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Bubbling solutions for an exponential nonlinearity in $\mathbb{R}^2$ ☆

Shengbing Deng, Monica Musso *

Departamento de Matemática, Pontificia Universidad Catolica de Chile, Avda. Vicuña Mackenna 4860, Macul, Chile

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Abstract

We study the following boundary value problem

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter and $0 < p < 2$. We construct bubbling solutions to problem (0.1) using a Lyapunov–Schmidt reduction procedure.

Keywords: Bubbling solutions; Lyapunov–Schmidt reduction procedure

1. Introduction

We consider the following boundary value problem

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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* Corresponding author.
E-mail addresses: shbdeng65@gmail.com (S. Deng), mmusso@mat.puc.cl (M. Musso).

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where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter and $0 < p \leq 2$. This problem is the Euler–Lagrange equation for the functional

$$J_\lambda^p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{p}\int_\Omega e^{u^p}, \quad u \in H_0^1(\Omega). \tag{1.2}$$

If $p = 1$, problem (1.1) becomes

$$\begin{cases} \Delta u + \lambda e^u = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which can be called the Liouville equation after [24]. This problem is related to Berger's problem concerning the existence of metrics in a given Riemannian surface with prescribed Gaussian curvatures. We refer the reader to the book of T. Aubin [6] for the description of the links between this equation and possible geometric applications.

There are many results about the behavior and existence of solution to (1.3). Thanks to the works of H. Brezis and F. Merle [9], Y.Y. Li and I. Shafrir [23], L. Ma and J. Wei [25], K. Nagasaki and T. Suzuki [29], the asymptotic behavior of solutions to problem (1.3) has been well understood. More precisely, it is by now known that if u_λ is an unbounded family of solutions to (1.1) for which $\lambda \int_\Omega e^{u_\lambda}$ remains uniformly bounded as $\lambda \to 0$, then there is an integer $k \geq 1$, such that necessarily

$$\lim_{\lambda \to 0} \lambda \int_\Omega e^{u_\lambda} = 8k\pi. \tag{1.4}$$

Moreover, there are k distinct points ξ_j, $j = 1, \cdots, k$, in Ω, separated uniformly from each other and from he boundary $\partial\Omega$, such that, as $\lambda \to 0$, u_λ peaks to infinity in each one of them, and remains bounded away from them, that is, the solutions u_λ to problem (1.3) remain uniformly bounded on $\Omega \setminus \bigcup_{j=1}^k B_\delta(\xi_j)$ and

$$\sup_{B_\delta(\xi_j)} u_\lambda \to +\infty, \quad \text{as } \lambda \to 0,$$

for any $\delta > 0$. The location of the blow-up points $\xi_1, \cdots, \xi_k$ is such that, after passing to a subsequence, it converges to a critical point of the function

$$\varphi_k(\xi_1, \cdots, \xi_k) = \sum_{j=1}^k H(\xi_j, \xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j), \tag{1.5}$$

where $G(x, y)$ is the Green's function of the problem

$$\begin{cases} -\Delta_x G(x, y) = 8\pi \delta_y(x), & x \in \Omega; \\ G(x, y) = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

and $H(\cdot,\cdot)$ is its regular part defined as

$$H(x, y) = G(x, y) - 4 \log \frac{1}{|x-y|}. \tag{1.7}$$

Conversely, many authors constructed blow-up solutions to problem (1.3) with property (1.4). In [8], S. Baraket and F. Pacard considered problem (1.3) in an open bounded subset Ω of $\mathbb{C}$, and they showed that given a non-degenerate critical point $(\xi_1, \dots, \xi_k)$ of the function φ_k defined in (1.5), there is a sequence u_λ of solutions to (1.3), that converges to a function u^* in $C^{2,\alpha}_{loc}(\Omega \backslash \{\xi_1, \cdots, \xi_k\})$, where u^* is the solution of

$$\begin{cases} -\Delta u^* = \sum_{j=1}^{k} 8\pi \delta_{\xi_j}, & \text{in } \Omega; \\ u^* = 0 & \text{on } \partial\Omega. \end{cases}$$

P. Esposito, M. Grossi, A. Pistoia [20] generalized this result relaxing the assumption of non-degenerate critical point for φ_k to that of *stable critical point* for φ_k. By using the notion of *topologically non-trivial critical value* for φ_k, that we will recall later on, M. del Pino, M. Kowalczyk, M. Musso [14] could establish the following general result: *If the domain Ω is not simply connected, and given any integer $k \geq 1$, there exist k points $\xi_1, \dots, \xi_k$ in Ω and a family of solutions u_λ to (1.3), satisfying (1.4) and bubbling at exactly those k points. The shape of these solutions is given by*

$$u_\lambda(x) = \sum_{j=1}^{k} G(x, \xi_j) + o(1), \quad \text{as } \lambda \to 0 \tag{1.8}$$

where $o(1) \to 0$ as $\lambda \to 0$ uniformly in compact sets contained in $\Omega \setminus \{\xi_1, \dots, \xi_k\}$. Furthermore

$$J^1_\lambda(u_\lambda) = -16k\pi + 8k\pi \log 8 - 8k\pi \log \lambda - 4\pi \varphi_k(\xi) + o(1) \tag{1.9}$$

where φ_k is defined in (1.5) and $o(1) \to 0$ as $\lambda \to 0$.

If $p = 2$, problem (1.1) becomes

$$\begin{cases} \Delta u + \lambda u e^{u^2} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

This problem is the Euler–Lagrange equation for the functional J^2_λ (see (1.2)) which corresponds to the free energy associated to the critical Trudinger embedding (in the sense of Orlicz spaces) [35,27,34]

$$H^1_0(\Omega) \ni u \longmapsto e^{u^2} \in L^p(\Omega) \quad \forall p \geq 1,$$

which is connected to the critical Trudinger–Moser inequality

$$C(\Omega) = \sup\left\{ \int_\Omega e^{4\pi u^2} / u \in H^1_0(\Omega), \int_\Omega |\nabla u|^2 = 1 \right\} < +\infty,$$

[28]. Observe that, in general, critical points of the above constrained variational problem satisfy, after a simple scaling, an equation of the form (1.10). The Trudinger–Moser embedding is critical, involving loss of compactness in $H_0^1(\Omega)$ for the functionals J_λ^2 which translates into the presence of non-convergent Palais–Smale (PS) sequences. Let us consider for instance a sequence $\lambda_n \to \lambda_0 \geq 0$, and a sequence u_n with $\nabla J_{\lambda_n}^2(u_n) \to 0$, $J_{\lambda_n}^2(u_n) \to c$. For the Trudinger–Moser functional J_λ^2, a classification of all PS sequences for J_λ does not seem possible after the results in [2]. Actually PS holds as long as $c < 2\pi$, see [1,13]. On the other hand, for *solutions* more is known. From the result in [16] (see also [3,4,16,30]), we have the following fact:

Assume that u_n solves problem (1.10) for $\lambda = \lambda_n$, with $J_{\lambda_n}^2(u_n)$ bounded as $\lambda_n \to 0$. Then, passing to a subsequence, there is an integer $k \geq 0$ such that

$$J_{\lambda_n}^2(u_n) = 2k\pi + o(1). \tag{1.11}$$

When $k = 1$ a more precise answer is obtained in [3]: the solution u_n has for large n only one isolated maximum, which blows up around a point $x_0 \in \Omega$ which is characterized as a critical point of Robin's function $x \mapsto H(x,x)$. When $k > 1$, such a description for the behavior of u_n is not known and it seems to be still an open problem.

It is natural to ask whether or not solutions satisfying (1.11) exist. From the result in [2], it follows that there is a $\lambda_0 > 0$ such that a solution to (1.10) exists whenever $0 < \lambda < \lambda_0$ (this is in fact true for a larger class of nonlinearities with *critical exponential growth*). By construction this solution falls, as $\lambda \to 0$, into the bubbling category (1.11) with $k = 1$. In the case of a domain with a sufficiently small hole, Struwe in [32] built a solution taking advantage of the presence of topology. M. del Pino, M. Musso and B. Ruf in [15] established a general result concerning existence and multiplicity of solutions of problem (1.10).

In order to state this result, let us introduce the following function of k distinct points $\xi_1, \xi_2, \cdots, \xi_k \in \Omega$ and k positive numbers $m_1, m_2, \cdots, m_k$,

$$\varphi_{k,2}(\xi, m) = a \sum_{j=1}^{k} m_j^2 + 2 \sum_{j=1}^{k} m_j^2 \log m_j^2 + \sum_{j=1}^{k} m_j^2 H(\xi_j, \xi_j) + \sum_{i \neq j} m_i m_j G(\xi_i, \xi_j), \tag{1.12}$$

where $a > 0$ is an absolute constant, and $G(x, y)$ is the Green's function defined in (1.6) and $H(\cdot,\cdot)$ its regular part. The authors in [15] established that, if $\varphi_{k,2}$ has a *topologically non-trivial critical value*, with corresponding critical point $(\xi_1, \ldots, \xi_k, m_1, \ldots, m_k) \in \Omega^k \times \mathbb{R}_+^k$, then there exists a solution u_λ of (1.10) with the shape

$$u_\lambda(x) = \sqrt{\lambda}\left[\sum_{j=1}^{k} m_j G(x, \xi_j) + o(1)\right], \quad \text{as } \lambda \to 0, \tag{1.13}$$

where $o(1) \to 0$ as $\lambda \to 0$ uniformly on compact sets of $\Omega \setminus \{\xi_1, \ldots, \xi_k\}$. Furthermore,

$$J_\lambda^2(u_\lambda) = 2k\pi + \alpha\lambda + 4\pi\lambda\varphi_{k,2}(\xi, m) + \lambda o(1)$$

where α is an absolute constant, $\varphi_{k,2}$ is defined in (1.12) and $o(1) \to 0$ as $\lambda \to 0$. In particular, in the case Ω is not simply connected they constructed the solution u_λ of (1.10), with two bubbling points, namely satisfying

$$u_\lambda(x) = \sqrt{\lambda}\left[\sum_{j=1}^{2} m_j G(x, \xi_j) + o(1)\right], \quad \text{as } \lambda \to 0,$$

where (m_1, m_2, ξ_1, ξ_2) is a critical point of $\varphi_{2,2}$ defined in (1.12), and $o(1) \to 0$ as $\lambda \to 0$ uniformly on compact sets of $\Omega \setminus \{\xi_1, \xi_2\}$.

The above result shows a difference between the behavior of finite-energy solutions to problem (1.3) (or problem (1.1) with $p = 1$) and those to problem (1.10) (or problem (1.1) with $p = 2$): far away from the concentration points $\xi_1, \ldots, \xi_k$, solutions to (1.3) are at main order sums of Green's functions centered at ξ_j (see (1.8)), while solutions to (1.10) are at main order sum of Green's functions centered at ξ_j but with different positive weights $\sqrt{\lambda} m_j$ whose values depend on the location of the concentration points $\xi_1, \ldots, \xi_k$ (see (1.13)). In other words: To construct solutions to (1.10), one not only needs to choose carefully the concentration points $\xi_1, \ldots, \xi_k$, as for problem (1.3), but one has to carefully choose the correct weights $m_1, \ldots, m_k$.

This paper is motivated to understand the solutions to problem (1.1) when p is between 1 and 2. In fact, we obtain existence results for (1.1) in the whole range $0 < p < 2$, and we find that in this range problem (1.1) behaves as in the case $p = 1$, in the sense described above. Let us state our result.

Let us define

$$\mathcal{M} = \left\{(\xi_1, \cdots, \xi_k) \in \Omega^k : \mathrm{dist}(\xi_j, \partial\Omega) \geq \delta,\ |\xi_i - \xi_j| \geq \delta \text{ for } i \neq j\right\}$$

for some $\delta > 0$. Let ε be a parameter, which depends on λ, defined as

$$p\lambda\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{2(p-2)}{p}} = 1. \tag{1.14}$$

Observe that, as $\lambda \to 0$, then $\varepsilon \to 0$, and $\lambda = \varepsilon^2$ if $p = 1$. Our result states as follows.

Theorem 1.1. *Let $0 < p < 2$ and k be an integer with $k \geq 1$. If Ω is not simply connected, then there exists $\lambda_0 > 0$ so that, for any $0 < \lambda < \lambda_0$ problem (1.1) has a solution u_λ. Furthermore*

$$\lim_{\lambda \to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega e^{u_\lambda^p} = 8k\pi, \tag{1.15}$$

where ε satisfies (1.14). Moreover, there exists a k-tuple $\xi^\lambda = (\xi_1^\lambda, \cdots, \xi_k^\lambda) \in \mathcal{M}$ such that as $\lambda \to 0$

$$\nabla\varphi_k\left(\xi_1^\lambda, \cdots, \xi_k^\lambda\right) \to 0,$$

and

$$u_\lambda(x) = \frac{1}{p}\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{1-p}{p}}\left(\sum_{j=1}^{k} G\left(x, \xi_j^\lambda\right) + o(1)\right) \tag{1.16}$$

where $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\bar{\Omega} \setminus \{\xi_1^\lambda, \cdots, \xi_k^\lambda\}$. Furthermore

$$J_\lambda^p(u_\lambda) = \lambda\varepsilon^{\frac{2(p-2)}{p}}\left[\frac{8k\pi}{(2-p)p}[-2+p\log 8] - \frac{16k\pi}{p}\log\varepsilon - \frac{4\pi}{2-p}\varphi_k\big(\xi^\lambda\big) + O\big(|\log\varepsilon|^{-1}\big)\right] \tag{1.17}$$

where $O(1)$ is uniformly bounded as $\lambda \to 0$.

In [31], T. Ogawa and T. Suzuki investigated the asymptotic behavior of the blow-up solutions for problem (1.1) when $0 < p \le 2$ and $\Omega = B(0,1)$. Every smooth positive solution of this problem must be radially symmetric and decreasing in $|x|$ by the result of Gidas, Ni and Nirenberg [22], then $u(0) = \|u\|_{L^\infty}$. Suppose u_λ is a solution satisfying $\|u_\lambda\|_{L^\infty} \to \infty$ as $\lambda \to 0$, then $u_\lambda(x) \to 0$ locally uniformly on $\bar{B}\backslash\{0\}$, as $\lambda \to 0$. Thus, if we consider problem (1.1) in the unit disk of $\mathbb{R}^2$, suppose u is the solution of (1.1), then u blow up at origin as $\lambda \to 0$. We also mention that problems involving Laplacian in bounded domain in $\mathbb{R}^2$ with more generality exponential nonlinearity have been studied in [5,33]. In particular, the author in [33] considered the existence of solution to

$$-\Delta u = \lambda u e^{u^p}, \qquad u > 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B,$$

where $\lambda > 0$ and $B \subset \mathbb{R}^2$ is the unit ball and $1 \le p \le 2$.

We will prove Theorem 1.1 as a consequence of a more general theorem, in a spirit similar to the one used in [14]. To do so, we need to recall the notion of *topologically non-trivial critical level* for φ_k. Let us consider an open set $\mathcal{D}$ compactly contained in the domain of the functional φ_k. We recall that φ_k *links in $\mathcal{D}$ at critical level $\mathcal{C}$ relative to B* and B_0 if B and B_0 are closed subsets of $\bar{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that the following conditions hold: Let us set Γ to be the class of the maps $\Phi \in C(B, \mathcal{D})$ with the property that there exists a function $\Psi \in C([0,1] \times B, \mathcal{D})$ such that

$$\Psi(0,\cdot) = Id_B, \qquad \Psi(1,\cdot) = \Phi, \qquad \Psi(t,\cdot)|_{B_0} = Id_{B_0} \quad \text{for } \forall t \in [0,1].$$

We assume

$$\sup_{\xi\in B_0}\varphi_k(\xi) < \mathcal{C} := \inf_{\Phi\in\Gamma}\sup_{\xi\in B}\varphi_k\big(\Phi(\xi)\big), \tag{1.18}$$

and for all $\xi \in \partial\mathcal{D}$ such that $\varphi_k(\xi) = \mathcal{C}$, there exists a vector τ tangent to $\partial\mathcal{D}$ at ξ such that

$$\nabla\varphi_k(\xi)\cdot\tau \ne 0. \tag{1.19}$$

Under these conditions a critical point $\bar{\xi} \in \mathcal{D}$ with $\varphi_k(\bar{\xi}) = \mathcal{C}$ exists, as a standard deformation argument involving the negative gradient flow of φ_k shows. Condition (1.18) is a general way of describing a change of topology in the level sets $\{\varphi_k \le c\}$ in $\mathcal{D}$ taking place at $c = \mathcal{C}$, while (1.19) prevents intersection of the level set $\mathcal{C}$ with the boundary. It is easy to check that the above conditions hold if

$$\inf_{\xi\in\mathcal{D}}\varphi_k(\xi) < \inf_{\xi\in\partial\mathcal{D}}\varphi_k(\xi), \quad \text{or} \quad \sup_{\xi\in\mathcal{D}}\varphi_k(\xi) > \sup_{\xi\in\partial\mathcal{D}}\varphi_k(\xi),$$

namely the case of (possibly degenerate) local minimum or maximum points of φ_k. The level $\mathcal{C}$ may be taken in these cases respectively as that of the minimum and the maximum of φ_k in $\mathcal{D}$. These hold also if φ_k is C^1-close to a function with a non-degenerate critical point in $\mathcal{D}$. We call $\mathcal{C}$ a non-trivial critical level of φ_k in $\mathcal{D}$.

Theorem 1.2. *For $0 < p < 2$, let $k \geq 1$, assume that φ_k defined by (1.5) has a topologically non-trivial critical level $\mathcal{C}$ in $\mathcal{D}$, then the problem (1.1) has a family solutions u_λ for λ small enough, such that*

$$\lim_{\lambda\to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega e^{u_\lambda^p} = 8k\pi, \tag{1.20}$$

where ε satisfies (1.14). Moreover, there exists a k-tuple $\xi^\lambda = (\xi_1^\lambda, \cdots, \xi_k^\lambda) \in \mathcal{M}$ such that as $\lambda \to 0$

$$\varphi_k\big(\xi_1^\lambda, \cdots, \xi_k^\lambda\big) \to c,$$

and

$$u_\lambda(x) = \frac{1}{p}\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{1-p}{p}} \left(\sum_{j=1}^k G\big(x, \xi_j^\lambda\big) + o(1)\right) \tag{1.21}$$

where $o(1) \to 0$ on each compact subset of $\bar\Omega \backslash \{\xi_1^\lambda, \cdots, \xi_k^\lambda\}$. Furthermore

$$\begin{aligned} J_\lambda^p(u_\lambda) = \lambda\varepsilon^{\frac{2(p-2)}{p}} \bigg[& \frac{8k\pi}{(2-p)p}[-2 + p\log 8] - \frac{16k\pi}{p}\log\varepsilon \\ & - \frac{4\pi}{2-p}\varphi_k\big(\xi^\lambda\big) + O\big(|\log\varepsilon|^{-1}\big)\bigg] \end{aligned} \tag{1.22}$$

where $O(1)$ is uniformly bounded as $\lambda \to 0$.

The proof of our result relies on a Lyapunov–Schmidt reduction procedure, introduced in [7,21] and used in many different contexts, see for instance [14,15,17–20]. The key step is to find the ansatz for the solution. Usually, the ansatz is built as a sum of terms, which turns out to be solutions of the associate limit problem, which are properly scaled and translated. For our problem, our approximate solution is built by using the following “basic cells”: the radially symmetric solutions of the following Liouville equation

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w < +\infty,$$

which are given by

$$w_\mu(z) := \log\frac{8\mu^2}{(\mu^2 + |z|^2)^2}, \qquad w_\mu(z-\xi) := \log\frac{8\mu^2}{(\mu^2 + |z-\xi|^2)^2} \tag{1.23}$$

where μ is any positive number and ξ any point in $\mathbb{R}^2$ (see [12]). If we use a sum of the above basic cells, properly scaled, and centered at several points of the domain as our approximate solution, we get a very good approximation of a solution in a region far away from the points, which unfortunately turns out to be not good enough close to these points. Thus we need to improve the approximation, at least near the points, and we do this adding two other terms in the expansion of the solution. This can be done in a very natural way, which has first been used in [17] for studying the following problem

$$\begin{cases} \Delta u + u^p = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.24}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, and p is a large exponent. Later on, this method has been applied in other contexts, see [10,18,19,26]. Observe that this method allows to improve the approximation near the points, but it is not useful to improve the approximation far away from these points. Nevertheless, as already mentioned, the approximation we build for this problem is sufficiently accurate in a regime far from the points. After the approximate solution is build, we find an actual solution to (1.1) as a small perturbation of the approximation.

The paper is organized as follows: Section 2 is devoted to describing a first approximation solution to problem (1.1) and estimating the error. Furthermore, problem (1.1) is written as a fixed point problem, involving a linear operator. In Section 3, we study the invertibility of the linear problem. In Section 4, we study the nonlinear problem. In Section 5, we study the variational reduction, we prove Theorems 1.1 and 1.2 in Section 6.

In this paper, the symbol C denotes a generic positive constant independent of λ, it could be changed from one line to another. The symbols $O(t)$ (respectively $o(t)$) will denote quantities for which $\frac{O(t)}{|t|}$ stays bounded (respectively, $\frac{o(t)}{|t|}$ tends to zero) as parameter λ goes to zero. In particular, we will often use the notation $o(1)$ standing for a quantity which tends to zero as $\lambda \to 0$.

2. Preliminaries and ansatz for the solution

In this first section we describe the approximate solution for problem (1.1) and then we estimate the error of such approximation in appropriate norms.

Let us consider k distinct points $\xi_1, \xi_2, \cdots, \xi_k$ in Ω; we choose a sufficiently small but fixed number $\delta > 0$ and assume that for $j = 1, 2, \cdots, k$,

$$\operatorname{dist}(\xi_j, \partial\Omega) \geq \delta, \qquad |\xi_i - \xi_j| \geq \delta \quad \text{for } i \neq j. \tag{2.1}$$

Furthermore, we consider k positive numbers μ_j such that

$$\delta < \mu_j < \delta^{-1}, \quad \text{for all } j = 1, \ldots, k. \tag{2.2}$$

The parameters μ_j will be chosen properly later on. Define the function

$$U_{\mu_j,\xi_j}(x) = \log \frac{8\mu_j^2}{(\mu_j^2\varepsilon^2 + |x - \xi_j|^2)^2}.$$

Let us denote $PU_{\mu_j,\xi_j}(x)$ the projection of U_{μ_j,ξ_j} into the space $H_0^1(\Omega)$, in other words, $PU_{\mu_j,\xi_j}(x)$ is the unique solution of

$$\begin{cases} \Delta PU_{\mu_j,\xi_j} = \Delta U_{\mu_j,\xi_j}, & \text{in } \Omega; \\ PU_{\mu_j,\xi_j} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Lemma 2.1. *Assume (2.1) and (2.2). We have*

$$PU_{\mu_j,\xi_j}(x) = U_{\mu_j,\xi_j}(x) + H(x,\xi_j) - \log\big(8\mu_j^2\big) + O\big(\mu_j^2\varepsilon^2\big) \tag{2.4}$$

in $C^1(\bar{\Omega})$ as $\varepsilon \to 0$, and

$$PU_{\mu_j,\xi_j}(x) = G(x,\xi_j) + O\big(\mu_j^2\varepsilon^2\big) \tag{2.5}$$

in $C^1_{loc}(\bar{\Omega}\backslash\{\xi_j\})$ as $\varepsilon \to 0$, where $G(\cdot,\cdot)$ and $H(\cdot,\cdot)$ are Green's function and its regular part as defined in (1.6) and (1.7).

Proof. Let $z(x) = PU_{\mu_j,\xi_j}(x) - U_{\mu_j,\xi_j}(x) + \log(8\mu_j^2)$, then $z(x)$ satisfies

$$\begin{cases} \Delta z(x) = 0 & \text{in } \Omega; \\ z(x) = 2\log\big(\mu_j^2\varepsilon^2 + |x-\xi_j|^2\big) & \text{on } \partial\Omega. \end{cases}$$

On the other hand, we note that $\eta(x) = H(x,\xi_j)$ satisfies

$$\begin{cases} \Delta \eta(x) = 0 & \text{in } \Omega; \\ \eta(x) = 2\log|x-\xi_j|^2 & \text{on } \partial\Omega. \end{cases}$$

Then we get

$$\begin{cases} \Delta\big(z(x) - \eta(x)\big) = 0 & \text{in } \Omega; \\ z(x) - \eta(x) = -2\log\dfrac{|x-\xi_j|^2}{\mu_j^2\varepsilon^2 + |x-\xi_j|^2} & \text{on } \partial\Omega. \end{cases}$$

Since $|x-\xi_j| > \delta$ for $x \in \partial\Omega$, then by the maximum principle we get

$$\max_{\bar{\Omega}}\big|z(\cdot) - \eta(\cdot)\big| = \max_{x\in\partial\Omega}\big|z(\cdot) - \eta(\cdot)\big| = O\big(\mu_j^2\varepsilon^2\big),$$

as $\varepsilon \to 0$, uniformly in Ω. Then we obtain the C^0-estimate in (2.4). Analogous computations give the C^1-closeness and hence the validity of (2.4). By (2.4) we deduce (2.5). □

We shall show later on that $PU_{\mu_j,\xi_j}(x)$ is a good approximation for a solution to (1.1) far from the points ξ_j, but unfortunately it is not good enough for our construction close to the points ξ_j. This is the reason why we need to further adjust $PU_{\mu_j,\xi_j}(x)$. To do this, we need to introduce the following functions w_j^0 and w_j^1.

Let w_{μ_j} be defined as in (1.23). Define the function w_j^i to be radial solution of

$$\Delta w_j^i + e^{w_{\mu_j}} w_j^i = e^{w_{\mu_j}} f^i \quad \text{in } \mathbb{R}^2, \text{ for } i = 0, 1, \tag{2.6}$$

and

$$f^0 = -\left(w_{\mu_j} + \frac{1}{2}(w_{\mu_j})^2\right), \tag{2.7}$$

$$\begin{aligned} f^1 = -\bigg(& w_j^0 + \frac{p-2}{2(p-1)}(w_{\mu_j})^2 + \frac{1}{2}(w_j^0)^2 + \frac{1}{8}(w_{\mu_j})^4 \\ &+ 2w_{\mu_j} w_j^0 + \frac{1}{2}(w_{\mu_j})^3 + \frac{1}{2} w_j^0 (w_{\mu_j})^2\bigg). \end{aligned} \tag{2.8}$$

In fact, as shown in [17] (see also [11]), there exist radially symmetric solutions with the properties that

$$w_j^i(y) = C_{ij} \log \frac{|y|}{\mu_j} + O\left(\frac{1}{|y|}\right) \quad \text{as } |y| \to \infty, \tag{2.9}$$

for some explicit constants C_{ij}, which can be explicitly computed. In particular, when $i = 0$, the constant C_{0j} is given by

$$\begin{aligned} C_{0j} &= -8 \int_0^{+\infty} t \frac{t^2-1}{(t^2+1)^3} \left[\log \frac{8\mu_j^{-2}}{(1+t^2)^2} + \frac{1}{2}\left(\log \frac{8\mu_j^{-2}}{(1+t^2)^2}\right)^2\right] dt \\ &= -4 \int_0^{+\infty} \frac{t^2-1}{(t^2+1)^3} \left[\log \frac{8\mu_j^{-2}}{(1+t^2)^2} + \frac{1}{2}\left(\log \frac{8\mu_j^{-2}}{(1+t^2)^2}\right)^2\right] d\left(t^2\right) \\ &\quad \text{set } r = t^2 + 1 \\ &= -4 \int_1^{+\infty} \frac{r-2}{r^3} \bigg[\log\left(8\mu_j^{-2}\right) - 2\log r \\ &\quad + \frac{1}{2}\left(\log\left(8\mu_j^{-2}\right)\right)^2 - 2\log\left(8\mu_j^{-2}\right)\log r + 2(\log r)^2\bigg] dr. \end{aligned}$$

Since

$$\int_1^{+\infty} \frac{r-2}{r^3} dr = 0, \quad \int_1^{+\infty} \frac{r-2}{r^3} \log r \, dr = \frac{1}{2}, \quad \text{and} \quad \int_1^{+\infty} \frac{r-2}{r^3} (\log r)^2 dr = \frac{3}{2},$$

we get

$$C_{0j} = 4\log 8 - 8 - 8\log \mu_j. \tag{2.10}$$

Let us define

$$w^0_{\mu_j,\xi_j}(x) := w^0_j\left(\frac{x-\xi_j}{\varepsilon}\right), \qquad w^1_{\mu_j,\xi_j}(x) := w^1_j\left(\frac{x-\xi_j}{\varepsilon}\right) \quad \text{for } x \in \Omega.$$

Let $Pw^0_{\mu_j,\xi_j}$ and $Pw^1_{\mu_j,\xi_j}$ denote the projections into $H^1_0(\Omega)$ of $w^0_{\mu_j,\xi_j}$ and $w^1_{\mu_j,\xi_j}$, respectively. By (2.9), we have that

$$\begin{aligned} P\left(w^i_{\mu_j,\xi_j}(x)\right) &= P\left(w^i_j\left(\frac{y-\xi'_j}{\mu_j}\right)\right) \\ &= w^i_j\left(\frac{y-\xi'_j}{\mu_j}\right) - \frac{C_{ij}}{4}H(x,\xi_j) + C_{ij}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon) \end{aligned} \tag{2.11}$$

in $C^1(\bar{\Omega})$ as $\varepsilon \to 0$, and

$$P\left(w^i_{\mu_j,\xi_j}(x)\right) = P\left(w^i_j\left(\frac{y-\xi'_j}{\mu_j}\right)\right) = -\frac{C_{ij}}{4}G(x,\xi_j) + O(\mu_j\varepsilon) \tag{2.12}$$

in $C^1_{loc}(\bar{\Omega}\setminus\{\xi_j\})$ as $\varepsilon \to 0$.

We define

$$\begin{aligned} U_\lambda(x) = \frac{1}{p\gamma^{p-1}}\sum_{j=1}^k\Bigg(&PU_{\mu_j,\xi_j}(x) + \frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(x) \\ &+ \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_j,\xi_j}(x)\Bigg). \end{aligned} \tag{2.13}$$

From (2.5) and (2.12), one has, away from the points ξ_j,

$$U_\lambda(x) = \frac{1}{p\gamma^{p-1}}\sum_{j=1}^k G(x,\xi_j)\left(1 - \frac{p-1}{p}\frac{1}{\gamma^p}\frac{C_{0j}}{4} - \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\frac{C_{1j}}{4} + O\left(\varepsilon^2\right)\right). \tag{2.14}$$

Consider now the change of variables

$$v(y) = p\gamma^{p-1}u(\varepsilon y) - p\gamma^p, \quad \text{with } \gamma^p = -\frac{4}{p}\log\varepsilon.$$

By (1.14), then problem (1.1) reduces to

$$\begin{cases} \Delta v + g(v) = 0, \quad v > 0 & \text{in } \Omega_\varepsilon; \\ v = -p\gamma^p & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.15}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$, and

$$g(v) = \left(1 + \frac{v}{p\gamma^p}\right)^{p-1} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}. \tag{2.16}$$

Let us define the first approximation solution to (2.15) as

$$V_\lambda(y) = p\gamma^{p-1}U_\lambda(\varepsilon y) - p\gamma^p, \tag{2.17}$$

with U_λ defined by (2.13). We write $y = \varepsilon^{-1}x$, $\xi'_j = \varepsilon^{-1}\xi_j$. For $|x - \xi_j| < \delta$ with δ sufficiently small but fixed, by using (2.4), (2.5), (2.11), (2.12) and the fact that $U_{\mu_j,\xi_j}(\varepsilon y) - p\gamma^p = w_j(y - \xi'_j)$, we have

$$\begin{aligned}
V_\lambda(y) &= PU_{\mu_j,\xi_j}(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_j,\xi_j}(\varepsilon y) - p\gamma^p \\
&\quad + \sum_{i\neq j}^{k}\left(PU_{\mu_i,\xi_i}(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_i,\xi_i}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_i,\xi_i}(\varepsilon y)\right) \\
&= U_{\mu_j,\xi_j}(\varepsilon y) + H(\varepsilon y, \xi_j) - \log\left(8\mu_j^2\right) + O\left(\mu_j^2\varepsilon^2\right) - p\gamma^p \\
&\quad + \frac{p-1}{p}\frac{1}{\gamma^p}\left[w^0_j\left(\frac{y-\xi'}{\mu_j}\right) - \frac{C_{0j}}{4}H(\varepsilon y, \xi_j) + C_{0j}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon)\right] \\
&\quad + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[w^1_j\left(\frac{y-\xi'}{\mu_j}\right) - \frac{C_{1j}}{4}H(\varepsilon y, \xi_j) + C_{1j}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon)\right] \\
&\quad + \sum_{i\neq j}^{k}G(\xi_i,\xi_j)\left[1 - \frac{C_{0j}}{4}\frac{p-1}{p}\frac{1}{\gamma^p} - \frac{C_{1j}}{4}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right] + O\left(\varepsilon^2\right) \\
&= w_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}w^0_j(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w^1_j(y) \\
&\quad - \log\left(8\mu_j^2\right) + \left[1 - \frac{C_{0j}}{4}\frac{p-1}{p}\frac{1}{\gamma^p} - \frac{C_{1j}}{4}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right] \\
&\quad \times\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^{k}G(\xi_i,\xi_j)\right) \\
&\quad + \left[C_{0j}\frac{p-1}{p}\frac{1}{\gamma^p} + C_{1j}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right]\left(\log(\mu_j) + \log\varepsilon\right) \\
&\quad + O\left(\varepsilon\left|y - \xi'\right|\right) + O\left(\varepsilon^2\right),
\end{aligned} \tag{2.18}$$

where

$$w_j(y) := w_{\mu_j}\left(y - \xi'_j\right), \qquad w^0_j(y) := w^0_j\left(\frac{y-\xi'_j}{\mu_j}\right), \qquad w^1_j(y) := w^1_j\left(\frac{y-\xi'_j}{\mu_j}\right).$$

We now choose the parameters μ_j: we assume they are defined by the relation

$$\log(8\mu_j^2) = \left(H(\xi_j,\xi_j) + \sum_{i\neq j}^k G(\xi_i,\xi_j)\right) - \frac{p-1}{4}C_{0j}$$
$$- \frac{p-1}{p}\frac{1}{\gamma^p}\frac{C_{0j}}{4}\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^k G(\xi_i,\xi_j) + 4\log(\mu_j) - (p-1)\frac{C_{1j}}{C_{0j}}\right)$$
$$- \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\frac{C_{1j}}{4}\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^k G(\xi_i,\xi_j) + 4\log(\mu_j)\right). \tag{2.19}$$

Taking into account the explicit expression (2.10) of the constant C_{0j}, we observe that μ_j bifurcates, as λ goes to zero, from the value $\bar{\mu}_j$ defined by

$$\bar{\mu}_j = 8^{-\frac{p}{2(2-p)}} e^{\frac{p-1}{2-p}} e^{\frac{1}{2(2-p)}[H(\xi_j,\xi_j)+\sum_{i\neq j}^k G(\xi_i,\xi_j)]} \tag{2.20}$$

solution of equation

$$\log(8\mu_j^2) = \left(H(\xi_j,\xi_j) + \sum_{i\neq j}^k G(\xi_i,\xi_j)\right) - \frac{p-1}{4}C_{0j}. \tag{2.21}$$

Thus, μ_j is a perturbation of order $\frac{1}{\gamma^p}$ of the value $\bar{\mu}_j$, namely

$$\log(8\mu_j^2) = \left[\frac{2(p-1)}{2-p}(1-\log 8) + \frac{1}{2-p}\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^k G(\xi_i,\xi_j)\right)\right]\left(1 + O\left(\frac{1}{\gamma^p}\right)\right). \tag{2.22}$$

Then, by this choice of the parameters μ_j, we deduce that, if $|y-\xi_j'| < \delta/\varepsilon$ with δ sufficiently small but fixed, we can rewrite

$$V_\lambda(y) = w_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y) + \theta(y), \tag{2.23}$$

with

$$\theta(y) = O(\varepsilon|y-\xi_j'|) + O(\varepsilon^2).$$

We will look for solutions to (2.15) of the form

$$v = V_\lambda + \phi,$$

where V_λ is defined as in (2.17), and ϕ represents a lower order correction. We aim at finding a solution for ϕ small provided that the points ξ_j are suitably chosen. For small ϕ, we can rewrite problem (2.15) as a nonlinear perturbation of its linearization, namely,

$$\begin{cases} L(\phi) = -\big[E_\lambda + N(\phi)\big], & x \in \Omega_\varepsilon; \\ \phi = 0, & x \in \partial\Omega_\varepsilon, \end{cases} \tag{2.24}$$

where

$$L(\phi) := \Delta\phi + g'(V_\lambda)\phi, \tag{2.25}$$

$$E_\lambda := \Delta V_\lambda + g(V_\lambda), \tag{2.26}$$

$$N(\phi) := g(V_\lambda + \phi) - g(V_\lambda) - g'(V_\lambda)\phi. \tag{2.27}$$

We recall that $g(t) = (1 + \frac{t}{p\gamma^p})^{p-1} e^{\gamma^p[(1+\frac{t}{p\gamma^p})^p - 1]}$.

In order to solve the problem (2.24), first we have to study the invertibility properties of the linear operator L. In order to do this, we introduce a weighted L^∞-norm defined as

$$\|h\|_* := \sup_{y\in\Omega_\varepsilon} \left(\sum_{j=1}^k \big(1 + |y - \xi_j'|\big)^{-3} + \varepsilon^2 \right)^{-1} \big|h(y)\big| \tag{2.28}$$

for any $h \in L^\infty(\Omega_\varepsilon)$. With respect to this norm, the error term E_λ given in (2.26) can be estimated in the following way.

Lemma 2.2. *Let $\delta > 0$ be a small but fixed number and assume that the points ξ_j satisfy (2.1). There exists $C > 0$, such that we have*

$$\|E_\lambda\|_* \le \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3} \tag{2.29}$$

for all λ small enough.

Proof. Far away from the points ξ_j, namely for $|x - \xi_j| > \delta$, i.e. $|y - \xi_j'| > \frac{\delta}{\varepsilon}$, for all $j = 1, \cdots, k$, from (2.5) and (2.12) we have that

$$\Delta V_\lambda(y) = p\gamma^{p-1}\varepsilon^2 \Delta U(\varepsilon y) = O\big(\gamma^{p-1}\varepsilon^4\big).$$

On the other hand, in this region we have

$$1 + \frac{V_\lambda(y)}{p\gamma^p} = 1 + \frac{4\log\varepsilon + O(1)}{p\gamma^p} = \frac{O(1)}{|\log\varepsilon|} \tag{2.30}$$

where $O(1)$ denotes a smooth function, uniformly bounded, as $\varepsilon \to 0$, in the considered region. Hence

$$g(V_\lambda) = \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{p-1} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p - 1]}$$
$$\le \begin{cases} C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1 \le p < 2; \\ C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} e^{\gamma^p \frac{O(1)}{|\log\varepsilon|^p}} & \text{if } 0 < p < 1. \end{cases}$$

$$
= \begin{cases} C \dfrac{\varepsilon^{\frac{4}{p}}}{|\log \varepsilon|^{p-1}} & \text{if } 1 \le p < 2; \\ C \dfrac{\varepsilon^{\frac{4}{p}}}{|\log \varepsilon|^{p-1}} e^{\frac{O(1)}{|\log \varepsilon|^{p-1}}} & \text{if } 0 < p < 1. \end{cases}
$$

Thus if we are far away from the points ξ_j, or equivalently for $|y - \xi_j'| > \frac{\delta}{\varepsilon}$, the size of the error, measured with respect to the $\| \cdot \|_*$-norm, is relatively small. In other words, if we denote by 1_{outer} the characteristic function of the set $\{y : |y - \xi_j'| > \frac{\delta}{\varepsilon}, j = 1, \dots, k\}$, then in this region we have

$$
\begin{aligned}
\|E_\lambda 1_{\mathsf{outer}}\|_* &\le \begin{cases} C \dfrac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log \varepsilon|^{p-1}} & \text{if } 1 \le p < 2; \\ C \dfrac{\varepsilon^{\frac{2-p}{p}}}{|\log \varepsilon|^{p-1}} e^{\log \varepsilon^{\frac{2-p}{p}} + \frac{C}{|\log \varepsilon|^{p-1}}} & \text{if } 0 < p < 1. \end{cases} \\
&= \begin{cases} C \dfrac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log \varepsilon|^{p-1}} & \text{if } 1 \le p < 2; \\ C \dfrac{\varepsilon^{\frac{2-p}{p}}}{|\log \varepsilon|^{p-1}} e^{-\frac{2-p}{p}|\log \varepsilon| + C|\log \varepsilon|^{1-p}} & \text{if } 0 < p < 1. \end{cases} \\
&\le \begin{cases} C \dfrac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log \varepsilon|^{p-1}} & \text{if } 1 \le p < 2; \\ C \dfrac{\varepsilon^{\frac{2-p}{p}}}{|\log \varepsilon|^{p-1}} & \text{if } 0 < p < 1. \end{cases}
\end{aligned} \tag{2.31}
$$

Here we used that $-\frac{2-p}{p}|\log \varepsilon| + C|\log \varepsilon|^{1-p} < 0$ for $0 < p < 1$ and ε small.

Let us now fix the index j in $\{1, \cdots, k\}$; for $|y - \xi_j'| < \frac{\delta}{\varepsilon}$, we have

$$
\Delta V_\lambda(y) = -e^{w_j(y)} + \frac{p-1}{p} \frac{1}{\gamma^p} \Delta w_j^0(y) + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \Delta w_j^1(y) + O\big(\varepsilon^2\big). \tag{2.32}
$$

On the other hand, for any $R > 0$ large but fixed, in the ball $|y - \xi_j'| < R_\varepsilon := R|\log \varepsilon|^\alpha$, with $\alpha \ge 3$, we can use Taylor expansion to first get

$$
\begin{aligned}
\left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{p-1} &= 1 + \frac{p-1}{p} \frac{1}{\gamma^p} w_j + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \left[w_j^0 + \frac{p-2}{2(p-1)} (w_j)^2 \right] \\
&\quad + \left(\frac{p-1}{p}\right)^3 \frac{1}{\gamma^{3p}} \big(\log |y|\big), \\
\gamma^p \left[\left(1 + \frac{V_\lambda}{p\gamma^p}\right)^p - 1 \right] &= w_j + \left(\frac{p-1}{p}\right) \frac{1}{\gamma^p} \left[w_j^0 + \frac{(w_j)^2}{2} \right] \\
&\quad + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \big(w_j^1 + w_j w_j^0\big) + \frac{1}{\gamma^{3p}} \big(\log |y|\big)
\end{aligned}
$$

and

$$
e^{\gamma^p[(1+vV_\lambda \frac{p\gamma^p}{)}^p-1]} = e^{w_j}\left[1 + \left(\frac{p-1}{p}\right)\frac{1}{\gamma^p}\left[w_j^0 + \frac{(w_j)^2}{2}\right]\right.
$$
$$
\left. + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left[w_j^1 + w_j w_j^0 + \frac{1}{2}\left(w_j^0 + (w_j)^2\right)^2\right] + \frac{1}{\gamma^{3p}}\left(\log|y|\right)\right]
$$

Thus we obtain

$$
\begin{aligned}
g(V_\lambda) &:= \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{p-1} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} \\
&= e^{w_j}\left[1 + \left(\frac{p-1}{p}\right)\frac{1}{\gamma^p}\left[w_j^0 + \frac{(w_j)^2}{2} + w_j\right]\right. \\
&\quad + \left(\frac{p-1^2}{p}\right)\frac{1}{\gamma^{2p}}\left[w_j^1 + 2w_j w_j^0 + \frac{1}{2}\left(w_j^0 + \frac{(w_j)^2}{2}\right)^2 + w_j^0 + \frac{p-2}{2(p-1)}w_j^2 + \frac{w_j^3}{2}\right] \\
&\quad \left. + O\left(\frac{\log|y-\xi_j'|}{\gamma^{3p}}\right)\right].
\end{aligned}
$$

Thus, thanks to the fact that we have improved our original approximation with the terms w_j^0 and w_j^1, and the definition of $*$-norm, we get that

$$
\|E_\lambda 1_{B(\xi_j', R_\varepsilon)}\|_* \le \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3}, \quad \text{for any } j = 1, \dots, k. \tag{2.33}
$$

Here $1_{B(\xi_j', R_\varepsilon)}$ denotes the characteristic function of $B(\xi_j, R_\varepsilon)$. Finally, in the remaining region, namely where $R_\varepsilon < |y - \xi_j'| < \frac{\delta}{\varepsilon}$, for any $j = 1, \dots, k$, we have from one hand that $|\Delta V_\lambda(y)| \le Ce^{w_j(y)}$, and also $|g(V_\lambda(y))| \le Ce^{w_j(y)}$ as a consequence of (2.18). This fact, together with (2.33) and (2.31) gives the estimate (2.29). □

As in the above computation, we find that very close to the point ξ_j in Ω, we have

$$
\left\|g'(V_\lambda) - e^{w_j}\right\|_* \to 0 \quad \text{as } \lambda \to 0, \tag{2.34}
$$

and there exists some positive constant D_0 such that

$$
g'(V_\lambda) \le D_0 \sum_{j=1}^k e^{w_j}. \tag{2.35}
$$

Moreover, we can get

$$
\left\|g''(V_\lambda)\right\|_* \le C. \tag{2.36}
$$

Proof of (2.34) and (2.35). We have

$$\begin{aligned} g'(V_\lambda) &= \frac{p-1}{p}\frac{1}{\gamma^p}\left(1+\frac{V_\lambda}{p\gamma^p}\right)^{p-2} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} \\ &\quad + \left(1+\frac{V_\lambda}{p\gamma^p}\right)^{2(p-1)} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} \\ &:= I_a + I_b. \end{aligned}$$

Far away from the points ξ_j, namely for $|x-\xi_j|>\delta$, i.e. $|y-\xi_j'|>\frac{\delta}{\varepsilon}$, for all $j=1,\cdots,k$, a consequence of (2.30) is that

$$I_a = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}}O(1), \quad \text{and} \quad I_b = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{2(p-1)}}O(1)$$

Then we have

$$g'(V_\lambda)1_{\text{outer}} = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}}O(1) \tag{2.37}$$

On the other hand, fix the index j in $\{1,\cdots,k\}$, for $|y-\xi_j'|<R_\varepsilon$ with $R_\varepsilon = R|\log\varepsilon|$, for any $R>0$ large but fixed, we use Taylor expansion to get

$$\begin{aligned} I_a &= \frac{p-1}{p}\frac{1}{\gamma^p}\left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^{p-2} \\ &\quad \times e^{\gamma^p[(1+\frac{1}{p\gamma^p}(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)))^p-1]} \\ &= \frac{p-2}{p}\frac{1}{\gamma^p}\left(\frac{p-1}{p-2}+\frac{p-1}{p}\frac{1}{\gamma^p}w_j(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^0(y)\right. \\ &\quad \left. +\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}w_j^1(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\theta(y)\right) \\ &\quad \times e^{w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)}e^{(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)}e^{\theta(y)}e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)]^2}, \end{aligned}$$

and

$$\begin{aligned} I_b &= \left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^{2(p-1)} \\ &\quad \times e^{\gamma^p[(1+\frac{1}{p\gamma^p}(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)))^p-1]} \\ &= \left(1+\frac{2(p-1)}{p}\frac{1}{\gamma^p}w_j(y)+2\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^0(y)\right. \end{aligned}$$

$$+ 2\left(\frac{p-1}{p}\right)^3 \frac{1}{\gamma^{3p}} w_j^1(y) + \frac{2(p-1)}{p} \frac{1}{\gamma^p} \theta(y)\Bigg)$$
$$\times e^{w_j(y)} e^{\frac{p-1}{p}\frac{1}{\gamma^p} w_j^0(y)} e^{(\frac{p-1}{p})^2 \frac{1}{\gamma^{2p}} w_j^1(y)} e^{\theta(y)} e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p} w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}} w_j^1(y)+\theta(y)]^2}.$$

By the definition of w_j^0 and w_j^1, we get that

$$I_a 1_{B(\xi_j', R_\varepsilon)} = \frac{O(1)}{|\log\varepsilon|}, \qquad I_b 1_{B(\xi_j', R_\varepsilon)} - e^{w_j(y)} = \frac{O(1)}{|\log\varepsilon|} \tag{2.38}$$

Finally, in the remaining region, namely where for any $j = 1, \ldots, k$, we have $R_\varepsilon < |y - \xi_j'| < \frac{\delta}{\varepsilon}$, we have

$$|I_a| \le \frac{C}{|\log\varepsilon|} e^{w_j(y)}, \qquad |I_b| \le C e^{w_j(y)}. \tag{2.39}$$

Then, from (2.38) and the definition of $*$-norm, we find that very close to the point ξ_j in Ω, we have

$$\left\| g'(V_\lambda) - e^{w_j} \right\|_* = \frac{O(1)}{|\log\varepsilon|}$$

which implies (2.34). Combining (2.37), (2.38) with (2.39) we obtain estimate (2.35). $\square$

Proof of (2.36). We have

$$\begin{aligned} g''(V_\lambda) &= \frac{(p-1)(p-2)}{p^2} \frac{1}{\gamma^{2p}} \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{p-3} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p - 1]} \\ &= \frac{3(p-1)}{p} \frac{1}{\gamma^p} \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{2p-3} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p - 1]} \\ &\quad + \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{3(p-1)} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p - 1]} \\ &:= I_c + I_d + I_e. \end{aligned}$$

By a similar computation as above: Far away from the points ξ_j, namely for $|x - \xi_j| > \delta$, i.e. $|y - \xi_j'| > \frac{\delta}{\varepsilon}$, for all $j = 1, \cdots, k$, we have

$$I_c = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} O(1), \qquad I_d = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{2(p-1)}} O(1), \quad \text{and} \quad I_e = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{3(p-1)}} O(1)$$

Then

$$g''(V_\lambda) 1_{\text{outer}} = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} O(1) \tag{2.40}$$

where again $O(1)$ denotes a function which is uniformly bounded, as $\varepsilon \to 0$, in the considered region. Let us now fix the index j in $\{1, \cdots, k\}$; for $|y - \xi_j'| < R_\varepsilon$ with any $R_\varepsilon := R|\log\varepsilon|$ for some $R > 0$ large but fixed, by Taylor expansion, we have

$$\begin{aligned}
I_c = {} & \frac{(p-1)(p-2)}{p^2}\frac{1}{\gamma^{2p}}\bigg(1+\frac{1}{p\gamma^p}\bigg(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y) \\
& +\bigg(\frac{p-1}{p}\bigg)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\bigg)\bigg)^{p-3} \\
& \times e^{\gamma^p[(1+\frac{1}{p\gamma^p}(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)))^p-1]} \\
= {} & \frac{(p-2)(p-3)}{p^2}\frac{1}{\gamma^{2p}}\bigg(\frac{p-1}{p-3}+\frac{p-1}{p}\frac{1}{\gamma^p}w_j(y)+\bigg(\frac{p-1}{p}\bigg)^2\frac{1}{\gamma^{2p}}w_j^0(y) \\
& +\bigg(\frac{p-1}{p}\bigg)^3\frac{1}{\gamma^{3p}}w_j^1(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\theta(y)\bigg) \\
& \times e^{w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)}e^{(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)}e^{\theta(y)}e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)]^2},
\end{aligned}$$

$$\begin{aligned}
I_d = {} & \frac{3(p-1)}{p}\frac{1}{\gamma^{p}}\bigg(1+\frac{1}{p\gamma^p}\bigg(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y) \\
& +\bigg(\frac{p-1}{p}\bigg)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\bigg)\bigg)^{2p-3} \\
& \times e^{\gamma^p[(1+\frac{1}{p\gamma^p}(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)))^p-1]} \\
= {} & \frac{3(2p-3)}{p}\frac{1}{\gamma^{p}}\bigg(\frac{p-1}{2p-3}+\frac{p-1}{p}\frac{1}{\gamma^p}w_j(y)+\bigg(\frac{p-1}{p}\bigg)^2\frac{1}{\gamma^{2p}}w_j^0(y) \\
& +\bigg(\frac{p-1}{p}\bigg)^3\frac{1}{\gamma^{3p}}w_j^1(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\theta(y)\bigg) \\
& \times e^{w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)}e^{(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)}e^{\theta(y)}e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)]^2},
\end{aligned}$$

and

$$\begin{aligned}
I_e = {} & \bigg(1+\frac{1}{p\gamma^p}\bigg(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\bigg(\frac{p-1}{p}\bigg)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\bigg)\bigg)^{3(p-1)} \\
& \times e^{\gamma^p[(1+\frac{1}{p\gamma^p}(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)))^p-1]} \\
= {} & \bigg(1+\frac{3(p-1)}{p}\frac{1}{\gamma^p}w_j(y)+3\bigg(\frac{p-1}{p}\bigg)^2\frac{1}{\gamma^{2p}}w_j^0(y)+3\bigg(\frac{p-1}{p}\bigg)^3\frac{1}{\gamma^{3p}}w_j^1(y) \\
& +\frac{3(p-1)}{p}\frac{1}{\gamma^p}\theta(y)\bigg) \\
& \times e^{w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)}e^{(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)}e^{\theta(y)}e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)]^2}.
\end{aligned}$$

Therefore, we get

$$I_c 1_{B(\xi_j',R_\varepsilon)} = \frac{O(1)}{|\log\varepsilon|}, \qquad I_d 1_{B(\xi_j',R_\varepsilon)} = \frac{O(1)}{|\log\varepsilon|^2}, \qquad I_e 1_{B(\xi_j',R_\varepsilon)} = O(1). \tag{2.41}$$

Finally, for $R_\varepsilon < |y-\xi_j'| < \frac{\delta}{\varepsilon}$, for any j, we have

$$|I_c| \le \frac{C}{|\log\varepsilon|}, \qquad |I_d| \le \frac{C}{|\log\varepsilon|^2}, \qquad |I_e| = O(1) + Ce^{w_j(y)}. \tag{2.42}$$

From (2.40), (2.41) with (2.42), by the definition of $*$-norm, we obtain (2.36) holds. □

3. The linearized problem

In this section, we prove the bounded invertibility of the operator L. We observe that the operator L can be approximately regarded as a superposition of the linear operator

$$L_j(\phi) = \Delta\phi + e^{w_j}\phi = \Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2}\phi.$$

The key fact to develop a satisfactory solvability theory for the operator L is the nondegeneracy of w up to the natural invariances of the equation under translations and dilations, which translates into the fact that

$$z_{0j}(y) = \partial_{\mu_j} w_{\mu_j}(y), \qquad z_{ij}(y) = \partial_{y_i} w_{\mu_j}(y), \quad i = 1, 2,$$

satisfy the function $\Delta Z + e^{w_j} Z = 0$, see [8] for a proof. Define for $i = 0, 1, 2$ and $j = 1, 2, \cdots, k$,

$$Z_{ij}(y) := z_{ij}\big(y - \xi_j'\big), \quad i = 0, 1, 2. \tag{3.1}$$

Consider a large but fixed number $R_0 > 0$ and a radial and smooth cut-off function η with $\eta(r) = 1$ if $r < R_0$ and $\eta(r) = 0$ if $r > R_0 + 1$. Write

$$\eta_j(y) = \eta\big(\big|y - \xi_j'\big|\big). \tag{3.2}$$

Given $h \in L^\infty(\Omega_\varepsilon)$, we consider the problem of finding a function ϕ such that for certain scalars c_{ij}, $i = 1, 2$, $j = 1, 2, \cdots, k$, it satisfies

$$\begin{cases} L(\phi) = h + \displaystyle\sum_{i=1}^{2}\sum_{j=1}^{k} c_{ij} Z_{ij}\eta_j, & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \displaystyle\int_{\Omega_\varepsilon} \phi Z_{ij}\eta_j = 0 & \text{for } i = 1, 2,\ j = 1, \cdots, k. \end{cases} \tag{3.3}$$

Consider the norm

$$\|\phi\|_\infty = \sup_{y\in\Omega_\varepsilon} \left|\phi(y)\right|.$$

The main result of this section is the following:

Proposition 3.1. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \cdots, k$, in Ω, satisfying (2.1), μ_j given by (2.22), and $h \in L^\infty(\Omega_\varepsilon)$, there is a unique solution $\phi := T_\lambda(h)$ to problem (3.3) for all $\lambda \le \lambda_0$. Moreover,*

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*. \tag{3.4}$$

The proof will be split into a series of lemmas which we state and prove next.

Lemma 3.1. *The operator L satisfies the maximum principle in $\tilde\Omega_\varepsilon = \Omega_\varepsilon \setminus \bigcup_{j=1}^k B(\xi_j', R)$ for R large. Namely, if $L(\phi) \le 0$ in $\tilde\Omega_\varepsilon$ and $\phi \ge 0$ on $\partial\tilde\Omega_\varepsilon$, then $\phi \ge 0$ in $\tilde\Omega_\varepsilon$.*

Proof. Given $a > 0$, we consider the function

$$Z(y) = \sum_{j=1}^k z_0\big(a\left|y - \xi_j'\right|\big), \quad y \in \Omega_\varepsilon, \tag{3.5}$$

where $z_0(r) = \frac{r^2-1}{r^2+1}$ is the radial solution in $\mathbb{R}^2$ of

$$\Delta z_0 + \frac{8}{(1+r^2)^2} z_0 = 0.$$

First, we observe that, if $|y - \xi_j'| \ge R$ for $R > \frac{1}{a}$, then $Z(y) > 0$. By the definition of z_0 we have

$$\begin{aligned} -\Delta Z(y) &= \sum_{j=1}^k \frac{8a^2(a^2|y-\xi_j'|^2 - 1)}{(1 + a^2|y-\xi_j'|^2)^3} \ge \sum_{j=1}^k \frac{1}{3}\frac{8a^2}{(1+a^2|y-\xi_j'|^2)^2} \\ &\ge \sum_{j=1}^k \frac{4}{27}\frac{8}{a^2|y-\xi_j'|^4} \end{aligned}$$

provided $R > \frac{\sqrt{2}}{a}$. On the other hand, in the same region, we have

$$g'(V_\lambda)Z(x) \le D_0 \sum_{j=1}^k e^{w_j} Z(y) \le D_0 \sum_{j=1}^k \frac{C}{|y-\xi_j'|^4},$$

for some constant $C > 0$ and D_0 satisfies (2.35). Hence if a is taken small and fixed, and $R > 0$ is chosen sufficiently large depending on this a, then we have $L(Z) < 0$ in $\tilde\Omega_\varepsilon$. The function $Z(y)$ is what we are looking for. □

Let us fix such a number $R > 0$ which we may take large whenever it is needed. Define the "inner norm" of ϕ in the following way

$$\|\phi\|_i = \sup_{y\in\bigcup_{j=1}^k B(\xi_j', R)} |\phi(y)|.$$

Lemma 3.2. *There exists a uniform constant $C > 0$ such that if $L(\phi) = h$ in Ω_ε, $\phi = 0$ on $\partial\Omega_\varepsilon$, then*

$$\|\phi\|_\infty \le C\big[\|\phi\|_i + \|h\|_*\big], \tag{3.6}$$

for any $h \in L^\infty(\Omega_\varepsilon)$.

Proof. We will establish this estimate with the use of suitable barriers. Let M be large, such that $\Omega_\varepsilon \subset B(\xi_j', \frac{M}{\varepsilon})$ for all j. Consider the solution ψ_j of the following problem

$$\begin{cases} -\Delta\psi_j = \dfrac{2}{|y-\xi_j'|^3} + 2\varepsilon^2, \quad R < |y-\xi_j'| < \dfrac{M}{\varepsilon}; \\ \psi_j(y) = 0 \quad \text{for } |y-\xi_j'| = R, \quad |y-\xi_j'| = \dfrac{M}{\varepsilon}. \end{cases}$$

We observe that by the direct computation we have that

$$\psi_j(r) = \frac{1}{R} - \frac{1}{r} - \varepsilon^2(r-R) - \left[\frac{1}{R} - \frac{1}{r} - \varepsilon^2\left(\frac{M}{\varepsilon} - R\right)\right]\frac{\log\frac{r}{R}}{\log\frac{M}{\varepsilon R}}.$$

Therefore, this function is uniform bound independent of ε as long as $a < R < \frac{1}{2\varepsilon}$.

Define now the function

$$\tilde\phi(y) = 2\|\phi\|_i Z(y) + \|h\|_* \sum_{j=1}^k \psi_j(y),$$

where Z is the function defined in (3.5). First, observe that by the definition of Z, choosing R large if necessary,

$$\tilde\phi(y) \ge 2\|\phi\|_i Z(y) \ge \|\phi\|_i \ge |\phi(y)| \quad \text{for } |y-\xi_j'| = R,\ j = 1,\cdots,k,$$

and, by the positivity of $Z(y)$ and $\psi_j(y)$,

$$\tilde\phi(y) \ge 0 = \phi(y) \quad \text{for } y \in \partial\Omega_\varepsilon.$$

Finally, by the definition of $\|\cdot\|_*$ we have that

$$|h(y)| \le \left(\sum_{j=1}^k \big(1 + |y-\xi_j'|\big)^{-3} + \varepsilon^2\right)\|h\|_*,$$

then

$$
\begin{aligned}
L(\tilde{\phi}) &= 2\|\phi\|_i L(Z) + \|h\|_* L\left(\sum_{j=1}^{k} \psi_j\right) \le \|h\|_* \sum_{j=1}^{k} \left(\Delta\psi_j + g'(V_\lambda)\psi_j\right) \\
&= \|h\|_* \sum_{j=1}^{k} \left(-\frac{2}{|y-\xi_j'|^3} - 2\varepsilon^2 + g'(V_\lambda)\psi_j\right) \\
&\le \|h\|_* \sum_{j=1}^{k} \left(-\frac{2}{|y-\xi_j'|^3} - 2\varepsilon^2 + \frac{2kD_0}{R} e^{w_j}\right) \\
&\le -\|h\|_* \left(\sum_{j=1}^{k} \left(1+\left|y-\xi_j'\right|\right)^{-3} + \varepsilon^2\right) \\
&\le -\left|h(y)\right| \le \left|L(\phi)(y)\right|,
\end{aligned}
$$

provided R large enough. Hence, from Lemma 3.1, we obtain that

$$
\left|\phi(y)\right| \le \tilde{\phi}(y) \quad \text{for } y \in \tilde{\Omega}_\varepsilon,
$$

and, since $Z(y) \le 1$ we get

$$
\|\phi\|_\infty \le C\left[\|\phi\|_i + \|h\|_*\right]. \qquad \square
$$

Next we prove uniform a priori estimates for the problem (3.3) when ϕ satisfies additionally orthogonality under dilations. Specifically, we consider the problem

$$
\begin{cases}
L(\phi) = h, & \text{in } \Omega_\varepsilon; \\
\phi = 0 & \text{on } \partial\Omega_\varepsilon; \\
\int_{\Omega_\varepsilon} \eta_j Z_{ij}\phi = 0 & \text{for } i = 0,1,2,\ j = 1,\cdots,k,
\end{cases}
\tag{3.7}
$$

and prove the following estimate.

Lemma 3.3. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1,\cdots,k$, in Ω, satisfying (2.1), μ_j given by (2.22), and $h \in L^\infty(\Omega_\varepsilon)$, and any solution ϕ to problem (3.7), one has*

$$
\|\phi\|_\infty \le C\|h\|_*. \tag{3.8}
$$

Proof. We carry out the proof of lemma by a contradiction. If the result were false, then there would exist a sequence $\lambda_n \to 0$, points $\xi_j^n \in \Omega$, $j = 1,\cdots,k$ in Ω, satisfying (2.1), function h_n with $\|h_n\|_* \to 0$ and ϕ_n with $\|\phi_n\|_\infty = 1$,

$$\begin{cases} L(\phi_n) = h_n & \text{in } \Omega_{\varepsilon_n}; \\ \phi_n = 0 & \text{on } \partial\Omega_{\varepsilon_n}; \\ \int\limits_{\Omega_\varepsilon} \eta_j Z_{ij}\phi_n = 0 & \text{for all } i = 0, 1, 2, \ j = 1, \cdots, k. \end{cases} \tag{3.9}$$

Then from Lemma 3.2, we see that $\|\phi_n\|_i$ stays away from zero. Up to a subsequence, for one of the indices, say j, we can assume that there exists $R > 0$ such that,

$$\sup_{|y-(\xi_j^n)'|<R} \left|\phi_n(y)\right| \geq \kappa > 0 \quad \text{for all } n.$$

Let us set $\hat{\phi}_n(z) = \phi_n((\xi_j^n)' + z)$. Elliptic estimate allows us to assume that $\hat{\phi}_n$ converges uniformly over compact subsets of $\mathbb{R}^2$ to a bounded, nonzero solution $\hat{\phi}$ of

$$\Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\phi = 0.$$

This implies that $\hat{\phi}$ is a linear combination of the functions z_{ij}, $i = 0, 1, 2$. But orthogonality conditions over $\hat{\phi}_n$ pass to the limit thanks to $\|\hat{\phi}_n\|_\infty \leq 1$. The dominated convergence theorem then yields that $\int_{\mathbb{R}^2} \eta(z) z_{ij}\hat{\phi} = 0$ for $i = 0, 1, 2$, thus a contradiction with $\liminf_{n\to\infty} \|\phi_n\|_i > 0$. $\square$

Now we establish a priori estimates for the problem (3.7) with the orthogonality condition $\int_{\Omega_\varepsilon} \eta_j Z_{0j}\phi = 0$ dropped. We consider the problem

$$\begin{cases} L(\phi) = h & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int\limits_{\Omega_\varepsilon} \eta_j Z_{ij}\phi = 0 & \text{for } i = 1, 2, \ j = 1, \cdots, k. \end{cases} \tag{3.10}$$

Lemma 3.4. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points $\xi_j \in \Omega$, $j = 1, \cdots, k$, satisfying (2.1), μ_j given by (2.22), and $h \in L^\infty(\Omega_\varepsilon)$, and any solution ϕ to problem (3.10), one has*

$$\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*, \tag{3.11}$$

for all $\lambda < \lambda_0$.

Proof. The proof is already contained in [14] but we reproduce it here for the sake of completeness. Let $R > R_0 + 1$ be a large and fixed number, and $\hat{z}_0$ be the solution of the problem

$$\begin{cases} \Delta \hat{z}_{0j} + \dfrac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2} \hat{z}_{0j} = 0, & \\ \hat{z}_{0j}(y) = z_{0j}(R) & \text{for } |y - \xi_j'| = R, \\ \hat{z}_{0j}(y) = 0 & \text{for } |y - \xi_j'| = \dfrac{\delta}{3\varepsilon}. \end{cases}$$

By computation, this function is explicitly given by

$$\hat{z}_{0j}(y) = z_{0j}(y)\left[1 - \frac{\int_R^r \frac{ds}{s z_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{s z_{0j}^2(s)}}\right], \quad r = |y - \xi_j'|.$$

Next we consider the radial smooth cut-off functions χ_1 and χ_2 withe the following properties:

$$0 \le \chi_1 \le 1, \qquad \chi_1 \equiv 1 \quad \text{in } B(0, R), \qquad \chi_1 \equiv 0 \quad \text{in } B(0, R+1)^c; \quad \text{and}$$
$$0 \le \chi_2 \le 1, \qquad \chi_2 \equiv 1 \quad \text{in } B\left(0, \frac{\delta}{4\varepsilon}\right), \qquad \chi_1 \equiv 0 \quad \text{in } B\left(0, \frac{\delta}{3\varepsilon}\right)^c,$$

and $|\chi_2'(r)| \le C\varepsilon$, $|\chi_2''(r)| \le C\varepsilon^2$. Then we set

$$\chi_{1j}(y) = \chi_1(|y - \xi_j'|), \qquad \chi_{2j}(y) = \chi_2(|y - \xi_j'|),$$

and define the test function

$$\tilde{z}_{0j} = \chi_{1j} Z_{0j} + (1 - \chi_{1j})\chi_{2j}\hat{z}_{0j}.$$

Letting ϕ be a solution to (3.10), we will modify ϕ so that the extra orthogonality conditions with respect to Z_{0j} are satisfied. We set

$$\tilde{\phi} = \phi + \sum_{j=1}^k d_j \tilde{z}_{0j}$$

with the number d_j is defined as

$$d_j = -\frac{\int_{\Omega_\varepsilon} \eta_j Z_{0j}\phi}{\int_{\Omega_\varepsilon} \eta_j |Z_{0j}|^2}.$$

Then

$$L(\tilde{\phi}) = h + \sum_{j=1}^k d_j L(\tilde{z}_{0j}), \tag{3.12}$$

and the orthogonality condition

$$\int_{\Omega_\varepsilon} \eta_j Z_{0i} \tilde{\phi} = 0, \quad \text{for all } i = 0, 1, 2,$$

holds. Then from the previous lemma we have the following estimate

$$\|\tilde{\phi}\|_\infty \le C\left[\|h\|_* + \sum_{j=1}^k |d_j| \left\| L(\tilde{z}_{0j}) \right\|_*\right]. \tag{3.13}$$

Next, we show that

$$\left\| L(\tilde{z}_{0j}) \right\|_* \le \frac{C}{\log \frac{1}{\varepsilon}}, \quad \text{and} \quad |d_j| \le C\left(\log \frac{1}{\varepsilon}\right)^2 \|h\|_*. \tag{3.14}$$

Indeed, we have

$$\begin{aligned} L(\tilde{z}_{0j}) &= 2\nabla \chi_{1j} \nabla(Z_{0j} - \hat{z}_{0j}) + \Delta \chi_{1j}(Z_{0j} - \hat{z}_{0j}) \\ &\quad + 2\nabla \chi_{2j} \nabla \hat{z}_{0j} + \Delta \chi_{2j} \hat{z}_{0j} + O(\varepsilon^4). \end{aligned}$$

We consider the following four regions

$$\Omega_1 = \left\{y : \left|y - \xi_j'\right| \le R\right\}, \qquad \Omega_2 = \left\{y : R < \left|y - \xi_j'\right| < R + 1\right\},$$
$$\Omega_3 = \left\{y : R + 1 \le \left|y - \xi_j'\right| \le \frac{\delta}{4\varepsilon}\right\}, \qquad \Omega_4 = \left\{y : \frac{\delta}{4\varepsilon} < \left|y - \xi_j'\right| < \frac{\delta}{3\varepsilon}\right\}.$$

First, we note that $L(\tilde{z}_0) = O(\varepsilon^4)$ for $y \in \Omega_1 \cup \Omega_3$. For $y \in \Omega_2$, we have

$$\hat{z}_{0j} - Z_{0j} = -z_{0j}(r) \frac{\int_R^r \frac{ds}{s z_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{s z_{0j}^2(s)}},$$

so that

$$|\hat{z}_{0j} - Z_{0j}| \le \frac{C}{\log \frac{1}{\varepsilon}}.$$

Similarly, in this region, we have

$$\left|\hat{z}_{0j}' - Z_{0j}'\right| \le \frac{C}{\log \frac{1}{\varepsilon}}.$$

On the other hand, for $y \in \Omega_4$, we have

$$\hat{z}_{0j}(r) \le \frac{C}{\log \frac{1}{\varepsilon}}, \quad \text{and} \quad \hat{z}_{0j}'(r) \le \frac{C\varepsilon}{\log \frac{1}{\varepsilon}}.$$

Therefore, from the definition of the $*$-norm, we get

$$\left\| L(\tilde{z}_{0j}) \right\|_* \leq \frac{C}{\log \frac{1}{\varepsilon}}, \tag{3.15}$$

where the number C depends in principle on the chosen large constant R.

Next we show that the other inequality of (3.14) holds. Testing Eq. (3.12) against $\tilde{z}_{0l}$ we have

$$\langle \tilde{\phi}, L(\tilde{z}_{0l}) \rangle = \langle h, \tilde{z}_{0l} \rangle + d_l \langle L(\tilde{z}_{0l}), \tilde{z}_{0l} \rangle,$$

where $\langle f, g \rangle = \int_{\Omega_\varepsilon} fg$. This relation and (3.13) give us that

$$d_l \langle L(\tilde{z}_{0l}), \tilde{z}_{0l} \rangle \leq C \|h\|_* \left[1 + \left\| L(\tilde{z}_{0l}) \right\|_* \right] + C \sum_{j=1}^{k} |d_j| \left\| L(\tilde{z}_{0l}) \right\|_*^2. \tag{3.16}$$

We want to measure the size of $\langle L(\tilde{z}_{0l}), \tilde{z}_{0l} \rangle$. We decompose

$$\langle L(\tilde{z}_{0l}), \tilde{z}_{0l} \rangle = \int_{\Omega_2} L(\tilde{z}_{0l}) \tilde{z}_{0l} + \int_{\Omega_4} L(\tilde{z}_{0l}) \tilde{z}_{0l} + O(\varepsilon). \tag{3.17}$$

Since

$$\left| \int_{\Omega_4} L(\tilde{z}_{0l}) \tilde{z}_{0l} \right| \leq C \int |\nabla \chi_{2l}| |\nabla \hat{z}_{0l}| |\hat{z}_{0l}| + C \int |\Delta \chi_{2l}| |\hat{z}_{0l}|^2 + O(\varepsilon^2) \leq \frac{C}{(\log \frac{1}{\varepsilon})^2}. \tag{3.18}$$

Moreover, for $y \in \Omega_2$, we have

$$\begin{aligned} \int_{\Omega_2} L(\tilde{z}_{0l}) \tilde{z}_{0l} &= 2 \int \nabla \chi_{1l} \nabla (Z_{0l} - \hat{z}_{0l}) \hat{z}_0 + \int \Delta \chi_{1l} (Z_{0l} - \hat{z}_{0l}) \hat{z}_{0l} + O(\varepsilon) \\ &= \int \nabla \chi_{1l} \nabla (Z_{0l} - \hat{z}_{0l}) \hat{z}_{0l} - \int \nabla \chi_{1l} (Z_{0l} - \hat{z}_{0l}) \nabla \hat{z}_{0l} + O(\varepsilon), \end{aligned}$$

from the integration by parts. Now, we observe that in the considered region Ω_2, $|\hat{z}_{0l} - Z_{0l}| \leq \frac{C}{\log \frac{1}{\varepsilon}}$, while $|\hat{z}'_{0l}| \sim \frac{1}{R^3} + \frac{1}{R} \frac{1}{\log \frac{1}{\varepsilon}}$. Then, for R large but independent of ε we have

$$\left| \int \nabla \chi_{1l} (Z_{0l} - \hat{z}_{0l}) \nabla \hat{z}_{0l} \right| \leq \frac{C_1}{R^3} \frac{1}{\log \frac{1}{\varepsilon}},$$

with C_1 being a constant to be chosen independent of R. Moreover

$$\int \nabla\chi_{1l}\nabla(Z_{0l}-\hat{z}_{0l})\hat{z}_{0l} = 2\pi \int_R^{R+1} \chi_{1l}'(z_{0l}-\hat{z}_{0l})'\hat{z}_{0l} r\,dr$$
$$= \frac{2\pi}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{s z_{0l}^2}} \int_R^{R+1} \chi_{1l}'\left[1 - \frac{4\mu_l^2 r^2 z_{0l}\int_R^r \frac{ds}{s z_{0l}^2}}{(\mu_l^2+r^2)^2}\right] dr$$
$$= -\frac{C_2}{\log\frac{1}{\varepsilon}}\left[1 + O\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)\right],$$

where C_2 is a positive constant independent of ε. Thus, choosing R large enough, we get

$$\int_{\Omega_2} L(\tilde{z}_{0l})\tilde{z}_{0l} \sim -\frac{C_2}{\log\frac{1}{\varepsilon}}.$$

Combining this and (3.17), (3.18) we get

$$\langle L(\tilde{z}_{0l}), \tilde{z}_{0l}\rangle \le -\frac{C_2}{\log\frac{1}{\varepsilon}}\left[1 + O\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)\right]. \tag{3.19}$$

From (3.15), (3.16) and (3.18) we have

$$|d_j| \le C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_*.$$

We thus have from estimate (3.13) that

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*. \qquad \square$$

Proof of Proposition 3.1. We first establish the validity of the a priori estimate (3.4). The previous lemma yields

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\left[\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{k}|c_{ij}|\right]. \tag{3.20}$$

Let us consider the cut-off function χ_{2j} defined in the previous lemma. Multiplying the first equation of (3.3) by $Z_{ij}\chi_{2j}$, we find

$$\langle L(\phi), Z_{ij}\chi_{2j}\rangle = \langle h, Z_{ij}\chi_{2j}\rangle + c_{ij}\int_{\Omega_\varepsilon} \eta_j |Z_{ij}|^2. \tag{3.21}$$

We have

$$L(Z_{ij}\chi_{2j}) = \Delta\chi_{2j} Z_{ij} + 2\nabla Z_{ij}\nabla\chi_{2j} + \varepsilon O\left((1+r)^{-3}\right),$$

with $r = |y - \xi'_j|$. Since $\Delta\chi_{2j} = O(\varepsilon^2)$, $\nabla\chi_{2j} = O(\varepsilon)$, and $Z_{ij} = O(r^{-1})$, $\nabla Z_{ij} = O(r^{-2})$, we get

$$L(Z_{ij}\chi_{2j}) = O\big(\varepsilon^3\big)\varepsilon O\big((1+r)^{-3}\big).$$

Then we have

$$\big|\langle L(\phi), Z_{ij}\chi_{2j}\rangle\big| = \big|\langle \phi, L(Z_{ij}\chi_{2j})\rangle\big| \le C\varepsilon\|\phi\|_\infty.$$

Combining this with (3.20) and (3.21) we find

$$|c_{ij}| \le C\bigg[\|h\|_* + \varepsilon \log\frac{1}{\varepsilon}\sum_{l,m}|c_{lm}|\bigg]. \tag{3.22}$$

Then,

$$|c_{ij}| \le C\|h\|_*.$$

Combining this with (3.20) we obtain the estimate (3.4) holds.

Next prove the solvability of problem (3.3). We consider the Hilbert space

$$\mathbb{H} = \bigg\{\phi \in H_0^1(\Omega_\varepsilon) : \int\limits_{\Omega_\varepsilon} \phi Z_{ij}\eta_j = 0 \text{ for } i = 1, 2,\ j = 1, 2, \cdots, k\bigg\},$$

endowed with the usual inner product $\langle\phi, \psi\rangle = \int_{\Omega_\varepsilon} \nabla\phi\nabla\psi$. Problem (3.3), expressed in a weak form, is equivalent to find $\phi \in \mathbb{H}$ such that

$$\langle\phi, \psi\rangle = \int\limits_{\Omega_\varepsilon} (W\phi - h)\psi\, dx, \quad \text{for all } \psi \in \mathbb{H},$$

where $W = g'(V_\lambda)$. With the aid of Riesz's representation theorem, this equation gets rewritten in $\mathbb{H}$ in the operator form

$$(Id - K)\phi = \tilde{h}, \tag{3.23}$$

for certain $\tilde{h} \in \mathbb{H}$, where K is a compact operator in $\mathbb{H}$. The homogeneous equation $\phi = K\phi$ in $\mathbb{H}$, which is equivalent to (3.3) with $h \equiv 0$, has only the trivial solution in view of the a priori estimate (3.4). Now, Fredholm's alternative guarantees unique solvability of (3.23) for any $\tilde{h} \in \mathbb{H}$. This finishes the proof. □

The result of Proposition 3.1 implies that the unique solution $\phi = T_\lambda(h)$ of (3.3) defines a continuous linear map form the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$ into L^∞, with norm bounded uniformly in λ.

Lemma 3.5. *The operator T_λ is differentiable with respect to the variables $\xi_1, \ldots, \xi_k$ in Ω satisfying (2.1); one has the estimate*

$$\left\| \partial_{(\xi'_m)_l} T_\lambda(h) \right\|_\infty \le C\left(\log \frac{1}{\varepsilon}\right)^2 \|h\|_* \quad \text{for } l = 1, 2,\ m = 1, 2, \cdots, k, \tag{3.24}$$

for a given positive C, independent of ε, and for all ε small enough.

Proof. Differentiating Eq. (3.3), formally $Z := \partial_{(\xi'_m)_l}\phi$ should satisfy

$$L(Z) = -\partial_{(\xi'_m)_l}\big(g'(V_\lambda)\big)\phi + \sum_{i=1}^{2} c_{im}\partial_{(\xi'_m)_l}(\eta_m Z_{im}) + \sum_{i=1}^{2}\sum_{j=1}^{k} d_{ij} Z_{ij}\eta_j$$

with $d_{ij} = \partial_{(\xi'_m)_l} c_{ij}$, and the orthogonality conditions now become

$$\int_{\Omega_\varepsilon} Z_{im}\eta_m Z = -\int_{\Omega_\varepsilon} \partial_{(\xi'_m)_l}(Z_{lm}\eta_m)\phi.$$

We consider the constants b_{im} defined as

$$b_{im}\int_{\Omega_\varepsilon} \eta_m Z_{im}^2 = \int_{\Omega_\varepsilon} \partial_{(\xi'_m)_l}(Z_{im}\eta_m)\phi, \quad \text{for } l = 1, 2.$$

Define

$$\tilde{Z} = Z + \sum_{i=1}^{2} b_{im}\eta_m Z_{im},$$

and

$$f = -\partial_{(\xi'_m)_l}\big(g'(V_\lambda)\big)\phi + \sum_{i=1}^{2} c_{im}\partial_{(\xi'_m)_l}(Z_{im}\eta_m) + \sum_{i=1}^{2} b_{im} L(\eta_m Z_{im}).$$

We then have

$$\begin{cases} L(\tilde{Z}) = f + \displaystyle\sum_{i=1}^{2}\sum_{j=1}^{k} b_{im}\eta_m Z_{im}, & \text{in } \Omega_\varepsilon; \\ \tilde{Z} = 0 & \text{on } \partial\Omega_\varepsilon; \\ \displaystyle\int_{\Omega_\varepsilon} \eta_m Z_{im}\tilde{Z} = 0 & \text{for } i = 0, 1, 2. \end{cases}$$

Namely, $\tilde{Z} = T_\lambda(f)$. Using the result of Proposition 3.1 we find that

$$\|f\|_* \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*,$$

hence,

$$\left\|\partial_{(\xi'_m)_l} T_\lambda(h)\right\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_* \quad \text{for } l = 1, 2,\ m = 1, 2, \cdots, k. \qquad \square$$

4. The nonlinear problem

In what follows we keep the notation introduced in the previous sections. We recall that our goal is to solve problem (3.3). The strategy is to solve first the following problem

$$\begin{cases} L(\phi) = -\left[E_\lambda + N(\phi)\right] + \sum_{i=1}^{2}\sum_{j=1}^{k} c_{ij}\eta_j Z_{ij}, & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int\limits_{\Omega_\varepsilon} \eta_j Z_{ij}\phi = 0 & \text{for all } i = 1, 2,\ \ j = 1, 2, \cdots, k. \end{cases} \tag{4.1}$$

We have the following result.

Lemma 4.1. *Under the assumptions of Proposition 3.1, there exist positive numbers C and λ_0, such that problem (4.1) has a unique solution ϕ which satisfies*

$$\|\phi\|_\infty \le \frac{C}{|\log\varepsilon|^2},$$

for all $\lambda < \lambda_0$. Moreover, if we consider the map $\xi' \mapsto \phi$ into the space $C(\bar{\Omega}_\varepsilon)$, the derivative $D_{\xi'}\phi$ exists and defines a continuous function of ξ'. Besides, there is a constant $C > 0$, such that

$$\|D_{\xi'}\phi\|_\infty \le \frac{C}{|\log\varepsilon|}. \tag{4.2}$$

Proof. In terms of the operator T_λ defined in Proposition 3.1, problem (4.1) becomes

$$\phi = T_\lambda\big(-\big(N(\phi) + E_\lambda\big)\big) := A(\phi). \tag{4.3}$$

For a given number $M > 0$, let us consider the region

$$\mathcal{F}_M := \left\{\phi \in C(\bar{\Omega}) : \|\phi\|_\infty \le \frac{M}{|\log\varepsilon|^2}\right\}.$$

From Proposition 3.1, we get

$$\left\|A(\phi)\right\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\left[\left\|N(\phi)\right\|_* + \|E_\lambda\|_*\right].$$

By the definition of $N(\phi)$ in (2.27), we can write

$$\big|N(\phi)\big| \le C\big|g''(V_\lambda + s\phi)\big||\phi|^2 \le C\big|g''(V_\lambda + s\phi)\big|\|\phi\|_\infty^2$$

for some $0 < s < 1$. Thus, using the fact that $\|\phi\|_\infty \to 0$ as $\lambda \to 0$, and (2.36), we obtain

$$\big\|N(\phi)\big\|_* \le C\|\phi\|_\infty^2$$

Thus

$$\big\|A(\phi)\big\|_\infty \le C|\log\varepsilon|\bigg(C\|\phi\|_\infty^2 + \frac{1}{|\log\varepsilon|^3}\bigg).$$

We then get that $A(\mathcal{F}_M) \subset \mathcal{F}_M$ for a sufficiently large but fixed M and all small λ. Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_M$, one has

$$\big\|N(\phi_1) - N(\phi_2)\big\|_* \le C\Big(\max_{i=1,2}\|\phi_i\|_\infty\Big)\|\phi_1 - \phi_2\|_\infty.$$

In fact,

$$\begin{aligned}
N(\phi_1) - N(\phi_2) &= g(V_\lambda + \phi_1) - g(V_\lambda + \phi_2) - g'(V_\lambda)(\phi_1 - \phi_2) \\
&= \int_0^1 \frac{d}{dt} g\big(V_\lambda + \phi_2 + t(\phi_1 - \phi_2)\big)dt - g'(V_\lambda)(\phi_1 - \phi_2) \\
&= \int_0^1 g'\big(V_\lambda + \phi_2 + t(\phi_1 - \phi_2) - g'(V_\lambda)\big)dt(\phi_1 - \phi_2).
\end{aligned}$$

Thus, for a certain $t^* \in (0, 1)$, and $s \in (0, 1)$

$$\begin{aligned}
\big|N(\phi_1) - N(\phi_2)\big| &\le C\big|g'\big(V_\lambda + \phi_2 + t^*(\phi_1 - \phi_2) - g'(V_\lambda)\big)\big|\|\phi_1 - \phi_2\|_\infty \\
&\le C\big|g''\big(V_\lambda + s\phi_2 + t^*(\phi_1 - \phi_2)\big)\big|\big(\|\phi_1\|_\infty + \|\phi_2\|_\infty\big)\|\phi_1 - \phi_2\|_\infty.
\end{aligned}$$

Thanks to (2.36) and the fact that $\|\phi_1\|_\infty, \|\phi_2\|_\infty \to 0$ as $\lambda \to 0$, we conclude that

$$\begin{aligned}
\big\|N(\phi_1) - N(\phi_2)\big\|_* &\le C\big\|g''(V_\lambda)\big\|_*\big(\|\phi_1\|_\infty + \|\phi_2\|_\infty\big)\|\phi_1 - \phi_2\|_\infty \\
&\le C\big(\|\phi_1\|_\infty + \|\phi_2\|_\infty\big)\|\phi_1 - \phi_2\|_\infty.
\end{aligned}$$

Then we have

$$\big\|A(\phi_1) - A(\phi_2)\big\|_\infty \le C|\log\varepsilon|\big\|N(\phi_1) - N(\phi_2)\big\|_* \le C|\log\varepsilon|\Big(\max_{i=1,2}\|\phi_i\|_\infty\Big)\|\phi_1 - \phi_2\|_\infty.$$

Thus the operator A has a small Lipschitz constant in $\mathcal{F}_M$ for all small λ, and therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $\xi' = (\xi_1', \cdots, \xi_k') \mapsto \phi$. Assume for instance that the partial derivative $\partial_{(\xi_j')_i}\phi$ exists for $i = 1, 2$. Since $\phi = T_\lambda(-(N(\phi) + E_\lambda))$, formally it follows that

$$\partial_{(\xi_j')_i}\phi = (\partial_{(\xi_j')_i}T_\lambda)\big(-\big(N(\phi)+E_\lambda\big)\big) + T_\lambda\big(-\big(\partial_{(\xi_j')_i}N(\phi)+\partial_{(\xi_j')_i}E_\lambda\big)\big).$$

From Lemma 3.5, we have

$$\big\|\partial_{(\xi_j')_i}T_\lambda\big(-\big(N(\phi)+E_\lambda\big)\big)\big\|_\infty \le C|\log\varepsilon|^2\big\|N(\phi)+E_\lambda\big\|_* \le C\frac{1}{|\log\varepsilon|}.$$

On the other hand,

$$\begin{aligned}\partial_{(\xi_j')_i}N(\phi) &= \big[g'(V_\lambda+\phi)-g'(V_\lambda)-g''(V_\lambda)\phi\big]\partial_{(\xi_j')_i}V_\lambda + \partial_{(\xi_j')_i}\big[g'(V_\lambda)-e^{w_j}\big]\phi \\ &\quad + \big[g'(V_\lambda+\phi)-g'(V_\lambda)\big]\partial_{(\xi_j')_i}\phi + \big[g'(V_\lambda)-e^{w_j}\big]\partial_{(\xi_j')_i}\phi.\end{aligned}$$

Then,

$$\big\|\partial_{(\xi_j')_i}N(\phi)\big\|_* \le C\bigg\{\|\phi\|_\infty^2 + \frac{1}{|\log\varepsilon|}\|\phi\|_\infty + \|\partial_{(\xi_j')_i}\phi\|_\infty\|\phi\|_\infty + \frac{1}{|\log\varepsilon|}\|\partial_{(\xi_j')_i}\phi\|_\infty\bigg\}.$$

Since $\|\partial_{(\xi_j)_i}E_\lambda\|_* \le \frac{C}{|\log\varepsilon|^3}$, and by Proposition 3.1 we then have

$$\|\partial_{(\xi_j')_i}\phi\|_\infty \le \frac{C}{|\log\varepsilon|},$$

for all $i = 1, 2$, $j = 1, \cdots, k$. Then, the regularity of the map $\xi' \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (4.3). This concludes the proof of the lemma. □

5. Variational reduction

We have solved the nonlinear problem (4.1). In order to find a solution to the original problem we need to find ξ' such that

$$c_{ij}\big(\xi'\big) = 0 \quad \text{for all } i = 1, 2,\ j = 1, \cdots, k. \tag{5.1}$$

This problem is variational: indeed it is equivalent to finding critical points of a function of $\xi = \varepsilon\xi'$. Associated to (1.1), let us consider the energy functional J_λ given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{p}\int_\Omega e^{u^p}\,dx, \quad u \in H_0^1(\Omega), \tag{5.2}$$

and the finite-dimensional restriction

$$F_\lambda(\xi) = J_\lambda\big((U_\lambda + \tilde{\phi})(x,\xi)\big), \tag{5.3}$$

where

$$(U_\lambda + \tilde{\phi})(x,\xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right) \tag{5.4}$$

with V_λ defined in (2.17), ϕ is the unique solution to problem (4.1) given by Lemma 4.1. Critical points of F_λ correspond to solutions of (5.1) for a small λ, as the following result states.

Lemma 5.1. *Under the assumptions of Proposition 3.1, the functional $F_\lambda(\xi)$ is of class C^1. Moreover, for all $\lambda > 0$ sufficiently small, if $D_\xi F_\lambda(\xi) = 0$, then ξ satisfies (5.1).*

Proof. A direct consequence of the results obtained in Lemma 4.1 and the definition of function U_λ is the fact that the map $\xi \mapsto F_\lambda(\xi)$ is of class C^1. Define

$$I_\lambda(v) = \frac{1}{2}\int\limits_{\Omega_\varepsilon} |\nabla v|^2\, dy - \int\limits_{\Omega_\varepsilon} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}\, dy. \tag{5.5}$$

Let us differentiate the function $F_\lambda(\xi)$ with respect to ξ. Since

$$J_\lambda\big((U_\lambda + \tilde{\phi})(x,\xi)\big) = \frac{1}{p^2\gamma^{2(p-1)}} I_\lambda\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right), \tag{5.6}$$

we can differentiate directly $I_\lambda(V_\lambda(\xi) + \phi(\xi))$ under the integral sign. Let $m \in \{1, \dots, k\}$ and $l \in 1, 2$. We have

$$\begin{aligned}
\partial_{\xi_{m,l}} F_\lambda(\xi) &= \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1} DI_\lambda\big(V_\lambda(\xi) + \phi(\xi)\big)\big[\partial_{(\xi'_m)_l} V_\lambda(\xi) + \partial_{(\xi'_m)_l}\phi(\xi)\big] \\
&= \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1} \sum_{i=1}^{2}\sum_{j=1}^{k} \int\limits_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij}\big[\partial_{(\xi'_m)_l} V_\lambda(\xi) + \partial_{(\xi'_m)_l}\phi(\xi)\big] \\
&= \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\left[\sum_{i=1}^{2}\sum_{j=1}^{k} \int\limits_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij}\partial_{(\xi'_m)_l} V_\lambda(\xi) + \sum_{i=1}^{2}\sum_{j=1}^{k} \int\limits_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij}\partial_{(\xi'_m)_l}\phi(\xi)\right]
\end{aligned}$$

By the expansion of V_λ, we have

$$\begin{aligned}
\partial_{(\xi'_m)_l} V_\lambda &= \partial_{(\xi'_m)_l}\Bigg(\sum_{m=1}^{k}\bigg(PU_{\mu_m,\xi_m}(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p} Pw^0_{\mu_m,\xi_m}(\varepsilon y) \\
&\quad + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} Pw^1_{\mu_m,\xi_m}(\varepsilon y)\bigg) - p\gamma^p\Bigg) \\
&= \partial_{(\xi'_m)_l}\left(w_m(y) + \frac{p-1}{p}\frac{1}{\gamma^p} w^0_m(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} w^1_m(y) + \theta(y)\right)
\end{aligned}$$

$$= \partial_{(\xi'_m)_l} w_m(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\partial_{(\xi'_m)_l} w_m^0(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\partial_{(\xi'_m)_l} w_m^1(y) + \partial_{(\xi'_m)_l}\theta(y)$$

$$= -Z_{lm} + \frac{p-1}{p}\frac{1}{\gamma^p}\partial_{(\xi'_m)_l} w_m^0(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\partial_{(\xi'_m)_l} w_m^1(y) + \partial_{(\xi'_m)_l}\theta(y).$$

Hence, for $j \neq m$, we have

$$\int_{\Omega_\varepsilon} \eta_j Z_{ij} \partial_{(\xi'_m)_l} V_\lambda(\xi) = \left(-\int_{B(\xi'_j,R)} \eta_j Z_{ij} Z_{lm}\right)\left(1 + O\left(\frac{1}{\gamma^p}\right)\right) = O(\varepsilon),$$

while for $j = m$ and $i \neq l$, by symmetry we get

$$\int_{\Omega_\varepsilon} \eta_j Z_{ij} \partial_{(\xi'_m)_l} V_\lambda(\xi) = \left(-\int_{B(\xi'_j,R)} \eta_j Z_{ij}\left(Z_{lm} + \frac{p-1}{p}\frac{1}{\gamma^p}\partial_{(\xi'_m)_l} w_m^0(y)\right)\right)$$
$$\times\left(1 + O\left(\frac{1}{\gamma^{2p}}\right)\right) = O\left(\frac{1}{\gamma^p}\right).$$

If now $j = m$ and $i = l$, we get

$$\int_{\Omega_\varepsilon} \eta_j Z_{ij} \partial_{(\xi'_m)_l} V_\lambda(\xi) = \left(-\int_{B(\xi'_m,R)} \eta_m Z_{lm} Z_{lm}\right)\left(1 + O\left(\frac{1}{\gamma^p}\right)\right).$$

We thus conclude that

$$\sum_{i=1}^{2}\sum_{j=1}^{k}\int_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij} \partial_{(\xi'_m)_l} V_\lambda(\xi) = -c_{lm}\int_{B(\xi'_m,R)} \eta_m Z_{lm} Z_{lm} + O\left(\frac{1}{\gamma^p}\right).$$

On the other hand, given (4.2), we have that

$$\left|\sum_{i=1}^{2}\sum_{j=1}^{k}\int_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij} \partial_{(\xi'_m)_l}\phi(\xi)\right| \le C\sum_{i,j}|c_{ij}|\|\partial_{(\xi'_m)_l}\phi\|_\infty \le o(1)\sum_{i,j}|c_{ij}|.$$

Thus, if $D_\xi F_\lambda(\xi) = 0$, for $i, l = 1, 2$, $j = 1, 2, \cdots, k$, we then have

$$c_{lm}\left(\int_{\Omega_\varepsilon} \eta_m Z_{lm} Z_{lm}\right)\big(1 + o(1)\big) = 0, \quad m = 1, \ldots, k,\ l = 1, 2. \tag{5.7}$$

This concludes the proof of the lemma. □

Next we give an asymptotic estimate of $F_\lambda(\xi)$ defined in (5.3). We have the following result.

Lemma 5.2. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that μ_j are given by (2.17), the following expansion holds*

$$\lambda^{-1}\varepsilon^{\frac{2(2-p)}{p}}F_\lambda(\xi)=\frac{8k\pi}{(2-p)p}[-2+p\log 8]-\frac{16k\pi}{p}\log\varepsilon-\frac{4\pi}{2-p}\varphi_k(\xi)+|\log\varepsilon|^{-1}\theta_\lambda(\xi) \tag{5.8}$$

uniformly for any points ξ_j, $j=1,\cdots,k$ in Ω, satisfying (2.1), where

$$\varphi_k(\xi)=\varphi_k(\xi_1,\cdots,\xi_k)=\sum_{j=1}^{k}H(\xi_j,\xi_j)+\sum_{i\neq j}G(\xi_i,\xi_j). \tag{5.9}$$

Furthermore

$$\lambda^{-1}\varepsilon^{\frac{2(2-p)}{p}}\nabla_{(\xi_m)_l}F_\lambda(\xi)=-\frac{4\pi}{(2-p)p}\nabla_{(\xi_m)_l}\varphi_k(\xi)+|\log\varepsilon|^{-1}\theta_\lambda(\xi). \tag{5.10}$$

In (5.8) and (5.10), the function θ_λ denotes a smooth function of the points ξ, which is uniformly bounded, as $\lambda\to 0$, for points ξ satisfying (2.1).

Proof. We have

$$\begin{aligned}F_\lambda(\xi)&=J_\lambda\big(U_\lambda(\xi)+\tilde\phi(\xi)\big)\\&=\frac12\int_\Omega\big|\nabla\big(U_\lambda(\xi)+\tilde\phi(\xi)\big)\big|^2\,dx-\frac{\lambda}{p}\int_\Omega e^{(U_\lambda(\xi)+\tilde\phi(\xi))^p}\,dx.\end{aligned} \tag{5.11}$$

Using the change of variables (5.4), namely $(U_\lambda+\tilde\phi)(x,\xi)=\gamma+\frac{1}{p\gamma^{p-1}}((V_\lambda+\phi)(\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}))$, together with (5.5) and (5.6), we have that

$$J_\lambda\big(U_\lambda(\xi)+\tilde\phi(\xi)\big)-J_\lambda\big(U_\lambda(\xi)\big)=\frac{1}{p^2\gamma^{2(p-1)}}\big[I_\lambda(V_\lambda+\phi)-I_\lambda(V_\lambda)\big]$$

Since by construction $I_\lambda'(V_\lambda+\phi)[\phi]=0$, we have

$$\begin{aligned}&J_\lambda\big(U_\lambda(\xi)+\tilde\phi(\xi)\big)-J_\lambda\big(U_\lambda(\xi)\big)\\&=\frac{1}{p^2\gamma^{2(p-1)}}\int_0^1 D^2I_\lambda(V_\lambda+t\phi)\phi^2(1-t)\,dt\\&=\frac{1}{p^2\gamma^{2(p-1)}}\int_0^1\Big[\int_{\Omega_\varepsilon}\big(E_\lambda+N(\phi)\big)\phi+\int_{\Omega_\varepsilon}\big[g_\lambda'(V_\lambda)-g_\lambda'(V_\lambda+t\phi)\big]\phi^2\Big](1-t)\,dt\end{aligned}$$

Since $\|E_\lambda\|_*\le\frac{c}{|\log\varepsilon|^3}$, $\|\phi\|_\infty\le\frac{c}{|\log\varepsilon|^2}$, $\|N(\phi)\|_*\le\frac{c}{|\log\varepsilon|^4}$ and (2.36), we get that

$$\left|J_\lambda\big(U_\lambda(\xi)+\tilde{\phi}(\xi)\big)-J_\lambda\big(U_\lambda(\xi)\big)\right|\le\frac{C}{\gamma^{2(p-1)}|\log\varepsilon|^3}\tag{5.12}$$

Next we expand

$$J_\lambda\big(U_\lambda(\xi)\big)=\frac{1}{2}\int\limits_\Omega\left|\nabla\big(U_\lambda(\xi)\big)\right|^2dx-\frac{\lambda}{p}\int\limits_\Omega e^{(U_\lambda(\xi))^p}\,dx.\tag{5.13}$$

First we expand the term $\int_\Omega|\nabla U_\lambda|^2$. By (2.23) we have

$$\begin{aligned}\frac{1}{2}\int\limits_\Omega\left|\nabla\big(U_\lambda(\xi)\big)\right|^2=\frac{1}{2}\frac{1}{p^2\gamma^{2(p-1)}}\Bigg\{&\sum_{j=1}^k\int\limits_\Omega|\nabla PU_{\mu_j,\xi_j}|^2+\sum_{l\neq j}\int\limits_\Omega\nabla PU_{\mu_l,\xi_l}\nabla PU_{\mu_j,\xi_j}\\&+\frac{p-1}{p}\frac{1}{\gamma^p}\sum_{j=1}^k\int\limits_\Omega\nabla PU_{\mu_j,\xi_j}(x)\nabla Pw^0_{\mu_j,\xi_j}(x)\\&+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\sum_{j=1}^k\int\limits_\Omega\nabla PU_{\mu_j,\xi_j}\nabla Pw^1_{\mu_j,\xi_j}\\&+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[\sum_{j=1}^k\int\limits_\Omega\left|\nabla Pw^0_{\mu_j,\xi_j}\right|^2+\sum_{l\neq j}\int\limits_\Omega\nabla Pw^0_{\mu_l,\xi_l}\nabla Pw^0_{\mu_j,\xi_j}\right]\\&+\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\sum_{j=1}^k\int\limits_\Omega\nabla Pw^0_{\mu_j,\xi_j}\nabla Pw^1_{\mu_j,\xi_j}\\&+\left(\frac{p-1}{p}\right)^4\frac{1}{\gamma^{4p}}\left[\sum_{j=1}^k\int\limits_\Omega\left|\nabla w^1_{\mu_j,\xi_j}\right|^2+\sum_{l\neq j}\int\limits_\Omega\nabla Pw^1_{\mu_l,\xi_l}\nabla Pw^1_{\mu_j,\xi_j}\right]\Bigg\}.\end{aligned}\tag{5.14}$$

Let us estimate the first two terms. We observe that the remaining terms are $O(\frac{1}{\gamma^{2(p-1)}\gamma^p})$. First, we note that PU_{μ_j,ξ_j} satisfies

$$-\Delta PU_{\mu_j,\xi_j}=\varepsilon^2e^{U_{\mu_j,\xi_j}},\quad\text{in }\Omega,\qquad PU_{\mu_j,\xi_j}=0\quad\text{on }\partial\Omega.$$

Then we have

$$\begin{aligned}&\int\limits_\Omega\left|\nabla PU_{\mu_j,\xi_j}(x)\right|^2\\&\quad=\varepsilon^2\int\limits_\Omega e^{U_{\mu_j,\xi_j}}PU_{\mu_j,\xi_j}(x)\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon^2 \int_\Omega e^{U_{\mu_j,\xi_j}} \big(U_{\mu_j,\xi_j}(x) + H(x,\xi_j) - \log\big(8\mu_j^2\big) + O\big(\mu_j^2\varepsilon^2\big)\big) \\
&= \int_\Omega \frac{8\varepsilon^2\mu_j^2}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2} \bigg(\log \frac{1}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2} + H(x,\xi_j) + O\big(\mu_j^2\varepsilon^2\big)\bigg) \\
&= \int_\Omega \frac{8\varepsilon^{-2}\mu_j^{-2}}{(1+|\frac{x-\xi_j}{\varepsilon\mu_j}|^2)^2} \bigg(\log \frac{\varepsilon^{-4}\mu_j^{-4}}{(1+|\frac{x-\xi_j}{\varepsilon\mu_j}|^2)^2} + H(x,\xi_j) + O\big(\mu_j^2\varepsilon^2\big)\bigg) \\
&= \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} \bigg(\log \frac{1}{(1+|z|^2)^2} + H(\xi_j + \varepsilon\mu_j z, \xi_j) - 4\log(\varepsilon\mu_j)\bigg) + O\big(\mu_j^2\varepsilon^2\big) \\
&= \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} \log \frac{1}{(1+|z|^2)^2} + \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} \big(H(\xi_j + \varepsilon\mu_j z, \xi_j) - H(\xi_j,\xi_j)\big) \\
&\quad + \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} H(\xi_j,\xi_j) - 4\log(\varepsilon\mu_j) \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} + O\big(\mu_j^2\varepsilon^2\big).
\end{aligned} \tag{5.15}
$$

But

$$
\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} = 8\pi + O(\varepsilon), \tag{5.16}
$$

and

$$
\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} \log \frac{1}{(1+|y|^2)^2} = -16\pi + O(\varepsilon). \tag{5.17}
$$

Moreover,

$$
\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} \big(H(\xi_j + \varepsilon\mu_j y, \xi_j) - H(\xi_j,\xi_j)\big) = \int_{\Omega_{\varepsilon\mu_j}} \frac{1}{(1+|y|^2)^2} O\big(\varepsilon^\alpha |y|^\alpha\big) = O(\varepsilon). \tag{5.18}
$$

Therefore from (5.15)–(5.18), we have

$$
\begin{aligned}
&\int_\Omega \big|\nabla P U_{\mu_j,\xi_j}(x)\big|^2 dx \\
&\quad = -16\pi + 8\pi H(\xi_j,\xi_j) - 32\pi \log\varepsilon - 16\pi \log\big(8\mu_j^2\big) + 16\pi \log(8) + O\bigg(\frac{1}{\gamma^p}\bigg).
\end{aligned} \tag{5.19}
$$

Now, we calculate that

$$\sum_{l\neq j}\int_{\Omega}\nabla PU_{\mu_l,\xi_l}\nabla PU_{\mu_j,\xi_j}\,dx$$
$$=\sum_{l\neq j}\int_{\Omega}\varepsilon^2 e^{U_{\mu_l,\xi_l}}PU_{\mu_j,\xi_j}$$
$$=\sum_{l\neq j}\int_{\Omega}\frac{8\varepsilon^2\mu_l^2}{(\varepsilon^2\mu_l^2+|x-\xi_l|^2)^2}\biggl(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}+H(x,\xi_j)$$
$$-\log\bigl(8\mu_j^2\bigr)+O\bigl(\mu_j^2\varepsilon^2\bigr)\biggr)$$
$$=\sum_{l\neq j}\int_{\Omega}\frac{8\varepsilon^2\mu_l^2}{(\varepsilon^2\mu_l^2+|x-\xi_l|^2)^2}\biggl(\log\frac{1}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}+H(x,\xi_j)+O\bigl(\mu_j^2\varepsilon^2\bigr)\biggr)$$
$$=\sum_{l\neq j}\int_{\Omega_{\varepsilon\mu_l}}\frac{8}{(1+|z|^2)^2}\biggl(\log\frac{1}{(\varepsilon^2\mu_j^2+|\varepsilon\mu_l z+\xi_l-\xi_j|^2)^2}+H(\xi_l+\varepsilon\mu_l z,\xi_j)\biggr)+O\bigl(\mu_j^2\varepsilon^2\bigr)$$
$$=\sum_{l\neq j}\int_{\Omega_{\varepsilon\mu_l}}\frac{8}{(1+|z|^2)^2}G(\xi_l,\xi_j)+O\bigl(\mu_j^2\varepsilon^2\bigr)$$
$$=8\pi\sum_{l\neq j}G(\xi_l,\xi_j)+O\bigl(\mu_j^2\varepsilon^2\bigr). \tag{5.20}$$

Thus, from (5.14), (5.19), (5.20) and (2.22) we have

$$\frac{1}{2}\int_{\Omega}\bigl|\nabla U_\lambda(x)\bigr|^2\,dx=\frac{1}{p^2\gamma^{2(p-1)}}\biggl\{-8k\pi-16k\pi\log\varepsilon+8k\pi\log(8)-8k\pi\frac{2(p-1)}{2-p}(1-\log 8)$$
$$-\frac{4p\pi}{2-p}\biggl(\sum_{j=1}^{k}H(\xi_j,\xi_j)+\sum_{i\neq j}^{k}G(\xi_i,\xi_j)\biggr)+O\biggl(\frac{1}{|\log\varepsilon|}\biggr)\biggr\}. \tag{5.21}$$

Finally, let us evaluate the second term in the energy

$$\frac{\lambda}{p}\int_{\Omega}e^{(U_\lambda)^p}\,dx=\frac{\lambda}{p}\int_{\Omega}e^{\gamma^p(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon}))^p}\,dx$$
$$=\frac{\lambda}{p}\sum_{j=1}^{k}\int_{B(\xi_j,\tilde\delta)}e^{\gamma^p(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon}))^p}\,dx$$
$$+\frac{\lambda}{p}\int_{\Omega\setminus\bigcup_{j=1}^{k}B(\xi_j,\tilde\delta)}e^{\gamma^p(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon}))^p}\,dx$$
$$:=I+II. \tag{5.22}$$

First we observe that

$$II = \lambda \Theta_\lambda(\xi) \tag{5.23}$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda \to 0$. On the other hand,

$$\begin{aligned}
I &= \frac{1}{p^2\gamma^{2(p-1)}} \sum_{j=1}^{k} \int\limits_{B(\xi_j', \tilde\delta/\varepsilon)} e^{\gamma^p[(1+\frac{1}{p\gamma^p}(V_\lambda)(y))^p-1]}\, dy \\
&= \frac{1}{p^2\gamma^{2(p-1)}} \sum_{j=1}^{k} \int\limits_{B(\xi_j', \tilde\delta/\varepsilon)} e^{\{w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\}}\left(1+O\left(\frac{1}{\gamma^p}\right)\right) dy \\
&= \frac{1}{p^2\gamma^{2(p-1)}} \sum_{j=1}^{k} \int\limits_{B(0, \frac{\tilde\delta}{\mu_j\varepsilon})} \frac{8}{(1+|y|^2)^2}\left(1+O\left(\frac{1}{\gamma^p}\right)\right) dy \\
&= \frac{1}{p^2\gamma^{2(p-1)}} 8k\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right),
\end{aligned} \tag{5.24}$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda \to 0$. From (5.22)–(5.24) we get

$$\frac{\lambda}{p}\int\limits_{\Omega} e^{(U_\lambda)^p}\, dx = \frac{1}{p^2\gamma^{2(p-1)}} 8k\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right). \tag{5.25}$$

Therefore, from (5.11), (5.12), (5.13), (5.21), (5.25) and (1.14) and by the choice of the parameters μ_j in (2.22), and (1.14), we can write the whole asymptotic expansion of $F_\lambda(\xi)$, namely (5.8) holds.

Let us now prove the validity of (5.10). Fix $m \in \{1, \dots, k\}$ and $l \in \{1, 2\}$. Arguing as in the proof of Lemma 5.1, we have

$$\partial_{(\xi_m)_l} F_\lambda(\xi) = \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\left[\sum_{i=1}^{2}\sum_{j=1}^{k} c_{ij} \int\limits_{\Omega_\varepsilon} \eta_j Z_{ij}\partial_{(\xi_m')_l} V_\lambda\right]\left(1+O\left(\frac{1}{\gamma^p}\right)\right). \tag{5.26}$$

On the one hand, if we multiply equation in (4.1) against $\partial_{(\xi_m')_l} V_\lambda$, we get

$$\int\limits_{\Omega_\varepsilon} \left(\Delta\upsilon_\xi + g(\upsilon_\xi)\right)\partial_{(\xi_m')_l} V_\lambda = \sum_{i=1}^{2}\sum_{j=1}^{k} c_{ij} \int\limits_{\Omega_\varepsilon} \eta_j Z_{ij}\partial_{(\xi_m')_l} V_\lambda$$

where $\upsilon_\xi = (V_\lambda + \phi)(y, \xi') = (V_\lambda + \phi)(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon})$. On the other hand, we have that

$$\partial_{(\xi_m)_l} U_\lambda(x) = \frac{\varepsilon^{-1}}{p\gamma^{p-1}}\partial_{(\xi_m')_l} V_\lambda\left(\frac{x}{\varepsilon}\right).$$

Putting together these informations, we have that

$$\partial_{(\xi_m)_l} F_\lambda(\xi) = \left(\int_\Omega \left[\Delta(U_\lambda + \tilde{\phi}) + \lambda (U_\lambda + \tilde{\phi})^{p-1} e^{(U_\lambda + \tilde{\phi})^p} \right] \partial_{(\xi_m)_l} U_\lambda \right) \big(1 + o(1)\big).$$

Furthermore, since $\|\tilde{\phi}\|_\infty \le \frac{C}{\gamma^{p-1}} \|\phi\|_\infty$, by definition of U_λ we have that

$$(U + \tilde{\phi})(x) = U_\lambda(x) \left(1 + O\left(\frac{1}{\gamma^p} \right) \right) \quad \text{in } \Omega.$$

Hence, by means of integrations by parts, and the boundary conditions satisfied by U_λ, we get that

$$\partial_{(\xi_m)_l} F_\lambda(\xi) = \left(\int_\Omega \left[\Delta U_\lambda + \lambda U_\lambda^{p-1} e^{U_\lambda^p} \right] \partial_{(\xi_m)_l} U_\lambda \right) \left(1 + O\left(\frac{1}{\gamma^p} \right) \right),$$

where $O(1)$ here denotes a smooth function of the points ξ, which is uniformly bounded as $\lambda \to 0$. We thus conclude that

$$\begin{aligned} \partial_{(\xi_m)_l} F_\lambda(\xi) &= \left(\int_\Omega \left[-\nabla U_\lambda \nabla \partial_{(\xi_m)_l} U_\lambda + \lambda U_\lambda^{p-1} e^{U_\lambda^p} \partial_{(\xi_m)_l} U_\lambda \right] \right) \left(1 + O\left(\frac{1}{\gamma^p} \right) \right) \\ &= -\partial_{(\xi_m)_l} J_\lambda(U_\lambda) \left(1 + O\left(\frac{1}{\gamma^p} \right) \right). \end{aligned}$$

Computations analogous to the ones we performed to get expansion (5.8) give us the validity of (5.10). This concludes the proof of the lemma. □

6. Proof of the main results

In this section, we will prove the main result.

Proof of Theorem 1.2. From Lemma 5.1, the function

$$U_\lambda(\xi) + \tilde{\phi}(\xi) = \frac{1}{p\gamma^{p-1}} \left(p\gamma^p + (V_\lambda + \phi)\left(\frac{x}{\varepsilon} \right) \right)$$

where V_λ defined by (2.17) and $\phi(\xi)$ is the unique solution of problem (4.1), is a solution of problem (1.1) if we adjust ξ so that it is a critical point of $F_\lambda(\xi)$ defined by (5.3). This is equivalent to finding a critical point of

$$\tilde{F}_\lambda(\xi) := A\lambda^{-1} \varepsilon^{\frac{2(2-p)}{p}} F_\lambda(\xi) + B + C \log \varepsilon,$$

for suitable constants A, B and C. On the other hand, from Lemma 5.2, for $\xi \in \mathcal{M}$, we have that,

$$\tilde{F}_\lambda(\xi) = \varphi_k(\xi) + O\big(|\log \varepsilon|^{-1}\big) \Theta_\lambda(\xi),$$

where φ_k is given by (1.5), and $\Theta_\lambda(\xi)$ is uniformly bounded in the considered region as $\lambda \to 0$.

Let us observe that if $M > \mathcal{C}$, then assumptions (1.18), (1.19) still hold for the function $\min\{M, \varphi_k(\xi)\}$ as well as for $\min\{M, \varphi_k(\xi) + O(|\log\varepsilon|^{-1})\Theta_\lambda(\xi)\}$. It follows that the function $\min\{M, \tilde{F}(\xi)\}$ satisfies for all λ small assumptions (1.18), (1.19) in $\mathcal{D}$ and therefore has a critical value $\mathcal{C}_\lambda < M$ which is close to the value $\mathcal{C}$ in this region. If $\xi_\lambda \in \mathcal{D}$ is a critical point at this level for $\tilde{F}_\lambda(\xi) + \beta$, then since

$$\tilde{F}_\lambda(\xi_\lambda) \le \mathcal{C}_\lambda < M$$

we have that there exists a $\delta > 0$ such that $|\xi_{\lambda,j} - \xi_{\lambda,i}| > \delta$, $\operatorname{dist}(\xi_{\lambda,j}, \partial\Omega) > 0$. This implies C^1-closeness of $\tilde{F}_\lambda(\xi)$ and $\varphi_k(\xi)$ at this level, hence $\nabla\varphi_k(\xi_\lambda) \to 0$. The function $u_\lambda = U(\xi_\lambda) + \tilde{\phi}(\xi_\lambda)$ is therefore a solution as predicted by the theorem.

Expansion (1.20) follows from (1.14) and (5.25), while (1.21) holds as a direct consequence of the construction of U_λ. Expansion (1.22) is a consequence of (5.8). $\square$

Proof of Theorem 1.1. According to the result of Theorem 1.2, the proof of Theorem 1.2 reduces to show that, for any $k \ge 1$ the function φ_k has a non-trivial critical value in some open set $\mathcal{D}$, compactly contained in Ω^k. This fact has already been established in [14] for the function $(-\varphi_k)$ in the context of construction of solutions to the Liouville problem

$$\Delta u + \varepsilon^2 e^u = 0, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega$$

for a not simply connected domain Ω in $\mathbb{R}^2$. For completeness, we recall here the principal ingredients employed in the proof of the existence of a non-trivial critical value for $(-\varphi_k)$ and we refer the reader to [14] for a complete proof of each step.

Let $\mathcal{D}$ be given by

$$\mathcal{D} = \left\{x \in \Omega^k : \operatorname{dist}\left(x, \partial\Omega^k\right) > \delta\right\}$$

for some positive and small δ to be chosen. Let Ω_1 be a bounded non-empty component of $\mathbb{R}^2 \setminus \bar{\Omega}$ and let γ be a closed, smooth Jordan curve contained in Ω which encloses Ω_1. Let S be the image of γ, $B_0 = \emptyset$ and $B = S^k$. Define

$$\mathcal{C} = \inf_{\Phi\in\Gamma} \sup_{z\in B} (-\varphi_k)\bigl(\Phi(z)\bigr) \tag{6.1}$$

where

$$\Gamma = \left\{\Phi(z) = \Psi(1, z) : \Psi : [0,1] \times B \to \mathcal{D} \text{ continuous and } \Psi(0, z) = z\right\}.$$

Observe that, since $\sum_j H(\xi_j, \xi_j)$ is bounded in $\mathcal{D}$ and $\sum_{i\neq j} G(\xi_i, \xi_j)$ is bounded below, the function $(-\varphi_k)$ is bounded above in $\mathcal{D}$.

With an argument based on degree theory, in Lemma 7.1 in [14], it is proven that:

There exists $K > 0$, independent of δ in the definition of $\mathcal{D}$, such that $\mathcal{C} \ge -K$.

This fact ensures the validity of (1.18).

A delicate analysis of the behavior of H and G contained in Lemma 7.2 and Lemma 7.3 in [14] is the key step to show the validity of the following result:

Given $K > 0$, there exists $\delta > 0$ such that, if $(\xi_1, \dots, \xi_k) \in \partial\mathcal{D}$, and $|\varphi_k(\xi_1, \dots, \xi_k)| \le K$, then there exists a vector τ, tangent to $\partial\mathcal{D}$, such that $\nabla\varphi_k(\xi_1, \dots, \xi_k) \cdot \tau \neq 0$.

This fact is proved in Lemma 7.4 in [14] and it shows the validity of (1.19). Having established (1.18) and (1.19), we conclude that φ_k has a non-trivial critical value in $\mathcal{D}$, which gives the proof of Theorem 1.1. □

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