



Trace and inverse trace of Steklov eigenvalues

Yongjie Shi, Chengjie Yu ^{*,1}

Department of Mathematics, Shantou University, Shantou, Guangdong, 515063, China

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Abstract

In this paper, we obtain some new estimates for the trace and inverse trace of Steklov eigenvalues. The estimates generalize some previous results of Hersch–Payne–Schiffer [13], Brock [2], Raulot–Savo [21] and Dittmar [5].

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1. Introduction

Let (M^n, g) be a compact oriented Riemannian manifold with nonempty boundary. The Dirichlet-to-Neumann map or Steklov operator for differential forms sends a differential p -form $\omega \in A^p(\partial M)$ to $i_\nu d\hat{\omega}$. Here ν is the outward unit normal vector and $\hat{\omega}$ is the tangential harmonic extension of ω . That is,

$$\begin{cases} \Delta \hat{\omega} = 0 \\ \iota^* \hat{\omega} = \omega \\ i_\nu \hat{\omega} = 0 \end{cases} \quad (1.1)$$

^{*} Corresponding author.

E-mail addresses: yjshi@stu.edu.cn (Y. Shi), cjyu@stu.edu.cn (C. Yu).

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where $\Delta = d\delta + \delta d$ is the Hodge–Laplace operator and $\iota : \partial M \rightarrow M$ is the natural inclusion. This definition of Dirichlet-to-Neumann map for differential forms was introduced by Raulot and Savo in [22] in recent years. When $p = 0$, it is clear that the definition is the same as the classical one that was essentially introduced by Steklov [25]. Moreover, it was shown in [22] that, the same as the classical one (see [26]), the Dirichlet-to-Neumann map is a nonnegative self-adjoint first order elliptic pseudo differential operator. So, the eigenvalues of the Dirichlet-to-Neumann map for differential p -forms are nonnegative and discrete, and we can list them in ascending order (counting multiplicity) as

$$0 \leq \sigma_1^{(p)} \leq \sigma_2^{(p)} \leq \cdots \leq \sigma_k^{(p)} \leq \cdots. \quad (1.2)$$

They are called Steklov eigenvalues of (M, g) for differential p -forms.

There are two other definitions of Dirichlet-to-Neumann map introduced by Joshi–Lionheart [16] and Belishev–Sharafutdinov [1] considering geometric inverse problems. However, the definition of Raulot–Savo [22] given above is more suitable for spectral analysis because it is self-adjoint and elliptic. The Dirichlet-to-Neumann map for functions has been extensively studied because it is deeply related with physics (see [18]) and Calderón’s inverse problem in applied mathematics (see [4,27]).

There have been many works on the estimate of Steklov eigenvalues. For example, in [29], Weinstock obtained the following estimate for any simply connected planar domain Ω :

$$\sigma_2^{(0)} L(\partial\Omega) \leq 2\pi \quad (1.3)$$

where equality holds if and only if Ω is a disk. Here $L(\partial\Omega)$ means the length of the $\partial\Omega$. This estimate was later generalized by Hersch–Payne–Schiffer [13] in a much more general form by using the conjugate harmonic functions. More precisely, Hersch–Payne–Schiffer [13] obtained the following inequalities for any simply connected planar domain Ω :

$$\sigma_{p+1}^{(0)} \sigma_{q+1}^{(0)} L(\partial\Omega)^2 \leq \begin{cases} (p+q)^2 \pi^2 & p+q \text{ is even} \\ (p+q-1)^2 \pi^2 & p+q \text{ is odd.} \end{cases} \quad (1.4)$$

Letting $p = q$ in (1.4), one obtains

$$\sigma_{p+1}^{(0)} L(\partial\Omega) \leq 2p\pi. \quad (1.5)$$

This is a generalization of (1.3). (1.4) and (1.5) were later generalized by [7,10–12] for general surfaces. (1.4) was recently generalized by [31] for general manifolds using the theory of Raulot–Savo [21] on Steklov eigenvalues of differential forms and harmonic conjugate forms. There are many other interesting estimates of Steklov eigenvalues, see for example [2,3,6,8,15,17,19,21–23,28]. [9] is an excellent survey for recent progresses on the topic.

In this paper, motivated by Brock [2] and Hersch–Payne–Schiffer [13], by combining the tricks in [30,31] and [13], we consider lower bound estimate for inverse trace and upper bound estimate for trace of Steklov eigenvalues.

We first consider inverse trace of Steklov eigenvalues for functions on Riemannian surfaces. In this case, we obtain the following result.

Theorem 1.1. Let (M^2, g) be a compact oriented Riemannian surface with $\partial M \neq \emptyset$. Then

$$\sum_{i=1}^{2n} f\left(\frac{1}{\sigma_{m+i}^{(0)}}\right) \geq 2 \sum_{i=1}^n f\left(\frac{1}{\lambda_{b_1+2m+2i-2}^{1/2}}\right) \quad (1.6)$$

for any increasing convex function f , and any positive integer m , where b_1 is the first Betti number of M and

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \quad (1.7)$$

are the spectrum of ∂M for Laplacian operator.

In Theorem 1.1, when M is simply connected, that is $b_1 = 0$, if we choose $m = 1$ and $f(t) = t$, by noting that $\lambda_{2i}(\partial M) = \left(\frac{2i\pi}{L(\partial M)}\right)^2$ with $L(\partial M)$ the length of ∂M (∂M has only one connected component), we have

$$\frac{1}{\sigma_2^{(0)}} + \frac{1}{\sigma_3^{(0)}} + \dots + \frac{1}{\sigma_{2n+1}^{(0)}} \geq \frac{L(\partial M)}{\pi} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right). \quad (1.8)$$

This is the inverse trace estimate in [13]. Moreover, if we further choose $m = 1$ and $f(t) = t^2$ in Theorem 1.1, we have

$$\frac{1}{\sigma_2^{(0)^2}} + \frac{1}{\sigma_3^{(0)^2}} + \dots + \frac{1}{\sigma_{2n+1}^{(0)^2}} \geq \frac{L(\partial M)^2}{2\pi^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right). \quad (1.9)$$

By letting $n \rightarrow \infty$ in the last inequality, we have

$$\sum_{i=2}^{\infty} \frac{1}{\sigma_i^{(0)^2}} \geq \frac{L(\partial M)^2}{12}. \quad (1.10)$$

This is an estimate stronger than Theorem 3.3 in [5].

Next, we consider the inverse trace of Steklov eigenvalues for general manifolds. For this case, we obtain the following result.

Theorem 1.2. Let (M^n, g) be a compact oriented Riemannian manifold with nonempty boundary. Then

(1)

$$\sum_{i=1}^m f\left(\frac{1}{\sigma_{r+i}^{(0)}}\right) + \sum_{i=1}^m f\left(\frac{1}{\sigma_{b_{n-2}+s+i-1}^{(n-2)}}\right) \geq 2 \sum_{i=1}^m f\left(\frac{1}{\lambda_{b_{n-1}+r+s+i-1}^{1/2}}\right) \quad (1.11)$$

(2)

$$\sum_{i=1}^m f\left(\frac{1}{\sigma_{r+i}^{(0)} \sigma_{b_{n-2}+s+i-1}^{(n-2)}}\right) \geq \sum_{i=1}^m f\left(\frac{1}{\lambda_{b_{n-1}+r+s+i-1}}\right) \quad (1.12)$$

for any increasing convex function f and any positive integers r, s , where b_p is the p -th Betti number of M and

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \quad (1.13)$$

are the spectrum of ∂M for Laplacian operator.

When $n = 2$, the theorem also gives some estimates for Riemannian surfaces. For example, when $r = s$, we have

$$\sum_{i=1}^m f\left(\frac{1}{\sigma_{r+i}^{(0)}}\right) \geq \sum_{i=1}^m f\left(\frac{1}{\lambda_{b_1+2r+i-1}^{1/2}}\right). \quad (1.14)$$

This is weaker than [Theorem 1.1](#). However, it is also sharp when M is a disk. Moreover, when $m = 1$ and $f(t) = t$ in [Theorem 1.2](#), we obtain

$$\sigma_{1+r}^{(0)} \sigma_{b_{n-2}+s}^{(n-2)} \leq \lambda_{b_{n-1}+r+s}. \quad (1.15)$$

This is the higher dimensional generalization of (1.4) in [31].

Finally, we obtain an estimate for trace of Steklov eigenvalues.

Theorem 1.3. *Let (M^n, g) be a compact oriented Riemannian manifold with nonempty boundary. Let V be the space of parallel exact 1-forms on M . Suppose that $\dim V = m > 0$. Then*

$$\sum_{i=1}^{C_m^{p+1}} \sigma_{b_p+i}^{(p)} \leq \frac{C_{m-1}^p \text{Vol}(\partial M)}{\text{Vol}(M)} \quad (1.16)$$

for $p = 1, 2, \dots, m-1$.

This result implies the general estimate in [30] directly (see [Corollary 5.1](#)). Applying the result to domains in Euclidean spaces, one can obtain the estimates in [2,30] and [21] (see [Corollary 5.2](#) and [Remark 5.1](#)).

The proof of [Theorem 1.1](#) and [Theorem 1.2](#) is motivated by [13] using conjugate harmonic forms. The proof of [Theorem 1.3](#) is motivated by [2]. The organization of the remaining parts of this paper is as follows. In [Section 2](#), we recall some preliminaries in conjugate harmonic forms and matrix inequalities. In [Section 3](#), we prove [Theorem 1.1](#). In [Section 4](#), we prove [Theorem 1.2](#). Finally, in [Section 5](#), we prove [Theorem 1.3](#).

2. Preliminaries

In this section, we recall some preliminaries in conjugate harmonic forms and matrix inequalities that will be used in next sections.

First, we summarize the construction of conjugate harmonic forms of a harmonic function on a general manifold in [31] as the following lemma. The proof which can be found in [31] is a simple application of Hodge decomposition theorem for compact Riemannian manifolds with nonempty boundary (see [24]).

Lemma 2.1. *Let (M^n, g) be a compact oriented Riemannian manifold with nonempty boundary and u be a harmonic function on M . Suppose that*

$$*du \perp_{L^2(M)} \mathcal{H}_N^{(n-1)}(M). \quad (2.1)$$

Then, there is a unique $\omega \in A^{n-2}(M)$ such that

- (1) $d\omega = *du$;
- (2) $\delta\omega = 0$;
- (3) $i_\nu\omega = 0$ and
- (4) $\omega \perp_{L^2(\partial M)} \mathcal{H}_N^{n-2}(M)$.

Here

$$\mathcal{H}_N^p = \{\gamma \in A^p(M) \mid d\gamma = \delta\gamma = 0 \text{ and } i_\nu\gamma = 0\}. \quad (2.2)$$

ω is called the conjugate harmonic form of u .

Next, recall an inequality on the trace of eigenvalues for positive definite matrices that will be used in next sections. Because we can not find direct reference for the inequality, we also give the proof of the inequality.

Lemma 2.2. *Let A and B be two symmetric $n \times n$ matrices that are both positive definite. Let*

$$\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$$

and

$$\lambda_1(B) \leq \lambda_2(B) \leq \cdots \leq \lambda_n(B)$$

be eigenvalues of A and B respectively. Then, for any increasing convex function f ,

$$\sum_{i=1}^n f(\lambda_i(A)\lambda_i(B)) \geq \sum_{i=1}^n f(A(i, i)B(i, i)) \quad (2.3)$$

where $A(i, j)$ and $B(i, j)$ are the (i, j) -entries of A and B respectively. In particular, by letting $B = I_n$, we have

$$\sum_{i=1}^n f(\lambda_i(A)) \geq \sum_{i=1}^n f(A(i, i)). \quad (2.4)$$

Proof. Let $A \circ B$ be the Hadamard product of A and B . That is,

$$A \circ B(i, j) = A(i, j)B(i, j)$$

for $i, j = 1, 2, \dots, n$. Then, by basic majorization relations (see [14] or [32, Theorem 2.6]),

$$\begin{aligned} & \{\lambda_1(A \circ B), \lambda_2(A \circ B), \dots, \lambda_n(A \circ B)\} \\ & \prec_w \{\lambda_1(A)\lambda_1(B), \lambda_2(A)\lambda_2(B), \dots, \lambda_n(A)\lambda_n(B)\}. \end{aligned} \quad (2.5)$$

Moreover, by Schur's Theorem (see [32, Theorem 2.1]),

$$\begin{aligned} & \{A(1, 1)B(1, 1), A(2, 2)B(2, 2), \dots, A(n, n)B(n, n)\} \\ & \prec \{\lambda_1(A \circ B), \lambda_2(A \circ B), \dots, \lambda_n(A \circ B)\}. \end{aligned} \quad (2.6)$$

Combining (2.5) and (2.6), we have

$$\begin{aligned} & \{A(1, 1)B(1, 1), A(2, 2)B(2, 2), \dots, A(n, n)B(n, n)\} \\ & \prec_w \{\lambda_1(A)\lambda_1(B), \lambda_2(A)\lambda_2(B), \dots, \lambda_n(A)\lambda_n(B)\}. \end{aligned} \quad (2.7)$$

Then, by majorization principles (see [20] or [32, Theorem 2.3]),

$$\begin{aligned} & \{f(A(1, 1)B(1, 1)), f(A(2, 2)B(2, 2)), \dots, f(A(n, n)B(n, n))\} \\ & \prec_w \{f(\lambda_1(A)\lambda_1(B)), f(\lambda_2(A)\lambda_2(B)), \dots, f(\lambda_n(A)\lambda_n(B))\} \end{aligned} \quad (2.8)$$

for any increasing convex function f . This gives us the conclusion. \square

3. Proof of Theorem 1.1

In this section, by using Lemma 2.1 and the trick in [13], we prove Theorem 1.1.

Proof of Theorem 1.1. Let

$$\phi_1 \equiv 1, \phi_2, \dots, \phi_k, \dots$$

be a complete orthonormal system of eigenfunctions on ∂M corresponding to the spectrum

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$$

of ∂M . Moreover, let

$$\psi_1, \psi_2, \dots, \psi_k, \dots$$

be a complete orthonormal system of eigenfunctions for positive Steklov eigenvalues of functions corresponding to eigenvalues listed in ascending order.

Let $u_1 \neq 0$ be the harmonic extension of $c_2\phi_2 + \dots + c_{b_1+2m}\phi_{b_1+2m}$ such that $u_1 \perp_{L^2(M)} \mathcal{H}_N^1(M)$ and $u_1, u_2 \perp_{L^2(\partial M)} \psi_1, \psi_2, \dots, \psi_{m-1}$. Here u_2 is the conjugate harmonic function of u_1 as in Lemma 2.1. This can always be done since the restrictions are just $b_1 + 2m - 2$ homogeneous linear equations for $b_1 + 2m - 1$ unknowns $c_2, c_3, \dots, c_{b_1+2m}$.

Similarly, let $u_3 \neq 0$ be the harmonic extension of $c_2\phi_2 + \dots + c_{b_1+2m+2}\phi_{b_1+2m+2}$ such that

- (1) $*du_3 \perp_{L^2(M)} \mathcal{H}_N^1(M)$;
- (2) $u_3, u_4 \perp_{L^2(\partial M)} \psi_1, \psi_2, \dots, \psi_{m-1}$ and
- (3) $du_3 \perp_{L^2(M)} du_1, du_2$.

Here u_4 is the conjugate harmonic function of u_3 .

Continuing this process on, we can construct nonconstant harmonic functions $u_1, u_2, u_3, u_4, \dots, u_{2n-1}, u_{2n}$, such that

- (1) $u_{2i-1} \in \text{span}\{\hat{\phi}_2, \hat{\phi}_3, \dots, \hat{\phi}_{b_1+2m+2i-2}\}$;
- (2) u_{2i} is the conjugate harmonic function of u_{2i-1} ;
- (3) $u_1, u_2, \dots, u_{2n} \perp_{L^2(\partial M)} \psi_1, \psi_2, \dots, \psi_{m-1}$;
- (4) $du_{2i-1} \perp_{L^2(M)} du_1, du_2, \dots, du_{2i-2}$

for $i = 1, 2, \dots, n$.

Now, we check that

$$du_{2i} \perp_{L^2(M)} du_1, du_2, \dots, du_{2i-1}. \quad (3.1)$$

First,

$$\begin{aligned} \int_M \langle du_{2i}, du_{2i-1} \rangle dV_M &= \int_M \langle *du_{2i-1}, du_{2i-1} \rangle dV_M \\ &= - \int_M du_{2i-1} \wedge du_{2i-1} dV_M = 0. \end{aligned} \quad (3.2)$$

Moreover,

$$\begin{aligned} \int_M \langle du_{2i}, du_{2j} \rangle dV_M &= \int_M \langle *du_{2i-1}, *du_{2j-1} \rangle dV_M \\ &= \int_M \langle du_{2i-1}, du_{2j-1} \rangle dV_M = 0, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \int_M \langle du_{2i}, du_{2j-1} \rangle dV_M &= \int_M \langle *du_{2i-1}, du_{2j-1} \rangle dV_M \\ &= - \int_M \langle du_{2i-1}, *du_{2j-1} \rangle dV_M = - \int_M \langle du_{2i-1}, du_{2j} \rangle dV_M = 0 \end{aligned} \quad (3.4)$$

for any $j = 1, 2, \dots, i-1$.

Hence, we have shown that

$$\int_M \langle du_i, du_j \rangle dV_M = 0 \quad (3.5)$$

for any $i \neq j$. Let $V = \text{span}\{u_1, u_2, \dots, u_{2n-1}, u_{2n}\}$ and A be a linear transformation on V such that

$$\int_M \langle du, dv \rangle = \int_{\partial M} \langle Au, v \rangle dV_{\partial M} \quad (3.6)$$

for any $u, v \in V$. Let

$$\lambda_1(A) \leq \lambda_2(A) \leq \dots \leq \lambda_{2n}(A)$$

be the eigenvalues of A . Then, by the Courant–Fischer’s min–max principle,

$$\sigma_{m+i}^{(0)} \leq \lambda_i(A) \quad (3.7)$$

for $i = 1, 2, \dots, 2n$.

Let $v_i = \frac{u_i}{(\int_M \langle du_i, du_i \rangle dV_M)^{1/2}}$. We still denote the matrix of A under the basis of $\{v_1, v_2, \dots, v_{2n}\}$ as A . Then, by (3.6)

$$A^{-1}(i, j) = \frac{\int_{\partial M} \langle u_i, u_j \rangle dV_{\partial M}}{(\int_M \langle du_i, du_i \rangle dV_M \int_M \langle du_j, du_j \rangle dV_M)^{1/2}}. \quad (3.8)$$

Moreover, by Cauchy–Schwarz inequality,

$$\begin{aligned} &A^{-1}(2i-1, 2i-1)A^{-1}(2i, 2i) \\ &= \frac{\int_{\partial M} \langle u_{2i-1}, u_{2i-1} \rangle dV_{\partial M} \int_{\partial M} \langle u_{2i}, u_{2i} \rangle dV_{\partial M}}{\int_M \langle du_{2i-1}, du_{2i-1} \rangle dV_M \int_M \langle du_{2i}, du_{2i} \rangle dV_M} \\ &= \frac{\int_{\partial M} \langle u_{2i-1}, u_{2i-1} \rangle dV_{\partial M} \int_{\partial M} \langle u_{2i}, u_{2i} \rangle dV_{\partial M}}{\int_M \langle *du_{2i}, *du_{2i} \rangle dV_M \int_M \langle du_{2i}, du_{2i} \rangle dV_M} \\ &= \frac{\int_{\partial M} \langle u_{2i-1}, u_{2i-1} \rangle dV_{\partial M} \int_{\partial M} \langle u_{2i}, u_{2i} \rangle dV_{\partial M}}{(\int_M \langle du_{2i}, du_{2i} \rangle dV_M)^2} \end{aligned} \quad (3.9)$$

$$\begin{aligned}
&= \frac{\int_{\partial M} \langle u_{2i-1}, u_{2i-1} \rangle dV_{\partial M} \int_{\partial M} \langle u_{2i}, u_{2i} \rangle dV_{\partial M}}{(\int_{\partial M} \langle u_{2i}, i_v du_{2i} \rangle dV_{\partial M})^2} \\
&= \frac{\int_{\partial M} \langle u_{2i-1}, u_{2i-1} \rangle dV_{\partial M} \int_{\partial M} \langle u_{2i}, u_{2i} \rangle dV_{\partial M}}{(\int_{\partial M} \langle u_{2i}, i_v * du_{2i-1} \rangle dV_{\partial M})^2} \\
&\geq \frac{\int_{\partial M} \langle u_{2i-1}, u_{2i-1} \rangle dV_{\partial M}}{\int_{\partial M} \langle du_{2i-1}, du_{2i-1} \rangle dV_{\partial M}} \\
&\geq \frac{1}{\lambda_{b_1+2m+2i-2}}
\end{aligned}$$

for $i = 1, 2, \dots, n$.

Finally, by (3.7), (3.9), Lemma 2.2, and that f is increasing and convex,

$$\begin{aligned}
\sum_{i=1}^{2n} f\left(\frac{1}{\sigma_{m+i}^{(0)}}\right) &\geq \sum_{i=1}^{2n} f\left(\frac{1}{\lambda_i(A)}\right) \\
&\geq \sum_{i=1}^n f\left(A^{-1}(2i-1, 2i-1)\right) + f\left(A^{-1}(2i, 2i)\right) \\
&\geq 2 \sum_{i=1}^n f\left(\frac{A^{-1}(2i-1, 2i-1) + A^{-1}(2i, 2i)}{2}\right) \\
&\geq 2 \sum_{i=1}^n f\left(\left(A^{-1}(2i-1, 2i-1)A^{-1}(2i, 2i)\right)^{1/2}\right) \\
&\geq 2 \sum_{i=1}^n f\left(\frac{1}{\lambda_{b_1+2m+2i-2}^{1/2}}\right).
\end{aligned} \tag{3.10}$$

This completes the proof of the theorem. \square

4. Proof of Theorem 1.2

In this section, by a similar argument as in the proof of Theorem 1.1 using conjugate harmonic forms, we prove Theorem 1.2.

Proof of Theorem 1.2. Let

$$\phi_1 \equiv 1, \phi_2, \dots, \phi_k, \dots$$

and

$$\psi_1, \psi_2, \dots, \psi_k, \dots$$

be the same as in the proof of Theorem 1.1. Moreover, let

$$\epsilon_1, \epsilon_2, \dots, \epsilon_k, \dots$$

be a complete orthonormal system of eigenforms for positive Steklov eigenvalues of differential $(n-2)$ -forms corresponding to eigenvalues listed in ascending order.

Similarly as in the proof of [Theorem 1.1](#). We can find nonconstant harmonic functions u_1, u_2, \dots, u_m such that

- (1) $*du_i \perp_{L^2(M)} \mathcal{H}_N^{n-1}(M)$;
- (2) $u_i \perp_{L^2(\partial M)} \psi_1, \psi_2, \dots, \psi_{r-1}$;
- (3) $\omega_i \perp_{L^2(\partial M)} \epsilon_1, \epsilon_2, \dots, \epsilon_{s-1}$ where ω_i is the conjugate harmonic form of u_i as in [Lemma 2.1](#);
- (4) $u_i \in \text{span}\{\hat{\phi}_2, \hat{\phi}_3, \dots, \hat{\phi}_{b_{n-1}+r+s+i-1}\}$;
- (5) $\int_M \langle du_i, du_j \rangle dV_M = \delta_{ij}$

for $i, j = 1, 2, \dots, m$. Moreover, note that

$$\int_M \langle d\omega_i, d\omega_j \rangle dV_M = \int_M \langle du_i, du_j \rangle dV_M = \delta_{ij} \quad (4.1)$$

for $i, j = 1, 2, \dots, m$.

Let $V = \text{span}\{u_1, u_2, \dots, u_m\}$ and $W = \text{span}\{\omega_1, \omega_2, \dots, \omega_m\}$. Let $A : V \rightarrow V$ and $B : W \rightarrow W$ be linear transformations on V and W such that

$$\int_M \langle du, dv \rangle dV_M = \int_{\partial M} \langle Au, v \rangle dV_{\partial M} \quad (4.2)$$

for any $u, v \in V$ and

$$\int_M \langle d\alpha, d\beta \rangle = \int_{\partial M} \langle B\alpha, \beta \rangle dV_{\partial M} \quad (4.3)$$

for any $\alpha, \beta \in W$ respectively. Then, by Courant–Fischer’s min–max principle,

$$\sigma_{r+i}^{(0)} \leq \lambda_i(A) \quad (4.4)$$

and

$$\sigma_{b_{n-2}+s+i-1}^{(n-2)} \leq \lambda_i(B) \quad (4.5)$$

for $i = 1, 2, \dots, m$.

Denote the matrix of A and B under the bases $\{u_1, u_2, \dots, u_m\}$ and $\{\omega_1, \omega_2, \dots, \omega_m\}$ as A and B respectively. Then, by [\(4.2\)](#) and [\(4.3\)](#),

$$A^{-1}(i, j) = \int_{\partial M} \langle u_i, u_j \rangle dV_{\partial M} \quad (4.6)$$

and

$$B^{-1}(i, j) = \int_{\partial M} \langle \omega_i, \omega_j \rangle dV_{\partial M} \quad (4.7)$$

for $i, j = 1, 2, \dots, m$. Moreover, by Cauchy–Schwarz inequality,

$$\begin{aligned} & A^{-1}(i, i)B^{-1}(i, i) \\ &= \frac{\int_{\partial M} \langle u_i, u_i \rangle dV_{\partial M} \int_{\partial M} \langle \omega_i, \omega_i \rangle dV_{\partial M}}{\int_M \langle du_i, du_i \rangle dV_M \int_M \langle d\omega_i, d\omega_i \rangle dV_M} \\ &= \frac{\int_{\partial M} \langle u_i, u_i \rangle dV_{\partial M} \int_{\partial M} \langle \omega_i, \omega_i \rangle dV_{\partial M}}{\int_M \langle *d\omega_i, *d\omega_i \rangle dV_M \int_M \langle d\omega_i, d\omega_i \rangle dV_M} \\ &= \frac{\int_{\partial M} \langle u_i, u_i \rangle dV_{\partial M} \int_{\partial M} \langle \omega_i, \omega_i \rangle dV_{\partial M}}{(\int_M \langle d\omega_i, d\omega_i \rangle dV_M)^2} \\ &= \frac{\int_{\partial M} \langle u_i, u_i \rangle dV_{\partial M} \int_{\partial M} \langle \omega_i, \omega_i \rangle dV_{\partial M}}{(\int_{\partial M} \langle \omega_i, i_v d\omega_i \rangle dV_{\partial M})^2} \\ &= \frac{\int_{\partial M} \langle u_i, u_i \rangle dV_{\partial M} \int_{\partial M} \langle \omega_i, \omega_i \rangle dV_{\partial M}}{(\int_{\partial M} \langle \omega_i, i_v * du_i \rangle dV_{\partial M})^2} \\ &\geq \frac{\int_{\partial M} \langle u_i, u_i \rangle dV_{\partial M}}{\int_{\partial M} \langle du_i, du_i \rangle dV_{\partial M}} \\ &\geq \frac{1}{\lambda_{b_{n-1}+r+s+i-1}}. \end{aligned} \quad (4.8)$$

We are ready to prove the inequalities.

(1) By (4.4), (4.5), (4.8), Lemma 2.2, and that f is increasing and convex,

$$\begin{aligned} \sum_{i=1}^m f\left(\frac{1}{\sigma_{r+i}^{(0)}}\right) + \sum_{i=1}^m f\left(\frac{1}{\sigma_{b_{n-2}+s+i-1}^{(n-2)}}\right) &\geq \sum_{i=1}^m f\left(\frac{1}{\lambda_i(A)}\right) + \sum_{i=1}^m f\left(\frac{1}{\lambda_i(B)}\right) \\ &\geq \sum_{i=1}^m f\left(A^{-1}(i, i)\right) + f\left(B^{-1}(i, i)\right) \\ &\geq 2 \sum_{i=1}^m f\left((A^{-1}(i, i)B^{-1}(i, i))^{1/2}\right) \\ &\geq 2 \sum_{i=1}^m f\left(\frac{1}{\lambda_{b_{n-1}+r+s+i-1}^{1/2}}\right). \end{aligned} \quad (4.9)$$

(2) By (4.4), (4.5), (4.8) and Lemma 2.2, and that f is increasing and convex,

$$\begin{aligned} \sum_{i=1}^m f\left(\frac{1}{\sigma_{r+i}^{(0)} \sigma_{b_{n-2}+s+i-1}^{(n-2)}}\right) &\geq \sum_{i=1}^m f\left(\frac{1}{\lambda_i(A) \lambda_i(B)}\right) \\ &\geq \sum_{i=1}^m f\left(A^{-1}(i, i) B^{-1}(i, i)\right) \\ &\geq \sum_{i=1}^m f\left(\frac{1}{\lambda_{b_{n-1}+r+s+i-1}}\right). \quad \square \end{aligned} \quad (4.10)$$

5. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3 and some simple corollaries of it.

Proof of Theorem 1.3. Let $A^{(p+1)}$ be a linear transformation on $\wedge^{p+1} V$ such that

$$\int_M \langle A^{(p+1)} \xi, \eta \rangle dV_M = \int_{\partial M} \langle i_\nu \xi, i_\nu \eta \rangle dV_{\partial M} \quad (5.1)$$

for any $\xi, \eta \in \wedge^{p+1} V$. Then, by Lemma 2.1 in [30],

$$\sigma_{b_p+i}^{(p)} \leq \lambda_i(A^{(p+1)}) \quad (5.2)$$

for $i = 1, 2, \dots, C_m^{p+1}$. So,

$$\sum_{i=1}^{C_m^{p+1}} \sigma_{b_p+i}^{(p)} \leq \text{tr}(A^{(p+1)}). \quad (5.3)$$

Let $\xi_1, \xi_2, \dots, \xi_{C_m^{p+1}}$ be an orthonormal basis of $\wedge^{(p+1)} V$. That is

$$\langle \xi_i, \xi_j \rangle = \delta_{ij} \quad (5.4)$$

for $i, j = 1, 2, \dots, C_m^{p+1}$. By (5.1), we know that

$$\begin{aligned} \text{tr} A^{(p+1)} &= \frac{1}{\text{Vol}(M)} \sum_{i=1}^{C_m^{p+1}} \int_{\partial M} \|i_\nu \xi_i\|^2 dV_{\partial M} \\ &= \frac{1}{\text{Vol}(M)} \int_{\partial M} \sum_{i=1}^{C_m^{p+1}} \|i_{\nu^\top} \xi_i\|^2 dV_{\partial M} \end{aligned} \quad (5.5)$$

where ν^\top is the orthogonal projection of ν onto the dual of V .

Let e_1, e_2, \dots, e_m be an orthogonal basis of the dual of V and $\omega_1, \omega_2, \dots, \omega_m$ be their dual. Without loss of generality, we can assume that $v^\top = v_1 e_1$ with $0 < v_1 \leq 1$ and $\{\xi_1, \xi_2, \dots, \xi_m\}$ are just $\{\omega_{i_1} \wedge \omega_{i_2} \wedge \dots \wedge \omega_{i_{p+1}} \mid 1 \leq i_1 < i_2 < \dots < i_{p+1} \leq m\}$. Then

$$\sum_{i=1}^{C_m^{p+1}} \|i_{v^\top} \xi_i\|^2 = v_1^2 \sum_{2 \leq i_2 < i_3 < \dots < i_{p+1} \leq m} \|\omega_{i_2} \wedge \dots \wedge \omega_{i_{p+1}}\|^2 \leq C_{m-1}^p. \quad (5.6)$$

Combining (5.5) and (5.6), we get the conclusion. \square

As a corollary, we get the following estimate in [30].

Corollary 5.1. *Let (M^n, g) be a compact oriented Riemannian manifold with nonempty boundary. Let V be the space of parallel exact 1-forms on M . Suppose that $\dim V = m > 0$. Then*

$$\sigma_{b_p+i}^{(p)} \leq \frac{C_{m-1}^p}{C_m^{p+1} + 1 - i} \frac{\text{Vol}(\partial M)}{\text{Vol}(M)} \quad (5.7)$$

for $p = 1, 2, \dots, m-1$ and $i = 1, 2, \dots, C_m^{p+1}$.

Proof. By Theorem 1.3,

$$(C_m^{p+1} + 1 - i) \sigma_{b_p+i}^{(p)} \leq \sum_{k=i}^{C_m^{p+1}} \sigma_{b_p+k}^{(p)} \leq \frac{C_{m-1}^p \text{Vol}(\partial M)}{\text{Vol}(M)}. \quad (5.8)$$

This gives us the conclusion. \square

Applying Theorem 1.3 to the Euclidean case, we have the following inequality.

Corollary 5.2. *Let Ω be a bounded domain with smooth boundary in \mathbb{R}^n . Then*

$$\sum_{i=1}^{C_n^{p+1}} \sigma_{b_p+i}^{(p)} \leq \frac{C_{n-1}^p \text{Vol}(\partial \Omega)}{\text{Vol}(\Omega)} \quad (5.9)$$

for $p = 0, 1, 2, \dots, n-1$.

Remark 5.1. By Cauchy–Schwarz inequality and (5.9), it is not hard to see that

$$\sum_{i=1}^{C_n^{p+1}} \frac{1}{\sigma_{b_p+i}^{(p)}} \geq \frac{n C_n^{p+1} \text{Vol}(\Omega)}{(p+1) \text{Vol}(\partial \Omega)}. \quad (5.10)$$

This is a generalization of Theorem 1 in [2].

References

- [1] M. Belishev, V. Sharafutdinov, Dirichlet to Neumann operator on differential forms, *Bull. Sci. Math.* 132 (2) (2008) 128–145.
- [2] F. Brock, An isoperimetric inequality for eigenvalues of the Stekloff problem, *Z. Angew. Math. Mech.* 81 (1) (2001) 69–71.
- [3] B. Colbois, A. El Soufi, A. Girouard, Isoperimetric control of the Steklov spectrum, *J. Funct. Anal.* 261 (5) (2011) 1384–1399.
- [4] A.-P. Calderón, On an inverse boundary value problem, in: *Seminar on Numerical Analysis and Its Applications to Continuum Physics*, Rio de Janeiro, 1980, Soc. Brasil. Mat., Rio de Janeiro, 1980, pp. 65–73.
- [5] Bodo Dittmar, Sums of reciprocal Stekloff eigenvalues, *Math. Nachr.* 268 (2004) 44–49.
- [6] J. Escobar, A comparison theorem for the first non-zero Steklov eigenvalue, *J. Funct. Anal.* 178 (1) (2000) 143–155.
- [7] A. Fraser, R. Schoen, The first Steklov eigenvalue, conformal geometry, and minimal surfaces, *Adv. Math.* 226 (5) (2011) 4011–4030.
- [8] A. Fraser, R. Schoen, Sharp eigenvalue bounds and minimal surfaces in the ball, *Invent. Math.* 203 (3) (March 2016) 823–890.
- [9] A. Girouard, I. Polterovich, Spectral geometry of the Steklov problem, *J. Spectr. Theory* (2016), in press.
- [10] A. Girouard, I. Polterovich, Upper bounds for Steklov eigenvalues on surfaces, *Electron. Res. Announc. Math. Sci.* 19 (2012) 77–85.
- [11] Alexandre Girouard, Iosif Polterovich, Shape optimization for low Neumann and Steklov eigenvalues, *Math. Methods Appl. Sci.* 33 (4) (2010) 501–516.
- [12] A. Girouard, I. Polterovich, On the Hersch–Payne–Schiffer estimates for the eigenvalues of the Steklov problem, *Funktsional. Anal. i Prilozhen.* 44 (2) (2010) 33–47 (in Russian); translation in *Funct. Anal. Appl.* 44 (2) (2010) 106–117.
- [13] J. Hersch, L.E. Payne, M.M. Schiffer, Some inequalities for Stekloff eigenvalues, *Arch. Ration. Mech. Anal.* 57 (1975) 99–114.
- [14] Roger A. Horn, Charles R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, ISBN 0-521-30587-X, 1991, viii+607 pp.
- [15] S. Ilias, O. Makhoul, A Reilly inequality for the first Steklov eigenvalue, *Differential Geom. Appl.* 29 (5) (2011) 699–708.
- [16] M.S. Joshi, W.R.B. Lionheart, An inverse boundary value problem for harmonic differential forms, *Asymptot. Anal.* 41 (2) (2005) 93–106.
- [17] Mikhail A. Karpukhin, Bounds between Laplace and Steklov eigenvalues on nonnegatively curved manifolds, arXiv:1512.09038.
- [18] Nikolay Kuznetsov, Tadeusz Kulczycki, Mateusz Kwaśnicki, Alexander Nazarov, Sergey Poborchi, Iosif Polterovich, Bartłomiej Siudeja, The legacy of Vladimir Andreevich Steklov, *Notices Amer. Math. Soc.* 61 (1) (2014) 9–22.
- [19] K.-K. Kwong, Some sharp eigenvalue estimate for differential forms, private communication.
- [20] Albert W. Marshall, Ingram Olkin, *Inequalities: Theory of Majorization and Its Applications*, Math. Sci. Eng., vol. 143, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, London, ISBN 0-12-473750-1, 1979, xx+569 pp.
- [21] S. Raulot, A. Savo, On the spectrum of the Dirichlet-to-Neumann operator acting on forms of a Euclidean domain, *J. Geom. Phys.* 77 (2014) 1–12.
- [22] S. Raulot, A. Savo, On the first eigenvalue of the Dirichlet-to-Neumann operator on forms, *J. Funct. Anal.* 262 (3) (2012) 889–914.
- [23] S. Raulot, A. Savo, A Reilly formula and eigenvalue estimates for differential forms, *J. Geom. Anal.* 21 (3) (2011) 620–640.
- [24] G. Schwarz, *Hodge Decomposition—A Method for Solving Boundary Value Problems*, Lecture Notes in Math., vol. 1607, Springer-Verlag, Berlin, ISBN 3-540-60016-7, 1995, viii+155 pp.
- [25] W. Stekloff, Sur les problèmes fondamentaux de la physique mathématique, *Ann. Sci. Éc. Norm. Supér.* (3) 19 (1902) 191–259.
- [26] Michael E. Taylor, *Partial Differential Equations II. Qualitative Studies of Linear Equations*, second ed., Appl. Math. Sci., vol. 116, Springer, New York, ISBN 978-1-4419-7051-0, 2011, xxii+614 pp.
- [27] G. Ulmann, Electrical impedance tomography and Calderón’s problem, <http://www.math.washington.edu/~gunther/publications/Papers/calderoniprevised.pdf>.

- [28] Qiaoling Wang, Changyu Xia, Sharp bounds for the first non-zero Stekloff eigenvalues, *J. Funct. Anal.* 257 (8) (2009) 2635–2644.
- [29] R. Weinstock, Inequalities for a classical eigenvalue problem, *J. Ration. Mech. Anal.* 3 (1954).
- [30] Liangwei Yang, Chengjie Yu, Estimates for higher Steklov eigenvalues, [arXiv:1601.01882](#).
- [31] Liangwei Yang, Chengjie Yu, A higher dimensional generalization of the Hersch–Payne–Schiffer inequality for Steklov eigenvalues, [arXiv:1508.06026](#).
- [32] Xingzhi Zhan, *Matrix Inequalities*, Lecture Notes in Math., vol. 1790, Springer-Verlag, Berlin, ISBN 3-540-43798-3, 2002, viii+116 pp.