*+1} - \frac{C}{p^*+1}\big(1-n(p^*-1)\big)\int_{R_0}^{r} \frac{\theta_1(\tau)\big(G_q(\tau)\big)^{p^*+1}}{\tau}\,d\tau. \tag{50}$$

Introducing the function

$$f_1 : \tau \in [R_0, \infty) \to f_1(\tau) := \frac{\theta_1(\tau)\big(G_q(\tau)\big)^{p^*+1}}{\tau}$$

one derives

$$\begin{aligned} f_1'(\tau) &= \frac{\theta_1'(\tau)\big(G_q(\tau)\big)^{p^*+1}\tau + (p^*+1)\theta_1(\tau)\big(G_q(\tau)\big)^{p^*}G_q'(\tau)\tau - \theta_1(\tau)\big(G_q(\tau)\big)^{p^*+1}}{\tau^2} \\ &\geq \frac{\theta_1(\tau)\big(G_q(\tau)\big)^{p^*}}{\tau^2}\Big((p^*+1)G_q'(\tau)\tau - G_q(\tau)\Big) \qquad \big(\text{since } \theta_1'(\tau) \geq 0\big) \\ &\geq \frac{\theta_1(\tau)\big(G_q(\tau)\big)^{p^*}}{\tau^2}\Big((p^*+1)g_q(\tau) - G_q(\tau)\Big) \qquad \big(\text{since } G_q'(\tau)\tau = g_q(\tau)\big) \\ &\geq \Big(p^*+1-\frac{\log 2}{4}\Big)\frac{\theta_1(\tau)\big(G_q(\tau)\big)^{p^*}g_q(\tau)}{\tau^2} \geq 0, \end{aligned}$$

where we have used the relation $G_q(\tau) \leq \dfrac{\log 2}{4}g_q(\tau)$. Indeed, by recalling the estimate (40) we may conclude

$$G_q(R) \leq \frac{\log 2}{4}\int_{Q_R^*} \Phi_q\big(|u(t,x)|\big)\Big(\varphi^*\Big(\frac{t^2+|x|^4}{R^4}\Big)\Big)^{\nu q^*} d(x,t) = \frac{\log 2}{4}g_q(R) \tag{51}$$

for all $R \geq R_0$. This means $G_q(\tau) \leq \dfrac{\log 2}{4}g_q(\tau)$ for any $\tau \in [R_0, r]$. In such a way, we gain the increasing property of the function $f_1 = f_1(\tau)$. So, one may estimate

$$\int_{R_0}^{r} \frac{\theta_1(\tau)\big(G_q(\tau)\big)^{p^*+1}}{\tau}d\tau \leq \frac{\theta_1(r)\big(G_q(r)\big)^{p^*+1}}{r}(r-R_0) \leq \theta_1(r)\big(G_q(r)\big)^{p^*+1}. \tag{52}$$

Combining (50) and (52) we obtain

$$\text{RHS of (47)} \geq \frac{Cn(p^*-1)}{p^*+1}\theta_1(r)\big(G_q(r)\big)^{p^*+1}. \tag{53}$$

In order to estimate the left-hand side (LHS) of (47), it is obvious that $g_q'(\tau) \geq 0$ for any $\tau \in [R_0, r]$ since the function $g_q = g_q(\tau)$ is increasing. Therefore, we may conclude

$$\text{LHS of (47)} \leq G_p(r)G_q'(r) + \int_{R_0}^{r} \frac{G_p(\tau)G_q'(\tau)}{\tau}d\tau. \tag{54}$$

By introducing the function

$$f_2: \ \tau \in [R_0, \infty) \to f_2(\tau) := \frac{G_p(\tau)G_q'(\tau)}{\tau}$$

one has

$$\begin{aligned}
f_2'(\tau) &= \frac{G_p'(\tau)G_q'(\tau)\tau + G_p(\tau)G_q''(\tau)\tau - G_p(\tau)G_q'(\tau)}{\tau^2} \\
&= \frac{G_p'(\tau)G_q'(\tau)\tau + G_p(\tau)\big(g_q'(\tau) - G_q'(\tau)\big) - G_p(\tau)G_q'(\tau)}{\tau^2} \\
&\qquad \Big(\text{since } G_q''(\tau) = \frac{g_q'(\tau) - G_q'(\tau)}{\tau}\Big) \\
&\geq \frac{G_q'(\tau)\big(G_p'(\tau)\tau - 2G_p(\tau)\big)}{\tau^2} \qquad \big(\text{since } g_q'(\tau) \geq 0\big) \\
&\geq \frac{G_q'(\tau)\big(g_p(\tau) - 2G_p(\tau)\big)}{\tau^2} \qquad \big(\text{since } G_p'(\tau)\tau = g_p(\tau)\big) \\
&\geq \Big(1 - \frac{\log 2}{2}\Big)\frac{g_p(\tau)G_q'(\tau)}{\tau^2} \geq 0.
\end{aligned}$$

Here we notice that by an analogous argument to (51) we also derive the estimate $G_p(\tau) \leq \frac{\log 2}{4} g_p(\tau)$ for any $\tau \in [R_0, r]$, which comes into play in the last line of the above chain of estimates. As a result, we arrive at the increasing property of the function $f_2 = f_2(\tau)$ to estimate

$$\int_{R_0}^{r} \frac{G_p(\tau)G_q'(\tau)}{\tau}d\tau \leq \frac{G_p(r)G_q'(r)}{r}(r - R_0) \leq G_p(r)G_q'(r). \tag{55}$$

Collecting (54) and (55) gives

$$\text{LHS of (47)} \lesssim G_p(r)G_q'(r). \tag{56}$$

Consequently, from (47), (49), (53) and (56) it follows

$$G_p(r)G_q'(r) \geq C\theta_1(r)\big(G_q(r)\big)^{p^*+1},$$

that is,

$$G_p(r) \geq \frac{C\theta_1(r)\big(G_q(r)\big)^{p^*+1}}{G_q'(r)}. \tag{57}$$

By plugging (57) into (46), one gets

$$G_q'(r) \geq \frac{C\theta_2(r)\,\big(\theta_1(r)\big)^{q^*}\,\big(G_q(r)\big)^{q^*(p^*+1)}}{\big(G_q'(r)\big)^{q^*}},$$

which is equivalent to

$$\begin{aligned} G_q'(r) &\geq C\big(\theta_2(r)\big)^{\frac{1}{q^*+1}}\,\big(\theta_1(r)\big)^{\frac{q^*}{q^*+1}}\,\big(G_q(r)\big)^{\frac{q^*(p^*+1)}{q^*+1}} \\ &= \frac{C}{r}\big(\mu_1(C_0r^{-n})\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(C_0r^{-n})\big)^{\frac{1}{q^*+1}}\big(G_q(r)\big)^{\frac{q^*(p^*+1)}{q^*+1}}. \end{aligned}$$

Summarizing we have

$$\frac{C}{r}\big(\mu_1(C_0r^{-n})\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(C_0r^{-n})\big)^{\frac{1}{q^*+1}} \leq \frac{G_q'(r)}{\big(G_q(r)\big)^{\frac{q^*(p^*+1)}{q^*+1}}}.$$

Integrating two sides of the last estimate over $[R_0, R^*]$ leads to

$$\begin{aligned} &C\int_{R_0}^{R^*}\frac{1}{r}\big(\mu_1(C_0r^{-n})\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(C_0r^{-n})\big)^{\frac{1}{q^*+1}}\,dr \\ &\leq \int_{R_0}^{R^*}\frac{G_q'(r)}{\big(G_q(r)\big)^{\frac{q^*(p^*+1)}{q^*+1}}}\,dr \\ &= -\frac{q^*+1}{p^*q^*-1}\big(G_q(r)\big)^{-\frac{p^*q^*-1}{q^*+1}}\Big|_{r=R_0}^{r=R^*} \leq \frac{n}{2}\big(G_q(R_0)\big)^{-\frac{2}{n}}, \end{aligned}$$

where we note that $\dfrac{q^*+1}{p^*q^*-1} = \dfrac{n}{2}$. For this reason, we pass $R^* \to \infty$ to derive

$$C\int_{R_0}^{\infty}\frac{1}{r}\big(\mu_1(C_0r^{-n})\big)^{\frac{q^*}{q^*+1}}\big(\mu_2(C_0r^{-n})\big)^{\frac{1}{q^*+1}}\,dr \leq \frac{n}{2}\big(G_q(R_0)\big)^{-\frac{2}{n}}.$$

Finally, carrying out change of variables $s = C_0 r^{-n}$ gives

$$C \int_0^{C_0 R_0^{-n}} \frac{1}{s} \big(\mu_1(s)\big)^{\frac{q^*}{q^*+1}} \big(\mu_2(s)\big)^{\frac{1}{q^*+1}} \, ds \le \frac{n}{2} \big(G_q(R_0)\big)^{-\frac{2}{n}}.$$

This contradicts to the assumption (14). Summarizing, the proof of Theorem 1.2 is completed. □

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