On the stability of limit cycles for planar differential systems

H. Giacomini[a], M. Grau[b],[*],[1]

[a]*Laboratoire de Mathématiques et Physique Théorique, C.N.R.S. UMR 6083, Faculté des Sciences et Techniques, Université de Tours, Parc de Grandmont, 37200 Tours, France*
[b]*Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Spain*

Abstract

We consider a planar differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, where P and Q are $\mathscr{C}^1$ functions in some open set $\mathscr{U} \subseteq \mathbb{R}^2$, and $\dot{} = \frac{d}{dt}$. Let γ be a periodic orbit of the system in $\mathscr{U}$. Let $f(x, y) : \mathscr{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a $\mathscr{C}^1$ function such that

$$P(x, y) \frac{\partial f}{\partial x}(x, y) + Q(x, y) \frac{\partial f}{\partial y}(x, y) = k(x, y) f(x, y),$$

where $k(x, y)$ is a $\mathscr{C}^1$ function in $\mathscr{U}$ and $\gamma \subseteq \{(x, y) \mid f(x, y) = 0\}$. We assume that if $p \in \mathscr{U}$ is such that $f(p) = 0$ and $\nabla f(p) = 0$, then p is a singular point.

We prove that $\int_0^T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) (\gamma(t)) \, dt = \int_0^T k(\gamma(t)) \, dt$, where $T > 0$ is the period of γ. As an application, we take profit from this equality to show the hyperbolicity of the known algebraic limit cycles of quadratic systems.
© 2005 Elsevier Inc. All rights reserved.

Keywords: Limit cycle; Planar differential system; Stability; Hyperbolicity; Polynomial vector field

[*] Corresponding author.
  E-mail addresses: Hector.Giacomini@phys.univ-tours.fr (H. Giacomini), mtgrau@matematica.udl.es (M. Grau).

1. Introduction

We consider a *planar differential system*

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are $\mathcal{C}^1$ functions in some open set $\mathcal{U} \subseteq \mathbb{R}^2$, and $\dot{} = \frac{d}{dt}$. A singular point of system (1) is a point $p \in \mathcal{U}$ such that $P(p) = Q(p) = 0$. We assume that all the singular points of (1) are isolated.

Given a system (1), we can always consider its vector field representation $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$.

We will denote by $\mathrm{div}(x, y)$ the *divergence* of system (1), that is, $\mathrm{div} = \partial P/\partial x + \partial Q/\partial y$.

We also need to consider the *flow* of system (1), which we denote by $\Phi_t(p)$ and which represents the unique solution of system (1) passing through the point $p \in \mathcal{U} \subseteq \mathbb{R}^2$. We notice that for each $p \in \mathcal{U}$ there exists an $\varepsilon_p > 0$ (which may be $\varepsilon_p = +\infty$) such that $t \in (-\varepsilon_p, \varepsilon_p)$ is the maximal symmetric interval of existence of the solution of (1) passing through p. We have that $\frac{d\Phi_t}{dt}(p) = (P(\Phi_t(p)), Q(\Phi_t(p)))$, for all $p \in \mathcal{U}$ and $t \in (-\varepsilon_p, \varepsilon_p)$, and $\Phi_0(p) = p$. Given $p \in \mathcal{U}$, the function $\Phi(\cdot, p) : (-\varepsilon_p, \varepsilon_p) \to \mathbb{R}^2$, where $\Phi(t, p) := \Phi_t(p)$, defines a *solution curve* or *orbit* of (1) through the point p.

A *limit cycle* of system (1) is an isolated periodic orbit. Let γ be a limit cycle for system (1). We say that γ is *stable* if there exists a neighborhood $\mathcal{U}_\gamma \subseteq \mathcal{U}$ of γ such that for all $p \in \mathcal{U}_\gamma$, we have $\lim_{t\to+\infty} d(\Phi_t(p), \gamma) = 0$. As usual, the previous application d is the distance between sets in the Hausdorff sense. Analogously, we say that γ is *unstable* if there exists a neighborhood $\mathcal{U}_\gamma \subseteq \mathcal{U}$ of γ such that for all $p \in \mathcal{U}_\gamma$, we have $\lim_{t\to-\infty} d(\Phi_t(p), \gamma) = 0$.

There might be limit cycles which are neither stable nor unstable. Using the Jordan curve theorem, which states that any simple closed curve, as the limit cycle, γ separates any neighborhood $\mathcal{U}_\gamma$ of γ into two disjoint sets having γ as a boundary, we can consider $\mathcal{U}_\gamma$ as the disjoint union of $\mathcal{U}_i \cup \gamma \cup \mathcal{U}_e$, where $\mathcal{U}_i$ and $\mathcal{U}_e$ are open sets situated, respectively, in the interior and exterior of γ. When for any $p \in \mathcal{U}_i$ we have $\lim_{t\to+\infty} d(\Phi_t(p), \gamma) = 0$ whereas for any $q \in \mathcal{U}_e$ we have $\lim_{t\to-\infty} d(\Phi_t(q), \gamma) = 0$ (or, the other way round, for any $p \in \mathcal{U}_i$ we have $\lim_{t\to-\infty} d(\Phi_t(p), \gamma) = 0$ whereas for any $q \in \mathcal{U}_e$ we have $\lim_{t\to+\infty} d(\Phi_t(q), \gamma) = 0$), we say that γ is *semi-stable*.

Any limit cycle γ of a system (1) is either stable, unstable or semi-stable as it is stated in [16]. For a detailed description of the classical known results on limit cycles see also [16].

The following result, which is stated as a corollary in [16, p. 214], gives a formula to distinguish the stability of a limit cycle.

Theorem 1. *Let $\gamma(t)$ be a periodic orbit of system* (1) *of period T. Then, γ is a stable limit cycle if*

$$\int_0^T \mathrm{div}(\gamma(t)) \, dt < 0$$

and it is an unstable limit cycle if

$$\int_0^T \operatorname{div}(\gamma(t))\,dt > 0.$$

It may be stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles if this quantity is zero.

A sketch of the proof of this theorem is given after the forthcoming Theorem 3.

When the quantity $\int_0^T \operatorname{div}(\gamma(t))\,dt$ is different from zero, we say that the limit cycle γ is *hyperbolic*.

Since we are considering differential systems (1) in the class of functions $\mathcal{C}^1$, we may have a limit cycle γ belonging to a sequence of periodic orbits $\{\gamma_n\,,\ n \in \mathbb{N}\}$ with γ_{n+1} in the interior of γ_n, such that the sequence accumulates to a singular point, a periodic orbit or a graphic and such that every trajectory between γ_n and γ_{n+1} spirals towards γ_n or γ_{n+1} as $t \to \pm\infty$. This kind of phenomena does not exist for analytic systems.

In this work, we give another quantity which equals to $\int_0^T \operatorname{div}(\gamma(t))\,dt$ for a periodic orbit γ defined in an implicit way, as explained below. This is the main result of the article and it is stated in Theorem 2 in the following section. We can, therefore, distinguish the hyperbolicity of a limit cycle using two different quantities.

Given a planar system (1) (or equivalently its vector field representation $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$), we define an *invariant curve* as a curve given by $f(x, y) = 0$, where $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^1$ function in the open set $\mathcal{U}$, non-locally null and such that there exists a $\mathcal{C}^1$ function in $\mathcal{U}$, denoted by $k(x, y)$, for which

$$P(x, y)\,\frac{\partial f}{\partial x}(x, y) + Q(x, y)\,\frac{\partial f}{\partial y}(x, y) = k(x, y)\,f(x, y) \tag{2}$$

for all $(x, y) \in \mathcal{U}$. Identity (2) can be rewritten by $\nabla f \cdot \mathbf{F} = kf$. As usual, ∇f denotes the gradient vector related to $f(x, y)$, that is, $\nabla f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y))$, $\mathbf{F}(x, y)$ is the previously defined vector $(P(x, y), Q(x, y))$, and $\cdot$ denotes the scalar product. We will denote by $\dot f$ or by $\frac{df}{dt}$ the function $\nabla f \cdot \mathbf{F}$ once evaluated on a solution of system (1).

We will always assume that if $p \in \mathcal{U}$ is such that $f(p) = 0$ and $\nabla f(p) = 0$, then p is a singular point of system (1). This is a technical hypothesis which generalizes the notion of not having multiple factors for algebraic curves. For instance, if we had written that the periodic orbit γ was contained in $f^2(x, y) = 0$, then we would have that $\nabla(f^2)(p) = 0$ for all $p \in \gamma$, in contradiction with the hypothesis.

We notice that, as a particular case, we may have a function $f(x, y)$ given by a polynomial in $\mathbb{R}[x, y]$. In such a case, $f(x, y) = 0$ is called an *invariant algebraic curve*. When, in addition, the system is polynomial, that is, $P, Q \in \mathbb{R}[x, y]$, then the function $k(x, y)$ is a real polynomial called *cofactor*. When we consider an algebraic

curve, we can always assume that it is defined by a polynomial $f(x, y) = 0$ such that the decomposition of $f(x, y)$ has no multiple factors. The same assumption must be done for curves defined by $\mathcal{C}^1$ functions and it is equivalent to the assumption that if $p \in \mathcal{U}$ is such that $f(p) = 0$ and $\nabla f(p) = 0$, then p is a singular point of system (1). More explicitly, assume that system (1) has an invariant algebraic curve given by $f(x, y) = 0$ and assume that the decomposition of $f(x, y)$ in the ring $\mathbb{R}[x, y]$ has no multiple factors, that is, $f(x, y) = b_1(x, y)b_2(x, y) \ldots b_k(x, y)$ where $b_j(x, y)$ is an irreducible polynomial in $\mathbb{R}[x, y]$ and $b_i(x, y) \neq cb_j(x, y)$ for any $c \in \mathbb{R} - \{0\}$ if $i \neq j$. Let $p \in \mathcal{U}$ be such that $f(p) = 0$ and $\nabla f(p) = 0$. Since the decomposition of $f(x, y)$ has no multiple factors, we deduce that p is a singular point of the curve $f(x, y) = 0$ and, hence, it is a singular point of system (1).

Our main result, Theorem 2, can only be applied when the periodic orbit γ is given in an implicit way, that is, when there exists an invariant curve $f(x, y) = 0$ such that $\gamma \subseteq \{(x, y) \mid f(x, y) = 0\}$. For instance, let us consider the following $\mathcal{C}^1$ system defined in all $\mathbb{R}^2$:

$$
\dot{x} = (x + y)\cos(x) - y(x^2 + xy + 2y^2),
$$

$$
\dot{y} = (y - x)(\cos(x) - y^2) + \frac{x^2 + y^2}{2}\sin(x), \tag{3}
$$

which has $y^2 - \cos(x) = 0$ as invariant curve. We define $f(x, y) := y^2 - \cos(x)$ and we have that $f \in \mathcal{C}^1(\mathbb{R}^2)$ and that $\nabla f(x, y) = (\sin(x), 2y)$. Therefore, there is no $p \in \mathbb{R}^2$ such that both $f(p) = 0$ and $\nabla f(p) = 0$. Moreover, $f(x, y) = 0$ satisfies Eq. (2) with $k(x, y) = 2y(x - y) - (x + y)\sin(x)$. The divergence of this system is $\text{div}(x, y) = -4y^2 + 2\cos(x) - x\sin(x)$ and $V(x, y) = (x^2 + y^2)f(x, y)$ is an inverse integrating factor. We denote by γ_n, $n \in \mathbb{Z}$, the oval of $f(x, y) = 0$ belonging to the strip $-\pi/2 + 2\pi n \leqslant x \leqslant \pi/2 + 2\pi n$. Each oval γ_n, with $n \in \mathbb{Z}$, gives a periodic orbit of (3) with minimal period $T_n > 0$. The oval γ_0 is a hyperbolic stable limit cycle for system (3), which can be shown just applying Theorem 1. We have, after some easy computations, that

$$
\int_0^{T_n} \text{div}(\gamma_n(t))\,dt \;=\; -4\arctan\left(\frac{x}{\sqrt{\cos(x)}}\right)\Bigg|_{x = -\pi/2 + 2\pi n,}^{x = \pi/2 + 2\pi n,}
$$

which is zero when $n \neq 0$ and it is -4π for γ_0. Each one of the other ovals of $f(x, y) = 0$, γ_n with $n \neq 0$, belongs to the period annulus of a center as it can be shown from the fact that the function $H(x, y) = f(x, y)(x^2 + y^2)\exp\{2\arctan(y/x)\}$ is a first integral for system (3). Our result can be applied for any of the periodic orbit γ_n of this example.

When considering a polynomial system, as far as the authors know, only algebraic limit cycles are known in this implicit way. A limit cycle is said to be *algebraic* if its points belong to an invariant algebraic curve.

The paper is organized as follows. Section 2 contains the statement and proof of the main result of this work, i.e. Theorem 2. Using Theorem 2, in Section 3, we will show that all the known algebraic limit cycles of a quadratic system are hyperbolic.

2. Main result

Theorem 2. *Let us consider a system* (1) *and* $\gamma(t)$ *a periodic orbit of period* $T > 0$. *Assume that* $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ *is an invariant curve with* $\gamma \subseteq \{(x, y) \mid f(x, y) = 0\}$ *and let* $k(x, y)$ *be the* $\mathcal{C}^1$ *function given in* (2). *We assume that if* $p \in \mathcal{U}$ *is such that* $f(p) = 0$ *and* $\nabla f(p) = 0$, *then* p *is a singular point of system* (1). *Then,*

$$\int_0^T k(\gamma(t)) \, dt \;=\; \int_0^T \mathrm{div}(\gamma(t)) \, dt. \qquad (4)$$

In order to prove Theorem 2, we need to recall the definition and some properties of the Poincaré map. Let us consider γ a periodic orbit with minimal period $T > 0$ for system (1) and $p_0 \in \gamma$. Let $\mathcal{U}_\gamma \subseteq \mathcal{U}$ be a neighborhood of γ not containing any singular point and $\Sigma = \{q \in \mathcal{U}_\gamma \mid (q - p_0) \cdot \mathbf{F}(p_0) = 0\}$, where $\cdot$ denotes the scalar product between the vectors $q - p_0$ and $\mathbf{F}(p_0)$.

As stated and proved in [16, pp. 210–211], we have that there exists a $\delta > 0$ and a unique function $\tau : \Sigma \to \mathbb{R}$, which is defined continuously and differentiable for any $q \in \Sigma \cap \mathcal{B}_\delta(p_0)$ such that $\tau(p_0) = T$ and $\Phi_{\tau(q)}(q) \in \Sigma$. As usual, $\mathcal{B}_\delta(p_0)$ is the ball of center p_0 and radius δ. Then, for any $q \in \Sigma \cap \mathcal{B}_\delta(p_0)$, the function $\mathcal{P}(q) = \Phi_{\tau(q)}(q)$ is called the *Poincaré map* for γ at p_0. It is clear that fixed points of the Poincaré map, $\mathcal{P}(q) = q$, give rise to periodic orbits for system (1). Moreover, it can be shown that $\mathcal{P} : \Sigma \to \Sigma$ is a $\mathcal{C}^1$ diffeomorphism.

We consider Σ as a subset of $\mathcal{U} \subseteq \mathbb{R}^2$, so $\mathcal{P}$ is considered as a planar function from $\Sigma \subset \mathbb{R}^2$ to $\mathbb{R}^2$. Hence, we notice that the derivative of $\mathcal{P}$ at p_0, which is a point in Σ, can be represented by a 2×2 matrix, which we denote by $D\mathcal{P}(p_0)$. The following theorem, stated and proved in [2, p. 118], is very useful to establish the stability of γ.

Theorem 3. *Let* $\mathbf{v}$ *be a non-null vector normal to* $\mathbf{F}(p_0)$. *Then,*

$$\mathbf{v} \cdot D\mathcal{P}(p_0) = \exp \left(\int_0^T \mathrm{div}(\gamma(t)) \, dt \right) \mathbf{v}. \qquad (5)$$

Theorem 3 is proved by using the variational equations of first order related to system (1). If $\Phi_t(x, y)$ is the flow related to the vector field $\mathbf{F}(x, y)$, we have that $\frac{d}{dt}(D\Phi_t(x, y)) = D\mathbf{F}(\Phi_t(x, y)) \cdot D\Phi_t(x, y)$ with the initial condition $D\Phi_t(x, y)_{|t=0} = I$, where D means the differential with respect to the point (x, y) and I is the identity matrix. These equations with respect to the matrix $D\Phi_t(x, y)$ are the variational equations of first order. Since $\mathcal{P}(q) = \Phi_{\tau(q)}(q)$, the solution of the variational equations of first order allows the computation of $D\mathcal{P}(p_0)$ in a point $p_0 \in \gamma$.

In order to show that the stability of γ is determined by the value of $\mathbf{v} \cdot D\mathcal{P}(p_0)$, as stated in Theorem 1, we consider the *displacement function* and we follow the reasoning of [16, p. 213]. For any $q \in \Sigma \cap \mathcal{B}_\delta(p_0)$, we have that $q = p_0 + s\mathbf{v}$, with $s \in (-\delta/|\mathbf{v}|, \delta/|\mathbf{v}|)$. Since $\mathcal{P}(q) \in \Sigma$, we have that given $s \in (-\delta/|\mathbf{v}|, \delta/|\mathbf{v}|)$, there exists a $\sigma(s) \in \mathbb{R}$ such that $\mathcal{P}(p_0 + s\mathbf{v}) = p_0 + \sigma(s)\mathbf{v}$. Therefore, we have defined a $\mathcal{C}^1$ function $\sigma : (-\delta/|\mathbf{v}|, \delta/|\mathbf{v}|) \to \mathbb{R}$ and the displacement function is given by d : $(-\delta/|\mathbf{v}|, \delta/|\mathbf{v}|) \to \mathbb{R}$ with $d(s) = \sigma(s) - s$. It is clear that $d(0) = 0$, $d'(s) = \sigma'(s) - 1$ and $\mathbf{v} \cdot D\mathcal{P}(p_0 + s\mathbf{v}) = \sigma'(s)\mathbf{v}$. Since $d(s)$ is $\mathcal{C}^1$, we have that the sign of $d'(s)$ coincides with the sign of $d'(0)$ for $|s|$ sufficiently small as long as $d'(0) \neq 0$. By the mean value theorem, we have that given $|s|$ sufficiently small there exists a $\xi \in (0, s)$ such that $d(s) = d'(\xi)s$. Therefore, if $d'(0) > 0$, we have that $d(s) > 0$ for $s > 0$ and $d(s) < 0$ for $s < 0$, which implies that the periodic orbit γ is an unstable limit cycle. Similar reasonings show that if $\sigma'(0) > 1$ then γ is an unstable limit cycle and if $\sigma'(0) < 1$ then γ is a stable limit cycle. Theorem 1 clearly follows from Theorem 3 and the fact that $\sigma'(0)\mathbf{v} = \mathbf{v} \cdot D\mathcal{P}(p_0)$.

Lemma 4. *Let us consider a system* (1) *and let* $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a non-null* $\mathcal{C}^1(\mathcal{U})$-*function. There exists a* $\mathcal{C}^1$ *function* $k(x, y)$ *such that* $\nabla f(q) \cdot \mathbf{F}(q) = k(q)f(q)$ *for any* $q \in \mathcal{U}$ *if, and only if, for any* $q \in \mathcal{U}$ *and any* $t \in \mathbb{R}$ *such that* $\Phi_t(q) \in \mathcal{U}$, *the following identity is satisfied*:

$$f(\Phi_t(q)) = f(q) \ \exp\left(\int_0^t k(\Phi_s(q))\,ds\right). \tag{6}$$

Proof. Assume that $\nabla f(q) \cdot \mathbf{F}(q) = k(q)f(q)$ for any $q \in \mathcal{U}$. We fix a point $q \in \mathcal{U}$ and we define $\varphi(t) = f(\Phi_t(q))$ for any $t \in \mathbb{R}$ such that $\Phi_t(q) \in \mathcal{U}$. We have that t belongs to an open interval $(-\varepsilon_q, \varepsilon_q)$ with $\varepsilon_q > 0$ (and it may be that $\varepsilon_q = +\infty$). We have, using some of the properties of the flow and the fact $\dot{f}(\Phi_t(q)) = k(\Phi_t(q))\,f(\Phi_t(q))$, that:

$$\dot{\varphi}(t) = \nabla f(\Phi_t(q)) \cdot \frac{d\Phi_t}{dt}(q) = \nabla f(\Phi_t(q)) \cdot \mathbf{F}(\Phi_t(q)) = \dot{f}(\Phi_t(q)) = k(\Phi_t(q))\ \varphi(t).$$

We deduce that $\frac{d\varphi}{dt}(t) = k(\Phi_t(q))\ \varphi(t)$ and $\varphi(0) = f(q)$. Solving this linear equation in the function $\varphi(t)$ we get $\varphi(t) = f(q)\exp\left(\int_0^t k(\Phi_s(q))\,ds\right)$. As we can consider the same reasoning for any $q \in \mathcal{U}$, we obtain identity (6). The reciprocal is proved by the same reasoning. $\square$

Lemma 5. *Let us consider a system* (1) *and* $\gamma(t)$ *a periodic orbit of period* $T > 0$. *Assume that* $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ *is an invariant curve with* $\gamma \subseteq \{(x, y) \mid f(x, y) = 0\}$ *and let* $k(x, y)$ *be the* $\mathcal{C}^1$ *function given in* (2). *Take any* p_0 *in* γ. *Then,*

$$\nabla f(p_0) \cdot D\mathcal{P}(p_0) = \exp\left(\int_0^T k(\gamma(t))\,dt\right) \nabla f(p_0). \tag{7}$$

Proof. We consider the Poincaré map defined in an interval of the straight line Σ containing p_0, $\mathcal{P}(q) = \Phi_{\tau(q)}(q)$. Since $f(x, y) = 0$ is an invariant curve defined in $\mathcal{U} \subseteq \mathbb{R}^2$, it is clear that for any $q \in \mathcal{U}$ and any $t \in \mathbb{R}$ such that $\Phi_t(q) \in \mathcal{U}$, identity (6) is satisfied as proved in Lemma 4. Hence,

$$f(\mathcal{P}(q)) = f(q) \exp \left(\int_0^{\tau(q)} k(\Phi_s(q)) \, ds \right)$$

and differentiating this identity with respect to q we get

$$\nabla f(\mathcal{P}(q)) \cdot D\mathcal{P}(q) = \exp \left(\int_0^{\tau(q)} k(\Phi_s(q)) \, ds \right) \nabla f(q)$$

$$+ f(q) \exp \left(\int_0^{\tau(q)} k(\Phi_s(q)) \, ds \right)$$

$$\times \left[\int_0^{\tau(q)} (\nabla k)(\Phi_s(q)) \cdot D\Phi_s(q) \, ds + k(\mathcal{P}(q))\nabla\tau(q) \right],$$

where $D\mathcal{P}(q)$ and $D\Phi_s(q)$ stand for the Jacobian matrix with respect to q of the functions $\mathcal{P}$ and Φ_s, respectively, in the point q.

We evaluate the previous identity in $q = p_0$, taking into account that $f(p_0) = 0$ and $\tau(p_0) = T$, and we get identity (7). $\square$

Proof of Theorem 2. The vector $\nabla f(p_0)$ is a non-null vector that is normal to the vector $\mathbf{F}(p_0)$ since $f(x, y) = 0$ is an invariant curve that contains γ, and $p_0 \in \gamma$. The fact of $\nabla f(p_0)$ to be a non-null vector is ensured by the assumption that if $p \in \mathcal{U}$ is such that $f(p) = 0$ and $\nabla f(p) = 0$, then p is a singular point of system (1). Since p_0 belongs to a periodic orbit, it cannot be a singular point.

Therefore, the vector $\mathbf{v}$ in identity (5) of Theorem 3 can be replaced by $\nabla f(p_0)$. Using identity (7) of Lemma 5, we deduce that

$$\exp \left(\int_0^T \text{div}(\gamma(t)) \, dt \right) = \exp \left(\int_0^T k(\gamma(t)) \, dt \right),$$

from which (4) follows. $\square$

3. Hyperbolicity of the known algebraic limit cycles of quadratic systems

We consider the families of quadratic systems with algebraic limit cycles known at the time of composition of this paper. These families sweep all the algebraic limit cycles defined by polynomials of degrees 2 and 4 for a quadratic system, as it is

proved in [9]. In [12–14], it is shown that there are no algebraic limit cycles of degree 3 for a quadratic system. See [8] for a short proof. In [11], two examples of quadratic systems with an algebraic limit cycle of degree 5 and 6 are described. We study the hyperbolicity of all these limit cycles.

The following result is due to Ch'in Yuan-shün [10] and characterizes the algebraic limit cycles of degree 2 for a quadratic system.

Theorem 6 (*Yuan-shün Ch'in [10]*). *If a quadratic system has an algebraic limit cycle of degree* 2, *then after an affine change of variables, the limit cycle becomes the circle*

$$\Gamma := x^2 + y^2 - 1 = 0. \tag{8}$$

Moreover, Γ *is the unique limit cycle of the quadratic system which can be written in the form*

$$\begin{aligned} \dot{x} &= -y\,(ax + by + c) - (x^2 + y^2 - 1), \\ \dot{y} &= x\,(ax + by + c), \end{aligned} \tag{9}$$

with $a \neq 0$, $c^2 + 4(b+1) > 0$ *and* $c^2 > a^2 + b^2$.

We summarize the four families of algebraic limit cycles of degree 4 for quadratic systems in the following result, which is stated and proved in [9]. We remark that these families were encountered previously to the work [9], but in this work it was shown that there are no other algebraic limit cycle of degree 4 for a quadratic system. System (10) was first described in [17], system (12) in [15], system (14) in [5] and system (16) in [9].

Theorem 7 (*Chavarriga et al. [9]*). *After an affine change of variables the only quadratic systems having an algebraic limit cycle of degree* 4 *are*

(a) *Yablonskii's system*

$$\begin{aligned} \dot{x} &= -4abcx - (a+b)y + 3(a+b)cx^2 + 4xy, \\ \dot{y} &= (a+b)abx - 4abcy + (4abc^2 - \tfrac{3}{2}(a+b)^2 + 4ab)x^2 \\ &\quad + 8(a+b)cxy + 8y^2, \end{aligned} \tag{10}$$

with $abc \neq 0$, $a \neq b$, $ab > 0$ *and* $4c^2(a-b)^2 + (3a-b)(a-3b) < 0$. *This system has the invariant algebraic curve*

$$(y + c\,x^2)^2 + x^2\,(x - a)(x - b) = 0, \tag{11}$$

whose oval is a limit cycle for system (10).

(b) *Filipstov's system*

$$\dot{x} = 6(1 + a) x + 2 y - 6(2 + a) x^2 + 12 x y,$$
$$\dot{y} = 15(1 + a) y + 3 a (1 + a) x^2 - 2(9 + 5 a) x y + 16 y^2,$$

(12)

with $0 < a < \frac{3}{13}$. *This system has the invariant algebraic curve*

$$3(1 + a)(a x^2 + y)^2 + 2 y^2(2 y - 3(1 + a)x) = 0,$$

(13)

whose oval is a limit cycle for system (12).

(c) *Chavarriga's system*

$$\dot{x} = 5 x + 6 x^2 + 4(1 + a) x y + a y^2,$$
$$\dot{y} = x + 2 y + 4 x y + (2 + 3 a) y^2,$$

(14)

with $\frac{-71 + 17\sqrt{17}}{32} < a < 0$ *has the invariant algebraic curve*

$$x^2 + x^3 + x^2 y + 2 a x y^2 + 2 a x y^3 + a^2 y^4 = 0,$$

(15)

whose oval is a limit cycle for system (14).

(d) *Chavarriga et al.'s system*

$$\dot{x} = 2(1 + 2 x - 2 a x^2 + 6 x y),$$
$$\dot{y} = 8 - 3 a - 14 a x - 2 a x y - 8 y^2,$$

(16)

with $0 < a < \frac{1}{4}$ *has the invariant algebraic curve*

$$\frac{1}{4} + x - x^2 + a x^3 + x y + x^2 y^2 = 0,$$

(17)

whose oval is a limit cycle for system (16).

In a work due to Christopher et al. [11] two families of quadratic systems with an algebraic limit cycle of degrees five and six, respectively, are given. These two families are constructed by means of a birational transformation of system (16). A *birational transformation* is a rational change of variables such that its inverse is also rational. Moreover, they prove that there is also a birational transformation which converts Yablonskii's system (10) into the system with a limit cycle of degree 2, that is, system (9).

The fact of the limit cycle of degree 2 being hyperbolic is stated in [18] (see pp. 256–258) following the proof of [10]. As a consequence, and taking into account the forthcoming Lemma 9, one of the limit cycles of degree 4 (the one due

to Yablonskii) is also hyperbolic, because this limit cycle of degree 4 is birationally equivalent to the one of degree 2, as it is shown in [11]. Our contribution is the proof of the hyperbolicity of the other known limit cycles of quadratic systems.

Lemma 8. *Let us consider a differential system* (1) *and a change of variables* $x = F(u, v)$ *and* $y = G(u, v)$, *where* F, G *are* C^2 *functions in* $\mathcal{U}$. *We denote by* $\dot{u} = R(u, v)$, $\dot{v} = S(u, v)$ *the transformed differential system. Let*

$$
J(u, v) := \frac{\partial F}{\partial u}(u, v) \frac{\partial G}{\partial v}(u, v) - \frac{\partial F}{\partial v}(u, v) \frac{\partial G}{\partial u}(u, v),
$$

be the Jacobian of the transformation. Then,

$$
\frac{\partial P}{\partial x}(F(u, v), G(u, v)) + \frac{\partial Q}{\partial y}(F(u, v), G(u, v)) = \frac{\partial R}{\partial u}(u, v) + \frac{\partial S}{\partial v}(u, v)
$$

$$
+ \frac{1}{J(u, v)} \left(\frac{\partial J}{\partial u}(u, v) R(u, v) + \frac{\partial J}{\partial v}(u, v) S(u, v) \right). \tag{18}
$$

Lemma 8 is a computational result whose proof is clear after some easy manipulations. We use it to prove the following result which states that the value of the integral of the divergence on the limit cycle does not change under transformations of dependent variables.

Lemma 9. *Let us consider a differential system* (1) *with a periodic orbit* γ *of period* $T > 0$ *and a change of variables* $x = F(u, v)$ *and* $y = G(u, v)$ *which is well-defined in a neighborhood of* γ. *We denote by* $\dot{u} = R(u, v)$, $\dot{v} = S(u, v)$ *the transformed differential system and by* ϑ *the corresponding periodic orbit. Then,*

$$
\int_0^T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) (\gamma(t)) \, dt = \int_0^T \left(\frac{\partial R}{\partial u} + \frac{\partial S}{\partial v} \right) (\vartheta(t)) \, dt.
$$

Proof. Using the same notation as in Lemma 8, we have that the integral $\int_0^T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)$ $(\gamma(t)) \, dt$ becomes, under the transformation of dependent variables $x = F(u, v)$ and $y = G(u, v)$,

$$
\int_0^T \left(\frac{\partial P}{\partial x}(F(u, v), G(u, v)) + \frac{\partial Q}{\partial y}(F(u, v), G(u, v)) \right) (\vartheta(t)) \, dt
$$

which, by Lemma 8, equals to

$$\int_0^T \left(\frac{\partial R}{\partial u}(u, v) + \frac{\partial S}{\partial v}(u, v) \right) (\vartheta(t)) \, dt$$

$$+ \int_0^T \frac{1}{J(u, v)} \left(\frac{\partial J}{\partial u}(u, v) R(u, v) + \frac{\partial J}{\partial v}(u, v) S(u, v) \right) (\vartheta(t)) \, dt.$$

We notice that the integrand of the second integral in the former expression can be rewritten as $d(J(u, v))/J(u, v)$ and, since the change of variables is well defined in a neighborhood of γ, we have that this expression is a well defined, exact 1-form which is integrated over the closed curve ϑ, so $\oint_\vartheta d(J(u, v))/J(u, v) = 0$. $\quad \square$

Therefore, in order to prove that all these families of limit cycles are hyperbolic, we only need to study the stability of the limit cycles of systems (12), (14) and (16). The hyperbolicity of the two limit cycles described in [11] is shown by the fact that they are birationally equivalent to (16).

Theorem 10. *Each one of the limit cycles of systems* (12), (14) *and* (16) *is hyperbolic.*

Proof. In order to prove the hyperbolicity of the limit cycles of systems (12), (14) and (16) we use the same process for all of them. These systems depend on a parameter a which belongs to a certain open interval when the limit cycle γ exists. We denote by $T > 0$ the period of the limit cycle and by $\mathcal{D}(a)$ the value of the integral $\int_0^T \operatorname{div}(\gamma(t)) \, dt$ for any value of the parameter for which the limit cycle exists. This value decides the hyperbolicity character of the limit cycle γ in the system with parameter a. By virtue of Lemma 9, we may consider any birational transformation of these systems well defined in a neighborhood of the limit cycle and we may consider the transformed system instead of the previous one because the value of the integral $\int_0^T \operatorname{div}(\gamma(t)) \, dt$ does not change.

Using Theorem 2, we have that

$$\mathcal{D}(a) = \int_0^T \operatorname{div}(\gamma(t)) \, dt + w \left(\int_0^T \operatorname{div}(\gamma(t)) \, dt - \int_0^T k(\gamma(t)) \, dt \right),$$

where k is the cofactor of the invariant algebraic curve containing the limit cycle and w is any real number.

We show that the function $\mathcal{D}(a)$ has no zero when a belongs to the interval of existence of limit cycle by choosing an adequate $w \in \mathbb{R}$ and parameterizing the limit cycle γ. The way of choosing the adequate value of w is purely heuristic, although we expect that this choice is related to some geometric property. We find it very surprising that it is possible to choose $w = -3$ for each one of the three families of systems.

3.1. Hyperbolicity of the limit cycle given by the algebraic curve (17) for system (16)

The stability of the limit cycle γ is given by the following function of the parameter a of the system, $\mathcal{D}(a) := \int_0^T \text{div}(\gamma(t)) \, dt$, where $\text{div}(x, y) = 2(2 - 5ax - 2y)$ is the divergence of system (16) and $T > 0$ the period of the limit cycle. Theorem 2 gives

$$\int_0^T \text{div}(\gamma(t)) \, dt = \int_0^T k(\gamma(t)) \, dt,$$

where $k(x, y) = 4(2 - 3ax + 2y)$ is the cofactor of the invariant algebraic curve (17). So, given any real number w, we have that

$$\mathcal{D}(a) = \int_0^T \text{div}(\gamma(t)) \, dt + w \int_0^T (\text{div} - k)(\gamma(t)) \, dt$$

$$= \int_0^T ((1 + w)\text{div} - wk) (\gamma(t)) \, dt.$$

We consider the following parameterization of the oval of the algebraic curve (17):

$$x(\tau) = \tau, \quad y_{\pm}(\tau) = \frac{-1 \pm 2\sqrt{(-a)\tau(\tau - \tau_1)(\tau - \tau_2)}}{2\tau}, \tag{19}$$

where $\tau_1 = \frac{1 - \sqrt{1 - 4a}}{2a}$, $\tau_2 = \frac{1 + \sqrt{1 - 4a}}{2a}$ and the parameter $\tau \in (\tau_1, \tau_2)$. The positive sign $y_+(\tau)$ gives a half of the oval and the negative sign $y_-(\tau)$ the other half. One of the endpoints of both parameterizations is $(x_1, y_1) = (\frac{1 - \sqrt{1 - 4a}}{2a}, -\frac{1 + \sqrt{1 - 4a}}{4})$ and the other endpoint is $(x_2, y_2) = (\frac{1 + \sqrt{1 - 4a}}{2a}, \frac{-1 + \sqrt{1 - 4a}}{4})$. We have that the vector field in (x_1, y_1) is $(0, 6\sqrt{1 - 4a})$ and in (x_2, y_2) is $(0, -6\sqrt{1 - 4a})$, so the flow on the limit cycle is clockwise. The line $2ax = 1$ cuts the limit cycle in two points with ordinates $\pm\sqrt{\frac{1 - 4a}{2}} - a$, which are given, respectively, by $y_{\pm}(1/2a)$. We have the following relation between the differentials: $d\tau = P(x(\tau), y_{\pm}(\tau)) \, dt$ where $P(x, y) = 2(1 + 2x - 2ax^2 + 6xy)$. Then,

$$\mathcal{D}(a) = \int_0^T ((1 + w) \, \text{div} - wk) \, (\gamma(t)) \, dt$$

$$= \int_{\tau_1}^{\tau_2} \left(\frac{((1 + w) \, \text{div} - wk)}{P} \right) (\tau, y_+(\tau)) \, d\tau$$

$$+ \int_{\tau_2}^{\tau_1} \left(\frac{((1 + w) \, \text{div} - wk)}{P} \right) (\tau, y_-(\tau)) \, d\tau$$

$$= \int_{\tau_1}^{\tau_2} \left[\left(\frac{((1 + w) \, \text{div} - wk)}{P} \right) (\tau, y_+(\tau)) \right.$$

$$\left. - \left(\frac{((1 + w) \, \text{div} - wk)}{P} \right) (\tau, y_-(\tau)) \right] d\tau.$$

For $w = -3$ and substituting by the parameterization, we get,

$$\mathcal{D}(a) = 8 \int_{\tau_1}^{\tau_2} \frac{\sqrt{a\tau(\tau - \tau_1)(\tau_2 - \tau)}}{\tau(1 + 8\tau + a\tau^2)} \, d\tau.$$

Since $\tau_1 > 0$ and $\tau_2 > \tau_1$ for any $a \in (0, 1/4)$ and the integrand $\frac{\sqrt{a\tau(\tau-\tau_1)(\tau_2-\tau)}}{\tau(1+8\tau+a\tau^2)}$ is also strictly positive and well defined for any $\tau \in (\tau_1, \tau_2)$ and $a \in (0, 1/4)$, we have that $\mathcal{D}(a) > 0$ for all $a \in (0, 1/4)$, which implies that the limit cycle in system (16) is hyperbolic (and unstable).

3.2. Hyperbolicity of the limit cycle given by the algebraic curve (13) for system (12)

In order to simplify our computations, we consider the following birational change of the parameter, $a = 3c/(4 + 5c)$, with inverse $c = 4a/(3 - 5a)$ and we have that $c \in (0, 1/2)$.

We consider the birational change of variables $(x, y) \to (u, v)$ given by $(x, y) = (X(u, v), Y(u, v))$, where

$$X(u, v) = -\frac{2(1 + 2c)}{c^2(1 + u)^3} (c - 2(1 + c)u + cu^2 - 2\sqrt{1 + 2c}\, v),$$

$$Y(u, v) = -\frac{12(1 + 2c)^2}{c^2(4 + 5c)(1 + u)^4} (c - 2(1 + c)u + cu^2 - 2\sqrt{1 + 2c}\, v).$$

The inverse of this change is given by $(u, v) = (U(x, y), V(x, y))$ with

$$U = \frac{6(1 + 2c)x}{(4 + 5c)y} - 1,$$

$$V = \frac{\sqrt{1 + 2c}}{(4 + 5c)^3 y^3} \left[(1 + 2c)\left(54c^2 x^4 + 18c(4 + 5c)x^2 y \right.\right.$$

$$\left.\left. - 6(4 + 5c)^2 xy^2\right) \right] + \sqrt{1 + 2c}.$$

The Jacobian of this change of variables is

$$\frac{\partial U}{\partial x} \frac{\partial V}{\partial y} - \frac{\partial U}{\partial y} \frac{\partial V}{\partial x} = \frac{324\, c^2\, (1 + 2c)^{5/2}\, x^4}{(4 + 5c)^4\, y^5},$$

which can be seen to be well defined and different from zero in all the points of the oval of the curve given in (13).

We get a transformed system in which we reparameterize its time t multiplying by $c^2(4 + 5c)(1 + u)^3/(12(1 + 2c))$. This reparameterization does not affect the direction

of the flow on the limit cycle. The new system reads for:

$$
\begin{aligned}
\dot{u} &= -2u\,(cu+4+9c)(cu^2-u+c) - 2\sqrt{1+2c}\,(cu^2-(4+5c)u+2c)v, \\
\dot{v} &= -c^2\,\sqrt{1+2c}\,(u+1)^2(u-1)(3u+2) \\
&\quad -(cu+4+9c)(3cu^2-2u+c)v + 2\sqrt{1+2c}\,(4+5c-3cu)v^2
\end{aligned}
\tag{20}
$$

and the limit cycle is transformed to the real oval of the curve $v^2 + u(cu^2 - u + c) = 0$. This algebraic curve is invariant for system (20) with cofactor $k(u,v) = 4\sqrt{1+2c}\,(4+5c-3cu)v - 2(cu+9c+4)(3cu^2-2u+c)$. The divergence of system (20) is $\operatorname{div}(u,v) = 2\sqrt{1+2c}\,(12+15c-8cu)v - [11c^2u^3 + c(28+81c)u^2 + (5c^2-54c-24)u + 3c(4+9c)]$.

We consider the following parameterization of the oval of the algebraic curve $v^2 + u(cu^2 - u + c) = 0$:

$$
u(\tau) = \tau, \quad v_{\pm}(\tau) = \pm\sqrt{c\tau(\tau-\tau_1)(\tau_2-\tau)},
\tag{21}
$$

where $\tau_1 = \frac{1-\sqrt{1-4c^2}}{2c}$, $\tau_2 = \frac{1+\sqrt{1-4c^2}}{2c}$ and the parameter $\tau \in (\tau_1, \tau_2)$. We notice that for $c \in (0, 1/2)$, we have that $0 < \tau_1 < 1 < \tau_2$. The endpoints of both parameterizations are $(\tau_1, 0)$ and $(\tau_2, 0)$, and the vector field at the point $(\tau_i, 0)$ is $(0, c^2\sqrt{1+2c}\,(\tau_i + 1)^2(3\tau_i + 2)(1-\tau_i))$, for $i = 1, 2$. Therefore, the flow on the limit cycle is clockwise. We follow analogous arguments to the previous example to deduce that

$$
\mathcal{D}(c) = \int_{\tau_1}^{\tau_2}\left[\left(\frac{((1+w)\operatorname{div}-wk)}{P}\right)(\tau, v_+(\tau)) \right.
$$
$$
\left. -\left(\frac{((1+w)\operatorname{div}-wk)}{P}\right)(\tau, v_-(\tau))\right]d\tau,
$$

where $P(u,v)$ is the polynomial which defines $\dot{u} = P(u,v)$. For $w = -3$ and substituting by the parameterization, we get

$$
\mathcal{D}(c) = 8\sqrt{1+2c}\int_{\tau_1}^{\tau_2}\frac{\sqrt{c\tau(\tau-\tau_1)(\tau_2-\tau)}}{(\tau+1)(c\tau^2+(17c+8)\tau+4+8c)}\,d\tau.
$$

Since $0 < \tau_1 < 1 < \tau_2$ for any $c \in (0, 1/2)$ and the integrand is strictly positive and well defined for any $\tau \in (\tau_1, \tau_2)$ and $c \in (0, 1/2)$, we have that $\mathcal{D}(c) > 0$ for all $c \in (0, 1/2)$, which implies that the limit cycle in system (20) given by the real oval of $v^2 + u(cu^2 - u + c) = 0$ is hyperbolic (and unstable). Hence, using Lemma 9, we have that the limit cycle given by the oval of the curve (13) in system (12) is hyperbolic and unstable.

3.3. Hyperbolicity of the limit cycle given by the algebraic curve (15) for system (14)

We consider the following birational change of variables $(x,y) \to (u,v)$ with $(x,y) = (X(u,v), Y(u,v))$ where

$$
(X(u,v), Y(u,v)) = \frac{(-2, -2u)}{v+1+u+2au^2}.
$$

The inverse of this change is $(u, v) = (U(x, y), V(x, y))$ with

$$U(x, y) = \frac{y}{x}, \quad V(x, y) = -2a\frac{y^2}{x^2} - \frac{(y+2)}{x} - 1.$$

This change of variables is well defined in a neighborhood of the real oval of the curve (15) and its Jacobian is

$$\frac{\partial U}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial V}{\partial x} = -\frac{2}{x^3}.$$

The algebraic curve (15) is transformed to $v^2 + 4au^2(u - 1) - (u + 1)^2 = 0$. We consider the transformed system in which we make a reparameterization of its time t which consists on multiplying by $v + 1 + u + 2au^2$. This reparameterization reverses the direction of the flow on the transformed limit cycle and the new system is written as

$$
\begin{aligned}
\dot{u} &= (u + 1)^2 - 4au^2(u - 1) + (1 - 3u)v, \\
\dot{v} &= 2(u + 1)(3 + u + 2au - au^2) + (1 + 4au + u - 6au^2)v - 5v^2.
\end{aligned}
\tag{22}
$$

The divergence of this system is $\operatorname{div}(u, v) = 3(1 + u) + 12au - 18au^2 - 13v$. The algebraic curve $v^2 + 4au^2(u-1) - (u+1)^2 = 0$ is invariant for system (22) with cofactor $k(u, v) = 2(1 + u + 4au - 6au^2 - 5v)$. The real oval of this curve is a hyperbolic limit cycle for system (22) if, and only if, the real oval of the curve (15) is a hyperbolic limit cycle for system (14), by Lemma 9. We are going to compute the function $\mathcal{D}(a) = \int_0^T \operatorname{div}(\gamma(t))\, dt$, where γ is the real oval of the curve $v^2 + 4au^2(u - 1) - (u + 1)^2 = 0$. To do so, we parameterize this oval by

$$u(\tau) = \tau, \qquad v_\pm(\tau) = \pm\sqrt{4a\tau^2(1 - \tau) + (\tau + 1)^2},$$

where τ takes values between τ_1 and τ_2. The values τ_1 and τ_2 are the two smallest roots of the polynomial $g(a, \tau) := 4a\tau^2(1 - \tau) + (\tau + 1)^2$ in τ. We consider a in the interval between $(17\sqrt{17} - 71)/32$ and 0, which are the values of the parameter for which the limit cycle exists. Since the coefficient of the highest order term of $g(a, \tau)$ is $-4a$ which is strictly negative, $g(a, -(3 + \sqrt{17})/2) = 4(29 + 7\sqrt{17})\left[a - (17\sqrt{17} - 71)/32\right] > 0$ and $g(a, -1) = 8a < 0$, we deduce that $\tau_1 < -(3 + \sqrt{17})/2 < \tau_2 < -1$. We denote by $P(u, v)$ the polynomial which defines $\dot{u} = P(u, v)$ in system (22). We consider the point with coordinates

$$(u_0, v_0) = \left(-\frac{3 + \sqrt{17}}{2}, 2\sqrt{29 + 7\sqrt{17}}\sqrt{a + \frac{71 - 17\sqrt{17}}{32}}\right)$$

and we have that $v_0^2 - g(a, u_0) = 0$ and $P(u_0, v_0) = v_0[v_0 + (11 + 3\sqrt{17})/2]$. We deduce that $P(u_0, v_0) > 0$ and that the flow on the limit cycle is clockwise. Using analogous arguments as in the previous examples we conclude that:

$$\mathcal{D}(a) = \int_{\tau_1}^{\tau_2} \left[\left(\frac{((1 + w)\,\mathrm{div} - wk)}{P(u, v)} \right) (\tau, v_+(\tau)) \right.$$
$$\left. - \left(\frac{((1 + w)\,\mathrm{div} - wk)}{P(u, v)} \right) (\tau, v_-(\tau)) \right] d\tau.$$

For $w = -3$ and substituting by the parameterization, we get

$$\mathcal{D}(a) = 2 \int_{\tau_1}^{\tau_2} \frac{\sqrt{g(a, \tau)}}{(\tau - 1)\,\tau\,(a\tau + 2)}\, d\tau.$$

We have that $\tau_1 < -(3 + \sqrt{17})/2 < \tau_2 < -1$ and that $\tau - 1 < 0$ and $\tau < 0 < -2/a$ for any $\tau \in (\tau_1, \tau_2)$ and for any $a \in ((17\sqrt{17} - 71)/32, 0)$. Hence, the integrand is strictly positive and well defined for any $\tau \in (\tau_1, \tau_2)$ and $a \in ((17\sqrt{17} - 71)/32, 0)$. We deduce that $\mathcal{D}(a) > 0$ for all $a \in ((17\sqrt{17} - 71)/32, 0)$, which implies that the limit cycle in system (22) given by the real oval of $v^2 - g(a, u) = 0$ is hyperbolic (and unstable). Thus, using Lemma 9 and the sign in the change of time, we have that the limit cycle given by the oval of the curve (15) in system (14) is hyperbolic and stable. $\square$

Acknowledgments

Appendix A

The aim of this appendix is to present some relations among elliptic integrals which the authors obtained by using the identity given by Theorem 2 for systems (12) and (16).

Before the presented proof of Theorem 10, the authors got its proof for systems (12) and (16) by computing the corresponding integrals which give place to elliptic integrals. The identity given in Theorem 2 was used to encounter a Fuchs equation for the function $\mathcal{D}(a)$. After some thorough analysis of this Fuchs equation, we deduce the non-vanishing of the function $\mathcal{D}(a)$ for any value of the parameter in which the limit cycle exists. We are not going to give this proof, but we think that the relations among elliptic integrals obtained by the former reasoning are interesting by themselves. Hence, we give the identities obtained which, as far as we know, do not appear in any

book of tables of integrals and relations between classical functions. On the other hand, we also give the obtention of the Fuchs equation for the function $\mathcal{D}(a)$ in the case of system (16).

A.1. Identities among elliptic integrals

The functions involved in this subsection are the complete elliptic integrals of first, second and third kinds, denoted by $K(\omega)$, $E(\omega)$ and $\Pi(\kappa, \omega)$, respectively. We recall the definition of these functions

$$K(\omega) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \omega \sin^2(\theta)}} = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - \omega t^2)}},$$

$$E(\omega) = \int_0^{\pi/2} \sqrt{1 - \omega \sin^2(\theta)}\, d\theta = \int_0^1 \frac{\sqrt{1 - \omega t^2}}{\sqrt{(1 - t^2)}}\, dt,$$

$$\Pi(\kappa, \omega) = \int_0^{\pi/2} \frac{d\theta}{(1 - \kappa \sin^2(\theta))\sqrt{1 - \omega \sin^2(\theta)}}$$

$$= \int_0^1 \frac{dt}{(1 - \kappa t^2)\sqrt{(1 - t^2)(1 - \omega t^2)}}$$

and their derivatives

$$K'(\omega) = \frac{1}{2(1 - \omega)\omega} E(\omega) - \frac{1}{2\omega} K(\omega),$$

$$E'(\omega) = \frac{1}{2\omega} E(\omega) - \frac{1}{2\omega} K(\omega),$$

$$\frac{\partial \Pi(\kappa, \omega)}{\partial \kappa} = \frac{1}{2\kappa(\kappa - 1)} K(\omega) + \frac{1}{2(\kappa - 1)(\omega - \kappa)} E(\omega)$$

$$+ \frac{\kappa^2 - \omega}{2\kappa(\kappa - 1)(\omega - \kappa)} \Pi(\kappa, \omega),$$

$$\frac{\partial \Pi(\kappa, \omega)}{\partial \omega} = \frac{1}{2(\kappa - \omega)(\omega - 1)} E(\omega) + \frac{1}{2(\kappa - \omega)} \Pi(\kappa, \omega).$$

We use the following parameterization of the oval of the algebraic curve (17) to explicitly compute the integrals for system (16). We parameterize the oval by

$$x(\tau) = \tau, \quad y_{\pm}(\tau) = \frac{-1 \pm 2\sqrt{(-a)\tau(\tau - \tau_1)(\tau - \tau_2)}}{2\tau}, \tag{23}$$

where $\tau_1 = \frac{1-\sqrt{1-4a}}{2a}$, $\tau_2 = \frac{1+\sqrt{1-4a}}{2a}$ and the parameter $\tau \in (\tau_1, \tau_2)$. The positive sign $y_{+}(\tau)$ gives a half of the oval and the negative sign $y_{-}(\tau)$ the other half. The explicit computation of the integrals for system (16) gives that identity (4) stated in Theorem 2 reads for

$$-9\,\mathrm{K}(\omega_0) + c_{+}\,\Pi(\omega_{+}, \omega_0) + c_{-}\,\Pi(\omega_{-}, \omega_0) \equiv 0, \tag{24}$$

which is valid for $a \in (0, 1/4)$, where

$$\omega_0 = \frac{2\sqrt{1-4a}}{1 + \sqrt{1-4a}}, \quad \omega_{\pm} = \frac{2\sqrt{1-4a}}{9 + \sqrt{1-4a} \pm 2\sqrt{16-a}},$$

$$c_{\pm} = \frac{9 - \sqrt{1-4a}}{2} \pm \sqrt{16-a}.$$

The derivative of the expression in (24) with respect to a gives place to the same identity (24). In fact, when computing the derivative with respect to a of the expression given in (24), using the described formulas of derivation for these elliptic integrals, we get $-1/(1 - 4a + \sqrt{1-4a})$ times the same expression (24). This simple factor is different from zero when $a \in (0, 1/4)$.

In the same way, we can explicitly compute the integrals involved in the identity (4) stated in Theorem 2 for system (12), via using the parameterization of the oval of (13) given by

$$x_{\pm}(\tau) = -\frac{2(1+2c)}{c^2(1+\tau)^3}(c - 2\tau - 2c\tau + c\tau^2 \pm 2\sqrt{1+2c}\sqrt{\tau(-c + \tau - c\tau^2)}),$$

$$y_{\pm}(\tau) = \frac{-12(1+2c)^2}{c^2(4+5c)(1+\tau)^4}(c - 2\tau - 2c\tau + c\tau^2 \pm 2\sqrt{1+2c}\sqrt{\tau(-c + \tau - c\tau^2)}), \tag{25}$$

where $\tau \in (\tau_1, \tau_2)$ with $\tau_1 = \frac{1-\sqrt{1-4c^2}}{2c}$ and $\tau_2 = \frac{1+\sqrt{1-4c^2}}{2c}$. It is clear that $0 < \tau_1 < 1 < \tau_2$ for $c \in (0, 1/2)$. We define $g(c, \tau) = -\tau(c - \tau + c\tau^2) = c\tau(\tau - \tau_1)(\tau_2 - \tau)$, which is strictly positive for all $\tau \in (\tau_1, \tau_2)$. The explicit computation of the integrals involved in the identity (4) gives

$$5\,\mathrm{K}(\varsigma_0) + C_{+}\,\Pi(\varsigma_{+}, \varsigma_0) + C_{-}\,\Pi(\varsigma_{-}, \varsigma_0) \equiv 0, \tag{26}$$

which is valid for $c \in (0, 1/2)$, where

$$\varsigma_0 = \frac{2\sqrt{1 - 4c^2}}{1 + \sqrt{1 - 4c^2}}, \quad \varsigma_\pm = \frac{2\sqrt{1 - 4c^2}}{9 + 17c + \sqrt{1 - 4c^2} \pm \sqrt{64(1 + 2c)^2 + c^2}},$$

$$C_\pm = \frac{-2(24 + 47c) \pm 3\sqrt{64(1 + 2c)^2 + c^2}}{9 + 17c + \sqrt{1 - 4c^2} \pm \sqrt{64(1 + 2c)^2 + c^2}}.$$

The derivative of expression (26) with respect to c gives place to the same identity (26).

The authors have not been able to give an analogous identity related to system (14) due to the fact that the corresponding integrals require much more computations to be identified with the elliptic integrals.

A.2. Fuchs equation for $\mathcal{D}(a)$ in system (16)

In this part of the appendix we develop the way we obtained a Fuchs equation for the function $\mathcal{D}(a)$ in system (16), via using relation (24). We think that the fact of obtaining a Fuchs equation satisfied by this function is interesting to further understand the stability of algebraic limit cycles for polynomial systems. We obtained a similar Fuchs equation for system (12), but we do not state it because the equation itself does not give any further information about the properties of system (12) and the way it was obtained is completely analogous to the way Eq. (27) for system (16) is obtained.

Let us consider system (16) and we parameterize the oval which contains the limit cycle by (23). Taking the notation described in the previous subsection:

$$\omega_0 = \frac{2\sqrt{1 - 4a}}{1 + \sqrt{1 - 4a}}, \quad \omega_\pm = \frac{2\sqrt{1 - 4a}}{9 + \sqrt{1 - 4a} \pm 2\sqrt{16 - a}},$$

$$c_\pm = \frac{9 - \sqrt{1 - 4a}}{2} \pm \sqrt{16 - a}$$

and

$$\mu = \sqrt{1 + \sqrt{1 - 4a}}, \quad b_\pm = 2(4 \pm \sqrt{16 - a}) c_\pm,$$

we explicitly compute the value of $\mathcal{D}(a)$

$$\mathcal{D}(a) = \int_0^T \mathrm{div}(\gamma(t)) \, dt$$

$$= \frac{\sqrt{2}}{\mu\sqrt{16 - a}} \left[-34\sqrt{16 - a} \, \mathrm{K}(\omega_0) + b_+ \, \Pi(\omega_+, \omega_0) - b_- \, \Pi(\omega_-, \omega_0) \right].$$

We compute the successive derivatives of $\mathcal{D}(a)$

$$\mathcal{D}'(a) = \frac{-4\sqrt{2}}{\mu\,(16-a)^{3/2}}\left[\sqrt{16-a}\,\mathrm{K}(\omega_0) + \frac{2\mu^2\sqrt{16-a}}{a}\,\mathrm{E}(\omega_0)\right.$$

$$\left. -\,c_+\,\Pi(\omega_+,\omega_0) + c_-\,\Pi(\omega_-,\omega_0)\right],$$

$$\mathcal{D}''(a) = \frac{6\sqrt{2}}{\mu\,(16-a)^{5/2}}\left[\frac{(10a^2+33a-64)\sqrt{16-a}}{3a(1-4a)}\,\mathrm{K}(\omega_0)\right.$$

$$+\frac{(73a^2-420a+128)\mu^2\sqrt{16-a}}{6a^2(1-4a)}\,\mathrm{E}(\omega_0)$$

$$\left. +\,c_+\,\Pi(\omega_+,\omega_0) - c_-\,\Pi(\omega_-,\omega_0)\right],$$

$$\mathcal{D}'''(a) = \frac{-\sqrt{2}}{\mu(16-a)^{7/2}}\left[\frac{(180a^4+1347a^3-9685a^2+25664a-4096)\sqrt{16-a}}{a^2(1-4a)^2}\,\mathrm{K}(\omega_0)\right.$$

$$+\frac{(1812a^4-20259a^3+102164a^2-60544a+8192)\mu^2\sqrt{16-a}}{2a^3(1-4a)^2}\,\mathrm{E}(\omega_0)$$

$$\left. -\,15c_+\,\Pi(\omega_+,\omega_0) + 15c_-\Pi(\omega_-,\omega_0)\right].$$

By elimination of independent functions and using identity (24) we obtain the following third-order homogeneous differential equation of Fuchs type for $\mathcal{D}(a)$:

$$8(a-16)a(4a-1)(17a+8)\,\mathcal{D}'''(a) + 4(612a^3-4119a^2-2600a$$

$$+512)\,\mathcal{D}''(a) + 6(a-2)(289a+528)\,\mathcal{D}'(a) + 3(17a+64)\,\mathcal{D}(a) = 0. \quad (27)$$

An easy computation shows that $\mathcal{D}(1/4) = 0$, $\mathcal{D}'(1/4) = -8\sqrt{2}\pi/9$ and $\mathcal{D}''(1/4) = 98\sqrt{2}\pi/27$. Hence, Eq. (27) univocally determines the function $\mathcal{D}(a)$ defined in $a \in (0, 1/4]$. A thorough analysis of the properties of $\mathcal{D}(a)$ gives that $\mathcal{D}(a) > 0$ for $a \in (0, 1/4)$.

  We remark that using identity (24) we get a Fuchs equation of order 3 for $\mathcal{D}(a)$. If we did not have this relation, we would get an equation of order 4, which would make the analysis of properties much more difficult. We notice that this Fuchs equation is an interesting alternative method to prove the hyperbolicity of the limit cycle in system (16). This kind of equation may exist for all algebraic limit cycle of a planar polynomial system and may let distinguish its hyperbolic character.

References

[2] A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.G. Maĭer, Theory of bifurcations of dynamic systems on a plane, Halsted Press (A division of John Wiley & Sons), New York, Toronto, Ont.; Israel Program for Scientific Translations, Jerusalem, London, 1973 (translated from the Russian).

[5] J. Chavarriga, A new example of quartic algebraic limit cycle for a quadratic system, preprint, Universitat de Lleida, 1998.

[8] J. Chavarriga, J. Llibre, J. Moulin-Ollagnier, On a result of Darboux, LMS J. Comput. Math. 4 (2001) 197–210 (electronic).

[9] J. Chavarriga, J. Llibre, J. Sorolla, Algebraic limit cycles of degree 4 for quadratic systems, J. Differential Equations 200 (2004) 206–244.

[10] Yuan-shün Ch'in, On algebraic limit cycles of degree 2 of the differential equation $\frac{dy}{dx} = \frac{\sum_{0 \le i+j \le 2} a_{ij} x^i y^j}{\sum_{0 \le i+j \le 2} b_{ij} x^i y^j}$, Sci. Sinica 7 (1958) 934–945, Acta Math. Sinica 8 (1958) 23–35.

[11] C. Christopher, J. Llibre, G. Świrszcz, Invariant algebraic curves of large degree for quadratic systems, J. Math. Anal. Appl. 303 (2005) 450–461.

[12] R.M. Evdokimenco, Construction of algebraic paths and the qualitative investigation in the large of the properties of integral curves of a system of differential equations, Differential Equations 6 (1970) 1349–1358.

[13] R.M. Evdokimenco, Behavior of integral curves of a dynamic system, Differential Equations 9 (1974) 1095–1103.

[14] R.M. Evdokimenco, Investigation in the large of a dynamic system with a given integral curve, Differential Equations 15 (1979) 215–221.

[15] V.F. Filiptsov, Algebraic limit cycles, Differential Equations 9 (1973) 983–988.

[16] L. Perko, Differential Equations and Dynamical Systems, third ed., Texts in Applied Mathematics, vol. 7, Springer, New York, 2001.

[17] A.I. Yablonskii, On limit cycles of certain differential equations, Differential Equations 2 (1966) 164–168.

[18] Ye Yan Qian, Cai Sui Lin, Chen Lan Sun, Huang Ke Cheng, Luo Ding Jun, Ma Zhi En, Wang Er Nian, Wang Ming Shu, Yang Xin An, Theory of Limit Cycles, second ed., Translations of Mathematical Monographs, vol. 66, American Mathematical Society, Providence, RI, 1986 (translated from the Chinese by Chi Y. Lo).