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Local and global stability for Lotka–Volterra systems with distributed delays and instantaneous negative feedbacks

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Abstract

This paper addresses the local and global stability of n-dimensional Lotka–Volterra systems with distributed delays and instantaneous negative feedbacks. Necessary and sufficient conditions for local stability independent of the choice of the delay functions are given, by imposing a weak nondelayed diagonal dominance which cancels the delayed competition effect. The global asymptotic stability of positive equilibria is established under conditions slightly stronger than the ones required for the linear stability. For the case of monotone interactions, however, sharper conditions are presented. This paper generalizes known results for discrete delays to systems with distributed delays. Several applications illustrate the results.

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Keywords: Lotka–Volterra system; Delayed population model; Distributed delays; Global asymptotic stability; Local asymptotic stability; Instantaneous negative feedback

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1. Introduction

Delay differential equations have been extensively used as models in population dynamics, neural networks, disease modelling and other important areas of science. Rather than considering ordinary differential equations (ODEs), it is often more realistic to use retarded functional differential equations (FDEs) to describe such models. In fact, the use of time-delays in differential equations arises naturally in mathematical models in biology, to account for the maturation period of biological species, synaptic transmission time among neurons, incubation time in epidemic models, and various other situations.

In this paper, we consider linear FDEs in $\mathbb{R}^n$ with undelayed diagonal terms, given by

$$x_i'(t) = -\left[b_i x_i(t) + \sum_{j=1}^{n} l_{ij} \int_{-\tau}^{0} x_j(t+\theta)\, d\eta_{ij}(\theta)\right], \quad i = 1, \ldots, n, \tag{1.1}$$

and multiple species Lotka–Volterra type models of the form

$$x_i'(t) = r_i(t)x_i(t)\left[1 - b_i x_i(t) - \sum_{j=1}^{n} l_{ij} \int_{-\tau}^{0} x_j(t+\theta)\, d\eta_{ij}(\theta)\right], \quad i = 1, \ldots, n. \tag{1.2}$$

Here, $b_i, l_{ij} \in \mathbb{R}$, $\tau > 0$, $r_i(t)$ are positive continuous functions and $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions. In biological terms, only *positive* solutions of the Lotka–Volterra system (1.2) are meaningful, and therefore admissible. Special attention to the autonomous case $r_i(t) \equiv r_i > 0$ in (1.2) will be given.

A most interesting topic in population dynamics is the stability of equilibria. For linear FDEs (1.1), we give in this paper sufficient conditions for asymptotic stability, and investigate whether such conditions are sharp, or, in other words, whether they are necessary and sufficient conditions for the asymptotic stability of (1.1) independently of the choices of delay functions η_{ij}, in a sense to be precised later.

From the point of view of applications, it is particularly important to study the stability and attractivity of a positive equilibrium of the multiple species Lotka–Volterra equation (1.2), if it exists. This is the main purpose of this paper. Therefore, when studying (1.2) we shall always assume that

(H1) there is a *positive* equilibrium $x^* = (x_1^*, \ldots, x_n^*)$ of (1.2).

When $r_i(t) \equiv r_i > 0$, the linearization of (1.2) about x^* has the form in (1.1) (with coefficients b_i, l_{ij} multiplied by x_i^*, $1 \leqslant i \leqslant n$), so the local asymptotic stability of x^* is given by the stability of (1.1).

In general, large delays are not harmless, and induce instability of equilibria, oscillations and even existence of unbounded solutions. If the delays are small enough, they are expected to be negligible, so that an FDE should behave mainly like an ODE. For Lotka–Volterra systems (1.2) without undelayed intraspecific competitions, i.e., where all b_i are zero, the general approach is to study the attractivity of the positive equilibrium x^* (if it exists) by imposing constraints of the size of the delays in the intraspecific terms. This line of investigation has been especially fruitful in the case of scalar equations since the pioneering work of Wright [28], and an extensive literature on the so-called 3/2-*type conditions* has been produced since then. Some valuable works (see e.g. [7,8,10,11,14]) have extended this study to n-dimensional delayed Lotka–Volterra systems without negative feedbacks, a much more difficult situation even for the case of $n = 2$ with discrete delays.

More recently, Tang and Zou [25] considered Lotka–Volterra systems with *distributed* delays of the form

$$\dot{x}_i(t) = r_i(t)x_i(t)\left[1 - \int_{-\tau_{ii}}^{0} x_i(t+\theta)\,d\eta_{ii}(\theta) - \sum_{j\neq i}^{n} l_{ij}\int_{-\tau_{ij}}^{0} x_j(t+\theta)\,d\eta_{ij}(\theta)\right], \quad i = 1,\dots,n, \tag{1.3}$$

where $r_i(t)$ are as in (1.2), $l_{ij} \geqslant 0$, $\tau_{ij} \geqslant 0$, and η_{ij} are *non-decreasing* bounded normalized functions. Eq. (1.3) can be seen as a particular case of (1.2), where all the operators $\psi \mapsto l_{ij}\int_{-\tau_{ij}}^{0}\psi(\theta)\,d\eta_{ij}(\theta)$, $\psi \in C([-\tau_{ij},0];\mathbb{R})$, are positive. In [25], the primary goal of the authors is to deal with the "pure-delay-type" situation $\tau_{ii} > 0$ in (1.3), although the situation where instantaneous negative feedbacks are present can be included in their setting. We further remark that [25] generalizes results and techniques previously established by the same authors in [24], for a 2-dimensional system with discrete delays (see also [14]). Several 3/2-type criteria for the global attractivity of the positive equilibrium of (1.3) are given in [25], by using a "sandwiching" technique, which extends to systems Wright's method [28] for scalar equations.

In many situations, however, it is not realistic to assume that the delays are very small. An alternative setting to study stability of n-dimensional Lotka–Volterra systems (1.2), pursued by many authors (see e.g. [9,12,13,15,17–21]) and followed here, is to assume that the so-called intraspecific competitions without delay $b_i x_i(t)$ dominate, in some sense, the delayed intraspecific competitions and interspecific interactions. The question is to establish sufficient conditions of diagonal dominance of the instantaneous negative feedbacks over the total competition matrix, so that the stability of (1.2) follows for all the choices of delay functions η_{ij}.

The work presented in this paper was strongly motivated by Faria [2], where the *scalar* equations (1.1) and (1.2) were studied, and Hofbauer and So [9] and Campbell [1], who dealt with n-dimensional systems with *discrete* delays.

In [2], a criterion for the global asymptotic stability of the delayed logistic type equation $x'(t) = r(t)[1 - b_0x(t) - L_0(x_t)]$, where $r(t)$ is continuous and positive, $b_0 \in \mathbb{R}$ and $L_0 : C([-\tau,0];\mathbb{R}) \to \mathbb{R}$ is a linear bounded operator, was established. Furthermore, it was also shown that such a criterion is sharp, in the sense that it provides a necessary and sufficient condition for the asymptotic stability *globally in the delays* of the linear scalar FDE $x'(t) = -[b_0x(t) + L_0(x_t)]$ (cf. [2] also for definitions).

Hofbauer and So [9] considered the particular case of autonomous systems with discrete delays of the form

$$x_i'(t) = x_i(t)\left[r_i - \sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij})\right], \quad i = 1,\dots,n, \tag{1.4}$$

where $r_i > 0$, $a_{ij} \in \mathbb{R}$, $\tau_{ij} \geqslant 0$ and $a_{ii} > 0$, $\tau_{ii} = 0$. Assuming (H1), they gave necessary and sufficient conditions for the global asymptotic stability of x^*, for all the choices of delays $\tau_{ij} \geqslant 0$, $i \neq j$. Moreover, such conditions coincide with the ones required for its asymptotic stability, independently of the choices of the delays. Note that in (1.4) there are no delayed intraspecific competitions. Later on, Campbell [1] studied the local and global stability of the trivial equilibrium of an FDE used to model artificial neural networks with discrete delays, without the restriction $\tau_{ii} = 0$:

$$x_i'(t) = -b_i x_i(t) + \sum_{j=1}^{n} x_{ij} g_j\bigl(x_j(t-\tau_{ij})\bigr), \quad i = 1, \dots, n.$$

Here, our goal is to extend both the results in [2] to n-dimensional equations and the results in [1,9] to a general situation with distributed delays.

There is an extensive literature dealing with local and global stability of Lotka–Volterra systems with delays. Related to the results presented here, besides the above cited works [1,2,9,24, 25] we mention the monographs of Gopalsamy [5], Kuang [10] and Smith [21], the papers of He [8], Kuang [11,12], Kuang and Smith [13], Saito and Takeuchi [20], So et al. [22,23], and references therein. We emphasize however that, in the literature, the usual approach to study the global stability of equilibria for systems (1.2) and other non-linear FDEs relies on the use of Lyapunov functionals or Razumikhin methods. In general, constructing a Lyapunov functional for a concrete n-dimensional FDE is not an easy task. Frequently, a new Lyapunov functional for each model under consideration is required. Contrary to the usual, our techniques (see also [2,3, 24,25]) do not involve Lyapunov functionals, and our method applies to general Lotka–Volterra systems (1.2), or even to broader frameworks.

We now give some definitions and set some notation.

Definition 1.1. An equilibrium x^* of (1.2) is said to be *globally asymptotically stable* (in the set of all positive solutions) if it is stable and is a global attractor of all positive solutions of (1.2).

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we say that $x > 0$ if $x_i > 0$ for $i = 1, \dots, n$, and that $x \geqslant 0$ if $x_i \geqslant 0$ for $i = 1, \dots, n$. For $x = (x_1, \dots, x_n) > 0$, x^{-1} is the vector given by $x^{-1} = (x_1^{-1}, \dots, x_n^{-1})$. We denote by $|\cdot|_\infty$ or simply $|\cdot|$ the supremum norm in $\mathbb{R}^n$, $|x|_\infty = \max_{1\leqslant i\leqslant n} |x_i|$. If $d = (d_1, \dots, d_n) > 0$, we also consider the norm in $\mathbb{R}^n$ given by $|x|_d = \max_{1\leqslant i\leqslant n}(d_i|x_i|)$. We use $\|\cdot\|_\infty$ or simply $\|\cdot\|$, respectively $\|\cdot\|_d$, to denote the supremum norm in $C_n := C([-\tau, 0]; \mathbb{R}^n)$ relative to the norm $|\cdot|_\infty$, respectively $|\cdot|_d$, in $\mathbb{R}^n$: $\|\varphi\|_\infty = \max_{-\tau\leqslant\theta\leqslant 0} |\varphi(\theta)|_\infty$ and $\|\varphi\|_d = \max_{-\tau\leqslant\theta\leqslant 0} |\varphi(\theta)|_d$. For a bounded linear functional $L : C_n \to \mathbb{R}$, where C_n is equipped with the norm $\|\cdot\|_\infty$, respectively $\|\cdot\|_d$, we denote the usual operator norm by $\|\cdot\|$, respectively $\|\cdot\|_d$.

For $d \in \mathbb{R}^n$ ($n \geqslant 1$), we use d to denote both the real vector and the constant function $\varphi(\theta) = d$ in C_n. C_n is supposed to be partially ordered with

$$\varphi \geqslant \psi \quad \text{if and only if} \quad \varphi_i(\theta) \geqslant \psi_i(\theta), \quad \theta \in [-\tau, 0],\ i = 1, \dots, n.$$

Recall now some concepts from matrix analysis.

Definition 1.2. If $D = [d_{ij}]$ is a square matrix with non-positive off-diagonal entries, i.e., $d_{ij} \leqslant 0$ for all $i \neq j$, we say that D is an *M-matrix* if all the eigenvalues of D have a non-negative real part, or, equivalently, if all the principal minors of D are non-negative; D is a *non-singular M-matrix* if all the eigenvalues of D have positive real part, or, equivalently, if all the principal minors of D are positive.

The latter is also equivalent to saying that D is an M-matrix and $\det D \neq 0$. M-matrices and non-singular M-matrices are also often referred to as matrices in *class* K_0 and *class* K, respectively. Definition 1.2 agrees with the notation in [1,22,23]. In some works [9], M-matrices are called *weakly diagonally dominant*. On the other hand, some authors [8,12,17] use the term "M-matrix" to denote a "non-singular M-matrix" as defined above, a situation the reader should be aware of, in order to avoid conceptual misunderstandings. For properties of M-matrices, we refer the reader to [4, Chapter 5].

For an $n \times n$ matrix $D = [d_{ij}]$, in the sequel we denote by $\tilde{D}$ the matrix $\tilde{D} = [\tilde{d}_{ij}]$, where $\tilde{d}_{ij} = -|d_{ij}|$ for $j \neq i$, $\tilde{d}_{ii} = d_{ii}$, $i, j = 1, \dots, n$.

The remainder of this paper is organized as follows: In Section 2, a criterion for the exponential asymptotic stability of linear FDEs (1.1) is presented. With $b_i > 0$ ($1 \leqslant i \leqslant n$), our criterion is shown to be optimal, in the sense that it gives necessary and sufficient conditions on the coefficients b_i, l_{ij} for the linear FDE (1.1) to be stable *independently of the delay functions* η_{ij}. These results generalize the ones in [1,9], concerning linear equations with discrete delays. In Section 3, we give sufficient conditions for boundedness of solutions and for the global asymptotic stability of the positive equilibrium x^* (if it exists) of (1.2), again improving known results in the literature. Such conditions are slightly stronger than the ones required for linear stability. Finally, in Section 4 we consider models (1.2) with non-decreasing delay functions η_{ij}, and present sharper criteria for boundedness of solutions and for the global stability of x^*. Throughout the paper, we illustrate the results with some well-known models.

2. Asymptotic stability for linear FDEs

Let $C_n := C([-\tau, 0]; \mathbb{R}^n)$ be equipped with the supremum norm $\|\cdot\|_\infty$ or any equivalent norm. In the phase space C_n, consider an autonomous system of linear FDEs of the form

$$x_i'(t) = -\bigl[b_i x_i(t) + L_i(x_t)\bigr], \quad i = 1, \dots, n, \tag{2.1}$$

where $b_i \in \mathbb{R}$, $L_i : C_n \to \mathbb{R}$ are linear bounded operators, $i = 1, \dots, n$. As usual, x_t denotes the function in C_n defined by $x_t(\theta) = x(t + \theta)$, $-\tau \leqslant \theta \leqslant 0$. Equivalently, one can write L_i as

$$L_i(\varphi) = \sum_{j=1}^{n} L_{ij}(\varphi_j), \quad L_{ij}(\varphi_j) = l_{ij} \int_{-\tau}^{0} \varphi_j(\theta)\, d\eta_{ij}(\theta), \quad \varphi = (\varphi_1, \dots, \varphi_n) \in C_n, \tag{2.2}$$

for some $l_{ij} \in \mathbb{R}$ and some normalized functions of bounded variation η_{ij}, $\eta_{ij} \in BV([-\tau, 0]; \mathbb{R})$ with $\operatorname{Var}_{[-\tau,0]} \eta_{ij} = 1$, so that (2.1) reads as

$$x_i'(t) = -\left[b_i x_i(t) + \sum_{j=1}^{n} l_{ij} \int_{-\tau}^{0} x_j(t + \theta)\, d\eta_{ij}(\theta)\right], \quad i = 1, \dots, n. \tag{2.3}$$

Set $a_{ij} = L_{ij}(1)$. From (2.2), $a_{ij} = l_{ij}(\eta_{ij}(0) - \eta_{ij}(-\tau))$ and $|l_{ij}| = \|L_{ij}\|$. Let $B = \operatorname{diag}(b_1, \dots, b_n)$, $A = [a_{ij}]$ and $C = [l_{ij}]$, and define the matrices

$$M = B + A, \qquad N = B + C.$$

In the sequel, consider also the matrices $\tilde{M} = B + \tilde{A}$, $\hat{N} = B + \hat{C}$, where $\tilde{A} = [\tilde{a}_{ij}]$, $\hat{C} = [\hat{l}_{ij}]$, for $\tilde{a}_{ij} = -|a_{ij}|$ for $j \neq i$, $\tilde{a}_{ii} = a_{ii}$, $\hat{l}_{ij} = -|l_{ij}|$ for $i, j = 1, \ldots, n$:

$$\tilde{M} = \begin{pmatrix} b_1 + a_{11} & -|a_{12}| & \ldots & -|a_{1n}| \\ & & \ldots & \\ -|a_{n1}| & -|a_{n2}| & \ldots & b_n + a_{nn} \end{pmatrix},$$
$$\hat{N} = \begin{pmatrix} b_1 - |l_{11}| & -|l_{12}| & \ldots & -|l_{1n}| \\ & & \ldots & \\ -|l_{n1}| & -|l_{n2}| & \ldots & b_n - |l_{nn}| \end{pmatrix}. \tag{2.4}$$

Note that all the off-diagonal entries of $\tilde{M}$, $\hat{N}$ are non-positive.

For studying the stability of (2.1), we first translate an algebraic property of the matrix $\hat{N}$ into an analytical condition on the linear operators L_i.

Lemma 2.1. *For $d = (d_1, \ldots, d_n) > 0$, then $\hat{N}d \geqslant 0$ if and only if $\|L_i\|_{d^{-1}} \leqslant d_i b_i$, $i = 1, \ldots, n$.*

Proof. Let L_i, L_{ij} be as in (2.2). Then $\|L_{ij}\| = |l_{ij}|$, and

$$\|L_i\|_{d^{-1}} = \sum_{j=1}^{n} d_j |l_{ij}|$$

for each $d = (d_1, \ldots, d_n) > 0$. On the other hand, $\hat{N}d \geqslant 0$ is equivalent to

$$\sum_{j=1}^{n} d_j |l_{ij}| \leqslant d_i b_i, \quad i = 1, \ldots, n. \qquad \square \tag{2.5}$$

Lemma 2.2. *Let $\tau > 0$, $b_i \in \mathbb{R}$ and $L_i : C_n \to \mathbb{R}$ be linear bounded operators, $i = 1, \ldots, n$. With the previous notation, suppose that*

(H2) *there is $d = (d_1, \ldots, d_n) > 0$ such that $\|L_i\|_{d^{-1}} \leqslant d_i b_i$, $i = 1, \ldots, n$.*

Then, all the characteristic roots λ of (2.1) have negative real parts, with the possible exception of $\lambda = 0$. If in addition $\det M \neq 0$, then (2.1) is exponentially asymptotically stable.

Proof. Write L_i as $L_i(\varphi) = \sum_{j=1}^{n} L_{ij}(\varphi_j)$, for $\varphi = (\varphi_1, \ldots, \varphi_n) \in C_n = C([-\tau, 0]; \mathbb{R}^n)$. The characteristic equation for (2.1) is

$$\det \Delta(\lambda) = 0, \quad \text{for } \Delta(\lambda) = \lambda I + B + \left[\left(L_{ij}\left(e^{\lambda \cdot}\right)\right)_{i,j=1}^{n}\right]. \tag{2.6}$$

Let $\lambda = \alpha + i\beta \neq 0$ be a root of (2.6), and consider $v \in \mathbb{C}^n$, $v \neq 0$, such that $\Delta(\lambda)v = 0$. For $d > 0$ as in (H2), let k be such that $|v|_{d^{-1}} = d_k^{-1}|v_k|$. We may suppose $v_k \in \mathbb{R}$, $v_k > 0$. We have

$$(\alpha + b_k)v_k = -\operatorname{Re} L_k\left(e^{\lambda\theta} v\right), \qquad \beta v_k = -\operatorname{Im} L_k\left(e^{\lambda\theta} v\right), \tag{2.7}$$

where we abuse the notation and write $L_k(e^{\lambda\theta} v)$ for $L_k(e^{\lambda \cdot} v)$.

Suppose now that $\alpha \geqslant 0$. Since $\|L_k\|_{d^{-1}} \leqslant d_k b_k$, then $|L_k(e^{\lambda\theta}v)| \leqslant d_k b_k \|e^{\lambda\cdot}v\|_{d^{-1}} \leqslant d_k b_k |v|_{d^{-1}} = b_k v_k$, hence

$$\big(\operatorname{Re} L_k\big(e^{\lambda\theta}v\big)\big)^2 + \big(\operatorname{Im} L_k\big(e^{\lambda\theta}v\big)\big)^2 \leqslant b_k^2 v_k^2. \tag{2.8}$$

If $\operatorname{Im} L_k(e^{\lambda\theta}v) = 0$, from (2.7) we have $\beta = 0$ and $\lambda = \alpha$, with

$$(\alpha + b_k)v_k = -L_k\big(e^{\alpha\theta}v\big) \leqslant b_k v_k,$$

implying that $\alpha \leqslant 0$, and therefore $\lambda = \alpha = 0$.

If $\operatorname{Im} L_k(e^{\lambda\theta}v) \neq 0$, from (2.7), (2.8) we obtain

$$(\alpha + b_k)v_k = -\operatorname{Re} L_k\big(e^{\lambda\theta}v\big) < \big|L_k\big(e^{\lambda\theta}v\big)\big| \leqslant b_k v_k,$$

and we conclude that $\alpha < 0$, a contradiction. Thus, all the roots of (2.6) have negative real parts, with the possible exception of zero.

Finally, note that $\Delta(0) = B + A = M$. If $\det M \neq 0$, then $\lambda = 0$ is not a root of the characteristic equation (2.6). $\square$

Theorem 2.3. *Let $\tau > 0$, $b_i, l_{ij} \in \mathbb{R}$ and $\eta_{ij} \in BV([-\tau, 0]; \mathbb{R})$ with $\operatorname{Var}_{[-\tau,0]} \eta_{ij} = 1$, $i, j = 1, \ldots, n$, be given. With the previous notation, suppose that $\det M \neq 0$ and $\hat{N}$ is an M-matrix. Then,* (2.3) *is exponentially asymptotically stable. Moreover, $b_i + a_{ii} > 0$, $i = 1, \ldots, n$.*

Proof. Let $L_i(\varphi) = \sum_{j=1}^n L_{ij}(\varphi_j)$ be as in (2.2). We consider separately the cases of $\hat{N}$ irreducible and reducible.

Case 1. If $\hat{N}$ is irreducible, then there is $d = (d_1, \ldots, d_n) > 0$ such that $\hat{N}d \geqslant 0$ [4, Theorem 5.9]. In consequence of Lemma 2.1, hypothesis (H2) is satisfied, and the asymptotic stability of (2.3) follows from Lemma 2.2. From (2.5), we also have $b_i + a_{ii} \geqslant b_i - |l_{ii}| \geqslant 0$, $i = 1, \ldots, n$; and if $b_i + a_{ii} = 0$ for some $i \in \{1, \ldots, n\}$, then $0 = d_i(b_i - |l_{ii}|) = \sum_{1 \leqslant j \leqslant n,\, j \neq i} d_j |l_{ij}|$, thus $l_{ij} = a_{ij} = 0$ for $1 \leqslant j \leqslant n$, $j \neq i$. This together with $b_i + a_{ii} = 0$ implies that the ith row of M is zero, which is not possible since $\det M \neq 0$.

Case 2. If $\hat{N}$ is reducible, after a permutation of rows and columns, which amounts to a permutation of the variables $x_1, \ldots, x_n$ in (2.3), we may suppose that

$$\hat{N} = \begin{pmatrix} \hat{N}_{11} & \ldots & \hat{N}_{1\ell} \\ & \ddots & \\ 0 & \ldots & \hat{N}_{\ell\ell} \end{pmatrix}, \tag{2.9}$$

where $\hat{N}_{km}$ are $n_k \times n_m$ matrices, with $\hat{N}_{kk}$ irreducible or zero $n_k \times n_k$ blocks, $\sum_{k=1}^{\ell} n_k = n$. Accordingly to (2.9), we have

$$l_{ij} = 0, \quad \text{for } n_1 + \cdots + n_k + 1 \leqslant i \leqslant n_1 + \cdots + n_{k+1},\ 1 \leqslant j \leqslant n_1 + \cdots + n_k,\ 1 \leqslant k \leqslant \ell - 1. \tag{2.10}$$

From (2.2) and (2.10), it follows that $[(L_{ij}(e^{\lambda\cdot}))_{i,j=1}^n]$ as well as the characteristic matrix $\Delta(\lambda)$ in (2.6) are also upper block triangular matrices. With the obvious notation, we write

$$\Delta(\lambda)=\lambda I+\operatorname{diag}(B_1,\dots,B_\ell)+\begin{pmatrix}\mathcal{L}_{11}(\lambda) & \dots & \mathcal{L}_{1\ell}(\lambda)\\ & \ddots & \\ 0 & \dots & \mathcal{L}_{\ell\ell}(\lambda)\end{pmatrix},$$

where $B_k=\operatorname{diag}(b_{1+N(k)},\dots,b_{N(k+1)})$ for $N(k)=\sum_{1\leqslant m\leqslant k-1}n_m$ and $\mathcal{L}_{km}(\lambda)$ are $n_k\times n_m$ blocks.

Let $\lambda=\alpha+i\beta$ be a root of the characteristic equation (2.6). This means that $\det\Delta(\lambda)=0$, or equivalently, $\det(\lambda I_{n_k}+B_k+\mathcal{L}_{kk}(\lambda))=0$, for some $k\in\{1,\dots,\ell\}$ (where I_{n_k} is the identity matrix of dimension n_k).

If the block $\hat N_{kk}$ is irreducible, from Case 1 we conclude that $\alpha=\operatorname{Re}\lambda<0$. Now, suppose that $\hat N_{kk}=0$ and $\alpha\geqslant 0$. Without loss of generality, we may assume that $k=1$, so that

$$b_i=|l_{ii}|,\quad 1\leqslant i\leqslant n_1,\quad \text{and}\quad l_{ij}=0,\quad 1\leqslant i,j\leqslant n_1,\ i\neq j.$$

The corresponding block $\lambda I_{n_1}+B_1+\mathcal{L}_{11}(\lambda)$ of $\Delta(\lambda)$ is a diagonal matrix, with diagonal entries $\lambda+|l_{ii}|+L_{ii}(e^{\lambda\cdot})$, $1\leqslant i\leqslant n_1$. Recall that $|L_{ii}(e^{\lambda\cdot})|=|l_{ii}\int_{-\tau}^0 e^{\lambda\theta}\,d\eta_{ii}(\theta)|\leqslant|l_{ii}|$.

If $\det(\lambda I_{n_1}+B_1+\mathcal{L}_{11}(\lambda))=0$, then $\lambda+|l_{ii}|+L_{ii}(e^{\lambda\cdot})=0$ for some $i\in\{1,\dots,n_1\}$, and in particular we get $\alpha\leqslant 0$. If $\alpha=0$, then $|l_{ii}|+\operatorname{Re}L_{ii}(e^{\lambda\cdot})=0$, implying that $\beta=-\operatorname{Im}L_{ii}(e^{\lambda\cdot})=0$, which is a contradiction, since $\Delta(0)=M$ and $\det M\neq 0$ imply that $\lambda\neq 0$. We therefore conclude that (2.3) is exponentially asymptotically stable.

We show now that $b_i+a_{ii}>0$, $i=1,\dots,n$, for a reducible matrix $\hat N$. Up to a permutation, $\hat N$ has the form (2.9). For irreducible diagonal blocks $\hat N_{kk}$, from Case 1 we derive that the diagonal entries b_i+a_{ii} of M are positive. If the block $\hat N_{kk}$ is zero, then, for $1+N(k)\leqslant i\leqslant N(k+1)$, we have $b_i=|l_{ii}|$ and the corresponding block M_{kk} of M is a diagonal matrix with b_i+a_{ii} as diagonal entries. On the other hand, these diagonal entries b_i+a_{ii} are non-zero, otherwise $\det M=0$, hence they are positive. □

We have also shown that:

Corollary 2.4. *Let $\tau>0$, $b_i,l_{ij}\in\mathbb{R}$ and $\eta_{ij}\in BV([-\tau,0];\mathbb{R})$ with $\operatorname{Var}_{[-\tau,0]}\eta_{ij}=1$, $i,j=1,\dots,n$, be given. If $\hat N$ is an M-matrix, then all the roots λ of the characteristic equation* (2.6) *have negative real parts with the possible exception of $\lambda=0$.*

Remark 2.1. If $\hat N$ is an M-matrix, then $b_i-|l_{ii}|\geqslant 0$, $i=1,\dots,n$. For $\hat N$ an irreducible M-matrix, one can even conclude that $b_i-|l_{ii}|>0$, $i=1,\dots,n$. In fact, under these assumptions, $\hat N$ satisfies (H2); as in the proof of Theorem 2.3, $b_i-|l_{ii}|=0$ implies now $l_{ij}=0$ for $j=1,\dots,n$, $j\neq i$, meaning that the ith row of $\hat N$ is zero, which is not possible for an irreducible matrix.

Lemma 2.5. *Let $b_i>0$, $l_{ij}\in\mathbb{R}$, $i,j=1,\dots,n$, be given, and define $N,\hat N$ as above. If $\det N\neq 0$ and $\hat N$ is not an M-matrix, then there exist $\tau_{ij}\geqslant 0$ such that, for η_{ij} defined as the Heaviside functions $\eta_{ij}(\theta)=0$ for $-\tau\leqslant\theta\leqslant-\tau_{ij}$, $\eta_{ij}(\theta)=1$ for $-\tau_{ij}<\theta\leqslant 0$ and $\tau=\max\{\tau_{ij}\colon i,j=1,\dots,n\}$, the characteristic equation for* (2.3) *has a root λ with $\operatorname{Re}\lambda>0$.*

Proof. The proof is given in [1, Lemmas 2.4–2.5] (see also [9]), and is omitted. □

Theorem 2.6. *Let $b_i > 0$, $l_{ij} \in \mathbb{R}$, $i, j = 1, \ldots, n$, be given. Then, Eq.* (2.3) *is exponentially asymptotically stable for all the choices of $\tau > 0$ and sets of functions $\eta = (\eta_{ij}) \subset BV([-\tau, 0]; \mathbb{R})$ with $\mathrm{Var}_{[-\tau,0]}\, \eta_{ij} = 1$, $i, j = 1, \ldots, n$, and such that $\det M_\eta \neq 0$, if and only if $\hat{N}$ is an M-matrix. Here, M_η is defined by $M_\eta = B + [a_{ij}]$ for $a_{ij} = l_{ij}(\eta_{ij}(0) - \eta_{ij}(-\tau))$.*

Proof. For a given $\eta = (\eta_{ij}) \subset BV([-\tau, 0]; \mathbb{R})$ with $\mathrm{Var}_{[-\tau,0]}\, \eta_{ij} = 1$, then $M_\eta = \Delta(0)$, where $\det \Delta(\lambda) = 0$ is the characteristic equation (2.6), and hence $\det M_\eta \neq 0$ if and only if $\lambda = 0$ is not a root of (2.6). Also, for $\eta = (\eta_{ij})$ with η_{ij} as in the statement of Lemma 2.5, we have $M_\eta = N$. Now, the sufficiency is given by Theorem 2.3 and the necessity condition by Lemma 2.5. □

In applications, (2.1) often takes the form (2.3) with *non-decreasing* normalized bounded variation functions η_{ij}. Clearly, in this case

$$\int_{-\tau}^{0} d\eta_{ij}(\theta) = 1, \qquad \|L_{ij}\| = |l_{ij}|, \qquad a_{ij} = l_{ij}, \quad i, j = 1, \ldots, n,$$

and in particular $M = N$. In this situation, the above theorem translates as:

Theorem 2.7. *Let $b_i > 0$, $l_{ij} \in \mathbb{R}$, $i, j = 1, \ldots, n$, be given. Then,* (2.3) *is exponentially asymptotically stable for all the choices of $\tau > 0$ and non-decreasing functions $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ with $\int_{-\tau}^{0} d\eta_{ij}(\theta) = 1$, $i, j = 1, \ldots, n$, if and only if $\det M \neq 0$ and $\hat{M}$ is an M-matrix. In particular, if $\det M \neq 0$ and $\hat{M}$ is an M-matrix, then the equation*

$$x_i'(t) = -\left[b_i x_i(t) + \sum_{1 \leqslant j \leqslant n} l_{ij} x_j(t - \tau_{ij})\right], \quad i = 1, \ldots, n, \tag{2.11}$$

is exponentially asymptotically stable for all the choices of discrete delays $\tau_{ij} \geqslant 0$, $i, j = 1, \ldots, n$.

Remark 2.2. Eq. (2.11) was studied in [9], with the restriction $\tau_{ii} = 0$, and later in [1] without such constraint. With our notation, for (2.11) we have $M = N$, and $\tilde{M} = \hat{M}$ if all the diagonal delays are zero. In terms of the linear asymptotic stability, our Theorems 2.3 and 2.6 generalize the results in [1,9] to the situation with distributed delays. In fact, for (2.11) with $\tau_{ii} = 0$ Hofbauer and So [9] proved its asymptotic stability independently of the choices of delays $\tau_{ij} \geqslant 0$ if and only if $l_{ii} > 0$ $(1 \leqslant i \leqslant n)$, $\det M \neq 0$ and $\hat{M}$ is an M-matrix, while Campbell [1] proved the same result without the constraint $\tau_{ii} = 0$. We further note that So et al. [22] considered (2.11) for the "pure-delay-type" situation, i.e., with all $b_i = 0$. They established the asymptotic stability of (2.11) with $b_i = 0$ by imposing that $[\tilde{l}_{ij}]$, where

$$\tilde{l}_{ij} = -\frac{1 + \frac{1}{9} l_{ii}\tau_{ii}(3 + 2a_{ii}\tau_{ii})}{1 - \frac{1}{9} l_{ii}\tau_{ii}(3 + 2a_{ii}\tau_{ii})}|l_{ij}| \quad \text{for } j \neq i,\ \tilde{l}_{ii} = l_{ii},$$

is a non-singular M-matrix, together with the 3/2-type condition $l_{ii}\tau_{ii} < 3/2$, $i = 1, \dots, n$. For generalization of [22] to non-autonomous linear systems $x_i'(t) = -\sum_{1\leqslant j\leqslant n} l_{ij}(t)x_j(t-\tau_{ij}(t))$, $i = 1, \dots, n$, see [23].

Example 2.1. Consider a *scalar* linear FDE on $C_1 = C([-\tau, 0]; \mathbb{R})$ of the form

$$x'(t) = -\big[b_0x(t) + L_0(x_t)\big],$$

where $b_0 \in \mathbb{R}$ and $L_0 : C_1 \to \mathbb{R}$ is a linear bounded operator. We write $L_0(\varphi) = l_0 \int_{-\tau}^0 \varphi(\theta)\, d\eta(\theta)$, for $|l_0| = \|L_0\|$ and some normalized bounded variation function $\eta : [-\tau, 0] \to \mathbb{R}$. From Theorem 2.6, the following result is derived:

Corollary 2.8. *Let $b_0, l_0 \in \mathbb{R}$ be given. Then, the scalar linear FDE*

$$x'(t) = -\left[b_0x(t) + l_0 \int_{-\tau}^{0} x(t+\theta)\, d\eta(\theta)\right] \tag{2.12}$$

is exponentially asymptotically stable for all the choices of $\tau > 0$ and $\eta \in BV([-\tau, 0]; \mathbb{R})$ with $\mathrm{Var}_{[-\tau,0]}\, \eta = 1$ *if and only if*

$$b_0 + l_0 \int_{-\tau}^{0} d\eta(\theta) > 0, \quad b_0 \geqslant |l_0|. \tag{2.13}$$

Remark 2.3. The above result was established in [2], where the general case of a linear scalar FDE $x'(t) = -L(x_t)$, $L : C_1 \to \mathbb{R}$ a linear bounded operator, was studied. Moreover, it was proven in [2] that if $L(1) > 0$ and L satisfies the hypothesis

(H2*) for all $\varphi \in C_1$ such that $|\varphi(\theta)| < \varphi(0)$ for $\theta \in [-r, 0)$, then $L(\varphi) > 0$,

then L has the form

$$L(\varphi) = b_0\varphi(0) + L_0(\varphi), \quad \varphi \in C_1, \tag{2.14}$$

for some $b_0 > 0$ and (non-atomic at zero) linear bounded operator $L_0 : C_1 \to \mathbb{R}$, for which (2.13) holds. Conversely, if (2.13) holds, then L given by (2.14) satisfies $L(1) > 0$ and (H2*). In the next section, the relevance of assumption (H2*), translated to the general framework of n-dimensional FDEs $x' = f(t, x_t)$, will become clear.

Example 2.2. Consider the following model for a ring of neurons with distributed delays

$$u_i'(t) = -b_iu_i(t) + \alpha_{ii}g_i(u_{t,i}) + \alpha_{i,i-1}g_{i-1}(u_{t,i-1}), \quad i = 1, \dots, n, \tag{2.15}$$

with the convention $i-1=n$ for $i=1$, where $g_i : C([-\tau,0];\mathbb{R}) \to \mathbb{R}$ are smooth functions with $g_i(0)=0$ and rescaled so that $g_i'(0)(1)=1$, $i=1,\dots,n$. The particular case of (2.15) with discrete delays,

$$u_i'(t) = -b_i u_i(t) + \alpha_{ii} g_i\big(u_i(t-\tau_i)\big) + \alpha_{i,i-1} g_{i-1}\big(u_{i-1}(t-\tau_{i-1})\big), \quad i=1,\dots,n,$$

$g_i : \mathbb{R} \to \mathbb{R}$, was studied in [1]. More generally, most of the literature on Hopfield neural networks with delays addresses models that take the form

$$u_i'(t) = -b_i u_i(t) + \sum_{j=1}^{n} a_{ij} g_j\big(u_j(t-\tau_{ij})\big), \quad i=1,\dots,n, \tag{2.16}$$

where $\tau_{ij} \geqslant 0$ are the synaptic transmission time-delays, $b_i > 0$ is related to the input capacity of neuron i, $A=[a_{ij}]$ is the connection matrix and $g_i : \mathbb{R} \to \mathbb{R}$ are C^1 sigmoidal-type activation functions, for which we may suppose (after translating an equilibrium to the origin and a scaling) that $g_i(0)=0$, $g_i'(0)=1$. For several criteria on local and global stability for such models, see e.g. [1,6,16,26,27], also for other relevant references.

For the concrete model (2.15), the next result generalizes [1, Theorem 4.1] to the situation with distributed delays.

Theorem 2.9. *Suppose that $g_i : C_1 \to \mathbb{R}$ are C^1-functions such that $g_i(0)=0$ and $g_i'(0)(1)=1$. For $\gamma_i = \|g_i'(0)\|$, if*

$$\prod_{i=1}^{n} (b_i + \alpha_{ii}) > \prod_{i=1}^{n} \alpha_{i,i-1} \tag{2.17}$$

and

$$|\alpha_{ii}|\gamma_i \leqslant b_i, \quad i=1,\dots,n, \qquad \left| \prod_{i=1}^{n} \alpha_{i,i-1}\gamma_{i-1} \right| \leqslant \prod_{i=1}^{n} \big(b_i - |\alpha_{ii}|\gamma_i\big), \tag{2.18}$$

then the trivial equilibrium of (2.15) *is asymptotically stable.*

Proof. The linearized equation about zero has the form (2.1), with $L_{ii} = \alpha_{ii} g_i'(0)$, $L_{i,i-1} = \alpha_{i,i-1} g_{i-1}'(0)$ and $L_{ij}=0$ for $j \neq i$, $j \neq i-1$. From Theorem 2.3, $\det M \neq 0$ and $\hat{N}$ is an M-matrix imply the asymptotic stability of the trivial solution of (2.15). Here, $M = B + A$, for $B = \operatorname{diag}(b_1,\dots,b_n)$ and $A=(a_{ij})$, where $a_{ij} = -\alpha_{ij}$ for $j=i,i-1$ and $a_{ij}=0$ for $j \neq i$, $j \neq i-1$; and $\hat{N} = B - |C|$, for $C=(c_{ij})$, with $c_{ij} = -\alpha_{ij}\gamma_j$ for $j=i,i-1$, and zero otherwise. It is easy to check that (2.18) is equivalent to saying that $\hat{N}$ is an M-matrix. Together with (2.18), (2.17) means that $\det M \neq 0$. □

3. Global stability for Lotka–Volterra systems

The results in this section concern global stability for n species delayed Lotka–Volterra models. We consider autonomous systems given by

$$x_i'(t) = r_i x_i(t)\big[1 - b_i x_i(t) - L_i(x_t)\big], \quad i = 1, \dots, n, \tag{3.1}$$

where $b_i \in \mathbb{R}$, $r_i > 0$ and $L_i : C \to \mathbb{R}$ are linear bounded operators. More generally, we shall also consider non-autonomous systems of FDEs of the form

$$x_i'(t) = r_i(t) x_i(t)\big[\alpha_i - b_i x_i(t) - L_i(x_t)\big], \quad i = 1, \dots, n,$$

where b_i, L_i are as in (3.1), and $\alpha_i \in \mathbb{R}$, $r_i : [0, \infty) \to (0, \infty)$ are continuous functions. For the sake of simplicity, we take $\alpha_i = 1$, $i = 1, \dots, n$, and write

$$x_i'(t) = r_i(t) x_i(t)\big[1 - b_i x_i(t) - L_i(x_t)\big], \quad i = 1, \dots, n. \tag{3.2}$$

As in Section 2, we write L_i as in (2.2), for some $l_{ij} \in \mathbb{R}$ and $\eta_{ij} \in BV([-\tau, 0], \mathbb{R})$ with $\mathrm{Var}_{[-\tau,0]}\, \eta_{ij} = 1$, and denote $a_{ij} = L_{ij}(1)$, $i, j = 1, \dots, n$. Again, $B = \mathrm{diag}(b_1, \dots, b_n)$, $M = B + [a_{ij}]$, $N = B + [l_{ij}]$ and $\tilde{M}$, $\hat{N}$ are as in (2.4).

In the sequel, for (3.2) the following hypotheses will be considered:

(H1) there is a vector $x^* = (x_1^*, \dots, x_n^*) > 0$ such that $Mx^* = [1, \dots, 1]^T$, i.e., x^* is a *positive* equilibrium of (3.2);
(H2) there is $d = (d_1, \dots, d_n) > 0$ such that $\|L_i\|_{d^{-1}} \leqslant d_i b_i$, $i = 1, \dots, n$;
(H3) $\det \tilde{M} \neq 0$;
(H4) $r_i(t)$ is uniformly bounded on $[0, \infty)$ and $\int^\infty r_i(t)\, dt = \infty$, $i = 1, \dots, n$.

If $x^* = (x_1^*, \dots, x_n^*)$ is a positive equilibrium of (3.2), for $y_i(t) = x_i(t) - x_i^*$ system (3.2) becomes

$$y_i'(t) = -r_i(t)\big(y_i(t) + x_i^*\big)\big[b_i y_i(t) + L_i(y_t)\big], \quad i = 1, \dots, n. \tag{3.3}$$

Due to the biological interpretation of the model, we restrict our attention to *positive* solutions of (3.2). Therefore, for (3.2) we take the set of *admissible initial conditions* as the set

$$C_{\hat{0}} = \big\{\varphi = (\varphi_1, \dots, \varphi_n) \in C_n\colon\ \varphi_i(\theta) \geqslant 0 \text{ for } \theta \in [-\tau, 0),\ \varphi_i(0) > 0,\ i = 1, \dots, n\big\},$$

and only consider solutions of (3.2) with initial conditions

$$x_{t_0} = \varphi, \quad \varphi \in C_{\hat{0}}, \tag{3.4}$$

for some $t_0 \geqslant 0$. The solution of (3.2)–(3.4) is denoted by $x(t, t_0, \varphi)$; for $t_0 = 0$, we also write $x(t, 0, \varphi) = x(t, \varphi)$. We often suppose that the initial condition (3.4) is fixed, and write simply $x(t)$ for $x(t, t_0, \varphi)$. Since $x_i(t, t_0, \varphi) = x_i(t_0)\exp(\int_{t_0}^t r_i(s)[1 - b_i x_i(s) - L_i(x_s)]\, ds) > 0$, it is clear that a solution $x(t, t_0, \varphi)$ with initial condition in $C_{\hat{0}}$ is an *admissible solution*, in the sense that $x_t(t_0,\varphi) \in C_{\hat{0}}$, whenever it is defined. Accordingly, if (H1) holds, the set of *admissible* initial conditions for (3.3) is the set $C_{-x^*} = C_{\hat{0}} - x^*$,

$$C_{-x^*} = \big\{(\varphi_1,\dots,\varphi_n) \in C\colon\ \varphi_i(\theta) \geqslant -x_i^* \text{ for } \theta \in [-\tau,0),\ \varphi_i(0) > -x_i^*,\ i=1,\dots,n\big\},$$

and the solutions $y_t(t_0,\varphi)$ of (3.3) with initial conditions $y_{t_0} = \varphi \in C_{-x^*}$ are *admissible* solutions.

In this section, we study the global asymptotic stability of the positive equilibrium x^* of (3.1), or (3.2), if it exists. If in addition $\det M \neq 0$, then the positive equilibrium of (3.2) is unique. For (3.1), its local stability is deduced from Theorem 2.3:

Theorem 3.1. *Suppose that x^* is a positive equilibrium of the autonomous system* (3.1). *If* $\det M \neq 0$ *and $\hat{N}$ is an M-matrix, then x^* is asymptotically stable.*

Next, we prove some auxiliary results, for which it is convenient to write (H2) in a more suitable form. From Lemma 2.1, (H2) is equivalent to saying that there is $d=(d_1,\dots,d_n)>0$ such that (2.5) holds, i.e., $\hat{N}d \geqslant 0$. (H2) implies the inequalities

$$d_i(b_i + a_{ii}) \geqslant \sum_{j\neq i} d_j|a_{ij}|, \quad i=1,\dots,n. \tag{3.5}$$

It also implies that $\hat{N}$ is an M-matrix [4]. In general, the reverse is not true for $n \geqslant 2$: the matrix $D = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 1\end{smallmatrix}\right)$ is an M-matrix but there is no $d>0$ such that $Dd \geqslant 0$. On the other hand, since $\tilde{M} \geqslant \hat{N}$, if $\hat{N}$ is an M-matrix, the same happens to $\tilde{M}$; together with $\det \tilde{M} \neq 0$, this means that $\tilde{M}$ is a non-singular M-matrix, thus there is $c=(c_1,\dots,c_n)>0$ such that $\tilde{M}c>0$ (see [4]). However, if (H2) and (H3) hold, one cannot conclude that $\tilde{M}d>0$, for the same vector $d>0$ as in (H2). Also, we recall that if $\tilde{M}$ is a non-singular M-matrix, then $\det M \neq 0$ [4, Theorem 5.17]; conversely, for any $n \geqslant 2$, we might have $\det M \neq 0$ and $\tilde{M}$ a singular M-matrix. In particular, we observe that, under (H1)–(H3), x^* is the unique positive equilibrium of (3.1), or (3.2).

By effecting the change $z_i(t) = d_i^{-1}y_i(t)$, $i=1,\dots,n$, where $d_1,\dots,d_n>0$ are as in (H2), (3.3) becomes

$$z_i'(t) = -r_i(t)\big(z_i(t) + d_i^{-1}x_i^*\big)\big[\hat{b}_i z_i(t) + \hat{L}_i(z_t)\big], \quad i=1,\dots,n, \tag{3.6}$$

with $\hat{b}_i = b_i d_i$, $\hat{a}_{ij} = a_{ij}d_j$, $\hat{L}_i(\phi) = L_i((d_j\phi_j)_{j=1}^n) = \sum_{j=1}^n d_j L_{ij}(\phi_j)$.

With the previous notations, we get $\|\hat{L}_i\| = \|L_i\|_{d^{-1}}$. Consequently, if hypothesis (H2) holds for system (3.3), then for (3.6) we have

$$\|\hat{L}_i\| \leqslant \hat{b}_i.$$

Assuming (H2), one may therefore assume without loss of generality that after translating x^* to the origin and a scaling of the variables, (3.2) is transformed into (3.3), with $\|L_i\| \leqslant b_i$, $i=1,\dots,n$.

A first lemma is stated in the more general framework of $\mathbb{R}^n$ with a norm $|\cdot|_d$, for some $d \in \mathbb{R}^n$, $d>0$. Naturally, for FDEs in $\mathbb{R}^n$ for which a set $S \subset C_n = C([-\tau,0];\mathbb{R}^n)$ is chosen as the set of admissible initial conditions, a solution $y(t)$ with initial condition $y_{t_0} = \varphi \in S$ is said to be admissible if $y_t \in S$ for $t>t_0$ whenever y_t is defined.

Lemma 3.2. *Choose a set $S \subset C_n$ as the set of admissible initial conditions for*

$$y'(t) = f(t, y_t), \quad t \geqslant t_0, \tag{3.7}$$

where $f : [t_0, \infty) \times S \to \mathbb{R}^n$ is continuous, $f = (f_1, \dots, f_n)$. Let $\mathbb{R}^n$ be equipped with a norm $|\cdot|_d$, for some $d = (d_1, \dots, d_n)$ with $d_i > 0$, and assume that f satisfies

(H2*) *for all $t \geqslant t_0$ and $\varphi \in S$ such that $|\varphi(\theta)|_d < |\varphi(0)|_d$ for $\theta \in [-\tau, 0)$, then $\varphi_i(0) f_i(t, \varphi) < 0$, for some $i \in \{1, \dots, n\}$ such that $|\varphi(0)|_d = d_i|\varphi_i(0)|$.*

Then, all admissible solutions of (3.7) *are defined and bounded for $t \geqslant t_0$. Moreover, if $y(t) = y(t, t_0, \varphi)$ $(\varphi \in S)$ is an admissible solution of* (3.7) *and $|y(t)|_d \leqslant K$ for $t \in [t_0 - \tau, t_0]$, then $|y(t)|_d \leqslant K$ for $t \geqslant t_0$.*

Proof. Let $y(t)$ be an admissible solution of (3.7) on $[t_0 - \tau, a)$ for some $a > t_0$, with $|y(t)|_d \leqslant K$ for $t \in [t_0 - \tau, t_0]$. Suppose that there is $t_1 > t_0$ such that $|y(t_1)|_d > K$, and define

$$T = \min\Big\{t \in [t_0, t_1]:\ \max_{s \in [t_0, t_1]} \big|y(s)\big|_d = \big|y(t)\big|_d\Big\}.$$

We have $|y(T)|_d > K$ and

$$\big|y(t)\big|_d < \big|y(T)\big|_d \quad \text{for } t \in [t_0, T).$$

Hence $|y_T(\theta)|_d = |y(T + \theta)|_d < |y(T)|_d$ for $-\tau \leqslant \theta < 0$. By (H2*), there is $i \in \{1, \dots, n\}$ such that $|y(T)|_d = d_i|y_i(T)|$ and $y_i(T) f_i(t, y_T) < 0$ for all $t \geqslant t_0$. Suppose that $y_i(T) > 0$ (the situation $y_i(T) < 0$ is analogous). Since $d_i y_i(t) \leqslant |y(t)|_d < d_i y_i(T)$ for $t_0 - \tau \leqslant t < T$, then $y_i'(T) \geqslant 0$. On the other hand, from (3.7) we have $y_i'(T) = f_i(T, y_T) < 0$, a contradiction. This proves that $y(t)$ is extensible to $[t_0 - \tau, \infty)$, with $|y(t)|_d \leqslant K$ for all $t > t_0$. □

Theorem 3.3. *Let $x_i^* > 0$, $r_i(t) > 0$ for $t \geqslant 0$, $i = 1, \dots, n$, and $S = C_{-x^*}$. If $\det M \neq 0$ and* (H2) *holds, then* (3.3) *satisfies* (H2*) *on* $[0, \infty)$. *In particular, all* (*admissible*) *solutions of* (3.2) *are defined and bounded on* $[0, \infty)$.

Proof. As observed above, we may assume that (3.3) satisfies the condition $\|L_i\| \leqslant b_i$, $i = 1, \dots, n$. Eq. (3.3) reads as (3.7), for $f_i(t, \varphi) = -r_i(t)(\varphi_i(0) + x_i^*)(b_i\varphi_i(0) + L_i(\varphi))$, $i = 1, \dots, n$. Let $t \geqslant 0$, $\varphi \in S$ and suppose $|\varphi(\theta)|_\infty < |\varphi(0)|_\infty$ for $\theta \in [-\tau, 0)$. Set $K = |\varphi(0)|_\infty$. Consider the partition $I = I_1 \cup I_2 \cup I_3$ of $I := \{1, \dots, n\}$, where

$$I_1 = \big\{i \in I:\ \varphi_i(0) = K\big\}, \qquad I_2 = \big\{i \in I:\ \varphi_i(0) = -K\big\}, \qquad I_3 = \big\{i \in I:\ \big|\varphi_i(0)\big| < K\big\}.$$

Define

$$-\gamma_1 = \min_{i \in I_1} \min_{\theta \in [-\tau, 0]} \varphi_i(\theta) > -K,$$

$$\gamma_2 = \max_{i \in I_2} \max_{\theta \in [-\tau, 0]} \varphi_i(\theta) < K,$$

$$\gamma_3 = \max_{i \in I_3} \max_{\theta \in [-\tau, 0]} \big|\varphi_i(\theta)\big| < K,$$

and $\varepsilon_0 = \min_{1\leqslant k\leqslant 3}(K-\gamma_k)/2$. Consider

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n, \quad \text{with } \varepsilon_i = \begin{cases} \varepsilon_0, & i \in I_1, \\ -\varepsilon_0, & i \in I_2, \\ 0, & i \in I_3. \end{cases}$$

For $\#I_k = n_k$, $k = 1, 2, 3$, we may suppose that I is ordered in such a way that

$$I_1 = \{1, \dots, n_1\}, \qquad I_2 = \{n_1+1, \dots, n_1+n_2\}, \qquad I_3 = \{n_1+n_2+1, \dots, n\},$$

so that ε reads as $\varepsilon = \varepsilon_0(1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$, with the obvious notation for dots.

From the definition of ε_0, it is easy to check that $|\varphi_i(\theta) - \varepsilon_i| \leqslant K - \varepsilon_0$ for all $i \in I$, hence $\|\varphi - \varepsilon\|_\infty \leqslant K - \varepsilon_0$ and $|L_i(\varphi - \varepsilon)| \leqslant b_i(K - \varepsilon_0)$, $1 \leqslant i \leqslant n$.

For $i \in I_1$, from (H2) we have

$$\begin{aligned} b_i\varphi_i(0) + L_i(\varphi) &= \varepsilon_0 b_i + \big(\varphi_i(0) - \varepsilon_0\big)b_i + L_i(\varphi - \varepsilon) + L_i(\varepsilon) \\ &\geqslant \varepsilon_0 b_i + L_i(\varepsilon) = \varepsilon_0\Bigg[(b_i + a_{ii}) + \sum_{j\in I_1,\, j\neq i} a_{ij} - \sum_{j\in I_2} a_{ij}\Bigg]. \end{aligned} \tag{3.8}$$

Analogously, for $i \in I_2$ we obtain

$$\begin{aligned} b_i\varphi_i(0) + L_i(\varphi) &= -\varepsilon_0 b_i + \big(\varphi_i(0) + \varepsilon_0\big)b_i + L_i(\varphi - \varepsilon) + L_i(\varepsilon) \\ &\leqslant -\varepsilon_0 b_i + L_i(\varepsilon) = \varepsilon_0\Bigg[-(b_i + a_{ii}) + \sum_{j\in I_1} a_{ij} - \sum_{j\in I_2,\, j\neq i} a_{ij}\Bigg]. \end{aligned} \tag{3.9}$$

From (3.5), (3.8) and (3.9), we conclude that

$$\varphi_i(0)\big(b_i\varphi_i(0) + L_i(\varphi)\big) \geqslant 0, \quad i \in I_1 \cup I_2.$$

If there is $i \in I_1 \cup I_2$ such that $\varphi_i(0)(b_i\varphi_i(0) + L_i(\varphi)) > 0$, then (H2*) holds. If $\varphi_i(0)(b_i\varphi_i(0) + L_i(\varphi)) = 0$ for all $i \in I_1 \cup I_2$, from (3.8) and (3.9) we deduce that

$$\sum_{j\in I_3} |a_{ij}| = 0, \quad i \in I_1 \cup I_2,$$

i.e., $a_{ij} = 0$, $i \in I_1 \cup I_2$, $j \in I_3$. (Note that this includes the case $I_3 = \emptyset$; however, $I_1 \cup I_2 \neq \emptyset$.) Hence, one can write

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \tag{3.10}$$

with M_{ij} matrices of dimensions $n_i \times n_j$, $i, j = 1, 2, 3$, and $M_{13} = 0$, $M_{23} = 0$. Again from (3.5), (3.8) and (3.9), and the definition of the vector ε, we have

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \eta = 0,$$

where $\varepsilon = (\eta, 0)$ and η is an $(n_1 + n_2) \times 1$ vector. But this is a contradiction since $\det M \neq 0$, and $M_{13} = 0$, $M_{23} = 0$ in (3.10) imply that $\det\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \neq 0$. $\square$

After having established the boundedness of positive solutions of (3.2), we are in a position to prove the main theorem in this section. In fact, our main result shows the asymptotic constancy of bounded solutions for a system more general than (3.2), as follows:

Theorem 3.4. *Consider the system*

$$x_i'(t) = r_i(t)x_i(t)\big[\alpha_i - b_i x_i(t) - L_i(x_t) - h_i(t)\big], \quad i = 1, \dots, n, \tag{3.11}$$

where $\alpha_i \in \mathbb{R}$, $b_i > 0$, $L_i : C_n \to \mathbb{R}$ are linear bounded operators, $r_i : [0, \infty) \to (0, \infty)$, $h_i : [0, \infty) \to \mathbb{R}$ are continuous functions, $i = 1, \dots, n$, with

$$h_i(t) \to 0, \quad t \to \infty,\ i = 1, \dots, n. \tag{3.12}$$

With the above notation, assume (H2)–(H4) *and that there is $x^* = (x_1^*, \dots, x_n^*) > 0$ such that $Mx^* = [\alpha_1, \dots, \alpha_n]^T$. Then, any positive solution $x(t)$ of* (3.11) *defined and bounded on* $[0, \infty)$ *satisfies $x(t) \to x^*$ as $t \to \infty$.*

Proof. By translating x_* to the origin by the change $y(t) = x(t) - x^*$, (3.11) becomes

$$y_i'(t) = -r_i(t)\big(y_i(t) + x_i^*\big)\big[b_i y_i(t) + L_i(y_t) + h_i(t)\big], \quad i = 1, \dots, n, \tag{3.13}$$

for which C_{-x^*} is the set of admissible initial conditions. As in the proof of Theorem 3.3, after a scaling we may assume (H2) with $d = (1, \dots, 1)$, i.e.,

$$\|L_i\| \leqslant b_i, \quad i \in \{1, \dots, n\} := I.$$

Let $y(t) = (y_i(t))_{i=1}^n$ be an admissible solution to (3.13), defined and bounded for $t \geqslant 0$. Set

$$-v_i = \liminf_{t\to\infty} y_i(t), \qquad u_i = \limsup_{t\to\infty} y_i(t), \quad i \in I,$$

and

$$v = \max_{1\leqslant i\leqslant n} v_i, \qquad u = \max_{1\leqslant i\leqslant n} u_i.$$

Note that $-x_i^* \leqslant -v_i \leqslant u_i < \infty$, $i \in I$.

It is sufficient to prove that $\max(u, v) = 0$. Assume e.g. that $|v| \leqslant u$, so that $\max(u, v) = u$. (The situation is analogous for $|u| \leqslant v$.)

Consider the decomposition of I, $I = I_1 \cup I_2 \cup I_3$, where

$$I_1 = \{i \in I\colon\ u_i = u\}, \qquad I_2 = \{i \in I\colon\ v_i = u,\ u_i < u\}, \qquad I_3 = \{i \in I\colon\ -u < -v_i \leqslant u_i < u\}.$$

Since $|v| \leqslant u$, then $I_1 \neq \emptyset$. Observe that the situation where one or both sets I_2, I_3 are empty is included in our setting. The proof is divided in several steps.

Claim 1. *For each $i \in I_1 \cup I_2$, there is a sequence (t_k^i) with $t_k^i \nearrow \infty$, $b_i y_i(t_k^i) + L_i(y_{t_k^i}) \to 0$, and $y_i(t_k^i) \to u$ if $i \in I_1$, $y_i(t_k^i) \to -u$ if $i \in I_2$, as $k \to \infty$.*

To prove Claim 1, for each $i \in I_1 \cup I_2$ we shall consider separately the cases of $y_i(t)$ eventually monotone and not eventually monotone.

Case 1. Assume that $y_i(t)$ is not eventually monotone.

Let $i \in I_1$, and consider (t_k^i) with $t_k^i \nearrow \infty$ as $k \to \infty$, a sequence of local maximum points so that $y_i(t_k^i) \to u_i = u$. Clearly, $y_i'(t_k^i) = 0 = b_i y_i(t_k^i) + L_i(y_{t_k^i}) + h_i(t_k^i) = \lim(b_i y_i(t_k^i) + L_i(y_{t_k^i}))$. For $i \in I_2$, the claim follows by considering a sequence of local minimum points (t_k^i) with $t_k^i \nearrow \infty$, $y_i(t_k^i) \to -u$ as $k \to \infty$.

Case 2. Assume that $y_i(t)$ is eventually monotone.

Let $i \in I_1 \cup I_2$. In this case,

$$\lim_{t\to\infty} y_i(t) = u \quad \text{if } i \in I_1 \quad \text{and} \quad \lim_{t\to\infty} y_i(t) = -u \quad \text{if } i \in I_2, \tag{3.14}$$

and for t large, either $y_i'(t) \leqslant 0$ or $y_i'(t) \geqslant 0$. If $y_i'(t) \geqslant 0$ for t large, then $b_i y_i(t) + L_i(y_t) + h_i(t) \leqslant 0$, hence

$$\limsup_{t\to\infty}\big(b_i y_i(t) + L_i(y_t) + h_i(t)\big) = \limsup_{t\to\infty}\big(b_i y_i(t) + L_i(y_t)\big) := c \leqslant 0.$$

If $c < 0$, then there is $t_1 > 0$ such that $b_i y_i(t) + L_i(y_t) + h_i(t) < c/2$ for $t \geqslant t_1$, implying that $y_i'(t) \geqslant -c r_i(t)(y_i(t) + x_i^*)/2$ and

$$y_i(t) + x_i^* \geqslant \big(y_i(t_1) + x_i^*\big)\exp\left(-\frac{c}{2}\int_{t_1}^{t} r_i(s)\,ds\right), \quad t \geqslant t_1.$$

From (H4) and the above inequality, we obtain $y_i(t) \to \infty$ as $t \to \infty$, which is not possible. Thus $c = 0$, which proves the claim.

If $y_i'(t) \leqslant 0$ for t large, in a similar way we get

$$\liminf_{t\to\infty}\big(b_i y_i(t) + L_i(y_t) + h_i(t)\big) = \liminf_{t\to\infty}\big(b_i y_i(t) + L_i(y_t)\big) := d \geqslant 0.$$

Suppose that $d > 0$. For any $\varepsilon > 0$, there is t_2 such that for $t \geqslant t_2$ we have $b_i y_i(t) + L_i(y_t) + h_i(t) > d/2$ and $\|y_t\| \leqslant u + \varepsilon$. Then, for $t \geqslant t_2$

$$0 < y_i(t) + x_i^* \leqslant \big(y_i(t_2) + x_i^*\big)\exp\left(-\frac{d}{2}\int_{t_2}^{t} r_i(s)\,ds\right) \to 0 \quad \text{as } t \to \infty.$$

We therefore conclude that

$$x_i^* + \lim_{t\to\infty} y_i(t) = 0. \tag{3.15}$$

Since we have assumed $u \geqslant 0$, (3.14) and (3.15) imply that $i \notin I_1$; and for $i \in I_2$, then $u = x_i^*$. But, for $t \geqslant t_2$

$$0 < d/2 \leqslant b_i y_i(t) + L_i(y_t) + h_i(t) \leqslant b_i y_i(t) + b_i(u+\varepsilon) + h_i(t) \to b_i\varepsilon, \quad t \to \infty.$$

Since $\varepsilon > 0$ is arbitrary, this is a contradiction. Hence $d = 0$, and Claim 1 is proven.

Claim 2. *For $i \in I_1 \cup I_2$, there is a sequence (t_k^i), $t_k^i \nearrow \infty$, such that $y_{t_k^i} \to \varphi^i = (\varphi_1^i, \dots, \varphi_n^i) \in C_n$ as $k \to \infty$, with*

$$b_i\varphi_i^i(0) + L_i(\varphi^i) = 0, \quad \varphi_i^i(0) = \begin{cases} u & \text{if } i \in I_1, \\ -u & \text{if } i \in I_2, \end{cases} \quad \text{and}$$
$$-v_j \leqslant \varphi_j^i(\theta) \leqslant u_j, \quad 1 \leqslant j \leqslant n,\ -\tau \leqslant \theta \leqslant 0.$$

Suppose that $i \in I_1$ (the situation $i \in I_2$ is treated in an analogous way). From Claim 1, let (t_k^i) be a sequence with $t_k^i \nearrow \infty$, $b_i y_i(t_k^i) + L_i(y_{t_k^i}) \to 0$ and $y_i(t_k^i) \to u$ as $k \to \infty$. Consider $(y_{t_k^i}) \subset C_n$, and fix $\varepsilon > 0$. Clearly $(y_{t_k^i})$ is uniformly bounded with $\|y_{t_k^i}\| \leqslant u + \varepsilon$ for $k \geqslant k_0$. On the other hand, from (3.12) and (H4) it follows that $y'(t)$ is uniformly bounded for $t \geqslant 0$, thus $(y_{t_k^i})$ is equicontinuous. By Ascoli–Arzelà theorem, for a subsequence, still denoted by $(y_{t_k^i})$, we have $y_{t_k^i} \to \varphi^i$ for some $\varphi^i = (\varphi_1^i, \dots, \varphi_n^i) \in C_n$. By letting $k \to \infty$ and $\varepsilon \to 0$, we conclude that φ^i satisfies all the requirements in Claim 2.

In the remaining proof, sequences (t_k^i) as in Claim 2 are supposed to be fixed, and φ^i denotes the limit in C_n of $(y_{t_k^i})$.

Observe that for $i \in I_1 \cup I_2$ and $j \in I_2 \cup I_3$, we have $\max_{\theta \in [-\tau,0]} \varphi_j^i(\theta) < u$. Now, define

$$J^i = \Big\{ j \in I_1\colon \min_{\theta \in [-\tau,0]} \varphi_j^i(\theta) = -u,\ \max_{\theta \in [-\tau,0]} \varphi_j^i(\theta) = u \Big\}, \quad i \in I_1 \cup I_2.$$

Claim 3. *If $u > 0$, then $J^i = \emptyset$ for all $i \in I_1 \cup I_2$.*

Let $u > 0$, and fix $\varepsilon > 0$ small. For some t_0, we have $\|y_t\| \leqslant u + \varepsilon$ and $|h_j(t)| \leqslant \varepsilon b_j$ for all $j \in I$ and $t \geqslant t_0$. Consider e.g. $i \in I_1$ and $j \in J^i$. Let $\theta_1, \theta_2 \in [-\tau, 0]$ be such that

$$u = \varphi_j^i(\theta_1) = \lim_k y_j(t_k^i + \theta_1), \qquad -u = \varphi_j^i(\theta_2) = \lim_k y_j(t_k^i + \theta_2).$$

Case 1. $\theta_2 < \theta_1$. From (H2) we obtain

$$y_j'(t) \leqslant b_j r_j(t)\big(y_j(t) + x_j^*\big)\big(u + 2\varepsilon - y_j(t)\big).$$

By integrating over an interval $[s,t] \subset [t_0, \infty)$, we derive

$$\big(y_j(t) + x_j^*\big)\big(u + 2\varepsilon - y_j(s)\big) \leqslant \big(y_j(s) + x_j^*\big)\big(u + 2\varepsilon - y_j(t)\big) \times \exp\left(\big(x_j^* + u + 2\varepsilon\big)b_j \int_s^t r_j(\sigma)\,d\sigma\right), \quad t \geqslant s \geqslant t_0. \quad (3.16)$$

From (H4), there is $\beta > 0$ such that $r_i(t) \leqslant \beta$, $t \geqslant 0$. For $t = t_k^i + \theta_1$, $s = t_k^i + \theta_2$ in (3.16), by letting $k \to \infty$ we conclude that

$$\big(u + x_j^*\big)(2u + 2\varepsilon) \leqslant 2\varepsilon\big(-u + x_j^*\big)\exp\big(\big(x_j^* + u + 2\varepsilon\big)b_j\beta\tau\big).$$

Since $\varepsilon > 0$ is arbitrarily small, we conclude that $u = 0$, which contradicts our assumption.

Case 2. $\theta_1 < \theta_2$. For this situation, we first prove that $u < x_j^*$. Since $|b_j y_j(t) + L_j(y_t) + h_j(t)| \leqslant b_j(2u + 3\varepsilon)$ for t large, then

$$y_j'(t) \geqslant -b_j(2u + 3\varepsilon)r_j(t)\big(y_j(t) + x_j^*\big),$$

leading to

$$\big(y_j(t) + x_j^*\big) \geqslant \big(y_j(s) + x_j^*\big)\exp\left(-b_j(2u + 3\varepsilon)\int_s^t r_j(\sigma)\,d\sigma\right), \quad t \geqslant s \geqslant t_0, \qquad (3.17)$$

for some t_0 large. With $t = t_k^i + \theta_2$, $s = t_k^i + \theta_1$ in (3.17), by letting $k \to \infty$, $\varepsilon \to 0^+$, we get

$$\big(-u + x_j^*\big) \geqslant \big(u + x_j^*\big)\exp(-2b_j u\beta\tau) > 0,$$

and hence $u < x_j^*$.

Now, let $\varepsilon > 0$ be small so that $u + 2\varepsilon < x_j^*$. For $t \geqslant t_0$, we have

$$y_j'(t) \geqslant -b_j r_j(t)\big(y_j(t) + x_j^*\big)\big(u + 2\varepsilon + y_j(t)\big),$$

and integration over an interval $[s, t] \subset [t_0, \infty)$ yields

$$\begin{aligned}&\big(y_j(t) + u + 2\varepsilon\big)\big(y_j(s) + x_j^*\big)\\ &\quad \geqslant \big(y_j(s) + u + 2\varepsilon\big)\big(y_j(t) + x_j^*\big)\exp\big(-\big(x_j^* - u - 2\varepsilon\big)b_j\beta(t - s)\big), \quad t \geqslant s \geqslant t_0. \qquad (3.18)\end{aligned}$$

From (3.18) with $t = t_k^i + \theta_2$, $s = t_k^i + \theta_1$, by letting $k \to \infty$, $\varepsilon \to 0^+$, we obtain

$$0 \geqslant 2u\big(x_j^* - u\big)\exp\big(-b_j\big(x_j^* - u\big)\beta\tau\big),$$

and therefore conclude that $u = 0$, which is a contradiction.

For $i \in I_2$, the proof of $J^i = \emptyset$ is similar.

Claim 4. *$y(t) \to 0$ as $t \to \infty$.*

Recall that we are considering the case $|v| \leqslant u$. For the sake of contradiction, assume that $u > 0$.

Fix $i \in I_1 \cup I_2$, and choose $\varphi^i \in C_n$ as in Claim 2. Since $J^i = \emptyset$ from Claim 3, the definition of I_j, $j = 1, 2, 3$, leads to

$$\text{either} \quad \min_{\theta \in [-\tau, 0]} \varphi_j^i(\theta) > -u \quad \text{or} \quad \max_{\theta \in [-\tau, 0]} \varphi_j^i(\theta) < u, \quad j \in I.$$

Consider now the partition of I

$$I = I_1^i \cup I_2^i \cup I_3,$$

where I_3 is as above and

$$I_1^i = \Big\{ j \in I_1 \cup I_2\colon \min_{\theta \in [-\tau,0]} \varphi_j^i(\theta) > -u \Big\}, \qquad I_2^i = \Big\{ j \in I_1 \cup I_2\colon \min_{\theta \in [-\tau,0]} \varphi_j^i(\theta) = -u \Big\}.$$

Note that the set I_3 does not depend on i; also, $i \in I_1^i$ if $i \in I_1$ and $i \in I_2^i$ if $i \in I_2$.

We now adapt the procedure followed in the proof of Theorem 3.3. For $i \in I_1 \cup I_2$, define

$$-\gamma_1^i = \min_{j \in I_1^i} \min_{-\tau \leqslant \theta \leqslant 0} \varphi_j^i(\theta) > -u,$$

$$\gamma_2^i = \max_{j \in I_2^i} \max_{-\tau \leqslant \theta \leqslant 0} \varphi_j^i(\theta) < u,$$

$$\gamma_3^i = \max_{j \in I_3} \max_{-\tau \leqslant \theta \leqslant 0} \big|\varphi_j^i(\theta)\big| < u,$$

and $\varepsilon_0^i = \min_{1 \leqslant k \leqslant 3} (u - \gamma_k^i)/2$. Consider

$$e^i = \big(e_1^i, \dots, e_n^i\big) \in \mathbb{R}^n, \quad \text{with } e_j^i = \begin{cases} \varepsilon_0^i, & j \in I_1^i, \\ -\varepsilon_0^i, & j \in I_2^i, \\ 0, & j \in I_3. \end{cases}$$

From the definition of ε_0^i, we have $\|\varphi^i - e^i\|_\infty \leqslant u - \varepsilon_0^i$. For $i \in I_1$, from $\|L_i\| \leqslant b_i$ and Claim 2, we get

$$\begin{aligned} 0 = b_i \varphi_i^i(0) + L_i\big(\varphi^i\big) &= \varepsilon_0^i b_i + \big(\varphi_i^i(0) - \varepsilon_0^i\big) b_i + L_i\big(\varphi^i - e^i\big) + L_i\big(e^i\big) \\ &\geqslant \varepsilon_0^i b_i + L_i\big(e^i\big) = \varepsilon_0^i \Big[b_i + a_{ii} + \sum_{j \in I_1^i,\, j \neq i} a_{ij} - \sum_{j \in I_2^i} a_{ij} \Big]. \end{aligned} \tag{3.19}$$

Analogously, for $i \in I_2$ we obtain

$$0 = b_i \varphi_i^i(0) + L_i\big(\varphi^i\big) \leqslant \varepsilon_0^i \Big[-(b_i + a_{ii}) + \sum_{j \in I_1^i} a_{ij} - \sum_{j \in I_2^i,\, j \neq i} a_{ij} \Big]. \tag{3.20}$$

Now, from (2.5) (with $d_1 = \cdots = d_n = 1$), (3.19) and (3.20) we conclude that

$$\sum_{j \in I_3} |a_{ij}| = \sum_{j \in I_3} |l_{ij}| = 0, \quad i \in I_1 \cup I_2,$$

or, equivalently,

$$a_{ij} = l_{ij} = 0 \quad \text{for } i \in I_1 \cup I_2,\ j \in I_3, \tag{3.21}$$

and

$$b_i = \sum_{j \in I} |a_{ij}| = \sum_{j \in I} |l_{ij}|, \quad i \in I_1 \cup I_2. \tag{3.22}$$

At this stage, after a permutation of I, we may suppose that I is ordered in such a way that

$$I_1 = \{1, \ldots, n_1\}, \qquad I_2 = \{n_1 + 1, \ldots, n_1 + n_2\}, \qquad I_3 = \{n_1 + n_2 + 1, \ldots, n_1 + n_2 + n_3\},$$

with $n_1 + n_2 + n_3 = n$. Recall that n_2, n_3 may be zero. According to this ordering, $\hat{N}$ has the form

$$\hat{N} = \big((\hat{N}_{ij})_{i,j=1}^3\big),$$

where $\hat{N}_{ij}$ are $n_i \times n_j$ matrices, $i, j = 1, 2, 3$. If $I_3 \neq \emptyset$, from (3.21) we have $\hat{N}_{j3} = 0$ for $j = 1, 2$. Next, from (3.21)–(3.22) one writes M in the form (3.10) with $M_{13} = M_{23} = 0$, and concludes that

$$\tilde{M}_0 \eta = 0, \quad \text{where } \tilde{M}_0 = \begin{pmatrix} \tilde{M}_{11} & -|M_{12}| \\ -|M_{21}| & \tilde{M}_{22} \end{pmatrix},$$

where $\tilde{M}_{ii}$ are $n_i \times n_i$ matrices, $i = 1, 2$, and $\eta = (1, \ldots, 1)$ is an $(n_1 + n_2)$-vector. This is not possible however, since $\det \tilde{M} \neq 0$ and $M_{13} = M_{23} = 0$ imply that $\det \tilde{M}_0 \neq 0$.

The above arguments show that $u = 0$, hence $v = 0$ as well. This ends the proof of the theorem. $\square$

We finally state our main result on the global asymptotic stability of the equilibrium x^* of (3.2).

Theorem 3.5. *Assume* (H1)–(H4). *Then the positive equilibrium of* (3.2) *is globally asymptotically stable* (*in the set of all positive solutions*).

Proof. By translating x_* to the origin, (3.2) becomes (3.3). As already noticed, (H2) and (H3) imply that $\det M \neq 0$. From Theorem 3.3, all admissible solutions of (3.3) are defined and *bounded* for $t \geqslant 0$, and the trivial equilibrium of (3.3) is uniformly stable (in the set $S = C_{-x^*}$ of all admissible solutions). From Theorem 3.4, we conclude that zero is globally attractive in S. $\square$

Some immediate consequences of this result are given below.

Corollary 3.6. *Assume* (H1), (H3), (H4) *and that $\hat{N}$ is an irreducible M-matrix. Then, the equilibrium x^* of* (3.2) *is globally asymptotically stable* (*in the set of all positive solutions*).

Proof. If $\hat{N}$ is irreducible, then $\hat{N}$ is in an M-matrix if and only if (H2) holds (see [4]). $\square$

Corollary 3.7. *Assume* (H1), (H4) *and that $\hat{N}$ is a non-singular M-matrix. Then, x^* is globally asymptotically stable* (*in the set of all positive solutions of* (3.2)).

Proof. If $\hat{N}$ is a non-singular M-matrix, then there is $d = (d_1, \ldots, d_n) > 0$ such that $\hat{N}d > 0$, so (H2) holds. Since $\tilde{M} \geqslant \hat{N}$, then $\tilde{M}$ is a non-singular M-matrix as well (see [4]). □

Corollary 3.8. *Assume* (H1), (H2), (H4) *and that* $a_{ii} > 0$ *for* $i = 1, \ldots, n$. *Then* x^* *is globally asymptotically stable* (*in the set of all positive solutions of* (3.2)).

Proof. For $d = (d_1, \ldots, d_n)$ as in (H2), we have

$$d_i b_i \geqslant \sum_{1 \leqslant j \leqslant n} d_j |a_{ij}|, \quad i = 1, \ldots, n,$$

hence $\tilde{M}d \geqslant 2\,\mathrm{diag}(a_{11}, \ldots, a_{nn})d > 0$. From [4, Theorems 5.1 and 5.7], $\tilde{M}$ is a non-singular M-matrix. □

Remark 3.1. For the class of n-neuron Hopfield networks with *discrete* delays (2.16), Campbell [1] proved its global asymptotic stability if $\hat{M}$ is a non-singular M-matrix, as in the above Corollary 3.7. Note that for FDEs with discrete delays (2.16), our matrices M and N coincide. We emphasize however that Corollary 3.7 deals with the general situation of *distributed* delays. We further remark that Tang and Zou [25] gave stability results for Lotka–Volterra systems with *distributed* delays of the form

$$\dot{x}_i(t) = r_i(t)x_i(t)\left[1 - \int_{-\tau_{ii}}^{0} x_i(t+\theta)\,d\eta_{ii}(\theta) - \sum_{j \neq i}^{n} l_{ij} \int_{-\tau_{ij}}^{0} x_j(t+\theta)\,d\eta_{ij}(\theta)\right], \quad i = 1, \ldots, n, \tag{3.23}$$

where $r_i(t)$ satisfy (H4), η_{ij} are *non-decreasing* bounded normalized functions, and the constants l_{ij} are *non-negative*. In particular, in (3.23) all the operators L_{ij} are *positive* (cf. Section 4, also for comparison of results). In [25], the authors are primarily interested in the situation $\tau_{ii} > 0$, $i = 1, \ldots, n$, where instantaneous negative feedbacks are absent, although the situation of zero diagonal delays is included in their setting. Several criteria for the global attractivity of the positive equilibrium of (3.23) (if it exists) are established, by imposing 3/2-type constraints on the diagonal delays τ_{ii}, and M-matrix-type conditions. Namely, for $M = [l_{ij}]$, where l_{ij}, $j \neq i$, are as in (3.23) and $l_{ii} = 1$, the following conditions are assumed in [25]: either (DD1) M is diagonal dominant, i.e., $1 > \sum_{j \neq i} l_{ij}$, $i = 1, \ldots, n$, or (DD2) $\hat{M}$ is a non-singular M-matrix.

Observe that hypothesis (H2), which for $n \geqslant 2$ is strictly stronger than having $\hat{N}$ an M-matrix, was used throughout the proof of Theorem 3.4. Also (H2) was essential to derive the global asymptotic stability result in Theorem 3.5, since we invoked Theorem 3.3 to conclude that admissible solutions of (3.2) are bounded. For system (3.2), written as

$$x_i'(t) = r_i(t)x_i(t)\left[1 - b_i x_i(t) - \sum_{j=1}^{n} l_{ij} \int_{-\tau}^{0} x_j(t+\theta)\,d\eta_{ij}(\theta)\right], \quad i = 1, \ldots, n, \tag{3.24}$$

it is interesting to investigate situations for which the criterion for the global asymptotic stability of the positive equilibrium x^* is sharp, in the sense that it coincides with the necessary and sufficient conditions, established in Section 2 for the situation $r_i(t) \equiv r_i > 0$, for the local asymptotic stability independently of τ and η_{ij} in (3.24). Though this is in general an open problem, the goal of the next section is to give partial answers to this question.

Next, we give sufficient conditions for x^* to be a global attractor of all *bounded* solutions of (3.2).

Theorem 3.9. *Assume* (H4) *and suppose that* $\det \tilde{M} \neq 0$ *and* $\hat{N}$ *is an M-matrix. If there is a positive equilibrium* x^* *of* (3.2)*, then* $x(t) \to x^*$ *as* $t \to \infty$ *for every bounded solution* $x(t)$ *of* (3.2) *with initial condition* $x_0 = \varphi \in C_{\hat{0}}$.

Proof. If $\hat{N}$ is an irreducible M-matrix, the result follows from Theorem 3.5. If $\hat{N}$ is reducible, as in the proof of Theorem 2.3, by reordering the variables x_i, $\hat{N}$ is written as (2.9), with the diagonal blocks $\hat{N}_{kk}$ irreducible or zero, $k = 1, \dots, \ell$. We prove the result for $\ell = 2$. The general result follows by induction.

Suppose that $n_1 + n_2 = n$, $l_{ij} = 0$ for $n_1 + 1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n_1$, and write accordingly

$$M = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}, \qquad \hat{N} = \begin{pmatrix} \hat{N}_{11} & \hat{N}_{12} \\ 0 & \hat{N}_{22} \end{pmatrix},$$

where M_{ij}, $\hat{N}_{ij}$ are $n_i \times n_j$ blocks and $\hat{N}_{ii}$ are irreducible or zero matrices.

Now, consider a positive *bounded* solution $x(t)$, $t \geqslant 0$, of (3.2). Write $x(t) = (y(t), z(t))$, $x^* = (y^*, z^*)$, with $y(t), y^* \in \mathbb{R}^{n_1}$, $z(t), z^* \in \mathbb{R}^{n_2}$, so that (3.2) reads as

$$y_i'(t) = r_i(t) y_i(t) \left[1 - b_i y_i(t) - \sum_{j=1}^{n_1} L_{ij}(y_{j,t}) - \sum_{p=1}^{n_2} L_{i(n_1+p)}(z_{p,t}) \right], \quad i = 1, \dots, n_1, \tag{3.25}$$

$$z_k'(t) = r_{n_1+k}(t) z_k(t) \left[1 - b_{n_1+k} z_k(t) - \sum_{p=1}^{n_2} L_{(n_1+k)(n_1+p)}(z_{p,t}) \right], \quad k = 1, \dots, n_2, \tag{3.26}$$

where $y_{j,t}(\theta) = y_j(t+\theta)$, $z_{p,t}(\theta) = z_p(t+\theta)$ for $t \geqslant 0$, $\theta \in [-\tau, 0]$ and $j = 1, \dots, n_1$, $p = 1, \dots, n_2$. Note that $\det \tilde{M}_{ii} \neq 0$, $i = 1, 2$, and that $\hat{N}_{11}$, $\hat{N}_{22}$ satisfy (H2).

For (3.26), from Theorem 3.5 we have $z_k(t) \to z_k^*$ as $t \to \infty$, for $k = 1, \dots, n_2$. Hence, (3.25) can be written as

$$y_i'(t) = r_i(t) y_i(t) \left[\alpha_i - b_i y_i(t) - \sum_{j=1}^{n_1} L_{ij}(y_{j,t}) - h_i(t) \right], \quad i = 1, \dots, n_1, \tag{3.27}$$

where $\alpha_i = 1 - \sum_{p=1}^{n_2} a_{i(n_1+p)} z_p^*$ and

$$h_i(t) = \sum_{p=1}^{n_2} L_{i(n_1+p)}(z_{p,t}) - \sum_{p=1}^{n_2} a_{i(n_1+p)} z_p^* \to 0 \quad \text{as } t \to \infty,\ i = 1, \dots, n_1.$$

Note that (3.27) has the form (3.11), for which (H2)–(H4) hold, and that $M_{11}y^* = [\alpha_1, \dots, \alpha_n]^T$. From Theorem 3.4, we conclude that $y(t) \to y^*$ as $t \to \infty$, for $i = 1, \dots, n_1$. $\square$

Another interesting aspect of the analogy of the qualitative behaviour between delayed Lotka–Volterra systems and their corresponding ODE models is given below.

Theorem 3.10. *Consider the Lotka–Volterra system* (3.2), *where now $r_i(t)$ are defined, continuous and positive on $\mathbb{R}$, $i = 1, \dots, n$. Assume* (H1)–(H3) *and*

(H4$^\pm$) *$r_i(t)$ is uniformly bounded on $(-\infty, \infty)$ and $\int^{\pm\infty} r_i(t)\,dt = \infty$, $i = 1, \dots, n$.*

Then the only positive solution of (3.2) *which is defined, bounded and bounded away from zero on $(-\infty, \infty)$ is the constant solution $x(t) = x^*$.*

Proof. By translating x^* to the origin, write (3.2) in the form (3.3) with $r_i(t)$ defined for $t \in \mathbb{R}$. Let $y(t)$ be a global bounded solution of (3.3), with $y(t) - x^* \geqslant m$, $t \in \mathbb{R}$, for some $m > 0$. We conclude that $y(t) \to 0$ as $t \to -\infty$ by adjusting the arguments in the proof of Theorem 3.4, so details are not presented. Now, fix any $\varepsilon > 0$ and suppose that $|y(t)| < \varepsilon$ for $t \leqslant t_0$. From Lemma 3.2 and Theorem 3.3, it follows that $|y(t)| < \varepsilon$ on the entire real line, hence $y(t)$ must be zero. $\square$

Remark 3.2. In fact, under conditions (H1)–(H3) and (H4$^\pm$), the existence of a positive heteroclinic $\chi(t)$ of (3.2) connecting the equilibria $\chi(-\infty) = 0$ to $\chi(\infty) = x^*$ is possible, therefore in the above lemma it is essential to assume that solutions are not only bounded, but also bounded away from zero on $\mathbb{R}$. As referred to, the proof of Theorem 3.10 follows closely the proof of Theorem 3.4; namely, Claims 1 and 2 hold with $t_k^i \to \infty$ replaced by $t_k^i \to -\infty$ and u_i, v_i defined by $u_i = \limsup_{t\to-\infty} y_i(t)$, $-v_i = \liminf_{t\to-\infty} y_i(t)$. However, for the proof of Claim 1, if $y_i(t)$ is eventually monotone as $t \to -\infty$ with $y_i'(t) \leqslant 0$ for t in the vicinity of $-\infty$, we can only conclude that $c := \limsup_{t\to-\infty}(b_i y_i(t) + L_i(y_t)) = 0$ if $v_i < x_i^*$, otherwise the situation $c < 0$ is possible.

We finalize this section with some applications.

Example 3.1. Consider the scalar delayed logistic equation

$$x'(t) = r(t)x(t)\big[1 - b_0 x(t) - L_0(x_t)\big], \quad t \geqslant 0, \tag{3.28}$$

where $b_0 \in \mathbb{R}$, $r : [0, \infty) \to (0, \infty)$ is continuous and $L_0 : C_1 \to \mathbb{R}$ is a linear bounded operator. Note that for (3.28), (H1)–(H3) translate as

$$b_0 + L_0(1) > 0, \quad b_0 \geqslant \|L_0\|. \tag{3.29}$$

Theorem 3.5 applied to the particular case $n = 1$ gives the following result:

Corollary 3.11. *For* (3.28), *suppose that* (H4) *and* (3.29) *are satisfied. Then the positive equilibrium $x^* = (b_0 + L_0(1))^{-1}$ of* (3.28) *is globally asymptotically stable* (*in the set of all admissible solutions*).

The above criterion was already established in [2]. Note that (3.29) is exactly the necessary and sufficient condition for the asymptotic stability of (2.12) in the statement of Corollary 2.8.

Example 3.2. Consider the following Lotka–Volterra system with distributed delays and symmetry:

$$x_1'(t) = x_1(t)\left[r_1 - ax_1(t) + \alpha \int_{-\tau}^{0} x_1(t+\theta)\,d\eta_{11}(\theta) + b_{12}\int_{-\tau}^{0} x_2(t+\theta)\,d\eta_{12}(\theta)\right],$$

$$x_2'(t) = x_2(t)\left[r_2 - ax_2(t) + b_{21} \int_{-\tau}^{0} x_1(t+\theta)\,d\eta_{21}(\theta) + \alpha\int_{-\tau}^{0} x_2(t+\theta)\,d\eta_{22}(\theta)\right]. \tag{3.30}$$

Here, τ, r_1, r_2, a, α, b_{12}, b_{21} are constants, $\tau, r_1, r_2, a > 0$, and $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are non-decreasing functions with $\eta_{ij}(0) - \eta_{ij}(-\tau) = 1$, $i, j = 1, 2$, and

$$\text{either} \quad b_{21} = -b_{12} \quad \text{or} \quad b_{21} = b_{12}.$$

The first situation models a predator–prey system (cf. [18,19]), while the second one is used to describe a cooperative or competition model (cf. [20]).

Theorem 3.12. *Consider the predator–prey system with symmetry* (3.30)*, where* $b_{21} = -b_{12} := \beta$. *If*

$$\max\left(\frac{r_2\beta}{r_1}, -\frac{r_1\beta}{r_2}\right) < a - \alpha, \tag{3.31}$$

then there exists a positive equilibrium $x^* = (x_1^*, x_2^*)$*. Additionally, if*

$$|\beta| < a - \alpha \quad \textit{and} \quad |\beta| \leqslant a + \alpha, \tag{3.32}$$

then $x(t) \to x^*$ *as* $t \to \infty$ *for every admissible solution* $x(t)$ *of* (3.30).

Proof. With $b_{21} = -b_{12} := \beta$, (3.31) is equivalent to saying that the equilibrium $x^* = (x_1^*, x_2^*)$,

$$x_1^* = \frac{r_1(a-\alpha) - r_2\beta}{(a-\alpha)^2 + \beta^2}, \qquad x_2^* = \frac{r_2(a-\alpha) + r_1\beta}{(a-\alpha)^2 + \beta^2},$$

is positive. Here $M = N = \begin{pmatrix} (a-\alpha)/r_1 & \beta/r_1 \\ -\beta/r_2 & (a-\alpha)/r_2 \end{pmatrix}$. With the previous notation, $\hat{M}$ is an M-matrix if and only if $|\alpha| + |\beta| \leqslant a$; for this situation, this is equivalent to (H2). And $\det \tilde{M} \neq 0$ means that $|\beta| \neq |a - \alpha|$. Under these circumstances, (H2)–(H3) translate as (3.32). □

We observe that the predator–prey situation $b_{21} = -b_{12} := \beta$ with discrete and distributed delays in (3.30) was addressed in [19] and [18], respectively, where the authors proved the global asymptotic stability of x^* (assuming its existence) under the weaker requirement

$$\sqrt{\alpha^2 + \beta^2} \leqslant a.$$

However, in both papers, the following restrictive assumption in the symmetry was imposed:

$$\eta_{11} = \eta_{21} := \mu, \qquad \eta_{12} = \eta_{22} := \nu. \tag{3.33}$$

To be more precise, [19] studied the equation with discrete delays

$$\begin{aligned} x_1'(t) &= x_1(t)\big[r_1 - ax_1(t) + \alpha x_1(t-\tau_1) - \beta x_2(t-\tau_2)\big],\\ x_2'(t) &= x_2(t)\big[r_2 - ax_2(t) + \beta x_1(t-\tau_1) + \alpha x_2(t-\tau_2)\big], \end{aligned}$$

whereas [18] dealt with the distributed delays situation

$$\begin{aligned} x_1'(t) &= x_1(t)\left[r_1 - ax_1(t) + \alpha\int_{-\tau}^{0} x_1(t+\theta)\,d\mu(\theta) - \beta\int_{-\tau}^{0} x_2(t+\theta)\,d\nu(\theta)\right],\\ x_2'(t) &= x_2(t)\left[r_2 - ax_2(t) + \beta\int_{-\tau}^{0} x_1(t+\theta)\,d\mu(\theta) + \alpha\int_{-\tau}^{0} x_2(t+\theta)\,d\nu(\theta)\right]. \end{aligned}$$

For a cooperative or competition model with symmetry, in a similar way we deduce:

Theorem 3.13. *Consider* (3.30) *with $b_{21} = b_{12} := \beta$, suppose that*

$$a - \alpha > \max\left(-\frac{r_2\beta}{r_1}, -\frac{r_1\beta}{r_2}\right),$$

and condition (3.32) *is satisfied. Then, there exists a positive equilibrium $x^* = (x_1^*, x_2^*)$, which is globally asymptotically stable.*

Theorem 3.13 was already obtained by Saito and Takeuchi [20], by using Lyapunov functionals. Here, we have used models (3.30) to illustrate the advantage of our approach, which enables us to obtain the global stability of general Lotka–Volterra type models (3.1), without having to construct specific Lyapunov functionals to each model under consideration, normally a rather difficult task. For the particular case of (3.30) with $b_{12} = \pm b_{21}$, from Theorems 2.7 and 3.5, one easily checks that the local and global stability of x^*, independently of the choices of the delay functions η_{ij}, coincide.

4. Monotone operators and sharp conditions for global stability

For the particular case of autonomous systems with discrete delays of the form

$$x_i'(t) = r_i x_i(t)\left[1 - \sum_{j=1}^{n} \alpha_{ij} x_j(t-\tau_{ij})\right], \quad i = 1, \dots, n, \tag{4.1}$$

where $r_i > 0$, $\alpha_{ij} \in \mathbb{R}$, $\tau_{ij} \geqslant 0$ and $\alpha_{ii} > 0$, $\tau_{ii} = 0$, Hofbauer and So [9] proved that the positive equilibrium x^*, if it exists, is globally asymptotically stable for all the choices of delays $\tau_{ij} \geqslant 0$,

$i \neq j$, if and only if $\det M \neq 0$ and $\hat{M}$ is an M-matrix. In the previous notation, for (4.1) we have $b_i = \alpha_{ii}$, $a_{ij} = l_{ij} = \alpha_{ij}$, $i \neq j$, $a_{ii} = 0$, $B = \mathrm{diag}(\alpha_{11}, \ldots, \alpha_{nn})$, $M = B + [a_{ij}]$ and $\tilde{M} = \hat{M} = B - |[a_{ij}]|$. As already noticed in Remark 3.1, later Campbell [1] overcame the restriction $\tau_{ii} = 0$ in (4.1), and considered an additive neural network with discrete delays $\tau_{ij} \geqslant 0$ written (after a translation) as (2.16), for g_j smooth increasing functions with $g_j(0) = 0$, $g_j'(0) = 1$, and showed the global attractivity of the trivial equilibrium of (2.16) if $\hat{M}$ is a *non-singular* M-matrix. (Note that this implies that $\tilde{M}$ is a non-singular matrix as well [4].)

In this section, our major aim is to identify a class of Lotka–Volterra systems (3.1), for which the optimal conditions for the local asymptotic stability of the positive equilibrium (cf. Theorems 2.6 and 2.7) are also sufficient conditions for its global asymptotic stability. In particular, we want to replace (H2) by the weaker condition of $\hat{N}$ being an M-matrix in Theorem 3.5.

An important class of Lotka–Volterra models (3.2), which includes the discrete delay system (4.1) (without the restriction $\tau_{ii} = 0$), is the one where the operators L_{ij} in (2.2) are all monotone.

We recall that a linear bounded operator $L : C_1 \to \mathbb{R}$ is *monotone* (relative to the order in C_1) if L is given by

$$L\varphi = \ell \int_{-\tau}^{0} \varphi(\theta)\, d\mu(\theta), \quad \varphi \in C_1,$$

for some $\ell \in \mathbb{R}$ and non-decreasing function $\mu : [-\tau, 0] \to \mathbb{R}$, $\mu(0) - \mu(-\tau) = 1$. If $\ell \geqslant 0$ (respectively $\ell \leqslant 0$), then L is said to be *positive* (respectively *negative*); this means that L is non-decreasing (respectively non-increasing), that is, $L\varphi \geqslant 0$ for all $\varphi \geqslant 0$ (respectively $\varphi \leqslant 0$).

For the general autonomous situation of systems (3.24) with a positive equilibrium x^*, we conjecture that x^* is globally asymptotically stable for all the choices of $\tau > 0$ and normalized *non-decreasing* functions η_{ij}, $i, j = 1, \ldots, n$, if and only if $\det M \neq 0$ and $\hat{M}$ is an M-matrix. If this conjecture is true, then the local and global asymptotic stabilities of x^* *independently* of the (distributed) delays coincide. Here, we consider monotone operators L_{ij}, and establish some criteria for the global stability of (3.2), namely that the above conjecture is valid if all L_{ij} are monotone and $a_{ii} > 0$. In what follows, stability is referred to the set of admissible solutions.

Theorem 4.1. *Consider Eq.* (3.2)*, and suppose that the operators L_{ij} in* (2.2) *are all negative, $i, j = 1, \ldots, n$. Assume* (H4) *and that M is a non-singular M-matrix. Then there exists a positive equilibrium of* (3.2)*, which is globally asymptotically stable.*

Proof. The operators L_{ij} are all negative, thus they are given by (2.2), for non-decreasing functions $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ with $\eta_{ij}(0) - \eta_{ij}(-\tau) = 1$ and $l_{ij} \leqslant 0$, $i, j = 1, \ldots, n$. With the previous notation, we have $a_{ij} = l_{ij}$ and

$$M = N = \tilde{M} = \hat{N} = \mathrm{diag}(b_1, \ldots, b_n) + [a_{ij}].$$

Since M is a non-singular M-matrix, hypothesis (H2) is satisfied; moreover, $M^{-1} \geqslant 0$ (see [4, Theorems 5.1 and 5.3]). Let $x^* = (x_1^*, \ldots, x_n^*)$ be the solution of $Mx = [1, \ldots, 1]^T$. Since $M^{-1} \geqslant 0$, then $x^* \geqslant 0$; and $x_i^* = 0$ if and only if all the entries of the ith row of M^{-1} are zero, which is not possible. The conclusion follows now from Theorem 3.5. □

If all the operators L_{ij} in (2.2) are monotone, Corollary 3.7 gives the following criterion:

Corollary 4.2. *Consider Eq.* (3.2)*, where the operators L_{ij} in* (2.2) *are all monotone and the functions $r_i(t)$ satisfy* (H4)*, $i, j = 1, \dots, n$. Assume also that there exists $d = (d_1, \dots, d_n) > 0$ such that $\hat{M}d > 0$, where $\hat{M} = B - |[l_{ij}]|$. Then, the positive equilibrium x^* of* (3.2) (*if it exists*) *is globally asymptotically stable.*

Remark 4.1. As mentioned before, [25] addresses the question of global attractivity for pure delay systems (3.23). For the situation $\tau_{ii} = 0$, $1 \leqslant i \leqslant n$, (3.23) reads as

$$\dot{x}_i(t) = r_i(t)x_i(t)\left[1 - \sum_{j \neq i}^{n} l_{ij} \int_{-\tau_{ij}}^{0} x_j(t+\theta)\, d\eta_{ij}(\theta)\right], \quad i = 1, \dots, n, \tag{4.2}$$

with $l_{ij}, \tau_{ij} \geqslant 0$ and η_{ij} non-decreasing bounded functions, normalized so that $\int_{-\tau_{ij}}^{0} d\eta_{ij}(\theta) = 1$. From the criterion established in [25, Theorem 2.3], it follows that if (H1) and (H4) are satisfied, and there exists $d = (d_1, \dots, d_n) > 0$ such that $\hat{M}d > 0$, where $\hat{M} = [\hat{l}_{ij}]$ with $\hat{l}_{ij} = -l_{ij}$ for $j \neq i$, $\hat{l}_{ii} = 1$, then the positive equilibrium is a global attractor of admissible solutions to (4.2).

Lemma 4.3. *Assume* (H4) *and that $\hat{N}$ is an M-matrix. Suppose that one of the following conditions holds*:

(i) *the operators L_{ij} are all positive for $i \neq j$, and L_{ii} are all monotone, $i, j = 1, \dots, n$, and* $\det M \neq 0$;
(ii) *the operators L_{ij} are all monotone, with $l_{ii} := L_{ii}(1) > 0$, $i, j = 1, \dots, n$.*

Then, all (*admissible*) *solutions of* (3.2) *are defined and bounded on* $[0, \infty)$.

Proof. (i) Let $x(t)$ be an admissible solution of (3.2) defined on $[-\tau, b)$, with $b > 0$ or $b = \infty$. If all the operators $L_{ij}, i \neq j$, are positive, then $L_{ij}(x_{t,j}) \geqslant 0$ for $i \neq j$. For $i = 1, \dots, n$, write L_{ii} as in (2.2), with η_{ii} non-decreasing, and either $l_{ii} \geqslant 0$ if L_{ii} is positive, or $l_{ii} \leqslant 0$ if L_{ii} is negative. Then,

$$x_i'(t) \leqslant r_i(t)x_i(t)g_i(x_{t,i}), \quad i = 1, \dots, n, \tag{4.3}$$

where $x_{t,i}$ is the ith coordinate of x_t and $g_i : C_1 \to \mathbb{R}$ is given by

$$g_i(\psi) = 1 - b_i\psi(0) \quad \text{if } l_{ii} \geqslant 0,$$

$$g_i(\psi) = 1 - b_i\psi(0) - l_{ii}\int_{-\tau}^{0} \psi(\theta)\, d\eta_{ii}(\theta) \quad \text{if } l_{ii} \leqslant 0,\ \psi \in C_1.$$

The positive solutions of the logistic ODEs $u_i'(t) = r_i(t)u_i(t)[1 - b_iu_i(t)]$ are bounded on $[0, \infty)$. On the other hand, from Theorem 2.3 and Remark 2.1, conditions $\det M \neq 0$ and $\hat{N}$ is an

M-matrix imply that $b_i + l_{ii} > 0$ and $b_i \geqslant |l_{ii}|$, $i = 1, \dots, n$. From the results for the scalar case in [2] (cf. Corollary 3.11), we derive that the positive solutions of

$$u_i'(t) = r_i(t)u_i(t)\left[1 - b_i u_i(t) - l_{ii}\int_{-\tau}^{0} u_i(t+\theta)\,d\eta_{ii}(\theta)\right]$$

are defined and bounded for $t \geqslant 0$, $i = 1, \dots, n$. As the function $g = (g_1, \dots, g_n)$ on the right-hand side of (4.3) satisfies the quasimonotone condition in [21, p. 78], by (4.3) and comparison results (see [21, Theorem 5.1.1]) the same happens to $x(t)$.

(ii) If the operators L_{ij} are all monotone, then they are as in (2.2), for normalized non-decreasing functions $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ and coefficients $l_{ij} \in \mathbb{R}$. Note that $a_{ij} = L_{ij}(1) = l_{ij}$, and the matrices M, N coincide. Let $x(t) = x(t, \varphi)$, $\varphi \in C_{\hat{0}}$, be a solution of (3.2). Then,

$$x_i'(t) \leqslant r_i(t)x_i(t)\left[1 - b_i x_i(t) - \sum_{j=1}^{n} l_{ij}^* \int_{-\tau}^{0} x_j(t+\theta)\,d\eta_{ij}(\theta)\right], \quad i = 1, \dots, n, \tag{4.4}$$

where $l_{ij}^* = 0$ if $l_{ij} > 0$ and $l_{ij}^* = l_{ij}$ if $l_{ij} \leqslant 0$. (Since $l_{ii} > 0$, then $l_{ii}^* = 0$.)

With our notation, here the matrices $\tilde{M}^*$ and $\hat{N}^*$ are given by $\tilde{M}^* = \hat{N}^* = M^* := B + [l_{ij}^*]$. Now, $M^* \geqslant \hat{N} + \varepsilon I$, where $\varepsilon = \min\{l_{ii}\colon 1 \leqslant i \leqslant n\} > 0$ and I is the $n \times n$ identity matrix. This means that M^* is a non-singular M-matrix, and from Theorem 4.1 we derive that system

$$u_i'(t) = r_i(t)u_i(t)\left[1 - b_i u_i(t) - \sum_{j=1}^{n} l_{ij}^* \int_{-\tau}^{0} u_j(t+\theta)\,d\eta_{ij}(\theta)\right], \quad i = 1, \dots, n,$$

has a unique positive equilibrium u^*, which is globally asymptotically stable. Again, the function on the right-hand side of the above system is quasimonotone. From (4.4) and [21, Theorem 5.1.1], all admissible solutions of (3.2) are defined and bounded on $[0, \infty)$. □

Theorem 4.4. *Consider system* (3.24)*, where $\tau > 0$, $b_i > 0$, $l_{ij} \in \mathbb{R}$, $r_i : [0, \infty) \to (0, \infty)$ are continuous, and $\eta_{ij} \in BV([-\tau, 0]; \mathbb{R})$,* $\mathrm{Var}_{[-\tau,0]}\,\eta_{ij} = 1$*, with η_{ij} non-decreasing for all $i, j = 1, \dots, n$, and $l_{ij} \geqslant 0$ for $i \neq j$. Define $B = \mathrm{diag}(b_1, \dots, b_n)$, $\tilde{M} = B + [\tilde{l}_{ij}]$, $\hat{M} = B + [\hat{l}_{ij}]$, where $\tilde{l}_{ij} = \hat{l}_{ij} = -l_{ij}$ for $j \neq i$, $\tilde{l}_{ii} = l_{ii}$, $\hat{l}_{ii} = -|l_{ii}|$. Assume* (H4)*, that* $\det \tilde{M} \neq 0$ *and $\hat{M}$ is an M-matrix. If there exists a positive equilibrium x^* of* (3.24)*, then x^* is globally asymptotically stable.*

Proof. Since $\tilde{M}$ is a non-singular M-matrix, then $\det M \neq 0$ [4, Theorem 5.17]. Lemma 4.3(i) implies that all positive solutions of (3.24) are bounded. The result is now a consequence of Theorem 3.9. □

Example 4.1. In [21, pp. 94–98], Smith considered the autonomous case of system (3.24), with all $l_{ij} \geqslant 0$ and η_{ij} normalized non-decreasing functions—or, in other words, system (3.1) with all operators L_{ij} being positive. For this situation, under the condition

$$\sum_{j=1}^{n} l_{ij} b_j^{-1} < 1, \quad i = 1, \dots, n, \tag{4.5}$$

Smith proved the existence of a globally attractive positive equilibrium. We note that (4.5) implies that $\hat{M} d > 0$, for $d^{-1} = (b_1, \dots, b_n)$. In particular, from (4.5) we deduce that $\hat{M}$ is a non-singular M-matrix, so also $\det \tilde{M} \neq 0$. Therefore, the criterion in Theorem 4.4 generalizes the one in [21].

The next result presents a sharp condition for global asymptotic stability.

Theorem 4.5. *Let $r_i > 0$, $b_i > 0$, $l_{ij} \in \mathbb{R}$ with $l_{ii} > 0$, $i, j = 1, \dots, n$, be given, and define $B = \operatorname{diag}(b_1, \dots, b_n)$, $M = B + [l_{ij}]$, $\hat{M} = B - |[l_{ij}]|$. Suppose that there is a positive vector x^* such that $M x^* = [1, \dots, 1]^T$. With $r_i(t) \equiv r_i$, then x^* is a globally asymptotically stable equilibrium of the autonomous Lotka–Volterra system* (3.24) *for all the choices of $\tau > 0$ and non-decreasing functions $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ with $\eta_{ij}(0) - \eta_{ij}(-\tau) = 1$, $i, j = 1, \dots, n$, if and only if* $\det M \neq 0$ *and $\hat{M}$ is an M-matrix.*

Proof. Take (3.24) with $r_i(t) \equiv r_i > 0$. For the sufficiency condition, suppose that $\det M \neq 0$ and $\hat{M}$ is an M-matrix. The boundedness of admissible solutions follows from Lemma 4.3(ii). Now, if $l_{ii} > 0$, $i = 1, \dots, n$, then $\tilde{M} = \hat{M} + 2 \operatorname{diag}(l_{11}, \dots, l_{nn})$ is non-singular [4, Theorems 5.1 and 5.3]. Theorem 3.9 yields the conclusion.

For the necessity, let x^* be globally asymptotically stable for all the choices of non-decreasing functions η_{ij}. From [1,9] (cf. Lemma 2.5), if $\det M \neq 0$ and $\hat{M}$ is not an M-matrix, for some choices of Heaviside functions η_{ij}, the characteristic equation for (2.3) has a root with positive real part. Thus, there is a non-trivial unstable manifold for the equilibrium x^* of (3.24). This implies that x^* is not a global attractor. On the other hand, if $\det M = 0$, then (3.24) has an infinity of positive equilibria, which contradicts the global attractivity of x^*. □

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