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On the solutions of a model equation for shallow water waves of moderate amplitude

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ABSTRACT

This paper is concerned with the Cauchy problem of a model equation for shallow water waves of moderate amplitude, which was proposed by A. Constantin and D. Lannes [The hydrodynamical relevance of the Camassa–Holmand Degasperis–Procesi equations, Arch. Ration. Mech. Anal. 192 (2009) 165–186]. First, the local well-posedness of the model equation is obtained in Besov spaces $B^s_{p,r}$, $p, r \in [1, \infty]$, $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$ (which generalize the Sobolev spaces H^s) by using Littlewood–Paley decomposition and transport equation theory. Second, the local well-posedness in critical case (with $s = \frac{3}{2}$, $p = 2$, $r = 1$) is considered. Moreover, with analytic initial data, we show that its solutions are analytic in both variables, globally in space and locally in time. Finally, persistence properties on strong solutions are also investigated.

1. Introduction

In this paper, we consider the following Cauchy problem for a model equation for shallow water waves of moderate amplitude [19]

$$\begin{cases} \eta_t + \eta_x + \dfrac{3}{2}\varepsilon\eta\eta_x - \dfrac{3}{8}\varepsilon^2\eta^2\eta_x + \dfrac{3}{16}\varepsilon^3\eta^3\eta_x + \mu(\alpha\eta_{xxx} + \beta\eta_{xxt}) & \\ \quad = \varepsilon\mu(\gamma\eta\eta_{xxx} + \delta\eta_x\eta_{xx}), & t > 0,\ x \in \mathbb{R}, \\ \eta(0, x) = \eta_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

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Here α, γ, δ and $\beta < 0$ are parameters, μ is shallowness parameter and ε is amplitude parameter. In terms of the two fundamental parameters μ and ε, the shallow water regime of waves of small amplitude (proper to KdV) is characterized by $\mu \ll 1$ and $\varepsilon = O(\mu)$, while the regime of shallow water waves of moderate amplitude (proper to CH) corresponds to $\mu \ll 1$ and $\varepsilon = O(\sqrt{\mu})$, see [19,1]. Since quantities of order $\varepsilon = O(\sqrt{\mu})$ are also of order $\varepsilon = O(\mu)$ for $\mu \ll 1$, the regime of moderate amplitude captures a wider range of wave profiles. In particular, within this regime one expects to obtain equations that model surface water wave profiles that develop singularities in finite time in the form of breaking waves. Recently, Constantin and Lannes [19] proposed the model equation for the evolution of the surface elevation (1.1). The local well-posedness of (1.1) for any initial data $\eta_0 \in H^{s+1}(\mathbb{R})$ with $s > \frac{3}{2}$ was proved in [19]. In [26], author proved the local well-posedness of Eq. (1.1) for initial data in $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ by using Kato's semigroup approach for quasi-linear equations. Note that, unlike KdV or CH, Eq. (1.1) does not have a bi-Hamiltonian integrable structure (see [10]). Nevertheless, the equation possesses solitary wave profiles that resemble those of CH, analyzed in [14], and present similarities with the shape of the solitary waves for the governing equations for water waves discussed in [21,15], as proved in [32].

For $\alpha = \frac{1}{12}, \beta = -\frac{1}{12}, \gamma = -\frac{7}{24}$ and $\sigma = -\frac{7}{12}$, Eq. (1.1) becomes the following equation

$$\begin{cases} \eta_t + \eta_x + \dfrac{3}{2}\varepsilon\eta\eta_x - \dfrac{3}{8}\varepsilon^2\eta^2\eta_x + \dfrac{3}{16}\varepsilon^3\eta^3\eta_x + \dfrac{\mu}{12}(\eta_{xxx} - \eta_{xxt}) & \\ \quad = -\dfrac{7}{24}\varepsilon\mu(\eta\eta_{xxx} + 2\eta_x\eta_{xx}), & t > 0,\ x \in \mathbb{R}, \\ \eta(0,x) = \eta_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

In [19], it is proved that if the maximal existence time of (1.2) is finite blow-up occurs in the form of wave breaking. Orbital stability and existence of solitary waves for Eq. (1.2) was recently obtained in [32,27].

One of the closest relatives of Eq. (1.1) is the Camassa–Holm equation

$$\begin{cases} u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

modelling the unidirectional propagation of shallow water waves over a flat bottom, $u(t,x)$ stands for the fluid velocity at time t in the spatial direction x. It is a well-known integrable equation describing the velocity dynamics of shallow water waves. This equation spontaneously exhibits emergence of singular solutions from smooth initial conditions. It has a bi-Hamilton structure [28] and is completely integrable [5,8]. In particular, it possesses an infinity of conservation laws and is solvable by its corresponding inverse scattering transform. After the birth of the Camassa–Holm equation, many works have been carried out to probe its dynamic properties. Such as, Eq. (1.2) has travelling wave solutions of the form $ce^{-|x-ct|}$, called peakons, which describes an essential feature of the travelling waves of largest amplitude (see [9,14,22,16]). It is shown in [20,12,17] that the inverse spectral or scattering approach is a powerful tool to handle the Camassa–Holm equation and analyze its dynamics. It is worthwhile to mention that Eq. (1.2) gives rise to geodesic flow of a certain invariant metric on the Bott–Virasoro group [18,55], and this geometric illustration leads to a proof that the Least Action Principle holds. It is shown in [13] that the blow-up occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time. Moreover, the Camassa–Holm equation has global conservative solutions [3,47] and dissipative solutions [4,48]. For other methods to handle the problems relating to various dynamic properties of the Camassa–Holm equation and other shallow water equations, the reader is referred to [2,21,7,39–45,36,37,29–31, 51–54,57,33–35] and the references therein.

Motivated by the references cited above, the goal of the present paper is to establish the local well-posedness for the strong solutions to the Cauchy problem (1.1) in Besov spaces. The proof of the local well-posedness is motivated by that in Danchin's celebrated paper [23–25] in the study of the local well-posedness to the Camassa–Holm equation. However, one problematics issue is that we here deal with the equation with a higher order nonlinearity in the Besov spaces, making the proof of several required nonlinear estimates somewhat delicate. These difficulties are nevertheless overcome by carefully estimates for each iterative approximation of solutions to (1.1). Moreover, we also prove the analyticity of its solutions in both variables, with x in $\mathbb{R}$ and t in an interval around zero, provided that the initial profile u_0 is an analytic function on the real line. Hence, this analytic result can be viewed as a Cauchy–Kowalevski theorem for (1.1). Analyticity is inherent to travelling water waves (see [16]). Finally, persistence properties on strong solutions are also investigated.

Our main results could be stated as follows, where the definition of Besov spaces $B^s_{p,r}$, $E^s_{p,r}(T)$ and E_{s_0} will be given in Sections 2 and 4.

Theorem 1.1. *Let $p, r \in [1,\infty]$ and $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$. Assume that $u_0 \in B^s_{p,r}$. There exist a time $T > 0$ and a unique solution $u \in E^s_{p,r}(T)$ to the Cauchy problem (2.1) such that the map $u_0 \mapsto u : B^s_{p,r} \mapsto C([0,T]; B^{s'}_{p,r}) \cap C^1([0,T]; B^{s'-1}_{p,r})$ is continuous for every $s' < s$ when $r = \infty$ and $s' = s$ whereas $r < \infty$.*

Theorem 1.2. *Let u_0 be in $B^{\frac{3}{2}}_{2,1}$. There exists a time $T > 0$ such that Eq. (2.1) has a unique solution $u \in C([0,T]; B^{\frac{3}{2}}_{2,1}) \cap C^1([0,T]; B^{\frac{1}{2}}_{2,1})$. Moreover, the solution depends continuously on the initial data, i.e., the mapping $\Phi : u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $B^{\frac{3}{2}}_{2,1}$ into $C([0,T]; B^{\frac{3}{2}}_{2,1}) \cap C^1([0,T]; B^{\frac{1}{2}}_{2,1})$.*

Remark 1.1. Note that for every $s \in \mathbb{R}$, $B^s_{2,2} = H^s$. Theorem 1.1 holds true in Sobolev spaces H^s with $s > \frac{3}{2}$, which covers the corresponding result in [19,26] proved by using Kato's semigroup theory.

Remark 1.2. In particular, we have obtained the local well-posedness of Eq. (2.1) in the case $B^{\frac{3}{2}}_{2,1}$. However, this is not true in the case $B^{\frac{3}{2}}_{2,\infty}$ in view of the proof of Proposition 4 in [25]. Noting that $B^{\frac{3}{2}}_{2,1} \hookrightarrow H^{\frac{3}{2}} \hookrightarrow B^{\frac{3}{2}}_{2,\infty}$, one can see that $s = \frac{3}{2}$ is the critical index.

The study of analytic regularity of solutions of the Camassa–Holm equation by A. Himonas and G. Misiolek [44] using an abstract Cauchy–Kowalevski theorem led us to investigate the analytic regularity of the Cauchy problem (2.1) and prove the following theorem.

Theorem 1.3. *If the initial data u_0 is real analytic on the line $\mathbb{R}$ and belongs in a space E_{s_0}, for some $0 < s_0 \leqslant 1$, then there exist an $\varepsilon > 0$ and a unique solution u to the Cauchy problem (2.1) that is analytic on $(-\varepsilon, \varepsilon) \times \mathbb{R}$.*

Remark 1.3. We would like to note that the analyticity properties of the solutions to Eq. (2.1) and Camassa–Holm equations are quite different from those of the Korteweg–de Vries equation whose solutions are analytic in the space variable for all time [56] but are not analytic in the time variable [50].

Theorem 1.4. *Assume that $u_0(x) \in H^s$ with $s > \frac{3}{2}$ satisfies that for some $\theta \in (0,1)$*

$$\left|u_0(x)\right|, \left|u_{0x}(x)\right| \sim O\left(e^{-\theta x}\right) \quad \text{as } x \uparrow \infty.$$

Then the corresponding strong solution $u(x) \in C([0,T]; H^s)$ to (2.1) satisfies that

$$\left|u(x)\right|, \left|u_x(x)\right| \sim O\left(e^{-\theta x}\right) \quad \text{as } x \uparrow \infty,$$

uniformly in the time interval $[0,T]$.

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Remark 1.4. In fact, Theorem 1.4 tells us the strong solution $u(x,t)$ corresponding to initial data with fast decay at infinity will behave asymptotically in the x-variable at infinity in its lifespan.

The rest of this paper is organized as follows. In Section 2, we prove the local well-posedness of the Cauchy problem (1.1) in the Besov spaces $B^s_{p,r}$, $p,r\in[1,\infty]$, $s>\max\{\frac{3}{2},1+\frac{1}{p}\}$. In Section 3, the local well-posedness in critical case (with $s=\frac{3}{2}$, $p=2$, $r=1$) is considered. Section 4 is devoted to the study of the analyticity of the Cauchy problem (1.1) based on a contraction type argument in a suitably chosen scale of the Banach spaces. Finally, persistence properties on strong solutions are also investigated.

2. Local well-posedness in $B^s_{p,r}$, $p,r\in[1,\infty]$, $s>\max\{\frac{3}{2},1+\frac{1}{p}\}$

In this section, we shall establish local well-posedness for the Cauchy problem (1.1) in the Besov spaces.

We prove here the well-posedness of the general class of equations

$$\begin{cases} u_t+u_x+\dfrac{3}{2}\varepsilon uu_x+\iota\varepsilon^2u^2u_x+\kappa\varepsilon^3u^3u_x+\mu(\alpha u_{xxx}+\beta u_{xxt}) & \\ \quad =\varepsilon\mu(\gamma uu_{xxx}+\delta u_x\eta_{xx}), & t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R}, \end{cases}$$

with $\iota,\kappa\in\mathbb{R}$, in particular, above equation coincides with (1.1) if one takes $\iota=-\frac{3}{8},\kappa=\frac{3}{16}$. Since $\beta<0$ and μ is so small, we also assume that $|\mu\beta|<1$. Using the Green function $G(x)\triangleq\frac{1}{2\sqrt{|\mu\beta|}}e^{-|\frac{x}{\sqrt{|\mu\beta|}}|}$, we have $(1+\mu\beta\partial_x^2)^{-1}f=G*f$ for all the $f\in L^2$, and $G*(u+\mu\beta u_{xx})=u$, where we denote by $*$ the convolution. Then we can rewrite the above Cauchy problem as follows

$$\begin{cases} u_t+\left(\dfrac{\alpha}{\beta}-\dfrac{\varepsilon\gamma}{\beta}u\right)u_x=P(D)\left[\left(1-\dfrac{\alpha}{\beta}\right)u+\left(\dfrac{3\varepsilon-2}{4}-\dfrac{1}{2\mu\beta}\right)u^2\right. & \\ \qquad\left.+\dfrac{\varepsilon^2\iota}{3}u^3+\dfrac{\varepsilon^3\kappa}{4}u^4+\dfrac{3\varepsilon\mu\gamma-\varepsilon\mu\delta-\mu\beta}{2}u_x^2\right], & x>t,\ x\in\mathbb{R},\\ u(x,0)=u_0(x), & x\in\mathbb{R}, \end{cases}\tag{2.1}$$

with the operator $P(D)\triangleq-\partial_x(1+\mu\beta\partial_x^2)^{-1}$.

For the convenience of the readers, we recall some facts on the Littlewood–Paley decomposition and some useful lemmas.

Notation. $\mathcal{S}$ stands for the Schwartz space of smooth functions over $\mathbb{R}^d$ whose derivatives of all order decay at infinity. The set $\mathcal{S}'$ of temperate distributions is the dual set of $\mathcal{S}$ for the usual pairing. We denote the norm of the Lebesgue space $L^p(\mathbb{R})$ by $\|\cdot\|_{L^p}$ with $1\leqslant p\leqslant\infty$, and the norm in the Sobolev space $H^s(\mathbb{R})$ with $s\in\mathbb{R}$ by $\|\cdot\|_{H^s}$.

Proposition 2.1 *(Littlewood–Paley decomposition). (See [6].) Let $\mathcal{B}\doteq\{\xi\in\mathbb{R}^d,\ |\xi|\leqslant\frac{4}{3}\}$ and $\mathcal{C}\doteq\{\xi\in\mathbb{R}^d,\ \frac{4}{3}\leqslant|\xi|\leqslant\frac{8}{3}\}$. There exist two radial functions $\chi\in C_c^\infty(\mathcal{B})$ and $\varphi\in C_c^\infty(\mathcal{C})$ such that*

$$\chi(\xi)+\sum_{q\geqslant0}\varphi(2^{-q}\xi)=1,\quad\forall\xi\in\mathbb{R}^d,$$

$$|q-q'|\geqslant2\quad\Rightarrow\quad\operatorname{Supp}\varphi(2^{-q}\cdot)\cap\operatorname{Supp}\varphi(2^{-q'}\cdot)=\emptyset,$$

$$q \geqslant 1 \quad \Rightarrow \quad \operatorname{Supp} \chi(\cdot) \cap \operatorname{Supp} \varphi\big(2^{-q'} \cdot\big) = \emptyset,$$

$$\frac{1}{3} \leqslant \chi(\xi)^2 + \sum_{q \geqslant 0} \varphi\big(2^{-q}\xi\big)^2 \leqslant 1, \quad \forall \xi \in \mathbb{R}^d.$$

Furthermore, let $h \doteq \mathcal{F}^{-1}\varphi$ and $\tilde{h} \doteq \mathcal{F}^{-1}\chi$. Then for all $f \in \mathcal{S}'(\mathbb{R}^d)$, the dyadic operators Δ_q and S_q can be defined as follows

$$\Delta_q f \doteq \varphi\big(2^{-q}D\big) f = 2^{qd} \int_{\mathbb{R}^d} h\big(2^q y\big) f(x-y)\, dy \quad \textit{for } q \geqslant 0,$$

$$S_q f \doteq \chi\big(2^{-q}D\big) f = \sum_{-1 \leqslant k \leqslant q-1} \Delta_k = 2^{qd} \int_{\mathbb{R}^d} \tilde{h}\big(2^q y\big) f(x-y)\, dy,$$

$$\Delta_{-1} f \doteq S_0 f \quad \textit{and} \quad \Delta_q f \doteq 0 \quad \textit{for } q \leqslant -2.$$

Hence,

$$f = \sum_{q \geqslant 0} \Delta_q f \quad \textit{in } \mathcal{S}'\big(\mathbb{R}^d\big),$$

where the right-hand side is called the nonhomogeneous Littlewood–Paley decomposition of f.

Lemma 2.1 *(Bernstein's inequality). (See [24].) Let $\mathcal{B}$ be a ball with center 0 in $\mathbb{R}^d$ and $\mathcal{C}$ a ring with center 0 in $\mathbb{R}^d$. A constant C exists so that, for any positive real number λ, any nonnegative integer k, any smooth homogeneous function σ of degree m and any couple of real numbers (a, b) with $b \geqslant a \geqslant 1$, there hold*

$$\operatorname{Supp} \hat{u} \subset \lambda\mathcal{B} \quad \Rightarrow \quad \sup_{|\alpha|=k} \big\|\partial^\alpha u\big\|_{L^a} \leqslant C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a},$$

$$\operatorname{Supp} \hat{u} \subset \lambda\mathcal{C} \quad \Rightarrow \quad C^{-k-1}\lambda^k \|u\|_{L^a} \leqslant \sup_{|\alpha|=k} \big\|\partial^\alpha u\big\|_{L^a} \leqslant C^{k+1}\lambda^k \|u\|_{L^a},$$

$$\operatorname{Supp} \hat{u} \subset \lambda\mathcal{C} \quad \Rightarrow \quad \big\|\sigma(D)u\big\|_{L^b} \leqslant C_{\sigma,m} \lambda^{m+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a},$$

for any function $u \in L^a$.

Definition 2.1 *(Besov space).* Let $s \in \mathbb{R}$, $1 \leqslant p, r \leqslant \infty$. The inhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ ($B^s_{p,r}$ for short) is defined by

$$B^s_{p,r} \doteq \big\{ f \in \mathcal{S}'\big(\mathbb{R}^d\big);\ \|f\|_{B^s_{p,r}} < \infty \big\},$$

where

$$\|f\|_{B^s_{p,r}} \doteq \begin{cases} (\sum_{q \in \mathbb{Z}} 2^{qsr} \|\Delta_q f\|^r_{L_p})^{\frac{1}{r}}, & \text{for } r < \infty, \\ \sup_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q f\|_{L_p}, & \text{for } r = \infty. \end{cases}$$

If $s = \infty$, $B^\infty_{p,r} \doteq \bigcap_{s \in \mathbb{R}} B^s_{p,r}$.

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Proposition 2.2. *(See [24].) Suppose that $s \in \mathbb{R}$, $1 \leqslant p, r, p_i, r_i \leqslant \infty$ $(i = 1, 2)$. We have*

(1) *Topological properties*: $B^s_{p,r}$ *is a Banach space which is continuously embedded in* $\mathcal{S}'$.
(2) *Density*: C^∞_c *is dense in* $B^s_{p,r} \Leftrightarrow 1 \leqslant p, r \leqslant \infty$.
(3) *Embedding*: $B^s_{p_1,r_1} \hookrightarrow B^{s-n(\frac{1}{p_1})-\frac{1}{p_2}}_{p_2,r_2}$, *if* $p_1 \leqslant p_2$ *and* $r_1 \leqslant r_2$.

$B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ *locally compact, if* $s_1 < s_2$.

(4) *Algebraic properties*: $\forall s > 0$, $B^s_{p,r} \cap L^\infty$ *is an algebra. Moreover,* $B^s_{p,r}$ *is an algebra, provided that* $s > \frac{n}{p}$ *or* $s \geqslant \frac{n}{p}$ *and* $r = 1$.
(5) *Complex interpolation*:

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leqslant C\|u\|^{\theta}_{B^{s_1}_{p,r}} \|u\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall u \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}, \ \forall \theta \in [0, 1].$$

(6) *Fatou lemma*: *If* $(u_n)_{n\in\mathbb{N}}$ *is bounded in* $B^s_{p,r}$ *and* $u_n \to u$ *in* $\mathcal{S}'$, *then* $u \in B^s_{p,r}$ *and*

$$\|u\|_{B^s_{p,r}} \leqslant \liminf_{n\to\infty} \|u_n\|_{B^s_{p,r}}.$$

(7) *Let* $m \in \mathbb{R}$ *and* f *be an* S^m*-multiplier (i.e.,* $f : \mathbb{R}^d \to \mathbb{R}$ *is smooth and satisfies that* $\forall \alpha \in \mathbb{N}^d$*, there exists a constant* C_α*, s.t.* $|\partial^\alpha f(\xi)| \leqslant C_\alpha(1 + |\xi|^{m-|\alpha|})$ *for all* $\xi \in \mathbb{R}^d$*). Then the operator* $f(D)$ *is continuous from* $B^s_{p,r}$ *to* $B^{s-m}_{p,r}$.
(8) *The paraproduct is continuous from* $B^{-\frac{1}{p}}_{P,1} \times (B^{\frac{1}{p}}_{p,\infty} \cap L^\infty)$ *to* $B^{-\frac{1}{p}}_{P,\infty}$*, i.e.,*

$$\|fg\|_{B^{-\frac{1}{p}}_{p,\infty}} \leqslant C\|f\|_{B^{-\frac{1}{p}}_{p,1}} \cap \|g\|_{B^{\frac{1}{p}}_{p,\infty}\cap L^\infty}.$$

(9) *A logarithmic interpolation inequality*

$$\|f\|_{B^{\frac{1}{p}}_{p,1}} \leqslant C\|f\|_{B^{\frac{1}{p}}_{p,\infty}} \ln\left(e + \frac{\|f\|_{B^{1+\frac{1}{p}}_{p,\infty}}}{\|f\|_{B^{\frac{1}{p}}_{p,\infty}}}\right).$$

Now we state some useful results in the transport equation theory, which are crucial to the proofs of our main theorems later.

Lemma 2.2. *(See [23,24].) Suppose that* $(p, r) \in [1, +\infty]^2$ *and* $s > -\frac{d}{p}$*. Let* v *be a vector field such that* ∇v *belongs to* $L^1([0, T]; B^{s-1}_{p,r})$ *if* $s > 1 + \frac{d}{p}$ *or to* $L^1([0, T]; B^{\frac{d}{p}}_{p,r} \cap L^\infty)$ *otherwise. Suppose also that* $f_0 \in B^s_{p,r}$*,* $F \in L^1([0, T]; B^s_{p,r})$ *and that* $f \in L^\infty(L^1([0, T]; B^s_{p,r}) \cap C([0, T]; \mathcal{S}')$ *solves the* d*-dimensional linear transport equations*

$$\begin{cases} \partial_t f + v \cdot \nabla f = F, \\ f|_{t=0} = f_0. \end{cases} \tag{T}$$

Then there exists a constant C *depending only on* s, p *and* d *such that the following statements hold*:

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(1) *If* $r = 1$ *or* $s \neq 1 + \frac{d}{p}$, *then*

$$\|f\|_{B^s_{p,r}} \leqslant \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}}\, d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{B^s_{p,r}}\, d\tau,$$

or

$$\|f\|_{B^s_{p,r}} \leqslant e^{CV(t)} C\left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}}\, d\tau\right) \tag{2.2}$$

holds, where $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{\frac{d}{p}}_{p,r}\cap L^\infty}\, d\tau$ *if* $s < 1 + \frac{d}{p}$ *and* $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{s-1}_{p,r}}\, d\tau$ *else.*

(2) *If* $s \leqslant 1 + \frac{d}{p}$ *and* $\nabla f_0 \in L^\infty$, $\nabla f \in L^\infty([0,T]\times\mathbb{R}^d)$ *and* $\nabla F \in L^1([0,T]; L^\infty)$, *then*

$$\|f\|_{B^s_{p,r}} + \|\nabla f\|_{L^\infty} \leqslant e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \|\nabla f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}} + \|\nabla F(\tau)\|_{L^\infty}\, d\tau\right)$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{\frac{d}{p}}_{p,r}\cap L^\infty}\, d\tau$.

(3) *If* $f = v$, *then for all* $s > 0$, *the estimate* (2.2) *holds with* $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{s-1}_{p,r}}\, d\tau$.

(4) *If* $r < +\infty$, *then* $f \in C([0,T]; B^s_{p,r})$. *If* $r = +\infty$, *then* $f \in C([0,T]; B^{s'}_{p,r})$ *for all* $s' < s$.

Lemma 2.3 *(Existence and uniqueness). (See [23,24].) Let* $(p, p_1, r) \in [1, +\infty]^3$ *and* $s > -d\min\{\frac{1}{p_1}, \frac{1}{p'}\}$ *with* $p' \doteq (1 - \frac{1}{p})^{-1}$. *Assume that* $f_0 \in B^s_{p,r}$, $F \in L^1([0,T]; B^s_{p,r})$. *Let* v *be a time dependent vector field such that* $v \in L^\rho([0,T]; B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$, $M > 0$ *and* $\nabla v \in L^1([0,T]; B^{\frac{d}{p}}_{p,r} \cap L^\infty)$ *if* $s < 1 + \frac{d}{p_1}$ *and* $\nabla v \in L^1([0,T]; B^{s-1}_{p_1,r})$ *if* $s > 1 + \frac{d}{p}$ *or* $s = 1 + \frac{d}{p_1}$ *and* $r = 1$. *Then the transport equations* (T) *have a unique solution* $f \in L^\infty([0,T]; B^s_{p,r}) \cap (\bigcap_{s'<s} C[0,T]; B^{s'}_{p,1})$ *and the inequalities in Lemma 2.2 hold true. Moreover,* $r < \infty$, *then we have* $f \in C([0,T]; B^s_{p,1})$.

Lemma 2.4 *(1-D Morse-type estimates). (See [23,24].) Assume that* $1 \leqslant p, r \leqslant +\infty$, *the following estimates hold*:

(i) *For* $s > 0$,

$$\|fg\|_{B^s_{p,r}} \leqslant C\big(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|g\|_{B^s_{p,r}}\|f\|_{L^\infty}\big);$$

(ii) $\forall s_1 \leqslant \frac{1}{p} < s_2$ ($s_2 \geqslant \frac{1}{p}$ *if* $r = 1$) *and* $s_1 + s_2 > 0$, *we have*

$$\|fg\|_{B^{s_1}_{p,r}} \leqslant C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}};$$

(iii) *In Sobolev spaces* $H^s = B^s_{2,2}$, *we have for* $s > 0$,

$$\|f\partial_x g\|_{H^s} \leqslant C\big(\|f\|_{H^{s+1}}\|g\|_{L^\infty} + \|\partial_x g\|_{H^s}\|f\|_{L^\infty}\big),$$

where C *is a positive constant independent of* f *and* g.

Definition 2.2. For $T > 0$, $s \in \mathbb{R}$ and $1 \leqslant p \leqslant +\infty$, we set

$$E^s_{p,r}(T) \triangleq C\big([0,T]; B^s_{p,r}\big) \cap C^1\big([0,T]; B^{s-1}_{p,r}\big) \quad \text{if } r \leqslant +\infty,$$
$$E^s_{p,\infty}(T) \triangleq L^\infty\big([0,T]; B^s_{p,\infty}\big) \cap lip^1\big([0,T]; B^{s-1}_{p,\infty}\big)$$
$$\text{and} \quad E^s_{p,r} \triangleq \bigcap_{T>0} E^s_{p,r}(T).$$

In the following, we denote $C > 0$ a generic constant only depending on p, r, s. Uniqueness and continuity with respect to the initial data are an immediate consequence of the following result.

Proposition 2.3. *Assume that $p, r \in [1,\infty]$ and $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$. Let $u, v \in L^\infty([0,T); B^s_{p,r}) \cap C([0,T); S')$ be two given solutions to Eq. (2.1) with the initial data $u_0, v_0 \in B^s_{p,r}$. Then for every $t \in [0,T]$, we have*

(1) *if $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$ but $s \neq 2 + \frac{1}{p}$, then*

$$\big\|u(t) - v(t)\big\|_{B^{s-1}_{p,r}} \leqslant \|u_0 - v_0\|_{B^{s-1}_{p,r}} \exp\left(C \int_0^t \sum_{i=1}^4 \big(\|u\|^i_{B^s_{p,r}} + \|v\|^i_{B^s_{p,r}}\big)\, d\tau\right); \tag{2.3}$$

(2) *if $s = 2 + \frac{1}{p}$ and $\theta \in [0,1]$ then*

$$\big\|u(t) - v(t)\big\|_{B^{s-1}_{p,r}} \leqslant C\|u_0 - v_0\|^\theta_{B^{s-1}_{p,r}} \exp\left(\theta C \int_0^t \sum_{i=1}^4 \big(\|u\|^i_{B^s_{p,r}} + \|v\|^i_{B^s_{p,r}}\big)\, d\tau\right) \times \big(\big\|u(t)\big\|_{B^s_{p,r}} + \big\|v(t)\big\|_{B^s_{p,r}}\big)^{1-\theta}.$$

Proof. For $s \neq 2 + \frac{1}{p}$, let $w = v - u$. It is obvious that w solves the transport equation

$$\begin{cases} w_t + \left(\dfrac{\alpha}{\beta} - \dfrac{\varepsilon\gamma}{\beta} u\right) w_x = \dfrac{\varepsilon\gamma}{\beta} v_x w + f, & t > 0,\ x \in \mathbb{R}, \\ w(x,0) = w_0(x) = u_0 - v_0, & x \in \mathbb{R}, \end{cases} \tag{2.4}$$

where

$$f = P(D)\left[\left(1 - \frac{\alpha}{\beta}\right) w + \left(\frac{3\varepsilon - 2}{4} - \frac{1}{2\mu\beta}\right)(u+v) w + \frac{\varepsilon^2 \iota}{3}\big(u^2 + uv + v^2\big) w + \frac{\varepsilon^3 \kappa}{4}\big(u^3 + u^2 v + u v^2 + v^3\big) w + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}(u_x + v_x) w_x\right].$$

In view of Lemma 2.2, it follows that

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$$e^{-C\int_0^t \|\partial_x[-\frac{\varepsilon\gamma}{\beta}u](\tau')\|_{B^{s-2}_{p,r}}\,d\tau'}\big\|w(t)\big\|_{B^{s-1}_{p,r}}$$
$$\leqslant \|w_0\|_{B^{s-1}_{p,r}}+C\int_0^t e^{-C\int_0^\tau \|\partial_x[-\frac{\varepsilon\gamma}{\beta}u](\tau')\|_{B^{s-2}_{p,r}}\,d\tau'}\left(\left\|\frac{\varepsilon\gamma}{\beta}v_xw\right\|_{B^{s-1}_{p,r}}+\|f\|_{B^{s-1}_{p,r}}\right)d\tau. \tag{2.5}$$

Due to $s>\max\{\frac{3}{2},1+\frac{1}{p}\}$, by virtue of Proposition 2.2, $B^{s-1}_{p,r}\subset L^\infty$ is an algebra. Thus we obtain

$$\left\|\frac{\varepsilon\gamma}{\beta}v_xw\right\|_{B^{s-1}_{p,r}}\leqslant C\|v_x\|_{B^{s-1}_{p,r}}\|w\|_{B^{s-1}_{p,r}}\leqslant C\|v\|_{B^{s}_{p,r}}\|w\|_{B^{s-1}_{p,r}},$$
$$\left\|\partial_x\left(\frac{\alpha}{\beta}-\frac{\varepsilon\gamma}{\beta}u^{(n)}\right)(\tau')\right\|_{B^{s-2}_{p,r}}\leqslant C\|\partial_x\big(u^{(n)}\big)(\tau')\|_{B^{s-2}_{p,r}}. \tag{2.6}$$

Since $P(D)$ is S^{-1}-multipliers, if $\max\{\frac{3}{2},1+\frac{1}{p}\}<s<2+\frac{1}{p}$, applying Proposition 2.2 and Lemma 2.4, we have

$$\begin{aligned}\|f\|_{B^{s-1}_{p,r}}&\leqslant C\bigg\|\left(1-\frac{\alpha}{\beta}\right)w+\left(\frac{3\varepsilon-2}{4}-\frac{1}{2\mu\beta}\right)(u+v)w+\frac{\varepsilon^2\iota}{3}\big(u^2+uv+v^2\big)w\\
&\quad+\frac{\varepsilon^3\kappa}{4}\big(u^3+u^2v+uv^2+v^3\big)w+\frac{3\varepsilon\mu\gamma-\varepsilon\mu\delta-\mu\beta}{2}(u_x+v_x)w_x\bigg\|_{B^{s-2}_{p,r}}\\
&\leqslant C\|w\|_{B^{s-2}_{p,r}}+\big\|(u+v)w\big\|_{B^{s-2}_{p,r}}+\big\|\big(u^2+uv+v^2\big)w\big\|_{B^{s-2}_{p,r}}\\
&\quad+\big\|\big(u^3+u^2v+uv^2+v^3\big)w\big\|_{B^{s-2}_{p,r}}+\big\|(u_x+v_x)w_x\big\|_{B^{s-2}_{p,r}}\\
&\leqslant C\sum_{i=1}^4\big(\|u\|^i_{B^s_{p,r}}+\|v\|^i_{B^s_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}.\end{aligned}$$

If $s>2+\frac{1}{p}$, we know that $B^{s-2}_{p,r}$ is an algebra. Thus, we also have

$$\|f\|_{B^{s-1}_{p,r}}\leqslant C\sum_{i=1}^4\big(\|u\|^i_{B^s_{p,r}}+\|v\|^i_{B^s_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}.$$

Therefore, inserting the above estimates to (2.6) we obtain

$$e^{-C\int_0^t \|\partial_xu(\tau')\|_{B^{s-2}_{p,r}}\,d\tau'}\big\|w(t)\big\|_{B^{s-1}_{p,r}}$$
$$\leqslant \|w_0\|_{B^{s-1}_{p,r}}+C\int_0^t e^{-C\int_0^\tau \|\partial_xu(\tau')\|_{B^{s-2}_{p,r}}\,d\tau'}\sum_{i=1}^4\big(\|u\|^i_{B^s_{p,r}}+\|v\|^i_{B^s_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}\,d\tau.$$

Hence, applying the Gronwall's inequality, we reach (2.3).

For the critical case $s=2+\frac{1}{p}$, we here use the interpolation method to deal with it. Indeed, if we choose $s_1\in(\max(1+\frac{1}{p},\frac{3}{2})-1,s-1)$, $s_2\in(s-1,s)$ and $\theta=\frac{s_2-(s-1)}{s_2-s_1}\in(0,1)$, then $s-1=\theta s_1+(1-\theta)s_2$. According complex interpolation and the consequence of case (1), we have

$$\begin{aligned}
\|w(t)\|_{B^{s-1}_{p,r}} &\leqslant \|w(t)\|^{\theta}_{B^{s_1}_{p,r}} \|w(t)\|^{1-\theta}_{B^{s_2}_{p,r}} \\
&\leqslant \|w_0\|^{\theta}_{B^{s_1}_{p,r}} \exp\left(\theta C \int_0^T \sum_{i=1}^{4} \left(\|u\|^i_{B^s_{p,r}} + \|v\|^i_{B^s_{p,r}}\right) d\tau\right) \left(\|u(t)\|_{B^{s_2}_{p,r}} + \|v(t)\|_{B^{s_2}_{p,r}}\right)^{1-\theta} \\
&\leqslant \|w_0\|^{\theta}_{B^{s-1}_{p,r}} \exp\left(\theta C \int_0^T \sum_{i=1}^{4} \left(\|u\|^i_{B^s_{p,r}} + \|v\|^i_{B^s_{p,r}}\right) d\tau\right) \left(\|u(t)\|_{B^{s}_{p,r}} + \|v(t)\|_{B^{s}_{p,r}}\right)^{1-\theta}.
\end{aligned}$$

Hence, we get the desired result. □

Now let us start the proof of Theorem 1.2, which is motivated by the proof of local existence theorem about the Camassa–Holm equation in [23]. Firstly, we shall use the classical Friedrichs regularization method to construct the approximate solutions to the Cauchy problem (2.1).

Lemma 2.5. *Assume that $u^{(0)} = 0$. Let $1 \leqslant p, r \leqslant +\infty$, $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$, and $u_0 \in B^s_{p,r}$. Then there exists a sequence of smooth functions $\{u^{(n)}\}_{n\in\mathbb{N}} \in C(R^+; B^{\infty}_{p,r})$ solving the following linear transport equation by induction*:

$$\begin{cases}
\left(\partial_t + \left(\frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta} u^{(n)}\right)\partial_x\right) u^{(n+1)} \\
\quad = P(D)\left[\left(1 - \frac{\alpha}{\beta}\right)u^{(n)} + \left(\frac{3\varepsilon - 2}{4} - \frac{1}{2\mu\beta}\right)\left(u^{(n)}\right)^2 + \frac{\varepsilon^2\iota}{3}\left(u^{(n)}\right)^3 \right. \\
\qquad \left. + \frac{\varepsilon^3\kappa}{4}\left(u^{(n)}\right)^4 + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}\left(u^{(n)}_x\right)^2\right], \\
u^{n+1}(x, 0) = u^{n+1}_0(x) = S_{n+1}u_0,
\end{cases} \tag{2.7}$$

where the operator $P(D) = -\partial_x(1 - \partial_x^2)^{-1}$. Moreover, there is a maximal existence time $T > 0$ such that the solutions $u^{(n)}$ satisfy the following conditions:

(i) $\{u^{(n)}\}_{n\in\mathbb{N}}$ *is uniformly bounded in $E^s_{p,r}(T)$,*
(ii) $\{u^{(n)}\}_{n\in\mathbb{N}}$ *is a Cauchy sequence in $C([0, T]; B^{s-1}_{p,r})$.*

Proof. Since all data $S_{n+1}u_0 \in B^{\infty}_{p,r}$, Lemma 2.3 enables us to show by induction that for all $n \in \mathbb{N}$, Eq. (2.1) has a global solution which belongs to $C(\mathbb{R}^+; B^{\infty}_{p,r})$. Thanks to $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$, we find $B^{s-1}_{p,r}$ is an algebra. From this, one obtains

$$\begin{aligned}
&\left\| P(D)\left[\left(1 - \frac{\alpha}{\beta}\right)u^{(n)} + \left(\frac{3\varepsilon - 2}{4} - \frac{1}{2\mu\beta}\right)\left(u^{(n)}\right)^2 + \frac{\varepsilon^2\iota}{3}\left(u^{(n)}\right)^3 \right.\right. \\
&\qquad \left.\left. + \frac{\varepsilon^3\kappa}{4}\left(u^{(n)}\right)^4 + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}\left(u^{(n)}_x\right)^2\right]\right\|_{B^s_{p,r}} \\
&\leqslant C\left(\|u^{(n)}\|_{B^s_{p,r}} + \|u^{(n)}\|^2_{B^s_{p,r}} + \|u^{(n)}\|^3_{B^s_{p,r}} + \|u^{(n)}\|^4_{B^s_{p,r}}\right) \\
&= C\sum_{i=1}^{4} \|u^{(n)}\|^i_{B^s_{p,r}},
\end{aligned}$$

$$\left\|\partial_x\left(\frac{\alpha}{\beta}-\frac{\varepsilon\gamma}{\beta}u^{(n)}\right)(\tau')\right\|_{B^{s-1}_{p,r}}\leqslant C\left\|\partial_x\left(u^{(n)}\right)(\tau')\right\|_{B^{s-1}_{p,r}}. \tag{2.8}$$

In view of Lemma 2.1 and the proof of Proposition 2.3, it follows that

$$\begin{aligned}&e^{-C\int_0^t\|\partial_x(u^{(n)})(\tau')\|_{B^{s-1}_{p,r}}d\tau'}\left\|u^{(n+1)}(t)\right\|_{B^s_{p,r}}\\&\quad\leqslant\|S_{n+1}u_0\|_{B^s_{p,r}}+C\int_0^t e^{-C\int_0^\tau\|\partial_x u^{(n)}(\tau')\|_{B^{s-1}_{p,r}}d\tau'}\left(\sum_{i=1}^4\left\|u^{(n)}\right\|^i_{B^s_{p,r}}\right)d\tau.\end{aligned} \tag{2.9}$$

Hence, we get

$$\begin{aligned}\left\|u^{(n+1)}(t)\right\|_{B^s_{p,r}}\leqslant{}& e^{C\int_0^t\|\partial_x u^{(n)}(\tau')\|_{B^{s-1}_{p,r}}d\tau'}\|u_0\|_{B^s_{p,r}}\\&+C\int_0^t e^{C\int_\tau^t\|\partial_x u^{(n)}(\tau)\|_{B^{s-1}_{p,r}}d\tau}\left(\sum_{i=1}^4\left\|u^{(n)}\right\|^i_{B^s_{p,r}}\right)d\tau.\end{aligned} \tag{2.10}$$

Let us choose a $T>0$ such that $6C\|u_0\|^3_{B^s_{p,r}}T<1$, and suppose by induction that for all $t\in[0,T]$

$$\left\|u^{(n)}(t)\right\|_{B^s_{p,r}}\leqslant\frac{\|u_0\|_{B^s_{p,r}}}{(1-6C\|u_0\|^3_{B^s_{p,r}}t)^{\frac13}}. \tag{2.11}$$

Indeed, since $B^{s-1}_{p,r}$ is an algebra, one obtains from (2.12) that

$$\begin{aligned}C\int_\tau^t\left\|\partial_x u^{(n)}(\tau')\right\|_{B^{s-1}_{p,r}}d\tau'&\leqslant C\int_\tau^t\left(\left\|u^{(n)}(\tau')\right\|_{B^s_{p,r}}+\left\|u^{(n)}(\tau')\right\|^3_{B^s_{p,r}}\right)d\tau'\\&\leqslant C\left(\int_\tau^t\frac{\|u_0\|_{B^s_{p,r}}}{(1-6C\|u_0\|^3_{B^s_{p,r}}\tau')^{\frac13}}d\tau'+\int_\tau^t\frac{\|u_0\|^3_{B^s_{p,r}}}{1-6C\|u_0\|^3_{B^s_{p,r}}\tau'}d\tau'\right)\\&=\frac{1}{4\|u_0\|^2_{B^s_{p,r}}}\left[\left(1-6C\|u_0\|^3_{B^s_{p,r}}\tau\right)^{\frac23}-\left(1-6C\|u_0\|^3_{B^s_{p,r}}t\right)^{\frac23}\right]\\&\quad+\frac16\ln\left(1-6C\|u_0\|^3_{B^s_{p,r}}\tau\right)-\frac16\ln\left(1-6C\|u_0\|^3_{B^s_{p,r}}t\right).\end{aligned} \tag{2.12}$$

And then inserting the above inequality and (2.12) into (2.11) leads to

$$\begin{aligned}\left\|u^{(n+1)}(t)\right\|_{B^s_{p,r}}\leqslant{}&\frac{e^{\frac{1}{4\|u_0\|^2_{B^s_{p,r}}}}\|u_0\|_{B^s_{p,r}}}{(1-6C\|u_0\|^3_{B^s_{p,r}}t)^{\frac16}}+\frac{Ce^{\frac{1}{4\|u_0\|^2_{B^s_{p,r}}}}}{(1-6C\|u_0\|^3_{B^s_{p,r}}t)^{\frac16}}\\&\times\int_0^t\left(1-6C\|u_0\|^3_{B^s_{p,r}}\tau\right)^{\frac16}\left(\frac{\|u_0\|^4_{B^s_{p,r}}}{(1-6C\|u_0\|^3_{B^s_{p,r}}\tau)^{\frac43}}+\frac{\|u_0\|^3_{B^s_{p,r}}}{1-6C\|u_0\|^3_{B^s_{p,r}}\tau}\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{\|u_0\|^2_{B^s_{p,r}}}{(1-6C\|u_0\|^2_{B^s_{p,r}}\tau)^{\frac{2}{3}}}+\frac{\|u_0\|_{B^s_{p,r}}}{(1-6C\|u_0\|_{B^s_{p,r}}\tau)^{\frac{1}{3}}}\Bigg)\,d\tau \\
&\leqslant \frac{e^{\frac{1}{4\|u_0\|^2_{B^s_{p,r}}}}\|u_0\|_{B^s_{p,r}}}{(1-6C\|u_0\|^3_{B^s_{p,r}}t)^{\frac{1}{6}}}\left(1+C\int_0^t \frac{\|u_0\|^3_{B^s_{p,r}}}{(1-6C\|u_0\|^3_{B^s_{p,r}}\tau)^{\frac{7}{6}}}\,d\tau\right) \\
&= \frac{e^{\frac{1}{4\|u_0\|^2_{B^s_{p,r}}}}\|u_0\|_{B^s_{p,r}}}{(1-6C\|u_0\|^3_{B^s_{p,r}}t)^{\frac{1}{3}}}. \qquad (2.13)
\end{aligned}
$$

Thus $\{u^n\}_{\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^s_{p,r})$. Similarly we deduce that $\partial_t u^{n+1}\in C([0,T];B^{s-1}_{p,r})$ uniformly bounded. Thus we get (i).

Next we show (ii). By Eq. (2.8), for all $m,n\in\mathbb{N}$, we obtain

$$
\partial_t\big(u^{(m+n+1)}-u^{(n+1)}\big)+\left(\frac{\alpha}{\beta}-\frac{\varepsilon\gamma}{\beta}u^{(n+m)}\right)\partial_x\big(u^{(m+n+1)}-u^{(n+1)}\big)=f' \qquad (2.14)
$$

where

$$
\begin{aligned}
f'=\;&\frac{\varepsilon\gamma}{\beta}u_x^{(n+1)}\big(u^{(m+n)}-u^{(n)}\big)+P(D)\Bigg[\left(1-\frac{\alpha}{\beta}\right)\big(u^{(m+n)}-u^{(n)}\big) \\
&+\left(\frac{3\varepsilon-2}{4}-\frac{1}{2\mu\beta}\right)\big(u^{(m+n)}+u^{(n)}\big)\big(u^{(m+n)}-u^{(n)}\big) \\
&+\frac{\varepsilon^2\iota}{3}\big(\big(u^{(m+n)}\big)^2+u^{(m+n)}u^{(n)}+\big(u^{(n)}\big)^2\big)\big(u^{(m+n)}-u^{(n)}\big) \\
&+\frac{\varepsilon^3\kappa}{4}\big(\big(u^{(m+n)}\big)^3+\big(u^{(m+n)}\big)^2u^{(n)}+u^{(m+n)}\big(u^{(n)}\big)^2+\big(u^{(n)}\big)^3\big)\big(u^{(m+n)}-u^{(n)}\big) \\
&+\frac{3\varepsilon\mu\gamma-\varepsilon\mu\delta-\mu\beta}{2}\big(u_x^{(m+n)}-u_x^{(n)}\big)\big(u_x^{(m+n)}-u_x^{(n)}\big)\Bigg].
\end{aligned}
$$

Similar to the proof of Proposition 2.3, one can deduce that

$$
\|f'\|_{B^{s-1}_{p,r}}\leqslant C\left(\sum_{i=1}^{4}\big(\|u^{(n+m)}\|^i_{B^s_{p,r}}+\|u^{(n)}\|^i_{B^s_{p,r}}\big)+\|u^{(n+1)}\|_{B^s_{p,r}}\right)\|u^{(m+n)}-u^{(n)}\|_{B^{s-1}_{p,r}}.
$$

Note that

$$
\begin{aligned}
\|u_0^{n+l}-u_0^{n+1}\|_{B^{s-1}_{p,r}}&=\|S_{n+l+1}u_0-S_{n+1}u_0\|_{B^s_{p,r}} \\
&=\left\|\sum_{k=n+l}^{l+n}\Delta_k u_0\right\|_{B^s_{p,r}}\leqslant C2^{-n}. \qquad (2.15)
\end{aligned}
$$

In the case of $s>\max\{\frac{3}{2},1+\frac{1}{p}\}$ but $s\neq 2+\frac{1}{p}$, in view of Lemmas 2.2 and 2.3, for all $t\in[0,T]$, we obtain

$$\begin{aligned}
&\big\|\big(u^{(n+m+1)}-u^{n+1}\big)(t)\big\|_{B^{s-1}_{p,r}}\\
&\leqslant e^{C\int_0^t\|u^{(n)}(\tau)\|_{B^s_{p,r}}\,d\tau}\Big(\big\|u_0^{(n+m)}-u_0^{(n)}\big\|_{B^{s-1}_{p,r}}+\int_0^t e^{-C\int_0^\tau\|u^{(n+m)}(\tau)\|_{B^s_{p,r}}\,d\tau}\\
&\quad\times C\left(\sum_{i=1}^4\big(\big\|u^{(n+m)}\big\|^i_{B^s_{p,r}}+\big\|u^{(n)}\big\|^i_{B^s_{p,r}}\big)+\big\|u^{(n+1)}\big\|_{B^s_{p,r}}\right)\big\|u^{(m+n)}-u^{(n)}\big\|_{B^{s-1}_{p,r}}\,d\tau \qquad (2.16)\\
&\leqslant C\left(2^{-n}+\int_0^t\big\|\big(u^{(n+m)}-u^{(n)}\big)(\tau)\big\|_{B^{s-1}_{p,r}}\,d\tau\right). \qquad (2.17)
\end{aligned}$$

By induction, with respect to the index m, one can easily get

$$\big\|\big(u^{(n+m+1)}-u^{(n+1)}\big)(t)\big\|_{L^\infty_T(B^{s-1}_{p,r})}\leqslant\frac{TC}{(n+1)!}\big(\big\|u^{(m)}\big\|_{L^\infty_T(B^s_{p,r})}\big)+C\sum_{k=0}^n 2^{k-n}\frac{(TC)^k}{k!}.$$

As $\|u^{(m)}\|_{L^\infty_T(B^{s-1}_{p,r})}$ and C are bounded independently of m, there exists constant C_1 independent of m, n such that

$$\big\|u^{(n+m+1)}-u^{(n+1)}(t)\big\|_{L^\infty_T([0,T];B^{s-1}_{p,r})}\leqslant C_1 2^{-n}.$$

Thus $u^{(n)}_{n\in N}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$.

On the other hand, for the critical points $s=2+\frac{1}{p}$, we can apply the interpolation method which has been used in the proof of Proposition 2.3 to show that $(u^{(n)})_{n\in\mathbb{R}}$ is also a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$ for this critical case. Therefore, we have completed the proof of Lemma 2.5. □

Proof of Theorem 1.1. Thanks to Lemma 2.5, we obtain that $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^s_{p,r})$, so it converges to some function $u\in C([0,T];B^{s-1}_{p,r})$. We now have to check that u belongs to $E^s_{p,r}(T)$ and solves the Cauchy problem (2.1). Since $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;B^s_{p,r})$ according to Lemma 2.5, the Fatou property for the Besov spaces (Proposition 2.2) guarantees that u also belongs to $L^\infty(0,T;B^s_{p,r})$.

On the other hand, as $\{u^{(n)}\}_{n\in\mathbb{N}}$ converges to u in $C([0,T];B^{s-1}_{p,r})$, an interpolation argument ensures that the convergence holds in $C([0,T];B^{s'}_{p,r})$, for any $s'<s$. It is then easy to pass to the limit in Eq. (2.1) and to conclude that u is indeed a solution to the Cauchy problem (2.1). Thanks to the fact that u belongs to $L^\infty(0,T;B^s_{p,r})$, the right-hand side of the equation

$$\begin{aligned}
u_t+\left(\frac{\alpha}{\beta}-\frac{\varepsilon\gamma}{\beta}u\right)u_x=P(D)\bigg[\left(1-\frac{\alpha}{\beta}\right)u+\left(\frac{3\varepsilon-2}{4}-\frac{1}{\mu\beta}\right)u^2+\frac{\varepsilon^2\iota}{3}u^3\\
+\frac{\varepsilon^3\kappa}{4}u^4+\frac{3\varepsilon\mu\gamma-\varepsilon\mu\delta-\mu\beta}{2}u_x^2\bigg]
\end{aligned}$$

belongs to $L^\infty(0,T;B^s_{p,r})$. In particular, for the case $r<\infty$, Lemma 2.3 enables us to conclude that $u\in C([0,T];B^{s'}_{p,r})$ for any $s'<s$. Finally, using the equation again, we see that $\partial_t u\in C([0,T];B^{s'}_{p,r})$ if $r<\infty$, and in $L^\infty(0,T;B^{s-1}_{p,r})$ otherwise. Moreover, a standard use of a sequence of viscosity approximate solutions $(u_\varepsilon)_\varepsilon>0$ for the Cauchy problem (2.1) which converges uniformly in $C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})$ gives the continuity of the solution u in $E^s_{p,r}$. □

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3. Local well-posedness in critical Besov spaces $B^{\frac{3}{2}}_{2,1}$

In this section, we shall establish the local well-posedness of Eq. (2.1) in critical Besov spaces. The remainder of this section is devoted to the proof of Theorem 1.2. To this end, we first prove the uniqueness of the solution.

Lemma 3.1. *Let u and v be solutions to (2.1) with initial datum u_0 and v_0 respectively. Assume that u_0 and v_0 belong to $B^{\frac{3}{2}}_{2,\infty} \cap lip$, and that u and v belong to $L^\infty([0,T]; B^{\frac{3}{2}}_{2,\infty} \cap Lip) \cap C([0,T]; B^{\frac{1}{2}}_{2,\infty})$. Let $w = v - u$ and $w_0 = v_0 - u_0$. There exists a constant C such that if for some $t_0 \leqslant T$*

$$\sup_{t\in[0,T]} \left(e^{-C\int_0^t \|\partial_x u(\tau)\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau} \left\| w(t) \right\|_{B^{\frac{1}{2}}_{2,\infty}} \right) \leqslant 1, \tag{3.1}$$

then, denoting $F(z) = z\ln(e+z)$ the following inequality holds true for $t \in [0, t_0]$

$$\frac{\|w(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}{e} \leqslant e^{C\int_0^t \|\partial_x u(\tau)\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau} \left(\frac{\|w_0\|_{B^{\frac{1}{2}}_{2,\infty}}}{e} \right)^{\exp[-C\int_0^t F[\sum_{i=1}^4 (\|u\|^i_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} + \|v\|^i_{B^{\frac{3}{2}}_{2,\infty}\cap Lip})]\, d\tau]}. \tag{3.2}$$

In particular, (3.2) holds true to $[0,T]$ provided that

$$\|w_0\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant e^{1-e^{C\int_0^t F[\sum_{i=1}^4 (\|u\|^i_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} + \|v\|^i_{B^{\frac{3}{2}}_{2,\infty}\cap Lip})]\, dt}}. \tag{3.3}$$

Proof. Obviously, w solves the transport equations

$$\begin{cases} w_t + \left(\dfrac{\alpha}{\beta} - \dfrac{\varepsilon\gamma}{\beta} u\right) w_x = f, & t > 0, \quad x \in \mathbb{R}, \\ w(x,0) = w_0(x) = u_0 - v_0, & x \in \mathbb{R}, \end{cases} \tag{3.4}$$

where

$$f = \frac{\varepsilon\gamma}{\beta} v_x w + P(D)\left[\left(1 - \frac{\alpha}{\beta}\right) w + \left(\frac{3\varepsilon - 2}{4} - \frac{1}{2\mu\beta} \right)(u+v)w + \frac{\varepsilon^2 \iota}{3}\left(u^2 + uv + v^2\right) w \right.$$
$$\left. + \frac{\varepsilon^3 \kappa}{4}\left(u^3 + u^2 v + u v^2 + v^3\right) w + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}(u_x + v_x) w_x \right].$$

Note that $P(D) \in Op(S^{-1})$. Applying Proposition 2.2 and the fact that

$$\|fg\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant \|f\|_{L^\infty} \|g\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant C\|f\|_{B^{\frac{1}{2}}_{2,1}} \|g\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty},$$

one can deduce that

$$\|f\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant C\|w\|_{B^{\frac{1}{2}}_{2,1}} \sum_{i=1}^{4}\left(\|u\|^{i}_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} + \|v\|^{i}_{B^{\frac{3}{2}}_{2,\infty}\cap Lip}\right),$$

$$\left\|\partial_x\left(\frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta}u\right)(\tau')\right\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} \leqslant C\left\|\partial_x u(\tau')\right\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}.$$

By Lemma 2.2 and Proposition 2.2, we obtain

$$\begin{aligned}\left\|w(t)\right\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant{}& \|w_0\|_{B^{\frac{1}{2}}_{2,\infty}} e^{C\int_0^t \|\partial_x u(\tau')\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau'} + C\int_0^t e^{C\int_\tau^t \|\partial_x u(\tau')\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau'} \\ &\times \left[\sum_{i=1}^{4}\left(\|u\|^{i}_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} + \|v\|^{i}_{B^{\frac{3}{2}}_{2,\infty}\cap Lip}\right)\right] \ln\left(e + \frac{\|w\|_{B^{\frac{3}{2}}_{2,\infty}}}{\|w\|_{B^{\frac{1}{2}}_{2,\infty}}}\right) d\tau. \qquad (3.5)\end{aligned}$$

Denote

$$W(t) = e^{-C\int_0^t \|\partial_x u(\tau')\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau'} \|w\|_{B^{\frac{1}{2}}_{2,\infty}},$$

$$H(t) = \sum_{i=1}^{4}\left(\|u\|^{i}_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} + \|v\|^{i}_{B^{\frac{3}{2}}_{2,\infty}\cap Lip}\right).$$

Because for $x \in (0, 1]$ and $\alpha > 0$,

$$\ln\left(e + \frac{\alpha}{x}\right) \leqslant \ln(e + \alpha)(1 - \ln x)$$

is true, inequality (3.4) can be rewritten as

$$W(t) \leqslant W(0) + C\int_0^t H(\tau)\ln\bigl(e + H(\tau)\bigr)W(\tau)\bigl(1 - \ln W(\tau)\bigr)\,d\tau$$

provided that $W \leqslant 1$ on $[0, t]$. Because of the hypothesis, using a Gronwall type argument yields

$$\frac{W(t)}{e} \leqslant \left(\frac{W(0)}{e}\right)^{\exp(-C\int_0^t H(\tau)\ln(e+H(\tau))\,d\tau)},$$

which yields the desired result. Note that (3.3) implies (3.1) with $t_0 = T$. This completes the proof of the lemma. □

The second step is to prove the existence of the solution.

Lemma 3.2. *Let the initial data u_0 be in $B^{\frac{3}{2}}_{2,1}$. Then there exists a $T > 0$ such that Eq. (2.1) has a solution $u \in C([0, T]; B^{\frac{3}{2}}_{2,1}) \cap C^1([0, T]; B^{\frac{1}{2}}_{2,1})$.*

Proof. Here we assume for the sake of simplicity that $B^{\frac{1}{2}}_{2,\infty} \hookrightarrow Lip$ and use a standard iterative process to build a solution. Introduce a nonnegative mollifier $\rho \in C_0^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \rho = 1$ and denote $\rho^{(n)}(x) = n^d \rho(nx)$. We choose $u^0 = 0$ and define a sequence of smooth functions $(u^{(n)})_{n\in\mathbb{N}}$ solving the following transport equation

$$\begin{cases} \left(\partial_t + \left(\frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta}u^{(n)}\right)\partial_x\right)u^{(n+1)} \\ \quad = P(D)\left[\left(1-\frac{\alpha}{\beta}\right)u^{(n)} + \left(\frac{3\varepsilon-2}{4} - \frac{1}{2\mu\beta}\right)\left(u^{(n)}\right)^2 \right. \\ \qquad \left. + \frac{\varepsilon^2\iota}{3}\left(u^{(n)}\right)^3 + \frac{\varepsilon^3\kappa}{4}\left(u^{(n)}\right)^4 + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}\left(u_x^{(n)}\right)^2\right], \\ u^{n+1}(x,0) = u_0^{n+1}(x) = \rho^{n+1} * u_0. \end{cases} \tag{3.6}$$

Note that $P(D) \in Op(S^{-1})$ and $B^{\frac{1}{2}}_{2,1}$ is an algebra, one can easily get

$$\begin{aligned} &\left\| P(D)\left[\left(1-\frac{\alpha}{\beta}\right)u^{(n)} + \left(\frac{3\varepsilon-2}{4} - \frac{1}{2\mu\beta}\right)\left(u^{(n)}\right)^2 + \frac{\varepsilon^2\iota}{3}\left(u^{(n)}\right)^3 \right.\right. \\ &\qquad \left.\left. + \frac{\varepsilon^3\kappa}{4}\left(u^{(n)}\right)^4 + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}\left(u_x^{(n)}\right)^2\right]\right\|_{B^{\frac{3}{2}}_{2,1}} \\ &\quad \leqslant C\left(\left\|u^{(n)}\right\|_{B^{\frac{3}{2}}_{2,1}} + \left\|u^{(n)}\right\|^2_{B^{\frac{3}{2}}_{2,1}} + \left\|u^{(n)}\right\|^3_{B^{\frac{3}{2}}_{2,1}} + \left\|u^{(n)}\right\|^4_{B^{\frac{3}{2}}_{2,1}}\right) \\ &\quad = C\sum_{i=1}^{4}\left\|u^{(n)}\right\|^i_{B^{\frac{3}{2}}_{2,1}}, \\ &\left\|\partial_x\left(\frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta}u^{(n)}\right)(\tau')\right\|_{B^{\frac{1}{2}}_{2,1}} \leqslant C\left\|\partial_x\left(u^{(n)}\right)(\tau')\right\|_{B^{\frac{1}{2}}_{2,1}}. \end{aligned} \tag{3.7}$$

By virtue of Lemma 2.2, we deduce

$$\begin{aligned} &e^{-C\int_0^t \|\partial_x(u^{(n)})(\tau')\|_{B^{\frac{1}{2}}_{2,1}} d\tau'}\left\|u^{(n+1)}(t)\right\|_{B^{\frac{3}{2}}_{2,1}} \\ &\quad \leqslant \|S_{n+1}u_0\|_{B^{\frac{3}{2}}_{2,1}} + C\int_0^t e^{-C\int_0^\tau \|\partial_x u^{(n)}(\tau')\|_{B^{\frac{1}{2}}_{2,1}} d\tau'}\left(\sum_{i=1}^{4}\left\|u^{(n)}\right\|^i_{B^{\frac{3}{2}}_{2,1}}\right) d\tau. \end{aligned} \tag{3.8}$$

Hence, we get

$$\begin{aligned} \left\|u^{(n+1)}(t)\right\|_{B^{\frac{3}{2}}_{2,1}} \leqslant{}& e^{C\int_0^t \|\partial_x u^{(n)}(\tau')\|_{B^{\frac{1}{2}}_{2,1}} d\tau'}\|u_0\|_{B^{\frac{3}{2}}_{2,1}} \\ &+ C\int_0^t e^{C\int_\tau^t \|\partial_x u^{(n)}(\tau)\|_{B^{\frac{1}{2}}_{2,1}} d\tau}\left(\sum_{i=1}^{4}\left\|u^{(n)}\right\|^i_{B^{\frac{3}{2}}_{2,1}}\right) d\tau\, d\tau. \end{aligned} \tag{3.9}$$

Let us choose a $T>0$ such that $6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} T<1$, and suppose by induction that for all $t\in[0,T]$

$$\big\|u^{(n)}(t)\big\|_{B^{\frac{3}{2}}_{p,r}} \leqslant \frac{\|u_0\|_{B^{\frac{3}{2}}_{p,r}}}{(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{p,r}} t)^{\frac{1}{3}}}. \tag{3.10}$$

Indeed, $B^{\frac{1}{2}}_{2,1}$ is an algebra, one obtains from (2.10) that

$$\begin{aligned}
C\int_\tau^t \big\|\partial_x u^{(n)}(\tau')\big\|_{B^{\frac{1}{2}}_{p,r}} d\tau' &\leqslant C\int_\tau^t \big(\big\|u^{(n)}(\tau')\big\|_{B^{\frac{3}{2}}_{p,r}} + \big\|u^{(n)}(\tau')\big\|^3_{B^{\frac{3}{2}}_{p,r}}\big)\,d\tau' \\
&\leqslant C\left(\int_\tau^t \frac{\|u_0\|_{B^{\frac{3}{2}}_{p,r}}}{(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} \tau')^{\frac{1}{3}}}\,d\tau' + \int_\tau^t \frac{\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}}}{1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} \tau'}\,d\tau'\right) \\
&= \frac{1}{4\|u_0\|^2_{B^{\frac{3}{2}}_{2,1}}}\big[\big(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} \tau\big)^{\frac{2}{3}} - \big(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} t\big)^{\frac{2}{3}}\big] \\
&\quad + \frac{1}{6}\ln\big(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} \tau\big) - \frac{1}{6}\ln\big(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} t\big).
\end{aligned} \tag{3.11}$$

And then inserting the above inequality and (3.10) into (3.9) leads to

$$\begin{aligned}
\big\|u^{(n+1)}(t)\big\|_{B^{\frac{3}{2}}_{2,1}} &\leqslant \frac{e^{\frac{1}{4\|u_0\|^2_{B^{\frac{3}{2}}_{2,1}}}}\|u_0\|_{B^{\frac{3}{2}}_{2,1}}}{(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} t)^{\frac{1}{6}}} + \frac{Ce^{\frac{1}{4\|u_0\|^2_{B^{\frac{3}{2}}_{2,1}}}}}{(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} t)^{\frac{1}{6}}} \\
&\quad \times \int_0^t \big(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} \tau\big)^{\frac{1}{6}}\left(\frac{\|u_0\|^4_{B^{\frac{3}{2}}_{2,1}}}{(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} \tau)^{\frac{4}{3}}} + \frac{\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}}}{1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} \tau}\right. \\
&\quad \left. + \frac{\|u_0\|^2_{B^{\frac{3}{2}}_{2,1}}}{(1-6C\|u_0\|^2_{B^{\frac{3}{2}}_{2,1}} \tau)^{\frac{2}{3}}} + \frac{\|u_0\|_{B^{\frac{3}{2}}_{2,1}}}{(1-6C\|u_0\|_{B^{\frac{3}{2}}_{2,1}} \tau)^{\frac{1}{3}}}\right)d\tau \\
&\leqslant \frac{e^{\frac{1}{4\|u_0\|^2_{B^{\frac{3}{2}}_{2,1}}}}\|u_0\|_{B^{\frac{3}{2}}_{2,1}}}{(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} t)^{\frac{1}{6}}}\left(1+C\int_0^t \frac{\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}}}{(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}} \tau)^{\frac{7}{6}}}\,d\tau\right)
\end{aligned}$$

$$=\frac{e^{\frac{1}{4\|u_0\|^2_{B^{\frac{3}{2}}_{2,1}}}}\|u_0\|_{B^{\frac{3}{2}}_{2,1}}}{(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}}t)^{\frac{1}{3}}}. \tag{3.12}$$

Hence, one can see that

$$\left\|u^{(n+1)}(t)\right\|_{B^{\frac{3}{2}}_{2,1}} \leqslant \frac{\|u_0\|_{B^{\frac{3}{2}}_{2,1}}}{(1-6C\|u_0\|^3_{B^{\frac{3}{2}}_{2,1}}t)^{\frac{1}{3}}},$$

which implies that $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $C([0;T];B^{\frac{3}{2}}_{2,1})$. Using Eq. (3.6), one can easily prove that $(\partial_t u^{(n)})_{l\in\mathbb{N}}$ is uniformly bounded in $C([0;T];B^{\frac{3}{2}}_{2,1})$.

Thanks to Proposition 2.2, the Arzela–Ascoli theorem and a standard diagonal process, we infer that, up to an extraction, $(u^{(n)})_{n\in N}$ tends to a limit u in $C([0;T];(B^{\frac{3}{2}}_{2,1})_{loc})$. Besides, by using the uniform bounds of $u^{(n)}$ and Proposition 2.2, we gather that $u\in L^\infty(0,T;B^{\frac{3}{2}}_{2,1})$. According to Proposition 2.2, one can deduce that $(u^{(n)})_{n\in N}$ tends to u in $C([0;T];(B^{s}_{2,1})_{loc})$ for any $s<\frac{3}{2}$. Taking limit in (3.6), one can see that u is indeed a solution to Eq. (2.1). Thanks to $u\in L^\infty(0,T;B^{\frac{3}{2}}_{2,1})$, the equations in Eq. (2.1) and Lemma 2.2, we have $u\in C([0;T];B^{\frac{1}{2}}_{2,1})$. Using Eq. (2.1) itself again, we can easily infer that $\partial_t u\in C([0;T];B^{\frac{1}{2}}_{2,1})$. Therefore, we obtained solution $u\in C([0,T];B^{\frac{3}{2}}_{2,1})\cap C^1([0,T];B^{\frac{1}{2}}_{2,1})$. This completes the proof of the lemma. □

The third step is to prove the continuity with respect to initial data.

Corollary 3.1. *Let us fix a $u_0\in B^{\frac{3}{2}}_{2,1}$ and an $r>0$. Then there exist $T,M>0$ such that for all $u_0'\in B^{\frac{3}{2}}_{2,1}$ with $\|u_0'-u_0\|_{B^{\frac{3}{2}}_{2,1}}\leqslant r$, Eq. (2.1) has a solution $\Phi(u_0')\in C([0,T];B^{\frac{3}{2}}_{2,1})\cap C^1([0,T];B^{\frac{1}{2}}_{2,1})$ satisfying $\|u'\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1})}\leqslant M$.*

Proof. Noting that $\|u_0'\|_{B^{\frac{3}{2}}_{2,1}}\leqslant\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+r$, (3.10) and (3.12), one can choose some suitable constant C, such that $T=C/6(\|z_0\|_{B^{\frac{3}{2}}_{2,1}}+r)^3$ and $M=2(\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+r)$, one can complete the proof of Corollary 3.1. □

Lemma 3.3. *(See [25].) Denote $\bar{\mathbb{N}}=\mathbb{N}\cup\{\infty\}$. Let $(v^{(l)})_{l\in\bar{\mathbb{N}}}$ be a sequence of functions belonging to $C([0,T];B^{\frac{1}{2}}_{2,1})$. Assume that $v^{(l)}$ is the solution to*

$$\begin{cases}\partial_t v^{(l)}+u^{(l)}\partial_x v^{(l)}=f,\\ v^{(l)}\big|_{t=0}=v_0,\end{cases} \tag{3.13}$$

with $v_0\in B^{\frac{1}{2}}_{2,1}$, $f\in L^1(0,T;B^{\frac{1}{2}}_{2,1})$ and that, for some $\alpha\in L^1(0,T)$,

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$$\sup_{l\in\overline{\mathbb{N}}}\left\|\partial_x u^{(l)}(t)\right\|_{B^{\frac{1}{2}}_{2,1}}\leqslant\alpha(t).$$

If in addition $u^{(l)}$ tends to $u^{(\infty)}$ in $L^1(0,T;B^{\frac{1}{2}}_{2,1})$, then $v^{(l)}$ tends to $v^{(\infty)}$ in $C([0,T];B^{\frac{1}{2}}_{2,1})$.

Lemma 3.4. *Let $u_0\in B^{\frac{3}{2}}_{2,1}$ and u be the corresponding solution to Eq. (2.1), which is guaranteed by Lemma 3.2. Then there exist a time $T>0$ and a neighborhood V of u_0 in $B^{\frac{3}{2}}_{2,1}$ such that $v\in V$, which is the solution of Eq. (2.1) with the initial datum v_0, the map $\Phi: v_0\to v(\cdot,v_0): V\subset B^{\frac{3}{2}}_{2,1}\to C([0,T];B^{\frac{3}{2}}_{2,1})\cap C^1([0,T];B^{\frac{1}{2}}_{2,1})$ is continuous.*

Proof. We divide the proof into several steps as follows.

Step 1. Continuity in $C([0,T];B^{\frac{1}{2}}_{2,1})$.

Applying Lemma 3.1 and Corollary 3.1, one can get that

$$\frac{\|\Phi(u_0')-\Phi(u_0')\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1})}}{e}\leqslant e^{CMT}\left(\frac{\|u_0'-z_0\|_{B^{\frac{1}{2}}_{2,\infty}}}{e}\right)^{\exp(-CMT\ln(e+M))}$$

provided that

$$\left\|u_0'-u_0\right\|_{B^{\frac{1}{2}}_{2,\infty}}\leqslant e^{1-\exp(CMT\ln(e+M))}.$$

In view of the uniform bounds in $C([0,T];B^{\frac{3}{2}}_{2,1})$ and an interpolation argument, we infer the mapping Φ is continuous from $B^{\frac{3}{2}}_{2,1}$ into $C([0,T];B^{\frac{1}{2}}_{2,1})$.

Step 2. Continuity in $C([0,T];B^{\frac{3}{2}}_{2,1})$.

Let $u_0^{(\infty)}\in B^{\frac{3}{2}}_{2,1}$ and $(u_0^{(n)})_{n\in N}$ tends to $u_0^{(\infty)}$ in $B^{\frac{3}{2}}_{2,1}$. Denote by $u^{(n)}$ the solution corresponding to datum $u_0^{(n)}$. By Corollary 3.1, we can find $T,M>0$ such that for all $n\in\overline{\mathbb{N}}$, $u^{(n)}$ is defined on $[0,T]$ and

$$\sup_{n\in\overline{\mathbb{N}}}\left\|u^{(n)}\right\|_{L^\infty(0,T;B^{\frac{3}{2}}_{2,1})}\leqslant M. \tag{3.14}$$

In order to prove $u^{(n)}$ tends to $u^{(\infty)}$ in $C([0,T];B^{\frac{3}{2}}_{2,1})$, according to Step 1, we only need to show that $v^{(n)}=\partial_x u^{(n)}$ tends to $v^{(\infty)}=\partial_x u^{(\infty)}$ in $C([0,T];B^{\frac{1}{2}}_{2,1})$. Note that $v^{(n)}$ solves the following transport equation

$$\begin{cases}\partial_t v^{(n)}+\left(\dfrac{\alpha}{\beta}-\dfrac{\varepsilon\gamma}{\beta}u^{(n)}\right)\partial_x v^{(n)}=f^{(n)},\\ v^{(n)}\big|_{t=0}=\partial_x u_0^{(n)}\end{cases}$$

with

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$$f^{(n)} = \frac{\varepsilon\gamma}{\beta}\big(u_x^{(n)}\big)^2 + \frac{1}{\mu\beta}\bigg[\Big(1-\frac{\alpha}{\beta}\Big)u^{(n)} + \Big(\frac{3\varepsilon-2}{4} - \frac{1}{2\mu\beta}\Big)\big(u^{(n)}\big)^2 + \frac{\varepsilon^2\iota}{3}\big(u^{(n)}\big)^3 + \frac{\varepsilon^3\kappa}{4}\big(u^{(n)}\big)^4$$
$$+ \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}\big(u_x^{(n)}\big)^2\bigg] - \frac{1}{\mu\beta}G(x) * \bigg[\Big(1-\frac{\alpha}{\beta}\Big)u^{(n)} + \Big(\frac{3\varepsilon-2}{4} - \frac{1}{2\mu\beta}\Big)\big(u^{(n)}\big)^2$$
$$+ \frac{\varepsilon^2\iota}{3}\big(u^{(n)}\big)^3 + \frac{\varepsilon^3\kappa}{4}\big(u^{(n)}\big)^4 + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}\big(u_x^{(n)}\big)^2\bigg].$$

Following Kato [49], we decompose $v^{(n)}$ into $v^{(n)} = z^{(n)} + w^{(n)}$ with

$$\begin{cases} \partial_t z^{(n)} + \Big(\dfrac{\alpha}{\beta} - \dfrac{\varepsilon\gamma}{\beta}u^{(n)}\Big)\partial_x z^{(n)} = f^{(n)} - f^{(\infty)}, \\ z^{(n)}\big|_{t=0} = \partial_x u_0^{(n)} - \partial_x u_0^{(\infty)} \end{cases}$$

and

$$\begin{cases} \partial_t w^{(n)} + \Big(\dfrac{\alpha}{\beta} - \dfrac{\varepsilon\gamma}{\beta}u^{(n)}\Big)\partial_x w^{(n)} = f^{(\infty)}, \\ w^{(n)}\big|_{t=0} = \partial_x u_0^{(\infty)} \end{cases}$$

one can easily check that $(f^{(n)})_{n\in\bar{\mathbb{N}}}$ is uniformly bounded in $C([0,T]; B^{\frac{1}{2}}_{2,1})$. Moreover, we can obtain the following inequalities

$$\big\|f^{(n)} - f^{(\infty)}\big\|_{B^{\frac{1}{2}}_{2,1}} \leqslant C\Bigg(\sum_{i=1}^{3}\Big(\big\|u^{(n)}\big\|^i_{B^{\frac{3}{2}}_{2,1}} + \big\|u^{(\infty)}\big\|^i_{B^{\frac{3}{2}}_{2,1}}\Big)\Bigg)$$
$$\times\Big(\big\|u^{(n)} - u^{(\infty)}\big\|_{B^{\frac{1}{2}}_{2,1}} + \big\|\partial_x u^{(n)} - \partial_x u^{(\infty)}\big\|_{B^{\frac{1}{2}}_{2,1}}\Big).$$

Applying Lemma 2.2, one can deduce that

$$\big\|z^{(n)}(t)\big\|_{B^{\frac{1}{2}}_{2,1}} \leqslant e^{C\int_0^t \|\partial_x(\frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta}u^{(n)})(\tau')\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau'}\big\|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\big\|_{B^{\frac{1}{2}}_{2,1}}$$
$$+ C\int_0^t e^{-C\int_\tau^t \|\partial_x(\frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta}u^{(n)})(\tau')\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau'}\Big(\big\|f^{(n)} - f^{(\infty)}\big\|_{B^{\frac{1}{2}}_{2,1}}\Big)\,d\tau$$
$$\leqslant e^{C\int_0^t \|u^{(n)}(\tau')\|_{B^{\frac{3}{2}}_{2,1}}\,d\tau'}\big\|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\big\|_{B^{\frac{1}{2}}_{2,1}}$$
$$+ C\int_0^t e^{-C\int_\tau^t \|u^{(n)}(\tau')\|_{B^{\frac{3}{2}}_{2,1}}\,d\tau'}\Bigg(\sum_{i=1}^{3}\Big(\big\|u^{(n)}\big\|^i_{B^{\frac{3}{2}}_{2,1}} + \big\|u^{(\infty)}\big\|^i_{B^{\frac{3}{2}}_{2,1}}\Big)\Bigg)$$
$$\times\Big(\big\|u^{(n)} - u^{(\infty)}\big\|_{B^{\frac{1}{2}}_{2,1}} + \big\|\partial_x u^{(n)} - \partial_x u^{(\infty)}\big\|_{B^{\frac{1}{2}}_{2,1}}\Big)\,d\tau. \tag{3.15}$$

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Lemma 3.3 tells us $w^{(n)}$ tends to $v^{(\infty)} = \partial_x u^{(\infty)}$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$. Let $\varepsilon > 0$, combining the above estimates (3.14) and (3.15), when $n \in \mathbb{N}$ is large enough, we can conclude

$$\|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{B^{\frac{1}{2}}_{2,1}} \leqslant \epsilon + CMe^{CMT}\Bigg(\epsilon + \|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \big(\|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{B^{\frac{1}{2}}_{2,1}} + \|u^{(n)} - u^{(\infty)}\|_{B^{\frac{1}{2}}_{2,1}}\big)\, d\tau\Bigg).$$

As $u^{(n)}$ tends to $u^{(\infty)}$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$, when n is very large, the last term is less than ε. Hence, using Gronwall's inequality, we get

$$\|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{B^{\frac{1}{2}}_{2,1}} \leqslant C\big(\epsilon + \|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B^{\frac{1}{2}}_{2,1}}\big),$$

here the constant C depends only on M and T, which yields the continuity in $C([0,T]; B^{\frac{3}{2}}_{2,1})$.

Step 3. Continuity in $C^1([0,T]; B^{\frac{3}{2}}_{2,1})$.

Using (3.14), the equations in Eq. (2.1) and Step 2, we reach Step 3. Consequently, we have completed the proof of the lemma. □

4. Analyticity of solutions

In this section, we will show the existence and uniqueness of analytic solutions to the system (1.1) on the line $\mathbb{R}$.

For the proof of Theorem 1.3, we will need a suitable scale of Banach spaces as follows. For any $s > 0$, we set

$$E_s = \left\{u \in C^\infty(\mathbb{R}):\ |||u|||_s = \sup_{k \in \mathbb{N}_0} \frac{s^k \|\partial^k u\|_{H^2}}{k!/(k+1)^2} < \infty\right\},$$

where $H^2(\mathbb{R})$ is the Sobolev space of order two on the real line and $\mathbb{N}_0$ is the set of nonnegative integers. One can easily verify that E_s equipped with the norm $|||\cdot|||_s$ is a Banach space and that, for any $0 < s' < s$, E_s is continuously embedded in $E_{s'}$ with

$$|||u|||_{s'} \leqslant |||u|||_s.$$

Another simple consequence of the definition is that any u in E_s is a real analytic function on $\mathbb{R}$. Crucial for our purposes is the fact that each E_s forms an algebra under pointwise multiplication of functions.

Lemma 4.1. *(See [44].) Let $0 < s < 1$. There is a constant $C > 0$, independent of s, such that for any u and v in E_s we have*

$$|||uv|||_s \leqslant C|||u|||_s|||v|||_s.$$

Lemma 4.2. *(See [44].) There is a constant $C > 0$ such that for any $0 < s' < s < 1$, we have $|||\partial_x u|||_{s'} \leqslant \frac{C}{s-s'}|||u|||_s$, and $|||(1-\partial_x^2)^{-1}u|||_{s'} \leqslant |||u|||_s$, $|||(1-\partial_x^2)^{-1}\partial_x u|||_{s'} \leqslant |||u|||_s$.*

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Theorem 4.1. *(See [6].) Let $\{X_s\}_{0<s<1}$ be a scale of decreasing Banach spaces, namely for any $s' < s$ we have $X_s \subset X_{s'}$ and $|||\cdot|||_{s'} \leqslant |||\cdot|||_s$. Consider the Cauchy problem*

$$\begin{cases} \dfrac{du}{dt} = F\big(t, u(t)\big), \\ u(0) = 0. \end{cases} \tag{4.1}$$

Let T, R and C be positive constants and assume that F satisfies the following conditions:

1) *If for $0 < s' < s < 1$ the function $t \mapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| \leqslant T$ with values in X_s and*

$$\sup_{|t|\leqslant T} \big|\big|\big|u(t)\big|\big|\big|_s < R,$$

then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{s'}$.

2) *For any $0 < s' < s < 1$ and any $u, v \in X_s$ with $|||u|||_s < R$, $|||v|||_s < R$,*

$$\sup_{|t|\leqslant T} \big|\big|\big|F(t,u) - F(t,v)\big|\big|\big|_{s'} \leqslant \frac{C}{s-s'} |||u-v|||_s.$$

3) *There exists $M > 0$ such that for any $0 < s < 1$,*

$$\sup_{|t|\leqslant T} \big|\big|\big|F(t,0)\big|\big|\big|_s \leqslant \frac{M}{1-s}.$$

Then there exist a $T_0 \in (0, T)$ and a unique function $u(t)$, which for every $s \in (0, 1)$ is holomorphic in $|t| < (1-s)T_0$ with values in X_s, and is a solution to the Cauchy problem (2.1).

We restate the Cauchy problem (2.1) in a more convenient form. Let $u_1 = u, u_2 = u_x$. Then the problem (2.1) can be written as a system for u_1 and u_2

$$\begin{cases} \partial_t u_1 = -\dfrac{\alpha}{\beta} u_2 + \dfrac{\varepsilon\gamma}{2\beta} \partial_x(u_1^2) - \partial_x G(x) * \bigg[\Big(1-\dfrac{\alpha}{\beta}\Big)u_1 + \Big(\dfrac{3\varepsilon-2}{4} - \dfrac{1}{2\mu\beta}\Big)u_1^2 \\ \qquad + \dfrac{\varepsilon^2\iota}{3}u_1^3 + \dfrac{\varepsilon^3\kappa}{4}u_1^4 + \dfrac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}u_2^2\bigg] = F_1(u_1, u_2), \\ \partial_t u_2 = \partial_x\Big(-\dfrac{\alpha}{\beta}u_2 + \dfrac{\varepsilon\gamma}{\beta}u_1 u_2\Big) + \dfrac{1}{\mu\beta}\bigg[\Big(1-\dfrac{\alpha}{\beta}\Big)u_1 + \Big(\dfrac{3\varepsilon-2}{4} - \dfrac{1}{2\mu\beta}\Big)u_1^2 \\ \qquad + \dfrac{\varepsilon^2\iota}{3}u_1^3 + \dfrac{\varepsilon^3\kappa}{4}u_1^4 + \dfrac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}u_2^2\bigg] - \dfrac{1}{\mu\beta}G(x) * \bigg[\Big(1-\dfrac{\alpha}{\beta}\Big)u_1 \\ \qquad + \Big(\dfrac{3\varepsilon-2}{4} - \dfrac{1}{2\mu\beta}\Big)u_1^2 + \dfrac{\varepsilon^2\iota}{3}u_1^3 + \dfrac{\varepsilon^3\kappa}{4}u_1^4 + \dfrac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}u_2^2\bigg] = F_2(u_1, u_2), \\ u_1(x, 0) = u_0(x), u_2(x, 0) = u_0'(x). \end{cases} \tag{4.2}$$

Now we are ready to prove Theorem 1.3. We will show that all three conditions of the abstract version of the Cauchy–Kowalevski theorem (Theorem 4.1) hold for Eq. (2.1) on the scale $\{X_s\}_{0<s<1}$.

Proof of Theorem 1.3. Let $u = (u_1, u_2)$ and $F = (F_1, F_2)$ in (4.2) and let X_s be a scale of decreasing Banach spaces defined as $X_s = E_s \times E_s$. Then we only need to verify the first two conditions of the abstract Cauchy–Kowalevski theorem since the map $F(u_1, u_2)$ does not depend on t explicitly.

Clearly, $t \mapsto F(t, u(t)) = (F_1(u_1, u_2), F_2(u_1, u_2))$ is holomorphic if $t \mapsto u_1(t)$ and $t \mapsto u_2(t)$ are both holomorphic. Therefore, to verify the first condition of the abstract theorem, we only need to show that for $s' < s$, $F_1(u_1, u_2)$ and $F_2(u_1, u_2)$ are in $u_1, u_2 \in E_s$. We begin with estimates on F_1 and F_2. By Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned}
\left|\!\left|\!\left| F_1(u_1,u_2) \right|\!\right|\!\right|_{s'} &= \left|\!\left|\!\left| -\frac{\alpha}{\beta}u_2 + \frac{\varepsilon\gamma}{2\beta}\partial_x\big(u_1^2\big) - \partial_x G(x) * \left[\left(1-\frac{\alpha}{\beta}\right)u_1 + \left(\frac{3\varepsilon-2}{4} - \frac{1}{2\mu\beta}\right)u_1^2 \right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. + \frac{\varepsilon^2\iota}{3}u_1^3 + \frac{\varepsilon^3\kappa}{4}u_1^4 + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}u_2^2\right] \right|\!\right|\!\right|_{s'} \\
&= \frac{C}{s-s'}|\!|\!|u_1|\!|\!|_s^2 + C\big(|\!|\!|u_1|\!|\!|_{s'} + |\!|\!|u_1|\!|\!|_{s'}^2 + |\!|\!|u_1|\!|\!|_{s'}^3 + |\!|\!|u_1|\!|\!|_{s'}^4 + |\!|\!|u_2|\!|\!|_{s'}^2\big), \\
\left|\!\left|\!\left| F_2(u_1,u_2) \right|\!\right|\!\right|_{s'} &= \left|\!\left|\!\left| \partial_x\left(-\frac{\alpha}{\beta}u_2 + \frac{\varepsilon\gamma}{\beta}u_1u_2\right) + \frac{1}{\mu\beta}\left[\left(1-\frac{\alpha}{\beta}\right)u_1 + \left(\frac{3\varepsilon-2}{4} - \frac{1}{2\mu\beta}\right)u_1^2 \right.\right.\right.\right. \\
&\quad + \frac{\varepsilon^2\iota}{3}u_1^3 + \frac{\varepsilon^3\kappa}{4}u_1^4 + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}u_2^2\bigg] - \frac{1}{\mu\beta}G(x) * \left[\left(1-\frac{\alpha}{\beta}\right)u_1\right. \\
&\quad \left.\left.\left.\left. + \left(\frac{3\varepsilon-2}{4} - \frac{1}{2\mu\beta}\right)u_1^2 + \frac{\varepsilon^2\iota}{3}u_1^3 + \frac{\varepsilon^3\kappa}{4}u_1^4 + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}u_2^2\right] \right|\!\right|\!\right|_{s'} \\
&= \frac{C}{s-s'}\big(|\!|\!|u_1|\!|\!|_s^2 + |\!|\!|u_1|\!|\!|_s|\!|\!|u_2|\!|\!|_s\big) + C\big(|\!|\!|u_1|\!|\!|_{s'} + |\!|\!|u_1|\!|\!|_{s'}^2 + |\!|\!|u_1|\!|\!|_{s'}^3 \\
&\quad + |\!|\!|u_1|\!|\!|_{s'}^4 + |\!|\!|u_2|\!|\!|_{s'}^2\big).
\end{aligned}$$

We proceed to establish the second condition of the abstract Cauchy–Kowalevski theorem. First we show that for some C independent of t the estimates

$$\left|\!\left|\!\left| F_1(u_1,u_2) - F_1(v_1,v_2) \right|\!\right|\!\right|_{s'} \leqslant \frac{C}{s-s'}|\!|\!|u-v|\!|\!|_{X_s} \tag{4.3}$$

and

$$\left|\!\left|\!\left| F_2(u_1,u_2) - F_2(v_1,v_2) \right|\!\right|\!\right|_{s'} \leqslant \frac{C}{s-s'}|\!|\!|u-v|\!|\!|_{X_s} \tag{4.4}$$

hold.

In order obtain the estimate (4.3) we use the triangle inequality and Lemma 4.1 with Lemma 4.2, we have

$$\begin{aligned}
&\left|\!\left|\!\left| F_1(u_1,u_2) - F_1(v_1,v_2) \right|\!\right|\!\right|_{s'} \\
&\quad = \left|\!\left|\!\left| -\frac{\alpha}{\beta}(u_2-v_2) + \frac{\varepsilon\gamma}{2\beta}\partial_x\big(u_1^2 - v_1^2\big) - \partial_x G(x) * \left[\left(1-\frac{\alpha}{\beta}\right)(u_1-v_1)\right.\right.\right.\right. \\
&\qquad + \left(\frac{3\varepsilon-2}{4} - \frac{1}{2\mu\beta}\right)\big(u_1^2 - v_1^2\big) + \frac{\varepsilon^2\iota}{3}\big(u_1^3 - v_1^3\big) + \frac{\varepsilon^3\kappa}{4}\big(u_1^4 - v_1^4\big) \\
&\qquad \left.\left.\left.\left. + \frac{3\varepsilon\mu\gamma - \varepsilon\mu\delta - \mu\beta}{2}\big(u_2^2 - v_2^2\big)\right] \right|\!\right|\!\right|_{s'} \\
&\quad \leqslant C|\!|\!|u_2 - v_2|\!|\!|_{s'} + \frac{C}{s-s'}|\!|\!|u_2 - v_2|\!|\!|_{s'}|\!|\!|u_2 + v_2|\!|\!|_{s'} + C|\!|\!|u_1 - v_1|\!|\!|_{s'}
\end{aligned}$$

$$
\begin{aligned}
&+ C|||u_1+v_1|||_{s'}|||u_1-v_1|||_{s'} + C\left|\left|\left|u_1^2+u_1v_1+v_1^2\right|\right|\right|_{s'}|||u_1-v_1|||_{s'}\\
&+ C\left|\left|\left|u_1^3+u_1^2v_1+u_1v_1+u_1v_1^2+v_1^3\right|\right|\right|_{s'}|||u_1-v_1|||_{s'}\\
&+ C|||u_2-v_2|||_{s'}|||u_2+v_2|||_{s'}\\
\leqslant\ & \frac{C}{s-s'}|||u-v|||_{X_s}.
\end{aligned}
$$

In a similar way to what we just did, we can show that the estimate (4.4) holds

The conditions (1) through (3) above are now easily verified once our system (4.3) is transformed into a new system with zero initial data as in (4.2). The proof of Theorem 1.3 is complete. □

5. Persistence property

In this section, we shall investigate persistence properties on the solution to Eq. (2.1) in L^∞-space. For the Camassa–Holm equation a solution u such that $u-u_{xx}$ has compact support preserves this feature at future times, although u with compact support is not preserved (see [11,38]). This property is best expressed by means of a decay statement in [46], which motivates the considerations made in Section 5.

Notation.

$$
\left|f(x)\right| \sim O\left(\left|g(x)\right|\right) \quad \text{as } x\uparrow\infty \text{ if } \lim_{x\to\infty}\frac{|f(x)|}{|g(x)|}=L,
$$

where L is a nonnegative constant. In order to shorten the presentation in the sequel, we introduce

$$
f(u)=\left(1-\frac{\alpha}{\beta}\right)u+\left(\frac{3\varepsilon-2}{4}-\frac{1}{2\mu\beta}\right)u^2+\frac{\varepsilon^2\iota}{3}u^3+\frac{\varepsilon^3\kappa}{4}u^4+\frac{3\varepsilon\mu\gamma-\varepsilon\mu\delta-\mu\beta}{2}u_x^2.
$$

Proof of Theorem 1.4. Firstly, we will give out the estimates on $\|u(x,t)\|_{L^\infty}$. Multiplying the first equation of (2.1) by u^{2n-1} with $n\in\mathbb{Z}^+$ and integrating the result with respect to x, one can get

$$
\int_{\mathbb{R}} u^{2n-1}u_t\,dx+\int_{\mathbb{R}} u^{2n-1}\left(\frac{\alpha}{\beta}-\frac{\varepsilon\gamma}{\beta}u\right)u_x\,dx+\int_{\mathbb{R}} u^{2n-1}\partial_xG*f(u)\,dx=0. \tag{5.1}
$$

The first term of the above inequality is

$$
\int_R u^{2n-1}u_t\,dx=\frac{1}{2n}\frac{d}{dt}\left\|u(t)\right\|_{L^{2n}}^{2n}=\left\|u(t)\right\|_{L^{2n}}^{2n-1}\frac{d}{dt}\left\|u(t)\right\|_{L^{2n}}.
$$

For the estimates of the other two terms in (5.1), we have

$$
\left|\int_{\mathbb{R}} u^{2n-1}\left(\frac{\alpha}{\beta}-\frac{\varepsilon\gamma}{\beta}u\right)u_x\,dx\right|=\left|\frac{\varepsilon\gamma}{\beta}\int_{\mathbb{R}} u^{2n-1}uu_x\,dx\right|\leqslant\left|\frac{\varepsilon\gamma}{\beta}\right|\left\|u_x(t)\right\|_{L^\infty}\left\|u(t)\right\|_{L^{2n}}^{2n}
$$

and

$$
\left|\int_{\mathbb{R}} u^{2n-1}\partial_xG*f(u)\,dx\right|\leqslant\left\|u(t)\right\|_{L^{2n}}^{2n-1}\left\|\partial_xG*f(u)\right\|_{L^{2n}}.
$$

Putting all the inequalities above into (5.1), we can get

$$\frac{d}{dt}\left\|u(t)\right\|_{L^{2n}} \leqslant \left|\frac{\varepsilon\gamma}{\beta}\right|\left\|u_x(t)\right\|_{L^\infty}\left\|u(t)\right\|_{L^{2n}} + \left\|\partial_x G * f(u)\right\|_{L^{2n}}. \tag{5.2}$$

Using the Sobolev embedding theorem, there exists a constant $M > 0$ such that

$$\frac{d}{dt}\left\|u(t)\right\|_{L^{2n}} \leqslant M\left\|u(t)\right\|_{L^{2n}} + \left\|\partial_x p * f(u)\right\|_{L^{2n}}. \tag{5.3}$$

By Gronwall's inequality, we obtain

$$\left\|u(t)\right\|_{L^{2n}} \leqslant e^{Mt}\left(\left\|u(0)\right\|_{L^{2n}} + \int_0^t \left\|\partial_x G * f(u)\right\|_{L^{2n}}\,d\tau\right). \tag{5.4}$$

For any $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we know that

$$\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}.$$

Taking limits in (5.4), we get

$$\left\|u(t)\right\|_{L^\infty} \leqslant e^{Mt}\left(\left\|u(0)\right\|_{L^\infty} + \int_0^t \left\|\partial_x p * f(u)\right\|_{L^\infty}\,d\tau\right).$$

Next, we will establish the estimates on $\|u_x(x,t)\|_{L^\infty}$ with the same method as above. Differentiating the first equation in (2.1) with respect to x-variable produces the following equation

$$u_{xt} + \left(\frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta}u\right)u_{xx} - \frac{\varepsilon\gamma}{\beta}u_x^2 + \partial_x^2 G * f(u) = 0. \tag{5.5}$$

Multiplying the above identity by u_x^{2n-1} and considering the second term with integration by parts

$$\int_{\mathbb{R}} u_x^{2n-1}\left(\frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta}u\right)u_{xx}\,dx = -\frac{\varepsilon\gamma}{\beta}\int_{\mathbb{R}} u_x^{2n-1}uu_{xx}\,dx = -\frac{\varepsilon\gamma}{\beta}\int_{\mathbb{R}} u\left(\frac{u_x^{2n}}{2n}\right)_x dx = \frac{\varepsilon\gamma}{2n\beta}\int_{\mathbb{R}} u_x^{2n}u_x\,dx,$$

we get

$$\int_R u_x^{2n-1}u_{xt}\,dx + \frac{\varepsilon\gamma}{2n\beta}\int_R u_x^{2n}u_x\,dx - \frac{\varepsilon\gamma}{\beta}\int_R u_x^{2n-1}u_x^2\,dx + \int_R u_x^{2n-1}\partial_x^2 G * f(u)\,dx = 0.$$

Similarly, one can obtain

$$\left\|u_x(t)\right\|_{L^\infty} \leqslant e^{2Mt}\left(\left\|u_x(0)\right\|_{L^\infty} + \int_0^t \left\|\partial_x^2 G * f(u)\right\|_{L^\infty}\,d\tau\right). \tag{5.6}$$

In order to get the desired result, we introduce the following weight function which is independent of t

$$\Phi_N(x)=\begin{cases}1, & x\leqslant 1,\\ e^{\theta x}, & x\in(1,N),\\ e^{\theta N}, & x\geqslant N,\end{cases}$$

where $N\in\mathbb{Z}^+$. Then from the first equation in (2.1) and (5.5), we get

$$\Phi_N u_t+\Phi_N\left(\frac{\alpha}{\beta}-\frac{\varepsilon\gamma}{\beta}u\right)u_x+\Phi_N\partial_x G*f(u)=0,$$

$$\Phi_N u_{xt}+\Phi_N\left(\frac{\alpha}{\beta}-\frac{\varepsilon\gamma}{\beta}u\right)u_{xx}-\frac{\varepsilon\gamma}{\beta}\Phi_N u_x^2+\Phi_N\partial_x^2 G*f(u)=0.$$

In order to get the estimate on $u_x\Phi_N$, we need to remove the second derivatives, by using integration by parts we obtain

$$\begin{aligned}\left|\frac{\varepsilon\gamma}{\beta}\int_{\mathbb{R}}(u_x\Phi_N)^{2n-1}\Phi_N u u_{xx}\,dx\right| &= \left|\frac{\varepsilon\gamma}{\beta}\right|\left|\int_{\mathbb{R}}(u_x\Phi_N)^{2n-1}u\big((\Phi_N u_x)_x-\Phi_N' u_x\big)\,dx\right|\\ &= \left|\frac{\varepsilon\gamma}{\beta}\right|\left|\int_{\mathbb{R}}u\left(\frac{(u_x\Phi_N)^{2n}}{2n}\right)_x dx-\int_{\mathbb{R}}u(u_x\Phi_N)^{2n-1}\Phi_N' u_x\,dx\right|\\ &\leqslant 2\left|\frac{\varepsilon\gamma}{\beta}\right|\big(\|u_x\|_\infty+\|u\|_\infty\big)\|u_x\Phi_N\|_{L^{2n}}^{2n},\end{aligned}$$

where we use the fact $0\leqslant\Phi_N'(x)\leqslant\Phi_N(x), x\in R$. Therefore, with these preparations, it holds that

$$\begin{aligned}&\big\|u(x,t)\Phi_N\big\|_{L^\infty}+\big\|u_x(x,t)\Phi_N\big\|_{L^\infty}\\ &\quad\leqslant e^{2Mt}\big(\big\|u_0(x)\Phi_N\big\|_{L^\infty}+\big\|u_{0x}(x)\Phi_N\big\|_{L^\infty}\big)\\ &\qquad+e^{2Mt}\int_0^t\big(\big\|\Phi_N\partial_x\big(G*f(u)\big)\big\|_{L^\infty}+\big\|\Phi_N\partial_x^2\big(G*f(u)\big)\big\|_{L^\infty}\big)\,d\tau.\end{aligned}\tag{5.7}$$

On the other hand, computing the integral we see there exists a $C_{(\theta,\mu,\beta)}>0$, depending only on $\theta\in(0,1),\mu,\beta$ such that

$$\Phi_N(x)\int_R e^{-\frac{|x-y|}{\sqrt{|\mu\beta|}}}\frac{1}{\Phi_N(y)}\Phi_N(y)\,dy\leqslant C_{(\theta,\mu,\beta)}.\tag{5.8}$$

Therefore, for suitable functions f and g, one obtains,

$$\big|\Phi_N\partial_x G*f(x)g(x)\big|\leqslant\frac{1}{2|\mu\beta|}\Phi_N(x)\int_R \operatorname{sgn}(x-y)e^{-\frac{|x-y|}{\sqrt{|\mu\beta|}}}\frac{1}{\Phi_N(y)}\Phi_N(y)f(y)g(y)\,dy$$

$$\leqslant \frac{1}{2|\mu\beta|} \| f\Phi_N\|_{L^\infty} \|g\|_{L^\infty} \Phi_N(x) \int_R e^{-\frac{|x-y|}{\sqrt{|\mu\beta|}}} \frac{1}{\Phi_N(y)}\, dy$$
$$\leqslant C_{(\theta,\mu,\beta)} \| f\Phi_N\|_{L^\infty} \|g\|_{L^\infty}. \tag{5.9}$$

Similarly, we can get

$$\left|\Phi_N \partial_x^2\big(G * f(x)g(x)\big)\right| \leqslant C_{(\theta,\mu,\beta)} \| f\Phi_N\|_{L^\infty} \|g\|_{L^\infty}. \tag{5.10}$$

Then combining (5.9), (5.10) with (5.7), it follows that there exists a constant $\widetilde{C} = \widetilde{C}(M, T, C_{(\theta,\mu,\beta)})$ such that

$$\|u\Phi_N\|_{L^\infty} + \|u_x\Phi_N\|_{L^\infty}$$
$$\leqslant \widetilde{C}\left(\big\|u(0)\Phi_N\big\|_{L^\infty} + \big\|u_x(0)\Phi_N\big\|_{L^\infty} + \int_0^t \big(\big\|u(\tau)\Phi_N\big\|_{L^\infty} + \big\|u_x(\tau)\Phi_N\big\|_{L^\infty}\big)\, d\tau\right). \tag{5.11}$$

Then for any $N \in \mathbb{Z}^+$, any $t \in [0, T]$ and $x > 0$ we have by Gronwall's inequality

$$\big\|u(t)\Phi_N\big\|_{L^\infty} + \|u_x\Phi_N\|_{L^\infty} \leqslant \widetilde{C}\big\|u(0)\Phi_N\big\|_{L^\infty} + \big\|u_x(0)\Phi_N\big\|_{L^\infty}$$
$$\leqslant \widetilde{C}\big(\big\|u(0)e^{\theta x}\big\|_{L^\infty} + \big\|u_x(0)e^{\theta x}\big\|_{L^\infty}\big).$$

Finally, passing limit as N goes to infinity in the above inequality, we obtain

$$\big\|u(t,x)e^{\theta x}\big\|_{L^\infty} + \big\|u_x(t,x)e^{\theta x}\big\|_{L^\infty} \leqslant \widetilde{C}\big\|u(0)e^{\theta x}\big\|_{L^\infty} + \big\|u_x(0)e^{\theta x}\big\|_{L^\infty}. \tag{5.12}$$

We complete the proof. □

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