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Global behavior of spherically symmetric Navier–Stokes equations with density-dependent viscosity ☆

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Abstract

In this paper, we study a free boundary problem for compressible spherically symmetric Navier–Stokes equations without a solid core. Under certain assumptions imposed on the initial data, we obtain the global existence and uniqueness of the weak solution, give some uniform bounds (with respect to time) of the solution and show that it converges to a stationary one as time tends to infinity. Moreover, we obtain the stabilization rate estimates of exponential type in L^∞-norm and weighted H^1-norm of the solution by constructing some Lyapunov functionals. The results show that such system is stable under the small perturbations, and could be applied to the astrophysics.

Keywords: Compressible Navier–Stokes equations; Density-dependent viscosity; Free boundary; Existence; Uniqueness; Asymptotic behavior

1. Introduction

We consider the compressible Navier–Stokes equations with density-dependent viscosity in $\mathbb{R}^n$ $(n \geqslant 2)$, which can be written in Eulerian coordinates as

$$\begin{cases} \partial_\tau \rho + \nabla \cdot (\rho \vec{u}) = 0, \\ \partial_\tau (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla P = \operatorname{div}\big(\mu\big(\nabla \vec{u} + \nabla \vec{u}^\top\big)\big) + \nabla(\lambda \operatorname{div} \vec{u}) - \rho \vec{f}. \end{cases} \tag{1.1}$$

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Here ρ, P, $\vec{u}=(u_1,\dots,u_n)$ and $\vec{f}$ are the density, pressure, velocity and the external force, respectively; $\mu=\mu(\rho)$ and $\lambda=\lambda(\rho)$ are two viscosity coefficients.

In this paper, the initial conditions are

$$\rho(\vec{\xi},0)=\rho_0(r), \quad r\in[0,b], \tag{1.2}$$

$$\vec{u}(\vec{\xi},0)=u_0(r)\frac{\vec{\xi}}{r}, \quad r\in(0,b], \qquad \vec{u}(\vec{\xi},0)|_{\vec{\xi}=0}=u_0(0)=0, \tag{1.3}$$

where $r=|\vec{\xi}|=\sqrt{\xi_1^2+\cdots+\xi_n^2}$ and $b>0$ is a constant, the boundary condition is

$$\big\{(P-\lambda\operatorname{div}\vec{u})\mathrm{Id}-\mu\big(\nabla\vec{u}+\nabla\vec{u}^{\top}\big)\big\}\cdot\vec{n}=P_\Gamma\vec{n}, \quad \vec{\xi}\in\partial\Omega_\tau, \tag{1.4}$$

where $\partial\Omega_\tau=\psi(\partial\Omega_0,\tau)$ is a free boundary, $\vec{n}$ is the unit outward normal vector of $\partial\Omega_\tau$ and $P_\Gamma>0$ is a external pressure. Here $\partial\Omega_0=\{\vec{\xi}\in\mathbb{R}^n\colon |\vec{\xi}|=b\}$ is the initial boundary and ψ is the flow of $\vec{u}$:

$$\begin{cases} \partial_\tau\psi(\vec{\xi},\tau)=\vec{u}\big(\psi(\vec{\xi},\tau),\tau\big), \quad \vec{\xi}\in\mathbb{R}^n, \\ \psi(\vec{\xi},0)=\vec{\xi}. \end{cases} \tag{1.5}$$

To simplify the presentation, we only consider the famous polytropic model, i.e. $P(\rho)=A\rho^\gamma$ with $\gamma>1$ and $A>0$ being constants. And we assume that the viscosity coefficients μ and λ are proportional to ρ^θ, i.e. $\mu(\rho)=c_1\rho^\theta$ and $\lambda(\rho)=c_2\rho^\theta$ where c_1, c_2 and θ are three constants.

For the initial-boundary value problem (1.1)–(1.4), we are looking for a spherically symmetric solution $(\rho,\vec{u})$:

$$\rho(\vec{\xi},\tau)=\rho(r,\tau), \qquad \vec{u}(\vec{\xi},\tau)=u(r,\tau)\frac{\vec{\xi}}{r},$$

with the spherically symmetric external force

$$\vec{f}=f(m,r,\tau)\frac{\vec{\xi}}{r}, \qquad m(\rho,r)=\int_0^r\rho(s,\tau)s^{n-1}\,ds, \quad r>0,$$

and $\partial\Omega_\tau=\{\vec{\xi}\in\mathbb{R}^n\colon |\vec{\xi}|=b(\tau),\, b(0)=b,\, b'(\tau)=u(b(\tau),\tau)\}$.

Then $(\rho,u)(r,\tau)$ is determined by

$$\begin{cases} \partial_\tau\rho+\partial_r(\rho u)+\dfrac{n-1}{r}\rho u=0, \\ \rho(\partial_\tau u+u\partial_r u)+\partial_r P \\ \quad =(\lambda+2\mu)\left(\partial_{rr}^2 u+\dfrac{n-1}{r}\partial_r u-\dfrac{n-1}{r^2}u\right)2\partial_r\mu\partial_r u+\partial_r\lambda\left(\partial_r u+\dfrac{n-1}{r}u\right)-\rho f, \end{cases} \tag{1.6}$$

where $(r,\tau)\in(0,b(\tau))\times(0,\infty)$, with the initial data

$$(\rho,u)|_{\tau=0}=(\rho_0,u_0)(r), \quad 0\leqslant r\leqslant b, \tag{1.7}$$

the fixed boundary condition

$$u|_{r=0} = 0, \tag{1.8}$$

and the free boundary condition

$$\left\{P - 2\mu\partial_r u - \lambda\left(\partial_r u + \frac{n-1}{r}u\right)\right\}\bigg|_{r=b(\tau)} = P_\Gamma, \tag{1.9}$$

where $b(0) = b$, $b'(\tau) = u(b(\tau), \tau)$.

Additionally, we assume the external force $f(m, r, \tau)$ and external pressure $P_\Gamma(\tau) \in C^1(\mathbb{R}_+)$ satisfy

$$P_\Gamma(\tau) = P_\infty + \Delta P(\tau), \qquad f(m, r, \tau) = f_\infty(m, r) + \Delta f(m, r, \tau), \tag{1.10}$$

for all $r \geqslant 0$ and $\tau \geqslant 0$, with

$$f_\infty(m, r) = \frac{Gm}{r^{n-1}}, \qquad m(\rho, r) = \int_0^r \rho s^{n-1}\, ds, \quad \Delta f(m, r, \tau) \in C^1(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+), \tag{1.11}$$

$$\left\|\Delta f(\cdot, \cdot, \tau)\right\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}_+)} \leqslant f_1(\tau), \qquad \left\|(\partial_r \Delta f, \partial_\tau \Delta f)(\cdot, \cdot, \tau)\right\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}_+)} \leqslant f_2(\tau), \tag{1.12}$$

$$f_1, \Delta P \in L^\infty \cap L^1(\mathbb{R}_+), \quad (\Delta P)', f_2 \in L^2(\mathbb{R}_+), \tag{1.13}$$

where $\mathbb{R}_+ = [0, \infty)$, P_∞ and G are two positive constants, the perturbations $(\Delta P, \Delta f)$ tend to 0 as $\tau \to \infty$ in some weak sense. f_∞ is the precise expression for its own gravitational force and Δf expresses the influence of the outside gravitational force, in the astrophysical case (with spherical symmetry). P_Γ also could express the influence of the surface tension force on the free boundary. This system can be treated as a simple model of one fluid in Ω_τ, whose evolution is influenced by the gravitational force and the external pressure generated by the other substance in $\mathbb{R}^n \setminus \Omega_\tau$. We study the stabilization problem of such system, which could be applied to the astrophysics.

Now, we consider the stationary problem, namely

$$\big(P(\rho_\infty)\big)_r = -\rho_\infty f_\infty\big(m(\rho_\infty, r), r\big) \tag{1.14}$$

in an interval $r \in (0, l_\infty)$ with the end l_∞ satisfying

$$P\big(\rho_\infty(l_\infty)\big) = P_\infty, \tag{1.15}$$

$$\int_0^{l_\infty} \rho_\infty r^{n-1}\, dr = M := \int_0^b \rho_0 r^{n-1}\, dr. \tag{1.16}$$

The unknown quantities are the stationary density $\rho_\infty \geqslant 0$ and free boundary $l_\infty > 0$. If

$$\gamma = \frac{2n-2}{n} \quad \text{and} \quad Gn^{\frac{2-n}{n}} M^{\frac{2}{n}} < 2A \tag{1.17}$$

or

$$\gamma > \frac{2n-2}{n}, \tag{1.18}$$

from Proposition 2.5, we know that there exists a unique solution (ρ_∞, l_∞) to the stationary system (1.14)–(1.16), satisfying $0 < \underline{\rho} \leqslant \rho_\infty(r) \leqslant \overline{\rho} < \infty$, $(\rho_\infty)_r(r) < 0$, $0 < r < l_\infty$ with $l_\infty < +\infty$.

To handle the free boundary problem (1.6)–(1.9), it is convenient to reduce the problem in Eulerian coordinates (r, τ) to the problem in Lagrangian coordinates (x, t), via the transformation:

$$x = \int_0^r y^{n-1}\rho(y,\tau)\,dy, \quad t = \tau. \tag{1.19}$$

Then the fixed boundary $r = 0$ and the free boundary $r = b(\tau)$ become

$$x = 0 \quad \text{and} \quad x = \int_0^{b(\tau)} y^{n-1}\rho(y,\tau)\,dy = \int_0^b y^{n-1}\rho_0(y)\,dy = M,$$

where M is the total mass initially. So that the region $\{(r,\tau)\colon\ 0 \leqslant r \leqslant b(\tau),\ \tau \geqslant 0\}$ under consideration is transformed into the region $\{(x,t)\colon\ 0 \leqslant x \leqslant M,\ t \geqslant 0\}$.

Under the coordinate transformation (1.19), Eqs. (1.6)–(1.9) are transformed into

$$\begin{cases} \partial_t \rho(x,t) = -\rho^2 \partial_x\big(r^{n-1}u\big), \\ \partial_t u(x,t) = r^{n-1}\Big\{\partial_x\big[\rho(\lambda+2\mu)\partial_x\big(r^{n-1}u\big) - P\big] - 2(n-1)\dfrac{u}{r}\partial_x\mu\Big\} - f(x,r,t), \\ r^n(x,t) = n\displaystyle\int_0^x \rho^{-1}(y,t)\,dy, \end{cases} \tag{1.20}$$

where $(x,t) \in (0, M) \times (0, \infty)$, with the initial data

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x), \qquad r|_{t=0} = r_0(x) = \left(n\int_0^x \rho_0^{-1}(y)\,dy\right)^{\frac{1}{n}} \tag{1.21}$$

and the boundary conditions

$$u(0,t) = 0, \tag{1.22}$$

$$\Big\{P - \rho(\lambda+2\mu)\partial_x\big(r^{n-1}u\big) + 2(n-1)\mu\frac{u}{r}\Big\}\Big|_{x=M} = P_\Gamma, \quad t > 0. \tag{1.23}$$

It is standard that if we can solve the problem (1.20)–(1.23), then the free boundary problem (1.1)–(1.4) has a solution.

From (1.14)–(1.16), it is easy to see that $\rho_\infty(x)$ is the solution to the stationary system

$$Ar_\infty^{n-1}\left(\rho_\infty^\gamma\right)_x = -f_\infty(x, r_\infty), \qquad r_\infty^n(x) = n\int_0^x \rho_\infty^{-1}(y)\,dy, \quad x \in (0, M), \tag{1.24}$$

$$\rho_\infty(M) = \left(\frac{P_\infty}{A}\right)^{\frac{1}{\gamma}}. \tag{1.25}$$

The results in [6,18] show that the compressible Navier–Stokes system with the constant viscosity coefficient have the singularity at the vacuum. Considering the modified Navier–Stokes system in which the viscosity coefficient depends on the density, Liu, Xin and Yang in [9] proved that such system is local well-posedness. It is motivated by the physical consideration that in the derivation of the Navier–Stokes equations from the Boltzmann equation through the Chapman–Enskog expansion to the second order, cf. [4], the viscosity coefficient is a function of the temperature. If we consider the case of isentropic fluids, this dependence is reduced to the dependence on the density function.

Since $n \geqslant 2$ and the viscosity coefficient μ depends on ρ, the nonlinear term $2(n-1)\frac{1}{r}u\partial_x\mu$ in $(1.20)_2$ makes the analysis significantly different from the one-dimensional case [9,14,17,19,20]. Considering the compressible spherically symmetric Navier–Stokes equations without a solid core, the techniques in the case of similar system with a solid core [1,2,11,13,21] failed to be of use in our case, so we need obtain some new *a priori* estimates.

For spherically symmetric solutions of the Navier–Stokes equations with constant viscosity, in [7], the author gave an information near the origin that the solution may develop vacuum region about the origin. The difficulty of this problem is to obtain the lower bound of the density ρ and the upper bound of the term $\frac{1}{r}u$. When the initial data are small in some sense, using some new *a priori* estimates on the solution, we can obtain the lower bound of the density and the upper bound of the term $\frac{1}{r}u$. The key ideas are using the classical continuity method and the result of Claim 1. In Claim 1, we want to prove that there is a small positive constant ϵ_1, such that, for any $T > 0$, if

$$I(t) = \left\|\rho(\cdot, t) - \rho_\infty\right\|_{L^\infty} + \left\|\frac{u}{r}(\cdot, t)\right\|_{L^\infty} \leqslant 2\epsilon_1, \quad \forall t \in [0, T],$$

then

$$I(t) \leqslant \epsilon_1, \quad \forall t \in [0, T].$$

Let

$$B[\rho, u, r] = \int_0^M \bigg[(\rho - \rho_\infty)^2 + r^{2n-2+\alpha}(\rho - \rho_\infty)_x^2 + \frac{u^2}{r^2} + r^{2n-2}u_x^2 + r^{2n-2+\alpha}\left(\rho^{1+\theta}\left(r^{n-1}u\right)_x\right)_x^2\bigg]\,dx,$$

where $\alpha = \frac{3}{2} - n$. In Lemmas 3.3–3.8, we get some uniform *a priori* estimates (with respect to time) on the solution in the weighted Sobolev space and the upper bound of $B[\rho, u, r]$. Using the bound of $B[\rho, u, r]$ and Sobolev's embedding theorem, we can finish the proof of Claim 1. Then, we will construct a weak solution by using the finite difference approximation. Our results show that: such system does not develop vacuum states or concentration states for all time, and the interface $\partial\Omega_\tau$ propagates with finite speed. Since these estimates of the solution are uniform in time, we could show that the solution converges to a stationary one as time tends to infinity. Moreover, we construct various Lyapunov functionals and obtained the stabilization rate estimates of exponential type.

We now briefly review the previous works in this direction. For the related free boundary problem of one-dimensional isentropic fluids with density-dependent viscosity (like $\mu(\rho) = c\rho^\theta$), see [9,14,17,19,20] and the references therein. For the spherically symmetric solutions of the Navier–Stokes equations with a free boundary, see [1,2,11,13,21], etc. Ducomet [2], Zlotnik [21] studied the similar system with a solid core and without the nonlinear term $2(n-1)\frac{1}{r}u\partial_x\mu$. Also see Lions [8] and Vaigant and Kazhikhov [16] for multidimensional isentropic fluids. For the related stabilization rate estimates in the one-dimensional case, see [3,10,12,15,20], etc.

Main assumptions on c_1, c_2, θ and γ can be stated as follows:

(A1) condition (1.17) or (1.18) holds;
(A2) $\theta \geqslant 0$. c_1 and c_2 satisfy that

$$c_1 > 0, \qquad 2c_1 + nc_2 > 0$$

and

$$\bigl[2c_1\alpha + c_2(2n-2+\alpha)\bigr]^2 - 4(2c_1+c_2)\bigl[2c_1(n-1) + c_2(n-1)(n-1+\alpha)\bigr] < 0, \tag{1.26}$$

where $\alpha = \frac{3}{2} - n$.

Under the above assumptions (A1), (A2), we will prove the existence of a global weak solution to the initial–boundary value problem (1.20)–(1.23) in the sense of the following definition.

Definition 1.1. A pair of functions $(\rho, u, r)(x, t)$ is called a global weak solution to the initial–boundary value problem (1.20)–(1.23), if for any $T > 0$,

$$\begin{gathered} \rho, u \in L^\infty\bigl([0, M] \times [0, T]\bigr) \cap C^1\bigl([0, T]; L^2\bigl([0, M]\bigr)\bigr), \\ r \in C^1\bigl([0, T]; L^\infty\bigl([0, M]\bigr)\bigr), \\ \bigl(r^{n-2}u\bigr)_x, \bigl(r^{n-1}\bigr)_x \in L^\infty\bigl([0, T]; L^{n-\frac{1}{2}}\bigl([0, M]\bigr)\bigr), \end{gathered}$$

and

$$\bigl(r^{n-1}u\bigr)_x \in L^\infty\bigl([0, M] \times [0, T]\bigr) \cap C^{\frac{1}{2}}\bigl([0, T]; L^2\bigl([0, M]\bigr)\bigr).$$

Furthermore, the following equations hold:

$$\rho_t + \rho^2\big(r^{n-1}u\big)_x = 0, \qquad \rho(x,0) = \rho_0(x) \quad \text{a.e.}$$

$$r_t = u, \qquad r^n(x,t) = n\int_0^x \rho^{-1}(y,t)\,dy, \qquad r(x,0) = r_0(x) \quad \text{a.e.}$$

and

$$\begin{aligned}
&\int_0^\infty\int_0^M \big[u\psi_t + \big(P - \rho(\lambda+2\mu)\big(r^{n-1}u\big)_x\big)\big(r^{n-1}\psi\big)_x \\
&\qquad + 2(n-1)\mu\big(r^{n-2}u\psi\big)_x - f(x,r,t)\psi\big]\,dx\,dt \\
&\quad = \int_0^\infty P_\Gamma\big(r^{n-1}\psi\big)(M,t)\,dt - \int_0^M u_0(x)\psi(x,0)\,dx,
\end{aligned}$$

for any test function $\psi(x,t) \in C_0^\infty(\Omega)$ with $\Omega = \{(x,t)\colon\ 0 < x \leqslant M,\ t \geqslant 0\}$.

In what follows, we always use $C(C_i)$ to denote a generic positive constant depending only on the initial data, independent of the given time T.

We now state the main theorems in this paper. Let $\underline{\rho} = \min_{x\in[0,M]}\rho_\infty$ and $\overline{\rho} = \max_{x\in[0,M]}\rho_\infty$.

Theorem 1.1. *Under the conditions* (1.10)–(1.12) *and* (A1), (A2)*, there exists a positive constant* $\epsilon_0 > 0$*, such that if*

$$\big\|(f_1, \Delta P)\big\|_{L^\infty\cap L^1} + \big\|(\Delta P)'\big\|_{L^2} + \|f_2\|_{L^2} \leqslant \epsilon_0, \tag{1.27}$$

$$\|\rho_0 - \rho_\infty\|_{L^\infty}^2 + B[\rho_0, u_0, r_0] \leqslant \epsilon_0^2, \tag{1.28}$$

then the system (1.20)–(1.23) *has a unique global weak solution* (ρ, u, r) *satisfying*

$$\rho(x,t) \in \left[\frac{1}{2}\underline{\rho}, \frac{3}{2}\overline{\rho}\right], \qquad r^n(x,t) \in \big[C^{-1}x, Cx\big], \tag{1.29}$$

$$\left\|\frac{u}{r}(\cdot,t)\right\|_{L^\infty} \leqslant C\big\|\partial_x\big(r^{n-1}u\big)(\cdot,t)\big\|_{L^\infty} \leqslant C\epsilon_0, \tag{1.30}$$

$$B[\rho, u, r] \leqslant C\epsilon_0^2, \tag{1.31}$$

for all $t \geqslant 0$ *and* $x \in [0, M]$*. Furthermore, we have*

$$\lim_{t\to+\infty}\int_0^M \big\{x^{\frac{2n-2+\alpha}{n}}u_x^2 + x^{\frac{2n-2+\alpha}{n}}\big[\big(\rho^\theta\big)_x - \big(\rho_\infty^\theta\big)_x\big]^2\big\}\,dx = 0,$$

$$\lim_{t\to+\infty}\big\|u(\cdot,t)\big\|_{L^\infty} + \big\|\rho(\cdot,t) - \rho_\infty(\cdot)\big\|_{L^\infty} + \big\|r(\cdot,t) - r_\infty(\cdot)\big\|_{L^\infty} = 0.$$

Remark 1.1. In fact, assumption (1.26) gives a restriction on $\frac{\lambda}{\mu}$, i.e.

$$\begin{aligned}&\frac{-18+8n+8n^2-8\sqrt{3}(n-1)\sqrt{4n-3}}{9-12n+4n^2}\\&\quad<\frac{\lambda}{\mu}<\frac{-18+8n+8n^2+8\sqrt{3}(n-1)\sqrt{4n-3}}{9-12n+4n^2}.\end{aligned}$$

If $n=3$, we can choose $\frac{\lambda}{\mu}=\frac{c_2}{c_1}\in(\frac{2}{3}(13-8\sqrt{3}),\frac{2}{3}(13+8\sqrt{3}))$.

Remark 1.2. We can choose the constant ϵ_0 as in (3.75).

The proof of the uniqueness part of Theorem 1.1 also shows that the continuous dependence of the solution on the initial data holds. We may state the following result without a proof.

Theorem 1.2. *For each $i=1,2$, let (ρ_i,u_i,r_i) be the solution to the system* (1.20)–(1.23) *with the initial data $(\rho_{0i},u_{0i},r_{0i})$, which satisfy regularity conditions* (1.29)–(1.31). *Then we have*

$$\begin{aligned}&\int_0^M\big[(u_1-u_2)^2+(\rho_1-\rho_2)^2+x^{-\frac{2}{n}}(r_1-r_2)^2\big](x,t)\,dx\\&\quad\leqslant Ce^{Ct}\int_0^M\big[(u_{01}-u_{02})^2+(\rho_{01}-\rho_{02})^2+x^{-\frac{2}{n}}(r_{01}-r_{02})^2\big]\,dx,\end{aligned}$$

for all $t\geqslant 0$.

Theorem 1.3. *Under the assumptions of Theorem* 1.1 *and*

$$f_1(t)+f_2(t)+\big|\Delta P(t)\big|+\big|(\Delta P)'(t)\big|\leqslant Ce^{-a_0t},\tag{1.32}$$

where a_0 is a positive constant, then we have

$$\int_0^M\big\{r^{2n-2+\alpha}(\rho-\rho_\infty)_x^2+r^{2n-2+\alpha}\big[\partial_x\big(\rho^{1+\theta}\partial_x\big(r^{n-1}u\big)\big)\big]^2+r^\alpha u_t^2\big\}\,dx\leqslant Ce^{-at},$$

$$\left\|\left(\frac{u}{r},\big(r^{n-1}u\big)_x\right)(\cdot,t)\right\|_{L^\infty}+\big\|\rho(\cdot,t)-\rho_\infty(\cdot)\big\|_{L^\infty}+\big\|r(\cdot,t)-r_\infty(\cdot)\big\|_{L^\infty}\leqslant Ce^{-at},$$

for all $t\geqslant 0$, where a is a positive constant.

Remark 1.3. Considering the general case that $(\mu,\lambda)(\rho)\in C(\mathbb{R}_+)\cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+)$, under the conditions (1.10)–(1.12), (A1) and

$$\begin{gathered}\mu(\rho)>0,\qquad 2\mu(\rho)+n\lambda(\rho)>0,\\ \big[2\mu\alpha+\lambda(2n-2+\alpha)\big]^2-4(2\mu+\lambda)\big[2\mu(n-1)+\lambda(n-1)(n-1+\alpha)\big]<0,\end{gathered}$$

for all $\rho\in[\frac{1}{2}\underline{\rho},\frac{3}{2}\overline{\rho}]$, we can obtain the same results.

Remark 1.4. In this paper, we study the case of $\gamma > 1$ and prove the main results in this case only, since the case of $\gamma = 1$ can be discussed through the similar process. The main difference is that (2.10) is replaced by

$$S[V] = \int_0^M \left(A \ln V_x + P_\infty V_x + \int_1^V Gx(nh)^{\frac{2-2n}{n}}\, dh \right) dx,$$

when $\gamma = 1$ and $n = 2$.

The rest of this paper is organized as follows. First, we obtain the existence and uniqueness of the solution to the stationary problem in Section 2. In Section 3, we will prove some *a priori* estimates which will be used to obtain global existence of the weak solutions. In Section 4, using the finite difference approximation and *a priori* estimates obtained in Section 3, we prove the existence part of Theorem 1.1. In Section 5, we will prove the uniqueness of the weak solution. In Section 6, we show that the solution of the free boundary problem tends to a stationary one, as $t \to +\infty$. In Section 7, we will obtain the stabilization rate estimates of exponential type on the solution by constructing some Lyapunov functionals.

2. The stationary problem

We start with a proof of the existence of a positive solution to the Lagrangian stationary problem. Zlotnik and Ducomet [21] studied the stationary problem with a solid core $r \geqslant r_0 > 0$. Using similar arguments in [21], we can obtain the following results for the stationary problem without a solid core.

Proposition 2.1. *If*

$$\gamma > \frac{2n-2}{n} \tag{2.1}$$

or

$$\gamma = \frac{2n-2}{n} \quad \textit{and} \quad Gn^{\frac{2-n}{n}} M^{\frac{2}{n}} < 2A, \tag{2.2}$$

or

$$0 < \gamma < \frac{2n-2}{n} \quad \textit{and} \quad P_\infty + \frac{G}{2} n^{\frac{2-n}{n}} M^{\frac{2}{n}} \delta_3^{\frac{2n-2}{n}} \leqslant A\delta_3^\gamma, \tag{2.3}$$

where

$$\delta_3 = \left(\frac{A\gamma n^{\frac{2n-2}{n}}}{(n-1)GM^{\frac{2}{n}}} \right)^{\frac{n}{2n-2-n\gamma}},$$

then the Lagrangian stationary problem (1.24)–(1.25) *has a positive solution* $\rho_\infty \in W^{1,\beta}([0, M])$, *where* $\beta \in [1, \frac{n}{n-2})$ *is a constant.*

Proof. We introduce the nonlinear operator

$$I : K \to W^{1,\beta}\big([0,M]\big),$$

where $K = \{f \in C([0,M]):\ \min_{x\in[0,M]} f(x) \geqslant (\frac{P_\infty}{A})^{\frac{1}{\gamma}}\}$, by setting

$$I(f)(x) = \left(\frac{P_\infty + \int_x^M G \frac{y}{r_f^{2n-2}(y)}\, dy}{A}\right)^{\frac{1}{\gamma}}$$

with $r_f^n(x) = n\int_0^x f^{-1}(y)\,dy$, $x \in [0,M]$. We can restate the problem (1.24)–(1.25) as the fixed-point problem

$$\rho_\infty = I(\rho_\infty). \tag{2.4}$$

For all $f \in K_\delta = \{f \in K:\ f \leqslant \delta\}$ with $\delta > (\frac{P_\infty}{A})^{\frac{1}{\gamma}}$, we have

$$nx\delta^{-1} \leqslant r_f^n(x)$$

and

$$\begin{aligned} P_\infty \leqslant A\big(I(f)\big)^\gamma &\leqslant P_\infty + G\delta^{\frac{2n-2}{n}} n^{-\frac{2n-2}{n}} \int_0^M x^{\frac{2-n}{n}}\, dx \\ &= P_\infty + \frac{G}{2}\delta^{\frac{2n-2}{n}} n^{\frac{2-n}{n}} M^{\frac{2}{n}}. \end{aligned}$$

If $\gamma > \frac{2n-2}{n}$, then $I(K_{\delta_1}) \subset K_{\delta_1}$, where δ_1 is a positive constant satisfying $P_\infty + \frac{G}{2}\delta_1^{\frac{2n-2}{n}} \times n^{\frac{2-n}{n}} M^{\frac{2}{n}} \leqslant A\delta_1^\gamma$. And one can immediately verify that I is a compact operator on K_{δ_1}. Since K_{δ_1} is a convex closed bounded nonempty subset of $C([0,M])$, the problem (2.4) has a solution $\rho \in K_{\delta_1}$ by Schauder's fixed point theorem.

If $\gamma = \frac{2n-2}{n}$ and $Gn^{\frac{2-n}{n}} M^{\frac{2}{n}} < 2A$, then $I(K_{\delta_2}) \subset K_{\delta_2}$, where δ_2 is a positive constant satisfying $P_\infty + \frac{G}{2}\delta_2^{\frac{2n-2}{n}} n^{\frac{2-n}{n}} M^{\frac{2}{n}} \leqslant A\delta_2^\gamma$.

If $\gamma < \frac{2n-2}{n}$ and

$$P_\infty + \frac{G}{2} n^{\frac{2-n}{n}} M^{\frac{2}{n}} \left(\frac{A\gamma n^{\frac{2n-2}{n}}}{(n-1)GM^{\frac{2}{n}}}\right)^{\frac{2n-2}{2n-2-n\gamma}} \leqslant A\left(\frac{A\gamma n^{\frac{2n-2}{n}}}{(n-1)GM^{\frac{2}{n}}}\right)^{\frac{n\gamma}{2n-2-n\gamma}}$$

then $I(K_{\delta_3}) \subset K_{\delta_3}$, where

$$\delta_3 = \left(\frac{A\gamma n^{\frac{2n-2}{n}}}{(n-1)GM^{\frac{2}{n}}}\right)^{\frac{n}{2n-2-n\gamma}}.$$

We can finish the proof of the theorem immediately. □

Letting $V_\infty = \frac{r_\infty^n}{n}$, using the equality $\frac{1}{\rho_\infty} = (V_\infty)_x$, one can eliminate the function ρ_∞ from the Lagrangian stationary problem (1.24)–(1.25) and obtain an equivalent boundary value problem for a nonlinear second-order ODE:

$$\big(A(V_\infty)_x^{-\gamma}\big)_x = -Gxn^{\frac{2-2n}{n}} V_\infty^{\frac{2-2n}{n}}, \quad x \in (0, M), \tag{2.5}$$

$$V_\infty(0) = 0, \qquad (V_\infty)_x(M) = \left(\frac{A}{P_\infty}\right)^{\frac{1}{\gamma}}, \tag{2.6}$$

for a function $V_\infty \in C^1([0, M])$ such that $(V_\infty)_x > 0$.

In accordance with the method of small perturbations, we replace V_∞ by $V = V_\infty + W$ with small W and linearize the operator in the last problem:

$$\begin{aligned}
&\big(A(V)_x^{-\gamma}\big)_x + Gxn^{\frac{2-2n}{n}} V^{\frac{2-2n}{n}} \\
&\quad = \big(-\gamma A(V_\infty)_x^{-\gamma-1} W_x\big)_x + (2-2n)Gx(nV_\infty)^{\frac{2-3n}{n}} W + \cdots, \quad x \in (0, M),
\end{aligned}$$

$$V(0) = 0 + W(0), \qquad A(V_x)^{-\gamma}\big|_{x=M} - P_\infty = -\gamma A\big\{(V_\infty)_x^{-\gamma-1} W_x\big\}\big|_{x=M} + \cdots,$$

up to the terms of the second order of smallness with respect to W. We define the linearized operator

$$L[W] = \big(-\gamma A\rho_\infty^{\gamma+1} W_x\big)_x + (2-2n)Gx(nV_\infty)^{\frac{2-3n}{n}} W, \quad W \in K_0, \tag{2.7}$$

where $K_0 = \{W \in C^1([0, M])\colon\ W(0) = 0,\ W_x(M) = 0\}$. It is easy to get

$$\big(L[W], W\big) = \int_0^M \big(\gamma A(\rho_\infty)^{1+\gamma} W_x^2 - (2n-2)Gx(nV_\infty)^{\frac{2-3n}{n}} W^2\big)\, dx, \quad W \in K_0.$$

Let

$$J[W] := \int_0^M \big(\gamma A(\rho_\infty)^{1+\gamma} W_x^2 - (2n-2)Gx(nV_\infty)^{\frac{2-3n}{n}} W^2\big)\, dx, \tag{2.8}$$

for $W \in K_1 = \{f \in C^1([0, M])\colon\ f(0) = 0\}$.

We say a stationary solution V_∞ is *statically stable* if

$$J[W] \geqslant \delta_3\big(\|W_x(x)\|^2_{L^2(0,M)} + \|x^{-1}W(x)\|^2_{L^2(0,M)}\big), \tag{2.9}$$

for some $\delta_3 > 0$ and all $W \in K_1$.

Now, the static potential energy takes the following form:

$$S[V] = \int_0^M \left(\frac{A}{\gamma-1}(V_x)^{1-\gamma} + P_\infty V_x + \int_1^V Gx(nh)^{\frac{2-2n}{n}}\, dh\right) dx. \tag{2.10}$$

We call $V \in K_2 = \{f \in C^1([0,M]):\ f(0)=0,\ \min(f_x) > 0\}$ is a point of *local quadratic minimum of S* if

$$S[V+W] - S[V] \geqslant \delta_4\big(\|W_x(x)\|^2_{L^2(0,M)} + \|x^{-1}W(x)\|^2_{L^2(0,M)}\big), \tag{2.11}$$

for all $W \in K_1$ and $\|W\|_{C^1([0,M])} \leqslant \delta_5$, for some $\delta_4 > 0$ and $\delta_5 > 0$.

We can clarify the variational sense of the definition of *statically stable* as follows.

Proposition 2.2. *A function $V \in K_2$ is a point of local quadratic minimum of S if and only if $V = V_\infty$ is a solution of the problem* (2.5)–(2.6) *and satisfies static stability condition* (2.9).

Proof. Let $V \in K_2$, $W \in K_1$ and $\|W\|_{C^1([0,M])} = 1$. Using Taylor's formula, we have

$$S[V+\epsilon W] = S[V] + \delta S[V](\epsilon W) + \frac{1}{2}\frac{d^2}{d\tau^2}S[V+\tau\epsilon W]\bigg|_{\tau=\tilde{\tau}},$$

where

$$\delta S[V](\epsilon W) = \int_0^M \big(-A(V_x)^{-\gamma}\epsilon W_x + P_\infty \epsilon W_x + Gx(nV)^{\frac{2-2n}{n}}\epsilon W\big)\,dx$$

and

$$\frac{d^2}{d\tau^2}S[V+\tau\epsilon W] = \int_0^M \big(\gamma A(V_x+\tau\epsilon W_x)^{-1-\gamma}(\epsilon W_x)^2 - (2n-2)Gx\big(n(V+\tau\epsilon W)\big)^{\frac{2-3n}{n}}(\epsilon W)^2\big)\,dx,$$

for all $|\epsilon| < \frac{1}{\min V_x}$ and some $\tilde{\tau} \in [0,1]$. If (2.11) holds, we have

$$\frac{d^2}{d\tau^2}S[V+\tau\epsilon W] \leqslant C\epsilon^2\big(\|W_x(x)\|^2_{L^2(0,M)} + \|x^{-1}W(x)\|^2_{L^2(0,M)}\big)$$

and

$$C\epsilon^2\big(\|W_x(x)\|^2_{L^2(0,M)} + \|x^{-1}W(x)\|^2_{L^2(0,M)}\big) + \epsilon\delta S[V](W) > 0,$$

for all $|\epsilon| \in (0, \min(\delta_5, \frac{1}{\min V_x}))$ and $\|W\|_{C^1([0,M])} = 1$. Thus, we obtain

$$\delta S[V](W) = 0,$$

i.e.

$$\int_0^M \big(-A(V_x)^{-\gamma}W_x + P_\infty W_x + Gx(nV)^{\frac{2-2n}{n}}W\big)\,dx = 0,$$

for all $W \in K_1$ and $\|W\|_{C^1[0,M]} = 1$, that is, V is a stationary point of S and a solution of the problem (2.5)–(2.6). We can rewrite $\frac{d^2}{d\tau^2} S[V + \tau\epsilon W]$ as follows

$$\frac{d^2}{d\tau^2} S[V + \tau\epsilon W] = \delta^2 S[V](\epsilon W) + S_1,$$

where $\delta^2 S[V](\epsilon W) = \frac{d^2}{d\tau^2} S[V + \tau\epsilon W]|_{\tau=0}$ and

$$\begin{aligned}|S_1| &= \left|\frac{d^2}{d\tau^2} S[V + \tau\epsilon W] - \delta^2 S[V](\epsilon W)\right| \\ &\leqslant C\epsilon\big(\|\epsilon W_x(x)\|^2_{L^2([0,M])} + \|x^{-1}\epsilon W(x)\|^2_{L^2([0,M])}\big).\end{aligned}$$

Thus, we obtain

$$\delta^2 S[V](\epsilon W) \geqslant (\delta_4 - C\epsilon)\big(\|\epsilon W_x(x)\|^2_{L^2([0,M])} + \|x^{-1}\epsilon W(x)\|^2_{L^2([0,M])}\big),$$

for all $\epsilon \in (0, \min(\delta_5, \frac{1}{\min V_x}, \frac{\delta_4}{2C}))$ and $\|W\|_{C^1([0,M])} = 1$. Moreover, we have

$$J[W] := \delta^2 S[V](W) \geqslant \frac{\delta_4}{2}\big(\|W_x(x)\|^2_{L^2([0,M])} + \|x^{-1} W(x)\|^2_{L^2([0,M])}\big), \tag{2.12}$$

for all $W \in K_1$.

If $V = V_\infty$ is a solution of the problem (2.5)–(2.6) and satisfies static stability condition (2.9), we can prove V_∞ is a point of local quadratic minimum of P easily. □

Proposition 2.3. *If $V = V_\infty$ is a solution of the problem* (2.5)–(2.6) *and $\gamma \geqslant \frac{2n-2}{n}$, then* (2.9) *and* (2.11) *hold.*

Proof. From $(A\rho_\infty^\gamma)_x = -G\frac{x}{r_\infty^{2n-2}} = -Gx(nV_\infty)^{\frac{2-2n}{n}}$, using integration by parts, we have

$$\begin{aligned}
J[W] &= \int_0^M \big(\gamma A(\rho_\infty)^{1+\gamma} W_x^2 - (2n-2)Gx(nV_\infty)^{\frac{2-3n}{n}} W^2\big)\,dx \\
&= \int_0^M \big(\gamma A(\rho_\infty)^{1+\gamma} W_x^2 + (2n-2)A\big(\rho_\infty^\gamma\big)_x (nV_\infty)^{-1} W^2\big)\,dx \\
&= \int_0^M \left(\gamma A(\rho_\infty)^{1+\gamma} W_x^2 - 2(2n-2)A\rho_\infty^\gamma (nV_\infty)^{-1} W W_x + \frac{2n-2}{n} A\rho_\infty^{\gamma-1} V_\infty^{-2} W^2\right) dx \\
&\quad + (2n-2)P_\infty\left(\frac{W^2}{nV_\infty}\right)(M) \\
&:= I_0[W] + (2n-2)P_\infty\left(\frac{W^2}{nV_\infty}\right)(M), \quad \text{for all } W \in K_0.
\end{aligned} \tag{2.13}$$

If $\gamma \geqslant \frac{2n-2}{n}$, we have

$$I_0[W] \geqslant \int_0^M \frac{2n-2}{n} A\rho_\infty^{1+\gamma} \left(W_x - \frac{W}{\rho_\infty V_\infty}\right)^2 dx. \tag{2.14}$$

If (2.9) not holds, we have for any integer $m > 1$, there exists $W_m \in K_0$ and $\|W_m\|_{C^1([0,M])} = 1$ such that

$$J[W_m] < \frac{1}{m}\left(\left\|(W_m)_x(x)\right\|^2_{L^2(0,M)} + \left\|x^{-1}W_m(x)\right\|^2_{L^2(0,M)}\right). \tag{2.15}$$

Then, there is a subsequence $m \to \infty$ for which

$$W_m \to W \quad \text{in } C\big([0,M]\big),$$
$$(W_m)_x \rightharpoonup W_x \quad \text{in } L^2\big([0,M]\big).$$

From (2.13)–(2.15), we have

$$W_x = \frac{W}{\rho_\infty V_\infty}, \quad x \in (0,M),$$

and $W(0) = W(M) = 0$. Thus, we obtain $W \equiv 0$. It is a contradiction.

Therefore, if $\gamma \geqslant \frac{2n-2}{n}$, then (2.9) holds. From Propositions 2.1 and 2.2, we can obtain (2.11) immediately. □

Now, we shall use the shooting method to prove the uniqueness of the solution.

Proposition 2.4. *Under the assumptions* (2.1)–(2.2)*, the Lagrangian stationary problem* (1.24)–(1.25) *has a unique positive solution* ρ_∞.

Proof. We consider the Cauchy problem

$$\left(A\rho_\infty^\gamma\right)_x = -Gx(nV_\infty)^{\frac{2-2n}{n}}, \qquad (V_\infty)_x = \rho_\infty^{-1}, \quad x \in (0,M), \tag{2.16}$$

$$\rho_\infty|_{x=0} = \sigma, \qquad V_\infty|_{x=0} = 0, \tag{2.17}$$

for the unknown functions $\rho_\infty(\sigma,x)$ and $V_\infty(\sigma,x)$, where $\sigma > 0$ is the shooting parameter. For each $\sigma > 0$, using similar arguments in Proposition 2.1, we can obtain the existence of the solution to this problem, satisfying

$$\rho_\infty(\sigma,x) \in \left[\frac{\sigma}{2}, \sigma\right], \qquad V_\infty(\sigma,x) \in \left[\frac{x}{\sigma}, \frac{2x}{\sigma}\right], \quad x \in [0,M_0], \tag{2.18}$$

$$\rho_\infty \in W^{1,\beta}\big([0,M_0]\big), \qquad V_\infty \in C^1\big([0,M_0]\big), \tag{2.19}$$

where M_0 is a positive constant satisfying $A\sigma^\gamma - \sigma^{\frac{2n-2}{n}}\frac{G}{2}n^{\frac{2-n}{n}}M_0^{\frac{2}{n}} \geqslant A(\frac{\sigma}{2})^\gamma$ and $M_0 \leqslant M$. If there exist two solutions (ρ_1, V_1) and (ρ_2, V_2) to this problem satisfying

$$\rho_i \in W^{1,\beta}\big([0, M_i]\big), \quad x \in [0, M_i], \tag{2.20}$$

where $M_i \in (0, M]$, $i = 1, 2$. From (2.20), there exists a positive constant $M_3 \in (0, \min\{M_1, M_2\})$ such that

$$\rho_i(x) \in \left[\frac{\sigma}{2}, \sigma\right] \quad \text{and} \quad V_i(x) \in \left[\frac{x}{\sigma}, \frac{2x}{\sigma}\right], \quad x \in [0, M_3],\ i = 1, 2.$$

Then, we have

$$A\rho_1^\gamma - A\rho_2^\gamma = \int_0^x Gyn^{\frac{2-2n}{n}}\big(V_2^{\frac{2-2n}{n}} - V_1^{\frac{2-2n}{n}}\big)\,dy \leqslant C\int_0^x y^{\frac{2-2n}{n}}\int_0^y \big|\rho_1^{-1} - \rho_2^{-1}\big|(z)\,dz\,dy,$$

and

$$\|\rho_1 - \rho_2\|_{L^\infty([0,\epsilon])} \leqslant C\|\rho_1 - \rho_2\|_{L^\infty([0,\epsilon])}\int_0^\epsilon y^{\frac{2-n}{n}}\,dy \leqslant C_\sigma \epsilon^{\frac{2}{n}}\|\rho_1 - \rho_2\|_{L^\infty([0,\epsilon])},$$

for all $x, \epsilon \in (0, M_3]$. Choosing $\epsilon < C_\sigma^{-\frac{n}{2}}$, we have

$$\rho_1 = \rho_2, \quad \text{for all } x \in [0, \epsilon].$$

Considering the Cauchy problem

$$\big(A\rho_\infty^\gamma\big)_x = -Gx(nV_\infty)^{\frac{2-2n}{n}}, \qquad (V_\infty)_x = \rho_\infty^{-1}, \quad x \in \left(\frac{\epsilon}{2}, M\right), \tag{2.21}$$

$$\rho_\infty|_{x=\frac{\epsilon}{2}} = \rho_1\left(\sigma, \frac{\epsilon}{2}\right), \qquad V_\infty|_{x=\frac{\epsilon}{2}} = \int_0^{\frac{\epsilon}{2}} \rho_1^{-1}(\sigma, y)\,dy, \tag{2.22}$$

using the classical ODE theory, we have $\rho_1(x) = \rho_2(x)$, $x \in [\frac{\epsilon}{2}, \min\{M_1, M_2\}]$. Thus, for each $\sigma > 0$, there exists a unique solution to the problem (2.16)–(2.17) satisfying $\rho_\infty(x, \sigma) > 0$ for $x \in [0, M_\sigma)$, where either $\rho_\infty|_{x=M_\sigma} = 0$ and $M_\sigma \in (0, M)$ or $M_\sigma = M$.

Clearly, if ρ_∞ is a solution to the problem (1.24)–(1.25), then ρ_∞ satisfies (2.16)–(2.17) for some $\sigma > 0$. We will show that this can be possible only for one value of σ. Using similar arguments in the above part and in [5, §V.3], we obtain that $(\partial_\sigma\rho_\infty^\gamma, \partial_\sigma V_\infty)$ is well defined and satisfies the linear Cauchy problem

$$A\big(\partial_\sigma\rho_\infty^\gamma\big)_x = (2n-2)Gx(nV_\infty)^{\frac{2-3n}{n}}\partial_\sigma V_\infty, \qquad (\partial_\sigma V_\infty)_x = -\frac{1}{\gamma}\rho_\infty^{-\gamma-1}\partial_\sigma\rho_\infty^\gamma, \tag{2.23}$$

where $x \in [0, M_\sigma)$,

$$\partial_\sigma \rho_\infty^\gamma\big|_{x=0} = 1, \qquad \partial_\sigma V_\infty|_{x=0} = 0. \tag{2.24}$$

It is easy to see that

$$\partial_\sigma \rho_\infty^\gamma > 0, \qquad (\partial_\sigma V_\infty)_x < 0, \qquad \partial_\sigma V_\infty < 0$$

hold on $[0, M_4)$, where either $\partial_\sigma \rho_\infty^\gamma|_{x=M_4} = 0$ and $M_4 \in (0, M_\sigma)$ or $M_4 = M_\sigma$. We claim that only $M_4 = M_\sigma$ can occur.

Assume that $M_4 \in (0, M_\sigma)$. Letting $\phi = A\rho_\infty^\gamma(\partial_\sigma V_\infty)_x + \frac{n}{2n-2}A\partial_\sigma \rho_\infty^\gamma (V_\infty)_x$, from (2.16) and (2.23), we have

$$\int_0^{M_4} \phi\, dx = \left\{ A\rho_\infty^\gamma \partial_\sigma V_\infty + \frac{n}{2n-2} A\partial_\sigma \rho_\infty^\gamma V_\infty \right\}\bigg|_0^{M_4}.$$

By the estimates $\rho_\infty(\sigma, M_4) > 0$, $\partial_\sigma \rho_\infty^\gamma|_{x=M_4} = 0$, $\partial_\sigma V_\infty|_{x=M_4} < 0$ and the initial conditions (2.17) and (2.24), we get

$$\int_0^{M_4} \phi\, dx < 0.$$

On the other hand, from (2.16) and (2.23), we have

$$\phi = A\rho_\infty^{-1} \partial_\sigma \rho_\infty^\gamma \left(\frac{n}{2n-2} - \frac{1}{\gamma} \right) \geqslant 0, \quad x \in (0, M_4).$$

It is a contradiction.

Thus, we obtain

$$\rho_\infty > 0, \qquad \partial_\sigma \rho_\infty > 0, \quad x \in (0, M_\sigma),$$

and M_σ is nondecreasing on $\sigma \in (0, \infty)$. Therefore, for each fixed point $x \in [0, \sup_{\sigma>0} M_\sigma)$, the function $\rho_\infty(\sigma, x)$ is strictly increasing on $\sigma > (\frac{P_\infty}{A})^{\frac{1}{\gamma}}$, and satisfies $A\rho_\infty^\gamma|_{x=M} = P_\infty$ for at most one value of σ. $\square$

Using the properties of the transformation (1.19) and Propositions 2.1–2.4, we can obtain the following proposition immediately.

Proposition 2.5. *Under the assumptions* (2.1)–(2.2)*, the Eulerian stationary problem* (1.14)–(1.16) *has a unique positive solution* (ρ_∞, l_∞)*, satisfying* $0 < \underline{\rho} \leqslant \rho_\infty(r) \leqslant \overline{\rho} < \infty$, $(\rho_\infty)_r(r) < 0$, $0 < r < l_\infty$ *with* $l_\infty < +\infty$.

3. *A priori* estimates

From (1.11), (1.20) and (1.24), we could obtain the following lemma easily.

Lemma 3.1. *Under the assumptions of Theorem* 1.1*, we have*

$$r_t = u, \tag{3.1}$$

$$A\rho_\infty^\gamma(x) = P_\infty + \int_x^M \frac{Gy}{r_\infty^{2n-2}(y)}\,dy, \tag{3.2}$$

$$\left(\frac{P_\infty}{A}\right)^{\frac{1}{\gamma}} \leqslant \rho_\infty \leqslant \overline{\rho} < \infty, \qquad r_\infty^n(x) \in \left[C^{-1}x, Cx\right], \tag{3.3}$$

$$\frac{d}{dx}\left(A\rho_\infty^\gamma(x)\right) = -G\frac{x}{r_\infty^{2n-2}}, \tag{3.4}$$

for all $x \in [0, M]$.

Lemma 3.2. *Under the assumptions of Theorem* 1.1*, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_0^M \left(\frac{1}{2}u^2 + \frac{A\rho^{\gamma-1}}{\gamma-1} + \frac{P_\infty}{\rho} + \int_1^r G\frac{x}{s^{n-1}}\,ds\right)dx \\
&\quad + \int_0^M \left\{\left(\frac{2}{n}c_1 + c_2\right)\rho^{1+\theta}\left[\left(r^{n-1}u\right)_x\right]^2 + \frac{2(n-1)}{n}c_1\rho^{1+\theta}\left(r^{n-1}u_x - \frac{u}{r\rho}\right)^2\right\}dx \\
&= -\int_0^M \Delta f u\,dx - \Delta P\left(ur^{n-1}\right)(M,t).
\end{aligned} \tag{3.5}$$

Proof. Multiplying $(1.20)_2$ by u, integrating the resulting equation over $[0, M]$, using integration by parts and the boundary conditions (1.22)–(1.23), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^M \frac{1}{2}u^2\,dx - \int_0^M A\rho^\gamma\,\partial_x\left(r^{n-1}u\right)dx \\
&\quad + \int_0^M \left\{(2c_1 + c_2)\rho^{1+\theta}\left[\left(r^{n-1}u\right)_x\right]^2 - 2c_1(n-1)\rho^\theta\left(r^{n-2}u^2\right)_x\right\}dx \\
&= -P_\Gamma\left(ur^{n-1}\right)(M,t) - \int_0^M fu\,dx.
\end{aligned} \tag{3.6}$$

From (1.20), we have

$$-\int_0^M A\rho^\gamma \partial_x\big(r^{n-1}u\big)\,dx = \frac{d}{dt}\int_0^M \frac{A}{\gamma-1}\rho^{\gamma-1}\,dx, \tag{3.7}$$

$$\begin{aligned}-P_\Gamma\big(ur^{n-1}\big)(M,t) &= -P_\infty\big(r_t r^{n-1}\big)(M,t) - \Delta P\big(ur^{n-1}\big)(M,t)\\ &= -\frac{d}{dt}\left\{P_\infty\frac{r^n(M,t)}{n}\right\} - \Delta P\big(ur^{n-1}\big)(M,t)\\ &= -\frac{d}{dt}\int_0^M \frac{P_\infty}{\rho}\,dx - \Delta P\big(ur^{n-1}\big)(M,t),\end{aligned} \tag{3.8}$$

$$-\int_0^M fu\,dx = -\frac{d}{dt}\int_0^M\int_1^r G\frac{x}{s^{n-1}}\,ds\,dx - \int_0^M \Delta f u\,dx, \tag{3.9}$$

and

$$\begin{aligned}&(2c_1+c_2)\rho^{1+\theta}\big(r^{n-1}u\big)_x^2 - 2c_1(n-1)\rho^\theta\big(r^{n-2}u^2\big)_x\\ &\quad= \left(\frac{2}{n}c_1+c_2\right)\rho^{1+\theta}\big(r^{n-1}u\big)_x^2 + \frac{2(n-1)}{n}c_1\rho^{1+\theta}\left(r^{n-1}u_x - \frac{u}{r\rho}\right)^2.\end{aligned} \tag{3.10}$$

From (3.6)–(3.10), we obtain (3.5) immediately. □

Claim 1. *Under the assumptions of Theorem* 1.1*, there is a small positive constant ϵ_1, such that, for any $T>0$, if*

$$I(t) = \big\|\rho(\cdot,t)-\rho_\infty\big\|_{L^\infty} + \left\|\frac{u}{r}(\cdot,t)\right\|_{L^\infty} \leqslant 2\epsilon_1, \quad \forall t\in[0,T], \tag{3.11}$$

then

$$I(t)\leqslant \epsilon_1, \quad \forall t\in[0,T].$$

Using the results in Lemmas 3.3–3.8, we can give the definition of ϵ_1 in (3.74) and finish the proof of Claim 1.

Lemma 3.3. *Under the assumptions of Theorem* 1.1 *and* (3.11)*, if ϵ_1 is small enough, we obtain*

$$\rho(x,t)\in\left[\frac{1}{2}\underline{\rho}, \frac{3}{2}\overline{\rho}\right], \tag{3.12}$$

$$r^n(x,t)\in\big[C^{-1}x, Cx\big], \tag{3.13}$$

$$\big\|u(\cdot,t)\big\|_{L^2} + \big\|\rho(\cdot,t)-\rho_\infty\big\|_{L^2} + \big\|r_\infty^{-n}\big(r^n-r_\infty^n\big)\big\|_{L_x^2} \leqslant C_1\epsilon_0, \tag{3.14}$$

$$\int_0^t \int_0^M \left(u^2 + r^{2n-2}u_x^2 + \frac{u^2}{r^2}\right) dx\, ds \leqslant C_1 \epsilon_0^2, \tag{3.15}$$

for all $t \in [0, T]$ and $x \in [0, M]$.

Proof. From Lemma 3.1 and (3.11), we can easily obtain the estimate (3.12) when $2\epsilon_1 \leqslant \frac{1}{2}\underline{\rho}$. From $(1.20)_3$ and (3.12), we can obtain (3.13) immediately. From (2.10), (3.2) and (3.5), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^M \frac{1}{2}u^2\, dx + S[V] - S[V_\infty]\right) \\
&\quad + \int_0^M \left\{\left(\frac{2}{n}c_1 + c_2\right)\rho^{1+\theta}\left(r^{n-1}u\right)_x^2 + \frac{2(n-1)}{n}c_1\rho^{1+\theta}\left(r^{n-1}u_x - \frac{u}{r\rho}\right)^2\right\} dx \\
&= -\int_0^M \Delta f u\, dx - \Delta P\left(ur^{n-1}\right)(M,t),
\end{aligned} \tag{3.16}$$

where $V_\infty = \frac{r_\infty^n}{n}$ and $V = \frac{r^n}{n}$. From (1.27), (2.11), (3.12)–(3.13) and Proposition 2.3, we have

$$\begin{aligned}
&C^{-1}\int_0^M \left[(\rho - \rho_\infty)^2 + \frac{(V - V_\infty)^2}{V_\infty^2}\right] dx \\
&\quad \leqslant S[V] - S[V_\infty] \leqslant C\int_0^M \left[(\rho - \rho_\infty)^2 + \frac{(V - V_\infty)^2}{V_\infty^2}\right] dx,
\end{aligned} \tag{3.17}$$

when $\|V - V_\infty\|_{C^1([0,M])} \leqslant C_2\epsilon_1 \leqslant \delta_5$, and

$$\left|\Delta P\left(ur^{n-1}\right)(M,t)\right| \leqslant C\epsilon_0\left(\int_0^M \left|\partial_x\left(r^{n-1}u\right)\right|^2 dx\right)^{\frac{1}{2}}. \tag{3.18}$$

From (1.27)–(1.28), (3.12) and (3.16)–(3.18), we obtain

$$\begin{aligned}
&\int_0^M \left(u^2 + (\rho - \rho_\infty)^2 + r_\infty^{-2n}\left(r^n - r_\infty^n\right)^2\right) dx + \int_0^t \int_0^M \left\{\left(r^{n-1}u\right)_x^2 + \left(r^{n-1}u_x - \frac{u}{r\rho}\right)^2\right\} dx\, ds \\
&\quad \leqslant C\epsilon_0^2 + C\int_0^t f_1(s)\left\|u(\cdot,s)\right\|_{L^2} ds,
\end{aligned} \tag{3.19}$$

using Gronwall's inequality and (1.27), we can obtain (3.14)–(3.15) immediately. □

Lemma 3.4. *Under the assumptions of Lemma 3.3, if ϵ_0 is small enough, we obtain*

$$\int_0^t \int_0^M \left[(\rho-\rho_\infty)^2 + r_\infty^{-2n}\left(r^n - r_\infty^n\right)^2\right] dx\, ds \leqslant C_3\epsilon_0^2, \tag{3.20}$$

for all $t \in [0, T]$.

Proof. Multiplying $(1.20)_2$ by $r^{1-n}(\frac{r^n}{n} - \frac{r_\infty^n}{n})$, integrating the resulting equation over $[0, M]$, using integration by parts and the boundary conditions (1.22)–(1.23), we obtain

$$\begin{aligned}
&\int_0^M A\left(\rho_\infty^\gamma - \rho^\gamma\right)\left(\rho^{-1} - \rho_\infty^{-1}\right) + Gx\left(r^{2-2n} - r_\infty^{2-2n}\right)\left(\frac{r^n}{n} - \frac{r_\infty^n}{n}\right) dx \\
&\quad = -\int_0^M \frac{u_t}{r^{n-1}}\left(\frac{r^n}{n} - \frac{r_\infty^n}{n}\right) dx + \Delta P\left\{\frac{r^n}{n} - \frac{r_\infty^n}{n}\right\}\bigg|_{x=M} \\
&\qquad - \int_0^M \Delta f \frac{r^{1-n}}{n}\left(r^n - r_\infty^n\right) dx + \int_0^M 2c_1(n-1)\rho^\theta\left(\frac{u}{r}\left(\frac{r^n}{n} - \frac{r_\infty^n}{n}\right)\right)_x dx \\
&\qquad + \int_0^M (2c_1 + c_2)\rho^{1+\theta}\partial_x\left(r^{n-1}u\right)\left(\rho_\infty^{-1} - \rho^{-1}\right) dx \\
&\quad := \sum_{i=1}^5 I_i.
\end{aligned} \tag{3.21}$$

We can rewrite the left-hand side of (3.21) as follows

$$\begin{aligned}
\text{L.H.S. of (3.21)} = &\int_0^M \left[\gamma A\rho_\infty^{\gamma+1}\left(\rho^{-1} - \rho_\infty^{-1}\right)^2 - (2n-2)Gxr_\infty^{2-3n}\left(\frac{r^n}{n} - \frac{r_\infty^n}{n}\right)^2\right] dx \\
&+ \int_0^M \left[g_1\left(\rho^{-1} - \rho_\infty^{-1}\right)^2 + g_2 r_\infty^{-2n}\left(\frac{r^n}{n} - \frac{r_\infty^n}{n}\right)^2\right] dx,
\end{aligned}$$

where

$$|g_1| = \left|\frac{A(\rho_\infty^\gamma - \rho^\gamma)}{\rho^{-1} - \rho_\infty^{-1}} - \gamma A\rho_\infty^{1+\gamma}\right| \leqslant C_4\epsilon_1$$

and

$$|g_2| = \left|Gxr_\infty^{2n}\left(r^{2-2n} - r_\infty^{2-2n}\right)\left(\frac{r^n}{n} - \frac{r_\infty^n}{n}\right)^{-1} + (2n-2)Gxr_\infty^{2-n}\right| \leqslant C_4\epsilon_1.$$

From (2.9), we have

$$\text{L.H.S. of (3.21)} \geqslant (2C_5 - C_4\epsilon_1)\int_0^M \left[\left(\rho^{-1}-\rho_\infty^{-1}\right)^2 + r_\infty^{-2n}\left(\frac{r^n}{n}-\frac{r_\infty^n}{n}\right)^2\right]dx$$
$$\geqslant C_5\int_0^M \left[\left(\rho^{-1}-\rho_\infty^{-1}\right)^2 + r_\infty^{-2n}\left(\frac{r^n}{n}-\frac{r_\infty^n}{n}\right)^2\right]dx, \tag{3.22}$$

when $C_4\epsilon_1 \leqslant C_5$.

From (3.11) and (3.12)–(3.13), using integration by parts, we can estimate I_i as follows:

$$I_1 = -\frac{d}{dt}\int_0^M \frac{u}{r^{n-1}}\left(\frac{r^n}{n}-\frac{r_\infty^n}{n}\right)dx + \int_0^M u^2\left(\frac{1}{n}+\frac{(n-1)r_\infty^n}{nr^n}\right)dx$$
$$\leqslant -\frac{d}{dt}\int_0^M \frac{u}{r^{n-1}}\left(\frac{r^n}{n}-\frac{r_\infty^n}{n}\right)dx + C\int_0^M u^2\,dx, \tag{3.23}$$

$$I_2 = \Delta P\int_0^M \left(\rho^{-1}-\rho_\infty^{-1}\right)dx \leqslant \frac{C_5}{10}\int_0^M \left(\rho^{-1}-\rho_\infty^{-1}\right)^2 dx + C|\Delta P|^2, \tag{3.24}$$

$$I_3 \leqslant \frac{C_5}{10}\int_0^M r_\infty^{-2n}\left(\frac{r^n}{n}-\frac{r_\infty^n}{n}\right)^2 dx + Cf_1^2, \tag{3.25}$$

$$I_4 \leqslant \frac{C_5}{10}\int_0^M \left[\left(\rho^{-1}-\rho_\infty^{-1}\right)^2 + r_\infty^{-2n}\left(\frac{r^n}{n}-\frac{r_\infty^n}{n}\right)^2\right]dx + C\int_0^M \left(\left[\left(r^{n-1}u\right)_x\right]^2 + \frac{u^2}{r^2}\right)dx \tag{3.26}$$

and

$$I_5 \leqslant \frac{C_5}{10}\int_0^M \left(\rho^{-1}-\rho_\infty^{-1}\right)^2 dx + C\int_0^M \left(r^{n-1}u\right)_x^2\,dx. \tag{3.27}$$

From (3.21)–(3.27), we get

$$\frac{d}{dt}\int_0^M \frac{u}{nr^{n-1}}\left(r^n - r_\infty^n\right)dx + C\int_0^M \left[\left(\rho^{-1}-\rho_\infty^{-1}\right)^2 + r_\infty^{-2n}\left(\frac{r^n}{n}-\frac{r_\infty^n}{n}\right)^2\right]dx$$
$$\leqslant C\int_0^M \left(r^{2n-2}u_x^2 + \frac{u^2}{r^2}\right)dx + C\left(|\Delta P|^2 + f_1^2\right), \tag{3.28}$$

and from (3.12)–(3.15), we obtain (3.20) immediately. □

From now on, we study the case of $\theta > 0$ and prove the main results in this case only, since the case of $\theta = 0$ can be discussed through the similar process.

Lemma 3.5. *Under the assumptions of Lemma* 3.3, *if ϵ_1 is small enough, we obtain*

$$\int_0^M \big[r^{2n-2}(\rho-\rho_\infty)_x^2\big](x,t)\,dx + \int_0^t\int_0^M \big[r^{2n-2}(\rho-\rho_\infty)_x^2\big](x,s)\,dx\,ds \leqslant C_6\epsilon_0^2, \tag{3.29}$$

for all $t\in[0,T]$.

Proof. From (1.20), we have

$$\begin{aligned}
&\partial_t H + \frac{A\gamma\rho^{\gamma-\theta}}{2c_1+c_2}H \\
&\quad= \frac{A\gamma}{2c_1+c_2}\rho^{\gamma-\theta}u + \left(\frac{2c_1+c_2}{\theta} - 2c_1\right)(n-1)r^{n-2}u\big(\rho^\theta\big)_x - f(x,r,t) \\
&\qquad - \frac{(n-1)(2c_1+c_2)}{\theta}r^{n-2}u\big(\rho_\infty^\theta\big)_x - \frac{A\gamma\rho^{\gamma-\theta}r^{n-1}}{\theta}\big(\rho_\infty^\theta\big)_x,
\end{aligned} \tag{3.30}$$

where $H = u + \frac{2c_1+c_2}{\theta}r^{n-1}(\rho^\theta-\rho_\infty^\theta)_x$. Multiplying (3.30) by H, integrating the resulting equation over $[0,M]$, using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^M H^2(x,t)\,dx + C_7\int_0^M H^2(x,t)\,dx \\
&\quad\leqslant C\int_0^M \left(\big|H\rho^{\gamma-\theta}u\big| + \left|\frac{u}{r}H^2\right| + \left|\frac{u^2}{r}H\right| + |\Delta f H|\right)dx \\
&\qquad + \int_0^M \left|G\frac{x}{r^{n-1}} + \frac{\rho^{\gamma-\theta}r^{n-1}}{\rho_\infty^{\gamma-\theta}}\big(A\rho_\infty^\gamma\big)_x\right||H|\,dx + C\int_0^M \big|r^{n-2}u\big(\rho_\infty^\gamma\big)_x H\big|\,dx \\
&\quad\leqslant C\int_0^M \left(u^2 + \frac{u^4}{r^2} + \frac{x^2r^{2n-4}}{r_\infty^{4n-4}}u^2\right)dx + \left(\frac14 + C_8\epsilon_1\right)C_7\int_0^M H^2\,dx \\
&\qquad + C\int_0^M \left|G\frac{x}{r^{n-1}} + \frac{\rho^{\gamma-\theta}r^{n-1}}{\rho_\infty^{\gamma-\theta}}\big(A\rho_\infty^\gamma\big)_x\right|^2 dx + Cf_1^2.
\end{aligned} \tag{3.31}$$

From (3.4) and (3.12), we have

$$\int_0^M \left|G\frac{x}{r^{n-1}} + \frac{\rho^{\gamma-\theta}r^{n-1}}{\rho_\infty^{\gamma-\theta}}\big(A\rho_\infty^\gamma\big)_x\right|^2 dx$$

$$= C \int_0^M \left| G \frac{x}{r^{n-1}} - G \frac{x \rho^{\gamma-\theta} r^{n-1}}{\rho_\infty^{\gamma-\theta} r_\infty^{2n-2}} \right|^2 dx$$

$$\leqslant C \int_0^M \left[(r - r_\infty)^2 + (\rho - \rho_\infty)^2 \right] dx. \tag{3.32}$$

Then, if $\epsilon_1 \leqslant 1$ and $C_8 \epsilon_1 \leqslant \frac{1}{4}$, from (3.31)–(3.32), we obtain

$$\frac{d}{dt} \int_0^M H^2(x,t)\, dx + \frac{C_7}{2} \int_0^M H^2(x,t)\, dx$$

$$\leqslant C \int_0^M \left(u^2 + (r - r_\infty)^2 + (\rho - \rho_\infty)^2 \right) dx + C f_1^2. \tag{3.33}$$

From (3.12)–(3.15), (3.20) and (3.33), we obtain (3.29) immediately. □

Lemma 3.6. *Under the assumptions of Lemma* 3.3, *if ϵ_1 is small enough, we obtain*

$$\left(\rho(M,t) - \rho_\infty(M) \right)^2 + \int_0^t \left(\rho(M,s) - \rho_\infty(M) \right)^2 ds \leqslant C_{10} \epsilon_0^2, \tag{3.34}$$

$$\int_0^t \int_0^M r^{\frac{1}{2}-m} (\rho - \rho_\infty)^2 \, dx\, ds \leqslant C_{11} \epsilon_0^2, \tag{3.35}$$

$$\int_0^M \left(r^{\frac{1}{2}-m} u^2 \right)(x,t)\, dx + \int_0^t \int_0^M r^{\frac{1}{2}-m} \left(r^{2n-2} u_x^2 + \frac{u^2}{r^2} \right) dx\, ds \leqslant C_{12} \epsilon_0^2, \tag{3.36}$$

$$\int_0^M \left(r^{2n-2+\frac{1}{2}-m} (\rho - \rho_\infty)_x^2 \right) dx + \int_0^t \int_0^M \left(r^{2n-2+\frac{1}{2}-m} (\rho - \rho_\infty)_x^2 \right) dx\, ds \leqslant C_{15} \epsilon_0^2, \tag{3.37}$$

$$\left\| \rho(\cdot,t) - \rho_\infty(\cdot) \right\|_{L^\infty} + \int_0^M \left| (\rho - \rho_\infty)_x \right| (x,t)\, dx \leqslant C_{16} \epsilon_0, \tag{3.38}$$

$$\left| r(x,t) - r_\infty(x) \right| \leqslant C_{16} \epsilon_0 x^{\frac{1}{n}}, \quad x \in [0, M], \tag{3.39}$$

for all $t \in [0,T]$ and $m = 0, 1, \ldots, n-1$.

Proof. From $(1.20)_1$ and the boundary condition (1.23), we have

$$A\rho^\gamma(M,t) - P_\Gamma + \frac{(2c_1+c_2)}{\theta}\partial_t\big(\rho^\theta\big)(M,t) = -2c_1(n-1)\left(\rho^\theta\frac{u}{r}\right)(M,t).$$

Multiplying the above equality by $\rho^\theta(M,t) - \rho^\theta_\infty(M)$, we obtain

$$\begin{aligned}&\frac{2c_1+c_2}{2\theta}\frac{d}{dt}\big(\rho^\theta - \rho^\theta_\infty\big)^2\Big|_{x=M} + \big(\rho^\theta(M,t) - \rho^\theta_\infty(M)\big)\big(A\rho^\gamma(M,t) - P_\infty\big)\\
&\quad= -2c_1(n-1)\left[\rho^\theta\frac{u}{r}\big(\rho^\theta - \rho^\theta_\infty\big)\right]\Big|_{x=M} + \Delta P\big(\rho^\theta(M,t) - \rho^\theta_\infty(M)\big).\end{aligned}$$

Combining (3.12)–(3.13), using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}&\frac{d}{dt}\big(\rho^\theta(M,t) - \rho^\theta_\infty(M)\big)^2 + C^{-1}\big(\rho^\theta(M,t) - \rho^\theta_\infty(M)\big)^2\\
&\quad\leqslant C|\Delta P|^2 + C\big(u^2r^n\big)(M,t) = C|\Delta P|^2 + C\int_0^M \partial_x\big(u^2r^n\big)\,dx\\
&\quad\leqslant C|\Delta P|^2 + C\int_0^M\left(r^{2n-2}u_x^2 + \frac{u^2}{r^2}\right)dx.\end{aligned}\tag{3.40}$$

Integrating the above inequality over $[0,t]$, using the estimates (3.12) and (3.15), we can obtain (3.34).

From (3.14)–(3.15), (3.20) and (3.29), we know that the estimates (3.35)–(3.37) hold with $m=0$.

Claim 2. *If* (3.35)–(3.37) *hold with* $m \leqslant k$, $k \in [0, n-2]$, *then the estimates* (3.35)–(3.37) *hold with* $m = k+1$.

We could prove Claim 2 as follows. Let $\alpha_k = \frac{1}{2} - k - 1$. Using Hölder's inequality, we have

$$\begin{aligned}\int_0^M r^{\alpha_k}(\rho-\rho_\infty)^2\,dx &= \int_0^M r^{\alpha_k}\left(\rho(M,s) - \rho_\infty(M) - \int_x^M \partial_x(\rho-\rho_\infty)\,dy\right)^2 dx\\
&\leqslant C\big(\rho(M,s) - \rho_\infty(M)\big)^2\\
&\quad + C\int_0^M r^{\alpha_k}\int_x^M r^{2n-2+\alpha_{k-1}}(\rho-\rho_\infty)_x^2\,dy\int_x^M r^{2-2n-\alpha_{k-1}}\,dy\,dx\\
&\leqslant C\big(\rho(M,s) - \rho_\infty(M)\big)^2 + C\int_0^M r^{2n-2+\alpha_{k-1}}(\rho-\rho_\infty)_x^2\,dx.\end{aligned}\tag{3.41}$$

From (3.34), (3.37) ($m = k$) and (3.41), we obtain (3.35) ($m = k + 1$).

Multiplying $(1.20)_2$ by ur^{α_k}, integrating the resulting equation over $[0, M]$, using integration by parts and the boundary conditions (1.22)–(1.23), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^M \frac{1}{2} r^{\alpha_k} u^2\,dx - \int_0^M \frac{\alpha_k}{2} r^{\alpha_k - 1} u^3\,dx \\
&= -\int_0^M \bigl[(2c_1 + c_2)\rho^{1+\theta}\bigl(r^{n-1}u\bigr)_x\bigl(r^{n-1+\alpha_k}u\bigr)_x - 2c_1(n-1)\rho^\theta\bigl(r^{n-2+\alpha_k}u^2\bigr)_x\bigr]\,dx \\
&\quad + \int_0^M A\bigl(\rho^\gamma - \rho_\infty^\gamma\bigr)\bigl(r^{n-1+\alpha_k}u\bigr)_x\,dx - \int_0^M \Delta f u r^{\alpha_k}\,dx \\
&\quad - \Delta P\bigl(ur^{n-1+\alpha_k}\bigr)(M,t) + \int_0^M \bigl(r^{n-1+\alpha_k}u\bigr)_x \int_x^M \left(\frac{Gy}{r_\infty^{2n-2}} - \frac{Gy}{r^{2n-2}}\right) dy\,dx \\
&= \sum_{i=1}^5 L_i.
\end{aligned} \tag{3.42}$$

We can estimate L_1 as follows

$$\begin{aligned}
-L_1 = \int_0^M \bigl\{&(2c_1 + c_2) r^{2n-2+\alpha_k}\rho^{1+\theta}u_x^2 \\
&+ \bigl[2\alpha_k c_1 + c_2(2n - 2 + \alpha_k)\bigr]\rho^\theta r^{n-2+\alpha_k} u u_x \\
&+ \bigl[2c_1(n-1) + c_2(n-1)(n-1+\alpha_k)\bigr]\rho^{\theta-1} r^{\alpha_k - 2}u^2\bigr\}\,dx.
\end{aligned} \tag{3.43}$$

Since

$$\bigl[2\alpha c_1 + c_2(2n-2+\alpha)\bigr]^2 - 4(2c_1+c_2)\bigl[2c_1(n-1) + c_2(n-1)(n-1+\alpha)\bigr] < 0,$$

where $\alpha = \frac{3}{2} - n$, and

$$\bigl[c_2(2n-2)\bigr]^2 - 4(2c_1+c_2)\bigl[2c_1(n-1) + c_2(n-1)^2\bigr] < 0,$$

we have

$$\bigl[2\alpha_k c_1 + c_2(2n-2+\alpha_k)\bigr]^2 - 4(2c_1+c_2)\bigl[2c_1(n-1) + c_2(n-1)(n-1+\alpha_k)\bigr] < 0.$$

Then there exists a positive constant C_{13} such that

$$-L_1 \geqslant C_{13} \int_0^M \left(r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 + \rho^{\theta-1} r^{\alpha_k-2} u^2\right) dx. \tag{3.44}$$

From (3.12)–(3.13), using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} L_2 &= \int_0^M A\left(\rho^\gamma - \rho_\infty^\gamma\right) r^{n-1+\alpha_k} u_x \, dx + \int_0^M A(n-1+\alpha_k) r^{\alpha_k-1} u \frac{\rho^\gamma - \rho_\infty^\gamma}{\rho} \, dx \\ &\leqslant \frac{C_{13}}{8} \int_0^M \left(r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 + \rho^{\theta-1} r^{\alpha_k-2} u^2\right) dx + C \int_0^M r^{\alpha_k} (\rho - \rho_\infty)^2 \, dx, \end{aligned} \tag{3.45}$$

$$L_3 \leqslant \frac{1}{8} C_{13} \int_0^M \rho^{\theta-1} r^{\alpha_k-2} u^2 \, dx + C f_1^2, \tag{3.46}$$

$$\begin{aligned} L_4 &= -\int_0^M \Delta P \partial_x \left(r^{n-1+\alpha_k} u\right) dx \\ &\leqslant \frac{1}{8} C_{13} \int_0^M \left(r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 + \rho^{\theta-1} r^{\alpha_k-2} u^2\right) dx + C|\Delta P|^2 \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} L_5 &\leqslant \frac{1}{8} C_{13} \int_0^M \left(r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 + \rho^{\theta-1} r^{\alpha_k-2} u^2\right) dx \\ &\quad + C \int_0^M r^{\alpha_k} \left[\int_x^M \left(\frac{Gy}{r_\infty^{2n-2}} - \frac{Gy}{r^{2n-2}} \right) dy \right]^2 dx \\ &\leqslant \frac{C_{13}}{8} \int_0^M \left(r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 + \rho^{\theta-1} r^{\alpha_k-2} u^2\right) dx + C \int_0^M r^{\alpha_k} (\rho - \rho_\infty)^2 \, dx. \end{aligned} \tag{3.48}$$

From (3.11), (3.12)–(3.13) and (3.42)–(3.48), we obtain

$$\begin{aligned} &\frac{d}{dt} \int_0^M \left(r^{\alpha_k} u^2\right)(x,t) \, dx + 2C_{14}^{-1} \int_0^M r^{\alpha_k} \left(r^{2n-2} u_x^2 + \frac{u^2}{r^2} \right) dx \\ &\quad \leqslant C\left(f_1^2 + |\Delta P|^2\right) + C_{14} \epsilon_1 \int_0^M r^{\alpha_k} \frac{u^2}{r^2} \, dx + C \int_0^M r^{\alpha_k} (\rho - \rho_\infty)^2 \, dx. \end{aligned}$$

When $C_{14}^2\epsilon_1 \leqslant \frac{1}{2}$, using the estimate (3.35) $(m = k+1)$, we can get

$$\frac{d}{dt}\int_0^M \big(r^{\alpha_k}u^2\big)(x,t)\,dx + C_{14}^{-1}\int_0^M r^{\alpha_k}\left(r^{2n-2}u_x^2 + \frac{u^2}{r^2}\right)dx \leqslant C\big(f_1^2 + |\Delta P|^2\big) + C\int_0^M r^{\alpha_k}(\rho-\rho_\infty)^2\,dx \tag{3.49}$$

and (3.36) $(m = k+1)$ holds.

From (1.20), we have

$$\begin{aligned}
&\partial_t H_1 + \frac{A\gamma}{2c_1+c_2}\rho^{\gamma-\theta}H_1 \\
&= \frac{\alpha_k}{2}r^{\frac{\alpha_k}{2}-1}u^2 + \frac{A\gamma}{2c_1+c_2}\rho^{\gamma-\theta}r^{\frac{\alpha_k}{2}}u - \frac{A\gamma\rho^{\gamma-\theta}r^{n-1+\frac{\alpha_k}{2}}}{\theta}\big(\rho_\infty^\theta\big)_x \\
&\quad + \left[\frac{2c_1+c_2}{\theta}\left(n-1+\frac{\alpha_k}{2}\right) - 2c_1(n-1)\right]r^{n-2+\frac{\alpha_k}{2}}u\big(\rho^\theta\big)_x \\
&\quad - \frac{(n-1+\frac{\alpha_k}{2})(2c_1+c_2)}{\theta}r^{n-2+\frac{\alpha_k}{2}}u\big(\rho_\infty^\theta\big)_x - f(x,r,t)r^{\frac{\alpha_k}{2}},
\end{aligned} \tag{3.50}$$

where $H_1 = r^{\frac{\alpha_k}{2}}u + \frac{2c_1+c_2}{\theta}r^{n-1+\frac{\alpha_k}{2}}(\rho^\theta - \rho_\infty^\theta)_x$. Multiplying (3.50) by H_1, integrating the resulting equation over $[0, M]$, using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^M H_1^2(x,t)\,dx + C_{18}\int_0^M H_1^2(x,t)\,dx \\
&\leqslant C\int_0^M \left(\big|H_1\rho^{\gamma-\theta}r^{\frac{\alpha_k}{2}}u\big| + \left|\frac{u}{r}\right|H_1^2 + r^{\frac{\alpha_k}{2}-1}u^2|H_1| + r^{\frac{\alpha_k}{2}}|\Delta f H_1|\right)dx \\
&\quad + C\int_0^M \big|r^{n-2+\frac{\alpha_k}{2}}u\big(\rho_\infty^\gamma\big)_x H_1\big| + r^{\frac{\alpha_k}{2}}\left|\frac{Gx}{r^{n-1}} + \frac{\rho^{\gamma-\theta}r^{n-1}}{\rho_\infty^{\gamma-\theta}}\big(A\rho_\infty^\gamma\big)_x\right||H_1|\,dx \\
&\leqslant C\int_0^M \left(r^{\alpha_k}u^2 + r^{\alpha_k-2}u^4 + \frac{x^2r^{2n-4+\alpha_k}}{r_\infty^{4n-4}}u^2\right)dx + Cf_1^2 \\
&\quad + \left(\frac{1}{4} + C_{17}\epsilon_1\right)C_{18}\int_0^M H_1^2\,dx + C\int_0^M r^{\alpha_k}\left|\frac{Gx}{r^{n-1}} + \frac{\rho^{\gamma-\theta}r^{n-1}}{\rho_\infty^{\gamma-\theta}}\big(\rho_\infty^\gamma\big)_x\right|^2 dx.
\end{aligned} \tag{3.51}$$

If $\epsilon_1 \leqslant 1$, $C_{17}\epsilon_1 \leqslant \frac{1}{4}$, using the estimates (3.11) and (3.12)–(3.13), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_0^M H_1^2(x,t)\,dx+\frac{C_{18}}{2}\int_0^M H_1^2(x,t)\,dx\\
&\quad\leqslant C\int_0^M\left[r^{\alpha_k}u^2+r^{\alpha_k}(\rho-\rho_\infty)^2+r^{\alpha_k}(r-r_\infty)^2\right]dx+Cf_1^2\\
&\quad\leqslant C\int_0^M\left[r^{\alpha_k}u^2+r^{\alpha_k}(\rho-\rho_\infty)^2\right]dx+Cf_1^2,
\end{aligned}
\tag{3.52}
$$

and from (3.35)–(3.36) ($m=k+1$), we have

$$
\int_0^M H_1^2(x,t)\,dx+\int_0^t\int_0^M H_1^2(x,s)\,dx\,ds\leqslant C\epsilon_0^2.
$$

From (3.12) and (3.35)–(3.36) ($m=k+1$), we obtain (3.37) ($m=k+1$) immediately and finish the proof of Claim 2.

From Claim 2, we obtain that the estimates (3.35)–(3.37) ($m=0,\dots,n-1$) hold. From (3.13) and (3.37), using Hölder's inequality, we obtain

$$
\int_0^M\left|(\rho-\rho_\infty)_x\right|dx\leqslant\left(\int_0^M r^{2n-2+\alpha}(\rho-\rho_\infty)_x^2\,dx\right)^{\frac12}\left(\int_0^M r^{-2n+2-\alpha}\,dx\right)^{\frac12}\leqslant C\epsilon_0. \tag{3.53}
$$

From (3.13)–(3.14) and (3.53), using Sobolev's embedding theorem, we could obtain (3.38)–(3.39) immediately. □

Lemma 3.7. *Under the assumptions of Lemma* 3.3*, if* ϵ_1 *is small enough, we obtain*

$$
\int_0^M\left(\frac{u^2}{r^2}+r^{2n-2}u_x^2\right)(x,t)\,dx+\int_0^t\int_0^M u_t^2(x,s)\,dx\,ds\leqslant C_9\epsilon_0^2\left(1+\left\|\left(r^{n-1}u\right)_x\right\|_{L^\infty_{tx}}\right), \tag{3.54}
$$

for all $t\in[0,T]$.

Proof. Multiplying $(1.20)_2$ by u_t, integrating the resulting equation over $[0,M]$, using integration by parts and the boundary conditions (1.22)–(1.23), we obtain

$$
\begin{aligned}
&\int_0^M u_t^2\,dx+\int_0^M(2c_1+c_2)\rho^{1+\theta}\left(r^{n-1}u\right)_x\left(r^{n-1}u_t\right)_x\,dx\\
&\quad=\int_0^M A\rho^\gamma\left(r^{n-1}u_t\right)_x\,dx-P_\Gamma\left(r^{n-1}u_t\right)(M,t)
\end{aligned}
$$

$$+\int_0^M 2c_1(n-1)\rho^\theta\big(r^{n-2}uu_t\big)_x\,dx-\int_0^M fu_t\,dx$$
$$:=\sum_{i=1}^4 N_i. \tag{3.55}$$

From (3.11), (3.12) and (3.15), using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\int_0^M (2c_1+c_2)\rho^{1+\theta}\big(r^{n-1}u\big)_x\big(r^{n-1}u_t\big)_x\,dx\\
&\quad=\frac{d}{dt}\int_0^M \frac{2c_1+c_2}{2}\rho^{1+\theta}\big[\big(r^{n-1}u\big)_x\big]^2\,dx\\
&\qquad-\int_0^M (2c_1+c_2)(n-1)\rho^{1+\theta}\big(r^{n-1}u\big)_x\big(r^{n-2}u^2\big)_x\,dx\\
&\qquad+\int_0^M \frac{(2c_1+c_2)}{2}(1+\theta)\rho^{2+\theta}\big[\big(r^{n-1}u\big)_x\big]^3\,dx\\
&\quad\geqslant\frac{d}{dt}\int_0^M \frac{2c_1+c_2}{2}\rho^{1+\theta}\big[\big(r^{n-1}u\big)_x\big]^2\,dx\\
&\qquad-C\bigg(\big\|\big(r^{n-1}u\big)_x\big\|^2_{L^2_x}+\bigg\|\frac{u}{r}\bigg\|^2_{L^2_x}\bigg)\big(1+\big\|\big(r^{n-1}u\big)_x\big\|_{L^\infty_{t,x}}\big),
\end{aligned}\tag{3.56}$$

$$\begin{aligned}
N_1&=\frac{d}{dt}\int_0^M A\rho^\gamma\big(r^{n-1}u\big)_x\,dx+\int_0^M A\gamma\rho^{\gamma+1}\big[\big(r^{n-1}u\big)_x\big]^2\,dx\\
&\quad-\int_0^M 2A(n-1)\rho^\gamma\frac{u}{r}\big(r^{n-1}u\big)_x\,dx+\int_0^M An(n-1)\rho^{\gamma-1}\frac{u^2}{r^2}\,dx\\
&\leqslant\frac{d}{dt}\int_0^M A\rho^\gamma\big(r^{n-1}u\big)_x\,dx+C\bigg(\big\|\big(r^{n-1}u\big)_x\big\|^2_{L^2_x}+\bigg\|\frac{u}{r}\bigg\|^2_{L^2_x}\bigg),
\end{aligned}\tag{3.57}$$

$$N_2=-\frac{d}{dt}\int_0^M \big(P_\infty+\Delta P(t)\big)\big(r^{n-1}u\big)_x\,dx$$

$$+\int_0^M (n-1)(P_\infty+\Delta P)\big(r^{n-2}u^2\big)_x\,dx\,ds+(\Delta P)'\int_0^M \big(r^{n-1}u\big)_x\,dy\,ds$$

$$\leqslant -\frac{d}{dt}\int_0^M \big(P_\infty+\Delta P(t)\big)\big(r^{n-1}u\big)_x\,dx$$

$$+C\left(\big\|\big(r^{n-1}u\big)_x\big\|_{L_x^2}^2+\left\|\frac{u}{r}\right\|_{L_x^2}^2+\big|(\Delta P)'\big|^2\right), \tag{3.58}$$

$$N_3=\frac{d}{dt}\int_0^M c_1(n-1)\rho^\theta\big(r^{n-2}u^2\big)_x\,dx+\int_0^M 2\theta c_1(n-1)\rho^{\theta+1}\frac{u}{r}\big[\big(r^{n-1}u\big)_x\big]^2\,dx$$

$$-\int_0^M \theta c_1 n(n-1)\rho^\theta\frac{u^2}{r^2}\big(r^{n-1}u\big)_x\,dx+\int_0^M 2nc_1(n-1)(n-2)\rho^{\theta-1}\frac{u^3}{r^3}\,dx$$

$$-\int_0^M 3c_1(n-1)(n-2)\rho^\theta\frac{u^2}{r^2}\big(r^{n-1}u\big)_x\,dx$$

$$\leqslant\frac{d}{dt}\int_0^M c_1(n-1)\rho^\theta\big(r^{n-2}u^2\big)_x\,dx+C\left(\big\|\big(r^{n-1}u\big)_x\big\|_{L_x^2}^2+\left\|\frac{u}{r}\right\|_{L_x^2}^2\right) \tag{3.59}$$

and

$$N_4\leqslant-\frac{d}{dt}\int_0^M G\frac{xu}{r^{n-1}}\,dx+\int_0^M (1-n)Gxr^{-n}u^2\,dx+\frac{1}{2}\int_0^M u_t^2\,dx+Cf_1^2. \tag{3.60}$$

From (3.55)–(3.60), using the fact that

$$\int_0^M\left\{\frac{1}{2}(2c_1+c_2)\rho^{1+\theta}\big[\big(r^{n-1}u\big)_x\big]^2-c_1(n-1)\rho^\theta\big(r^{n-2}u^2\big)_x\right\}dx$$

$$=\int_0^M\left\{\frac{1}{2}\left(\frac{2}{n}c_1+c_2\right)\rho^{1+\theta}\big[\big(r^{n-1}u\big)_x\big]^2+\frac{(n-1)}{n}c_1\rho^\theta\left(r^{n-1}u_x-\frac{u}{r\rho}\right)^2\right\}dx,$$

we have

$$\int_0^M\frac{1}{2}u_t^2\,dx+\frac{d}{dt}\int_0^M\left\{\frac{1}{2}\left(\frac{2}{n}c_1+c_2\right)\rho^{1+\theta}\big(r^{n-1}u\big)_x^2+\frac{(n-1)}{n}c_1\rho^\theta\left(r^{n-1}u_x-\frac{u}{r\rho}\right)^2\right\}dx$$

$$\leqslant \frac{d}{dt}\left\{\int_0^M\left[\left(A\rho^\gamma - A\rho_\infty^\gamma - \Delta P\right)\left(r^{n-1}u\right)_x + \left(r^{n-1}u\right)_x\int_x^M\left(\frac{Gy}{r_\infty^{2n-2}} - \frac{Gy}{r^{2n-2}}\right)dy\right]dx\right\}$$

$$+ C\left(1 + \left\|\left(r^{n-1}u\right)_x\right\|_{L_x^\infty}\right)\left(\left\|\left(r^{n-1}u\right)_x\right\|_{L_x^2}^2 + \left\|\frac{u}{r}\right\|_{L_x^2}^2 + f_1^2 + \left|(\Delta P)'\right|^2\right). \tag{3.61}$$

Integrating (3.61) over $[0,t]$, using the estimates (3.13)–(3.15) and the Cauchy–Schwarz inequality, we can obtain (3.54). $\square$

Lemma 3.8. *Under the assumptions of Lemma* 3.3*, we obtain*

$$\int_0^M \left(r^\alpha u_t^2\right)(x,t)\,dx + \int_0^t\int_0^M \left(r^{2n-2+\alpha}u_{xt}^2 + r^{\alpha-2}u_t^2\right)dx\,ds \leqslant C_{19}\epsilon_0^2, \tag{3.62}$$

$$\left\|\frac{u}{r}(\cdot,t)\right\|_{L^\infty} + \left\|\left(r^{n-1}u\right)_x(\cdot,t)\right\|_{L^\infty} \leqslant C_{20}\epsilon_0, \tag{3.63}$$

$$\int_0^M\left(\frac{u^2}{r^2} + r^{2n-2}u_x^2\right)(x,t)\,dx + \int_0^t\int_0^M u_t^2(x,s)\,dx\,ds \leqslant C_{21}\epsilon_0^2, \tag{3.64}$$

for all $t \in [0,T]$*, where* $\alpha = \frac{3}{2} - n$*.*

Proof. We differentiate Eq. $(1.20)_2$ with respect to t, multiply it by $u_t r^\alpha$ and integrate it over $[0,M]$, using the boundary conditions (1.22)–(1.23), then derive

$$\begin{aligned}
&\frac{d}{dt}\int_0^M \frac{1}{2}r^\alpha u_t^2\,dx - \frac{\alpha}{2}\int_0^M r^{\alpha-1}uu_t^2\,dx \\
&= -\int_0^M\left[(2c_1+c_2)\rho^{1+\theta}\left(r^{n-1}u\right)_x - A\rho^\gamma + P_\infty - 2c_1(n-1)\rho^\theta\frac{u}{r}\right] \\
&\quad\times\left((n-1)r^{n-2+\alpha}uu_t\right)_x dx - \int_0^M \partial_t\left[(2c_1+c_2)\rho^{1+\theta}\left(r^{n-1}u\right)_x - A\rho^\gamma\right. \\
&\quad\left. + A\rho_\infty^\gamma - 2c_1(n-1)\rho^\theta\frac{u}{r}\right]\left(r^{n-1+\alpha}u_t\right)_x dx \\
&\quad+ \int_0^M 2c_1(n-1)\partial_t\left(r^{n-1}\rho^\theta\partial_x\left(\frac{u}{r}\right)\right)r^\alpha u_t\,dx\,ds - \int_0^M f_t r^\alpha u_t\,dx \\
&\quad- \left[(n-1)\Delta P\left(r^{n-2+\alpha}uu_t\right)(M,t) + (\Delta P)'\left(r^{n-1+\alpha}u_t\right)(M,t)\right] \\
&:= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{3.65}$$

From (3.12), (3.35) and (3.36), using the Cauchy–Schwarz inequality, we obtain

$$J_1 \leqslant \epsilon \int_0^M \left(r^{\alpha-2}u_t^2 + r^{2n-2+\alpha}u_{xt}^2\right) dx$$
$$+ C_\epsilon\left(1 + \left\|\left(r^{n-1}u\right)_x\right\|_{L_x^\infty}^2\right) \int_0^M \left[r^{2n-2+\alpha}u_x^2 + r^{\alpha-2}u^2 + r^\alpha(\rho-\rho_\infty)^2\right] dx. \quad (3.66)$$

From (3.12)–(3.13), using the same argument in the proof of (3.44) and the Cauchy–Schwarz inequality, we get

$$J_2 + J_3 = -\int_0^M \left[(2c_1+c_2)\rho^{1+\theta}\left(r^{n-1}u_t\right)_x\left(r^{n-1+\alpha}u_t\right)_x - 2c_1(n-1)\rho^\theta\left(r^{n-2+\alpha}u_t^2\right)_x\right] dx$$
$$+ \int_0^M \left\{(2c_1+c_2)(1+\theta)\rho^{\theta+2}\left[\left(r^{n-1}u\right)_x\right]^2 - (n-1)(2c_1+c_2)\rho^{1+\theta}\left(r^{n-2}u^2\right)_x\right.$$
$$\left. - \gamma\rho^{\gamma+1}\left(r^{n-1}u\right)_x - 2c_1(n-1)\theta\rho^{\theta+1}\left(r^{n-1}u\right)_x\frac{u}{r} - 2c_1(n-1)\rho^\theta\frac{u^2}{r^2}\right\}$$
$$\times\left[(n-1+\alpha)\frac{r^{\alpha-1}u_t}{\rho} + r^{n-1+\alpha}u_{tx}\right] dx$$
$$+ 2c_1(n-1)\int_0^M \left\{(n-1)r^{n-2+\alpha}u\rho^\theta\left(\frac{u}{r}\right)_x u_t\right.$$
$$\left. - \theta r^{n-1+\alpha}\rho^{\theta+1}\left(r^{n-1}u\right)_x\left(\frac{u}{r}\right)_x u_t - r^{n-1+\alpha}\rho^\theta\left(\frac{u^2}{r^2}\right)_x u_t\right\} dx$$
$$\leqslant -(C_{22}-\epsilon)\int_0^M \left(r^{2n-2+\alpha}u_{xt}^2 + r^{\alpha-2}u_t^2\right) dx$$
$$+ C_\epsilon\left(1 + \left\|\left(r^{n-1}u\right)_x\right\|_{L_x^\infty}^2\right) \int_0^M \left[r^{2n-2+\alpha}u_x^2 + r^{\alpha-2}u^2 + r^\alpha(\rho-\rho_\infty)^2\right] dx, \quad (3.67)$$

$$J_4 \leqslant \epsilon\int_0^M r^{\alpha-2}u_t^2\,dx + C_\epsilon\int_0^M \left((1-n)Gxr^{-n}u + \partial_r\Delta f u + \partial_t\Delta f\right)^2 r^{2+\alpha}\,dx$$
$$\leqslant \epsilon\int_0^M r^{\alpha-2}u_t^2\,dx\,ds + C_\epsilon\left(f_2^2 + \int_0^M r^{2+\alpha}u^2\,dx\right) \quad (3.68)$$

and

$$J_5 = -\int_0^M \big[(\Delta P)'\big(r^{n-1+\alpha}u_t\big)_x + (n-1)\Delta P\big(r^{n-2+\alpha}uu_t\big)_x\big]\,dx$$
$$\leqslant \epsilon \int_0^M \big(r^{\alpha-2}u_t^2 + r^{2n-2+\alpha}u_{xt}^2\big)\,dx + C_\epsilon\big(|\Delta P|^2 + \big|(\Delta P)'\big|^2\big). \tag{3.69}$$

Let $\epsilon = \frac{1}{8}C_{22}$, from (3.65)–(3.69), we have

$$\frac{d}{dt}\int_0^M r^\alpha u_t^2\,dx + \frac{C_{22}}{2}\int_0^M \big(r^{2n-2+\alpha}u_{xt}^2 + r^{\alpha-2}u_t^2\big)\,dx$$
$$\leqslant C\big(1 + \big\|\big(r^{n-1}u\big)_x\big\|_{L_x^\infty}^2\big)\int_0^M \big[r^{2n-2+\alpha}u_x^2 + r^{\alpha-2}u^2 + r^\alpha(\rho-\rho_\infty)^2\big]\,dx$$
$$+ C\big(f_2^2 + |\Delta P|^2 + \big|(\Delta P)'\big|^2\big) + C_{23}\epsilon_1\frac{C_{22}}{2}\int_0^M r^{\alpha-2}u_t^2\,dx.$$

If $C_{23}\epsilon_1 \leqslant \frac{1}{2}$, from (3.35) and (3.36), we can obtain

$$\frac{d}{dt}\int_0^M r^\alpha u_t^2\,dx + \frac{C_{22}}{4}\int_0^M \big(r^{2n-2+\alpha}u_{xt}^2 + r^{\alpha-2}u_t^2\big)\,dx$$
$$\leqslant C\big(1 + \big\|\big(r^{n-1}u\big)_x\big\|_{L_x^\infty}^2\big)\int_0^M \big[r^{2n-2+\alpha}u_x^2 + r^{\alpha-2}u^2 + r^\alpha(\rho-\rho_\infty)^2\big]\,dx$$
$$+ C\big(f_2^2 + |\Delta P|^2 + \big|(\Delta P)'\big|^2\big) \tag{3.70}$$

and

$$\int_0^M \big(r^\alpha u_t^2\big)(x,t)\,dx + \int_0^t\int_0^M \big(r^{2n-2+\alpha}u_{xt}^2 + r^{\alpha-2}u_t^2\big)(x,s)\,dx\,ds$$
$$\leqslant C\epsilon_0^2\big(1 + \big\|\big(r^{n-1}u\big)_x\big\|_{L_{tx}^\infty}^2\big). \tag{3.71}$$

From Eq. $(1.20)_2$, we have

$$(2c_1+c_2)r^{n-1}\partial_x\big(\rho^{1+\theta}\partial_x\big(r^{n-1}u\big)\big) = u_t + Ar^{n-1}\big(\rho^\gamma\big)_x + 2c_1(n-1)r^{n-2}u\big(\rho^\theta\big)_x + f,$$

and using the estimates (3.11), (3.12)–(3.13), (3.35)–(3.37) and (3.71), conclude that

$$\int_0^M r^{2n-2+\alpha}\left[\partial_x\left(\rho^{1+\theta}\partial_x\left(r^{n-1}u\right)\right)\right]^2 dx \leqslant C\epsilon_0^2\left(1+\left\|\left(r^{n-1}u\right)_x\right\|_{L_{tx}^\infty}^2\right)$$

and

$$\int_0^M \left|\partial_x\left(\rho^{1+\theta}\partial_x\left(r^{n-1}u\right)\right)\right| dx \leqslant C\epsilon_0\left(1+\left\|\left(r^{n-1}u\right)_x\right\|_{L_{tx}^\infty}\right), \tag{3.72}$$

for all $t \in [0,T]$. From (3.12), (3.54) and (3.72), using Sobolev's embedding theorem $W^{1,1} \hookrightarrow L^\infty$, we can obtain

$$\left\|\partial_x\left(r^{n-1}u\right)\right\|_{L_{tx}^\infty} \leqslant C_{24}\epsilon_0\left(1+\left\|\left(r^{n-1}u\right)_x\right\|_{L_{tx}^\infty}\right). \tag{3.73}$$

If $C_{24}\epsilon_0 \leqslant \frac{1}{2}$, from (3.54), (3.71) and (3.73), we can get (3.62)–(3.64) immediately. □

Now, we can let

$$\epsilon_1 = (C_{16}+C_{20})\epsilon_0. \tag{3.74}$$

If $(1+\frac{4}{\underline{\rho}}+\frac{C_2}{\delta_5}+\frac{C_4}{C_5}+4C_8+2C_{14}^2+4C_{17}+2C_{23})\epsilon_1+2C_{24}\epsilon_0 \leqslant 1$, using the results in Lemmas 3.3–3.8, we finish the proof of Claim 1. Thus, we can let ϵ_0 be a positive constant satisfying

$$\left(1+\frac{4}{\underline{\rho}}+\frac{C_2}{\delta_5}+\frac{C_4}{C_5}+4C_8+2C_{14}^2+4C_{17}+2C_{23}\right)(C_{16}+C_{20})\epsilon_0+2C_{24}\epsilon_0 = 1. \tag{3.75}$$

Using the classical continuity method, we can obtain the following lemma.

Lemma 3.9. *Under the assumptions in Theorem* 1.1*, the solution* (ρ,u) *satisfies the estimates* (3.12)–(3.15)*,* (3.20)*,* (3.29)*,* (3.34)–(3.39)*,* (3.62)–(3.64) *for all* $t \geqslant 0$.

From Lemma 3.9, we can obtain the following lemma easily.

Lemma 3.10. *Under the assumptions in Theorem* 1.1*, if* ϵ_0 *is small enough, we have*

$$\begin{gathered}
\left\|\rho(\cdot,t_1)-\rho(\cdot,t_2)\right\|_{L^2} \leqslant C|t_1-t_2|,\\
\left\|u(\cdot,t_1)-u(\cdot,t_2)\right\|_{L^2} \leqslant C|t_1-t_2|,\\
\left\|r(\cdot,t_1)-r(\cdot,t_2)\right\|_{L^\infty} \leqslant C|t_1-t_2|,\\
\left\|\partial_x\left(r^{n-1}u\right)(\cdot,t_1)-\partial_x\left(r^{n-1}u\right)(\cdot,t_2)\right\|_{L^2} \leqslant C|t_1-t_2|^{\frac{1}{2}},\\
\left\|\left(\left(r^{n-2}u\right)_x,\left(r^{n-1}\right)_x\right)(\cdot,t)\right\|_{L^{n-\frac{1}{2}}} \leqslant C,
\end{gathered}$$

for all $t_1,t_2,t \geqslant 0$.

4. Difference scheme and approximate solutions

In this section, applying a discrete difference scheme as in [1], we construct approximate solutions to the initial–boundary value problem (1.20)–(1.23).

For any given positive integer N, let $h = \frac{1}{N}$ be an increment in x and $x_j = jh$ for $j \in \{0, \dots, N\}$. For each integer N, we construct the following time-dependent functions:

$$\big(\rho_j(t), u_j(t), r_j(t)\big), \quad j = 0, \dots, N,$$

that form a discrete approximation to $(\rho, u, r)(x_j, t)$ for $j = 0, \dots, N$.

First, $\rho_i(t)$, $u_j(t)$ and $r_{i+1}(t)$, $i = 0, \dots, N$, $j = 1, \dots, N$, are determined by the following system of $3N + 2$ differential equations:

$$\frac{d}{dt}\rho_i = -\rho_i^2 \delta\big(r_i^{n-1} u_i\big), \tag{4.1}$$

$$\frac{d}{dt}u_j = r_j^{n-1}\delta\sigma_j - 2(n-1)r_j^{n-2}u_j\delta(\mu_{j-1}) - f_j, \tag{4.2}$$

$$\frac{d}{dt}r_{i+1} = u_{i+1}, \tag{4.3}$$

with the boundary conditions:

$$u_0(t) = 0, \qquad r_0^n(t) = h, \tag{4.4}$$

$$P_N - \rho_N(\lambda_N + 2\mu_N)\delta\big(r_N^{n-1}u_N\big) + 2(n-1)\frac{u_{N+1}}{r_{N+1}}\mu_N = P_\Gamma, \tag{4.5}$$

and initial data

$$(\rho_j, u_j)(0) = \left(\frac{1}{h}\int_{(j-1)h}^{jh} \rho_0(y)\,dy, \frac{1}{h}\int_{(j-1)h}^{jh} u_0(y)\,dy\right), \quad j = 1, \dots, N, \tag{4.6}$$

$$\rho_0(0) = \rho_1(0), \qquad u_0(0) = 0, \qquad r_0^n(0) = h, \tag{4.7}$$

$$r_i^n(0) = h + n\sum_{l=0}^{i-1}\frac{h}{\rho_l(0)}, \quad i = 1, \dots, N+1, \tag{4.8}$$

and $u_{N+1}(0)$ satisfies

$$P_N(0) - \rho_N(0)\big(\lambda_N(0) + 2\mu_N(0)\big)\delta\big(r_N^{n-1}(0)u_N(0)\big) + 2(n-1)\frac{u_{N+1}(0)}{r_{N+1}(0)}\mu_N(0) = P_\Gamma(0), \tag{4.9}$$

where δ is the operator defined by $\delta w_j = (w_{j+1} - w_j)/h$, and

$$\sigma_j(t) = \rho_{j-1}(\lambda_{j-1} + 2\mu_{j-1})\delta\big(r_{j-1}^{n-1}u_{j-1}\big) - P_{j-1},$$

$$\lambda_j = \lambda(\rho_j), \qquad \mu_j = \mu(\rho_j), \qquad P_j = P(\rho_j),$$

$$f_j(t) = f(jh, r_j, t).$$

The boundary conditions (4.4)–(4.5) are consistent with the initial data. The condition (4.5) determines $u_{N+1}(t)$.

Let $(\rho_{\infty,i}, r^n_{\infty,i}) = (\rho_\infty(ih), h + r^n_\infty(ih))$, $i = 0, \dots, N$, we have

$$r^{n-1}_{\infty,j}\delta\big(A\rho^\gamma_{\infty,j-1}\big) = -\frac{Gjh}{r^{n-1}_{\infty,j}} + Q_{1j},$$

$$r^n_{\infty,j} = h + n\sum_{k=0}^{j-1}\frac{h}{\rho_{\infty,k}} + Q_{2j},$$

and

$$|Q_{1j}| \leqslant C(jh)^{\frac{1-n}{n}}h, \qquad |Q_{2j}| \leqslant C(jh)^{\frac{2}{n}}h, \quad j = 1, \dots, N.$$

Then, for any small h, the initial data $(\rho_i, u_i, r_i)(0)$ and the external force f_i, $i = 0, \dots, N$, satisfy

$$\max_{i\in\{0,\dots,N\}}\big|\rho_i(0) - \rho_{\infty,i}\big|^2 + \sum_{j=0}^{N-1}\big[r_j^{2n-2+\alpha}(\delta\rho_j - \delta\rho_{\infty,j})^2\big](0)h \leqslant C\epsilon_0^2, \tag{4.10}$$

$$C^{-1}(i+1)h \leqslant r_i^n(0) \leqslant C(i+1)h, \qquad \sum_{j=0}^{N}\big[r_j^{-2}u_j^2 + r_j^{2n-2}(\delta u_j)^2\big](0)h \leqslant C\epsilon_0^2, \tag{4.11}$$

$$\sum_{j=1}^{N}\big(r_j^{2n-2\alpha}\big[\delta\big(\rho_{j-1}^{1+\theta}\delta\big(r_{j-1}^{n-1}u_{j-1}\big)\big)\big]^2\big)(0)h \leqslant C\epsilon_0^2, \tag{4.12}$$

where $C > 0$ is independent of h.

The basic theory of differential equations guarantees the local existence of smooth solutions (ρ_i, u_i, r_i) $(i = 0, \dots, N)$ to the Cauchy problem (4.1)–(4.9) on an interval $[0, T^h)$, such that

$$0 < \rho_i(t) < \infty, \qquad \big|u_i(t)\big|, \big|r_i(t)\big| < \infty, \quad i = 0, \dots, N,$$

with the aid of (4.10)–(4.12).

For any fixed $T > 0$, by virtue of Lemmas 3.1–3.10 and using similar arguments as in [1,7], we can obtain the following lemma and prove that the Cauchy problem (4.1)–(4.9) has a unique solution for $t \in [0, T]$ when $h \leqslant h_{T,\epsilon_0}$, where $h_{T,\epsilon_0} > 0$ is a constant dependent on T and ϵ_0.

Lemma 4.1. *For any $h \in (0, h_{T,\epsilon_0}]$, there exists a positive constant C independent of h such that*

$$\rho_i(t) \in \left[\frac{1}{2}\underline{\rho}, \frac{3}{2}\overline{\rho}\right], \qquad \big\|\rho_i(t) - \rho_{\infty,i}\big\|_{L^\infty} \leqslant C\epsilon_0,$$

$$r_l^n(t) \in \big[C^{-1}(l+1)h, C(l+1)h\big],$$

$$\sum_{j=0}^{N}\big(u_j^2(t) + \big|\rho_j(t) - \rho_{\infty,j}\big|^2\big)h \leqslant C\epsilon_0^2,$$

$$\left|\frac{u_l(t)}{r_l(t)}\right| \leqslant C\epsilon_0,$$

$$\int_0^t \sum_{j=0}^N \left(u_j^2 + r_j^{2n-2}(\delta u_j)^2 + \frac{u_j^2}{r_j^2}\right)(s)h\,ds \leqslant C\epsilon_0^2,$$

$$\sum_{j=0}^N \left(\frac{u_j^2}{r_j^2} + r_j^{2n-2}(\delta u_j)^2\right)(t)h + \int_0^t \sum_{j=0}^N \left(\frac{d}{dt}u_j\right)^2(s)h\,ds \leqslant C\epsilon_0^2,$$

$$\int_0^t \sum_{j=0}^N \left[r_j^\alpha(\rho_j - \rho_{\infty,j})^2 + (r_j - r_{\infty,j})^2\right]h\,ds \leqslant C\epsilon_0^2,$$

$$\sum_{j=0}^N \left(r_j^\alpha u_j^2\right)(t)\,dx + \int_0^t \sum_{j=0}^N r_j^\alpha\left(r_j^{2n-2}(\delta u_j)^2 + \frac{u_j^2}{r_j^2}\right)h\,ds \leqslant C\epsilon_0^2,$$

$$\sum_{j=0}^{N-1} \left(r_j^{2n-2+\alpha}(\delta\rho_j - \delta\rho_{\infty,j})^2\right)(t)h + \int_0^t \sum_{j=0}^{N-1} \left(r_j^{2n-2+\alpha}(\delta\rho_j - \delta\rho_{\infty,j})^2\right)(s)h\,ds \leqslant C\epsilon_0^2,$$

$$\sum_{j=0}^{N-1} |\delta\rho_j - \delta\rho_{\infty,j}|(t)h \leqslant C\epsilon_0,$$

$$\sum_{j=0}^N \left[r_j^\alpha\left(\frac{d}{dt}u_j\right)^2\right](t)h \leqslant C\epsilon_0^2,$$

$$\left|\delta\left(r_i^{n-1}u_i\right)(t)\right| \leqslant C\epsilon_0,$$

$$\sum_{j=0}^N \left(\left|\rho_j(t_1) - \rho_j(t_2)\right|^2 + \left|u_j(t_1) - u_j(t_2)\right|^2\right)h \leqslant C|t_1 - t_2|^2,$$

$$\left|r_l(t_1) - r_l(t_2)\right| \leqslant C|t_1 - t_2|,$$

$$\sum_{j=0}^N \left|\delta\left(r_j^{n-1}u_j\right)(t_1) - \delta\left(r_j^{n-1}u_j\right)(t_2)\right|^2 h \leqslant C|t_1 - t_2|,$$

$$\sum_{j=0}^N \left(\left|\delta\left(r_j^{n-2}u_j\right)\right|^{n-\frac{1}{2}} + \left|\delta\left(r_j^{n-1}\right)\right|^{n-\frac{1}{2}}\right)h \leqslant C,$$

for all $t_1, t_2, t \in [0, T]$, $i \in \{0, \ldots, N\}$ *and* $l \in \{1, \ldots, N+1\}$.

Now, we can define our approximate solutions $(\rho^N, u^N, r^N)(x,t)$ for the Cauchy problem (1.20)–(1.23). For each fixed N and $t \in [0,T]$, we define piecewise linear continuous functions $(\rho^N, u^N, r^N)(x,t)$ with respect to x as follows: when $x \in [[xN], [xN]+1]$

$$\rho^N(x,t)=\rho_{[xN]}(t)+\big(xN-[xN]\big)\big(\rho_{[xN]+1}(t)-\rho_{[xN]}(t)\big),$$
$$u^N(x,t)=u_{[xN]}(t)+\big(xN-[xN]\big)\big(u_{[xN]+1}(t)-u_{[xN]}(t)\big),$$
$$r^N(x,t)=\big(r^n_{[xN]}(t)+\big(xN-[xN]\big)\big(r^n_{[xN]+1}(t)-r^n_{[xN]}(t)\big)\big)^{\frac{1}{n}}.$$

From Lemma 4.1, using similar arguments as in [1,7], we can obtain the compactness of approximate solutions (ρ^N, u^N, r^N) and prove the existence part of Theorem 1.1. Since the constant C in Lemma 4.1 is independent of T, we can obtain the regularity estimates (1.29)–(1.31) easily.

5. Uniqueness

In this section, applying energy method, we will prove the uniqueness of the solution in Theorem 1.1. Let $(\rho_1,u_1,r_1)(x,t)$ and $(\rho_2,u_2,r_2)(x,t)$ be two solutions in Theorem 1.1. Then we have, $i=1,2$, $(x,t)\in[0,M]\times[0,T]$,

$$\rho_i(x,t)\in\left[\frac{1}{2}\underline{\rho},\frac{3}{2}\overline{\rho}\right],\qquad C^{-1}x^{\frac{1}{n}}\leqslant r_i(x,t)\leqslant Cx^{\frac{1}{n}},\tag{5.1}$$
$$\big|x^{-\frac{1}{n}}u_i(x,t)\big|+\big|x^{\frac{n-1}{n}}\partial_x u_i(x,t)\big|\leqslant C.\tag{5.2}$$

For simplicity, we may assume that $(\rho_1,u_1,r_1)(x,t)$ and $(\rho_2,u_2,r_2)(x,t)$ are suitably smooth since the following estimates are valid for the solutions with the regularity indicated in Theorem 1.1 by using the Friedrichs mollifier.

Let

$$\varrho=\rho_1-\rho_2,\qquad w=u_1-u_2,\qquad R=r_1-r_2.$$

From (3.1), we have

$$\begin{aligned}\frac{d}{dt}\int_0^M x^{-\frac{2}{n}}R^2(x,t)\,dt&=2\int_0^M x^{-\frac{2}{n}}Rw\,dx\\&\leqslant\epsilon\int_0^M x^{-\frac{2}{n}}w^2\,dx+C_\epsilon\int_0^M x^{-\frac{2}{n}}R^2\,dx.\end{aligned}\tag{5.3}$$

From (1.20) and (5.1)–(5.2), we have

$$\begin{aligned}\frac{d}{dt}\int_0^M \varrho^2(x,t)\,dt&=2\int_0^M\varrho\partial_t(\rho_1-\rho_2)\,dx\\&=2\int_0^M\varrho\left(-\rho_1^2r_1^{n-1}\partial_x u_1+\rho_2^2r_2^{n-1}\partial_x u_2-(n-1)\frac{\rho_1u_1}{r_1}+(n-1)\frac{\rho_2u_2}{r_2}\right)dx\\&\leqslant\epsilon\int_0^M\big(x^{\frac{2n-2}{n}}w_x^2+x^{-\frac{2}{n}}w^2\big)\,dx+C_\epsilon\int_0^M\big(\varrho^2+x^{-\frac{2}{n}}R^2\big)\,dx.\end{aligned}\tag{5.4}$$

From Eq. $(1.20)_2$ and boundary conditions (1.22)–(1.23), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_0^M \frac{1}{2}w^2(x,t)\,dx\\
&\quad+\int_0^M \left\{\left(\frac{2}{n}c_1+c_2\right)\rho_1^{1+\theta}\left[\left(r_1^{n-1}w\right)_x\right]^2+\frac{2(n-1)}{n}c_1\rho_1^{1+\theta}\left(r_1^{n-1}w_x-\frac{w}{r_1\rho_1}\right)^2\right\}dx\\
&=-\int_0^M \partial_x\left(r_1^{n-1}w\right)\left[(2c_1+c_2)\left(\rho_1^{1+\theta}-\rho_2^{1+\theta}\right)\partial_x\left(r_1^{n-1}u_2\right)\right.\\
&\quad\left.+(2c_1+c_2)\rho_2^{1+\theta}\partial_x\left(\left(r_1^{n-1}-r_2^{n-1}\right)u_2\right)-\left(\rho_1^{\gamma}-\rho_2^{\gamma}\right)\right]dx\\
&\quad+\int_0^M 2c_1(n-1)\partial_x\left[r_1^{n-1}wu_2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\rho_1^{\theta}\,dx\\
&\quad+\int_0^M 2c_1(n-1)\partial_x\left[r_1^{n-1}w\left(\frac{u_2}{r_2}\right)\right]\left(\rho_1^{\theta}-\rho_2^{\theta}\right)dx\\
&\quad+\int_0^M 2c_1(n-1)\rho_2^{\theta}\partial_x\left[\left(r_1^{n-1}-r_2^{n-1}\right)w\frac{u_2}{r_2}\right]dx\\
&\quad-\int_0^M \partial_x\left(\left(r_1^{n-1}-r_2^{n-2}\right)w\right)\left[(2c_1+c_2)\rho_2^{1+\theta}\partial_x\left(r_2^{n-1}u_2\right)-\rho_2^{\gamma}\right]dx\\
&\quad+\int_0^M wGx\left(r_2^{1-n}-r_1^{1-n}\right)dx+\int_0^M w\left(\Delta f(x,r_2,t)-\Delta f(x,r_1,t)\right)dx.
\end{aligned}\tag{5.5}
$$

From (5.1)–(5.2) and (5.5), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_0^M \frac{1}{2}w^2(x,t)\,dx+C_{22}\int_0^M\left\{x^{\frac{2n-2}{n}}w_x^2+x^{-\frac{2}{n}}w^2\right\}dx\\
&\quad\leqslant C\int_0^M\left(x^{-\frac{2}{n}}R^2+\varrho^2+w^2\right)dx.
\end{aligned}\tag{5.6}
$$

From (5.3)–(5.4) and (5.6), letting $\epsilon = \frac{1}{4}C_{22}$, we obtain

$$\frac{d}{dt}\int_0^M \left[w^2+\varrho^2+x^{-\frac{2}{n}}R^2\right]dx \leqslant C\int_0^M \left(x^{-\frac{2}{n}}R^2+\varrho^2+w^2\right)dx.$$

Using Gronwall's inequality, we have for any $t \in [0,T]$,

$$\int_0^M \left[w^2+\varrho^2+x^{-\frac{2}{n}}R^2\right]dx = 0.$$

This proves the uniqueness of solution in Theorem 1.1.

6. Asymptotic behavior

In this section, we consider the asymptotic behavior of the solution to the free boundary problem (1.20)–(1.23). We will show that the solution to the free boundary problem tends to the stationary solution as $t \to +\infty$.

The following lemma is proved in [15].

Lemma 6.1. *Suppose that $y \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+)$ satisfies*

$$y = y_1' + y_2,$$

and

$$|y_2| \leqslant \sum_{i=1}^n \alpha_i, \qquad |y'| \leqslant \sum_{i=1}^n \beta_i, \quad \text{on } \mathbb{R}^+,$$

where $y_1 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+)$, and $\lim_{s\to+\infty} y_1(s) = 0$ and $\alpha_i, \beta_i \in L^{p_i}(\mathbb{R}^+)$ for some $p_i \in [1,\infty)$, $i = 1,\dots,m$. Then $\lim_{s\to+\infty} y(s) = 0$.

Proposition 6.1. *Under the assumptions of Theorem* 1.1*, the total kinetic energy*

$$E(t) := \int_0^M \frac{1}{2}u^2(x,t)\,dx \to 0 \quad \text{as } t \to +\infty.$$

Proof. From (3.15) and Lemma 3.9, we have $E(t) \in L^1(\mathbb{R}^+)$. Using the Cauchy–Schwarz inequality, we obtain

$$\left|E'(t)\right| \leqslant E(t) + \int_0^M u_t^2\,dx.$$

Taking into account the estimate (3.64) and Lemma 3.9, applying Lemma 6.1, we finish the proof. □

Proposition 6.2. *Under the assumptions of Theorem* 1.1*, we have*

$$\int_0^M (r - r_\infty)^2(x,t)\,dx \to 0 \quad \text{as } t \to +\infty.$$

Proof. From (3.20) and Lemma 3.9, we have $\int_0^M (r - r_\infty)^2(x,t)\,dx \in L^1(\mathbb{R}^+)$. Using the Cauchy–Schwarz inequality, we obtain

$$\left|\frac{d}{dt}\int_0^M (r - r_\infty)^2\,dx\right| = \left|2\int_0^M (r - r_\infty)u\,dx\right| \leqslant 2E(t) + \int_0^M (r - r_\infty)^2\,dx.$$

Taking into account the estimate $E(t) \in L^1(\mathbb{R}^+)$, applying Lemma 6.1, we finish the proof. □

Proposition 6.3. *Under the assumptions of Theorem* 1.1*, we have*

$$\int_0^M (\rho - \rho_\infty)^2(x,t)\,dx \to 0 \tag{6.1}$$

and

$$\left\|(\rho - \rho_\infty)(\cdot,t)\right\|_{L^q} \to 0, \quad q \in [1,\infty), \tag{6.2}$$

as $t \to +\infty$.

Proof. From (3.20) and Lemma 3.9, we have $\int_0^M (\rho - \rho_\infty)^2(x,t)\,dx \in L^1(\mathbb{R}^+)$. From (1.29), using the Cauchy–Schwarz inequality, we obtain

$$\left|\frac{d}{dt}\int_0^M (\rho - \rho_\infty)^2\,dx\right| \leqslant \int_0^M (\rho - \rho_\infty)^2\,dx + C\int_0^M \left(r^{n-1}u\right)_x^2\,dx.$$

Taking into account the estimate (3.15) and Lemma 3.9, applying Lemma 6.1, we obtain (6.1). From (1.29), (3.3) and (6.1), we can obtain (6.2) easily. □

Proposition 6.4. *Under the assumptions of Theorem* 1.1*, we have*

$$\int_0^M x^{\frac{2n-2+\alpha}{n}}\left(\left(\rho^\theta\right)_x - \left(\rho_\infty^\theta\right)_x\right)^2(x,t)\,dx \to 0, \quad \text{as } t \to +\infty.$$

Proof. From (1.29), (3.35), (3.37) and Lemma 3.9, we have

$$\int_0^M x^{\frac{2n-2+\alpha}{n}} \big((\rho^\theta)_x - (\rho_\infty^\theta)_x\big)^2(x,t)\,dx \in L^1(\mathbb{R}^+).$$

From (1.20), (1.29)–(1.30) and (3.3), using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\left|\frac{d}{dt}\int_0^M x^{\frac{2n-2+\alpha}{n}} \big((\rho^\theta)_x - (\rho_\infty^\theta)_x\big)^2\,dx\right| \\
&\quad = 2\theta\left|\int_0^M x^{\frac{2n-2+\alpha}{n}} \big((\rho^\theta)_x - (\rho_\infty^\theta)_x\big)\big(\rho^{\theta+1}\partial_x\big(r^{n-1}u\big)\big)_x\,dx\right| \\
&\quad = \frac{2\theta}{2c_1+c_2}\left|\int_0^M x^{\frac{2n-2+\alpha}{n}} \big((\rho^\theta)_x - (\rho_\infty^\theta)_x\big)\left(\frac{u_t}{r^{n-1}} + A\big(\rho^\gamma\big)_x\right.\right. \\
&\qquad \left.\left. + 2c_1(n-1)\frac{u(\rho^\theta)_x}{r} + \frac{f(x,r,t)}{r^{n-1}}\right)dx\right| \\
&\quad \leqslant C\int_0^M \Big[x^{\frac{2n-2+\alpha}{n}} \big((\rho^\theta)_x - (\rho_\infty^\theta)_x\big)^2 + r^\alpha u_t^2 + r^\alpha(r-r_\infty)^2 \\
&\qquad + r^\alpha(\rho-\rho_\infty)^2 + r^{\alpha-2}u^2\Big]\,dx + f_1^2.
\end{aligned}$$

Taking into account the estimates (1.29), (3.35)–(3.37), (3.62) and Lemma 3.9, applying Lemma 6.1, we end the proof. □

From Propositions 6.3 and 6.4, using Sobolev's embedding theorem, we can obtain the following corollary immediately.

Corollary 6.1. *Under the assumptions of Theorem* 1.1*, we have*

$$\big\|\rho(\cdot,t) - \rho_\infty(\cdot)\big\|_{L^\infty} + \big\|r(\cdot,t) - r_\infty(\cdot)\big\|_{L^\infty} \to 0, \quad \text{as } t \to +\infty.$$

Proposition 6.5. *Under the assumptions of Theorem* 1.1*, we have*

$$\int_0^M x^{\frac{2n-2+\alpha}{n}} u_x^2(x,t)\,dx \to 0, \quad \text{as } t \to +\infty.$$

Proof. From the estimates (1.29), (3.36) and Lemma 3.9, we have

$$\int_0^M x^{\frac{2n-2+\alpha}{n}} u_x^2(x,t)\,dx \in L^1(\mathbb{R}^+).$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\frac{d}{dt}\int_0^M x^{\frac{2n-2+\alpha}{n}} u_x^2\,dx\right| &= \left|2\int_0^M x^{\frac{2n-2+\alpha}{n}} u_x u_{xt}\,dx\right| \\ &\leqslant \int_0^M x^{\frac{2n-2+\alpha}{n}} u_x^2\,dx + \int_0^M x^{\frac{2n-2+\alpha}{n}} u_{xt}^2\,dx.\end{aligned}$$

Taking into account the estimates (1.29), (3.36), (3.62) and Lemma 3.9, applying Lemma 6.1, we end the proof. □

From Propositions 6.1 and 6.5, using Sobolev's embedding theorem, we can obtain the following corollary immediately.

Corollary 6.2. *Under the assumptions of Theorem* 1.1*, we have*

$$\left\|u(\cdot,t)\right\|_{L^\infty} \to 0, \quad \text{as } t \to +\infty.$$

Thus, we finish the proof of Theorem 1.1.

7. Stabilization rate estimates

Now we are in position to estimate the stabilization rate. We first state the following proposition which gives the stabilization rate estimates in $L^2([0, M])$-norm of the solution.

Proposition 7.1. *Under the assumptions of Theorem* 1.3*, we have*

$$\int_0^M \left(u^2 + (\rho-\rho_\infty)^2 + x^{-2}\left(r^n - r_\infty^n\right)^2\right)dx \leqslant Ce^{-a_1 t} \tag{7.1}$$

and

$$\left|\rho(M,t)-\rho_\infty(M)\right| + \left(\int_0^M r^{2n-2}(\rho-\rho_\infty)_x^2\,dx\right)^{\frac{1}{2}} + \left\|r(\cdot,t)-r_\infty(x)\right\|_{L^2} \leqslant Ce^{-a_1 t}, \tag{7.2}$$

for all $t \geqslant 0$, where a_1 is a positive constant.

Proof. Let

$$V_1 = \int_0^M \frac{1}{2} u^2\,dx + S[V] - S[V_\infty],$$

$$W_1 = \int_0^M \left\{ \left(r^{n-1}u\right)_x^2 + r^{2n-2}u_x^2 + \frac{u^2}{r^2} \right\} dx.$$

From (1.32), (3.16)–(3.18), we have

$$V_1' + 2C_{31}W_1 \leqslant C f_1 V_1^{\frac{1}{2}} + C|\Delta P|^2 \leqslant C e^{-a_0 t} V_1^{\frac{1}{2}} + C e^{-2a_0 t}, \tag{7.3}$$

$$C_{32}^{-1} \int_0^M \left(u^2 + (\rho - \rho_\infty)^2 + x^{-2}\left(r^n - r_\infty^n\right)^2\right) dx$$
$$\leqslant V_1 \leqslant C_{32} \int_0^M \left(u^2 + (\rho - \rho_\infty)^2 + x^{-2}\left(r^n - r_\infty^n\right)^2\right) dx, \tag{7.4}$$

and

$$C_{33} \left\| u(\cdot, t) \right\|_{L^2} \leqslant W_1. \tag{7.5}$$

From (3.28), we have

$$\int_0^M \left[(\rho - \rho_\infty)^2 + x^{-2}\left(r^n - r_\infty^n\right)^2 \right] dx$$
$$\leqslant -C_{38} \frac{d}{dt} \int_0^M \frac{u}{r^{n-1}} \left(\frac{r^n}{n} - \frac{r_\infty^n}{n} \right) dx + C_{38} W_1 + C e^{-2a_0 t}. \tag{7.6}$$

From (1.29), we obtain

$$\left| C_{38} \int_0^M \frac{u}{r^{n-1}} \left(\frac{r^n}{n} - \frac{r_\infty^n}{n} \right) dx \right| \leqslant C_{39} \int_0^M \left(u^2 + |\rho - \rho_\infty|^2 \right) dx. \tag{7.7}$$

Let

$$V_2 = V_1 + \epsilon C_{38} \int_0^M \frac{u}{r^{n-1}} \left(\frac{r^n}{n} - \frac{r_\infty^n}{n} \right) dx,$$

$$W_2 = C_{31} W_1 + \epsilon \int_0^M \left[\left(\rho^\gamma - \rho_\infty^\gamma \right)^2 + x^{-2} \left(r^n - r_\infty^n \right)^2 \right] dx,$$

where $\epsilon = \min\{\frac{C_{31}}{C_{38}}, \frac{1}{2C_{32}C_{39}}\}$. From (7.3) and (7.4)–(7.7), we have

$$V_2' + W_2 \leqslant Ce^{-2a_0 t}, \tag{7.8}$$

$$\begin{aligned} & C_{39}^{-1} \int_0^M \left(u^2 + (\rho - \rho_\infty)^2 + x^{-2} \left(r^n - r_\infty^n \right)^2 \right) dx \\ & \quad \leqslant V_2 \leqslant C_{39} \int_0^M \left(u^2 + (\rho - \rho_\infty)^2 + x^{-2} \left(r^n - r_\infty^n \right)^2 \right) dx, \end{aligned} \tag{7.9}$$

and

$$C_{40} \int_0^M \left(u^2 + (\rho - \rho_\infty)^2 + x^{-2} \left(r^n - r_\infty^n \right)^2 \right) dx \leqslant W_2. \tag{7.10}$$

Thus V_2 is a Lyapunov functional. From (1.32), we obtain the estimate (7.1). From (1.29), (3.33), (3.40) and (7.1), we can get (7.2) easily. □

Proposition 7.2. *Under the assumptions of Theorem* 1.3*, we obtain*

$$\int_0^M \left(\frac{u^2}{r^2} + r^{2n-2} u_x^2 \right)(x,t)\, dx \leqslant Ce^{-a_3 t}, \tag{7.11}$$

for all $t \geqslant 0$, where a_3 is a positive constant.

Proof. Let

$$\begin{aligned} V_3 = \int_0^M \bigg\{ & \frac{1}{2} \left(\frac{2}{n} c_1 + c_2 \right) \rho^{1+\theta} \left[\left(r^{n-1} u \right)_x \right]^2 + \frac{(n-1)}{n} c_1 \rho^\theta \left(r^{n-1} u_x - \frac{u}{r\rho} \right)^2 \\ & + \left(A\rho_\infty^\gamma - A\rho^\gamma + \Delta P \right) \left(r^{n-1} u \right)_x + u \left(G \frac{x}{r^{n-1}} - G \frac{x r^{n-1}}{r_\infty^{2n-2}} \right) \bigg\} dx. \end{aligned}$$

From (1.30) and (3.61), we have

$$V_3' + \int_0^M \frac{1}{2} u_t^2(x,s)\,dx \leqslant C_{41}\big(f_1^2 + \big|(\Delta P)'\big|^2 + W_2\big).$$

From (3.13)–(3.14), we have

$$V_3 \geqslant \int_0^M \left\{ C_{42}\left(\big(r^{n-1}u\big)_x^2 + \frac{u^2}{r^2} + r^{2n-2}u_x^2 \right) - C_{43}\big((\rho-\rho_\infty)^2 + |\Delta P|^2\big) \right\} dx,$$

$$V_3 \leqslant \int_0^M \left\{ C_{44}\left(\big(r^{n-1}u\big)_x^2 + \frac{u^2}{r^2} + r^{2n-2}u_x^2 \right) + C_{43}\big((\rho-\rho_\infty)^2 + |\Delta P|^2\big) \right\} dx.$$

Letting $V_4 = V_2 + \eta V_3 + \eta C_{43}|\Delta P|^2$, where $\eta = \min\{\frac{1}{2}, \frac{1}{4C_{39}C_{43}}, \frac{1}{2C_{41}}\}$. From (7.8)–(7.10), we have

$$CW_2 \geqslant V_4 \geqslant C^{-1}W_1$$

and

$$V_4' + C^{-1}W_2 \leqslant C\big(f_1^2 + |\Delta P|^2 + \big|(\Delta P)'\big|^2\big).$$

Thus V_4 is a Lyapunov functional. From (1.32), we can obtain the estimate (7.11). □

Proposition 7.3. *Under the assumptions of Theorem* 1.3*, we obtain*

$$\int_0^M r^{\frac{1}{2}-m}(\rho-\rho_\infty)^2\,dx\,ds \leqslant Ce^{-at}, \tag{7.12}$$

$$\int_0^M r^{\frac{1}{2}-m}(r-r_\infty)^2\,dx\,ds \leqslant Ce^{-at}, \tag{7.13}$$

$$\int_0^M \big(r^{\frac{1}{2}-m}u^2\big)(x,t)\,dx \leqslant Ce^{-at}, \tag{7.14}$$

$$\int_0^M \big(r^{2n-2+\frac{1}{2}-m}(\rho-\rho_\infty)_x^2\big)(x,t)\,dx \leqslant Ce^{-at}, \tag{7.15}$$

for all $t \geqslant 0$ and $m = 0, 1, \ldots, n-1$, where a is a positive constant.

Proof. From (7.1)–(7.2), we know that the estimates (7.12)–(7.15) hold with $m = 0$.

Claim 3. *If* (7.12)–(7.15) *hold with* $m \leqslant k$, $k \in [0, n-2]$, *then the estimates* (7.12)–(7.15) *hold with* $m = k+1$.

We could prove Claim 3 as follows. Let $\alpha_k = \frac{1}{2} - k - 1$. From (3.41), (7.1) and (7.15) ($m = k$), we have

$$\int_0^M r^{\alpha_k} (\rho - \rho_\infty)^2 \, dx \leqslant C e^{-at},$$

and (7.12) ($m = k+1$) holds. From (1.29) and (7.12), we can obtain (7.13) ($m = k+1$) easily.

From (3.49), we obtain

$$\frac{d}{dt} \int_0^M r^{\alpha_k} u^2 \, dx + C_{42} \int_0^M r^{\alpha_k} \left(r^{2n-2} u_x^2 + \frac{u^2}{r^2} \right) dx \leqslant C\big(f_1^2 + |\Delta P|^2\big) + C e^{-at}. \quad (7.16)$$

Thus $\int_0^M (r^{\alpha_k} u^2)(x,t) \, dx$ is a Lyapunov functional, and we obtain (7.14) ($m = k+1$) immediately.

From (3.52) and (7.12)–(7.14), we have

$$\begin{aligned} &\frac{d}{dt} \int_0^M H_1^2 \, dx + \frac{C_{18}}{2} \int_0^M H_1^2 \, dx \\ &\quad \leqslant C \int_0^M \big(r^{\alpha_k} u^2 + r^{\alpha_k} (\rho - \rho_\infty)^2 + r^{\alpha_k} (r - r_\infty)^2 \big) + C f_1^2 \leqslant C e^{-at}. \end{aligned}$$

Thus $\int_0^M H_1^2(x,t) \, dx$ is a Lyapunov functional. Using the estimates (1.29) and (7.12)–(7.14), we obtain (7.15) ($m = k+1$), finish the proof of Claim 3 and Proposition 7.3 immediately. □

From (1.29), (7.12) and (7.15), using Hölder's inequality and Sobolev's embedding theorem, we could obtain the following proposition.

Proposition 7.4. *Under the assumptions of Theorem* 1.3*, we obtain*

$$\big\| \rho(\cdot, t) - \rho_\infty(\cdot) \big\|_{L^\infty} + \big\| r(\cdot, t) - r_\infty(\cdot) \big\|_{L^\infty} \leqslant C e^{-at},$$

for all $t \geqslant 0$, *where* a *is a positive constant.*

Proposition 7.5. *Under the assumptions of Theorem* 1.3*, we obtain*

$$\int_0^M \big(r^\alpha u_t^2 \big)(x,t) \, dx \leqslant C e^{-at}, \quad (7.17)$$

for all $t \geqslant 0$, *where* $\alpha = \frac{3}{2} - n$ *and* a *is a positive constant.*

Proof. From (1.30) and (3.70), we have

$$\frac{d}{dt}\int_0^M r^\alpha u_t^2\,dx + C_{45}\int_0^M \big(r^{2n-2+\alpha}u_{xt}^2 + r^{\alpha-2}u_t^2\big)\,dx$$
$$\leqslant C_{46}\int_0^M \big[r^{2n-2+\alpha}u_x^2 + r^{\alpha-2}u^2 + r^\alpha(\rho-\rho_\infty)^2\big]\,dx$$
$$+ C\big(f_2^2 + |\Delta P|^2 + \big|(\Delta P)'\big|^2\big). \tag{7.18}$$

Let $V_4 = \int_0^M (r^\alpha u^2)(x,t)\,dx + \frac{C_{42}}{2C_{46}}\int_0^M (r^\alpha u_t^2)(x,t)\,dx$. From (1.32), (7.12), (7.16) ($k=n-2$) and (7.18), we have

$$V_4' + C^{-1}V_4 \leqslant Ce^{-a_0 t} + Ce^{-at}.$$

Thus V_4 is a Lyapunov functional, and we obtain (7.17) immediately. □

Proposition 7.6. *Under the assumptions of Theorem* 1.3*, we obtain*

$$\int_0^M r^{2n-2+\alpha}\big[\partial_x\big(\rho^{1+\theta}\partial_x\big(r^{n-1}u\big)\big)\big]^2\,dx + \left\|\left(\frac{u}{r}, \big(r^{n-1}u\big)_x\right)(\cdot,t)\right\|_{L^\infty} \leqslant Ce^{-at},$$

for all $t \geqslant 0$, where $\alpha = \frac{3}{2} - n$ and a is a positive constant.

Proof. From Eq. $(1.20)_2$, we have

$$(2c_1+c_2)r^{n-1}\partial_x\big(\rho^{1+\theta}\partial_x\big(r^{n-1}u\big)\big) = u_t + Ar^{n-1}\big(\rho^\gamma\big)_x + 2c_1(n-1)r^{n-2}u\big(\rho^\theta\big)_x + f,$$

and using the estimates (1.29)–(1.30), (1.32), (7.12)–(7.15) and (7.17), conclude that

$$\int_0^M r^{2n-2+\alpha}\big[\partial_x\big(\rho^{1+\theta}\partial_x\big(r^{n-1}u\big)\big)\big]^2\,dx \leqslant Ce^{-at}$$

and

$$\int_0^M \big|\partial_x\big(\rho^{1+\theta}\partial_x\big(r^{n-1}u\big)\big)\big|\,dx \leqslant Ce^{-at}. \tag{7.19}$$

From (1.29), (7.11) and (7.19), using Sobolev's embedding theorem $W^{1,1} \hookrightarrow L^\infty$, we can obtain

$$\big\|\partial_x\big(r^{n-1}u\big)(\cdot,t)\big\|_{L^\infty} \leqslant Ce^{-at}$$

and

$$\left\| \frac{u}{r}(\cdot, t) \right\|_{L^\infty} \leqslant Ce^{-at}. \qquad \square$$

References

[1] G.Q. Chen, M. Kratka, Global solutions to the Navier–Stokes equations for compressible heat-conducting flow with symmetry and free boundary, Comm. Partial Differential Equations 27 (5–6) (2002) 907–943.

[2] B. Ducomet, A.A. Zlotnik, Viscous compressible barotropic symmetric flows with free boundary under general mass force. I. Uniform-in-time bounds and stabilization, Math. Methods Appl. Sci. 28 (7) (2005) 827–863.

[3] B. Ducomet, A.A. Zlotnik, Lyapunov functional method for 1D radiative and reactive viscous gas dynamics, Arch. Ration. Mech. Anal. 177 (2005) 185–229.

[4] H. Grad, Asymptotic theory of the Boltzmann equation II, in: J. Laurmann (Ed.), Rarefied Gas Dynamics, vol. 1, Academic Press, New York, 1963, pp. 26–59.

[5] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.

[6] D. Hoff, D. Serre, The failure of continuous dependence on initial data for the Navier–Stokes equations of compressible flow, SIAM J. Appl. Math. 51 (4) (1991) 887–898.

[7] D. Hoff, Spherically symmetric solutions of the Navier–Stokes equations for compressible, isothermal flow with large, discontinuous initial data, Indiana Univ. Math. J. 41 (4) (1992) 1225–1302.

[8] P.L. Lions, Mathematical Topics in Fluid Mechanics, vols. 1–2, Oxford University Press, New York, 1996, 1998.

[9] T.P. Liu, Z.P. Xin, T. Yang, Vacuum states of compressible flow, Discrete Contin. Dyn. Syst. 4 (1) (1998) 1–32.

[10] A. Matsumura, S. Yanagi, Uniform boundedness of the solutions for a one-dimensional isentropic model system of a compressible viscous gas, Comm. Math. Phys. 175 (1996) 259–274.

[11] Š. Matušů-Nečasová, M. Okada, T. Makoni, Free boundary problem for the equation of spherically symmetric motion of viscous gas (II)–(III), Japan J. Indust. Appl. Math. 12 (1995) 195–203, Japan J. Indust. Appl. Math. 14 (1997) 199–213.

[12] P. Mucha, Compressible Navier–Stokes system in 1-D, Math. Methods Appl. Sci. 24 (2001) 607–622.

[13] M. Okada, T. Makino, Free boundary value problems for the equation of spherically symmetrical motion of viscous gas, Japan J. Indust. Appl. Math. 10 (1993) 219–235.

[14] M. Okada, Š. Matušů-Nečasová, T. Makino, Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity, Ann. Univ. Ferrara Sez. VII (N.S.) 48 (2002) 1–20.

[15] I. Straškraba, A.A. Zlotnik, Global behavior of 1d-viscous compressible barotropic fluid with a free boundary and large data, J. Math. Fluid Mech. 5 (2003) 119–143.

[16] V.A. Vaigant, A.V. Kazhikhov, On existence of global solutions to the two-dimensional Navier–Stokes equations for a compressible viscosity fluid, Siberian Math. J. 36 (1995) 1108–1141.

[17] S.W. Vong, T. Yang, C.J. Zhu, Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum (II), J. Differential Equations 192 (2) (2003) 475–501.

[18] Z. Xin, Blow-up of smooth solution to the compressible Navier–Stokes equations with compact density, Comm. Pure Appl. Math. 51 (3) (1998) 229–240.

[19] T. Yang, C.J. Zhu, Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum, Comm. Math. Phys. 230 (2) (2002) 329–363.

[20] T. Zhang, D.Y. Fang, Global behavior of compressible Navier–Stokes equations with a degenerate viscosity coefficient, Arch. Ration. Mech. Anal. 182 (2) (2006) 223–253.

[21] A.A. Zlotnik, B. Ducomet, The stabilization rate and stability of viscous compressible barotropic symmetric flows with a free boundary for a general mass force, Sb. Math. 196 (11–12) (2005) 1745–1799.