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ABSTRACT

In this paper we exhibit the dissipative mechanism of the Cahn–Hilliard equation in $H^1(\mathbb{R}^N)$. We show a weak form of dissipativity by showing that each individual solution is attracted, in some sense, by the set of equilibria. We also indicate that strong dissipativity, that is, asymptotic compactness in $H^1(\mathbb{R}^N)$, cannot be in general expected. Then we consider two types of perturbations: a nonlinear perturbation and a small linear perturbation. In both cases we show that, for the resulting equations, the dissipative mechanism becomes strong enough to obtain the existence of a compact global attractor.

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1. Introduction

In this paper we study the Cauchy problem for the Cahn–Hilliard equation in \mathbb{R}^N

$$u_t + \Delta^2 u + \Delta f(x, u) = 0, \quad t > 0, \quad x \in \mathbb{R}^N, \quad (1.1)$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^N, \quad (1.2)$$

under some mild assumptions on the nonlinear term $f: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$.

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This equation was originally derived in [11], with a cubic bistable nonlinearity, $f(x, u) = u - u^3$, as a phenomenological equation describing phase transition problems in binary metallic alloys. Since then, a large amount of literature has been produced about the Cahn–Hilliard equation which is considered a paradigmatic model describing a mechanism for pattern formation and space time coherence. Most of the existing references however deal with the case in which Cahn–Hilliard model is considered in a bounded domain, subject to suitable boundary conditions, see for example [40,36] and references therein. See also [23,14,32] for a system of Cahn–Hilliard equations, which appear naturally in the studies of multi-component alloys.

Hence, just a few references seem to have considered the problem in \mathbb{R}^N or in unbounded domains. For example [10] studied in dimension $N = 1$ the stability of a particular monotone increasing steady solution $u_0(x) = \tanh(\frac{x}{2})$, when $f(u) = \frac{u}{2} - \frac{u^3}{2}$. For initial data which are perturbations of $u_0(x)$ and using some weighted space, some stability and time decay estimates are obtained. In [33], the authors studied in \mathbb{R}^N the stability of constant solutions of (1.1), assuming some local growth conditions and regularity on $f = f(u)$ (i.e. independent of $x \in \mathbb{R}^N$). The perturbation of the constant initial data is assumed to be bounded and sufficiently small in $L^1(\mathbb{R}^N)$. Stability is obtained in $L^\infty(\mathbb{R}^N)$ and decay rates are obtained in some $L^p(\mathbb{R}^N)$. Sharper time decay rates for $N \geq 3$ were later obtained in [20].

On the other hand, several variations of (1.1) have been proposed in order to provide more complete models from the physical point of view. These models share some mathematical properties with (1.1) but also have some very different mathematical features. Without intending to be exhaustive, the so-called Viscous Cahn–Hilliard model can be found in [21,35,34,13]. A model with inertial terms has been studied in [24] (see also references therein), while several different perturbations have been analyzed in dimension $N = 1$ in [42,43]. Finally, stochastic perturbations of (1.1), called Cahn–Hilliard–Cook equations, can be found in [7,8]. All these references deal with the case of a bounded domain.

Concerning unbounded domains, in [9] the author considered the viscous Cahn–Hilliard model in a channel like unbounded domain in dimensions $N = 2, 3$. Using weighted spaces it was proved that the model has a finite dimensional attractor.

In [18] the authors considered in \mathbb{R}^N the viscous Cahn–Hilliard model in the Sobolev space $H^1(\mathbb{R}^N)$, and under some growth and smoothness restrictions on the nonlinear term, proved the existence of an attractor. These results include the case of a nonlinear dissipative modification of (1.1); see Section 8.1. Some extension for $N \geq 3$ was also developed in [19].

To the best of our knowledge the results in [18,19] and [10,33,20] mentioned above are the only ones in which the asymptotic behavior of (1.1) (or some close variant of it) is studied in an unbounded domain. This is probably due to the fact that (1.1) does not have a strong dissipative character as we show below.

As it is customary in problems in unbounded domains, the key for a dissipative mechanism relies on the ability of the solutions to become compact for large times. In the case of bounded domains the smoothing effect on the solutions is enough to guarantee such compactness but this is not longer true in unbounded ones. In the case of reaction–diffusion equations in unbounded domains, see (1.4) below, it was shown in [4] that both diffusion and reaction must collaborate in order to produce the compactness needed. Similar results for fourth-order problems of the form

$$u_t + \Delta^2 u = f(x, u), \quad t > 0, x \in \mathbb{R}^N,$$

have been obtained in [17], for which the results resemble pretty much those for (1.4).

However, as we show below, this sort of collaboration between the second-order nonlinear term and the fourth-order linear part in (1.1) is not strong enough as for the reaction–diffusion case (1.4). This difficulty in (1.1) can be overcome using suitable weighted spaces in which the linear part of (1.1) has compact resolvent which is enough to gain compactness for the solutions of the evolution problem.

However a natural space for (1.1) is the space $H^1(\mathbb{R}^N)$ since under some mild assumptions on f the energy functional $E : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} F(x, u), \quad (1.3)$$

where $F(x, u) = \int_0^u f(x, s) ds$ is decreasing along the solutions of (1.1).

Therefore, here we complete some of the previous analysis in several directions. After discussing local and global existence of solutions in $H^1(\mathbb{R}^N)$, we show a weak form of dissipativity of (1.1) by showing that each individual solution somehow approaches the set of equilibria as time goes to infinity. Despite this fact, we also show that strong dissipativity (more precisely, asymptotic compactness in the sense of [31]) cannot be in general expected. This stems from the fact that for linear equations (i.e. when $f(x, u)$ is replaced by a linear term in u) the spectrum of the elliptic operator always contains zero and therefore solutions of the linear parabolic equation do not converge exponentially to zero.

Then we consider two types of perturbations of (1.1) for which the resulting solutions are asymptotically compact. First, following ideas in [18], we consider a nonlinear perturbation of (1.1) but in a less restrictive setting than in that reference. Second we consider a small linear dissipative perturbation of (1.1). In both cases we show that for the resulting equations the dissipative mechanism is strong enough to obtain the missing compactness in (1.1).

In our approach we focus on a strong parallelism between (1.1) and another archetypical model for pattern formation, namely, the so-called Cahn–Allen or reaction–diffusion model

$$u_t - \Delta u = f(x, u), \quad t > 0, \quad x \in \mathbb{R}^N. \quad (1.4)$$

For example, both models (1.1) and (1.4) share the same energy functional (1.3), which acts as a Lyapunov functional for solutions, that is, the energy decreases with time along solutions. Also, both equations have the same set of equilibrium solutions which due to the decreasing energy, play an important role in the behavior of all solutions. At the level of existence of solutions, the growth allowed in the nonlinear terms is the same in both equations, see Theorem 2.1 and [4]. Also, the dissipative mechanism that we consider in this paper for (1.1), see (3.3)–(3.6), is pretty much the same as for (1.4), see [4], with small variations of the parameters, due to specific conditions for both problems. A crucial difference between both models however, stems from the fact that for (1.4) the maximum principle applies, but not for (1.1), combined with the fact that, as mentioned above, for linear equations, no exponential decay of solutions can be achieved. This explains why the dissipative character of (1.4) is much stronger than that of (1.1).

For the nonlinear term we will assume the general form

$$f(x, u) = g(x) + m(x)u + f_0(x, u), \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R}, \quad (1.5)$$

with

$$f_0(x, 0) = 0, \quad \frac{\partial f_0}{\partial u}(x, 0) = 0, \quad x \in \mathbb{R}^N, \quad (1.6)$$

$$f_0 : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \quad \text{locally Lipschitz in } u \in \mathbb{R} \text{ uniformly for } x \in \mathbb{R}^N, \quad (1.7)$$

$$m \in L^r_U(\mathbb{R}^N), \quad \max \left\{ \frac{N}{2}, 1 \right\} < r \leq \infty, \quad (1.8)$$

and

$$g \in L^p(\mathbb{R}^N) \quad \text{for some } 1 < p < \infty, \quad (1.9)$$

where the above space $L_U^r(\mathbb{R}^N)$ is defined, for $1 \leq r \leq \infty$, as

$$L_U^r(\mathbb{R}^N) \stackrel{\text{def}}{=} \left\{ \phi \in L_{\text{loc}}^r(\mathbb{R}^N) : \|\phi\|_{L_U^r(\mathbb{R}^N)} = \sup_{y \in \mathbb{R}^N} \|\phi\|_{L^r(B(y,1))} < \infty \right\}$$

(see [5,28] and note that $L_U^\infty(\mathbb{R}^N) := L^\infty(\mathbb{R}^N)$).

Sometimes we will also assume some growth restriction of the form

$$|f_0(x, u_1) - f_0(x, u_2)| \leq c|u_1 - u_2|(1 + |u_1|^{\rho-1} + |u_2|^{\rho-1}), \quad u_1, u_2 \in \mathbb{R}, \quad (1.10)$$

for some suitable $\rho > 1$ and $c > 0$.

For the functional setting we will consider the Bessel potential spaces, which we generically denote $H_p^\alpha(\mathbb{R}^N)$ (see [41]). When $p = 2$ we will denote these spaces as $H^\alpha(\mathbb{R}^N)$, which are Hilbert spaces.

The paper is organized as follows. First, in Section 2 we show that (1.1) can be solved locally in time for initial data in Bessel potential spaces $H_p^1(\mathbb{R}^N)$. This requires some restrictions in ρ in (1.10). The particular case of $p = 2$, i.e. of initial data in $H^1(\mathbb{R}^N)$ is also stated.

Then in Section 3, under suitable dissipativity assumptions on f , see (3.3)–(3.6) we prove that the energy (1.3) provides a good estimate of the $H^1(\mathbb{R}^N)$ -norm of the solutions.

With this estimate in Section 4 we prove the solutions of (1.1) are global in time and exploit (1.4) to obtain some preliminary information on the set of equilibria. Also, we show that in general, asymptotic compactness for (1.1) cannot be expected.

Then in Section 5 we obtain stronger estimates on the solutions in spaces $H_p^1(\mathbb{R}^N)$. In Section 6 we prove that each individual solution in $H^1(\mathbb{R}^N)$ approaches to the set of equilibria in $H_{\text{loc}}^s(\mathbb{R}^N)$ for any $s < 2$. Some examples of prototypical classes of nonlinearities are considered in Section 7.

In Section 8 we introduce some perturbations in the original Cahn–Hilliard equation (1.1)–(1.2) in such a way that the perturbed semigroup is asymptotically compact and therefore has a global attractor. The goal is then to understand possible mechanisms that make the equation dissipative. We consider first in Section 8.1 a nonlinear dissipative perturbation in a similar way as in [18] but with much less growth and regularity restrictions. Then in Section 8.2 we consider a small linear perturbation. In both cases we show that for the resulting equations the dissipative mechanism is strong enough to obtain the missing compactness in (1.1).

Some of the crucial technical results needed in this paper have been collected in two appendices. First, Appendix A contains several results for linear operators of the form $(-\Delta)^k + C(x)I$, with $k = 1, 2$, and suitable potential $C(x)$. Appendix B contains the proof of local existence of solutions invoked in Section 2.

2. Local well posedness in $H_p^1(\mathbb{R}^N)$

Our concern in this section are local solutions of (1.1)–(1.2), where $u_0 \in H_p^1(\mathbb{R}^N)$ for some $1 < p < \infty$.

Theorem 2.1. Assume (1.5)–(1.9). Then the problem (1.1)–(1.2) is locally well posed in $H_p^1(\mathbb{R}^N)$, provided that either

- (i) $1 > \frac{N}{p} > \frac{N}{r} - 1$,
- (ii) $1 = \frac{N}{p}$ and (1.10) holds with some $1 < \rho < \infty$,
- (iii) $1 < \frac{N}{p}$ and (1.10) holds with some $1 < \rho \leq \rho_c := \frac{N+p}{N-p} = 1 + \frac{2p}{N-p}$.

The proof of Theorem 2.1 is based on the analytic semigroup approach as in [26]. Actually, since we deal with critical exponents, some extension of this approach, developed in [2,3], will be used. For details see Appendix B. Note however that the growth allowed in the nonlinear term in Theorem 2.1 is the same as for the reaction–diffusion equation (1.4); see Proposition 4.5 below.

Remark 2.2.

- (i) The solution u through $u_0 \in H_p^1(\mathbb{R}^N)$ can be continued onto the maximal interval of existence $[0, \tau_{u_0})$ and satisfies

$$u \in C([0, \tau_0), H_p^1(\mathbb{R}^N)) \cap C((0, \tau_0), H_p^{2\beta^*(p)}(\mathbb{R}^N)) \cap C^1((0, \tau_0), H_p^s(\mathbb{R}^N)), \quad (2.1)$$

for $s < 2\beta^*(p)$, where

$$\beta^*(p) := 1 + \left(\frac{N}{2p} - \frac{N}{2r} \right)_-$$

and a_- denotes the negative part of $a \in \mathbb{R}$.

- (ii) The equation in (1.1) holds as $u_t = -\Delta(\Delta u + f(\cdot, u))$ as an equality in $H_p^{2(\beta^*(p)-1)}(\mathbb{R}^N)$. Notice that $\beta^*(p) < 1$ if $r < p$ and $\beta^*(p) = 1$ if $r \geq p$. In the latter case the equation actually holds in $L^p(\mathbb{R}^N)$ and for each $t > 0$, $\Delta u + f(\cdot, u) \in H_p^2(\mathbb{R}^N)$.
- (iii) On the other hand, the solution u satisfies the variation of constants formula

$$u(t) = e^{-\Delta^2 t} u_0 + \int_0^t e^{-\Delta^2(t-s)} (-\Delta)(f(\cdot, u(s))) ds, \quad t \in [0, \tau_0), \quad (2.2)$$

or, equivalently,

$$u(t) = e^{-\Delta^2 t} u_0 + \int_0^t (-\Delta) e^{-\Delta^2(t-s)} (f(\cdot, u(s))) ds, \quad t \in [0, \tau_0). \quad (2.3)$$

- (iv) Concerning the maximal interval of existence $[0, \tau_{u_0})$, whenever $\rho < \rho_c$ we have that

$$\tau_{u_0} < \infty \quad \text{implies} \quad \limsup_{t \rightarrow \tau_{u_0}^-} \|u(t)\|_{H_p^1(\mathbb{R}^N)} = \infty.$$

Thus, for $\rho < \rho_c$, an $H_p^1(\mathbb{R}^N)$ -estimate of the solution on finite time intervals is sufficient for the global existence, whereas for $\rho = \rho_c$ global existence generally follows from an estimate in a stronger norm; see (B.8).

- (v) Concerning additional smoothing action of the local solution, if $p \leq r$, $\rho < \rho_c$ then for any $\theta < \frac{1}{4}$ and $R > 0$, there exist $t_0(R)$ and $C(R)$ such that for any $0 < t \leq t_0$ and u_0, z_0 in the ball in $H_p^1(\mathbb{R}^N)$ of radius $R > 0$,

$$\begin{aligned} t^\theta \|u(t, u_0) - u(t, z_0)\|_{H_p^{1+4\theta}(\mathbb{R}^N)} &\leq c(R) \|u_0 - z_0\|_{H_p^1(\mathbb{R}^N)}, \\ t^\theta \|u(t, u_0)\|_{H_p^{1+4\theta}(\mathbb{R}^N)} &\leq c(R). \end{aligned} \quad (2.4)$$

Due to the lack of maximum principle, in further analysis of the Cahn–Hilliard problem below we have to rely on some “energy” type estimates of solutions. This is the reason why, although local existence for (1.1)–(1.2) can be done in more general spaces, the asymptotic behavior will be studied in the $H^1(\mathbb{R}^N)$ setting. For reader’s convenience we summarize below the results considering the case $r \geq p = 2$.

Corollary 2.3. Suppose that (1.5)–(1.7) is satisfied with $m \in L^r_U(\mathbb{R}^N)$, $r > \max\{\frac{N}{2}, 1\}$, $r \geq 2$, $g \in L^2(\mathbb{R}^N)$ and either

- (i) $N = 1$,
- (ii) $N = 2$ and (1.10) holds with some $1 < \rho < \infty$,
- (iii) $N \geq 3$ and (1.10) holds with some $1 < \rho \leq \rho_c := \frac{N+2}{N-2} = 1 + \frac{4}{N-2}$.

Then, for any $u_0 \in H^1(\mathbb{R}^N)$, there is a unique solution u to (1.1)–(1.2) defined on a maximal interval of existence $[0, \tau_{u_0})$. Furthermore,

$$u \in C([0, \tau_0), H^1(\mathbb{R}^N)) \cap C((0, \tau_0), H^2(\mathbb{R}^N)) \cap C^1((0, \tau_0), H^s(\mathbb{R}^N)), \quad s < 2,$$

$$\Delta u(t) + f(\cdot, u(t)) \in H^2(\mathbb{R}^N), \quad t \in [0, \tau_{u_0}),$$

and for $t \in (0, \tau_{u_0})$ we have

$$u_t + (-\Delta)(-\Delta u - f(x, u)) = 0 \quad \text{in } L^2(\mathbb{R}^N).$$

Also $\tau_{u_0} < \infty$ implies

$$\limsup_{t \rightarrow \tau_{u_0}^-} \|u(t)\|_{H^1(\mathbb{R}^N)} = \infty \quad \text{for } \rho < \rho_c.$$

Finally, if $\rho < \rho_c$ and $\theta < \frac{1}{4}$, then for any $R > 0$, there exist $t_0(R)$ and $C(R)$ such that for any $0 < t \leq t_0$ and for any u_0, z_0 in the ball in $H^1(\mathbb{R}^N)$ of radius $R > 0$,

$$t^\theta \|u(t, u_0) - u(t, z_0)\|_{H^{1+4\theta}(\mathbb{R}^N)} \leq c(R) \|u_0 - z_0\|_{H^1(\mathbb{R}^N)},$$

$$t^\theta \|u(t, u_0)\|_{H^{1+4\theta}(\mathbb{R}^N)} \leq c(R). \quad (2.5)$$

3. $H^1(\mathbb{R}^N)$ -estimate

In this section we derive $H^1(\mathbb{R}^N)$ -estimate of the solutions of (1.1)–(1.2) in Corollary 2.3. For this we will need some auxiliary lemmas.

Lemma 3.1.

- (i) Let $P_0 = -\Delta$ in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$. Then $P_0 : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ has the inverse P_0^{-1} defined on the range $P_0(H^2(\mathbb{R}^N))$ with values in $H^2(\mathbb{R}^N)$.
- (ii) Since $\sigma(P_0) = [0, \infty)$, then P_0 is not onto nor P_0^{-1} is continuous for the $L^2(\mathbb{R}^N)$ -norm.

Proof. If $\phi \in H^2(\mathbb{R}^N)$ and $P_0\phi = 0$ then $\int_{\mathbb{R}^N} |\nabla \phi|^2 = 0$. Hence ϕ is a constant and since $\phi \in H^2(\mathbb{R}^N)$ we infer that $\phi = 0$, which proves part (i).

For part (ii) observe that if P_0 was onto, by the closed graph theorem P_0^{-1} would be continuous contradicting that $0 \in \sigma(P_0)$.

Finally, assume that P_0^{-1} is continuous. Then there exists $c > 0$ such that for all $u \in H^2(\mathbb{R}^N)$ we have

$$\|u\|_{L^2(\mathbb{R}^N)} \leq c \|\Delta u\|_{L^2(\mathbb{R}^N)}.$$

Then take a smooth u with norm one in $L^2(\mathbb{R}^N)$ and consider $u_\lambda(x) = u(\lambda x)$, with $\lambda > 0$ in the inequality above. Then we get $\lambda^{N/2} \leq c \lambda^{N/2+2} \|\Delta u\|_{L^2(\mathbb{R}^N)}$, and letting $\lambda \rightarrow 0$ we get a contradiction. \square

Note that we can write (1.1) as

$$u_t + P_0(P_0 u - f(\cdot, u)) = 0. \quad (3.1)$$

Then Corollary 2.3 implies that $u_t \in P_0(H^2(\mathbb{R}^N))$ and by Lemma 3.1, (3.1) can be written as

$$P_0^{-1} u_t - \Delta u = f(\cdot, u). \quad (3.2)$$

As mentioned in (1.3) the functional below is a natural energy for the solutions of (1.1) as in Corollary 2.3.

Lemma 3.2. Assume the conditions for local existence as in Corollary 2.3. Then the functional E in (1.3),

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} F(x, u),$$

where $F(x, u) = \int_0^u f(x, s) ds$, is well defined from $H^1(\mathbb{R}^N)$ into \mathbb{R} and bounded on bounded sets of $H^1(\mathbb{R}^N)$.

Proof. Using the mean value theorem we have

$$\begin{aligned} \int_{\mathbb{R}^N} |F(x, \phi)| &= \int_{\mathbb{R}^N} |F(x, \phi) - F(x, 0)| = \int_{\mathbb{R}^N} |\phi| |f(x, \theta\phi)| \\ &\leq \int_{\mathbb{R}^N} |m| |\phi|^2 + \int_{\mathbb{R}^N} |\phi| |f_0(x, \theta\phi)| + \int_{\mathbb{R}^N} |\phi| |g| =: \mathcal{I}_1(\phi) + \mathcal{I}_2(\phi) + \mathcal{I}_3(\phi), \end{aligned}$$

where $\theta = \theta(x) \in (0, 1)$. By Lemma A.5 functional \mathcal{I}_1 is well defined and bounded on bounded subsets of $H^1(\mathbb{R}^N)$. The latter property also holds for \mathcal{I}_3 , since $\mathcal{I}_3(\phi) \leq \|g\|_{L^2(\mathbb{R}^N)} \|\phi\|_{L^2(\mathbb{R}^N)}$.

Considering next \mathcal{I}_2 we need to take into account cases (i)–(iii) in Corollary 2.3. If B is bounded in $H^1(\mathbb{R}^N)$ and (i) holds then using embedding $H^1(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ there is an interval $[-r, r]$ containing the set $\{\phi(x), x \in \mathbb{R}^N, \phi \in B\}$. Choosing a Lipschitz constant L_r for $f_0(x, s)$ with $s \in [-r, r]$ we then have $\mathcal{I}_2(\phi) \leq L_r \|\phi\|_{L^2(\mathbb{R}^N)}^2$, which gives the result. If (ii) or (iii) holds, then $\mathcal{I}_2(\phi) \leq c(\|\phi\|_{L^2(\mathbb{R}^N)}^2 + \|\phi\|_{L^{\rho+1}(\mathbb{R}^N)}^{\rho+1})$. Using the embedding $H^1(\mathbb{R}^N) \subset L^{\rho+1}(\mathbb{R}^N)$ where ρ is arbitrary large in the case (ii) and $\rho \leq \frac{N+2}{N-2}$ in the case (iii) we conclude that there exists a positive constant c_B such that $|\mathcal{I}_2(\phi)| \leq c_B$ for any $\phi \in B$. \square

In what follows we will assume a structure condition

$$vf(x, v) \leq C(x)v^2 + D(x)|v|, \quad x \in \mathbb{R}^N, v \in \mathbb{R}, \quad (3.3)$$

where

$$0 \leq D \in L^s(\mathbb{R}^N), \quad \max\left\{1, \frac{2N}{N+2}\right\} \leq s \leq 2 \quad (s > 1 \text{ if } N = 2), \quad (3.4)$$

and

$$C \in L_U^r(\mathbb{R}^N), \quad r > \max\left\{\frac{N}{2}, 1\right\}. \quad (3.5)$$

Note that from (3.4) we have $|g(x)| \leq D(x)$, $x \in \mathbb{R}$. We will also assume that the solutions of the linear problem

$$\begin{cases} u_t = \Delta u + C(x)u, & t > 0, x \in \mathbb{R}^N, \\ u(0) = u_0 \in L^2(\mathbb{R}^N) \end{cases} \quad (3.6)$$

are exponentially decaying as $t \rightarrow \infty$. Note that this happens if and only if there is a certain $\omega_0 > 0$ such that

$$\int_{\mathbb{R}^N} (|\nabla \phi|^2 - C(x)\phi^2) \geq \omega_0 \|\phi\|_{L^2(\mathbb{R}^N)}^2,$$

for all $\phi \in H^1(\mathbb{R}^N)$; see (A.2) in Appendix A.

Remark 3.3. Observe that (3.3) implies $|g(x)| \leq D(x)$. Thus, if g is as in Corollary 2.3 then (3.4) implies that $g \in L^q(\mathbb{R}^N)$ for every $q \in [s, 2]$.

Without going into full details, we merely mention that one can avoid this condition on g , by assuming in (3.3) that D decomposes into $D_1 + D_2$, where $D_1 \in L^s(\mathbb{R}^N)$, $\max\{1, \frac{2N}{N+2}\} \leq s < 2$ ($s > 1$ if $N = 2$) and $D_2 \in L^2(\mathbb{R}^N)$, which will not change the results below in any essential way.

Then we have the following additional properties of the energy in Lemma 3.2.

Lemma 3.4. Besides the conditions of Lemma 3.2, assume also the structure condition (3.3) with $C(\cdot)$, $D(\cdot)$ satisfying (3.4)–(3.6).

Then, there are positive constants c_1, c_2 such that

$$\|\phi\|_{H^1(\mathbb{R}^N)}^2 \leq c_1 E(\phi) + c_2, \quad \phi \in H^1(\mathbb{R}^N). \quad (3.7)$$

Proof. As a consequence of (3.3) we have

$$F(x, \theta) \leq \frac{1}{2} C(x)\theta^2 + D(x)|\theta|, \quad x \in \mathbb{R}^N, \theta \in \mathbb{R}, \quad (3.8)$$

and hence

$$2E(\phi) \geq \int_{\mathbb{R}^N} |\nabla \phi|^2 - \int_{\mathbb{R}^N} C(x)\phi^2 - 2 \int_{\mathbb{R}^N} D(x)|\phi|, \quad \phi \in H^1(\mathbb{R}^N).$$

Now observe that if s' denotes the conjugate exponent to s , we have the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{s'}(\mathbb{R}^N)$. Then using Hölder inequality and taking into account that the norm $\|\phi\|_{H^1(\mathbb{R}^N)}$ can be bounded by $c(\|\nabla \phi\|_{L^2(\mathbb{R}^N)} + \|\phi\|_{L^2(\mathbb{R}^N)})$, we get for any $\nu > 0$

$$\begin{aligned} 2E(\phi) &\geq \int_{\mathbb{R}^N} |\nabla \phi|^2 - \int_{\mathbb{R}^N} C(x)\phi^2 - 2\|D\|_{L^s(\mathbb{R}^N)}\|\phi\|_{L^{s'}(\mathbb{R}^N)} \\ &\geq \int_{\mathbb{R}^N} |\nabla \phi|^2 - \int_{\mathbb{R}^N} C(x)\phi^2 - \frac{4c^2}{\nu}\|D\|_{L^s(\mathbb{R}^N)}^2 - \frac{\nu}{2}(\|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2 + \|\phi\|_{L^2(\mathbb{R}^N)}^2) \\ &= \frac{\nu}{2}\|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} ((1-\nu)|\nabla \phi|^2 - C(x)\phi^2) - \frac{4c^2}{\nu}\|D\|_{L^s(\mathbb{R}^N)}^2 - \frac{\nu}{2}\|\phi\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

Since by the assumption below (3.6), (A.2) holds for some $\omega_0 > 0$, applying Lemma A.4(ii) with $\nu > 0$ chosen so small that $\omega(\nu) > \nu$, we obtain

$$\begin{aligned} 2E(\phi) &\geq \frac{\nu}{2} \|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2 + \left(\omega(\nu) - \frac{\nu}{2} \right) \|\phi\|_{L^2(\mathbb{R}^N)}^2 - \frac{4c^2}{\nu} \|D\|_{L^s(\mathbb{R}^N)}^2 \\ &\geq \frac{\nu}{2} (\|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2 + \|\phi\|_{L^2(\mathbb{R}^N)}^2) - \frac{4c^2}{\nu} \|D\|_{L^s(\mathbb{R}^N)}^2, \end{aligned}$$

which completes the proof. \square

Using Lemma 3.4 we will next show that the solution of (1.1)–(1.2), as in Corollary 2.3, can be estimated in $H^1(\mathbb{R}^N)$ uniformly for $t \geq 0$ and for u_0 in bounded sets of $H^1(\mathbb{R}^N)$.

Lemma 3.5.

- (i) If u is a solution of (1.1)–(1.2) as in Corollary 2.3, then as long as the solution exists we have that $u_t \in P_0(H^2(\mathbb{R}^N))$, thus $P_0^{-1}u_t$ is defined as in Lemma 3.1, and

$$\begin{aligned} \frac{d}{dt} E(u(t)) &= -\langle P_0^{-1}u_t, u_t \rangle_{L^2(\mathbb{R}^N)} = -\|\nabla(P_0^{-1}u_t)\|_{L^2(\mathbb{R}^N)}^2 \\ &= -\|\nabla(\Delta u + f(\cdot, u))\|_{L^2(\mathbb{R}^N)}^2. \end{aligned} \quad (3.9)$$

- (ii) Assume the conditions for local existence as in Corollary 2.3. Suppose also that the assumptions of Lemma 3.4 are satisfied.

Then local solution u of (1.1)–(1.2) through $u_0 \in H^1(\mathbb{R}^N)$, as long as it exists, satisfies the estimate

$$\|u(t)\|_{H^1(\mathbb{R}^N)}^2 \leq c_1 E(u(t)) + c_2 \leq c_1 E(u_0) + c_2, \quad (3.10)$$

where c_1, c_2 are certain positive constants. Furthermore, $E(u(t))$ is a decreasing function of t and

$$\begin{aligned} E(u(t)) - E(u(s)) &= -\int_s^t \|\nabla(P_0^{-1}u_t)\|_{L^2(\mathbb{R}^N)}^2 = -\int_s^t \|\nabla(\Delta u + f(\cdot, u))\|_{L^2(\mathbb{R}^N)}^2 \\ &\leq E(u_0) + \frac{c_2}{c_1}, \quad t \geq s \geq 0, \quad u_0 \in H^1(\mathbb{R}^N). \end{aligned} \quad (3.11)$$

Proof. (i) Note that from Corollary 2.3, $u_t, \Delta(\Delta u + f(\cdot, u)) \in L^2(\mathbb{R}^N)$ and $\Delta u + f(\cdot, u) \in H^2(\mathbb{R}^N)$. Thus, we multiply the left-hand side of (3.2) by u_t and the right-hand side by $\Delta(\Delta u + f(\cdot, u)) = u_t$ and integrate in \mathbb{R}^N , to get $\langle P_0^{-1}u_t, u_t \rangle_{L^2(\mathbb{R}^N)} = \langle \Delta u + f(\cdot, u), \Delta(\Delta u + f(\cdot, u)) \rangle_{L^2(\mathbb{R}^N)}$. Writing now $\langle P_0^{-1}u_t, u_t \rangle_{L^2(\mathbb{R}^N)}$ as $\langle P_0^{-1}u_t, -\Delta P_0^{-1}u_t \rangle_{L^2(\mathbb{R}^N)}$ and integrating by parts we get (3.9).

- (ii) Using (3.9) we have that $E(u(t))$ is non-increasing and

$$E(u(t)) \leq E(u_0), \quad (3.12)$$

and (3.10) follows from (3.7) and (3.12). Finally, (3.10) ensures that $E(u(s)) \geq -\frac{c_2}{c_1}$ and (3.11) is a consequence of (3.9) and (3.12). \square

4. Global solutions and equilibria in $H^1(\mathbb{R}^N)$

In this section our concern is proving that solutions of (1.1)–(1.2), where $u_0 \in H^1(\mathbb{R}^N)$, given in Corollary 2.3, are globally defined in time. We also derive some properties of the equilibria, i.e. stationary solutions. For this we will only consider hereafter subcritical cases $\rho < \rho_c$ if $N \geq 3$.

4.1. Semigroup of solutions in $H^1(\mathbb{R}^N)$

Due to the continuation properties stated in Corollary 2.3 the following result is an immediate consequence of the $H^1(\mathbb{R}^N)$ -estimate in Section 3.

Corollary 4.1. Assume the conditions in Corollary 2.3, where if $N \geq 3$, we assume in addition that $\rho < \rho_c$. Suppose also that the assumptions of Lemma 3.4 are satisfied.

Then the local solution of (1.1)–(1.2) through $u_0 \in H^1(\mathbb{R}^N)$ as in Corollary 2.3, exists globally in time. Consequently, there is a C^0 semigroup associated to (1.1)–(1.2) in $H^1(\mathbb{R}^N)$ defined as

$$S(t)u_0 = u(t; u_0), \quad t \geq 0, \quad u_0 \in H^1(\mathbb{R}^N),$$

and $\{S(t): t \geq 0\}$ has bounded orbits of bounded sets.

Remark 4.2. Note that the above does not apply in the critical case, $N \geq 3$ and $\rho = \rho_c$, which requires a stronger estimate on the solution.

On the other hand, note that Lemma 3.5 leads to the following conclusion.

Corollary 4.3. The energy functional in (1.3) is a Lyapunov function for the semigroup $\{S(t): t \geq 0\}$ from Corollary 4.1; that is,

- (i) $E: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ is bounded below,
- (ii) $E(u_0) \rightarrow \infty$ as $\|u_0\|_{H^1(\mathbb{R}^N)} \rightarrow \infty$,
- (iii) $E(S(t)u_0)$ is non-increasing in t for each $u_0 \in H^1(\mathbb{R}^N)$,
- (iv) if $u_0 \in H^1(\mathbb{R}^N)$ is such that $S(t)u_0$ is defined for all $t \in \mathbb{R}$ and $E(S(t)u_0) = E(u_0)$ for $t \in \mathbb{R}$ then u_0 is a stationary solution (or equilibrium point), i.e. $S(t)u_0 = u_0$ for $t \geq 0$.

Proof. Note that (i) and (ii) follow from (3.7) while (iii) follows from (3.9). For (iv), if $E(S(t)u_0)$ is constant for $t \geq 0$, then from (3.11), $u_t = 0$ and thus $u(t) = u_0$ for all $t \geq 0$. \square

Consider now the set $\mathcal{E} \subset H^1(\mathbb{R}^N)$ of all stationary solutions of (1.1)–(1.2), that is $u_0 \in H^1(\mathbb{R}^N)$ such that the corresponding solution as in Corollary 2.3 is constant in time, i.e. $S(t)u_0 = u(t; u_0) = u_0$, $t \geq 0$. Then we have

Proposition 4.4. Under the assumption of Corollary 4.1, the set \mathcal{E} consists of the elements of $H^2(\mathbb{R}^N)$ that satisfy

$$-\Delta u = f(x, u), \quad x \in \mathbb{R}^N.$$

Moreover, \mathcal{E} is bounded in $H^1(\mathbb{R}^N)$.

Proof. If $u \in \mathcal{E}$, then by Corollary 2.3, $u \in H^2(\mathbb{R}^N)$, $\Delta u + f(x, u) \in H^2(\mathbb{R}^N)$ and $\Delta(\Delta u + f(x, u)) = 0$ in $L^2(\mathbb{R}^N)$. Thus, $\mathcal{E} \subset H^2(\mathbb{R}^N)$ and by Lemma 3.1, $\Delta u + f(x, u) = 0$.

Now we prove \mathcal{E} is bounded in $H^1(\mathbb{R}^N)$. For this we multiply both sides of $-\Delta u = f(x, u)$ by u in $L^2(\mathbb{R}^N)$ and use the structure condition (3.3) to get, for any $0 < \nu < 1$,

$$\nu \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} ((1 - \nu)|\nabla u|^2 - C(x)u^2) \leq \int_{\mathbb{R}^N} D(x)|u|.$$

Since $C(\cdot)$, $D(\cdot)$ satisfy (3.4)–(3.6), by Theorem A.2(iv), Lemma A.4 and Hölder's inequality, we have

$$\nu \int_{\mathbb{R}^N} |\nabla u|^2 + \omega(\nu) \int_{\mathbb{R}^N} u^2 \leq \|D\|_{L^s(\mathbb{R}^N)} \|u\|_{L^{s'}(\mathbb{R}^N)}.$$

Since s is defined in (3.4), we have $H^1(\mathbb{R}^N) \hookrightarrow L^{s'}(\mathbb{R}^N)$ and then we obtain

$$\nu \int_{\mathbb{R}^N} |\nabla u|^2 + \omega(\nu) \int_{\mathbb{R}^N} u^2 \leq \frac{c^2}{\nu} \|D\|_{L^s(\mathbb{R}^N)}^2 + \frac{\nu}{2} (\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|u\|_{L^2(\mathbb{R}^N)}^2).$$

Note that choosing $\nu > 0$ small enough we have $\omega(\nu) > \nu$ which allows us to conclude that

$$\frac{\nu}{2} \left(\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} u^2 \right) \leq \frac{\nu}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \left(\omega(\nu) - \frac{\nu}{2} \right) \int_{\mathbb{R}^N} u^2 \leq \frac{c^2}{\nu} \|D\|_{L^s(\mathbb{R}^N)}^2,$$

and the proof is complete. \square

As noted in the Introduction and as can be seen from Proposition 4.4 equilibria of the Cahn–Hilliard equation (1.1) coincide with equilibria of the Cahn–Allen equation (1.4). Therefore some of their properties of the former can be obtained from the latter. Below we summarize some properties of the reaction–diffusion equation (1.4) and its equilibria.

Proposition 4.5. *Under the same assumptions of Corollary 4.1 the reaction–diffusion equation*

$$\begin{aligned} u_t - \Delta u &= f(x, u), \quad t > 0, x \in \mathbb{R}^N, \\ u(0) &= u_0 \in H^1(\mathbb{R}^N), \end{aligned} \tag{4.1}$$

defines a semigroup $\{\tilde{S}(t), t \geq 0\}$ in $H^1(\mathbb{R}^N)$ and (1.3) is also a Lyapunov functional for $\{\tilde{S}(t), t \geq 0\}$.

Furthermore, assume that in (3.3) we have, instead of (3.4),

$$0 \leq D \in L^\sigma(\mathbb{R}^N) \cap L^s(\mathbb{R}^N), \quad \sigma > N/2, \quad \max \left\{ 1, \frac{2N}{N+2} \right\} \leq s \leq 2.$$

Then, this semigroup is asymptotically compact and therefore, \mathcal{E} is nonempty and there is even a compact global attractor $\tilde{\mathcal{A}}$ for $\{\tilde{S}(t), t \geq 0\}$ in $H^1(\mathbb{R}^N)$.

Moreover, there are two ordered extremal equilibria φ_m, φ_M in $H^1(\mathbb{R}^N)$, minimal and maximal respectively, so that any equilibrium ψ in $H^1(\mathbb{R}^N)$ satisfies

$$\varphi_m(x) \leq \psi(x) \leq \varphi_M(x), \quad x \in \mathbb{R}^N.$$

Proof. Local existence of solutions follows as in [4, Theorem 3.1].

Multiplying (4.1) by u_t in $L^2(\mathbb{R}^N)$ it is easy to see that the energy (1.3) is also a Lyapunov functional for the solutions of (4.1). Hence, from Lemma 3.4, we obtain $H^1(\mathbb{R}^N)$ -estimates on the local solutions, which, using that $\rho < \rho_c$ implies that the solutions are global and we have a well defined semigroup $\{\tilde{S}(t): t \geq 0\}$.

With the additional assumptions on D , asymptotic compactness follows as in [4, Theorems 5.1–5.5], while the existence of extremal equilibria follows as in [39, Theorem 3.1]. \square

Observe that after Corollary 4.3 and Proposition 4.4, the results in [31] would conclude the existence of a global attractor for (1.1)–(1.2) provided the semigroup $\{S(t): t \geq 0\}$ from Corollary 4.1 is asymptotically compact in $H^1(\mathbb{R}^N)$.

However we show below that this property is not expected to hold in general.

4.2. Lack of asymptotic compactness

For reaction–diffusion equations (1.4) in unbounded domains, the dissipative mechanism is based on a suitable cooperation of the linear and nonlinear parts of the equation, where a major role in this is played by structure conditions of the type (3.3)–(3.6), see e.g. [4] and [17].

However in what follows we show for (1.1)–(1.2) that no asymptotic compactness can be expected in general.

Proposition 4.6. *A linear map $f(x, u) = -\varepsilon u$, with $\varepsilon > 0$, satisfies (3.3)–(3.6) but there is no global attractor for*

$$u_t + \Delta^2 u - \varepsilon \Delta u = 0, \quad t > 0, \quad x \in \mathbb{R}^N. \quad (4.2)$$

Consequently, the corresponding linear semigroup is not asymptotically compact.

Proof. If for a certain $\varepsilon \geq 0$ the linear semigroup $\{T(t): t \geq 0\}$ associated with (4.2) has a global attractor in, let us say $H^1(\mathbb{R}^N)$, then it has a global attractor also in $L^2(\mathbb{R}^N)$ because of the smoothing effect.

On the other hand, the attractor reduces to the single equilibrium $\{0\}$. To see this, just note that for (4.2) we have $\mathcal{E} = \{0\}$ and, using the Lyapunov functional $E(u) = \frac{1}{2}(\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \varepsilon\|u\|_{L^2(\mathbb{R}^N)}^2)$, a complete orbit $\gamma(u_0) = \{T(t)u_0, t \in \mathbb{R}\}$ lying on the attractor must be forward and backward in time convergent to zero. Then the Lyapunov functional is constant along this orbit which proves that such an orbit coincides with $\{0\}$.

Hence, if the global attractor exists for (4.2), then $\mathcal{A} = \{0\}$ and must attract bounded sets, in particular the unit sphere, so that

$$\|T(t)\|_{\mathcal{L}(L^2(\mathbb{R}^N))} = \sup_{\|u_0\|_{L^2(\mathbb{R}^N)}=1} \|T(t)u_0\|_{L^2(\mathbb{R}^N)} \xrightarrow{t \rightarrow \infty} 0.$$

From this we infer that the spectral radius $r(T(t)) \xrightarrow{t \rightarrow \infty} 0$. Since the spectral radius $r(T(t))$ is equal to $e^{-\omega_0 t}$, where the ‘growth bound’ ω_0 coincides with the ‘bottom spectrum bound’ of the generator of $\{T(t): t \geq 0\}$ (see [22, Proposition IV.2.2 and Corollary IV.3.12]), we thus have that $\omega_0 > 0$, and that $\operatorname{Re} \sigma(\Delta^2 - \varepsilon \Delta)$ is strictly positive. Consequently, for $\phi \in H^2(\mathbb{R}^N)$, $\|\phi\|_{L^2(\mathbb{R}^N)}$ is bounded by a multiple of $\|(\Delta^2 - \varepsilon \Delta)^{\frac{1}{2}} \phi\|_{L^2(\mathbb{R}^N)}$ and

$$\begin{aligned} \|\phi\|_{L^2(\mathbb{R}^N)}^2 &\leq c(\|\Delta \phi\|_{L^2(\mathbb{R}^N)}^2 - \varepsilon \langle \Delta \phi, \phi \rangle_{L^2(\mathbb{R}^N)}) \\ &\leq c\|\Delta \phi\|_{L^2(\mathbb{R}^N)}^2 + \frac{\varepsilon^2 c^2}{2} \|\Delta \phi\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} \|\phi\|_{L^2(\mathbb{R}^N)}^2, \quad \phi \in H^2(\mathbb{R}^N). \end{aligned}$$

But this implies that $\|\phi\|_{L^2(\mathbb{R}^N)}$ is bounded by a multiple of $\|\Delta\phi\|_{L^2(\mathbb{R}^N)}$, which implies P_0^{-1} is continuous and contradicts part (ii) in Lemma 3.1.

In particular, the semigroup associated to (4.2) cannot be asymptotically compact. If it was, since it has bounded orbits of bounded sets, one-point set of equilibria and a Lyapunov function, then it would have a global attractor, see [31]. \square

5. Uniform bootstrapping: smoothing action of the nonlinear semigroup

In this section we obtain estimates of the solution of (1.1)–(1.2) through $u_0 \in H^1(\mathbb{R}^N)$ in stronger norms than in $H^1(\mathbb{R}^N)$, see (3.10). Our approach is based on the smoothing properties of the semigroup in Corollary 4.1.

5.1. $H^2(\mathbb{R}^N)$ -estimate away from $t = 0$

We start with the following lemma which states that the solution of (1.1)–(1.2) is bounded in $H^s(\mathbb{R}^N)$, $s < 2$, uniformly on unbounded time intervals away from $t = 0$ and for u_0 in bounded subsets of $H^1(\mathbb{R}^N)$.

Lemma 5.1. *Let $\{S(t): t \geq 0\}$ be the semigroup from Corollary 4.1.*

Then, for each $t > 0$ and $s < 2$, $S(t)$ maps bounded sets of $H^1(\mathbb{R}^N)$ into bounded sets of $H^s(\mathbb{R}^N)$. Consequently, positive orbits of bounded subsets of $H^1(\mathbb{R}^N)$ are immediately bounded in $H^s(\mathbb{R}^N)$ for any $s < 2$.

In particular the set of equilibria \mathcal{E} is bounded in $H^s(\mathbb{R}^N)$ for any $s < 2$.

Proof. Take u_0 in arbitrarily fixed ball B in $H^1(\mathbb{R}^N)$. Then by Corollary 4.1, the orbit $\{S(t)u_0, u_0 \in B, t \geq 0\}$ is bounded in $H^1(\mathbb{R}^N)$ and so it remains inside a ball in $H^1(\mathbb{R}^N)$ of radius $R > 0$ for some R .

Now note that from (2.5), for any $\theta < \frac{1}{4}$ and for any v_0 in the ball in $H^1(\mathbb{R}^N)$ of radius $R > 0$, there exists $t_0 = t_0(R)$ such that for any $0 < t \leq t_0$, we have

$$t^\theta \|S(t)v_0\|_{H^{1+4\theta}(\mathbb{R}^N)} \leq c(R).$$

Therefore, for any $\theta < \frac{1}{4}$ and v_0 as above, we have

$$\|S(t_0)v_0\|_{H^{1+4\theta}(\mathbb{R}^N)} \leq \tilde{C}(R).$$

In particular, taking $v_0 = S(t)u_0$ with $t > 0$ and $u_0 \in B$ we get that the set $\{S(t)u_0, u_0 \in B, t \geq t_0 > 0\}$ is bounded in $H^{1+4\theta}(\mathbb{R}^N)$, for any $\theta < \frac{1}{4}$. In particular the set of equilibria is bounded in $H^{1+4\theta}(\mathbb{R}^N)$, for any $\theta < \frac{1}{4}$. Also observe that t_0 above can be chosen as small as we want. \square

We now strengthen the result of Lemma 5.1 as follows.

Lemma 5.2. *Let $\{S(t): t \geq 0\}$ be the semigroup from Corollary 4.1.*

Then, for each $t > 0$, $S(t)$ maps bounded sets of $H^1(\mathbb{R}^N)$ into bounded sets of $H^2(\mathbb{R}^N)$. Consequently, positive orbits of bounded subsets of $H^1(\mathbb{R}^N)$ are immediately bounded in $H^2(\mathbb{R}^N)$.

Proof. If B is bounded in $H^1(\mathbb{R}^N)$ and $t > 0$ then for any $\delta \in (0, t)$, by Lemma 5.1, $S(t - \delta)B$ is bounded in $H^{1+4\theta}(\mathbb{R}^N)$ for any $\theta < \frac{1}{4}$. On the other hand, from the proof of Theorem 2.1 in Appendix B, the nonlinear term $\Delta(f(\cdot, u))$ in (1.1) is Lipschitz continuous on bounded sets from $H^{1+4\bar{\varepsilon}}(\mathbb{R}^N)$ into $H^{-2}(\mathbb{R}^N)$, for some $\bar{\varepsilon}$ which is less although close to $\frac{1}{4}$. Thus, using Remark B.4(i) of Appendix B and applying Theorem 5 in [12], we observe that $S(\delta)$ takes bounded subsets of $H^{1+4\bar{\varepsilon}}(\mathbb{R}^N)$ into bounded subsets of $H^2(\mathbb{R}^N)$. Hence $S(t)B = S(\delta)(S(t - \delta)B)$ is bounded in $H^2(\mathbb{R}^N)$.

Note that Theorem 5 in [12] applies because in the above scenario Δ^2 is a sectorial operator in the space $\mathcal{X} = H^{-2}(\mathbb{R}^N)$ (see Theorem A.2), whereas the nonlinear term $\Delta(f(\cdot, u))$ is Lipschitz continuous

map from the fractional power space $\mathcal{X}^{\frac{3}{4}+\varepsilon} = H^{1+4\varepsilon}(\mathbb{R}^N)$ into $\mathcal{X} = H^{-2}(\mathbb{R}^N)$ (in terms of the scale from Theorem A.2 we have $H^{-2}(\mathbb{R}^N) = E_p^{-\frac{1}{2}}(2)$ and $H^{1+4\varepsilon}(\mathbb{R}^N) = E_p^{\frac{1}{4}+\varepsilon}(2)$). \square

The result in Lemma 5.2 can be rephrased as follows.

Corollary 5.3. *Under the assumptions of Corollary 4.1 the solutions $u(\cdot, u_0)$ of (1.1)–(1.2) through $u_0 \in H^1(\mathbb{R}^N)$ are bounded in $H^2(\mathbb{R}^N)$ uniformly on unbounded time intervals away from zero and for u_0 in bounded subsets of $H^1(\mathbb{R}^N)$.*

In particular, the set of equilibria, \mathcal{E} , is bounded in $H^2(\mathbb{R}^N)$.

On the other hand, we also obtain the following conclusion.

Corollary 5.4. *Under the assumptions of Corollary 4.1, if $u(t) = u(t, u_0)$ is the solution of (1.1)–(1.2) through $u_0 \in H^1(\mathbb{R}^N)$ then both Δu and $f(\cdot, u)$ are bounded in $L^2(\mathbb{R}^N)$ uniformly on unbounded time intervals away from zero and for u_0 in bounded subsets of $H^1(\mathbb{R}^N)$.*

Proof. We evidently have that $\|\Delta u\|_{L^2(\mathbb{R}^N)} \leq \|u\|_{H^2(\mathbb{R}^N)}$. Hence it is sufficient to prove boundedness of $\|f(\cdot, u)\|_{L^2(\mathbb{R}^N)}$.

First observe that if $N \leq 3$, we have $H^2(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ and thus, because of the $H^2(\mathbb{R}^N)$ -estimate in Lemma 5.2,

$$\forall_{t_0 > 0} \exists_{a_0 > 0} \forall_{t \geq t_0 > 0} \forall_{x \in \mathbb{R}^N} |u(t, x)| \leq a_0.$$

Choosing then a Lipschitz constant L_0 for $f_0(x, s)$ uniform for $x \in \mathbb{R}^N$ and $s \in [-a_0, a_0]$ we get

$$\|f_0(\cdot, u(t))\|_{L^2(\mathbb{R}^N)} = \|f_0(\cdot, u(t)) - f_0(\cdot, 0)\|_{L^2(\mathbb{R}^N)} \leq L_0 \|u(t)\|_{L^2(\mathbb{R}^N)}.$$

Now if $N \geq 4$, using decomposition $f_0 = f_{01} + f_{02}$ as in Lemma B.1, we obtain

$$\begin{aligned} \|f_{02}(\cdot, u(t))\|_{L^2(\mathbb{R}^N)} &= \|f_{02}(\cdot, u(t)) - f_{02}(\cdot, 0)\|_{L^2(\mathbb{R}^N)} \leq c \|u(t)\|_{L^2(\mathbb{R}^N)}^\rho \\ &= c \|u(t)\|_{L^{2\rho}(\mathbb{R}^N)}^\rho. \end{aligned} \quad (5.1)$$

From Sobolev embeddings $H^2(\mathbb{R}^N) \hookrightarrow L^{2\rho}(\mathbb{R}^N)$ for any $\rho \in (1, \infty)$ if $N = 4$ and for $\rho \leq \frac{N}{N-4}$ if $N > 4$, which hold by the growth assumptions in Corollary 2.3. Hence from (5.1) we get

$$\|f_{02}(\cdot, u(t))\|_{L^2(\mathbb{R}^N)} \leq c \|u(t)\|_{H^2(\mathbb{R}^N)}^\rho.$$

On the other hand, for f_{01} , due to (B.3), we have

$$\forall_{t_0 > 0} \exists_{L > 0} \forall_{t \geq t_0} \|f_{01}(\cdot, u(t))\|_{L^2(\mathbb{R}^N)} = \|f_{01}(\cdot, u(t)) - f_{01}(\cdot, 0)\|_{L^2(\mathbb{R}^N)} \leq L \|u(t)\|_{L^2(\mathbb{R}^N)}.$$

Recalling (1.5), we now need to ensure boundedness of $\|mu(t)\|_{L^2(\mathbb{R}^N)}$. For this we refer to Lemma A.1 which, due to the assumption that $r \geq 2$, $r > \max\{\frac{N}{2}, 1\}$, we apply here with $k = 1$ and $\beta = 0$ to get

$$\|mu(t)\|_{L^2(\mathbb{R}^N)} \leq \|m\|_{L^r_U(\mathbb{R}^N)} \|u(t)\|_{H^{2\alpha}(\mathbb{R}^N)},$$

for any $\alpha < 1$ close enough to 1.

From all the above it is clear that, whenever $t_0 > 0$ and B is bounded in $H^1(\mathbb{R}^N)$, there is a constant $C > 0$ independent of $t \geq t_0$ and $u_0 \in B$ such that $\|f(\cdot, u(t))\|_{L^2(\mathbb{R}^N)} \leq C$ for all $t \geq t_0$. \square

5.2. Estimates in $H_p^2(\mathbb{R}^N)$ with $p > 2$, away from $t = 0$

In Section 5.1 we assumed that g in (1.5) is in $L^2(\mathbb{R}^N)$. With further assumption on g the bootstrapping procedure allows us to obtain uniform bounds on the solutions in some other Bessel potential spaces, which depend as well on the local integrability properties of the ‘potential’ term m . Note that the results below use a combination of both Hilbert space techniques, that gives us an ‘energy’ estimate, and Banach space techniques, which enables to better exhibit uniform smoothing properties of the nonlinear semigroup. For similar results, see [27].

Lemma 5.5. *If besides the assumptions of Corollary 4.1 we have that*

$$g \in L^q(\mathbb{R}^N) \quad \text{for some } 2 < q \leq \infty,$$

then, for any $2 \leq p \leq \min\{r, q\}$, the solutions $u(\cdot, u_0)$ of (1.1)–(1.2) through $u_0 \in H^1(\mathbb{R}^N)$ are bounded in $H_p^2(\mathbb{R}^N)$ uniformly on unbounded time intervals away from $t = 0$ and for u_0 in bounded subsets of $H^1(\mathbb{R}^N)$.

Proof. Note that, by assumption, g belongs to $L^s(\mathbb{R}^N)$ for every $s \in [2, q]$. Also let $p_0 := 2$.

If $p_0 = \min\{r, q\}$ then the result follows from Lemma 5.2.

Thus assume that $p_0 < \min\{r, q\}$ and choose any $\tau > 0$ and B bounded in $H_{p_0}^1(\mathbb{R}^N)$. From Lemma 5.2 we have that $S(\tau)\gamma^+(B)$ is bounded in $H_{p_0}^2(\mathbb{R}^N) \hookrightarrow H_{p_1}^1(\mathbb{R}^N)$, where either $p_1 = \frac{Np_0}{N-p_0} > p_0$ if $N > p_0$, or p_1 is arbitrary from the interval $[p_0, \infty)$ if $N \leq p_0$.

When $p_1 < \min\{r, q\}$ we can restart the solutions using Theorem 2.1 with $p = p_1 > p_0$.

Then, using (2.4), analogously as in the proof of Lemma 5.1 we ensure that for each sufficiently small $\tilde{\tau}$ the set $S(\tau + \tilde{\tau})\gamma^+(B)$ is bounded in $H_{p_1}^s(\mathbb{R}^N)$ whenever $s < 2$. Consequently, using Remark B.4(i) of Appendix B and [12, Theorem 5], we deduce similarly as in the proof of Lemma 5.2 that for each sufficiently small $\hat{\tau}$ the set $S(\tau + \tilde{\tau} + \hat{\tau})\gamma^+(B)$ is bounded in $H_{p_1}^2(\mathbb{R}^N) \hookrightarrow H_{p_2}^1(\mathbb{R}^N)$, where $p_2 = \frac{Np_1}{N-p_1}$ if $N > p_1$, or p_2 is any number from $[p_1, \infty)$ if $N \leq p_1$.

When $p_2 < \min\{r, q\}$ we repeat the above procedure, which actually can be carried out as long as we get $p_k < \min\{r, q\}$. As for the values p_1, \dots, p_k , they will all satisfy the relation $p_k = \frac{Np_{k-1}}{N-p_{k-1}} > p_0$.

Actually, since p_k is strictly increasing with respect to k , there exists a certain $k_0 \geq 1$ such that $p_{k_0} \geq \min\{r, q\} > p_{k_0-1}$. Otherwise the sequence $\{p_k\}$ would have a limit p^* satisfying $p^* = \frac{Np^*}{N-p^*}$, which is impossible.

Consequently, choosing any $p \in [2, \min\{r, q\}]$ and proceeding as above, we ‘reach’ the bound in $H_p^2(\mathbb{R}^N)$ in a finite number of steps. \square

6. Convergence to equilibria

In this section we prove that even without the asymptotic compactness for the solutions of (1.1)–(1.2), see Section 4.2, still there is a weak form of dissipation in the equation that makes individual solutions to approach equilibria. More precisely, we have

Theorem 6.1. *Let $\{S(t) : t \geq 0\}$ be the semigroup in $H^1(\mathbb{R}^N)$ from Corollary 4.1.*

Then the set of equilibria \mathcal{E} is nonempty and for each $u_0 \in H^1(\mathbb{R}^N)$ and any sequence $t_n \rightarrow \infty$, there are a subsequence $\{t_{n_k}\}$ and an equilibrium $\psi \in \mathcal{E}$, such that as $k \rightarrow \infty$,

$$u(t_{n_k}) \rightarrow \psi \quad \text{in } H_{\text{loc}}^s(\mathbb{R}^N) \text{ and in } H_\phi^s(\mathbb{R}^N) \quad \text{for any } s < 2$$

where the weight is given by $\phi(x) = (1 + |x|^2)^{-\nu}$ with $\nu > \frac{N}{2}$.

Furthermore, as in Proposition 4.5, assume that in (3.3) we have, instead of (3.4),

$$D \in L^\sigma(\mathbb{R}^N) \cap L^s(\mathbb{R}^N), \quad \sigma > N/2, \quad \max\left\{1, \frac{2N}{N+2}\right\} \leq s \leq 2.$$

Then

- (i) there are two ordered extremal equilibria φ_m, φ_M in $H^1(\mathbb{R}^N)$, minimal and maximal respectively, so that any equilibrium ψ in $H^1(\mathbb{R}^N)$ satisfies

$$\varphi_m(x) \leq \psi(x) \leq \varphi_M(x), \quad x \in \mathbb{R}^N,$$

- (ii) the order interval $[\varphi_m, \varphi_M]_{H^1(\mathbb{R}^N)}$ attracts ‘pointwise asymptotic dynamics’ of (1.1) in the sense that for each $u_0 \in H^1(\mathbb{R}^N)$ and any sequence $t_n \rightarrow \infty$, there is a subsequence $\{t_{n_k}\}$ such that

$$\varphi_m(x) \leq \lim_{k \rightarrow \infty} u(t_{n_k}, x; u_0) \leq \varphi_M(x)$$

for a.e. $x \in \mathbb{R}^N$.

Proof. Note that once the first statement is proved, under the additional assumptions on $D(x)$, Proposition 4.5 allows to prove parts (i) and (ii).

We thus focus on the proof of statement about the convergence to equilibria, which resembles [6, §3.5].

Let us fix $u_0 \in H^1(\mathbb{R}^N)$ and a sequence of times $t_n \rightarrow \infty$. Let also u be the solution of (1.1) through u_0 . Our goal is to show that $\{u(t_n)\}$ has a subsequence convergent to equilibrium. This will be proved in two steps. First we will define some auxiliary sequence \tilde{t}_n and prove that this property holds for $\{u(\tilde{t}_n)\}$. After this is done we will transfer this information to the sequence $\{u(t_n)\}$.

Step 1. Let us define an auxiliary sequence \tilde{t}_n such that $\{u(\tilde{t}_n)\}$ will possess a subsequence convergent to equilibrium.

From (3.11) we have $\int_0^\infty \|\nabla(\Delta u + f(\cdot, u))\|_{L^2(\mathbb{R}^N)}^2 dt < \infty$ which implies that

$$\int_{t_n-1}^{t_n-\frac{1}{2}} \|\nabla(\Delta u + f(\cdot, u))\|_{L^2(\mathbb{R}^N)}^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, we can assume that for some sequence

$$\tilde{t}_n \in \left[t_n - 1, t_n - \frac{1}{2}\right], \quad n \in \mathbb{N}, \quad (6.1)$$

we have

$$\|\nabla G_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (6.2)$$

where $G_n = \Delta u(\tilde{t}_n) + f(\cdot, u(\tilde{t}_n))$. Also from Corollary 5.4 this sequence is bounded in $L^2(\mathbb{R}^N)$.

Using again that $u(\tilde{t}_n)$ is bounded in $H^2(\mathbb{R}^N)$, taking subsequences if necessary, we can assume that there exists $\varphi \in H^2(\mathbb{R}^N)$ such that

$$\begin{aligned} u(\tilde{t}_n) &\rightharpoonup \varphi, \quad \text{weakly in } H^2(\mathbb{R}^N), H^1(\mathbb{R}^N) \text{ and } L^2(\mathbb{R}^N), \\ \Delta u(\tilde{t}_n) &\rightharpoonup \Delta \varphi, \quad f(\cdot, u(\tilde{t}_n)) \rightharpoonup F, \quad \text{weakly in } L^2(\mathbb{R}^N), \\ G_n &\rightharpoonup \Delta \varphi + F \quad \text{weakly in } H^1(\mathbb{R}^N) \end{aligned}$$

for some $F \in L^2(\mathbb{R}^N)$. This and (6.2) imply that $\nabla(\Delta\varphi + F) = 0$ and since $\Delta\varphi + F \in L^2(\mathbb{R}^N)$ we get

$$\Delta\varphi + F = 0.$$

Now fix the ball $B(0, R)$ in \mathbb{R}^N and use that $u(\tilde{t}_n)$ is bounded in $H^2(B(0, R))$ and therefore for any subsequence of \tilde{t}_n and for any $s < 2$, there is another subsequence, that we denote $\tilde{t}_{n'}$, such that $u(\tilde{t}_{n'}) \rightarrow \varphi$ in $H^s(B(0, R))$. Since the limit is always φ , independent of the subsequence $\tilde{t}_{n'}$, we get in fact that

$$u(\tilde{t}_n) \rightarrow \varphi \quad \text{in } H^s(B(0, R)).$$

In particular, taking a subsequence if necessary, we can assume $u(\tilde{t}_n) \rightarrow \varphi$ a.e. in $B(0, R)$ and therefore, by the assumptions on f , we have

$$f(\cdot, u(\tilde{t}_n)) \rightarrow f(\cdot, \varphi) \quad \text{a.e. in } B(0, R)$$

for any $R > 0$. Hence we use $f(\cdot, u(\tilde{t}_n)) \rightarrow F$ weakly in $L^2(\mathbb{R}^N)$ and Lemma 4.8 in [29] to obtain that $F = f(\cdot, \varphi)$ in $B(0, R)$ for any $R > 0$.

In summary

$$u(\tilde{t}_n) \rightarrow \varphi \quad \text{in } H_{\text{loc}}^s(\mathbb{R}^N) \tag{6.3}$$

with $\Delta\varphi + f(\cdot, \varphi) = 0$ in \mathbb{R}^N and $\varphi \in H^2(\mathbb{R}^N)$. That is $\varphi \in \mathcal{E}$.

Step 2. Observe that in (6.3) local convergence is obtained for (a subsequence of) the sequence in (6.1). Now we show that this information can be transferred to the original sequence t_n . For this we set the solution of (1.1)–(1.2) in a larger weighted space, in which we can use the variation of constants formula. Hence we will now show that, from (6.1) and (6.3), whenever $s < 2$, we have

$$u(t_n) \rightarrow \varphi \quad \text{in } H_\phi^s(\mathbb{R}^N),$$

$\phi(x) = (1 + |x|^2)^{-\nu}$ and $\nu > \frac{N}{2}$.

Define $\tau_n = t_n - \tilde{t}_n$ which, from (6.1),

$$\tau_n \in \left[\frac{1}{2}, 1 \right].$$

Then observe that

$$u(t_n) - \varphi = u(\tau_n, u(\tilde{t}_n)) - \varphi. \tag{6.4}$$

Also, due to compactness of the embedding of $H^2(\mathbb{R}^N)$ into the weighted space $L_\phi^2(\mathbb{R}^N)$, where $\phi(x) = (1 + |x|^2)^{-\frac{\nu}{2}}$ and $\nu > \frac{N}{2}$ (see [5]), there is a subsequence of $\{u(\tilde{t}_n)\}$, which for simplicity of the notation we will again denote $\{u(\tilde{t}_n)\}$, such that

$$u(\tilde{t}_n) \rightarrow \varphi \quad \text{in } L_\phi^2(\mathbb{R}^N). \tag{6.5}$$

Now we rewrite (1.1) as

$$u_t + A^2 u - A(f(\cdot, u) + 2\lambda_0 u) + \lambda_0(f(\cdot, u) + \lambda_0 u) = 0$$

with $A = -\Delta + \lambda_0 I$ for some $\lambda_0 > 0$ and use initial data

$$u_{0n} = u(\tilde{t}_n), \quad (6.6)$$

so using the corresponding variation of constants formula, the solution $u_n(t) = u(t, u_{0n})$ reads

$$u_n(t) = e^{-A^2 t} u_{0n} + \int_0^t e^{-A^2(t-s)} (A(f(\cdot, u(s)) + 2\lambda_0 u(s)) - \lambda_0(f(\cdot, u(s)) + \lambda_0 u(s))) ds.$$

Also, since φ is an equilibrium,

$$\varphi = e^{-A^2 t} \varphi + \int_0^t e^{-A^2(t-s)} (A(f(\cdot, \varphi) + 2\lambda_0 \varphi) - \lambda_0(f(\cdot, \varphi) + \lambda_0 \varphi)) ds,$$

and we get, after some manipulation,

$$\begin{aligned} u_n(t) - \varphi &= e^{-A^2 t} (u_{0n} - \varphi) + \int_0^t A e^{-A^2(t-s)} (f(\cdot, u_n(s)) - f(\cdot, \varphi)) ds \\ &\quad + 2\lambda_0 \int_0^t A e^{-A^2(t-s)} (u_n(s) - \varphi) ds - \lambda_0 \int_0^t e^{-A^2(t-s)} (f(\cdot, u_n(s)) - f(\cdot, \varphi)) ds \\ &\quad - \lambda_0^2 \int_0^t e^{-A^2(t-s)} (u_n(s) - \varphi) ds. \end{aligned} \quad (6.7)$$

Note that $A = -\Delta + \lambda_0 I$ is sectorial in $L_\phi^2(\mathbb{R}^N)$ (see [5]) and that we have a freedom to choose $\lambda_0 > 0$ such that for the spectrum in $L_\phi^2(\mathbb{R}^N)$ we have $\operatorname{Re}(\sigma(-\Delta + \lambda_0 I)) > 0$. Also note that the square of this operator, $B = A^2$, is also sectorial in $L_\phi^2(\mathbb{R}^N)$ (see [30]). In fact, due to [5, Theorem 5.1 and Proposition 5.1], A in $L_\phi^2(\mathbb{R}^N)$ is of type (ω, M) in the sense of [30] for arbitrarily small positive ω . The results in [30] now imply that $B = A^2$ in $L_\phi^2(\mathbb{R}^N)$ will be of type $(\hat{\omega}, \hat{M})$ with $\hat{\omega} = 2\omega$, thus with $\hat{\omega} \in (0, \frac{\pi}{2})$, so that $B = A^2$ in $L_\phi^2(\mathbb{R}^N)$ is a sectorial operator as in [26].

Hence $-B$ generates a C^0 analytic semigroup $\{e^{-Bt} : t \geq 0\}$ in $L_\phi^2(\mathbb{R}^N)$ satisfying

$$\|e^{-Bt}\|_{\mathcal{L}(L_\phi^2(\mathbb{R}^N))} \leq M, \quad t \in [0, 1], \quad (6.8)$$

and

$$\|B^\alpha e^{-Bt}\|_{\mathcal{L}(L_\phi^2(\mathbb{R}^N))} \leq \frac{M}{t^\alpha}, \quad t \in (0, 1]. \quad (6.9)$$

On the other hand, from the results of [30] we also have

$$B^\alpha = A^{2\alpha}, \quad \alpha > 0 \quad (6.10)$$

(see also [15, (1.3.49)]). In particular $A = B^{1/2}$.

On the other hand, [5, Theorem 5.1] ensures that for any $s \in (0, 1)$, the norm $\|A^s v\|_{L^2_\phi(\mathbb{R}^N)}$ is equivalent to $\|\phi^{\frac{1}{2}} v\|_{H^{2s}(\mathbb{R}^N)}$. Hence, from (6.10) we get that $\|B^\alpha v\|_{L^2_\phi(\mathbb{R}^N)}$ is equivalent to $\|\phi^{\frac{1}{2}} v\|_{H^{4\alpha}(\mathbb{R}^N)}$ for $\alpha \in (0, \frac{1}{2})$ and in what follows we will use the equality of norms

$$\|B^\alpha v\|_{L^2_\phi(\mathbb{R}^N)} = \|\phi^{\frac{1}{2}} v\|_{H^{4\alpha}(\mathbb{R}^N)}, \quad \alpha \in \left(0, \frac{1}{2}\right). \quad (6.11)$$

Hence, using (6.10) with $\alpha = \frac{1}{2}$ we rewrite (6.7) as

$$\begin{aligned} u_n(t) - \varphi &= e^{-Bt}(u_{0n} - \varphi) + \int_0^t B^{\frac{1}{2}} e^{-B(t-s)} (f(\cdot, u_n(s)) - f(\cdot, \varphi)) ds \\ &\quad + 2\lambda_0 \int_0^t B^{\frac{1}{2}} e^{-B(t-s)} (u_n(s) - \varphi) ds - \lambda_0 \int_0^t e^{-B(t-s)} (f(\cdot, u_n(s)) - f(\cdot, \varphi)) ds \\ &\quad - \lambda_0^2 \int_0^t e^{-B(t-s)} (u_n(s) - \varphi) ds. \end{aligned} \quad (6.12)$$

We will now use (6.8), (6.9) and (6.12) to estimate $\|B^\alpha(u_n(t) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)}$ with any $\alpha \in (0, \frac{1}{2})$ and $t \in (0, 1]$, as follows

$$\begin{aligned} \|B^\alpha(u_n(t) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)} &\leq \frac{M}{t^\alpha} \|u_{0n} - \varphi\|_{L^2_\phi(\mathbb{R}^N)} + \int_0^t \frac{M}{(t-s)^{\frac{1}{2}+\alpha}} \|f(\cdot, u_n(s)) - f(\cdot, \varphi)\|_{L^2_\phi(\mathbb{R}^N)} ds \\ &\quad + 2\lambda_0 \int_0^t \frac{M}{(t-s)^{\frac{1}{2}+\alpha}} \|u_n(s) - \varphi\|_{L^2_\phi(\mathbb{R}^N)} ds \\ &\quad + \lambda_0 \int_0^t \frac{M}{(t-s)^\alpha} \|f(\cdot, u_n(s)) - f(\cdot, \varphi)\|_{L^2_\phi(\mathbb{R}^N)} ds \\ &\quad + \lambda_0^2 \int_0^t \frac{M}{(t-s)^\alpha} \|u_n(s) - \varphi\|_{L^2_\phi(\mathbb{R}^N)} ds. \end{aligned} \quad (6.13)$$

If $N \leq 3$ from the bounds in $H^2(\mathbb{R}^N)$ we have

$$\forall t \geq t_0 > 0 \quad \forall x \in \mathbb{R}^N \quad |u(t, x)|, |\varphi(x)| \leq a_0,$$

and choosing Lipschitz constant L for $f_0(x, s)$ with $s \in [-a_0, a_0]$ we get

$$\|f_0(\cdot, u_n(s)) - f_0(\cdot, \varphi)\|_{L^2_\phi(\mathbb{R}^N)} \leq L \|u_n(s) - \varphi\|_{L^2_\phi(\mathbb{R}^N)} \leq c \|B^\alpha(u_n(s) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)}.$$

If $N \geq 4$ we use the decomposition $f_0 = f_{01} + f_{02}$ as in Lemma B.1 to get both

$$\|f_{01}(\cdot, u_n(s)) - f_{01}(\cdot, \varphi)\|_{L^2_\phi(\mathbb{R}^N)} \leq L_1 \|u_n(s) - \varphi\|_{L^2_\phi(\mathbb{R}^N)} \leq c \|B^\alpha(u_n(s) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)}$$

and

$$\begin{aligned} \|f_{02}(\cdot, u_n(s)) - f_{02}(\cdot, \varphi)\|_{L^2_\phi(\mathbb{R}^N)} &\leq c \| |u_n(s) - \varphi| (|u_n(s)|^{\rho-1} + |\varphi|^{\rho-1}) \phi^{\frac{1}{2}} \|_{L^2(\mathbb{R}^N)} \\ &\leq c \|\phi^{\frac{1}{2}}(u_n(s) - \varphi)\|_{L^{\frac{2N}{N-8\alpha}}(\mathbb{R}^N)} \| |u_n(s)|^{\rho-1} + |\varphi|^{\rho-1} \|_{L^{\frac{N}{4\alpha}}(\mathbb{R}^N)}. \end{aligned}$$

From Sobolev embeddings we have $H^{4\alpha}(\mathbb{R}^N) \hookrightarrow L^{\frac{N(\rho-1)}{4\alpha}}(\mathbb{R}^N)$ whenever $2 \leq \frac{N(\rho-1)}{4\alpha} \leq \frac{2N}{N-8\alpha}$, which translates into the condition $1 + \frac{8\alpha}{N} \leq \rho \leq \frac{N}{N-8\alpha}$. Here we can assume $1 + \frac{8\alpha}{N} \leq \rho$ since otherwise ρ can be increased, if necessary, to a value close enough to $\rho_c = \frac{N+2}{N-2}$. On the other hand, for α suitably close to $\frac{1}{2}$ we have that $\rho \leq \frac{N}{N-8\alpha}$ since $\rho_c < \frac{N}{N-4}$. Thus, for $\alpha < \frac{1}{2}$ close enough to $\frac{1}{2}$ we obtain

$$\begin{aligned} &\|f_0(\cdot, u_n(s)) - f_0(\cdot, \varphi)\|_{L^2_\phi(\mathbb{R}^N)} \\ &\leq c \|\phi^{\frac{1}{2}}(u_n(s) - \varphi)\|_{H^{4\alpha}(\mathbb{R}^N)} (\|u_n(s)\|_{H^{4\alpha}(\mathbb{R}^N)}^{\rho-1} + \|\varphi\|_{H^{4\alpha}(\mathbb{R}^N)}^{\rho-1}). \end{aligned} \quad (6.14)$$

Again from the bounds in $H^2(\mathbb{R}^N)$, see Lemma 5.2, for any $\alpha \in (0, \frac{1}{2})$ we have the estimate of the form

$$\forall t \geq t_0 > 0 \quad \|u(t)\|_{H^{4\alpha}(\mathbb{R}^N)}^{\rho-1} + \|\varphi\|_{H^{4\alpha}(\mathbb{R}^N)}^{\rho-1} \leq \tilde{c}$$

and using (6.11) we get from (6.14)

$$\|f_0(\cdot, u_n(s)) - f_0(\cdot, \varphi)\|_{L^2_\phi(\mathbb{R}^N)} \leq c\tilde{c} \|\phi^{\frac{1}{2}}(u_n(s) - \varphi)\|_{H^{4\alpha}(\mathbb{R}^N)} \leq \hat{c} \|B^\alpha(u_n(s) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)}.$$

Now for the linear term in f , note that if $r = \infty$, that is when $m \in L^\infty(\mathbb{R}^N)$, we immediately have

$$\begin{aligned} \|m(u_n(s) - \phi)\|_{L^2_\phi(\mathbb{R}^N)} &\leq \|m\|_{L^\infty(\mathbb{R}^N)} \|u_n(s) - \phi\|_{L^2_\phi(\mathbb{R}^N)} \\ &\leq c \|B^\alpha(u_n(s) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)}. \end{aligned}$$

If $r < \infty$ we will also obtain below that

$$\|m(u_n(s) - \phi)\|_{L^2_\phi(\mathbb{R}^N)} \leq c \|B^\alpha(u_n(s) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)}$$

although with a more delicate argument. Namely, we set $q(r) = \frac{2r}{r-2}$ if $\infty > r > 2$, $q(r) = \infty$ if $r = 2$ and cover \mathbb{R}^N with cubes Q_i , $i \in \mathbb{Z}^N$, centered at $i \in \mathbb{Z}^N$ and having unitary edges parallel to the axes. Thus we have $\mathbb{R}^N = \bigcup_{i \in \mathbb{Z}^N} \overline{Q_i}$, where $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and $H^{4\alpha}(Q_i) \hookrightarrow L^{q(r)}(Q_i)$ for $\alpha < \frac{1}{2}$ close enough to $\frac{1}{2}$ as $r > \frac{N}{2}$. Consequently, we obtain

$$\begin{aligned}
\|m(u_n(s) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)}^2 &= \|m(u_n(s) - \varphi)\phi^{\frac{1}{2}}\|_{L^2(\mathbb{R}^N)}^2 \\
&= \sum_{i \in \mathbb{Z}^N} \|m\|_{L^r(Q_i)}^2 \|(u_n(s) - \varphi)\phi^{\frac{1}{2}}\|_{L^{q(r)}(Q_i)}^2 \\
&\leq \|m\|_{L^r_U(\mathbb{R}^N)}^2 \sum_{i \in \mathbb{Z}^N} \|(u_n(s) - \varphi)\phi^{\frac{1}{2}}\|_{H^{4\alpha}(Q_i)}^2.
\end{aligned} \tag{6.15}$$

Since, analogously as in [4, Lemma 2.4] we have

$$\left(\sum_{i \in \mathbb{Z}^N} \|\chi\|_{H^{4\gamma}_q(Q_i)}^q \right)^{\frac{1}{q}} \leq c' \|\chi\|_{H^{4\gamma}_q(\mathbb{R}^N)}, \quad \gamma \in (0, 1), \quad q \in (1, \infty),$$

then (6.15) extends, using (6.11), to the estimate

$$\begin{aligned}
\|m(u_n(s) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)}^2 &\leq c' \|m\|_{L^r_U(\mathbb{R}^N)}^2 \|(u_n(s) - \varphi)\phi^{\frac{1}{2}}\|_{H^{4\alpha}(\mathbb{R}^N)}^2 \\
&\leq c \|B^\alpha(u_n(s) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)}.
\end{aligned} \tag{6.16}$$

Since we also have $1 < \frac{1}{(t-s)^2}$ for all $1 \geq t > s > 0$, then (6.13)–(6.16) imply, for $t \in [0, 1]$,

$$\|B^\alpha(u_n(t) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)} \leq \frac{M}{t^\alpha} \|u_{0n} - \varphi\|_{L^2_\phi(\mathbb{R}^N)} + \int_0^t \frac{C}{(t-s)^{\frac{1}{2}+\alpha}} \|B^\alpha(u_n(s) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)} ds,$$

and, using the Singular Gronwall Lemma, see [26, Lemma 7.1.1], we get

$$\sup_{t \in [\frac{1}{2}, 1]} t^\alpha \|B^\alpha(u_n(t) - \varphi)\|_{L^2_\phi(\mathbb{R}^N)} \leq \tilde{c} \|u_{0n} - \varphi\|_{L^2_\phi(\mathbb{R}^N)}, \tag{6.17}$$

for some constant $\tilde{c} > 0$ and $\alpha < \frac{1}{2}$ close enough to $\frac{1}{2}$.

Connecting now (6.4)–(6.6), (6.17) and (6.11) we conclude that for $\alpha < \frac{1}{2}$ close enough to $\frac{1}{2}$

$$\|\phi^{\frac{1}{2}}(u(t_n) - \varphi)\|_{H^{4\alpha}(\mathbb{R}^N)} \rightarrow 0.$$

This, in turn, implies the convergence in $H^s_\phi(\mathbb{R}^N)$, for $s < 2$ as in the statement. \square

We now conclude that, if Lemma 5.5 applies, then convergence to equilibrium in Theorem 6.1 actually holds in some stronger norms.

Corollary 6.2. *Let $\{S(t) : t \geq 0\}$ be the semigroup in $H^1(\mathbb{R}^N)$ from Corollary 4.1 and suppose that $g \in L^q(\mathbb{R}^N)$ for some $2 < q \leq \infty$.*

Then, in Theorem 6.1, $\{u(t_{n_k})\}$ converges to an equilibrium in $H^s_{p,\text{loc}}(\mathbb{R}^N)$ for any $s < 2$ and $2 \leq p \leq \min\{r, q\}$.

In particular, whenever $\min\{r, q\} > \frac{N}{2}$, we have that $\{u(t_{n_k})\}$ converges to an equilibrium in $C^\mu_{\text{loc}}(\mathbb{R}^N)$ for each $\mu \in (0, \mu_0)$, where $\mu_0 = 2$ if $\min\{r, q\} = \infty$ and $\mu_0 = 2 - \frac{N}{\min\{r, q\}}$ otherwise.

7. Examples

In this section, in order to show the scope of our results, we check out conditions (3.3)–(3.6) for some prototypical classes of nonlinearities.

Example 7.1. Assume the nonlinear term is of the form

$$f(x, v) = -Cv + f_0(v) + g(x), \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R},$$

with

$$vf_0(v) \leq 0, \quad g \in L^2(\mathbb{R}^N),$$

and $C > 0$.

Then, (1.5)–(1.7) are satisfied with $m(x) = C$, which is in $L^r_U(\mathbb{R}^N)$, for any $r > \max\{\frac{N}{2}, 1\}$. As for the structure condition (3.3), it holds with $C(x) = -C$, $D(x) = |g(x)| \in L^2(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (|\nabla \phi|^2 - C(x)\phi^2) \geq C \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^N), \quad \|\phi\|_{L^2(\mathbb{R}^N)} = 1.$$

Consequently, the solutions of the linear problem (3.6) are exponentially decaying as $t \rightarrow \infty$ (see Theorem A.2).

Thus, the results of Sections 4–6, thus in particular Theorem 6.1, apply.

Example 7.2. Consider now a ‘logistic’ type nonlinearity

$$f(x, v) = m(x)v - v|v|^{\rho-1}, \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R}, \quad \rho \in (1, \infty), \quad (7.1)$$

which includes the “bistable” model $m(x)v - v^3$, which is typical in the context of Cahn–Hilliard equation. In this case we will assume that

$$m(x) = m_1(x) + m_2(x), \quad x \in \mathbb{R}^N,$$

where

$$m_1, m_2 \in L^r_U(\mathbb{R}^N) \quad \text{for a certain } 2 \leq r \leq \infty$$

and some more conditions to be specified below.

Using Young’s inequality, we have

$$\begin{aligned} vf(x, v) &= m(x)v^2 - |v|^{\rho+1} = m_1(x)v^2 + m_2(x)|v|^{\frac{\rho-1}{\rho}}|v|^{\frac{\rho+1}{\rho}} - |v|^{\rho+1} \\ &\leq m_1(x)v^2 + \frac{\rho-1}{\rho}|m_2(x)|^{\frac{\rho}{\rho-1}}|v|, \quad v \in \mathbb{R}, \quad x \in \mathbb{R}^N, \end{aligned} \quad (7.2)$$

then (3.3) holds with

$$C(x) = m_1(x), \quad D(x) = \frac{\rho-1}{\rho}|m_2(x)|^{\frac{\rho}{\rho-1}}. \quad (7.3)$$

Consequently, the results in Sections 4–6 will apply provided that the semigroup generated by $\Delta + m_1(\cdot)I$ is exponentially decaying as $t \rightarrow \infty$ and

$$|m_2(x)|^{\frac{\rho}{\rho-1}} \in L^s(\mathbb{R}^N) \quad \text{for some } \max\left\{1, \frac{2N}{N+2}\right\} \leq s \leq 2 \quad (s > 1 \text{ if } N = 2).$$

In a similar way we can handle more general nonlinearities; for example

$$f(x, v) = m(x)v + l(x)p(v) - n(x)v|v|^{\rho-1}, \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R},$$

where $0 \leq n \in L^\infty(\mathbb{R}^N)$ and $p(v)$ is a polynomial of degree less than ρ .

8. Asymptotic compactness for perturbed Cahn–Hilliard equations

Recall that from Proposition 4.6 asymptotic compactness cannot be expected in general for the semigroup associated with (1.1) in $H^1(\mathbb{R}^N)$. Hence the existence of a global attractor in $H^1(\mathbb{R}^N)$ cannot be expected in general either. Although both linear and nonlinear parts of Eq. (1.1) cooperate, this mechanism is not so strong as in the case of the second- or fourth-order problems described in [4] and [17] respectively.

Therefore our goal in this section is to introduce some perturbations in the original Cahn–Hilliard equations (1.1)–(1.2) in such a way that the perturbed semigroup is asymptotically compact and therefore has a global attractor. The goal is then to understand possible mechanisms that make the equation dissipative.

8.1. Asymptotic compactness from a nonlinear perturbation

First, we consider the family of perturbed equations

$$u_t + (-\Delta + \delta I)((-\Delta + \varepsilon I)u - f(x, u)) = 0, \quad t > 0, \quad x \in \mathbb{R}^N \quad (\delta, \varepsilon \geq 0), \quad (8.1)$$

see [18], for a similar approach.

Note that (8.1) coincides with (1.1) when $\delta = \varepsilon = 0$ and, assuming the structure condition (3.3) with $C(\cdot)$, $D(\cdot)$ satisfying (3.4)–(3.6) we can (and will) hereafter always assume that $\varepsilon = 0$. Indeed, defining $\tilde{f}(x, s) = f(x, s) - \varepsilon s$, (8.1) reads

$$u_t + (-\Delta + \delta I)(-\Delta u - \tilde{f}(x, u)) = 0, \quad t > 0, \quad x \in \mathbb{R}^N,$$

and $\tilde{f}(x, u)$ still satisfies (3.3) with $\tilde{C}(x) = C(x) - \varepsilon$, $\tilde{D}(x) = D(x)$ which satisfy (3.4)–(3.6).

Observe then that (8.1) (with $\varepsilon = 0$) reads

$$u_t + \Delta^2 u + \Delta f(x, u) - \delta \Delta u - \delta f(x, u) = 0, \quad t > 0, \quad x \in \mathbb{R}^N, \quad (8.2)$$

which is a nonlinear dissipative perturbation of (1.1).

Then define the operator

$$P_\delta := -\Delta + \delta I, \quad \delta > 0,$$

and write (8.1) (with $\varepsilon = 0$) as

$$u_t + P_\delta(P_0 u - f(\cdot, u)) = 0, \quad t > 0, \quad (8.3)$$

see (3.1). Since $\delta > 0$, $P_\delta = -\Delta + \delta I$ is invertible (say in $L^2(\mathbb{R}^N)$), and (8.3) can be written as

$$P_\delta^{-1}u_t = \Delta u + f(x, u). \quad (8.4)$$

Notice that (8.4) is a nonlocal linear perturbation of (3.2).

Concerning existence of solutions of (8.3) we have similar results to Theorem 2.1 and Corollary 2.3. Moreover we have the following results. Observe that the assumptions on the nonlinear term below are the same as in Corollary 2.3 but assuming some growth condition when $N = 1$.

Proposition 8.1. *Suppose that (1.5)–(1.7) is satisfied with $m \in L^r_U(\mathbb{R}^N)$, $r > \max\{\frac{N}{2}, 1\}$, $r \geq 2$, $g \in L^2(\mathbb{R}^N)$ and for $N = 1, 2$ (1.10) holds with some $1 < \rho < \infty$, while for $N \geq 3$ (1.10) holds with some $1 < \rho < \rho_c := \frac{N+2}{N-2} = 1 + \frac{4}{N-2}$. Assume also the structure condition (3.3) with $C(\cdot)$, $D(\cdot)$ satisfying (3.4)–(3.6). Then*

- (i) *the perturbed problem (8.3), (1.2) is globally well posed in $H^1(\mathbb{R}^N)$,*
- (ii) *the associated semigroup $\{S_\delta(t): t \geq 0\}$ of global solutions has bounded positive orbits of bounded sets. Moreover the function $E: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ in (1.3) is a Lyapunov function which is bounded on bounded sets, and the solutions satisfy*

$$\begin{aligned} E(u(t)) - E(u(s)) &= - \int_s^t (\|\nabla(\Delta u + f(x, u))\|_{L^2(\mathbb{R}^N)}^2 + \delta \|\Delta u + f(x, u)\|_{L^2(\mathbb{R}^N)}^2) \\ &= - \int_s^t (\|\nabla(P_\delta^{-1}u_t)\|_{L^2(\mathbb{R}^N)}^2 + \delta \|P_\delta^{-1}u_t\|_{L^2(\mathbb{R}^N)}^2) \\ &\leq E(u_0) + \frac{c_2}{c_1}, \quad t \geq s \geq 0, \delta > 0, u_0 \in H^1(\mathbb{R}^N). \end{aligned} \quad (8.5)$$

Furthermore,

- (iii) *positive orbits of bounded subsets of $H^1(\mathbb{R}^N)$ under $\{S_\delta(t): t \geq 0\}$ are immediately bounded in $H^2(\mathbb{R}^N)$; consequently both Δu and $f(\cdot, u)$ remain bounded in $L^2(\mathbb{R}^N)$ -norm uniformly on unbounded time intervals away from zero and for u_0 in bounded subsets of $H^1(\mathbb{R}^N)$,*
- (iv) *the set of equilibria of $\{S_\delta(t): t \geq 0\}$ is independent of $\delta \geq 0$, coincides with the set \mathcal{E} of equilibria for (4.1),*
- (v) *\mathcal{E} is bounded in $H^2(\mathbb{R}^N)$ and both parts (i) and (ii) of Theorem 6.1 apply.*

Proof. Note that from (8.3) and (8.4) we obtain that

$$\begin{aligned} \langle P_\delta^{-1}u_t, u_t \rangle_{L^2(\mathbb{R}^N)} &= \|\nabla(\Delta u + f(x, u))\|_{L^2(\mathbb{R}^N)}^2 + \delta \|\Delta u + f(x, u)\|_{L^2(\mathbb{R}^N)}^2 \\ &= \|\nabla(P_\delta^{-1}u_t)\|_{L^2(\mathbb{R}^N)}^2 + \delta \|P_\delta^{-1}u_t\|_{L^2(\mathbb{R}^N)}^2. \end{aligned} \quad (8.6)$$

Then, multiplying (8.4) in $L^2(\mathbb{R}^N)$ by $-u_t$ and using (8.6), we get

$$\frac{d}{dt}[E(u)] = -\langle P_\delta^{-1}u_t, u_t \rangle_{L^2(\mathbb{R}^N)} = -(\|\nabla(P_\delta^{-1}u_t)\|_{L^2(\mathbb{R}^N)}^2 + \delta \|P_\delta^{-1}u_t\|_{L^2(\mathbb{R}^N)}^2) \leq 0.$$

Thus $E(u)$ is non-increasing with respect to time variable and, similarly as for the unperturbed problem, we obtain (3.10) and (8.5). This proves parts (i)–(ii).

Part (iii) is then a consequence of the argument used in the proofs of Lemmas 3.2, 5.1, 5.2 and Corollary 5.4.

Finally, part (iv) follows immediately from (8.4), whereas part (v) can be obtained from boundedness of \mathcal{E} in $H^1(\mathbb{R}^N)$ (see Proposition 4.4) and from (iii). \square

Now we prove below that because $\delta > 0$, the semigroup $\{S_\delta(t) : t \geq 0\}$ associated with (8.3) has a global attractor. Recall that \mathbf{A} is a global attractor for the semigroup $\{S_\delta(t) : t \geq 0\}$ in $H^1(\mathbb{R}^N)$ if \mathbf{A} is invariant under $\{S_\delta(t) : t \geq 0\}$, compact in $H^1(\mathbb{R}^N)$ and, for any B bounded in $H^1(\mathbb{R}^N)$, we have that

$$\sup_{b \in B} \inf_{a \in \mathbf{A}} \|S_\delta(t)b - a\|_{H^1(\mathbb{R}^N)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Theorem 8.2. *Under the assumptions in Proposition 8.1, assume the following additional condition on the nonlinear term: there exist $G(\cdot)$, $H(\cdot)$ such that*

$$\begin{aligned} 0 \leq H \in L^{\tilde{s}}(\mathbb{R}^N), \quad \max \left\{ 1, \frac{2N}{N+2} \right\} \leq \tilde{s} \leq 2 \quad (\tilde{s} > 1 \text{ if } N = 2), \\ G \in L^{\tilde{r}}_U(\mathbb{R}^N), \quad \tilde{r} > \frac{N}{2}, \quad \tilde{r} \geq 2, \end{aligned} \quad (8.7)$$

with

$$\int_{\mathbb{R}^N} (|\nabla \phi|^2 - G(x)\phi^2) \geq \tilde{\omega}_0 \|\phi\|_{L^2(\mathbb{R}^N)}^2, \quad (8.8)$$

for all $\phi \in H^1(\mathbb{R}^N)$ and some $\tilde{\omega}_0 > 0$ and

$$sf(x, s) - \tilde{\nu}F(x, s) \leq G(x)s^2 + H(x)|s|, \quad x \in \mathbb{R}^N, \quad s \in \mathbb{R}, \quad (8.9)$$

for sufficiently small $0 < \tilde{\nu}$.

Under these assumptions the semigroup $\{S_\delta(t) : t \geq 0\}$ in $H^1(\mathbb{R}^N)$ associated to (8.3), possesses a global attractor \mathbf{A} in $H^1(\mathbb{R}^N)$.

Proof. Due to Proposition 8.1 we only need to show that $\{S_\delta(t) : t \geq 0\}$ is asymptotically compact; that is, each sequence of the form $\{S_\delta(t_n)u_{0n}\}$, where $t_n \rightarrow \infty$ and $\{u_{0n}\}$ is bounded in $H^1(\mathbb{R}^N)$, has a subsequence convergent in $H^1(\mathbb{R}^N)$. Note that, by Proposition 8.1, the positive orbit of $\{u_{0n}\}$ will be bounded in $H^1(\mathbb{R}^N)$ and it will be immediately bounded in $H^2(\mathbb{R}^N)$.

The proof will follow in three steps.

Step 1. We will first show that for each B bounded in $H^1(\mathbb{R}^N)$ the tails of the solutions are uniformly small for large times. By this we mean that for arbitrarily chosen $\xi > 0$ there exist certain $\tau > 0$ and $R > 0$ such that

$$\sup_{u_0 \in B} \sup_{t \geq \tau} \|u\|_{L^2(\{|x| > R\})} < \xi. \quad (8.10)$$

For this, choose any smooth function $\chi_0 : [0, \infty) \rightarrow [0, 1]$ such that $\chi_0(x) = 0$ for $s \in [0, 1]$ and $\chi_0(s) = 1$ for $s \geq 2$. Let $\chi(x) = \chi_0^2(x)$ for $x \in \mathbb{R}^N$ and define

$$\psi_k(x) = \chi \left(\frac{|x|^2}{k^2} \right), \quad x \in \mathbb{R}^N,$$

and

$$\phi_k(x) = \psi_k^2(x), \quad x \in \mathbb{R}^N, \quad (8.11)$$

where $k = 1, 2, \dots$ can be chosen as large as we wish.

With this we define the “tail of the energy” (1.3) as

$$E_{\phi_k}(t) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k - \int_{\mathbb{R}^N} F(x, u) \phi_k.$$

Then we prove below that the tail of the energy is uniformly small for large times, see (8.24), and also that the tail of the energy controls the tail of the solution, see (8.27). From these, (8.10) will follow.

Hence, multiplying (8.4) by $\lambda u \phi_k$, where $\lambda > 0$ is arbitrary, we get

$$\lambda \int_{\mathbb{R}^N} u \phi_k P_{\delta}^{-1} u_t = -\lambda \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k - \lambda \int_{\mathbb{R}^N} u \nabla u \nabla \phi_k + \lambda \int_{\mathbb{R}^N} u f(x, u) \phi_k$$

and hence, using Cauchy inequality we have

$$\begin{aligned} 0 &\leq -\lambda \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k - \lambda \int_{\mathbb{R}^N} u \nabla u \nabla \phi_k \\ &\quad + \lambda \int_{\mathbb{R}^N} u f(x, u) \phi_k + \frac{\delta}{2} \int_{\mathbb{R}^N} |P_{\delta}^{-1} u_t|^2 \phi_k + \frac{\lambda^2}{2\delta} \int_{\mathbb{R}^N} u^2 \phi_k. \end{aligned} \quad (8.12)$$

Multiplying next (8.4) by $u_t \phi_k$, we obtain

$$\frac{d}{dt} E_{\phi_k}(t) + \int_{\mathbb{R}^N} u_t \phi_k P_{\delta}^{-1} u_t = - \int_{\mathbb{R}^N} u_t \nabla u \nabla \phi_k. \quad (8.13)$$

From (8.13), writing u_t as $P_{\delta} P_{\delta}^{-1} u_t$, we get

$$\frac{d}{dt} E_{\phi_k}(t) + \int_{\mathbb{R}^N} |\nabla P_{\delta}^{-1} u_t|^2 \phi_k + \delta \int_{\mathbb{R}^N} |P_{\delta}^{-1} u_t|^2 \phi_k = R_1(\phi_k), \quad (8.14)$$

where, from (8.4),

$$\begin{aligned} R_1(\phi_k) &= \int_{\mathbb{R}^N} (\Delta u + f(x, u)) \nabla (\Delta u + f(x, u)) \nabla \phi_k - \int_{\mathbb{R}^N} P_{\delta} P_{\delta}^{-1} u_t \nabla u \nabla \phi_k \\ &= \int_{\mathbb{R}^N} (\Delta u + f(x, u)) \nabla (\Delta u + f(x, u)) \nabla \phi_k - \int_{\mathbb{R}^N} \nabla P_{\delta}^{-1} u_t \Delta u \nabla \phi_k \\ &\quad - \int_{\mathbb{R}^N} \nabla P_{\delta}^{-1} u_t \nabla u \Delta \phi_k - \delta \int_{\mathbb{R}^N} P_{\delta}^{-1} u_t \nabla u \nabla \phi_k \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^N} (\Delta u + f(x, u)) \nabla (\Delta u + f(x, u)) \nabla \phi_k - \int_{\mathbb{R}^N} \nabla (\Delta u + f(x, u)) \Delta u \nabla \phi_k \\
&\quad - \int_{\mathbb{R}^N} \nabla (\Delta u + f(x, u)) \nabla u \Delta \phi_k - \delta \int_{\mathbb{R}^N} (\Delta u + f(x, u)) \nabla u \nabla \phi_k \\
&=: R_{11}(\phi_k) + R_{12}(\phi_k) + R_{13}(\phi_k) + R_{14}(\phi_k).
\end{aligned}$$

Summing up both sides in (8.12) and (8.14), adding and subtracting a multiple of $\int_{\mathbb{R}^N} F(x, u) \phi_k$, so that E_{ϕ_k} appears on the right-hand side, we have

$$\begin{aligned}
&\frac{d}{dt} E_{\phi_k}(t) + \int_{\mathbb{R}^N} |\nabla P_{\delta}^{-1} u_t|^2 \phi_k + \frac{\delta}{2} \int_{\mathbb{R}^N} |P_{\delta}^{-1} u_t|^2 \phi_k \\
&\leq -\lambda \left(1 - \frac{\tilde{\nu}}{2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k - \lambda \tilde{\nu} E_{\phi_k}(t) + \lambda \left(\int_{\mathbb{R}^N} u f(x, u) \phi_k - \tilde{\nu} \int_{\mathbb{R}^N} F(x, u) \phi_k \right) \\
&\quad + \frac{\lambda^2}{2\delta} \int_{\mathbb{R}^N} u^2 \phi_k + R_2(\phi_k),
\end{aligned} \tag{8.15}$$

where

$$R_2(\phi_k) = R_1(\phi_k) - \lambda \int_{\mathbb{R}^N} u \nabla u \cdot \nabla \phi_k.$$

We now omit the sum $\int_{\mathbb{R}^N} |\nabla P_{\delta}^{-1} u_t|^2 \phi_k + \frac{\delta}{2} \int_{\mathbb{R}^N} |P_{\delta}^{-1} u_t|^2 \phi_k$ in (8.15) and use (8.9) and Hölder inequality to get, for $\tilde{\nu} \in (0, \tilde{\nu}_0)$,

$$\begin{aligned}
&\frac{d}{dt} E_{\phi_k}(t) + \lambda \tilde{\nu} E_{\phi_k}(t) \leq -\lambda \left[\left(1 - \frac{\tilde{\nu}}{2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k - \int_{\mathbb{R}^N} G(x) |u|^2 \phi_k \right] \\
&\quad + \lambda \|H\phi_k^{\frac{1}{2}}\|_{L^{\tilde{s}}(\mathbb{R}^N)} \|u\phi_k^{\frac{1}{2}}\|_{L^{\tilde{s}'}(\mathbb{R}^N)} + \frac{\lambda^2}{2\delta} \int_{\mathbb{R}^N} u^2 \phi_k + R_2(\phi_k).
\end{aligned}$$

Consequently, since

$$\|u\phi_k^{\frac{1}{2}}\|_{L^{\tilde{s}'}(\mathbb{R}^N)} \leq c(\|\nabla(u\phi_k^{\frac{1}{2}})\|_{L^2(\mathbb{R}^N)} + \|u\phi_k^{\frac{1}{2}}\|_{L^2(\mathbb{R}^N)}), \tag{8.16}$$

we have

$$\begin{aligned}
&\frac{d}{dt} E_{\phi_k}(t) + \lambda \tilde{\nu} E_{\phi_k}(t) \\
&\leq -\lambda \left[\left(1 - \frac{\tilde{\nu}}{2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k - \int_{\mathbb{R}^N} G(x) |u|^2 \phi_k \right] \\
&\quad + c\lambda \|H\phi_k^{\frac{1}{2}}\|_{L^{\tilde{s}}(\mathbb{R}^N)} (\|\nabla(u\phi_k^{\frac{1}{2}})\|_{L^2(\mathbb{R}^N)} + \|u\phi_k^{\frac{1}{2}}\|_{L^2(\mathbb{R}^N)}) + \frac{\lambda^2}{2\delta} \int_{\mathbb{R}^N} u^2 \phi_k + R_2(\phi_k). \tag{8.17}
\end{aligned}$$

Recalling (8.11) we obtain

$$|\nabla(u\psi_k)|^2 = \phi_k |\nabla u|^2 + u^2 |\nabla \psi_k|^2 + 2\psi_k u \nabla u \nabla \psi_k. \quad (8.18)$$

From (8.17), (8.18), using Cauchy inequality with any $\gamma \in (0, 1)$ and restricting λ to the interval $(0, 1)$, we then have, with $\gamma, \lambda \in (0, 1)$,

$$\begin{aligned} & \frac{d}{dt} E_{\phi_k}(t) + \lambda \tilde{\nu} E_{\phi_k}(t) \\ & \leq -\lambda \left[\left(1 - \frac{\tilde{\nu}}{2}\right) \int_{\mathbb{R}^N} |\nabla(u\psi_k)|^2 - \int_{\mathbb{R}^N} G(x)(u\psi_k)^2 \right] + \frac{c^2}{\gamma} \|H\psi_k\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 \\ & \quad + \frac{\lambda^2 \gamma}{2} (\|\nabla(u\phi_k^{\frac{1}{2}})\|_{L^2(\mathbb{R}^N)}^2 + \|u\phi_k^{\frac{1}{2}}\|_{L^2(\mathbb{R}^N)}^2) + \frac{\lambda^2}{2\delta} \int_{\mathbb{R}^N} (u\psi_k)^2 + R_3(\psi_k) \\ & \leq -\lambda \left[\left(1 - \frac{\tilde{\nu}}{2}\right) \int_{\mathbb{R}^N} |\nabla(u\psi_k)|^2 - \int_{\mathbb{R}^N} G(x)(u\psi_k)^2 \right] + \frac{c^2}{\gamma} \|H\psi_k\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 \\ & \quad + \frac{\lambda \gamma}{2} \|\nabla(u\phi_k^{\frac{1}{2}})\|_{L^2(\mathbb{R}^N)}^2 + \frac{\lambda^2}{2} \|u\phi_k^{\frac{1}{2}}\|_{L^2(\mathbb{R}^N)}^2 + \frac{\lambda^2}{2\delta} \int_{\mathbb{R}^N} (u\psi_k)^2 + R_3(\psi_k) \\ & \leq -\lambda \left[\left(1 - \frac{\tilde{\nu}}{2} - \frac{\gamma}{2}\right) \int_{\mathbb{R}^N} |\nabla(u\psi_k)|^2 - \int_{\mathbb{R}^N} G(x)(u\psi_k)^2 \right] \\ & \quad + \frac{1}{2} \left(\frac{1}{\delta} + 1 \right) \lambda^2 \int_{\mathbb{R}^N} (u\psi_k)^2 + \frac{c^2}{\gamma} \|H\psi_k\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + R_3(\psi_k), \end{aligned} \quad (8.19)$$

where

$$R_3(\psi_k) = R_2(\psi_k^2) + 2\lambda \left(1 - \frac{\tilde{\nu}}{2}\right) \int_{\mathbb{R}^N} \psi_k u \nabla u \nabla \psi_k + \lambda \left(1 - \frac{\tilde{\nu}}{2}\right) \int_{\mathbb{R}^N} u^2 |\nabla \psi_k|^2$$

and $\tilde{\nu}$ is sufficiently small. By (8.8) and Lemma A.4 for any $0 < \gamma < \min\{\frac{\tilde{\nu}}{2}, 1\}$ there is a certain $\omega > 0$ such that

$$\left(1 - \frac{\tilde{\nu}}{2} - \frac{\gamma}{2}\right) \int_{\mathbb{R}^N} |\nabla(u\psi_k)|^2 - \int_{\mathbb{R}^N} G(x)(u\psi_k)^2 \geq \omega \int_{\mathbb{R}^N} (u\psi_k)^2. \quad (8.20)$$

Hence, applying (8.20) and choosing $\lambda \in (0, 1)$ so small that $\omega > \frac{\lambda}{2} \left(\frac{1}{\delta} + 1\right)$ we obtain from (8.19)

$$\begin{aligned} \frac{d}{dt} E_{\phi_k}(t) + \lambda \tilde{\nu} E_{\phi_k}(t) & \leq -\lambda \left(\omega - \frac{\lambda}{2} \left(\frac{1}{\delta} + 1 \right) \right) \int_{\mathbb{R}^N} (u\psi_k)^2 + \frac{c^2}{2\gamma} \|H\psi_k\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + R_3(\psi_k) \\ & \leq \frac{c^2}{\gamma} \|H\psi_k\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + R_3(\psi_k). \end{aligned} \quad (8.21)$$

We now remark that, for any $k \in \mathbb{N}$, the properties of ψ_k and the estimates in part (iii) of Proposition 8.1 ensure that for all ‘times’ away from zero, and uniformly on bounded sets of initial data, except for $R_{11}(\phi_k)$, $R_{12}(\phi_k)$, $R_{13}(\phi_k)$, for all the remaining terms in $R_1(\phi_k)$, $R_2(\phi_k)$ and $R_3(\psi_k)$ we have an upper estimate of the form

$$\frac{M_\tau}{k}, \quad t \geq \tau,$$

where τ is arbitrarily fixed and $M_\tau > 0$ is independent of $k \in \mathbb{N}$, $t \geq \tau$ and $u_0 \in B \subset H^1(\mathbb{R}^N)$. On the other hand, for $R_{11}(\phi_k)$, $R_{12}(\phi_k)$, $R_{13}(\phi_k)$, we also have that

$$|R_{11}(\phi_k)| + |R_{12}(\phi_k)| + |R_{13}(\phi_k)| \leq \frac{M_\tau}{k} + \frac{c}{k} \|\nabla(\Delta u + f(x, u))\|_{L^2(\mathbb{R}^N)}^2 =: \frac{M_\tau}{k} + \frac{c}{k} \tilde{R}(t), \quad t \geq \tau,$$

where, due to (8.5),

$$\int_\tau^\infty \tilde{R}(t) dt = \int_\tau^\infty \|\nabla(\Delta u + f(x, u))\|_{L^2(\mathbb{R}^N)}^2 dt \leq \sup_{u_0 \in B} E(u_0) + \frac{c_2}{c_1} =: \tilde{M}. \quad (8.22)$$

Hence (8.21) reads

$$\frac{d}{dt} E_{\phi_k}(t) + \lambda \tilde{v} E_{\phi_k}(t) \leq \frac{c^2}{\gamma} \|H\psi_k\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + \frac{2M_\tau}{k} + \frac{c}{k} \tilde{R}(t), \quad t \geq \tau,$$

and using Gronwall's inequality and (8.22) we get

$$\begin{aligned} E_{\phi_k}(t) &\leq E_{\phi_k}(\tau) e^{-\lambda \tilde{v}(t-\tau)} + \frac{c^2}{\gamma \lambda \tilde{v}} \|H\psi_k\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + \frac{2M_\tau}{\lambda \tilde{v} k} + \frac{c}{k} \int_\tau^t \tilde{R}(s) e^{-\lambda \tilde{v}(t-s)} ds \\ &\leq E_{\phi_k}(\tau) e^{-\lambda \tilde{v}(t-\tau)} + \frac{c^2}{\gamma \lambda \tilde{v}} \|H\psi_k\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + \frac{2M_\tau}{\lambda \tilde{v} k} + \frac{c}{k} \tilde{M}, \quad t \geq \tau. \end{aligned}$$

Note that boundedness of orbits in $H^1(\mathbb{R}^N)$ together with the boundedness of the energy E on bounded sets of $H^1(\mathbb{R}^N)$ (see Proposition 8.1(iii)) and boundedness of ϕ_k ensure that

$$E_{\phi_k}(\tau) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(\tau)|^2 \phi_k - \int_{\mathbb{R}^N} F(x, u(\tau)) \phi_k \leq N \quad (8.23)$$

where a constant $N > 0$ is uniform for $u_0 \in B \subset H^1(\mathbb{R}^N)$ and even uniform for $\tau \geq 0$. Using (8.22) and (8.23) we obtain

$$E_{\phi_k}(t) \leq N e^{-\lambda \tilde{v}(t-\tau)} + \frac{c^2}{2\gamma \lambda \tilde{v}} \|H\psi_k\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + \frac{2M_\tau}{\lambda \tilde{v} k} + \frac{c\tilde{M}}{k}, \quad t \geq \tau. \quad (8.24)$$

Hence, the tail of the energy is uniformly small for large times, as claimed.

On the other hand, we prove now that the ‘tail of the energy’ controls the ‘tail’ of solution. For this, using (3.8) and (8.18) we have

$$\begin{aligned}
2E_{\phi_k} &\geq \int_{\mathbb{R}^N} |\nabla(u\psi_k)|^2 - \int_{\mathbb{R}^N} (u^2 |\nabla\psi_k|^2 + 2\psi_k u \nabla u \nabla \psi_k) \\
&\quad - \int_{\mathbb{R}^N} C(x)(u\psi_k)^2 - 2 \int_{\mathbb{R}^N} D(x)\psi_k |u\psi_k|,
\end{aligned} \tag{8.25}$$

and then, using again properties of ψ_k and $H^1(\mathbb{R}^N)$ -estimate of the positive orbits,

$$\int_{\mathbb{R}^N} (u^2 |\nabla\psi_k|^2 + 2\psi_k u \nabla u \nabla \psi_k) \leq \frac{\hat{M}}{k}, \quad t \geq 0, \tag{8.26}$$

where \hat{M} does not depend on $t \geq 0$ and u_{0n} , $n \in \mathbb{N}$.

Combining (8.25), (8.26) and (8.16) we get for any $\nu > 0$

$$\begin{aligned}
2E_{\phi_k} + \frac{\hat{M}}{k} &\geq \int_{\mathbb{R}^N} |\nabla(u\psi_k)|^2 - \int_{\mathbb{R}^N} C(x)(u\psi_k)^2 - 2\|D\psi_k\|_{L^s(\mathbb{R}^N)} \|u\psi_k\|_{L^{s'}(\mathbb{R}^N)} \\
&\geq \int_{\mathbb{R}^N} |\nabla(u\psi_k)|^2 - \int_{\mathbb{R}^N} C(x)(u\psi_k)^2 - \frac{2c^2}{\nu} \|D\psi_k\|_{L^s(\mathbb{R}^N)}^2 \\
&\quad - \nu (\|\nabla(u\psi_k)\|_{L^2(\mathbb{R}^N)}^2 + \|u\psi_k\|_{L^2(\mathbb{R}^N)}^2) \\
&\geq (1-\nu) \int_{\mathbb{R}^N} |\nabla(u\psi_k)|^2 - \int_{\mathbb{R}^N} C(x)(u\psi_k)^2 - \frac{2c^2}{\gamma} \|D\psi_k\|_{L^s(\mathbb{R}^N)}^2 - \frac{\nu}{2} \|u\psi_k\|_{L^2(\mathbb{R}^N)}^2.
\end{aligned}$$

Applying Lemma A.4 (see Theorem A.2(iv)) with $\nu > 0$ chosen so small that $\omega(\nu) > \nu$ we obtain

$$2E_{\phi_k} + \frac{\hat{M}}{k} + \frac{2c^2}{\gamma} \|D\psi_k\|_{L^s(\mathbb{R}^N)}^2 \geq \frac{\nu}{2} \|u\psi_k\|_{L^2(\mathbb{R}^N)}^2. \tag{8.27}$$

Connecting (8.24) and (8.27) we obtain (8.10). The proof of Step 1 is thus complete.

Step 2. We now show that each sequence of the form $\{S_\delta(t_n)u_{0n}\}$, where $\{u_{0n}\}$ is bounded in $H^1(\mathbb{R}^N)$ and $t_n \rightarrow \infty$, has a subsequence convergent in $L^2(\mathbb{R}^N)$.

Since, by Proposition 8.1, almost all elements of the sequence $\{S_\delta(t_n)u_{0n}\}$ are in a certain bounded subset of $H^2(\mathbb{R}^N)$ there exists a subsequence, $\{S_\delta(t_{n_l})u_{0n_l}\}$, which converges in $L^2(\{|x| < k\})$ for any $k \in \mathbb{N}$. Choosing any $\xi > 0$, we know from (8.10) that there is a certain $k_0 \in \mathbb{N}$ such that

$$\|S_\delta(t_{n_l})u_{0n_l} - S_\delta(t_{n_m})u_{0n_m}\|_{L^2(\{|x| > k_0\})} \leq 2\xi \quad \text{for all } l, m \geq \hat{N}.$$

On the other hand, since $\{S_\delta(t_{n_l})u_{0n_l}\}$ converges in each $L^2(\{|x| < k\})$, we infer that

$$\|S_\delta(t_{n_l})u_{0n_l} - S_\delta(t_{n_m})u_{0n_m}\|_{L^2(\{|x| < k_0\})} \leq \xi \quad \text{for all } n, m \geq \tilde{N}.$$

Hence, $\{S_\delta(t_{n_l})u_{0n_l}\}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$, which completes the proof of Step 2.

Step 3. We now show that each sequence of the form $\{S_\delta(t_n)u_{0n}\}$, where $\{u_{0n}\}$ is bounded in $H^1(\mathbb{R}^N)$ and $t_n \rightarrow \infty$, has a subsequence convergent in $H^1(\mathbb{R}^N)$. Again, by Proposition 8.1, the set $\{S_\delta(t_n - 1)u_{0n} : n \geq n_0\}$ is bounded in $H^2(\mathbb{R}^N)$. From Step 2 there is then a subsequence $\{S_\delta(t_{n_l} - 1)u_{0n_l}\}$ convergent in $L^2(\mathbb{R}^N)$.

Similarly as in Remark B.4(ii) of Appendix B we now observe that for the perturbed problem (8.3), (1.2) one can use [15, Theorem 3.2.1], which will imply that $S_\delta(1)$ takes $\{S_\delta(t_{n_l} - 1)u_{0n_l}\}$ into a precompact subset of $H^{1+4\varepsilon}(\mathbb{R}^N)$ for any $\varepsilon < \frac{1}{4}$ close enough to $\frac{1}{4}$.

Indeed Theorem 3.2.1 in [15] applies because, after rewriting Eq. (8.3) as $u_t + P_0^2 u = (P_0 + \delta)f(\cdot, u) - \delta P_0 u =: \tilde{F}(u)$, we have that P_0^2 is a sectorial operator in the space $\mathcal{X} = H^{-2}(\mathbb{R}^N)$ (see Theorem A.2), whereas from the proof of Theorem 2.1 in Appendix B (see Lemmas B.1–B.3) the nonlinear term \tilde{F} is a Lipschitz continuous map from the fractional power space $\mathcal{X}^{\frac{3}{4}+\varepsilon} = H^{1+4\varepsilon}(\mathbb{R}^N) = E_2^{\frac{1}{4}+\varepsilon}(2)$ into $\mathcal{X} = H^{-2}(\mathbb{R}^N) = E_2^{-\frac{1}{2}}(2)$ for any $\varepsilon < \frac{1}{4}$ close enough to $\frac{1}{4}$. On the other hand, $\{S_\delta(t_{n_l} - 1)u_{0n_l}\}$ is bounded in $\mathcal{X}^{\frac{3}{4}+\varepsilon} = H^{1+4\varepsilon}(\mathbb{R}^N)$ and convergent in $L^2(\mathbb{R}^N) \hookrightarrow H^{-2}(\mathbb{R}^N)$. Consequently, by [15, Theorem 3.2.1], $\{S_\delta(1)S_\delta(t_{n_l} - 1)u_{0n_l}\} = \{S_\delta(t_{n_l})u_{0n_l}\}$ has compact closure in $H^{1+4\varepsilon}(\mathbb{R}^N)$ whenever ε is less than $\frac{1}{4}$ and close enough to $\frac{1}{4}$. \square

Remark 8.3. Observe that in fact, keeping in (8.27) some term $\int_{\mathbb{R}^N} |\nabla(u\psi_k)|^2$ and using (8.18), we can see that the “tail of the energy”, (8.24), controls the “tail” of the $H^1(\mathbb{R}^N)$ -norm of the solution.

Example 8.4. Observe that the additional condition on the nonlinear term, (8.9), is quite natural. For example, if f is as in Example 7.2 then we also have

$$vf(x, v) - \tilde{v}F(x, v) = \left(1 - \frac{\tilde{v}}{2}\right)m(x)v^2 - \left(1 - \frac{\tilde{v}}{\rho+1}\right)|v|^{\rho+1} \leq \left(1 - \frac{\tilde{v}}{2}\right)vf(v), \quad (8.28)$$

whenever $x \in \mathbb{R}^N$, $v \in \mathbb{R}$. From (7.2) and (8.28) it is thus clear that (8.9) holds with $G(x) = (1 - \frac{\tilde{v}}{2})C(x)$ and $H(x) = (1 - \frac{\tilde{v}}{2})D(x)$ for arbitrarily small $\tilde{v} > 0$, where $C(x)$ and $D(x)$ are as in (7.3). Therefore, for f in (7.1) Theorem 8.2 will then apply as well.

We also have the following

Corollary 8.5.

- (i) The attractor \mathbf{A} for the semigroup $\{S_\delta(t) : t \geq 0\}$ in $H^1(\mathbb{R}^N)$ in Theorem 8.2, is bounded in $H^2(\mathbb{R}^N)$ and, for any $s < 2$, it is compact in $H^s(\mathbb{R}^N)$ and attracts bounded subsets of $H^1(\mathbb{R}^N)$ with respect to the Hausdorff semidistance in $H^s(\mathbb{R}^N)$.
- (ii) The set of equilibria \mathcal{E} is nonempty and for each $u_0 \in H^1(\mathbb{R}^N)$ and any sequence $t_n \rightarrow \infty$, there are a subsequence $\{t_{n_k}\}$ and an equilibrium $\psi \in \mathcal{E}$, such that as $k \rightarrow \infty$,

$$u(t_{n_k}) \rightarrow \psi \quad \text{in } H^s(\mathbb{R}^N) \text{ for } s < 2.$$

Furthermore $\mathbf{A} = W^u(\mathcal{E})$ is the unstable set of equilibria.

- (iii) Assume additionally that in (3.3) we have, instead of (3.4),

$$D \in L^\sigma(\mathbb{R}^N) \cap L^s(\mathbb{R}^N), \quad \sigma > N/2, \quad \max\left\{1, \frac{2N}{N+2}\right\} \leq s \leq 2.$$

Then there are two ordered extremal equilibria φ_m, φ_M in $H^1(\mathbb{R}^N)$, minimal and maximal respectively, so that any equilibrium ψ in $H^1(\mathbb{R}^N)$ satisfies

$$\varphi_m(x) \leq \psi(x) \leq \varphi_M(x), \quad x \in \mathbb{R}^N,$$

and the order interval $[\varphi_m, \varphi_M]_{H^1(\mathbb{R}^N)}$ attracts ‘pointwise asymptotic dynamics’ of (8.1) in the sense that for each $u_0 \in H^1(\mathbb{R}^N)$ and any sequence $t_n \rightarrow \infty$, there is a subsequence $\{t_{n_k}\}$ such that

$$\varphi_m(x) \leq \lim_{k \rightarrow \infty} u(t_{n_k}, x; u_0) \leq \varphi_M(x)$$

for a.e. $x \in \mathbb{R}$.

Proof. (i) It suffices to note that in Step 3 of the proof of Theorem 8.2 above, we have actually shown that, given $s < 2$, each sequence of the form $\{S_\delta(t_n)u_{0n}\}$, where $\{u_{0n}\}$ is bounded in $H^1(\mathbb{R}^N)$ and $t_n \rightarrow \infty$, has a subsequence convergent in $H^s(\mathbb{R}^N)$. On the other hand, for any positive time, the semigroup is bounded on bounded sets from $H^1(\mathbb{R}^N)$ into $H^2(\mathbb{R}^N)$ (see Proposition 8.1). Thus the semigroup will thus have the attractor (which will be the same set) both in $H^1(\mathbb{R}^N)$ and in $H^s(\mathbb{R}^N)$ with any fixed $s \in (1, 2)$.

(ii) Since the semigroup $\{S_\delta(t): t \geq 0\}$ in $H^1(\mathbb{R}^N)$ is asymptotically compact then, due to Proposition 8.1, the ω -limit set $\omega(u_0)$ of any point $u_0 \in H^1(\mathbb{R}^N)$ is nonempty, compact, attracts u_0 and is contained in the set of equilibria \mathcal{E} . Actually, the attractor is the unstable set of \mathcal{E} (see [25]).

(iii) On the other hand, since the set of equilibria of $\{S_\delta(t): t \geq 0\}$ in $H^1(\mathbb{R}^N)$ coincides with the set of equilibria for (1.1) (or (4.1)), Theorem 6.1 applies. \square

Remark 8.6. The existence of a global attractor for a perturbed problem of the form

$$u_t + (-\Delta + \delta I)((-\Delta + \delta I)u - f(x, u)) = 0, \quad t > 0, x \in \mathbb{R}^N, \quad (8.29)$$

with $\delta > 0$ was shown in [18], although for smoother nonlinear term such that $f(x, \cdot) \in C^1(\mathbb{R})$ for $x \in \mathbb{R}^N$ and $|\frac{\partial f}{\partial s}(x, s)| \leq C(1 + |\alpha_3(x)| + |s|^{\frac{2}{N-2}})$ with some $\alpha_3 \in L^{\frac{N+}{2}}(\mathbb{R}^N)$, which we do not assume here. Also the authors assumed in [18] the structure conditions

$$\exists_{\mu \in (0, \frac{\delta}{2})} \forall_{s \in \mathbb{R}} \forall_{x \in \mathbb{R}^N} F(x, s) \leq \mu s^2 + \alpha_1(x)|s| + \gamma_1(x), \quad (8.30)$$

and

$$\exists_{l \in (0, \delta)} \exists_{k \in (0, \frac{\delta-l}{\mu})} \forall_{s \in \mathbb{R}} \forall_{x \in \mathbb{R}^N} sf(x, s) - kF(x, s) \leq ls^2 + \alpha_2(x)|s| + \gamma_2(x), \quad (8.31)$$

with certain nonnegative functions $\alpha_1, \alpha_2 \in L^2(\mathbb{R}^N)$ and $\gamma_1, \gamma_2 \in L^1(\mathbb{R}^N)$.

Since for a certain $\theta \in (0, 1)$ we can write $k = \theta \frac{\delta-l}{\mu}$, conditions (8.30)–(8.31) imply that

$$sf(x, s) \leq (\theta\delta + (1-\theta)l)|s|^2 + \left(\theta \frac{\delta-l}{\mu} \alpha_1(x) + \alpha_2(x)\right)|s| + \left(\theta \frac{\delta-l}{\mu} \gamma_1(x) + \gamma_2(x)\right).$$

Actually, with $\tilde{f}(x, u) := -\delta u + f(x, u)$, we can write (8.29) as in (8.3) with

$$s\tilde{f}(x, s) \leq C(x)|s|^2 + D(x)|s| + \left(\theta \frac{\delta-l}{\mu} \gamma_1(x) + \gamma_2(x)\right),$$

with $C(x) = -(1-\theta)(\delta-l) < 0$ (since from (8.31), $\delta-l > 0$) and $D(x) = \theta \frac{\delta-l}{\mu} \alpha_1(x) + \alpha_2(x)$.

Therefore, comparing the conditions above with (1.10), (3.8) and (8.9), we see that conditions in Theorem 8.2 apply to (8.29) with less regularity requirements and larger growth conditions on f .

8.2. Asymptotic compactness from a small linear term

In this section we show another mechanism to turn (1.1) into a dissipative equation by adding a very small linear perturbation. The equation we consider here is

$$u_t + \Delta^2 u + \Delta f(x, u) + \delta u = 0, \quad t > 0, x \in \mathbb{R}^N, \delta > 0; \quad (8.32)$$

hence the perturbation is ‘much weaker’ than in (8.2). However, the energy (1.3) is no longer decreasing along solutions of (8.32). Also, the set of equilibria of (8.32) does not coincide any longer with the set of equilibria of (4.1).

In any case, note that under the assumptions on the nonlinear term as in Corollary 2.3, but assuming some growth condition when $N = 1$, we obtain a completely analogous result to parts (i) and (iii) of Proposition 8.1.

Hence, in what follows we thus focus on exhibiting the dissipativeness mechanism of (8.32) in $H^1(\mathbb{R}^N)$. The first step in this direction will be finding an $H^1(\mathbb{R}^N)$ -estimate of the solutions, which is uniform in time and in bounded sets of initial data.

Let us suppose that the assumptions of Proposition 8.1 hold. Denoting as before $E = \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} F(x, u)$, multiplying (8.32) by $-(\Delta u + f(\cdot, u))$ in $L^2(\mathbb{R}^N)$ and using (3.3) together with the estimate

$$\int_{\mathbb{R}^N} D(x)|u| \leq \|D\|_{L^s(\mathbb{R}^N)} \|u\|_{H^1(\mathbb{R}^N)} \leq c \|D\|_{L^s(\mathbb{R}^N)} (\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|u\|_{L^2(\mathbb{R}^N)}^2)^{\frac{1}{2}},$$

we get, using Hölder inequality,

$$\begin{aligned} & \frac{d}{dt} E(t) + \|\nabla(\Delta u + f(\cdot, u))\|_{L^2(\mathbb{R}^N)}^2 + \delta \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \\ &= \delta \int_{\mathbb{R}^N} u f(\cdot, u) \\ &\leq \delta \left(\int_{\mathbb{R}^N} C(x) u^2 + \frac{c^2}{2\nu} \|D\|_{L^s(\mathbb{R}^N)}^2 + \frac{\nu}{2} (\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|u\|_{L^2(\mathbb{R}^N)}^2) \right). \end{aligned}$$

We then have

$$\begin{aligned} & \frac{d}{dt} E(t) + \|\nabla(\Delta u + f(\cdot, u))\|_{L^2(\mathbb{R}^N)}^2 + \delta \left(\int_{\mathbb{R}^N} (1 - \nu) |\nabla u|^2 - C(x) u^2 \right) + \frac{\delta \nu}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \\ &\leq \frac{\delta c^2}{2\nu} \|D\|_{L^s(\mathbb{R}^N)}^2 + \frac{\delta \nu}{2} \|u\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

Using next Lemma A.4, with $\omega_0 > 0$ because of (3.6) and Theorem A.2, we obtain, choosing sufficiently small $\nu \in (0, \frac{\omega_0}{2})$ such that in Lemma A.4(ii) we have $\omega(\nu) > \frac{\omega_0}{2}$,

$$\frac{d}{dt} E(t) + \|\nabla(\Delta u + f(\cdot, u))\|_{L^2(\mathbb{R}^N)}^2 + \frac{\delta \nu}{2} (\|u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^N)}^2) \leq \frac{\delta c^2}{2\nu} \|D\|_{L^s(\mathbb{R}^N)}^2. \quad (8.33)$$

On the other hand, from the proof of Lemma 3.4 we know that for $\phi \in H^1(\mathbb{R}^N)$,

$$\tilde{E}(\phi) := 2E(\phi) + \frac{4c^2}{\nu} \|D\|_{L^s(\mathbb{R}^N)}^2 \geq \frac{\nu}{2} (\|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2 + \|\phi\|_{L^2(\mathbb{R}^N)}^2), \quad (8.34)$$

whereas from the proof of Lemma 3.2, we have

$$0 \leq \tilde{E}(\phi) \leq a(\|\phi\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2)^{\frac{\rho+1}{2}} + b, \quad \phi \in H^1(\mathbb{R}^N),$$

for certain constants $a, b > 0$. Therefore for $\phi \in H^1(\mathbb{R}^N)$,

$$\tilde{E}^{\frac{2}{\rho+1}}(\phi) \leq \tilde{a}(\|\phi\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2) + \tilde{b},$$

for some $\tilde{a}, \tilde{b} > 0$.

Hence we get from (8.33)

$$\frac{d}{dt} \tilde{E}(t) + \frac{\delta v}{2\tilde{a}} \tilde{E}^{\frac{2}{\rho+1}}(t) \leq \frac{\delta c^2}{2v} \|D\|_{L^s(\mathbb{R}^N)}^2 + \frac{\tilde{b}\delta v}{2\tilde{a}}.$$

Consequently, denoting by z^+ the unique positive root of $\frac{\delta v}{2\tilde{a}} z^{\frac{2}{\rho+1}} = \frac{\delta c^2}{2v} \|D\|_{L^s(\mathbb{R}^N)}^2 + \frac{\tilde{b}\delta v}{2\tilde{a}}$, we have that

$$\tilde{E}(t) \leq \max\{\tilde{E}(u_0), z^+\} \quad (8.35)$$

and

$$\forall_{z > z^+} \forall_{r > 0} \exists_{t_0 > 0} \forall_{\|u_0\|_{H^1(\mathbb{R}^N)} < r} \tilde{E}(u) \in [0, z]. \quad (8.36)$$

In particular, we obtain the boundedness of positive orbits of bounded sets and the existence of an absorbing set in $H^1(\mathbb{R}^N)$ as shown in Theorem 8.7 below.

The above estimates lead to the following result.

Theorem 8.7. *Under the assumption of Proposition 8.1,*

- (i) *the problem (8.32), (1.2) is globally well posed in $H^1(\mathbb{R}^N)$; furthermore, the associated semigroup $\{T_\delta(t): t \geq 0\}$ in $H^1(\mathbb{R}^N)$ has bounded orbits of bounded sets and is bounded dissipative; that is, equivalently, there is B_0 bounded in $H^1(\mathbb{R}^N)$ such that for any set of initial data B bounded in $H^1(\mathbb{R}^N)$ there exists $T = T(B)$ such that*

$$\forall_{u_0 \in B} \forall_{t \geq T(B)} u(t; u_0) \in B_0,$$

- (ii) *for each $t > 0$, $T_\delta(t)$ maps bounded sets of $H^1(\mathbb{R}^N)$ into bounded sets of $H^2(\mathbb{R}^N)$. Consequently, positive orbits of bounded subsets of $H^1(\mathbb{R}^N)$ are immediately bounded in $H^2(\mathbb{R}^N)$ and there is a bounded set in $H^2(\mathbb{R}^N)$, which absorbs bounded subsets of $H^1(\mathbb{R}^N)$.*

Proof. Part (i) follows from the estimates (8.34) and (8.35)–(8.36). Then, following the argument used in the proofs of Lemma 5.1 we obtain that for each $t > 0$ and any $s < 2$, $T_\delta(t)$ maps bounded sets of $H^1(\mathbb{R}^N)$ into bounded sets of $H^s(\mathbb{R}^N)$.

To complete the proof similarly as in Remark B.4(ii) of Appendix B we now observe that for the perturbed problem (8.32), (1.2) one can use [12, Theorem 5], which will imply that, for any $\varepsilon < \frac{1}{4}$ close enough to $\frac{1}{4}$, any B bounded in $H^{1+4\varepsilon}(\mathbb{R}^N)$ and any $t_0 > 0$ close enough to 0, $T_\delta(t_0)B$ is bounded in $H^2(\mathbb{R}^N)$.

Indeed Theorem 5 in [12] applies because, after rewriting Eq. (8.3) as $u_t + P_0^2 u = P_0 f(\cdot, u) - \delta u =: \hat{\mathcal{F}}(u)$, we have that P_0^2 is a sectorial operator in the space $\mathcal{X} = H^{-2}(\mathbb{R}^N)$ (see Theorem A.2), whereas from the proof of Theorem 2.1 in Appendix B (see Lemmas B.1–B.3) the nonlinear term $\hat{\mathcal{F}}$

is a Lipschitz continuous map from the fractional power space $\mathcal{X}^{\frac{3}{4}+\varepsilon} = H^{1+4\varepsilon}(\mathbb{R}^N) = E_2^{\frac{1}{4}+\varepsilon}(2)$ into $\mathcal{X} = H^{-2}(\mathbb{R}^N) = E_2^{-\frac{1}{2}}(2)$ for any $\varepsilon < \frac{1}{4}$ close enough to $\frac{1}{4}$. Consequently, given B bounded in $H^1(\mathbb{R}^N)$ (hence, by part (i) with positive orbit $\gamma^+(B)$ bounded in $H^1(\mathbb{R}^N)$) we obtain via [12, Theorem 5] that for each $t_0 > 0$ sufficiently small $T_\delta(t_0)\gamma^+(B)$ is bounded in $H^2(\mathbb{R}^N)$.

Finally note that, if B_0 is bounded in $H^1(\mathbb{R}^N)$ and absorbing bounded sets of $H^1(\mathbb{R}^N)$ then $\gamma^+(B_0)$ has the same properties and from what was said above we infer that $T_\delta(t_0)\gamma^+(B_0)$ is bounded in $H^2(\mathbb{R}^N)$ and absorbs bounded subsets of $H^1(\mathbb{R}^N)$. \square

Remark 8.8. From (8.33)–(8.34) and (8.35) we obtain that for any fixed numbers $a > 0$ and $\tau \geq 0$ the following estimate holds for every $t \geq \tau$:

$$\int_{\tau}^t \|\nabla(\Delta u + f(\cdot, u))(s)\|_{L^2(\mathbb{R}^N)}^2 e^{-a(t-s)} ds \leq \frac{\delta c^2 \|D\|_{L^s(\mathbb{R}^N)}^2}{2av} + 2 \max\{\tilde{E}(u_0), z^+\}. \quad (8.37)$$

Indeed, by (8.33) and the definition of $\tilde{E}(u)$ in (8.34), the left-hand side of (8.37) is bounded by

$$\int_{\tau}^t \frac{\delta c^2}{2v} \|D\|_{L^s(\mathbb{R}^N)}^2 e^{-a(t-s)} ds - \int_{\tau}^t \frac{d}{ds} \tilde{E}(u(s)) e^{-a(t-s)} ds =: J_1 + J_2,$$

where $J_1 = \frac{\delta c^2 \|D\|_{L^s(\mathbb{R}^N)}^2}{2av} (1 - e^{-a(t-\tau)})$ whereas from integration by parts formula and (8.35) we have

$$\begin{aligned} J_2 &= -\tilde{E}(u(s)) e^{-a(t-s)} \Big|_{s=\tau}^{s=t} + a \int_{\tau}^t \tilde{E}(u(s)) e^{-a(t-s)} ds \\ &= -\tilde{E}(u(t)) + \tilde{E}(u(\tau)) e^{-a(t-\tau)} + a \int_{\tau}^t \tilde{E}(u(s)) e^{-a(t-s)} ds \\ &\leq \max\{\tilde{E}(u_0), z^+\} \left(1 + a \int_{\tau}^t e^{-a(t-s)} ds \right) = 2 \max\{\tilde{E}(u_0), z^+\}. \end{aligned}$$

We next prove the asymptotic compactness of $\{T_\delta(t): t \geq 0\}$.

Theorem 8.9. Under the assumptions of Theorem 8.7 if the nonlinear term satisfies (8.7)–(8.9), then the semi-group $\{T_\delta(t): t \geq 0\}$ in Theorem 8.7 is asymptotically compact in $H^1(\mathbb{R}^N)$ and has a global attractor.

Proof. Note that once it is shown that for each B bounded in $H^1(\mathbb{R}^N)$ and for arbitrarily chosen $\xi > 0$ there exist certain $\tau > 0$ and $R > 0$ such that

$$\sup_{u_0 \in B} \sup_{t \geq \tau} \|u\|_{L^2(\{|x| > R\})} < \xi, \quad (8.38)$$

then the result follows in the same manner as in Steps 2–3 of the proof of Theorem 8.2. Below we thus concentrate on proving (8.38), for which we follow a similar argument as in the proof of Theorem 8.2.

As before choose a smooth function as in (8.11). Multiply then (8.32) by $-(\Delta u + f(\cdot, u))\phi_k$ and integrate over $x \in \mathbb{R}^N$ to get that the tail of the energy (8.24) satisfies the equality

$$\frac{d}{dt} E_{\phi_k}(t) + \int_{\mathbb{R}^N} |\nabla(\Delta u + f(x, u))|^2 \phi_k + \delta \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k - \delta \int_{\mathbb{R}^N} u f(x, u) \phi_k = \mathcal{R}(\phi_k), \quad (8.39)$$

where

$$\begin{aligned} \mathcal{R}(\phi_k) &= - \int_{\mathbb{R}^N} u_t \nabla u \nabla \phi_k - \frac{1}{2} \int_{\mathbb{R}^N} \nabla(\Delta u + f(x, u))^2 \nabla \phi_k - \delta \int_{\mathbb{R}^N} u \nabla u \nabla \phi_k \\ &= \int_{\mathbb{R}^N} (-u_t) \nabla \nabla \phi_k + \frac{1}{2} \int_{\mathbb{R}^N} (\Delta u + f(x, u))^2 \Delta \phi_k - \delta \int_{\mathbb{R}^N} u \nabla u \nabla \phi_k \\ &= \int_{\mathbb{R}^N} (\Delta(\Delta u + f(x, u)) + \delta u) \nabla u \nabla \phi_k + \frac{1}{2} \int_{\mathbb{R}^N} (\Delta u + f(x, u))^2 \Delta \phi_k - \delta \int_{\mathbb{R}^N} u \nabla u \nabla \phi_k \\ &= - \int_{\mathbb{R}^N} \nabla(\Delta u + f(x, u)) (\Delta u \nabla \phi_k + \nabla u \Delta \phi_k) + \frac{1}{2} \int_{\mathbb{R}^N} (\Delta u + f(x, u))^2 \Delta \phi_k. \end{aligned}$$

Using (8.39) and adding and subtracting a multiple of $\int_{\mathbb{R}^N} F(x, u) \phi_k$, so that E_{ϕ_k} appears on the right-hand side, we have

$$\begin{aligned} \frac{d}{dt} E_{\phi_k}(t) + \int_{\mathbb{R}^N} |\nabla(\Delta u f(x, u))|^2 \phi_k + \delta \left(1 - \frac{\tilde{v}}{2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k \\ + \delta \tilde{v} \left(\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k - \int_{\mathbb{R}^N} F(x, u) \phi_k\right) - \delta \left(\int_{\mathbb{R}^N} u f(x, u) \phi_k - \tilde{v} \int_{\mathbb{R}^N} F(x, u) \phi_k\right) \\ = \mathcal{R}(\phi_k). \end{aligned}$$

Using (8.9) and dropping out the term $\int_{\mathbb{R}^N} |\nabla(\Delta u + f(x, u))|^2 \phi_k$, we get

$$\frac{d}{dt} E_{\phi_k}(t) + \delta \tilde{v} E_{\phi_k}(t) + \delta \left(1 - \frac{\tilde{v}}{2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \phi_k \leq \delta \int_{\mathbb{R}^N} G(x) |u|^2 \phi_k + \delta \int_{\mathbb{R}^N} H(x) |u| \phi_k + \mathcal{R}(\phi_k). \quad (8.40)$$

Since $H^1(\mathbb{R}^N) \hookrightarrow L^{\tilde{s}'}(\mathbb{R}^N)$ and using (8.18), we can rewrite (8.40) as

$$\begin{aligned} \frac{d}{dt} E_{\phi_k}(t) + \delta \tilde{v} E_{\phi_k}(t) + \delta \left(1 - \delta - \frac{\tilde{v}}{2}\right) \int_{\mathbb{R}^N} |\nabla(u \psi_k)|^2 + \delta^2 \int_{\mathbb{R}^N} |\nabla(u \psi_k)|^2 \\ \leq \delta \int_{\mathbb{R}^N} G(x) |u \psi_k|^2 + \frac{\delta}{2\gamma} \|\psi_k H\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + \frac{\delta\gamma}{2} \|\psi_k u\|_{L^{\tilde{s}'}(\mathbb{R}^N)}^2 + \mathcal{R}(\phi_k) + \hat{\mathcal{R}}(\phi_k) \\ \leq \delta \int_{\mathbb{R}^N} G(x) |u \psi_k|^2 + \frac{\delta}{2\gamma} \|\psi_k H\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + c \frac{\delta\gamma}{2} (\|\psi_k u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla(\psi_k u)\|_{L^2(\mathbb{R}^N)}^2) + \overline{\mathcal{R}}(\phi_k), \end{aligned}$$

where

$$\bar{\mathcal{R}}(\phi_k) = \mathcal{R}(\phi_k) + \hat{\mathcal{R}}(\phi_k) \quad \text{and} \quad \hat{\mathcal{R}}(\phi_k) = \delta \left(1 - \frac{\tilde{\nu}}{2}\right) \int_{\mathbb{R}^N} (|u \nabla \psi_k|^2 + 2\psi_k u \nabla u \nabla \psi_k).$$

Then, applying (8.8) and Lemma A.4, we obtain for suitably small parameters $\delta, \tilde{\nu} > 0$ that

$$\left(1 - \delta - \frac{\tilde{\nu}}{2}\right) \int_{\mathbb{R}^N} |\nabla(u \psi_k)|^2 - \int_{\mathbb{R}^N} G(x) |u \psi_k|^2 \geq \omega \|\psi_k u\|_{H^1(\mathbb{R}^N)}^2$$

and hence we have

$$\begin{aligned} \frac{d}{dt} E_{\phi_k}(t) + \delta \tilde{\nu} E_{\phi_k}(t) &\leq \frac{d}{dt} E_{\phi_k}(t) + \delta \tilde{\nu} E_{\phi_k}(t) + \bar{c} (\|\psi_k u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla(\psi_k u)\|_{L^2(\mathbb{R}^N)}^2) \\ &\leq \frac{\delta}{2\gamma} \|\psi_k H\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + \bar{\mathcal{R}}(\phi_k), \end{aligned} \quad (8.41)$$

where $\bar{c} := \min\{\delta\omega, \delta^2\} - c\frac{\delta\gamma}{2}$ and $\gamma > 0$ is small enough so that $\bar{c} > 0$. Using now the properties of ϕ_k and recalling from Theorem 8.7(ii) that positive orbits of bounded sets are immediately bounded in $H^2(\mathbb{R}^N)$ we obtain that

$$\bar{\mathcal{R}}(\phi_k) \leq \frac{M_\tau}{k} + \frac{c}{k} \int_{\mathbb{R}^N} |\nabla(\Delta u + f(x, u))|^2, \quad t \geq \tau. \quad (8.42)$$

Hence, from (8.41) and (8.42) and solving the resulting inequality, we infer that

$$\begin{aligned} E_{\phi_k}(t) &\leq E_{\phi_k}(\tau) e^{-\delta \tilde{\nu}(t-\tau)} + \frac{1}{2\tilde{\nu}\gamma} \|\psi_k H\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 \\ &\quad + \frac{M_\tau}{\delta \tilde{\nu} k} + \frac{c}{k} \int_\tau^t \|\nabla(\Delta u + f(\cdot, u))(s)\|_{L^2(\mathbb{R}^N)}^2 e^{-\delta \tilde{\nu}(t-s)} ds, \quad t \geq \tau. \end{aligned}$$

Recalling now (8.37) we obtain an estimate like in (8.23) and using then (8.27), we conclude that for each $t \geq \tau$

$$\begin{aligned} \frac{\nu}{2} \|u \psi_k\|_{L^2(\mathbb{R}^N)}^2 &\leq 2E_{\phi_k}(t) + \frac{\hat{M}}{k} + \frac{2c^2}{\gamma} \|D\psi_k\|_{L^s(\mathbb{R}^N)}^2 \\ &\leq 2Ne^{-\delta \tilde{\nu}(t-\tau)} + \frac{1}{\gamma \tilde{\nu}} \|\psi_k H\|_{L^{\tilde{s}}(\mathbb{R}^N)}^2 + \frac{2M_\tau}{\delta \tilde{\nu} k} + \frac{2c}{k} \left(\frac{c^2 \|D\|_{L^s(\mathbb{R}^N)}^2}{2\tilde{\nu}\nu} + 2 \max\{\tilde{E}(u_0), z^+\} \right) \\ &\quad + \frac{\hat{M}}{k} + \frac{2c^2}{\gamma} \|D\psi_k\|_{L^s(\mathbb{R}^N)}^2. \end{aligned}$$

Thus (8.38) holds and the rest of the proof is the same as in Theorem 8.2. \square

Remark 8.10. Note that both Proposition 8.1 and Theorem 8.2 actually hold for $N = 1$ without assuming (1.10). If the nonlinear term satisfies (8.7)–(8.9) then Theorem 8.7 also holds for $N = 1$ without assuming (1.10). Indeed, multiplying (8.32) by $-(\Delta u + f(\cdot, u))$, one can proceed as in the proof of Theorem 8.9 with $\psi_k = 1$ to obtain the $H^1(\mathbb{R}^N)$ bound on the solutions. Consequently, Theorem 8.9 also holds for $N = 1$ without assuming (1.10).

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Appendix A. Operators $(-\Delta)^k + C(x)I$ in Bessel potential spaces

In this section we collect some results concerning perturbations of a power of the Laplacian by a potential. We will consider the potentials C which satisfy some mild integrability assumptions uniformly on unit balls in \mathbb{R}^N . In general we omit the proofs, which can be carried out similarly as in [38] and [16].

We first consider a multiplication operator Q_C , which we define for any function $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$Q_C(\phi)(x) = m(x)\phi(x), \quad x \in \mathbb{R}^N.$$

Lemma A.1. Suppose that $k = 1, 2$, $C \in L^r_U(\mathbb{R}^N)$ with $r > \max\{\frac{N}{2k}, 1\}$, $p \in (1, \infty)$ and let β be any number from the interval

$$I(k, p) = (-\beta^*(k, p'), \beta^*(k, p) - 1] \subset (-1, 0],$$

where

$$\beta^*(k, p) := 1 + \left(\frac{N}{2kp} - \frac{N}{2kr} \right)_-$$

and $x_- = \min\{x, 0\}$ denotes the negative part of $x \in \mathbb{R}$.

Then, there is a certain interval $(\alpha_0, 1 + \beta)$ such that for any $\alpha \in (\alpha_0, 1 + \beta)$

$$Q_C \in \mathcal{L}(H_p^{2k\alpha}(\mathbb{R}^N), (H_{p'}^{-2k\beta}(\mathbb{R}^N))') \quad \text{and} \quad \|Q_C\|_{\mathcal{L}(H_p^{2k\alpha}(\mathbb{R}^N), (H_{p'}^{-2k\beta}(\mathbb{R}^N))')} \leq c \|C\|_{L^r_U(\mathbb{R}^N)}.$$

The above lemma is a crucial ingredient in the proof of the following result. Note that below we use the notation $\nabla^2 = \Delta$.

Theorem A.2. Suppose that $k = 1, 2$, $C \in L^r_U(\mathbb{R}^N)$ and $r > \max\{\frac{N}{2k}, 1\}$.

- (i) Then the operator $P_C^k = (-\Delta)^k - C(x)I$ is a sectorial operator in $L^p(\mathbb{R}^N)$ and $-P_C^k$ generates a C^0 analytic semigroup, $\{e^{-P_C^k t}: t \geq 0\}$, in $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$.
- (ii) The scale of fractional power spaces, $\{E_p^\alpha(k), \alpha \in \mathbb{R}\}$, associated to this operator, is given by

$$E_p^\alpha(k) = \begin{cases} H_p^{2k\alpha}(\mathbb{R}^N) & \text{for } 0 \leq \alpha \leq \beta^*(k, p) \leq 1, \\ (H_{p'}^{-2k\alpha}(\mathbb{R}^N))' & \text{for } -1 \leq -\beta_*(k, p) \leq \alpha < 0, \end{cases}$$

with $0 < \beta^*(k, p) = 1 + (\frac{N}{2kp} - \frac{N}{2kr})_- \leq 1$ and $\beta_*(k, p) = \beta^*(k, p') = 1 + (\frac{N}{2kp'} - \frac{N}{2kr})_-$, where we will also use the usual notation

$$H_p^s(\mathbb{R}^N) = (H_{p'}^{-s}(\mathbb{R}^N))', \quad s < 0.$$

(iii) On this scale of spaces, the analytic semigroup generated by $-P_C^k$ satisfies, for some $\omega \in \mathbb{R}$,

$$\|e^{-P_C^k t}\|_{\mathcal{L}(Y_p^\sigma, Y_p^\xi)} \leq M \frac{e^{-\omega t}}{t^{\xi-\sigma}}, \quad t > 0, \quad -\beta_*(k, p) \leq \sigma \leq \xi \leq \beta^*(k, p). \quad (\text{A.1})$$

(iv) Also, if $p = 2$ then (A.1) is satisfied for some $\omega > 0$ if and only if there is a certain $\omega_0 > 0$ such that

$$\int_{\mathbb{R}^N} (|\nabla^k \phi|^2 - C(x)\phi^2) \geq \omega_0 \|\phi\|_{L^2(\mathbb{R}^N)}^2, \quad (\text{A.2})$$

for all $\phi \in H^k(\mathbb{R}^N)$. We say then that the C^0 analytic semigroup $\{e^{-P_C^k t}; t \geq 0\}$ in $L^2(\mathbb{R}^N)$ is exponentially decaying as $t \rightarrow \infty$.

Remark A.3.

(i) Observe that for $k = 1, 2$, $\beta^*(k, p) = 1$ iff $r \geq p$ and, for all $1 < p < \infty$,

$$\beta^*(k, p) \geq 1 - \frac{N}{2kr} > 0.$$

Hence, the interval $[-\beta_*(k, p), \beta^*(k, p)]$ contains at least the symmetric interval

$$\left[-1 + \frac{N}{2kr}, 1 - \frac{N}{2kr}\right].$$

Also, the length of the interval $(-\beta_*(k, p), \beta^*(k, p))$ is $L = \beta^*(k, p) + \beta^*(k, p')$ and then

$$L = \begin{cases} 1 + \beta^*(k, p'), & \text{if } p' \geq r \geq p, \\ 1 + \beta^*(k, p), & \text{if } p \geq r \geq p', \\ 2, & \text{if } r \geq p, p', \\ 2 + \frac{N}{2k} - \frac{N}{kr}, & \text{if } p, p' \geq r. \end{cases}$$

Note that in any case $L > 1$ since $r > \max\{\frac{N}{2k}, 1\}$.

- (ii) The case $r \geq p$ reflects that the potential is suitable integrable with respect to the base space, $L^p(\mathbb{R}^N)$. Hence, in this case the potential can be naturally handled as a perturbation of the Laplacian or bi-Laplacian operator.
- (iii) On the other hand, when $r < p$, the potential is poorly integrable with respect to the base space and it is more difficult to handle as a perturbation of the Laplacian or bi-Laplacian.
- (iv) Note that it is implicit in (A.2) that since $C \in L_U^r(\mathbb{R}^N)$ with $r > \max\{\frac{N}{2k}, 1\}$ and $\phi \in H^k(\mathbb{R}^N)$, then $C\phi^2 \in L^1(\mathbb{R}^N)$.

Another useful result is the following.

Lemma A.4. Suppose that $k = 1, 2$ and $C \in L^r_U(\mathbb{R}^N)$ with $r > \max\{\frac{N}{2k}, 1\}$. Then

(i) The domain of $P_C^k = \Delta^k - C(x)I$ in $L^2(\mathbb{R}^N)$, $D_{L^2}(P_C^k)$, is included in $H^k(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} P_C^k \phi \psi = \int_{\mathbb{R}^N} \nabla^k \phi \nabla^k \psi - \int_{\mathbb{R}^N} C(x) \phi \psi = \int_{\mathbb{R}^N} \phi P_C^k \psi, \quad \phi, \psi \in D_{L^2}(P_C^k).$$

Furthermore, there exists $\omega_0 \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} (|\nabla^k \phi|^2 - C(x) \phi^2) \geq \omega_0 \|\phi\|_{L^2(\mathbb{R}^N)}^2 \quad \text{for each } \phi \in H^k(\mathbb{R}^N).$$

(ii) If ω_0 is as above, there is a continuous decreasing real-valued function $\omega(v)$ defined in a certain interval $[0, v_0]$ such that for $v \in [0, v_0]$,

$$\int_{\mathbb{R}^N} ((1-v)|\nabla^k \phi|^2 - C(x) \phi^2) \geq \omega(v) \int_{\mathbb{R}^N} \phi^2 \quad \text{for all } \phi \in H^k(\mathbb{R}^N),$$

and

$$\lim_{v \rightarrow 0^+} \omega(v) = \omega(0) = \omega_0.$$

Note that in fact the constant ω_0 in Lemma A.4 above, gives a lower bound of the bottom spectrum of P_C or P_C^2 in $L^2(\mathbb{R}^N)$. So part (ii) of the lemma above is a sort of continuity of the bottom spectrum with respect to the diffusion coefficient.

We now come back to the last statement of Remark A.3 proving a technical lemma below.

Lemma A.5. Suppose that $C \in L^r_U(\mathbb{R}^N)$, where $r > \max\{\frac{N}{2}, 1\}$.

Then, for any $\gamma > 0$ and a certain $c_\gamma > 0$ we have

$$\int_{\mathbb{R}^N} |C(x)| |\phi|^2 \leq \gamma \|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2 + c_\gamma \|\phi\|_{L^2(\mathbb{R}^N)}^2, \quad \phi \in H^1(\mathbb{R}^N).$$

Proof. Note that it is sufficient to consider the case $r < \infty$. Note also that if Q_i is an open cube in \mathbb{R}^N centered at $i \in \mathbb{Z}^N$ and with all edges unitary and parallel to the axes then $\mathbb{R}^N = \bigcup_{i \in \mathbb{Z}^N} \overline{Q_i}$ and $Q_i \cap Q_j = \emptyset$ for $i \neq j$.

Letting $r' = \frac{r}{r-1}$ and choosing $s \in (0, 1)$ such that $s - \frac{N}{2} \geq -\frac{N}{2r'}$, which is possible because $r > \max\{\frac{N}{2}, 1\}$, we have $H^s_2(Q_i) \hookrightarrow L^{2r'}(Q_i)$ and

$$\begin{aligned} \int_{\mathbb{R}^N} |C(x)| |\phi|^2 &= \sum_{i \in \mathbb{Z}^N} \int_{Q_i} |C(x)| |\phi|^2 \leq \sum_{i \in \mathbb{Z}^N} \|C\|_{L^r(Q_i)} \|\phi\|_{L^{2r'}(Q_i)}^2 \\ &\leq \sum_{i \in \mathbb{Z}^N} \|C\|_{L^r_U(\mathbb{R}^N)} \|\phi\|_{H^s(Q_i)}^2. \end{aligned} \tag{A.3}$$

With the aid of the interpolation inequality, $\|\phi\|_{H^s(Q_i)} \leq c\|\phi\|_{H^1(Q_i)}^s \|\phi\|_{L^2(Q_i)}^{1-s}$ (see [41, §2.4.2(11)]) and Hölder inequality we obtain from (A.3) that for each $\phi \in H^1(\mathbb{R}^N)$, every $\varepsilon > 0$ and $c_\varepsilon = \varepsilon^{-\frac{s}{1-s}} c^{\frac{1}{1-s}} \|C\|_{L_U^r(\mathbb{R}^N)}^{\frac{1}{1-s}}$

$$\int_{\mathbb{R}^N} C(x)|\phi|^2 \leq \sum_{i \in \mathbb{Z}^N} (\varepsilon \|\phi\|_{H^1(Q_i)}^2 + c_\varepsilon \|\phi\|_{L^2(Q_i)}^2) = \varepsilon \|\phi\|_{H^1(\mathbb{R}^N)}^2 + c_\varepsilon \|\phi\|_{L^2(\mathbb{R}^N)}^2. \quad (\text{A.4})$$

Substituting now $\|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2 + \|\phi\|_{L^2(\mathbb{R}^N)}^2$ in place of $\|\phi\|_{H^{\frac{1}{2}}(\mathbb{R}^N)}^2$ -norm in (A.4) we get the result. \square

Corollary A.6. Suppose that $C \in L_U^r(\mathbb{R}^N)$, where $r > \max\{\frac{N}{2}, 1\}$.

Then, for any $\gamma > 0$ there is a certain $\tilde{c}_\gamma > 0$ such that

$$\int_{\mathbb{R}^N} C(x)|\nabla \psi|^2 \leq \gamma \|\Delta \psi\|_{L^2(\mathbb{R}^N)}^2 + \tilde{c}_\gamma \|\psi\|_{H^1(\mathbb{R}^N)}^2, \quad \psi \in H^2(\mathbb{R}^N).$$

Proof. If $\psi \in H^2(\mathbb{R}^N)$, then summing (A.4) with $\phi = \psi'_{x_j}$, $j = 1, \dots, N$, we obtain

$$\int_{\mathbb{R}^N} C(x)|\nabla \psi|^2 \leq \varepsilon \sum_{j=1}^N \|\psi'_{x_j}\|_{H^1(\mathbb{R}^N)}^2 + c_\varepsilon \sum_{j=1}^N \|\psi'_{x_j}\|_{L^2(\mathbb{R}^N)}^2 \leq \varepsilon \|\psi\|_{H^2(\mathbb{R}^N)}^2 + c_\varepsilon \|\nabla \psi\|_{L^2(\mathbb{R}^N)}^2,$$

where $(\|\Delta \psi\|_{L^2(\mathbb{R}^N)} + \|\psi\|_{L^2(\mathbb{R}^N)})$ can equivalently replace $\|\psi\|_{H^2(\mathbb{R}^N)}$. The result now follows easily. \square

Appendix B. Proofs of the results of Section 2

With the results described in Appendix A we will rewrite (1.1)–(1.2) as

$$\begin{cases} \dot{u} + P_0^2 u = P_0(f(\cdot, u)) =: \mathcal{F}(u), & t > 0, \\ u(0) = u_0 \in H_p^1(\mathbb{R}^N) \end{cases}$$

and using properties of operators P_0^k , $k = 1, 2$, in scales of fractional powers (see [1]) we will look for the solutions that were originally defined in [2,3] as ε -regular solutions. Actually, following [3] the proof of Theorem 2.1 will be complete if we show that \mathcal{F} decomposes into a finite sum of maps \mathcal{F}_i which are ε_i -regular map relative to the pair of spaces $(E_p^{\frac{1}{4}}(2), E_p^{-\frac{3}{4}}(2))$; namely, there are constants $\rho_1 > 1$, $c_i > 0$, $\varepsilon_i \in (0, \frac{1}{\rho_i})$ and $\gamma(\varepsilon_i) \in [\rho_i \varepsilon_i, 1)$ such that

$$\|\mathcal{F}_i(v) - \mathcal{F}_i(w)\|_{E_p^{\gamma(\varepsilon_i) - \frac{3}{4}}(2)} \leq c_i \|v - w\|_{E_p^{\frac{1}{4} + \varepsilon_i}(2)} \left(1 + \|v\|_{E_p^{\frac{1}{4} + \varepsilon_i}(2)}^{\rho_i - 1} + \|w\|_{E_p^{\frac{1}{4} + \varepsilon_i}(2)}^{\rho_i - 1} \right) \quad (\text{B.1})$$

for any $v, w \in E_p^{\frac{1}{4} + \varepsilon_i}(2)$, where

$$\min\{\gamma(\varepsilon_i)\} =: \underline{\gamma} > \bar{\varepsilon} := \max\{\varepsilon_i\}. \quad (\text{B.2})$$

For this purpose we will suitably decompose the nonlinear term (see [16, Lemma 3.1]).

Lemma B.1. If f_0 satisfies (1.7), (1.10) and $f_0(\cdot, 0) = 0$ then there exists a decomposition

$$f_0(x, v) = f_{01}(x, v) + f_{02}(x, v), \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R},$$

such that

$$\begin{aligned} f_{01}(x, 0) &= f_{02}(x, 0) = 0, \\ f_{01} : \mathbb{R}^N \times \mathbb{R} &\rightarrow \mathbb{R} \text{ is a globally Lipschitz map} \end{aligned} \quad (\text{B.3})$$

and

$$|f_{02}(x, v_1) - f_{02}(x, v_2)| \leq c|v_1 - v_2|(|v_1|^{\rho-1} + |v_2|^{\rho-1}), \quad v_1, v_2 \in \mathbb{R}, \quad (\text{B.4})$$

for some $c > 0$.

Below we consider the maps

$$\mathcal{F}_1(u) = P_0(f_{01}(\cdot, u) + g), \quad \mathcal{F}_2(u) = P_0(f_{02}(\cdot, u)) \quad \text{and} \quad \mathcal{F}_3(u) = P_0(m(\cdot)u).$$

Due to (B.3) ε -regularity properties of \mathcal{F}_1 are much straightforward and in the next lemma we describe ε -regularity of \mathcal{F}_2 .

Lemma B.2. Assume (B.4) for some $\rho \in (1, \rho_c]$.

Then $\mathcal{F}_2(u) = P(f_{02}(\cdot, u))$ satisfies (B.1) with certain $c_2 > 0$, $\rho_2 = \rho$, $\varepsilon_2 \in (0, \frac{1}{\rho})$ and $\gamma(\varepsilon_2) \in [\rho\varepsilon_2, 1)$. In particular, (B.1) holds with $\rho_2 = \rho$ and

$$\gamma(\varepsilon_2(\rho)) = \frac{1}{4}, \quad \varepsilon_2(\rho) = \max \left\{ 0, \frac{\rho(N-p) - N}{4\rho p} \right\}.$$

Consequently whenever $\rho \leq \rho_c$, $P(f_{02}(\cdot, u))$ can be viewed as a Lipschitz continuous map on bounded sets from $E_p^{\frac{1}{4} + \frac{1}{4\rho}}(2) = H_p^{1 + \frac{1}{\rho}}(\mathbb{R}^N)$ into $E_p^{-\frac{1}{2}}(2) = H_p^{-2}(\mathbb{R}^N)$.

Proof. Note that

$$\|P_0(f_{02}(\cdot, v) - f_{02}(\cdot, w))\|_{E_p^{\gamma(\varepsilon_2) - \frac{3}{4}}(2)} \leq c \|P_0(f_{02}(\cdot, v) - f_{02}(\cdot, w))\|_{H_p^{4\gamma(\varepsilon_2) - 3}(\mathbb{R}^N)}$$

and hence,

$$\begin{aligned} & \|P_0(f_{02}(\cdot, v) - f_{02}(\cdot, w))\|_{E_p^{\gamma(\varepsilon_2) - \frac{3}{4}}(2)} \\ & \leq c' \|(P_0 + I)^{2\gamma(\varepsilon_2) - \frac{3}{2}}(P_0 + I - I)(f_{02}(\cdot, v) - f_{02}(\cdot, w))\|_{L^p(\mathbb{R}^N)} \\ & \leq c'' \|(P_0 + I)^{2\gamma(\varepsilon_2) - \frac{1}{2}}(f_{02}(\cdot, v) - f_{02}(\cdot, w))\|_{L^p(\mathbb{R}^N)} \\ & \leq c''' \|f_{02}(\cdot, v) - f_{02}(\cdot, w)\|_{H_p^{4\gamma(\varepsilon_2) - 1}(\mathbb{R}^N)}, \end{aligned}$$

where we consider $\gamma(\varepsilon_2)$ such that

$$4\gamma(\varepsilon_2) - 1 \leq 0.$$

We now use the embedding

$$H_p^{4\gamma(\varepsilon_2)-1}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad p \geq q \geq \frac{Np}{N + (1 - 4\gamma(\varepsilon_2))p}, \quad q > 1,$$

to get via Hölder inequality

$$\begin{aligned} & \|P_0(f_{02}(\cdot, v) - f_{02}(\cdot, w))\|_{E_p^{\gamma(\varepsilon_2)-\frac{3}{4}}(2)} \\ & \leq \tilde{c} \| |v_1 - v_2| (|v_1|^{\rho-1} + |v_2|^{\rho-1}) \|_{L^{\frac{Np}{N+(1-4\gamma(\varepsilon_2))p}}(\mathbb{R}^N)} \\ & \leq \tilde{c} \|v_1 - v_2\|_{L^{\frac{Np}{N-(1+4\varepsilon_2)p}}(\mathbb{R}^N)} \| (|v_1|^{\rho-1} + |v_2|^{\rho-1}) \|_{L^{\frac{N}{2+4\varepsilon_2-4\gamma(\varepsilon_2)}}(\mathbb{R}^N)}. \end{aligned}$$

It is easy to see that the right-hand side of the above inequality will be bounded by the right-hand side of (B.1) provided that

$$\frac{N(\rho - 1)}{2 + 4\varepsilon_2 - 4\gamma(\varepsilon_2)} \leq \frac{Np}{N - (1 + 4\varepsilon_2)p},$$

that is when

$$\rho \leq \frac{N + p - 4p\gamma(\varepsilon_2)}{N - (1 + 4\varepsilon_2)p}. \quad (\text{B.5})$$

Since $\gamma(\varepsilon_2) \in [\varepsilon_2\rho, 1)$ it is sufficient to ensure that $\rho \leq \frac{N+p-4p\rho\varepsilon_2}{N-4(1+\varepsilon_2)p}$, which is the case for $\rho \leq \rho_c$. On the other hand, when $\rho = \rho_c$ then one needs $\gamma(\varepsilon_2) = \varepsilon_2\rho$ to satisfy (B.5).

We remark that, whenever $\rho \in (1, \rho_c]$, the above argument can be carried out with $\gamma(\varepsilon_2(\rho)) = \frac{1}{4}$ and $\varepsilon_2(\rho) = \max\{0, \frac{\rho(N-p)-N}{4\rho p}\}$. We also remark that $\varepsilon_2(\rho_c) = \frac{1}{4\rho_c}$ and that $\varepsilon_2(\rho) = 0$ for $\rho < \frac{N}{N-p}$. \square

We now translate the properties of the multiplication operator from Lemma A.1 into the ε_3 -regularity of the map \mathcal{F}_3 .

Lemma B.3. Suppose that $m \in L_U^r(\mathbb{R}^N)$, $r > \max\{\frac{N}{2}, 1\}$ and $p \in (1, \infty)$ and either $\frac{N}{p} \geq 1$ or $1 > \frac{N}{p} > \frac{N}{r} - 1$. Then \mathcal{F}_3 satisfies

$$\|\mathcal{F}_3(v) - \mathcal{F}_3(w)\|_{E_p^{\gamma(\varepsilon_3)-\frac{3}{4}}(2)} \leq c \|v - w\|_{E_p^{\frac{1}{4}+\varepsilon_3}(2)}$$

with $\gamma(\varepsilon_3) = \frac{1}{4} + (\frac{N}{4p} - \frac{N}{4r})_- > 0$ and with any number $\varepsilon_3 > 0$ which is strictly less and arbitrarily close to $\frac{1}{4} + (\frac{N}{4p} - \frac{N}{4r})_-$. Consequently, (B.1) holds with a certain $\rho_3 > 1$ such that $\gamma(\varepsilon_3) > \rho_3\varepsilon_3$.

Proof. Note that $\frac{1}{4} + (\frac{N}{4p} - \frac{N}{4r})_- > 0$ as we have both $r > \frac{N}{2}$ and $\frac{N}{p} \geq 1$ or $1 > \frac{N}{p} > \frac{N}{r} - 1$.

As at the beginning of the proof of Lemma B.2 we obtain

$$\begin{aligned}
\|P_0(Q_m(v) - Q_m(w))\|_{E_p^{\gamma(\varepsilon_3) - \frac{3}{4}}(2)} &\leq c \|Q_m(v) - Q_m(w)\|_{H_p^{4\gamma(\varepsilon_3) - 1}(\mathbb{R}^N)} \\
&\leq c \|Q_m(v) - Q_m(w)\|_{H_p^{(\frac{N}{p} - \frac{N}{r})_-}(\mathbb{R}^N)} \\
&= c \|Q_m(v) - Q_m(w)\|_{H_p^{2(\beta^*(1,p) - 1)}(\mathbb{R}^N)}.
\end{aligned}$$

Using then Lemma A.1 with $\beta = \beta^*(1, p) - 1$ we have

$$\|P_0(Q_m(v) - Q_m(w))\|_{E_p^{\gamma(\varepsilon_3) - \frac{3}{4}}(2)} \leq c \|m\|_{L'_U(\mathbb{R}^N)} \|v - w\|_{H_p^{2\alpha}(\mathbb{R}^N)}$$

where α is strictly less and arbitrarily close to $\beta^*(1, p)$. Viewing now 2α as a sum $1 + 4\varepsilon_3$ we get

$$\|P_0(Q_m(v) - Q_m(w))\|_{E_p^{\gamma(\varepsilon_3) - \frac{3}{4}}(2)} \leq c \|m\|_{L'_U(\mathbb{R}^N)} \|v - w\|_{H_p^{1+4\varepsilon_3}(\mathbb{R}^N)} \leq c \|v - w\|_{E_p^{\frac{1}{4} + \varepsilon_3}(2)}$$

with ε_3 strictly less and arbitrarily close to $\frac{1}{4} + (\frac{N}{4p} - \frac{N}{4r})_-$. \square

Proof of Theorem 2.1. We will focus below on the case (iii).

Letting $\gamma(\varepsilon_1) = \frac{1}{4}$ and using (B.3) we immediately have

$$\begin{aligned}
\|P_0(f_{01}(\cdot, v) - f_{01}(\cdot, w))\|_{E_p^{\gamma(\varepsilon_1) - \frac{3}{4}}(2)} &\leq c \|f_{01}(\cdot, v) - f_{01}(\cdot, w)\|_{H_p^{4\gamma(\varepsilon_1) - 1}(\mathbb{R}^N)} \\
&\leq c' \|v - v\|_{L^p(\mathbb{R}^N)} \leq c' \|v - v\|_{E_p^{\frac{1}{4} + \varepsilon_1}(2)},
\end{aligned}$$

with any $\varepsilon_1 > 0$. From this and Lemmas B.1–B.3 the maps \mathcal{F}_i , $i = 1, 2, 3$, satisfy (B.1). Furthermore, we have

$$\underline{\gamma} = \frac{1}{4} + \left(\frac{N}{4p} - \frac{N}{4r} \right)_-$$

and

$$\bar{\varepsilon} = \max \left\{ 0, \frac{\rho(N-p) - N}{4\rho p}, \varepsilon_3 \right\},$$

for some ε_3 that is less but close to $\frac{1}{4} + (\frac{N}{4p} - \frac{N}{4r})_-$, so that $\underline{\gamma} > \varepsilon_3$. Note that $\underline{\gamma} > 0$ as we have both $r > \frac{N}{2}$ and $N > p$. Also note that $r > \frac{N}{2}$ and $N > p$ imply that $\underline{\gamma} > \frac{\rho(N-p) - N}{4\rho p}$. Thus (B.2) holds and the result follows from [3, Theorem 2.2].

We remark that in the case (i) letting $\gamma(\varepsilon_2) = \frac{1}{4}$ we have

$$\begin{aligned}
\|P_0(f_{02}(\cdot, v) - f_{02}(\cdot, w))\|_{E_p^{\gamma(\varepsilon_2) - \frac{3}{4}}(2)} &\leq c \|f_{01}(\cdot, v) - f_{01}(\cdot, w)\|_{H_p^{4\gamma(\varepsilon_2) - 1}(\mathbb{R}^N)} \\
&\leq c' \|v - v\|_{E_p^{\frac{1}{4} + \varepsilon_2}(2)},
\end{aligned}$$

with any $\varepsilon_2 > 0$. In the case (ii) we obtain

$$\begin{aligned} \|P_0(f_{02}(\cdot, v) - f_{02}(\cdot, w))\|_{E_p^{\gamma(\varepsilon_2) - \frac{3}{4}}(2)} &\leq c \|f_{01}(\cdot, v) - f_{01}(\cdot, w)\|_{H_p^{4\gamma(\varepsilon_2) - 1}(\mathbb{R}^N)} \\ &\leq c' \|v - w\|_{E_p^{\frac{1}{4} + \varepsilon_2}(2)} \left(1 + \|v\|_{E_p^{\frac{1}{4} + \varepsilon_2}(2)}^{\rho - 1} + \|w\|_{E_p^{\frac{1}{4} + \varepsilon_2}(2)}^{\rho - 1}\right), \end{aligned}$$

again with $\gamma(\varepsilon_2) = \frac{1}{4}$ and any $\varepsilon_2 > 0$. Thus in the case (i) or (ii) the result follows as well provided that we have $\underline{\gamma} = \frac{1}{4} + (\frac{N}{4p} - \frac{N}{4r})_- > 0$ which translates into the assumption

$$\beta^*(1, p) > \frac{1}{2}. \quad (\text{B.6})$$

Note that (B.6) is satisfied if $r \geq p$ or if $p \leq N$, since $r > \frac{N}{2}$. Also, (B.6) is satisfied for $p = 2$ since $r > \max\{\frac{N}{2}, 1\}$. Thus (B.6) evidently holds in the case (ii) whereas in the case (i) we have it assuming $\frac{N}{p} > \frac{N}{r} - 1$. \square

Remark B.4. The above proof of Theorem 2.1 yields that $\mathcal{F}(u) = P_0(f(\cdot, u))$ is Lipschitz continuous on bounded sets from $E_p^{\frac{1}{4} + \bar{\varepsilon}}$ into $E_p^{\underline{\gamma} - \frac{3}{4}}$, where $\bar{\varepsilon} < \underline{\gamma}$ due to (B.2). For $r \geq p$ we have $\underline{\gamma} = \frac{1}{4}$ and hence, for any $\varepsilon < \frac{1}{4}$ close enough to $\frac{1}{4}$, $P_0(f(\cdot, u))$ is then a Lipschitz map from $E_p^{\frac{1}{4} + \varepsilon} = H_p^{1+4\varepsilon}(\mathbb{R}^N)$ into $E_p^{-\frac{1}{2}} = H_p^{-2}(\mathbb{R}^N)$. Therefore, both Δ^2 can be viewed as a sectorial operator in the space $\mathcal{X} = \mathcal{X}^0 := H^{-2}(\mathbb{R}^N)$ (see Theorem A.2) and the nonlinear term $\Delta(f(\cdot, u))$ can be viewed as a Lipschitz map from the fractional power space $\mathcal{X}^{\frac{3}{4} + \varepsilon} = H_p^{1+4\varepsilon}(\mathbb{R}^N)$ into $\mathcal{X} = H_p^{-2}(\mathbb{R}^N)$ with any $\varepsilon < \frac{1}{4}$ close enough to $\frac{1}{4}$, from which we infer that

- (i) choosing any $\varepsilon < \frac{1}{4}$ close enough to $\frac{1}{4}$, due to [12, Theorem 5], for each B bounded in $H_p^{1+4\varepsilon}(\mathbb{R}^N)$ and for arbitrarily small $t_0 > 0$ we have $\sup_{u_0 \in B} \|u(t, u_0)\|_{H_p^2(\mathbb{R}^N)} < \infty$,
- (ii) due to [15, Theorem 3.2.1], if the solutions exist globally in time, then given any $\varepsilon < \frac{1}{4}$ close enough to $\frac{1}{4}$ and given any set B bounded in $H_p^{1+4\varepsilon}(\mathbb{R}^N)$ and having compact closure in $H_p^{-2}(\mathbb{R}^N)$ we have that $\{u(t, u_0), u_0 \in B\}$ has compact closure in $H_p^{1+4\varepsilon}(\mathbb{R}^N)$ for any $t > 0$.

Proof of Remark 2.2. Following the proof of Theorem 2.1 and using [3, Theorem 2.2] we infer that there is a non-continuable unique solution u through $u_0 \in E_p^{\frac{1}{4}}(2)$ satisfying the variation of constants formula

$$u(t) = e^{-P_0^2 t} u_0 + \int_0^t e^{-P_0^2(t-s)} \mathcal{F}(u(s)) ds, \quad t \in [0, \tau_0).$$

Furthermore, u depends continuously upon the initial condition and regularity results in [3, Theorem 2.2] ensure that

$$u \in C([0, \tau_0), E_p^{\frac{1}{4}}(2)) \cap C((0, \tau_0), E_p^{\frac{1}{4} + \underline{\gamma}}(2)) \cap C^1((0, \tau_0), E_p^s(2)) \quad \text{for each } s < \frac{1}{4} + \underline{\gamma}, \quad (\text{B.7})$$

that is (2.1) holds. For $\rho < \rho_c$ we have $\gamma(\varepsilon_i) > \rho_i \varepsilon_i$, $i = 1, 2, 3$, and the existence time is uniform on arbitrarily large balls in $E_p^{\frac{1}{4}}(2) = H_p^1(\mathbb{R}^N)$. Consequently, due to the results in [3] (see also [2]) an $H_p^1(\mathbb{R}^N)$ -estimate of the solution on finite time intervals, guarantees that the solution exists for all

$t \geq 0$. On the other hand, for $\rho = \rho_c$ we have that an $E_p^{\frac{1}{4}+\bar{\varepsilon}}(2) = H_p^{1+4\bar{\varepsilon}}(\mathbb{R}^N)$ -estimate of the solutions on finite time intervals guarantees its global existence. Thus $\tau_{u_0} < \infty$ implies

$$\limsup_{t \rightarrow \tau_{u_0}^-} \|u(t)\|_{E_p^{\frac{1}{4}}(2)} = \infty \quad \text{for } \rho < \rho_c,$$

and

$$\limsup_{t \rightarrow \tau_{u_0}^-} \|u(t)\|_{E_p^{\frac{1}{4}+\bar{\varepsilon}}(2)} = \infty \quad \text{for } \rho = \rho_c. \quad (\text{B.8})$$

Concerning additional smoothing action of the solution we have the following counterpart of [3, (2.6)]: whenever $\rho < \rho_c$ and $\theta < \underline{\gamma}$, for any u_0, z_0 in the ball in $E_p^{\frac{1}{4}}(2)$ of radius $R > 0$, there exist $t_0(R)$ and $C(R)$ such that for any $0 < t \leq t_0$

$$t^\theta \|u(t, u_0) - u(t, z_0)\|_{E_p^{\frac{1}{4}+\theta}(2)} \leq C(R) \|u_0 - z_0\|_{E_p^{\frac{1}{4}}(2)}, \quad \theta < \underline{\gamma}, \quad (\text{B.9})$$

and

$$t^\theta \|u(t, u_0)\|_{E_p^{\frac{1}{4}+\theta}(2)} \leq C(R). \quad (\text{B.10})$$

Recall that $\underline{\gamma} = \frac{1}{4} + (\frac{N}{4p} - \frac{N}{4r})_-$ and $\underline{\gamma} = \frac{1}{4}$ when $p \leq r$.

Actually, referring to [3, Theorem 2.2] we have together with (B.9) that

$$\lim_{t \rightarrow 0^+} t^\theta \|u(t, u_0)\|_{E_p^{\frac{1}{4}+\theta}(2)} = 0, \quad \theta < \underline{\gamma}. \quad (\text{B.11})$$

To prove (B.10) we now choose any u_0 in an arbitrarily fixed ball $B(z_0, R)$ in $H^1(\mathbb{R}^N)$ around $z_0 \in H^1(\mathbb{R}^N)$ of radius $R > 0$. It follows from (B.11) that given $\theta < \underline{\gamma}$ there exists $t_0 > 0$ such that for $t \in (0, t_0]$

$$t^\theta \|u(t, z_0)\|_{E_p^{\frac{1}{4}+\theta}(2)} \leq 1.$$

Using this and (B.9), we obtain for any fixed $t \in (0, t_0]$ and $\theta < \underline{\gamma}$ that

$$\begin{aligned} t^\theta \|u(t, u_0)\|_{E_p^{\frac{1}{4}+\theta}(2)} &\leq t^\theta \|u(t, z_0)\|_{E_p^{\frac{1}{4}+\theta}(2)} + t^\theta \|u(t, u_0) - u(t, z_0)\|_{E_p^{\frac{1}{4}+\theta}(2)} \\ &\leq 1 + C(R) \|u_0 - z_0\|_{E_p^{\frac{1}{4}}(2)}. \end{aligned}$$

We now indicate the space in which the equality $\dot{u} + P_0^2 u = \mathcal{F}(u)$ actually holds. We first focus on the case when $r \geq p$, that is when $\beta^*(p) = \beta^*(1, p) = 1$.

Since in the case $r \geq p$ we have $\underline{\gamma} = \frac{1}{4}$, the map \mathcal{F} is then Lipschitz on bounded sets from $E_p^{\frac{1}{4}+\bar{\varepsilon}}(2)$ into $E_p^{-\frac{1}{2}}(2)$. Thus for $r \geq p$ we infer that, as long as the solution exists,

$$\dot{u} + P_0^2 u = \mathcal{F}(u) \quad \text{in } E_p^{-\frac{1}{2}}(2).$$

On the other hand, letting

$$w = -P_0 u + f(\cdot, u),$$

we may rewrite $\dot{u} + P_0^2 u = \mathcal{F}(u)$ as $\dot{u} = P_0 w$. We observe from Lemmas B.1–B.3 that

$$f \text{ takes } H_p^2(\mathbb{R}^N) \text{ into } L^p(\mathbb{R}^N).$$

Since $\underline{\gamma} = \frac{1}{4}$, (B.7) implies that $\dot{u} \in L^p(\mathbb{R}^N)$ and due to what was said above we thus have $w \in E_p^0 = L^p(\mathbb{R}^N)$. Using next properties of P_0 we infer from the equation $\dot{u} = P_0 w$ that $w = (P_0 + I)^{-1}(\dot{u} + w) \in (P_0 + I)^{-1}(L^p(\mathbb{R}^N)) = H_p^2(\mathbb{R}^N)$. Thus the equality $\dot{u} = P_0 w$ holds in $L^p(\mathbb{R}^N)$.

When $r < p$, in which case $\beta^*(p) = \beta^*(1, p) < 1$, we have due to the proof of Theorem 2.1 that $\underline{\gamma} = \frac{1}{2}\beta^*(1, p) - \frac{1}{4} < \frac{1}{4}$, \mathcal{F} is Lipschitz on bounded sets from $E_p^{\frac{1}{4}+\bar{\varepsilon}}(2)$ into $E_p^{\frac{\gamma}{p}-\frac{3}{4}}(2)$ and, as long as the solution exists, the inequality $\dot{u} + P_0^2 u = \mathcal{F}(u)$ holds in $E_p^{\frac{\gamma}{p}-\frac{3}{4}}(2) = E_p^{\frac{1}{2}\beta^*(1, p)-1}(2)$. For completeness we now prove that $\dot{u} = P_0 w$ in $H_p^{2(\beta^*(p)-1)}(\mathbb{R}^N)$.

We have that $\underline{\gamma}$ is bigger than $\bar{\varepsilon}$ and thus bigger than a number ε_2 appearing in Lemma B.2. This implies $2\beta^*(1, p) > 1 + 4\bar{\varepsilon} > 1 + 4\varepsilon_2$. On the other hand, in Lemma B.2 we can choose $\gamma(\varepsilon_2(\rho)) = \frac{1}{4}$ and follow the proof of this lemma to conclude that f_{02} takes $E_p^{\frac{1}{4}+\varepsilon_2}(2) = H_p^{1+4\varepsilon_2}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$. Consequently, since $H_p^{2\beta^*(1, p)}(\mathbb{R}^N) \hookrightarrow H_p^{1+4\varepsilon_2}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N) \hookrightarrow H_p^{2\beta^*(1, p)-2}(\mathbb{R}^N)$, f_{02} and thus also f_0 take $H_p^{2\beta^*(1, p)}(\mathbb{R}^N)$ into $H_p^{2\beta^*(1, p)-2}(\mathbb{R}^N)$.

Applying Lemma A.1 with $\beta = \beta^*(1, p) - 1$ and α less but close enough to $\beta^*(1, p)$ we next have that the multiplication operator Q_m , $Q_m(\phi)(\cdot) = m(\cdot)\phi(\cdot)$, takes $H_p^{2\alpha}(\mathbb{R}^N)$ and thus also $H_p^{2\beta^*(1, p)}(\mathbb{R}^N)$ into $H_p^{2\beta^*(1, p)-2}(\mathbb{R}^N)$. This and the above property of f_{02} ensure together that

$$f \text{ takes } H_p^{2\beta^*(1, p)}(\mathbb{R}^N) \text{ into } H_p^{2\beta^*(1, p)-2}(\mathbb{R}^N). \quad (\text{B.12})$$

Summarizing, for $r < p$, as long as the solution exists, we have $\dot{u} + P_0^2 u = \mathcal{F}(u)$ in $H_p^{2\beta^*(1, p)-4}(\mathbb{R}^N)$, which can be viewed as $\dot{u} = P_0(-P_0 + f(\cdot, u)) = P_0 w$ with $u \in H_p^{2\beta^*(1, p)}(\mathbb{R}^N)$, $\dot{u} \in H_p^{2\beta^*(1, p)-2}(\mathbb{R}^N)$ (see (B.7)) and $w = -P_0 u + f(\cdot, u) \in H_p^{2\beta^*(1, p)-2}(\mathbb{R}^N)$ due to (B.12) and properties of P_0 . From the equation $\dot{u} = P_0 w$, using again properties of P_0 , we thus infer that $w = (P_0 + I)^{-1}(\dot{u} + w) \in (P_0 + I)^{-1}(H_p^{2\beta^*(1, p)-2}(\mathbb{R}^N)) = H_p^{2\beta^*(1, p)}(\mathbb{R}^N)$. Consequently, the equality $\dot{u} = P_0 w$ holds in $H_p^{2(\beta^*(1, p)-1)}(\mathbb{R}^N)$. \square

Proof of (2.3). We finally make a few remarks concerning (2.3). Note that Δ can be viewed as a closed operator in $H^{-2}(\mathbb{R}^N)$ with the domain $L^2(\mathbb{R}^N)$. Furthermore, Δ commutes with the resolvent operators of $-\Delta^2$; that is, whenever a is in the resolvent set $\rho(-\Delta^2)$, we have

$$\Delta(a + \Delta^2)^{-1}v = (a + \Delta^2)^{-1}\Delta v, \quad v \in L^2(\mathbb{R}^N). \quad (\text{B.13})$$

Indeed (B.13) is implied by the condition

$$v = (a + \Delta^2)\Delta^{-1}(a + \Delta^2)^{-1}\Delta v,$$

which is true as

$$v = (a + \Delta^2)\Delta^{-1}(a + \Delta^2)^{-1}\Delta v = \Delta^{-1}(a + \Delta^2)(a + \Delta^2)^{-1}\Delta v = v.$$

Combining this with the closedness of the operator Δ and with the exponential formula $\lim_{n \rightarrow \infty} (I + \frac{\tau}{n} \Delta^2)^{-n} v = e^{-\Delta^2 \tau} v$ (see [37, §1.8]) we conclude that

$$e^{-\Delta^2 \tau} \Delta v = \Delta e^{-\Delta^2 \tau} v, \quad v \in L^2(\mathbb{R}^N). \quad (\text{B.14})$$

Thus (2.3) is immediate from (2.2) and (B.14). \square

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