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# Asymptotic behavior for the Stokes flow and Navier–Stokes equations in half spaces

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## ABSTRACT

Using the solution formula in Ukai (1987) [27] for the Stokes equations, we find asymptotic profiles of solutions in  $L^1(\mathbb{R}_+^n)$  ( $n \geq 2$ ) for the Stokes flow and non-stationary Navier–Stokes equations. Since the projection operator  $P: L^1(\mathbb{R}_+^n) \rightarrow L_\sigma^1(\mathbb{R}_+^n)$  is unbounded, we use a decomposition for  $P(u \cdot \nabla u)$  to overcome the difficulty, and prove that the decay rate for the first derivatives of the strong solution  $u$  of the Navier–Stokes system in  $L^1(\mathbb{R}_+^n)$  is controlled by  $t^{-\frac{1}{2}}(1 + t^{-\frac{n+2}{2}})$  for any  $t > 0$ .

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## 1. Introduction and main results

We consider the asymptotic behavior in  $L^1$  of the Stokes flow and non-stationary Navier–Stokes equations in the upper-half space, respectively:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = 0 & \text{in } \mathbb{R}_+^n \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}_+^n \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial \mathbb{R}_+^n \times (0, \infty), \\ u(x, 0) = u_0 & \text{in } \mathbb{R}_+^n, \end{cases} \quad (1.1)$$

and

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$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}_+^n \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}_+^n \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial \mathbb{R}_+^n \times (0, \infty), \\ u(x, 0) = u_0 & \text{in } \mathbb{R}_+^n, \end{cases} \quad (1.2)$$

where  $n \geq 2$ , and  $\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$  is the upper-half space of  $\mathbb{R}^n$ ;  $u = (u_1(x, t), u_2(x, t), \dots, u_n(x, t))$  and  $p = p(x, t)$  denote unknown velocity vector and the pressure respectively, while  $u(x, 0) = u_0(x)$  is a given initial velocity vector field.

**Definition.**  $u$  is called a weak solution of (1.2) if

$$\begin{aligned} u \in L^\infty(0, \infty; L_\sigma^2(\mathbb{R}_+^n)) \quad \text{with } \nabla u \in L^2(0, \infty; L^2(\mathbb{R}_+^n)) \quad \text{satisfies} \\ - \int_0^\infty \int_{\mathbb{R}_+^n} u \partial_\tau v \, dx \, d\tau + \int_0^\infty \int_{\mathbb{R}_+^n} \nabla u \cdot \nabla v \, dx \, d\tau + \int_0^\infty \int_{\mathbb{R}_+^n} u \cdot \nabla u \cdot v \, dx \, d\tau \\ = \int_{\mathbb{R}_+^n} u_0 v(0) \, dx \quad \text{for all } v \in C_0^\infty([0, \infty); C_{0,\sigma}^\infty(\mathbb{R}_+^n)), \end{aligned}$$

where  $u_0 \in L_\sigma^2(\mathbb{R}_+^n)$  and  $\partial_\tau v = \frac{\partial}{\partial \tau} v(x, \tau)$ . Moreover, the energy inequality holds for almost all  $t \in [0, \infty)$ :

$$\|u(t)\|_{L^2(\mathbb{R}_+^n)}^2 + 2 \int_0^t \int_{\mathbb{R}_+^n} |\nabla u(x, s)|^2 \, dx \, ds \leq \|u_0\|_{L^2(\mathbb{R}_+^n)}^2.$$

In order to obtain the estimates of Navier–Stokes solutions, it is necessary to consider the estimates of Stokes solutions. In the whole space  $\mathbb{R}^n$ , the Stokes solution  $u$  behaves just like that of the heat equation with initial data  $u_0$ . Moreover, for all  $1 \leq q \leq \infty$ ,

$$\|\nabla u(t)\|_{L^q(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}} \|u_0\|_{L^q(\mathbb{R}^n)},$$

which is valid for the half space  $\mathbb{R}_+^n$  with  $1 < q < \infty$  (see [27]), and for the exterior domains with  $1 < q < n$  (see [9]). Schonbek [24,25] also considered the decay rates on the whole space  $\mathbb{R}^n$  under some restrictions on  $u_0$ . Dan and Shibata [10,11] established the  $L^q$ – $L^r$  estimates for the Stokes flow of (1.1) on the exterior domains. Giga, Matsui and Shimizu [16] showed the decay rate for the first derivatives of the Stokes flow  $u$  of (1.1) in  $L^1(\mathbb{R}_+^n)$ . That is,

$$\|\nabla u(t)\|_{L^1(\mathbb{R}_+^n)} \leq C t^{-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}_+^n)}. \quad (1.3)$$

Subsequently, Shimizu [26] estimated the decay rate for the Stokes flow of (1.1) in  $L^\infty(\mathbb{R}_+^n)$ .

If  $u_0 \in L^1(\mathbb{R}_+^n)$  is in some weighted spaces, and satisfies some conditions, Bae and Choe [3] showed the decay rate in  $L^q(\mathbb{R}_+^n)$  ( $1 < q < \infty$ ) of  $\nabla u$  is  $t^{-1}$ . If the initial data  $u_0$  lies in an appropriate weighted space, Bae [2] estimated the time decay rates in  $L^1$ , in the Hardy space and in  $L^1$  of the gradient of solutions of (1.1) on the half spaces. In addition, Bae [1] also established the decay rates for the Stokes flow  $u$  of (1.1) in  $L^1(\mathbb{R}_+^n)$ . Precisely, Bae [1] proved that: if  $\nabla \cdot u_0 = 0$  in  $\mathbb{R}_+^n$ , and  $\int_{\mathbb{R}^{n-1}} u_0(y', y_n) \, dy' = 0$  for a.e.  $y_n > 0$ , then for  $t > 0$

$$\|u(t)\|_{L^1(\mathbb{R}_+^n)} \leq Ct^{-1} \int_{\mathbb{R}_+^n} y_n |y'| |u_0(y)| dy \quad (1.4)$$

and

$$\|u(t)\|_{L^1(\mathbb{R}_+^n)} \leq Ct^{-\frac{1}{2}} \int_{\mathbb{R}_+^n} |y'| |u_0(y)| dy. \quad (1.5)$$

For the Stokes flow  $u$  of (1.1) with the initial data  $u_0$ , we don't expect  $\|u(t)\|_{L^1(\mathbb{R}_+^n)} \leq C\|u_0\|_{L^1(\mathbb{R}_+^n)}$ , which holds in the whole space. In fact, Desch, Hieber and Pruss [12] found a counterexample: there exists a function  $u_0 \in L^1(\mathbb{R}_+^n)$  such that the corresponding Stokes flow  $u$  does not belong to  $L^1(\mathbb{R}_+^n)$ .

Let  $\mathcal{F}$  be the Fourier transform in  $\mathbb{R}^n$ :

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx.$$

The Riesz operators  $R_j$  ( $j = 1, 2, \dots, n$ ),  $S_j$  ( $j = 1, 2, \dots, n-1$ ), and the operator  $\Lambda$  are defined by

$$\mathcal{F}(R_j f)(\xi) = \frac{i\xi_j}{|\xi|} \mathcal{F}f(\xi),$$

$$\mathcal{F}(S_j f)(\xi) = \frac{i\xi_j}{|\xi'|} \mathcal{F}f(\xi),$$

$$\mathcal{F}(\Lambda f)(\xi) = |\xi'| \mathcal{F}f(\xi),$$

where  $\xi = (\xi_1, \xi_2, \dots, \xi_{n-1}, \xi_n) = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ .

Set  $R' = (R_1, R_2, \dots, R_{n-1})$ ,  $S = (S_1, S_2, \dots, S_{n-1})$ , and define the operators  $V_1$  and  $V_2$  by  $V_1 u_0 = -S \cdot u'_0 + u_{0n}$ ,  $V_2 u_0 = u' + Su_{0n}$ , where  $u_0 = (u_{01}, u_{02}, \dots, u_{0n}) = (u'_0, u_{0n})$ .

Let  $r$  be the restriction operator from  $\mathbb{R}^n$  to  $\mathbb{R}_+^n$ , and  $e$  is the extension operator from  $\mathbb{R}_+^n$  to  $\mathbb{R}^n$ , which is defined by

$$ef(x) = \begin{cases} f(x) & \text{for } x_n \geq 0, \\ 0 & \text{for } x_n < 0. \end{cases}$$

We also define the operator  $U$  by  $Uf = rR' \cdot S(R' \cdot S + R_n)ef$ , and the operators  $E(t)$  and  $F(t)$  by

$$E(t)f(x) = \int_{\mathbb{R}_+^n} [G_t(x' - y', x_n - y_n) - G_t(x' - y', x_n + y_n)] f(y) dy$$

and

$$F(t)f(x) = \int_{\mathbb{R}_+^n} [G_t(x' - y', x_n - y_n) + G_t(x' - y', x_n + y_n)] f(y) dy,$$

where  $G_t$  is the Gauss kernel  $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ .

**Theorem 1.1.** Let  $u$  be the solution of (1.1) with initial data  $u_0 = (u'_0, u_{0n}) \in L^1(\mathbb{R}_+^n)$ ,  $u_0|_{\partial\mathbb{R}_+^n} = 0$ ,  $\nabla \cdot u_0 = 0$  in  $\mathbb{R}_+^n$  ( $n \geq 2$ ). Then for any  $t > 0$

$$u - a \in L^1(\mathbb{R}_+^n).$$

Moreover,

$$\|u(t) - a(t)\|_{L^1(\mathbb{R}_+^n)} \leq C \|u_0\|_{L^1(\mathbb{R}_+^n)},$$

where  $a = a(t) = (a'(t), a_n(t))$ , and

$$\begin{aligned} a_n(t) &= R' \cdot R'E(t)S \cdot u'_0 + R_n R' \cdot SE(t)u_{0n}, \\ a'(t) &= SE(t)u_{0n} - R_n R'E(t)S \cdot u'_0 + R' \cdot R' SE(t)u_{0n}. \end{aligned}$$

Using the solution formula for the Stokes equation in [27], we establish the decay rates of  $u$ ,  $\partial_t u$ ,  $\nabla u$ ,  $\nabla^2 u$  in  $L^1(\mathbb{R}_+^n)$  for the Stokes flow  $u$  of (1.1) under some conditions on initial data  $u_0$ .

**Theorem 1.2.** Let  $u$  be the solution of (1.1) with initial data  $u_0 \in L^1(\mathbb{R}_+^n)$ ,  $u_0|_{\partial\mathbb{R}_+^n} = 0$ ,  $\nabla \cdot u_0 = 0$  in  $\mathbb{R}_+^n$  ( $n \geq 2$ ). Set

$$v_0(z', z_n) \triangleq \int_{-\infty}^{z_\ell} u_0(z_1, z_2, \dots, z_{\ell-1}, y_\ell, z_{\ell+1}, \dots, z_{n-1}, z_n) dy_\ell \quad \text{with } 1 \leq \ell \leq n-1.$$

If  $v_0 \in L^1(\mathbb{R}_+^n)$  for some  $1 \leq \ell \leq n-1$ , then there is a constant  $C$  independent of  $u_0$  such that

$$\|u(t)\|_{L^1(\mathbb{R}_+^n)} \leq C t^{-\frac{1}{2}} \|v_0\|_{L^1(\mathbb{R}_+^n)}. \quad (1.6)$$

**Theorem 1.3.** Let  $(u, p)$  be the solution of (1.1) with initial data  $u_0 = (u'_0, u_{0n}) \in L^1(\mathbb{R}_+^n)$ ,  $u_0|_{\partial\mathbb{R}_+^n} = 0$ ,  $\nabla \cdot u_0 = 0$  in  $\mathbb{R}_+^n$  ( $n \geq 2$ ). Then there is a constant  $C$  independent of  $u_0$  such that

$$\begin{aligned} \|\nabla u(t)\|_{L^1(\mathbb{R}_+^n)} &\leq C t^{-1} \left( \int_{\mathbb{R}_+^n} y_n |u_0(y)| dy + \int_{\mathbb{R}_+^n} |y'| |u_0(y)| dy \right) \\ \text{if } \int_{\mathbb{R}^{n-1}} u_0(y', y_n) dy' &= 0 \text{ for a.e. } y_n > 0, \end{aligned} \quad (1.7)$$

$$\|\nabla^2 u(t)\|_{L^1(\mathbb{R}_+^n)} \leq C t^{-1} \|u_0\|_{L^1(\mathbb{R}_+^n)}, \quad (1.8)$$

$$\|\partial_t u(t)\|_{L^1(\mathbb{R}_+^n)} \leq C t^{-1} \|u_0\|_{L^1(\mathbb{R}_+^n)}, \quad (1.9)$$

$$\|\nabla p(t)\|_{L^1(\mathbb{R}_+^n)} \leq C t^{-1} \|u_0\|_{L^1(\mathbb{R}_+^n)}. \quad (1.10)$$

**Remark.** (1) Let  $A$  denote the Stokes operator  $-P\Delta$  in  $\mathbb{R}_+^n$ , where  $P$  is the projection:  $L^r(\mathbb{R}_+^n) \rightarrow L^r_\sigma(\mathbb{R}_+^n)$ ,  $1 < r < \infty$ . Then the solution  $u$  of (1.1) can be expressed by  $u(t) = e^{-tA}u_0$ . From (1.3), we have for  $t > 0$

$$\begin{aligned}\|\nabla^2 u(t)\|_{L^1(\mathbb{R}_+^n)} &\leq \sum_{i,j=1}^n \|\partial_i e^{-\frac{t}{2}A} \partial_j e^{-\frac{t}{2}A} u_0\|_{L^1(\mathbb{R}_+^n)} \leq Ct^{-\frac{1}{2}} \sum_{j=1}^n \|\partial_j e^{-\frac{t}{2}A} u_0\|_{L^1(\mathbb{R}_+^n)} \\ &\leq Ct^{-1} \|u_0\|_{L^1(\mathbb{R}_+^n)}.\end{aligned}$$

Inductively, we can prove that for any positive integer  $k$ ,

$$\|\nabla^k u(t)\|_{L^1(\mathbb{R}_+^n)} \leq C(k)t^{-\frac{k}{2}} \|u_0\|_{L^1(\mathbb{R}_+^n)}.$$

In addition, if  $u_0 \in L^1(\mathbb{R}_+^n)$  satisfies  $\nabla \cdot u = 0$  in  $\mathbb{R}_+^n$ , and  $\int_{\mathbb{R}^{n-1}} u_0(y', y_n) dy' = 0$  for a.e.  $y_n > 0$ , then from (1.3), (1.5), we have

$$\begin{aligned}\|\nabla u(t)\|_{L^1(\mathbb{R}_+^n)} &= \left\| \nabla e^{-\frac{t}{2}A} u\left(\frac{t}{2}\right) \right\|_{L^1(\mathbb{R}_+^n)} \leq Ct^{-\frac{1}{2}} \left\| u\left(\frac{t}{2}\right) \right\|_{L^1(\mathbb{R}_+^n)} \\ &\leq \begin{cases} Ct^{-\frac{3}{2}} \int_{\mathbb{R}_+^n} y_n |y'| |u_0(y)| dy, \\ Ct^{-1} \int_{\mathbb{R}_+^n} |y'| |u_0(y)| dy. \end{cases}\end{aligned}$$

(2) We are not sure whether the following inequality holds for the pressure  $p$ :

$$\|p(t)\|_{L^1(\mathbb{R}_+^n)} \leq Ct^{-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}_+^n)}. \quad (1.11)$$

From (1.1) and (1.8), (1.10), we can roughly believe that  $p, \nabla u$  should have the same decay rate in  $L^1(\mathbb{R}_+^n)$ . On the other hand, using the solution formula on the pressure  $p$  in [27], and after a detailed calculation, we have

$$\begin{aligned}p(x, t) &= -\frac{1}{t} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} \partial_n H(x' - y', x_n) G_t(y' - z', z_n) z_n u_{0n}(z) dz dy' \\ &\quad - \frac{C_n}{t} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}^{n-1}} \partial_n H(x' - y', x_n) G_t(y' - z', z_n) \\ &\quad \times \sum_{j=1}^n |z' - w'|^{-n} (z_j - w_j) u_{0j}(w', z_n) dw' dz dy',\end{aligned}$$

where  $C_n = 2^{1-\frac{n}{2}} \sqrt{\pi} \Gamma(\frac{n-1}{2})$ ,  $\Gamma$  is the gamma function,  $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ , and  $H(x)$  is the fundamental solution of the Laplace operator. Unfortunately, we are unable to verify that  $p \in L^1(\mathbb{R}_+^n)$  and the expected decay rate in (1.11).

There is a great literature on the decay rates for the Navier–Stokes flows of (1.2) on the exterior domains, see [4–8, 17–21, 23] and the references therein. Recently, Fujigaki and Miyakawa [13, 14] considered the decay rate of weak and strong solutions for (1.2) in  $L^q(\mathbb{R}_+^n)$  with  $1 < q < \infty$ .

It is well known that the projection operator  $P : L^r(\mathbb{R}_+^n) \rightarrow L_\sigma^r(\mathbb{R}_+^n)$  is unbounded for  $r = 1, \infty$ , which results in many difficulties in dealing with (1.2). As far as we know, there are no results on the decay properties for weak or strong solutions of (1.2) in  $L^r(\mathbb{R}_+^n)$  with  $r = 1, \infty$ . As in the Stokes' case, we also don't expect that weak or strong solutions of (1.2) belong to  $L^1(\mathbb{R}_+^n)$ , and we need to analyze the structure of solutions of (1.2) furthermore.

**Theorem 1.4.** Let  $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$  ( $n \geq 2$ ) satisfy

$$\int_{\mathbb{R}^n_+} (1 + y_n) |u_0(y)| dy < \infty.$$

Then there exists a weak solution  $u$  of (1.2) such that:

(i) For any  $t > 0$

$$\|u(t) - (a(t) + b(t))\|_{L^1(\mathbb{R}^n_+)} \leq C,$$

where  $C > 0$  is independent of  $t > 0$ ,  $a = (a', a_n)$  is given in Theorem 1.1, and  $b = (b', b_n)$  is defined by

$$\begin{aligned} b_n(t) &= \sum_{i,j=1}^n R' \cdot R' S \cdot \nabla' \int_0^t E(t-s) \mathcal{N} \partial_i \partial_j (u_i u_j)(s) ds \\ &\quad + \sum_{i,j=1}^n R_n R' \cdot S \int_0^t E(t-s) \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j)(s) ds, \\ b'(t) &= - \sum_{i,j=1}^n R_n R_n S \int_0^t E(t-s) \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j)(s) ds \\ &\quad - \sum_{i,j=1}^n R_n R' \cdot \nabla' S \int_0^t E(t-s) \mathcal{N} \partial_i \partial_j (u_i u_j)(s) ds \\ &\quad + S \int_0^t E(t-s) (u_n \partial_n u_n)(s) ds - R_n R_n S \int_0^t E(t-s) \partial_n (u_n u_n)(s) ds, \end{aligned}$$

where  $\nabla' = (\partial_1, \partial_2, \dots, \partial_{n-1})$  and the operator  $\mathcal{N}$  is defined in (3.2) below.

(ii) If  $u$  is the strong solution of (1.2), then

$$\|\nabla u(t)\|_{L^1(\mathbb{R}^n_+)} \leq C t^{-\frac{1}{2}} (1 + t^{-\frac{n+2}{2}}) \quad \text{for all } t > 0.$$

**Remark.** Compared with (1.3), the result (ii) in Theorem 1.4 shows that the decay rate of the first derivatives of the strong solution of (1.2) also can be controlled by  $C t^{-\frac{1}{2}}$ ,  $t > 1$ .

**Theorem 1.5.** Suppose  $u_0 \in L^2_\sigma(\mathbb{R}^n_+) \cap L^q(\mathbb{R}^n_+)$  ( $\frac{n}{2-k} < q < \infty$ ,  $k = 0, 1$ ,  $n \geq 2$ ) satisfies

$$\int_{\mathbb{R}^n_+} (1 + y_n) |u_0(y)| dy < \infty.$$

Let  $u$  be the strong solution of (1.2). Then for  $k = 0, 1$

$$\|\nabla^k u(t)\|_{L^\infty(\mathbb{R}^n_+)} \leq C t^{-\frac{n+k+1}{2}} (1 + t^{-\frac{n}{2}}) \quad \text{for all } t > 0.$$

**Remark.** We are not sure whether the result in Theorem 1.5 holds if  $k \geq 2$ , because our method fails in this case (see the proof of Theorem 1.5 in Section 3).

Throughout this paper, we denote the norm of  $L^r(\mathbb{R}_+^n)$  ( $1 \leq r < \infty$ ) by  $\|u\|_{L^r(\mathbb{R}_+^n)} = (\int_{\mathbb{R}_+^n} |u(x)| dx)^{\frac{1}{r}}$ , the norm of  $L^\infty(\mathbb{R}_+^n)$  by  $\|u\|_{L^\infty(\mathbb{R}_+^n)} = \text{ess sup}_{x \in \mathbb{R}_+^n} |u(x)|$ , and positive constants (possibly different line to line) by  $C$ .

## 2. Decay rates for the Stokes flow in $L^1(\mathbb{R}_+^n)$

In this section, before giving the proofs of Theorems 1.1–1.3, we introduce some notations and useful lemmas.

A function  $f \in L^1(\mathbb{R}^n)$  belongs to the Hardy space  $\mathcal{H}^1(\mathbb{R}^n)$  if  $\sup_{s>0} |G_s * f(x)| \in L^1(\mathbb{R}^n)$ , where the symbol  $*$  denotes the convolution with respect to the space variable  $x$ . The norm of  $f \in \mathcal{H}^1(\mathbb{R}^n)$  is defined by

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \triangleq \left\| \sup_{s>0} |G_s * f| \right\|_{L^1(\mathbb{R}^n)}.$$

It is known (see [15,22]) that an  $L^1$ -function  $f$  is in  $\mathcal{H}^1(\mathbb{R}^n)$  if and only if all its Riesz transforms  $R_j f$  are in  $L^1(\mathbb{R}^n)$  and that

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \approx \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j f\|_{L^1(\mathbb{R}^n)} \quad (\text{equivalent norm}).$$

For simplicity, we denote the operator norm of  $R_j$  on  $\mathcal{H}^1(\mathbb{R}^n)$  by  $\|R_j\|$ . We also need the Hardy space on the half space, denoted by  $\mathcal{H}^1(\mathbb{R}_+^n)$ , which norm is defined by

$$\|f\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \triangleq \inf \{ \|\tilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)} \mid \tilde{f} \in \mathcal{H}^1(\mathbb{R}^n), \tilde{f}|_{\mathbb{R}_+^n} = f \}.$$

By the solution formula in [27], the solution  $u$  of (1.1) is represented as

$$\begin{cases} u_n = U E(t) V_1 u_0, \\ u' = E(t) V_2 u_0 - S U E(t) V_1 u_0, \end{cases} \quad (2.1)$$

where the operators  $U, S, E, V_1, V_2$  are given in the introduction.

Note that the solution  $u$  of (1.1) is given as a restriction  $r\bar{u}$  of one vector field  $\bar{u} = (\bar{u}', \bar{u}_n)$ :

$$\begin{cases} \bar{u}_n = R' \cdot S (R' \cdot S + R_n) e E(t) V_1 u_0, \\ \bar{u}' = e E(t) V_2 u_0 - S U e E(t) V_1 u_0 = e E(t) V_2 u_0 - S \bar{u}_n. \end{cases} \quad (2.2)$$

**Lemma 2.1.** Assume that  $\nabla \cdot u_0 = 0$  in  $\mathbb{R}_+^n$ ,  $u_0|_{\partial\mathbb{R}_+^n} = 0$ . Let  $\bar{u} = (\bar{u}', \bar{u}_n)$  be given in (2.2). Then for any  $1 \leq j \leq n-1$ , we have

$$\begin{aligned} \bar{u}_n &= \sum_{k,m=1}^{n-1} R_k^2 \partial_m \Lambda^{-1} e E(t) u_{0m} + \sum_{m=1}^{n-1} R_n R_m e E(t) u_{0m} \\ &\quad - \sum_{k=1}^{n-1} R_k^2 e E(t) u_{0n} + \sum_{k=1}^{n-1} R_n R_k \partial_k \Lambda^{-1} e E(t) u_{0n}, \end{aligned} \quad (2.3)$$

$$\begin{aligned} \bar{u}_j = & eE(t)u_{0j} + \partial_j \Lambda^{-1} eE(t)u_{0n} + \sum_{m=1}^{n-1} R_j R_m eE(t)u_{0m} \\ & - \sum_{m=1}^{n-1} R_n R_m \partial_j \Lambda^{-1} eE(t)u_{0m} + \sum_{k=1}^{n-1} R_k^2 \partial_j \Lambda^{-1} eE(t)u_{0n} + R_n R_j eE(t)u_{0n}, \end{aligned} \quad (2.4)$$

$$\begin{aligned} \partial_t \bar{u}_n = & \sum_{k,m=1}^{n-1} \partial_k \partial_k \partial_m \Lambda^{-1} eE(t)u_{0m} + \sum_{m=1}^{n-1} R_n R_m e \partial_t E(t)u_{0m} \\ & - \sum_{k=1}^{n-1} R_k^2 e \partial_t E(t)u_{0n} + \sum_{k=1}^{n-1} \partial_n \partial_k \partial_k \Lambda^{-1} eE(t)u_{0n}, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \partial_t \bar{u}_j = & e \partial_t E(t)u_{0j} - \partial_j \partial_n eE(t)u_{0n} + \sum_{m=1}^{n-1} \partial_j \partial_m eE(t)u_{0m} \\ & + \sum_{m=1}^{n-1} \partial_n \partial_j \partial_m \Lambda^{-1} eE(t)u_{0m} - \sum_{m=1}^{n-1} \partial_n \partial_m \partial_j \Lambda^{-1} eE(t)u_{0m}. \end{aligned} \quad (2.6)$$

**Proof.** To show (2.3)–(2.6), it is convenient to use the Fourier transform for  $\bar{u}$  in (2.2). Note that the operators  $S_j$  and  $eE(t)$  commute, we have

$$\begin{aligned} \mathcal{F}(\bar{u}_n)(\xi) &= \frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi'|} \left( \frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi'|} + \frac{i\xi_n}{|\xi|} \right) \left( -\frac{i\xi'}{|\xi'|} \cdot \mathcal{F}(eE(t)u'_0) + \mathcal{F}(eE(t)u_{0n}) \right) \\ &= \frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi|} i\xi' |\xi'|^{-1} \cdot \mathcal{F}(eE(t)u'_0) + \frac{i\xi_n}{|\xi|} \frac{i\xi'}{|\xi|} \cdot \mathcal{F}(eE(t)u_{0n}) \\ &\quad - \frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi|} \cdot \mathcal{F}(eE(t)u_{0n}) + \frac{i\xi_n}{|\xi|} \frac{i\xi'}{|\xi|} \cdot i\xi' \cdot |\xi'|^{-1} \cdot \mathcal{F}(eE(t)u_{0n}), \end{aligned} \quad (2.7)$$

which implies (2.3).

From (2.2), (2.3), we get

$$\begin{aligned} \mathcal{F}(\bar{u}_j)(\xi) &= \mathcal{F}(eE(t)u_{0j}) + \mathcal{F}(S_j eE(t)u_{0n}) - \mathcal{F}(S_j \bar{u}_n) \\ &= \mathcal{F}(eE(t)u_{0j}) + \frac{i\xi_j}{|\xi'|} \mathcal{F}(eE(t)u_{0n}) - \frac{i\xi_j}{|\xi'|} \mathcal{F}(\bar{u}_n) \\ &= \mathcal{F}(eE(t)u_{0j}) + i\xi_j |\xi'|^{-1} \mathcal{F}(eE(t)u_{0n}) + \frac{i\xi_j}{|\xi|} \frac{i\xi'}{|\xi|} \cdot \mathcal{F}(eE(t)u'_0) \\ &\quad - \frac{i\xi_n}{|\xi|} \frac{i\xi'}{|\xi|} i\xi_j |\xi'|^{-1} \cdot \mathcal{F}(eE(t)u'_0) + \frac{i\xi'}{|\xi|} \frac{i\xi'}{|\xi|} i\xi_j |\xi'|^{-1} \mathcal{F}(eE(t)u_{0n}) \\ &\quad + \frac{i\xi_n}{|\xi|} \frac{i\xi_j}{|\xi|} \mathcal{F}(eE(t)u_{0n}), \end{aligned}$$

which implies (2.4).



Note that  $\partial_t eE(t)f = \Delta eE(t)f$ . From (2.3), we have

$$\begin{aligned} \partial_t \bar{u}_n &= \sum_{k,m=1}^{n-1} R_k^2 \partial_m \Lambda^{-1} \partial_t eE(t) u_{0m} + \sum_{m=1}^{n-1} R_n R_m \partial_t eE(t) u_{0m} \\ &\quad - \sum_{k=1}^{n-1} R_k^2 \partial_t eE(t) u_{0n} + \sum_{k=1}^{n-1} R_n R_k \partial_k \Lambda^{-1} \partial_t eE(t) u_{0n} \\ &= \sum_{k,m=1}^{n-1} R_k^2 \partial_m \Lambda^{-1} \Delta eE(t) u_{0m} + \sum_{m=1}^{n-1} R_n R_m \partial_t eE(t) u_{0m} \\ &\quad - \sum_{k=1}^{n-1} R_k^2 \partial_t eE(t) u_{0n} + \sum_{k=1}^{n-1} R_n R_k \partial_k \Lambda^{-1} \Delta eE(t) u_{0n}. \end{aligned} \quad (2.8)$$

Since

$$\begin{aligned} \mathcal{F} \left( \sum_{k,m=1}^{n-1} R_k^2 \partial_m \Lambda^{-1} \Delta eE(t) u_{0m} \right) &= \sum_{k,m=1}^{n-1} \frac{i\xi_k}{|\xi|} \frac{i\xi_k}{|\xi|} \frac{i\xi_m}{|\xi'|} (-|\xi|^2) \mathcal{F}(eE(t) u_{0m}) \\ &= - \sum_{k,m=1}^{n-1} i\xi_k i\xi_k i\xi_m |\xi'|^{-1} \mathcal{F}(eE(t) u_{0m}) \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} \mathcal{F} \left( \sum_{k=1}^{n-1} R_n R_k \partial_k \Lambda^{-1} \Delta eE(t) u_{0n} \right) &= \sum_{k=1}^{n-1} \frac{i\xi_n}{|\xi|} \frac{i\xi_k}{|\xi|} \frac{i\xi_k}{|\xi'|} (-|\xi|^2) \mathcal{F}(eE(t) u_{0n}) \\ &= - \sum_{k=1}^{n-1} i\xi_k i\xi_k i\xi_n |\xi'|^{-1} \mathcal{F}(eE(t) u_{0n}). \end{aligned} \quad (2.10)$$

Inserting (2.9), (2.10) into (2.8), we infer that (2.5) holds.

From (2.5), we have

$$\begin{aligned} \mathcal{F}(\partial_j \Lambda^{-1} \partial_t \bar{u}_n) &= \sum_{k,m=1}^{n-1} i\xi_j |\xi'|^{-1} i\xi_k i\xi_k i\xi_m |\xi'|^{-1} \mathcal{F}(eE(t) u_{0m}) \\ &\quad + \sum_{m=1}^{n-1} i\xi_j |\xi'|^{-1} \frac{i\xi_n}{|\xi|} \frac{i\xi_m}{|\xi|} (-|\xi|^2) \mathcal{F}(eE(t) u_{0m}) \\ &\quad - \sum_{k=1}^{n-1} i\xi_j |\xi'|^{-1} \frac{i\xi_k}{|\xi|} \frac{i\xi_k}{|\xi|} (-|\xi|^2) \mathcal{F}(eE(t) u_{0n}) \\ &\quad - \sum_{k=1}^{n-1} i\xi_j |\xi'|^{-1} i\xi_k i\xi_k i\xi_n |\xi'|^{-1} \mathcal{F}(eE(t) u_{0n}) \end{aligned}$$

$$\begin{aligned}
&= - \sum_{m=1}^{n-1} i\xi_j i\xi_m \mathcal{F}(eE(t)u_{0m}) - \sum_{m=1}^{n-1} i\xi_j i\xi_n i\xi_m |\xi'|^{-1} \mathcal{F}(eE(t)u_{0m}) \\
&\quad + \sum_{k=1}^{n-1} i\xi_j i\xi_k i\xi_k |\xi'|^{-1} \mathcal{F}(eE(t)u_{0n}) + i\xi_n i\xi_j \mathcal{F}(eE(t)u_{0n}),
\end{aligned}$$

which implies that

$$\begin{aligned}
\partial_j \Lambda^{-1} \partial_t \bar{u}_n &= - \sum_{k,m=1}^{n-1} \partial_j \partial_m eE(t)u_{0m} + \partial_j \partial_n eE(t)u_{0n} \\
&\quad - \sum_{m=1}^{n-1} \partial_j \partial_n \partial_m \Lambda^{-1} eE(t)u_{0m} + \sum_{k=1}^{n-1} \partial_j \partial_k \partial_k eE(t)u_{0n}, \tag{2.11}
\end{aligned}$$

$$\begin{aligned}
\partial_j \Lambda^{-1} \partial_t eE(t)u_{0n}(x) &= \partial_j \Lambda^{-1} \Delta eE(t)u_{0n}(x) \\
&= \sum_{k=1}^{n-1} \partial_j \Lambda^{-1} \partial_k \partial_k eE(t)u_{0n}(x) + \partial_j \Lambda^{-1} \partial_n \partial_n eE(t)u_{0n}(x) \\
&= \sum_{k=1}^{n-1} \partial_j \partial_k \partial_k \Lambda^{-1} eE(t)u_{0n}(x) \\
&\quad + c_n \partial_j \int_{\mathbb{R}^{n-1}} |x' - y'|^{-n+2} \partial_{x_n}^2 [eE(t)u_{0n}](y', x_n) dy' \\
&= \sum_{k=1}^{n-1} \partial_j \partial_k \partial_k \Lambda^{-1} eE(t)u_{0n}(x) + c_n \partial_j \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} |x' - y'|^{-n+2} \partial_{x_n}^2 \\
&\quad \times [G_t(y' - z', x_n - z_n) - G_t(y' - z', x_n + z_n)] u_{0n}(z) dz dy' \\
&= \sum_{k=1}^{n-1} \partial_j \partial_k \partial_k \Lambda^{-1} eE(t)u_{0n}(x) + c_n \partial_j \partial_n \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} |x' - y'|^{-n+2} \\
&\quad \times [G_t(y' - z', x_n - z_n) - G_t(y' - z', x_n + z_n)] \partial_n u_{0n}(z) dz dy' \\
&= \sum_{k=1}^{n-1} \partial_j \partial_k \partial_k \Lambda^{-1} eE(t)u_{0n}(x) + \sum_{m=1}^{n-1} c_n \partial_j \partial_n \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} |x' - y'|^{-n+2} \\
&\quad \times \partial_{z_m} [G_t(y' - z', x_n - z_n) + G_t(y' - z', x_n + z_n)] u_{0m}(z) dz dy' \\
&= \sum_{k=1}^{n-1} \partial_j \partial_k \partial_k \Lambda^{-1} eE(t)u_{0n}(x) - \sum_{m=1}^{n-1} \partial_j \partial_m \partial_n \Lambda^{-1} eF(t)u_{0m}(x) \quad \text{for } n \geq 3. \tag{2.12}
\end{aligned}$$

Here we have used the two facts:  $\nabla \cdot u_0 = 0$  in  $\mathbb{R}_+^n$ ,  $u_0|_{\partial\mathbb{R}_+^n} = 0$ ; and if  $n \geq 3$ , the integral kernel of  $\Lambda^{-1}$  is  $c_n |x'|^{-n+2}$  for some  $c_n > 0$ .

If  $n = 2$ . Then  $\partial_1 \Lambda^{-1} = S_1$  and

$$\begin{aligned}
 \partial_1 \Lambda^{-1} \partial_2 \partial_2 e E(t) u_{02}(x) &= S_1 \partial_2^2 e E(t) u_{02}(x) \\
 &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|y_1| > \epsilon} y_1^{-1} \partial_{x_2}^2 [e E(t) u_{02}](x_1 - y_1, x_2) dy_1 \\
 &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|y_1| > \epsilon} \int_{\mathbb{R}_+^2} y_1^{-1} \partial_{x_2}^2 [G_t(x_1 - y_1 - z_1, x_2 - z_2) \\
 &\quad - G_t(x_1 - y_1 - z_1, x_2 + z_2)] u_{02}(z) dz dy_1 \\
 &= \partial_{x_2} \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|y_1| > \epsilon} \int_{\mathbb{R}_+^2} y_1^{-1} [G_t(x_1 - y_1 - z_1, x_2 - z_2) \\
 &\quad + G_t(x_1 - y_1 - z_1, x_2 + z_2)] \partial_2 u_{02}(z) dz dy_1 \\
 &= \partial_{x_2} \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|y_1| > \epsilon} \int_{\mathbb{R}_+^2} y_1^{-1} \partial_{z_1} [G_t(x_1 - y_1 - z_1, x_2 - z_2) \\
 &\quad + G_t(x_1 - y_1 - z_1, x_2 + z_2)] u_{01}(z) dz dy_1 \\
 &= -\partial_1 \partial_2 \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|y_1| > \epsilon} y_1^{-1} [e F(t) u_{01}](x_1 - y_1, x_2) dy_1 \\
 &= -\partial_1 \partial_2 S_1 e F(t) u_{01}(x) = -\partial_1 \partial_2 \Lambda^{-1} e F(t) u_{01}(x).
 \end{aligned}$$

Whence (2.12) also holds for  $n = 2$ . From (2.2), (2.11), (2.12), we obtain

$$\begin{aligned}
 \partial_j \bar{u}_j &= e E(t) u_{0j} + \partial_j \Lambda^{-1} \partial_t e E(t) u_{0n} - \partial_j \Lambda^{-1} \partial_t \bar{u}_n \\
 &= e E(t) u_{0j} - \partial_j \partial_n e E(t) u_{0n} + \sum_{m=1}^{n-1} \partial_j \partial_m e E(t) u_{0m} \\
 &\quad + \sum_{m=1}^{n-1} \partial_n \partial_j \partial_m \Lambda^{-1} e E(t) u_{0m} - \sum_{m=1}^{n-1} \partial_n \partial_j \partial_m \Lambda^{-1} e F(t) u_{0m},
 \end{aligned}$$

which is (2.6).  $\square$

**Lemma 2.2.** For any  $1 \leq j \leq n$ ,  $1 \leq k, \ell \leq n - 1$ , we have for  $n \geq 2$

$$|\partial_j \partial_k \partial_\ell \Lambda^{-1} e G_t(x)| \leq C_{m, \ell_0} t^{\frac{\ell_0 + m - n - 2}{2}} |x'|^{-m} |x_n|^{-\ell_0},$$

where  $\ell_0 \geq 0$ , and  $0 \leq m \leq n + 1$  if  $1 \leq j \leq n - 1$ ;  $0 \leq m \leq n$  if  $j = n$ .

**Proof.** Note that for  $n \geq 3$

$$\partial_j \partial_k \partial_\ell \Lambda^{-1} e G_t(x) = c_n \partial_j \partial_k \partial_\ell \int_{\mathbb{R}^{n-1}} |x' - y'|^{-n+2} \theta(x_n) G_t(y', x_n) dy',$$

where  $\theta(x_n) = \begin{cases} 1 & \text{if } x_n > 0, \\ 0 & \text{if } x_n \leq 0. \end{cases}$

Set  $x = t^{\frac{1}{2}}z$ . Then

$$\begin{aligned} \partial_j \partial_k \partial_\ell \Lambda^{-1} eG_t(x) &= c_n t^{-1-\frac{n}{2}} \partial_{z_j} \partial_{z_k} \partial_{z_\ell} \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n+2} \theta(z_n) G_1(y', z_n) dy' \\ &= t^{-1-\frac{n}{2}} \partial_{z_j} \partial_{z_k} \partial_{z_\ell} \Lambda^{-1} eG_1(z). \end{aligned} \quad (2.13)$$

So it is sufficient to prove that

$$|\partial_{z_j} \partial_{z_k} \partial_{z_\ell} \Lambda^{-1} eG_1(z)| \leq C_{\ell_0, m} |z'|^{-\ell_0} |z_n|^{-m}, \quad (2.14)$$

where  $j, k, \ell, \ell_0, m$  are the same as in the above.

**Case 1.**  $n \geq 3$  and  $1 \leq j \leq n-1, 1 \leq k, \ell \leq n-1$ .

Let  $\psi_1 \in C_0^\infty(\mathbb{R}^{n-1})$  such that  $0 \leq \psi_1 \leq 1$ ,  $\text{supp } \psi_1 \subset \{x' \in \mathbb{R}^{n-1} \mid |x'| < 1\}$ , and  $\psi_1 \equiv 1$  on  $\{x' \in \mathbb{R}^{n-1} \mid |x'| < \frac{1}{2}\}$ . Set  $\psi_2 = 1 - \psi_1$ . Then

$$\begin{aligned} \partial_{z_j} \partial_{z_k} \partial_{z_\ell} \Lambda^{-1} eG_1(z') &= c_n (4\pi)^{-\frac{n-1}{2}} \partial_j \partial_k \partial_\ell \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n+2} e^{-\frac{|y'|^2}{4}} dy' \\ &= c_n (4\pi)^{-\frac{n-1}{2}} \partial_j \partial_k \partial_\ell \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n+2} \psi_1(z' - y') e^{-\frac{|y'|^2}{4}} dy' \\ &\quad + c_n (4\pi)^{-\frac{n-1}{2}} \partial_j \partial_k \partial_\ell \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n+2} \psi_2(z' - y') e^{-\frac{|y'|^2}{4}} dy' \\ &= I_1(z') + I_2(z'), \\ |I_1(z')| &= c_n (4\pi)^{-\frac{n-1}{2}} \left| \partial_j \partial_k \partial_\ell \int_{\mathbb{R}^{n-1}} |y'|^{-n+2} \psi_1(y') e^{-\frac{|z'-y'|^2}{4}} dy' \right| \\ &= c_n (4\pi)^{-\frac{n-1}{2}} \left| \int_{\mathbb{R}^{n-1}} |y'|^{-n+2} \psi_1(y') \left[ \frac{\delta_{j\ell}(z_k - y_k) + \delta_{k\ell}(z_j - y_j) + \delta_{kj}(z_\ell - y_\ell)}{4} \right. \right. \\ &\quad \left. \left. - \frac{(z_\ell - y_\ell)(z_j - y_j)(z_k - y_k)}{8} \right] e^{-\frac{|z'-y'|^2}{4}} dy' \right| \\ &\leq C \int_{|y'| \leq 1} |y'|^{-n+2} [|z'| + |z'|^3] e^{-\frac{1}{4}(|z'|^2 - 1)} dy' \\ &\leq C_{\ell_0} |z'|^{-\ell_0} \quad \text{for any } \ell_0 \geq 0, \end{aligned}$$

$$\begin{aligned}
 I_2(z') &= (4\pi)^{-\frac{n-1}{2}} c_n \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n+2} [\partial_{z_j} \partial_{z_k} \partial_{z_\ell} \psi_2(z' - y')] e^{-\frac{|y'|^2}{4}} dy' \\
 &\quad - (4\pi)^{-\frac{n-1}{2}} (n-2) c_n \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n} [(z_\ell - y_\ell) \partial_{z_j} \partial_{z_k} \psi_2(z' - y') \\
 &\quad + (z_k - y_k) \partial_{z_j} \partial_{z_\ell} \psi_2(z' - y') + (z_j - y_j) \partial_{z_\ell} \partial_{z_k} \psi_2(z' - y')] e^{-\frac{|y'|^2}{4}} dy' \\
 &\quad + (4\pi)^{-\frac{n-1}{2}} n(n-2) c_n \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n-2} [(z_k - y_k)(z_\ell - y_\ell) \partial_{z_j} \psi_2(z' - y') \\
 &\quad + (z_j - y_j)(z_\ell - y_\ell) \partial_{z_k} \psi_2(z' - y') + (z_j - y_j)(z_k - y_k) \partial_{z_\ell} \psi_2(z' - y')] e^{-\frac{|y'|^2}{4}} dy' \\
 &\quad - (4\pi)^{-\frac{n-1}{2}} (n-2) c_n \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n} [\delta_{k\ell} \partial_{z_j} \psi_2(z' - y') \\
 &\quad + \delta_{j\ell} \partial_{z_k} \psi_2(z' - y') + \delta_{jk} \partial_{z_\ell} \psi_2(z' - y')] e^{-\frac{|y'|^2}{4}} dy' \\
 &\quad + (4\pi)^{-\frac{n-1}{2}} n(n-2) c_n \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n-2} [\delta_{k\ell}(z_j - y_j) + \delta_{jk}(z_\ell - y_\ell) \\
 &\quad + \delta_{j\ell}(z_k - y_k)] \psi_2(z' - y') e^{-\frac{|y'|^2}{4}} dy' - (4\pi)^{-\frac{n-1}{2}} n(n-2)(n+2) c_n \\
 &\quad \times \int_{\mathbb{R}^{n-1}} |z' - y'|^{-n-4} (z_j - y_j)(z_k - y_k)(z_\ell - y_\ell) \psi_2(z' - y') e^{-\frac{|y'|^2}{4}} dy' \\
 &= J_1(z') + J_2(z') + J_3(z') + J_4(z') + J_5(z') + J_6(z'),
 \end{aligned}$$

$$\begin{aligned}
 |J_1(z')| &\leq C \|\nabla^3 \psi_2\|_{L^\infty(\mathbb{R}^{n-1})} \int_{\frac{1}{2} \leq |y'| \leq 1} |y'|^{-n+2} e^{-\frac{|z'-y'|^2}{4}} dy' \\
 &\leq C \int_{\frac{1}{2} \leq |y'| \leq 1} |y'|^{-n+2} e^{-\frac{1}{4}(\frac{|z'|^2}{2}-1)} dy' \leq C_{\ell_0} |z'|^{-\ell_0} \quad \text{for any } \ell_0 \geq 0,
 \end{aligned}$$

$$\begin{aligned}
 |J_2(z')| &\leq C \|\nabla^2 \psi_2\|_{L^\infty(\mathbb{R}^{n-1})} \int_{\frac{1}{2} \leq |y'| \leq 1} |y'|^{-n+1} e^{-\frac{|z'-y'|^2}{4}} dy' \\
 &\leq C \int_{\frac{1}{2} \leq |y'| \leq 1} |y'|^{-n+1} e^{-\frac{1}{4}(\frac{|z'|^2}{2}-1)} dy' \leq C_{\ell_0} |z'|^{-\ell_0} \quad \text{for any } \ell_0 \geq 0,
 \end{aligned}$$

$$\begin{aligned}
 |J_3(z') + J_4(z')| &\leq C \|\nabla \psi_2\|_{L^\infty(\mathbb{R}^{n-1})} \int_{\frac{1}{2} \leq |y'| \leq 1} |y'|^{-n} e^{-\frac{|z'-y'|^2}{4}} dy' \\
 &\leq C \int_{\frac{1}{2} \leq |y'| \leq 1} |y'|^{-n} e^{-\frac{1}{4}(\frac{|z'|^2}{2}-1)} dy' \leq C_{\ell_0} |z'|^{-\ell_0} \quad \text{for any } \ell_0 \geq 0,
 \end{aligned}$$

$$\begin{aligned}
|J_5(z') + J_6(z')| &\leq C \int_{|y'| \geq \frac{1}{2}} |z' - y'|^{-n-1} e^{-\frac{|y'|^2}{4}} dy' \\
&\leq C(\ell_0) |z'|^{-\ell_0} \int_{|y'| \geq \frac{1}{2}} (|z' - y'|^{\ell_0-n-1} + |y'|_0^\ell |z' - y'|^{-n-1}) e^{-\frac{|y'|^2}{4}} dy' \\
&\leq C_{\ell_0} |z'|^{\ell_0} \quad \text{for any } 0 \leq \ell_0 \leq n+1.
\end{aligned}$$

From the above arguments, we conclude that for any  $0 \leq \ell_0 \leq n+1$ ,  $m \geq 0$

$$|\partial_{z_j} \partial_{z_k} \partial_{z_\ell} \Lambda^{-1} e G_1(z)| = \theta(z_n) (4\pi)^{-\frac{1}{2}} e^{-\frac{|z_n|^2}{4}} |\partial_{z_j} \partial_{z_k} \partial_{z_\ell} \Lambda^{-1} G'_1(z')| \leq C_{\ell_0, m} |z'|^{-\ell_0} |z_n|^{-m}$$

which is (2.14) in Case 1. Here  $G'_1(z')$  is defined by  $G'_t(z') = (4\pi t)^{-\frac{n-1}{2}} e^{-\frac{|z'|^2}{4t}}$  at  $t = 1$ .

**Case 2.**  $n \geq 3$  and  $j = n$ ,  $1 \leq k, \ell \leq n-1$ .

$$|\partial_{z_j} \partial_{z_k} \partial_{z_\ell} \Lambda^{-1} e G_1(z)| = |\partial_{z_n} [\theta(z_n) (4\pi)^{-\frac{1}{2}} e^{-\frac{|z_n|^2}{4}}]| |\partial_{z_k} \partial_{z_\ell} \Lambda^{-1} G'_1(z')| \leq C_{\ell_0, m} |z'|^{-\ell_0} |z_n|^{-m}$$

which is (2.14) in Case 2. Here we have used the estimate (see [16]):

$$|\partial_{z_k} \partial_{z_\ell} \Lambda^{-1} G'_1(z')| \leq C(\ell_0) |z'|^{-\ell_0} \quad \text{for any } 0 \leq \ell_0 \leq n.$$

**Case 3.**  $n = 2$  and  $j = k = \ell = 1$ .

$$\begin{aligned}
\partial_{z_j} \partial_{z_k} \partial_{z_\ell} \Lambda^{-1} G'_t(x') &= \partial_1 \Lambda^{-1} \partial_1^2 G'_t(x_1) = S_1 \partial_1^2 G'_t(x_1) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|y_1| > \epsilon} y_1^{-1} \partial_{x_1}^2 G'_t(x_1 - y_1) dy_1 \\
&= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|y_1| > \epsilon} \partial_{x_1}^2 G'_t(x_1 - y_1) d \log |y_1| \\
&= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \left\{ [\partial_{x_1}^2 G'_t(x_1 - y_1) \log |y_1|] \Big|_{\epsilon}^{\infty} + [\partial_{x_1}^2 G'_t(x_1 - y_1) \log |y_1|] \Big|_{-\infty}^{-\epsilon} \right. \\
&\quad \left. - \int_{|y_1| > \epsilon} \log |y_1| \partial_{y_1} \partial_{x_1}^2 G'_t(x_1 - y_1) dy_1 \right\} \\
&= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \left\{ [\partial_{x_1}^2 G'_t(x_1 + \epsilon) - \partial_{x_1}^2 G'_t(x_1 - \epsilon)] \log \epsilon \right. \\
&\quad \left. + \partial_{x_1}^2 \int_{|y_1| > \epsilon} \log |y_1| \frac{x_1 - y_1}{2t} G'_t(x_1 - y_1) dy_1 \right\} \\
&= \frac{1}{\pi} \partial_{x_1}^2 \int_{-\infty}^{\infty} \log |y_1| \frac{x_1 - y_1}{2t} G'_t(x_1 - y_1) dy_1 \\
&= \frac{1}{\pi} t^{-\frac{3}{2}} \partial_{z_1}^2 \int_{-\infty}^{\infty} (\log t^{\frac{1}{2}} + \log |y_1|) \frac{z_1 - y_1}{2} G'_1(z_1 - y_1) dy_1
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\pi} t^{-\frac{3}{2}} \partial_{z_1}^2 \int_{-\infty}^{\infty} \log |y_1| \frac{z_1 - y_1}{2} G'_1(z_1 - y_1) dy_1 \\
 &= \frac{1}{\pi} t^{-\frac{3}{2}} \partial_{z_1}^2 \left( \int_{|y_1| < 1} + \int_{|y_1| > 1} \right) \log |y_1| \frac{z_1 - y_1}{2} G'_1(z_1 - y_1) dy_1 \\
 &= \frac{1}{\pi} t^{-\frac{3}{2}} (L_1(z_1) + L_2(z_1)), \tag{2.15}
 \end{aligned}$$

$$\begin{aligned}
 |L_1(z_1)| &= \left| \int_{|y_1| < 1} \log |y_1| \left[ \frac{3}{4}(z_1 - y_1) + \left( \frac{z_1 - y_1}{2} \right)^3 \right] G'_1(z_1 - y_1) dy_1 \right| \\
 &\leq C(1 + |z_1|^3) e^{-\frac{|z_1|^2}{8}} \int_{|y_1| < 1} |\log |y_1|| dy_1 \\
 &\leq C(\ell_0) |z_1|^{-\ell_0} \quad \text{for any } \ell_0 \geq 0. \tag{2.16}
 \end{aligned}$$

Note that

$$\begin{aligned}
 L_2(z_1) &= \partial_{z_1}^2 \int_{|y_1| > 1} \log |y_1| \frac{z_1 - y_1}{2} G'_1(z_1 - y_1) dy_1 = -\partial_{z_1}^2 \int_{|y_1| > 1} \log |y_1| \partial_{y_1} G'_1(z_1 - y_1) dy_1 \\
 &= -\partial_{z_1}^2 \left\{ [\log |y_1| G'_1(z_1 - y_1)]|_1^\infty + [\log |y_1| G'_1(z_1 - y_1)]|_{-\infty}^{-1} \right. \\
 &\quad \left. - \int_{|y_1| > 1} y_1^{-1} G'_1(z_1 - y_1) dy_1 \right\} \\
 &= \partial_{z_1}^2 \int_{|y_1| > 1} y_1^{-1} G'_1(z_1 - y_1) dy_1 = -\partial_{z_1} \int_{|y_1| > 1} y_1^{-1} \partial_{y_1} G'_1(z_1 - y_1) dy_1 \\
 &= -\partial_{z_1} \left\{ [y_1^{-1} G'_1(z_1 - y_1)]|_1^\infty + [y_1^{-1} G'_1(z_1 - y_1)]|_{-\infty}^{-1} + \int_{|y_1| > 1} y_1^{-2} G'_1(z_1 - y_1) dy_1 \right\} \\
 &= \partial_{z_1} \{ G'_1(z_1 - 1) + G'_1(z_1 + 1) \} + \int_{|y_1| > 1} y_1^{-2} \partial_{y_1} G'_1(z_1 - y_1) dy_1 \\
 &= \partial_{z_1} \{ G'_1(z_1 - 1) + G'_1(z_1 + 1) \} + [y_1^{-2} G'_1(z_1 - y_1)]|_1^\infty + [y_1^{-2} G'_1(z_1 - y_1)]|_{-\infty}^{-1} \\
 &\quad + 2 \int_{|y_1| > 1} y_1^{-3} G'_1(z_1 - y_1) dy_1 \\
 &= (4\pi)^{-\frac{1}{2}} \left\{ \frac{z_1 - 3}{2} e^{-\frac{|z_1 - 1|^2}{4}} + \frac{z_1 + 3}{2} e^{-\frac{|z_1 + 1|^2}{4}} \right\} + 2(4\pi)^{-\frac{1}{2}} \int_{|y_1| > 1} y_1^{-3} e^{-\frac{|z_1 - y_1|^2}{4}} dy_1.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 |L_2(z_1)| &\leq C(1 + |z_1|) \left( e^{-\frac{|z_1-1|^2}{4}} + e^{-\frac{|z_1+1|^2}{4}} \right) \\
 &\quad + C(\ell_0) |z_1|^{-\ell_0} \int_{|y_1|>1} (|y_1|^{\ell_0-3} + |z_1 - y_1|^{\ell_0} |y_1|^{-3}) e^{-\frac{|z_1-y_1|^2}{4}} dy_1 \\
 &\leq C(\ell_0) |z_1|^{-\ell_0} \quad \text{for any } \ell_0 \in [0, 3].
 \end{aligned} \tag{2.17}$$

Whence from (2.15)–(2.17), we obtain

$$\begin{aligned}
 |\partial_j \partial_k \partial_\ell \Lambda^{-1} e G_t(x)| &= (4\pi t)^{-\frac{1}{2}} \theta(x_2) e^{-\frac{|x_2|^2}{4t}} |\partial_1 \Lambda^{-1} \partial_1^2 G'_t(x_1)| \\
 &\leq C_{\ell_0, m} t^{-\frac{\ell_0+m-4}{2}} |x_1|^{-\ell_0} |x_2|^{-m} \quad \text{for any } \ell_0 \in [0, 3], m \geq 0.
 \end{aligned}$$

**Case 4.**  $n = 2$  and  $j = 2, k = \ell = 1$ .

$$\begin{aligned}
 |\partial_j \partial_k \partial_\ell \Lambda^{-1} e G_t(x)| &= |\partial_2 \partial_1^2 \Lambda^{-1} e G_t(x)| = |\partial_2 \Lambda e G_t(x)| \\
 &= (4\pi t)^{-\frac{1}{2}} \theta(x_2) \frac{|x_2|}{2t} e^{-\frac{|x_2|^2}{4t}} |\Lambda G'_t(x_1)| \\
 &\leq C_{\ell_0, m} t^{-\frac{\ell_0+m-4}{2}} |x_1|^{-\ell_0} |x_2|^{-m} \quad \text{for any } \ell_0 \in [0, 2], m \geq 0.
 \end{aligned}$$

Here we have used the estimate for  $n = 2$  (see [16]):

$$|\Lambda G'_t(x_1)| \leq C(\ell_0) t^{\frac{\ell_0-3}{2}} |x_1|^{-\ell_0} \quad \text{for any } \ell_0 \in [0, 2].$$

From the above arguments on the four cases, we complete the proof of Lemma 2.2.  $\square$

**Lemma 2.3.** Let  $a \in L^1(\mathbb{R}_+^n)$ . Then for any  $t > 0$

$$\begin{aligned}
 \|e \partial_t E(t) a\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leq C t^{-1} \|a\|_{L^1(\mathbb{R}_+^n)}, \\
 \|\partial_j \partial_k e E(t) a\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leq C t^{-1} \|a\|_{L^1(\mathbb{R}_+^n)} \quad \text{for any } 1 \leq j, k \leq n, \\
 \|\partial_j \partial_k \partial_\ell \Lambda^{-1} e E(t) a\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leq C t^{-1} \|a\|_{L^1(\mathbb{R}_+^n)} \quad \text{for any } 1 \leq j \leq n, 1 \leq k, \ell \leq n-1, \\
 \|\partial_j \partial_k \partial_\ell \Lambda^{-1} e F(t) a\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leq C t^{-1} \|a\|_{L^1(\mathbb{R}_+^n)} \quad \text{for any } 1 \leq j \leq n, 1 \leq k, \ell \leq n-1.
 \end{aligned}$$

**Proof.** For simplicity, we always denote the odd and even extensions of a function  $f$  from  $\mathbb{R}_+^n$  to  $\mathbb{R}^n$  by

$$f^*(x', x_n) = \begin{cases} f(x', x_n) & \text{if } x_n \geq 0, \\ -f(x', -x_n) & \text{if } x_n < 0, \end{cases}$$

and

$$f_*(x', x_n) = \begin{cases} f(x', x_n) & \text{if } x_n \geq 0, \\ f(x', -x_n) & \text{if } x_n < 0, \end{cases}$$

respectively.



Then  $e[\partial_t E(t)a^*](x) = \theta(x_n)[\partial_t G_t * a^*](x)$ , and so for any  $s > 0$

$$\begin{aligned} \int_{\mathbb{R}^n} |G_s * [e\partial_t E(t)a^*]|(w) dw &= \int_{\mathbb{R}^n} |G_s(w-x)\theta(x_n)[\partial_t G_t * a^*](x) dx| dw \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_s(w-x)\theta(x_n) |\partial_t G_t(x-y)| |a^*(y)| dy dx dw \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| -\frac{n}{2t} + \frac{|x-y|^2}{4t^2} \right| |G_t(x-y)| |a^*(y)| dy dx \\ &\leq Ct^{-1} \int_{\mathbb{R}^n} (1+|x|^2) G_1(x) dx \int_{\mathbb{R}^n} |a^*(y)| dy \\ &\leq Ct^{-1} \|a\|_{L^1(\mathbb{R}_+^n)}, \end{aligned}$$

which implies that

$$\|e\partial_t E(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|e\partial_t E(t)a^*\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \|a\|_{L^1(\mathbb{R}_+^n)}.$$

Observe that

$$\begin{aligned} \int_{\mathbb{R}^n} |G_s * [\partial_j \partial_k eE(t)a^*]|(w) dw &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} G_s(w-x) \partial_{x_j} \partial_{x_k} [\theta(x_n) G_t * a^*](x) dx \right| dw \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}^n} G_s(w-x) |\partial_{x_j} \partial_{x_k} G_t(x-y)| |a^*(y)| dy dx dw \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| -\frac{\delta_{kj}}{2t} + \frac{(x_j - y_j)(x_k - y_k)}{4t^2} \right| |G_t(x-y)| |a^*(y)| dy dx \\ &\leq Ct^{-1} \int_{\mathbb{R}^n} (1+|x|^2) G_1(x) dx \int_{\mathbb{R}^n} |a^*(y)| dy \\ &\leq Ct^{-1} \|a\|_{L^1(\mathbb{R}_+^n)}, \end{aligned}$$

which implies that

$$\|\partial_j \partial_k eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|\partial_j \partial_k eE(t)a^*\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \|a\|_{L^1(\mathbb{R}_+^n)}.$$

Using Lemma 2.2, we have for any  $1 \leq j \leq n$ ,  $1 \leq k, \ell \leq n-1$

$$\begin{aligned} &\int_{\mathbb{R}^n} |G_s * [\partial_j \partial_k \partial_\ell \Lambda^{-1} eE(t)a^*]|(w) dw \\ &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} G_s(w-x) [\partial_j \partial_k \partial_\ell \Lambda^{-1} eG_t * a^*](x) dx \right| dw \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_s(w-x) |\partial_{x_j} \partial_{x_k} \partial_{x_\ell} \Lambda^{-1} [\theta(x_n - y_n) G_t(x-y) a^*(y)]| dy dx dw \\
&\leq C_{\ell_0, m} t^{\frac{m+\ell_0-n-2}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x' - y'|^{-\ell_0} |x_n - y_n|^{-m} |a^*(y)| dy dx \\
&\leq C_{\ell_0, m} t^{\frac{m+\ell_0-n-2}{2}} \int_{\mathbb{R}^n} |x'|^{-\ell_0} |x_n|^{-m} dx \int_{\mathbb{R}^n} |a^*(y)| dy \leq C t^{-1} \|a\|_{L^1(\mathbb{R}_+^n)},
\end{aligned}$$

where  $0 \leq \ell_0 \leq n$ ,  $m \geq 0$ , and we have used the estimate (see [16]):

$$C_{\ell_0, m} t^{\frac{m+\ell_0-n-1}{2}} \int_{\mathbb{R}^n} |x'|^{-\ell_0} |x_n|^{-m} dx \leq C t^{-\frac{1}{2}} \quad \text{for any } 0 \leq \ell_0 \leq n, m \geq 0.$$

From which we get

$$\|\partial_j \partial_k \partial_\ell \Lambda^{-1} eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq \|\partial_j \partial_k \partial_\ell \Lambda^{-1} eE(t)a^*\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C t^{-1} \|a\|_{L^1(\mathbb{R}_+^n)}.$$

Similarly, if we use the even extension  $a_*$  of  $a$ , we also can prove that for any  $1 \leq j \leq n$ ,  $1 \leq k, \ell \leq n-1$

$$\|\partial_j \partial_k \partial_\ell \Lambda^{-1} eF(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C t^{-1} \|a\|_{L^1(\mathbb{R}_+^n)}. \quad \square$$

**Lemma 2.4.** Let  $a \in L^1(\mathbb{R}_+^n)$  ( $n \geq 2$ ),  $1 \leq \ell \leq n-1$ . Set

$$b(x', x_n) \triangleq \int_{-\infty}^{x_\ell} a(x_1, x_2, \dots, x_{\ell-1}, y_\ell, x_{\ell+1}, \dots, x_{n-1}, x_n) dy_\ell.$$

If  $b \in L^1(\mathbb{R}_+^n)$  for some  $1 \leq \ell \leq n-1$ , then

$$\begin{aligned}
&\|eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}} \|b\|_{L^1(\mathbb{R}_+^n)}, \\
&\|\partial_k \Lambda^{-1} eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}} \|b\|_{L^1(\mathbb{R}_+^n)} \quad \text{for any } 1 \leq k \leq n-1.
\end{aligned}$$

**Proof.** Note that

$$\begin{aligned}
E(t)a(x) &= \int_{\mathbb{R}_+^n} [G_t(x' - y', x_n - y_n) - G_t(x' - y', x_n + y_n)] a(y) dy \\
&= \int_{\mathbb{R}_+^n} [G_t(x' - y', x_n - y_n) - G_t(x' - y', x_n + y_n)] \partial_\ell b(y) dy \\
&= - \int_{\mathbb{R}_+^n} \partial_{y_\ell} [G_t(x' - y', x_n - y_n) - G_t(x' - y', x_n + y_n)] b(y) dy \\
&= \partial_\ell E(t)b(x).
\end{aligned}$$

We have for some  $1 \leq \ell \leq n-1$

$$eE(t)a = e\partial_\ell E(t)b = \partial_\ell eE(t)b.$$

It is not difficult to verify that  $E(t)a^* = G_t * a^*$  and

$$b^*(x', x_n) = \int_{-\infty}^{x_\ell} a^*(x_1, x_2, \dots, x_{\ell-1}, x_\ell, x_{\ell+1}, \dots, x_{n-1}, x_n) dy_\ell.$$

For any  $s > 0$ , we have

$$\begin{aligned} \int_{\mathbb{R}^n} |G_s * [eE(t)a^*]|(x) dx &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_s(x-y) \theta(y_n) G_t(y-z) \partial_{z_\ell} b^*(z) dy dz \right| dx \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_s(x-y) |\partial_{y_\ell} G_t(y-z)| |b^*(z)| dy dz dx \\ &= \int_{\mathbb{R}^n} G_s(w) dw \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| -\frac{y_\ell - z_\ell}{2t} G_t(y-z) \right| |b^*(z)| dy dz \\ &\leq ct^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y-z| G_t(y-z) |b^*(z)| dy dz \\ &\leq ct^{-\frac{1}{2}} \int_{\mathbb{R}^n} |w| G_1(w) dw \int_{\mathbb{R}^n} |b^*(z)| dz \leq ct^{-\frac{1}{2}} \|b\|_{L^1(\mathbb{R}_+^n)}, \end{aligned}$$

which implies that

$$\|eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|eE(t)a^*\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-\frac{1}{2}} \|b\|_{L^1(\mathbb{R}_+^n)}.$$

From Lemma 2.4 in [16], we know that

$$\|\partial_k \partial_\ell \Lambda^{-1} eE(t)b\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-\frac{1}{2}} \|b\|_{L^1(\mathbb{R}_+^n)} \quad \text{for any } 1 \leq k, \ell \leq n-1.$$

Whence,

$$\begin{aligned} \|\partial_k \Lambda^{-1} eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} &= \|\partial_k \Lambda^{-1} \partial_\ell eE(t)b\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|\partial_k \partial_\ell \Lambda^{-1} eE(t)b\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\leq Ct^{-\frac{1}{2}} \|b\|_{L^1(\mathbb{R}_+^n)}. \quad \square \end{aligned}$$

**Lemma 2.5.** Assume that  $a \in L^1(\mathbb{R}_+^n)$  ( $n \geq 2$ ). Then

$$\|\partial_j eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \int_{\mathbb{R}_+^n} y_n |a(y)| dy \quad \text{for any } 1 \leq j \leq n,$$

$$\|\partial_j \partial_k \Lambda^{-1} eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \int_{\mathbb{R}_+^n} y_n |a(y)| dy \quad \text{for any } 1 \leq j, k \leq n-1,$$

$$\|\Lambda eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \int_{\mathbb{R}_+^n} y_n |a(y)| dy.$$

Furthermore, if  $\int_{\mathbb{R}^{n-1}} a(y', y_n) dy' = 0$  for a.e.  $y_n > 0$ , then

$$\|\partial_j \partial_k \Lambda^{-1} eF(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \int_{\mathbb{R}_+^n} |y' a(y)| dy \quad \text{for any } 1 \leq j, k \leq n-1. \quad (2.18)$$

**Remark.** The condition in (2.18):  $\int_{\mathbb{R}^{n-1}} a(y', y_n) dy' = 0$  is the same to the assumption in Theorem 1.1 in [1]. As in [1], the aim of such assumption is also to overcome the estimation difficulty of the  $(n-1)$ -dimensional Riesz transform near boundary in the Stokes solution formula in  $\mathbb{R}_+^n$ . Such condition in (2.18) is stronger than the zero average assumption:  $\int_{\mathbb{R}_+^n} u_0(y) dy = 0$ . Because in  $\mathbb{R}_+^n$ , we do not know whether  $\int_{\mathbb{R}_+^n} u_0(y) dy = 0$ . It is shown in [22] that if  $\nabla \cdot u_0 = 0$  and  $u_0 \in L^1(\mathbb{R}^n)$ , then  $\int_{\mathbb{R}^n} u_0(y) dy = 0$ .

**Proof of Lemma 2.5.** The proofs of the first three estimates are given in Theorem 2.4, Lemmas 3.3, 3.4 in [2] respectively, we omit the details here. Now we give the proof of (2.18).

Note that  $a_*(y) = a(y', y_n)$  if  $y_n \geq 0$ , and  $a_*(y) = a(y', -y_n)$  if  $y_n < 0$ . So for any  $y \in \mathbb{R}^n$ ,

$$\begin{aligned} G_t * a_*(y) &= \int_{\mathbb{R}^n} G_t(y-z) a_*(z) dz \\ &= \int_{\mathbb{R}_+^n} G_t(y-z) a(z) dz + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 G_t(y'-z', y_n-z_n) a(z', -z_n) dz' dz_n \\ &= \int_{\mathbb{R}_+^n} G_t(y'-z', y_n-z_n) a(z) dz + \int_{\mathbb{R}^{n-1}} \int_0^\infty G_t(y'-z', y_n+z_n) a(z', z_n) dz' dz_n \\ &= \int_{\mathbb{R}_+^n} [G_t(y'-z', y_n-z_n) + G_t(y'-z', y_n+z_n)] a(z', z_n) dz' dz_n \\ &= \int_{\mathbb{R}_+^n} [G_t(y'-z', y_n-z_n) + G_t(y'-z', y_n+z_n)] a_*(z', z_n) dz' dz_n \\ &= F(t) a_*(y). \end{aligned}$$

In addition, since  $\int_{\mathbb{R}^{n-1}} a(y', y_n) dy' = 0$  for a.e.  $y_n > 0$ , we infer  $\int_{\mathbb{R}^{n-1}} a_*(y', y_n) dy' = 0$  for a.e.  $y_n \in \mathbb{R}^1$ . Therefore, for any  $s > 0$ , we have

$$\begin{aligned} &\{G_s * [\partial_j \partial_k \Lambda^{-1} (eF(t)a_*)]\}(x) \\ &= \{G_s * [\partial_j \partial_k \Lambda^{-1} (eG_t * a_*)]\}(x) \\ &= \partial_j \partial_k \Lambda^{-1} \{G_s * [eG_t * a_*]\}(x) \quad (\text{it can be verified by Fourier transform}) \end{aligned}$$

$$\begin{aligned}
&= \partial_j \partial_k \Lambda^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_s(x-y) \theta(y_n) [G_t(y' - z', y_n - z_n) - G_t(y', y_n - z_n)] a_*(z) dz dy \\
&= \partial_j \partial_k \Lambda^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 G_s(x-y) \theta(y_n) \nabla' G_t(y' - \tau z', y_n - z_n) \cdot (-z') a_*(z) d\tau dz dy.
\end{aligned}$$

Similar to the proof of Lemma 2.3, we also deduce that for any  $1 \leq j, k \leq n-1$

$$\|\partial_j \partial_k \Lambda^{-1} eF(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|\partial_j \partial_k \Lambda^{-1} eF(t)a_*\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \int_{\mathbb{R}_+^n} |y'| |a(y)| dy,$$

which is (2.18).  $\square$

**Proof of Theorem 1.1.** From Lemma 2.1, we conclude

$$\begin{aligned}
\|u_n - a_n\|_{L^1(\mathbb{R}_+^n)} &\leq \|u_n - a_n\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \\
&\leq \left\| \bar{u}_n - \sum_{k,m=1}^{n-1} R_k^2 eE(t) S_m u_{0m} - \sum_{k=1}^{n-1} R_n R_k S_k eE(t) u_{0n} \right\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
&\leq \sum_{m=1}^{n-1} \|R_n\| \|R_m\| \|eE(t) u_{0m}\|_{\mathcal{H}^1(\mathbb{R}^n)} + \sum_{k=1}^{n-1} \|R_k\|^2 \|eE(t) u_{0n}\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
&\leq C \|u_0\|_{L^1(\mathbb{R}_+^n)}.
\end{aligned}$$

For any  $1 \leq j \leq n-1$

$$\begin{aligned}
\|u_j - a_j\|_{L^1(\mathbb{R}_+^n)} &\leq \|u_j - a_j\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \\
&\leq \left\| \bar{u}_j - S_j eE(t) u_{0n} - \sum_{k=1}^{n-1} R_k^2 S_j eE(t) u_{0n} + \sum_{m=1}^{n-1} R_n R_m S_j eE(t) u_{0m} \right\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
&\leq \|eE(t) u_{0j}\|_{\mathcal{H}^1(\mathbb{R}^n)} + \|R_n\| \|R_j\| \|eE(t) u_{0n}\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
&\quad + \sum_{m=1}^{n-1} \|R_j\| \|R_m\| \|eE(t) u_{0m}\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
&\leq C \|u_0\|_{L^1(\mathbb{R}_+^n)}.
\end{aligned}$$

From the above arguments, we complete the proof of Theorem 1.1.  $\square$

**Proof of Theorem 1.2.** From Lemmas 2.1, 2.4, we have

$$\|\bar{u}_n\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq \sum_{k,m=1}^{n-1} \|R_k\|^2 \|\partial_m \Lambda^{-1} eE(t) u_{0m}\|_{\mathcal{H}^1(\mathbb{R}^n)} + \sum_{m=1}^{n-1} \|R_m\| \|R_n\| \|eE(t) u_{0m}\|_{\mathcal{H}^1(\mathbb{R}^n)}$$

$$\begin{aligned}
& + \sum_{k=1}^{n-1} \| \| R_k \| \|^2 \| eE(t)u_{0n} \|_{\mathcal{H}^1(\mathbb{R}^n)} + \sum_{k=1}^{n-1} \| \| R_k \| \| \| R_n \| \| \| \partial_k \Lambda^{-1} eE(t)u_{0n} \|_{\mathcal{H}^1(\mathbb{R}^n)} \\
& \leq Ct^{-\frac{1}{2}} \| v_0 \|_{L^1(\mathbb{R}_+^n)}.
\end{aligned}$$

In addition for any  $1 \leq j \leq n-1$

$$\begin{aligned}
\| \bar{u}_j \|_{\mathcal{H}^1(\mathbb{R}^n)} & \leq \| eE(t)u_{0j} \|_{\mathcal{H}^1(\mathbb{R}^n)} + \| \partial_j \Lambda^{-1} eE(t)u_{0n} \|_{\mathcal{H}^1(\mathbb{R}^n)} \\
& + \sum_{m=1}^{n-1} \| \| R_m \| \| \| R_j \| \| \| eE(t)u_{0m} \|_{\mathcal{H}^1(\mathbb{R}^n)} + \sum_{m=1}^{n-1} \| \| R_m \| \| \| R_n \| \| \| \partial_j \Lambda^{-1} eE(t)u_{0m} \|_{\mathcal{H}^1(\mathbb{R}^n)} \\
& + \sum_{k=1}^{n-1} \| \| R_k \| \|^2 \| \partial_j \Lambda^{-1} eE(t)u_{0n} \|_{\mathcal{H}^1(\mathbb{R}^n)} + \| \| R_j \| \| \| R_n \| \| \| eE(t)u_{0n} \|_{\mathcal{H}^1(\mathbb{R}^n)} \\
& \leq Ct^{-\frac{1}{2}} \| v_0 \|_{L^1(\mathbb{R}_+^n)}.
\end{aligned}$$

Since  $u = \bar{u}|_{\mathbb{R}_+^n}$ , we get

$$\| u \|_{L^1(\mathbb{R}_+^n)} \leq \| u \|_{\mathcal{H}^1(\mathbb{R}_+^n)} \leq \| \bar{u} \|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-\frac{1}{2}} \| v_0 \|_{L^1(\mathbb{R}_+^n)}. \quad \square$$

**Proof of Theorem 1.3.** The following two equalities can be found in [16]: For any  $1 \leq j \leq n$

$$\begin{aligned}
\partial_j \bar{u}_n & = -R_j \{ R' \cdot \Lambda eE(t)u'_0 - R_n \nabla' \cdot eE(t)u'_0 + R' \cdot \nabla' eE(t)u_{0n} + R_n \Lambda eE(t)u_{0n} \}, \\
\partial_j \bar{u}_j & = \partial_j (eE(t)u'_0) + w_j + R_j \{ R' (\nabla' \cdot eE(t)u'_0) - R_n \nabla' (\nabla' \Lambda^{-1} \cdot eE(t)u'_0) \\
& \quad - R' \Lambda eE(t)u_{0n} + R_n \nabla' eE(t)u_{0n} \},
\end{aligned}$$

where

$$w_j = \begin{cases} \partial_j \nabla' \Lambda^{-1} eE(t)u_{0n} & \text{if } 1 \leq j \leq n-1, \\ -\nabla' (\nabla' \cdot \Lambda^{-1} eE(t)u'_0) & \text{if } j = n. \end{cases}$$

Therefore from Lemma 2.5, we obtain

$$\begin{aligned}
\| \partial_j \bar{u}_n \|_{\mathcal{H}^1(\mathbb{R}^n)} & \leq \| \| R_j \| \| \left\{ \sum_{k=1}^{n-1} \| \| R_k \| \| \| \Lambda eE(t)u_{0k} \|_{\mathcal{H}^1(\mathbb{R}^n)} + \| \partial_k eE(t)u_{0n} \|_{\mathcal{H}^1(\mathbb{R}^n)} \right. \\
& \quad \left. + \| \| R_n \| \| \| \nabla' \cdot eE(t)u'_0 \|_{\mathcal{H}^1(\mathbb{R}^n)} + \| \Lambda eE(t)u_{0n} \|_{\mathcal{H}^1(\mathbb{R}^n)} \right\} \\
& \leq Ct^{-1} \int_{\mathbb{R}_+^n} y_n |u_0(y)| dy.
\end{aligned}$$

Similarly,

$$\|\partial_j \bar{u}'\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \left( \int_{\mathbb{R}_+^n} y_n |u_0(y)| dy + \int_{\mathbb{R}_+^n} |y'| |u_0(y)| dy \right).$$

Whence,

$$\|\nabla u\|_{L^1(\mathbb{R}_+^n)} \leq \|\nabla u\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \leq \|\nabla \bar{u}\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \left( \int_{\mathbb{R}_+^n} y_n |u_0(y)| dy + \int_{\mathbb{R}_+^n} |y'| |u_0(y)| dy \right).$$

From Lemmas 2.1, 2.3, we conclude

$$\begin{aligned} \|\partial_t \bar{u}_n\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leq \sum_{k,m=1}^{n-1} \|\partial_k \partial_m \Lambda^{-1} eE(t) u_{0m}\|_{\mathcal{H}^1(\mathbb{R}^n)} + \sum_{m=1}^{n-1} \|R_m\| \|R_n\| \|e\partial_t E(t) u_{0m}\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\quad + \sum_{k=1}^{n-1} \|R_k\|^2 \|e\partial_t E(t) u_{0n}\|_{\mathcal{H}^1(\mathbb{R}^n)} + \sum_{k=1}^{n-1} \|\partial_k \partial_m \Lambda^{-1} eE(t) u_{0n}\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\leq Ct^{-1} \|u_0\|_{L^1(\mathbb{R}_+^n)}. \end{aligned}$$

For any  $1 \leq j \leq n-1$

$$\begin{aligned} \|\partial_t \bar{u}_j\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leq \|e\partial_t E(t) u_{0j}\|_{\mathcal{H}^1(\mathbb{R}^n)} + \|\partial_j \partial_n eE(t) u_{0n}\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\quad + \sum_{m=1}^{n-1} \|\partial_j \partial_m eE(t) u_{0m}\|_{\mathcal{H}^1(\mathbb{R}^n)} + \sum_{m=1}^{n-1} \|\partial_n \partial_j \partial_m \Lambda^{-1} eF(t) u_{0m}\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\leq Ct^{-1} \|u_0\|_{L^1(\mathbb{R}_+^n)}. \end{aligned}$$

Therefore, we obtain

$$\|\partial_t u\|_{L^1(\mathbb{R}_+^n)} \leq \|\partial_t u\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \leq \|\partial_t \bar{u}\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-1} \|u_0\|_{L^1(\mathbb{R}_+^n)}.$$

From the equation in (1.1), we have

$$\|\nabla p\|_{L^1(\mathbb{R}_+^n)} \leq Ct^{-1} \|u_0\|_{L^1(\mathbb{R}_+^n)}. \quad \square$$

### 3. Decay rates for weak and strong solutions of (1.2)

The following result is well known, and its proof can be found in [13].

**Lemma 3.1.** Suppose that  $u_0 \in L^2_\sigma(\mathbb{R}_+^n)$  ( $n \geq 2$ ) satisfies

$$\int_{\mathbb{R}_+^n} (1 + y_n) |u_0(y)| dy < \infty.$$

(i) There exists a weak solution  $u$ , which is unique in case  $n = 2$ , such that for all  $t > 0$

$$\|u(t)\|_{L^2(\mathbb{R}_+^n)} \leq C(1+t)^{-\frac{1}{2}-\frac{n}{4}}.$$

(ii) Let  $u_0 \in L_\sigma^q(\mathbb{R}_+^n)$  for all  $1 < q < \infty$ . If  $u_0$  is small in  $L^n(\mathbb{R}_+^n)$ , there is a strong solution  $u$  of (1.2) for all  $t > 0$ , satisfying

$$\|u(t)\|_{L^q(\mathbb{R}_+^n)} \leq C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \quad \text{for all } 1 < q < \infty,$$

and

$$\|\nabla u(t)\|_{L^q(\mathbb{R}_+^n)} \leq Ct^{-1-\frac{n}{2}(1-\frac{1}{q})} \quad \text{for all } 1 < q < \infty.$$

The solution  $u$  of (1.2) given in Lemma 3.1 can be rewritten as follows

$$u(t) = e^{-tA}u_0 - \int_0^t w(t-s)ds, \quad (3.1)$$

where  $w(t-s) = e^{-(t-s)A}P(u(s) \cdot \nabla u(s))$  for any  $0 < s < t$ .

Let  $g = \mathcal{N}f$  denote the solution of the Neumann problem

$$\begin{cases} -\Delta g = f & \text{in } \mathbb{R}_+^n, \\ \partial_\nu g|_{\partial\mathbb{R}_+^n} = 0. \end{cases} \quad (3.2)$$

Since the operator  $F(t)$  satisfies

$$\begin{cases} \partial_t(Ff) - \Delta(Ff) = 0 & \text{in } \mathbb{R}_+^n \times (0, \infty), \\ \partial_\nu(Ff)|_{\partial\mathbb{R}_+^n \times (0, \infty)} = 0. \end{cases}$$

Observe that

$$\lim_{t \rightarrow \infty} [G_t(x' - y', x_n - y_n) + G_t(x' - y', x_n + y_n)] = 0 \quad \text{for any } x = (x', x_n), y = (y', y_n) \in \mathbb{R}^n,$$

and

$$\lim_{t \rightarrow 0} [G_t(x' - y', x_n - y_n) + G_t(x' - y', x_n + y_n)] = 0 \quad \text{for a.e. } x = (x', x_n), y = (y', y_n) \in \mathbb{R}^n.$$

By the Lebesgue convergence theorem, for any function  $f$ , which belongs to a suitable space,  $C^\infty(\overline{\mathbb{R}_+^n}) \cap L^1(\mathbb{R}_+^n)$  for example, we conclude

$$\lim_{t \rightarrow \infty} (Ff)(x, t) = \lim_{t \rightarrow \infty} \int_{\mathbb{R}_+^n} [G_t(x' - y', x_n - y_n) + G_t(x' - y', x_n + y_n)] f(y', y_n) dy' dy_n = 0$$



and

$$\lim_{t \rightarrow 0} (Ff)(x, t) = \lim_{t \rightarrow 0} \int_{\mathbb{R}_+^n} [G_t(x' - y', x_n - y_n) + G_t(x' - y', x_n + y_n)] f(y', y_n) dy' dy_n = 0.$$

Whence from the above equation which  $Ff$  satisfies, we deduce

$$\int_0^\infty \partial_t (Ff)(x, t) dt = \lim_{t \rightarrow \infty} (Ff)(x, t) - \lim_{t \rightarrow 0} (Ff)(x, t) = 0 \quad \text{in } \mathbb{R}_+^n,$$

and then

$$\begin{cases} -\Delta \left( \left( \int_0^\infty F dt \right) f \right) = 0 & \text{in } \mathbb{R}_+^n, \\ \left| \partial_\nu \left( \left( \int_0^\infty F dt \right) f \right) \right|_{\partial \mathbb{R}_+^n} = 0. \end{cases}$$

Together with (3.2), we conclude that

$$\mathcal{N} = \int_0^\infty F(\tau) d\tau. \quad (3.3)$$

For simplicity, we always assume that  $u$  is the strong solution of (1.2). Because if  $u$  is a weak solution of (1.2), we can use the approximate solution  $u_N$  as in [9] to replace  $u$ , and also verify part (i) in Theorem 1.4.

Since  $u = 0$  on  $\partial \mathbb{R}_+^n$ , from (3.2), we have

$$P(u \cdot \nabla u) = u \cdot \nabla u + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j). \quad (3.4)$$

Here we need to point out that (3.4) plays the fundamental role in the proof of Theorem 1.4, because the projection operator  $P : L^1(\mathbb{R}_+^n) \rightarrow L_\sigma^1(\mathbb{R}_+^n)$  is unbounded.

Using the solution formula in [27], and from (3.1), (3.4), we get for  $0 < s < t$

$$w(t-s) = (w', w_n) = e^{-(t-s)A} \left[ u \cdot \nabla u + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j) \right]$$

and

$$\begin{cases} w_n = UE(t-s)V_1 \left[ u \cdot \nabla u + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j) \right], \\ w' = E(t-s)V_2 \left[ u \cdot \nabla u + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j) \right] - Sw_n. \end{cases} \quad (3.5)$$

Note that the solution  $w$  of (3.5) is given as a restriction  $r\bar{w}$  of one vector field  $\bar{w} = (\bar{w}', \bar{w}_n)$ :

$$\begin{cases} \bar{w}_n = R' \cdot S(R' \cdot S + R_n)eE(t-s)V_1 \left[ u \cdot \nabla u + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j) \right], \\ \bar{w}' = eE(t-s)V_2 \left[ u \cdot \nabla u + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j) \right] - S\bar{w}_n. \end{cases} \quad (3.6)$$

**Lemma 3.2.** For any  $0 < s < t$  and  $1 \leq m \leq n-1$ , we have

$$\begin{cases} \bar{w}_n = \sum_{i=1}^4 I_i(t-s), \\ \bar{w}_m = \sum_{i=1}^{12} J_i(t-s), \end{cases}$$

where

$$\begin{aligned} I_1(t-s) &= \sum_{i,\ell=1}^{n-1} \sum_{k=1}^n R_i R_k \partial_\ell \partial_\ell \Lambda^{-1} eE(t-s)(u_i u_k) + \sum_{i=1}^{n-1} \sum_{k=1}^n R_n R_i \partial_k eE(t-s)(u_i u_k) \\ &\quad + \sum_{\ell=1}^{n-1} \sum_{k=1}^n R_n R_k \partial_\ell \partial_\ell \Lambda^{-1} eF(t-s)(u_n u_k) + \sum_{k=1}^n R_n R_n \partial_k eF(t-s)(u_n u_k), \\ I_2(t-s) &= \sum_{k,\ell=1}^{n-1} \sum_{i,j=1}^n R_\ell R_\ell \partial_k \partial_k \Lambda^{-1} eE(t-s) \mathcal{N} \partial_i \partial_j (u_i u_j) \\ &\quad + \sum_{k=1}^{n-1} \sum_{i,j=1}^n R_n R_k eE(t-s) \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j), \\ I_3(t-s) &= - \sum_{\ell=1}^{n-1} \sum_{i=1}^n R_\ell R_\ell eE(t-s)(u_i \partial_i u_n) + \sum_{i,k=1}^{n-1} R_n R_k \partial_k \partial_i \Lambda^{-1} eE(t-s)(u_i u_n) \\ &\quad + \sum_{k=1}^{n-1} R_n R_n \partial_k \partial_k \Lambda^{-1} eF(t-s)(u_n u_n), \\ I_4(t-s) &= - \sum_{\ell=1}^{n-1} \sum_{i,j=1}^n R_\ell R_\ell eE(t-s) \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j) \\ &\quad + \sum_{k=1}^{n-1} \sum_{i,j=1}^n R_n R_k \partial_k \Lambda^{-1} eE(t-s) \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j), \end{aligned}$$

$$\begin{aligned}
 J_1(t-s) &= \sum_{i=1}^n eE(t-s)(u_i \partial_i u_m) \\
 &\quad + \sum_{i=1}^{n-1} \partial_i \partial_m \Lambda^{-1} eE(t-s)(u_i u_n) + \sum_{i,j=1}^n eE(t-s) \partial_m \mathcal{N} \partial_i \partial_j (u_i u_j) \\
 &\quad + \sum_{i,j=1}^n S_m eE(t-s) \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j) + S_m eE(t-s)(u_n \partial_n u_n), \\
 J_2(t-s) &= \sum_{k=1}^n \sum_{i=1}^{n-1} R_i R_k \partial_m eE(t-s)(u_i u_k), \\
 J_3(t-s) &= - \sum_{k=1}^n \sum_{i=1}^{n-1} R_n R_k \partial_i \partial_m \Lambda^{-1} eE(t-s)(u_i u_k), \\
 J_4(t-s) &= \sum_{k=1}^n R_n R_k \partial_m eF(t-s)(u_n u_k), \\
 J_5(t-s) &= - \sum_{k=1}^n R_n R_n \partial_m \partial_k \Lambda^{-1} eF(t-s)(u_n u_k) - S_m R_n R_n \partial_n eF(t-s)(u_n u_n), \\
 J_6(t-s) &= - \sum_{i=1}^n R_m R_i \Lambda eE(t-s)(u_i u_n) - R_m R_n \Lambda eF(t-s)(u_n u_n), \\
 J_7(t-s) &= \sum_{i=1}^{n-1} R_n R_i \partial_m eE(t-s)(u_i u_n), \\
 J_8(t-s) &= R_n R_n \partial_m eF(t-s)(u_n u_n), \\
 J_9(t-s) &= - \sum_{k=1}^{n-1} \sum_{i,j=1}^n S_m R_n R_k \partial_k eE(t-s) \mathcal{N} \partial_i \partial_j (u_i u_j), \\
 J_{10}(t-s) &= \sum_{\ell=1}^{n-1} \sum_{i,j=1}^n S_m R_\ell R_\ell eE(t-s) \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j), \\
 J_{11}(t-s) &= - \sum_{k=1}^{n-1} \sum_{i,j=1}^n R_k R_k eE(t-s) \partial_m \mathcal{N} \partial_i \partial_j (u_i u_j), \\
 J_{12}(t-s) &= \sum_{i,j=1}^n R_n R_m eE(t-s) \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j).
 \end{aligned}$$

**Proof.** Note that the operators  $S_j$  and  $eE(t)$  commute, from (3.6), we have

$$\mathcal{F}(\bar{w}_n)(\xi) = \frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi'|} \left( \frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi'|} + \frac{i\xi_n}{|\xi|} \right)$$

$$\begin{aligned}
& \times \left\{ \left( -\frac{i\xi'}{|\xi'|} \right) \cdot \mathcal{F} \left[ eE(t-s) \left( \partial_n(u_n u') + \sum_{j=1}^{n-1} \partial_j(u_j u') \right) \right. \right. \\
& + \sum_{k,j=1}^n eE(t-s) (\nabla' \mathcal{N} \partial_k \partial_j(u_k u_j)) \left. \right] + \mathcal{F} \left[ eE(t-s) (u \cdot \nabla u_n) \right. \\
& \left. \left. + \sum_{k,j=1}^n eE(t-s) \partial_n \mathcal{N} \partial_k \partial_j(u_k u_j) \right] \right\} \\
& = \left( \frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi|} \frac{i\xi'}{|\xi'|} + \frac{i\xi_n}{|\xi|} \frac{i\xi'}{|\xi|} \right) \left\{ \sum_{j=1}^{n-1} i\xi_j \mathcal{F} [eE(t-s)(u_j u')] \right. \\
& \quad \left. + i\xi_n \mathcal{F} [eF(t-s)(u_n u')] + i\xi' \sum_{k,j=1}^n \mathcal{F} [eE(t-s)(\mathcal{N} \partial_k \partial_j(u_k u_j))] \right\} \\
& \quad + \left( -\frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi|} + \frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi'|} \frac{i\xi_n}{|\xi|} \right) \left\{ \sum_{j=1}^{n-1} i\xi_j \mathcal{F} [eE(t-s)(u_j u_n)] \right. \\
& \quad \left. + i\xi_n \mathcal{F} [eF(t-s)(u_n u_n)] + \sum_{k,j=1}^n \mathcal{F} [eE(t-s) \partial_n \mathcal{N} \partial_k \partial_j(u_k u_j)] \right\} \\
& = \mathcal{F} \left[ \sum_{i=1}^4 I_i(t-s) \right], \tag{3.7}
\end{aligned}$$

which implies that  $\bar{w}_n = \sum_{i=1}^4 I_i(t-s)$ .

From (3.6), we get for any  $1 \leq m \leq n-1$

$$\begin{aligned}
\bar{w}_m &= eE(t-s) \left\{ \sum_{j=1}^n \partial_j(u_j u_m) + \sum_{k,j=1}^n \partial_m \mathcal{N} \partial_k \partial_j(u_k u_j) \right. \\
& \quad \left. + S_m \left[ \sum_{j=1}^n \partial_j(u_j u_n) + \sum_{k,j=1}^n \partial_n \mathcal{N} \partial_k \partial_j(u_k u_j) \right] \right\} - S_m \bar{w}_n.
\end{aligned}$$

Whence,

$$\begin{aligned}
\mathcal{F}(\bar{w}_m)(\xi) &= \sum_{j=1}^{n-1} i\xi_j \mathcal{F} [eE(t-s)(u_j u_m)] + i\xi_n \mathcal{F} [eF(t-s)(u_n u_n)] \\
& \quad + \sum_{k,j=1}^n \mathcal{F} [eE(t-s) \partial_m \mathcal{N} \partial_k \partial_j(u_k u_j)] + \sum_{j=1}^{n-1} \frac{i\xi_m}{|\xi'|} i\xi_j \mathcal{F} [eE(t-s)(u_j u_n)] \\
& \quad + \frac{i\xi_m}{|\xi'|} i\xi_n \mathcal{F} [eF(t-s)(u_n u_n)] + \frac{i\xi_m}{|\xi'|} \sum_{k,j=1}^n \mathcal{F} [eE(t-s) \partial_n \mathcal{N} \partial_k \partial_j(u_k u_j)] \\
& \quad - \frac{i\xi_m}{|\xi'|} \mathcal{F}(\bar{w}_n). \tag{3.8}
\end{aligned}$$

Inserting (3.8) into (3.7), and after a direct calculation, we conclude that

$$\mathcal{F}(\bar{w}_m) = \mathcal{F}\left(\sum_{i=1}^{12} J_i(t-s)\right),$$

which implies that  $\bar{w}_m = \sum_{i=1}^{12} J_i(t-s)$  for  $1 \leq m \leq n-1$ .  $\square$

**Lemma 3.3.** *Let  $u$  be the strong solution of (1.2). Then for any  $t > 0$ :*

(i) *For any  $1 \leq k \leq n$ , we have  $\sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \in L^1(\mathbb{R}_+^n)$  and*

$$\left\| \sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j)(t) \right\|_{L^1(\mathbb{R}_+^n)} \leq C(\|u(t)\|_{L^2(\mathbb{R}_+^n)}^2 + \|\nabla u(t)\|_{L^2(\mathbb{R}_+^n)}^2);$$

$$(ii) \quad \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^2(\mathbb{R}_+^n)}^2 ds \leq \begin{cases} 2t^{\frac{1}{2}} \|u_0\|_{L^2(\mathbb{R}_+^n)}^2 & \text{if } 0 < t < 1, \\ Ct^{-\frac{1}{2}} & \text{if } t \geq 1. \end{cases}$$

**Proof.** From (3.3), we get

$$\begin{aligned} & \left\| \sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j)(t) \right\|_{L^1(\mathbb{R}_+^n)} \\ &= \left\| \sum_{i,j=1}^n \partial_k \int_0^\infty F(\tau) \partial_i \partial_j (u_i u_j)(t) d\tau \right\|_{L^1(\mathbb{R}_+^n)} \\ &= \left\| \sum_{i,j=1}^n \partial_k \left( \int_0^1 + \int_1^\infty \right) G_\tau * [\partial_i \partial_j (u_i u_j)]_*(t) d\tau \right\|_{L^1(\mathbb{R}^n)} \\ &\leq \left\| \sum_{i,j=1}^n \partial_k \int_0^1 G_\tau * [\partial_i \partial_j (u_i u_j)]_*(t) d\tau \right\|_{L^1(\mathbb{R}^n)} + \left\| \sum_{i,j=1}^n \partial_i \partial_j \partial_k \int_1^\infty G_\tau * [(u_i u_j)_*](t) d\tau \right\|_{L^1(\mathbb{R}^n)} \\ &\leq C \int_0^1 \|\partial_k G_\tau\|_{L^1(\mathbb{R}^n)} d\tau \|\nabla u(t)\|_{L^2(\mathbb{R}_+^n)}^2 + C \sum_{i,j=1}^n \int_1^\infty \|\partial_i \partial_j \partial_k G_\tau\|_{L^1(\mathbb{R}^n)} d\tau \|u(t)\|_{L^2(\mathbb{R}_+^n)}^2 \\ &\leq C \int_0^1 \tau^{-\frac{1}{2}} d\tau \|\partial_k G_1\|_{L^1(\mathbb{R}^n)} \|\nabla u(t)\|_{L^2(\mathbb{R}_+^n)}^2 + C \sum_{i,j=1}^n \int_1^\infty \tau^{-\frac{3}{2}} d\tau \|\partial_i \partial_j \partial_k G_1\|_{L^1(\mathbb{R}^n)} \|u(t)\|_{L^2(\mathbb{R}_+^n)}^2 \\ &\leq C(\|\nabla u(t)\|_{L^2(\mathbb{R}_+^n)}^2 + \|u(t)\|_{L^2(\mathbb{R}_+^n)}^2). \end{aligned}$$

Now we prove part (ii). Note that

$$\|u(t)\|_{L^2(\mathbb{R}_+^n)} \leq C(1+t)^{-\frac{1}{2}-\frac{n}{4}} \quad \text{for any } t > 0.$$

We have for any  $t \geq 1$

$$\begin{aligned} \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^2(\mathbb{R}_+^n)}^2 ds &\leq C \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) (t-s)^{-\frac{1}{2}} (1+s)^{-1-\frac{n}{2}} ds \\ &\leq Ct^{-\frac{1}{2}} \int_0^{\frac{t}{2}} (1+s)^{-1-\frac{n}{2}} ds + C(1+t)^{-1-\frac{n}{2}} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} ds \\ &\leq Ct^{-\frac{1}{2}}. \end{aligned}$$

If  $t \in (0, 1)$ , then we derive

$$\int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^2(\mathbb{R}_+^n)}^2 ds \leq \|u_0\|_{L^2(\mathbb{R}_+^n)}^2 \int_0^t (t-s)^{-\frac{1}{2}} ds = 2t^{\frac{1}{2}} \|u_0\|_{L^2(\mathbb{R}_+^n)}^2. \quad \square$$

**Proof of Theorem 1.4.** Let  $u$  be the weak solution of (1.2) given in Lemma 3.1. Set

$$\begin{aligned} \bar{b}_n(t) &= \sum_{\ell,k=1}^{n-1} \sum_{i,j=1}^n R_\ell R_k \partial_k \partial_k \Lambda^{-1} \int_0^t eE(t-s) \mathcal{N} \partial_i \partial_j (u_i u_j)(s) ds \\ &\quad + \sum_{k=1}^{n-1} \sum_{i,j=1}^n R_n R_k \partial_k \Lambda^{-1} \int_0^t eE(t-s) \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j)(s) ds. \end{aligned}$$

From Theorem 1.1 and Lemmas 3.2, 3.3, we deduce

$$\begin{aligned} &\left\| \int_0^t \bar{w}_n(t-s) ds - \bar{b}_n(t) \right\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\ &\leq \sum_{i,\ell=1}^{n-1} \sum_{k=1}^n \|R_i\| \|R_k\| \int_0^t \|\partial_\ell \partial_\ell \Lambda^{-1} eE(t-s)(u_i u_k)(s)\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\ &\quad + \sum_{i=1}^{n-1} \sum_{k=1}^n \|R_n\| \|R_i\| \int_0^t \|\partial_k eE(t-s)(u_i u_k)(s)\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\ &\quad + \sum_{\ell=1}^{n-1} \sum_{k=1}^n \|R_n\| \|R_k\| \int_0^t \|\partial_\ell \partial_\ell \Lambda^{-1} eF(t-s)(u_n u_k)(s)\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\ &\quad + \sum_{k=1}^n \|R_n\|^2 \int_0^t \|\partial_k eF(t-s)(u_n u_k)(s)\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^{n-1} \|R_n\| \|R_k\| \int_0^t \left\| eE(t-s) \sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j)(s) \right\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\
 & + \sum_{\ell=1}^{n-1} \sum_{i=1}^n \|R_\ell\|^2 \int_0^t \|eE(t-s)(u_i \partial_i u_n)(s)\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\
 & + \sum_{i,k=1}^{n-1} \|R_n\| \|R_k\| \int_0^t \|\partial_k \partial_i \Lambda^{-1} eE(t-s)(u_i u_n)(s)\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\
 & + \sum_{k=1}^{n-1} \|R_n\|^2 \int_0^t \|\partial_k \partial_k \Lambda^{-1} eF(t-s)(u_n u_n)(s)\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\
 & + \sum_{\ell=1}^{n-1} \|R_\ell\|^2 \int_0^t \left\| eE(t-s) \sum_{i,j=1}^n \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j)(s) \right\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\
 & \leq C \sum_{k=1}^n \int_0^t (t-s)^{-\frac{1}{2}} \|u_n u_k\|_{L^1(\mathbb{R}_+^n)} ds + C \int_0^t \|u \cdot \nabla u\|_{L^1(\mathbb{R}_+^n)} ds \\
 & \quad + C \sum_{k=1}^n \int_0^t \left\| \sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j)(s) \right\|_{L^1(\mathbb{R}_+^n)} ds \\
 & \leq C \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^2(\mathbb{R}_+^n)}^2 ds + C \int_0^t (\|u(s)\|_{L^2(\mathbb{R}_+^n)}^2 + \|\nabla u(s)\|_{L^2(\mathbb{R}_+^n)}^2) ds \\
 & \leq C(1+t)^{-\frac{1}{2}} + \int_0^t (1+s)^{-1-\frac{n}{2}} ds + C \int_0^t \|\nabla u(s)\|_{L^2(\mathbb{R}_+^n)}^2 ds \\
 & \leq C + C(1+t)^{-\frac{1}{2}}. \tag{3.9}
 \end{aligned}$$

For any  $1 \leq m \leq n-1$ , set

$$\begin{aligned}
 \bar{b}_m(t) = & - \sum_{i,j=1}^n R_n R_n S_m \int_0^t eE(t-s) \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j)(s) ds \\
 & - \sum_{i,j=1}^n R_n S_m R' \cdot \nabla' \int_0^t eE(t-s) \mathcal{N} \partial_i \partial_j (u_i u_j)(s) ds \\
 & - R_n R_n S_m \int_0^t eE(t-s) \partial_n (u_n u_n)(s) ds + S_m \int_0^t eE(t-s) (u_n \partial_n u_n)(s) ds.
 \end{aligned}$$

Then we have for any  $1 \leq m \leq n-1$

$$\begin{aligned}
& \left\| \int_0^t \bar{w}_m(t-s) ds - \bar{b}_m(t) \right\|_{\mathcal{H}^1(R^n)} ds \\
& \leq \sum_{i=1}^n \int_0^t \|eE(t-s)(u_i \partial_i u_m)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \sum_{i=1}^{n-1} \int_0^t \|\partial_i \partial_m \Lambda^{-1} eE(t-s)(u_i u_n)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \int_0^t \left\| eE(t-s) \sum_{i,j=1}^n \partial_m \mathcal{N} \partial_i \partial_j (u_i u_j)(s) \right\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \sum_{k=1}^n \sum_{i=1}^{n-1} \|R_i\| \|R_k\| \int_0^t \|\partial_m eE(t-s)(u_i u_k)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \sum_{k=1}^n \sum_{i=1}^{n-1} \|R_n\| \|R_k\| \int_0^t \|\partial_i \partial_m \Lambda^{-1} eE(t-s)(u_i u_k)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \sum_{k=1}^n \|R_n\| \|R_k\| \int_0^t \|\partial_m eF(t-s)(u_n u_k)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \sum_{k=1}^n \|R_n\|^2 \int_0^t \|\partial_m \partial_k \Lambda^{-1} eF(t-s)(u_n u_k)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \sum_{i=1}^n \|R_m\| \|R_i\| \int_0^t \|\Lambda eE(t-s)(u_i u_n)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \|R_m\| \|R_n\| \int_0^t \|\Lambda eF(t-s)(u_n u_n)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \sum_{i=1}^{n-1} \|R_n\| \|R_i\| \int_0^t \|\partial_m eE(t-s)(u_i u_n)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \|R_n\|^2 \int_0^t \|\partial_m eF(t-s)(u_n u_n)(s)\|_{\mathcal{H}^1(R^n)} ds \\
& \quad + \sum_{k=1}^{n-1} \|R_k\|^2 \int_0^t \left\| eE(t-s) \sum_{i,j=1}^n \partial_m \mathcal{N} \partial_i \partial_j (u_i u_j)(s) \right\|_{\mathcal{H}^1(R^n)} ds
\end{aligned}$$



$$\begin{aligned}
 & + \| \| R_n \| \| R_m \| \int_0^t \left\| eE(t-s) \sum_{i,j=1}^n \partial_n \mathcal{N} \partial_i \partial_j (u_i u_j)(s) \right\|_{\mathcal{H}^1(\mathbb{R}^n)} ds \\
 & \leq C \int_0^t \| u \cdot \nabla u_m \|_{L^1(\mathbb{R}_+^n)} ds + C \sum_{i,k=1}^n \int_0^t (t-s)^{-\frac{1}{2}} \| u_i u_k \|_{L^1(\mathbb{R}_+^n)} ds \\
 & \quad + C \sum_{k=1}^n \int_0^t \left\| \sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j)(s) \right\|_{L^1(\mathbb{R}_+^n)} ds \\
 & \leq C \int_0^t (t-s)^{-\frac{1}{2}} \| u(s) \|_{L^2(\mathbb{R}_+^n)}^2 ds + C \int_0^t (\| u(s) \|_{L^2(\mathbb{R}_+^n)}^2 + \| \nabla u(s) \|_{L^2(\mathbb{R}_+^n)}^2) ds \\
 & \leq C(1+t)^{-\frac{1}{2}} + \int_0^t (1+s)^{-1-\frac{n}{2}} ds + C \int_0^t \| \nabla u(s) \|_{L^2(\mathbb{R}_+^n)}^2 ds \\
 & \leq C + C(1+t)^{-\frac{1}{2}}.
 \end{aligned} \tag{3.10}$$

From Theorem 1.1 and (3.9), (3.10), we obtain

$$\begin{aligned}
 \| u(t) - (a(t) + b(t)) \|_{L^1(\mathbb{R}_+^n)} & \leq \| e^{-tA} u_0 - a(t) \|_{L^1(\mathbb{R}_+^n)} + \sum_{j=1}^n \left\| \int_0^t w_j(t-s) ds - b_j(t) \right\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \\
 & \leq C \| u_0 \|_{L^1(\mathbb{R}_+^n)} + \sum_{j=1}^n \left\| \int_0^t \bar{w}_j(t-s) ds - \bar{b}_j(t) \right\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C.
 \end{aligned}$$

If  $u$  is a weak solution of (1.2) obtained in Lemma 3.1, we don't know if the boundary value  $u \cdot \nabla u|_{\partial \mathbb{R}_+^n}$  exists even in some weak sense. Therefore we cannot apply (3.4) to weak solutions directly. To avoid this difficulty, we employ the approximate sequence of solutions  $\{u_N\}$  as introduced in [9], for which (3.4) holds. By passing a limit, we can verify that there exists one weak solution  $u$  of (1.2), for which part (i) in Theorem 1.4 holds, we omit the details here.

Now we prove the part (ii) for the strong solution  $u$  of (1.2). By (3.4) and Lemmas 3.1, 3.3, we obtain

$$\begin{aligned}
 \| \nabla u(t) \|_{L^1(\mathbb{R}_+^n)} & \leq \| \nabla e^{-tA} u_0 \|_{L^1(\mathbb{R}_+^n)} + \int_0^t \| \nabla e^{-(t-s)A} P(u \cdot \nabla u) \|_{L^1(\mathbb{R}_+^n)} ds \\
 & \leq Ct^{-\frac{1}{2}} \| u_0 \|_{L^1(\mathbb{R}_+^n)} + C \int_0^t (t-s)^{-\frac{1}{2}} \left( \| u \cdot \nabla u \|_{L^1(\mathbb{R}_+^n)} \right. \\
 & \quad \left. + \left\| \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^1(\mathbb{R}_+^n)} \right) ds
 \end{aligned}$$

$$\begin{aligned}
&\leq Ct^{-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}_+^n)} + C \int_0^t (t-s)^{-\frac{1}{2}} (\|u(s)\|_{L^2(\mathbb{R}_+^n)}^2 + \|\nabla u(s)\|_{L^2(\mathbb{R}_+^n)}^2) ds \\
&\leq Ct^{-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}_+^n)} + C \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^2(\mathbb{R}_+^n)}^2 ds \\
&\quad + C \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) (t-s)^{-\frac{1}{2}} \|\nabla u(s)\|_{L^2(\mathbb{R}_+^n)}^2 ds \\
&\leq Ct^{-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}_+^n)} + Ct^{-\frac{1}{2}} + Ct^{-\frac{1}{2}} \int_0^{\frac{t}{2}} \|\nabla u(s)\|_{L^2(\mathbb{R}_+^n)}^2 ds \\
&\quad + C \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} s^{-2-\frac{n}{2}} ds \\
&\leq Ct^{-\frac{1}{2}} (1 + t^{-\frac{n+2}{2}}) \quad \text{for any } t > 0. \quad \square
\end{aligned}$$

To proceed, the following estimates are needed, which can be found in [13].

**Lemma 3.4.** For any  $a \in L_\sigma^r(\mathbb{R}_+^n)$ ,

$$\|\nabla^k e^{-tA} a\|_{L^q(\mathbb{R}_+^n)} \leq C_k t^{-\frac{k}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})} \|a\|_{L^r(\mathbb{R}_+^n)}$$

with  $k = 0, 1, \dots$ , provided that  $1 \leq r < q \leq \infty$  or  $1 < r \leq q < \infty$ .

**Proof of Theorem 1.5.** The strong solution  $u$  of (1.2) given in (ii) of Lemma 3.1 with  $q = 2$  can be written as

$$u(t) = e^{-\frac{t}{2}A} u\left(\frac{t}{2}\right) - \int_{\frac{t}{2}}^t e^{-(t-s)A} P(u \cdot \nabla u) ds. \quad (3.11)$$

From Lemmas 3.1, 3.4 and (3.11), we obtain for  $k = 0, 1$

$$\begin{aligned}
\|\nabla^k u(t)\|_{L^\infty(\mathbb{R}_+^n)} &\leq \left\| \nabla^k e^{-\frac{t}{2}A} u\left(\frac{t}{2}\right) \right\|_{L^\infty(\mathbb{R}_+^n)} + \int_{\frac{t}{2}}^t \|\nabla^k e^{-(t-s)A} P(u \cdot \nabla u)\|_{L^\infty(\mathbb{R}_+^n)} ds \\
&\leq Ct^{-\frac{k}{2} - \frac{n}{4}} \left\| u\left(\frac{t}{2}\right) \right\|_{L^2(\mathbb{R}_+^n)} + C \int_{\frac{t}{2}}^t (t-s)^{-\frac{k}{2} - \frac{n}{4}} \|P(u \cdot \nabla u)\|_{L^2(\mathbb{R}_+^n)} ds \\
&\leq Ct^{-\frac{k}{2} - \frac{n}{4}} (1+t)^{-\frac{1}{2} - \frac{n}{2}(1-\frac{1}{2})}
\end{aligned}$$

$$\begin{aligned}
& + C \int_{\frac{t}{2}}^t (t-s)^{-\frac{k}{2}-\frac{n}{4}} \|u(s)\|_{L^4(\mathbb{R}_+^n)} \|\nabla u(s)\|_{L^4(\mathbb{R}_+^n)} ds \\
& \leq Ct^{-\frac{n+k+1}{2}} + C \int_{\frac{t}{2}}^t (t-s)^{-\frac{k}{2}-\frac{n}{4}} (1+s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{4})} s^{-1-\frac{n}{2}(1-\frac{1}{4})} ds \\
& \leq Ct^{-\frac{n+k+1}{2}} + C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{4})} t^{-1-\frac{n}{2}(1-\frac{1}{4})} \int_{\frac{t}{2}}^t (t-s)^{-\frac{k}{2}-\frac{n}{4}} ds \\
& \leq C(t^{-\frac{n+k+1}{2}} + t^{-\frac{2n+k+1}{2}}) \quad \text{for all } t > 0. \quad \square
\end{aligned}$$

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