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J. Differential Equations 262 (2017) 1850–1896

Journal of Differential Equations

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Configurations of periodic orbits for equations with delayed positive feedback

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Received 8 April 2016; revised 21 September 2016

Available online 10 November 2016

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

Abstract

We consider scalar delay differential equations of the form

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)),$$

where $\mu > 0$ and f is a nondecreasing C^1-function. If χ is a fixed point of $f_\mu : \mathbb{R} \ni u \mapsto f(u)/\mu \in \mathbb{R}$ with $f'_\mu(\chi) > 1$, then $[-1, 0] \ni s \mapsto \chi \in \mathbb{R}$ is an unstable equilibrium. A periodic solution is said to have large amplitude if it oscillates about at least two fixed points $\chi_- < \chi_+$ of f_μ with $f'_\mu(\chi_-) > 1$ and $f'_\mu(\chi_+) > 1$. We investigate what type of large-amplitude periodic solutions may exist at the same time when the number of such fixed points (and hence the number of unstable equilibria) is an arbitrary integer $N \geq 2$. It is shown that the number of different configurations equals the number of ways in which N symbols can be parenthesized. The location of the Floquet multipliers of the corresponding periodic orbits is also discussed.

MSC: 34K13; 37D05

Keywords: Delay differential equation; Positive feedback; Large-amplitude periodic solution; Slow oscillation; Floquet multiplier

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http://dx.doi.org/10.1016/j.jde.2016.10.031

1. Introduction

We study the delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

under the hypotheses

$(H0)$ $\mu > 0$,
$(H1)$ feedback function $f \in C^1(\mathbb{R}, \mathbb{R})$ is nondecreasing.

If $\chi \in \mathbb{R}$ is a fixed point of $f_\mu : \mathbb{R} \ni u \mapsto f(u)/\mu \in \mathbb{R}$, then $\hat{\chi} \in C = C([-1, 0], \mathbb{R})$, defined by $\hat{\chi}(s) = \chi$ for all $s \in [-1, 0]$, is an equilibrium of the semiflow. In this paper we assume that

$(H2)$ if χ is a fixed point of f_μ, then $f'_\mu(\chi) \neq 1$.

This hypothesis guarantees that all equilibria are hyperbolic. It is well known that if χ is an unstable fixed point of f_μ (that is, if $f'_\mu(\chi) > 1$), then $\hat{\chi}$ is an unstable equilibrium. If χ is a stable fixed point of f_μ (that is, if $f'_\mu(\chi) < 1$), then $\hat{\chi}$ is also stable (exponentially stable). The stable and unstable equilibria alternate in pointwise ordering.

Mallet-Paret and Sell have verified a Poincaré–Bendixson type result for (1.1) in the case when $f'(u) > 0$ for all $u \in \mathbb{R}$ [17]. Krisztin, Walther and Wu obtained further detailed results on the structure of the solutions (see e.g. [9,7,8,12–14]). They have characterized the geometrical and topological properties of the closure of the unstable set of an unstable equilibrium, the so-called Krisztin–Walther–Wu attractor. If there is only one unstable equilibrium, sufficient conditions can be given for the closure of the unstable set to be the global attractor.

The chief motivation for the present work comes from the paper [17] of Mallet-Paret and Sell. They have shown that if $f'(u) > 0$ for all $u \in \mathbb{R}$, then

$$\pi_2 : C \ni \varphi \mapsto (\varphi(0), \varphi(-1)) \in \mathbb{R}^2$$

maps different (nonconstant and constant) periodic orbits of (1.1) onto disjoint sets in $\mathbb{R}^2$, and the images of nonconstant periodic orbits are simple closed curves in $\mathbb{R}^2$. They have also shown that a nonconstant periodic solution $p : \mathbb{R} \to \mathbb{R}$ of (1.1) oscillates about a fixed point χ of f_μ if and only if $\pi_2\hat{\chi} = (\chi, \chi)$ is in the interior of $\pi_2\{p_t : t \in \mathbb{R}\}$. See Fig. 1.1. These results give a strong restriction on what type of periodic solutions the equation may have for the same feedback function f: Suppose that $p^1 : \mathbb{R} \to \mathbb{R}$ and $p^2 : \mathbb{R} \to \mathbb{R}$ are periodic solutions of equation (1.1). For both $i \in \{1, 2\}$, let E_i be the set of those fixed points of f_μ about which p^i oscillates. Then either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$ or $E_1 \cap E_2 = \emptyset$. We can easily extend these assertions to the case when $f'(u) \geq 0$ for all $u \in \mathbb{R}$, see Proposition 3.4 in Section 3.

This paper considers large-amplitude periodic solutions: periodic solutions oscillating about at least two unstable fixed points of f_μ. Fig. 1.2 lists all configurations of large-amplitude periodic solutions allowed by the previously cited results of Mallet-Paret and Sell in case there are three and four unstable equilibria, respectively. It is a natural question whether all of them indeed exist for some nonlinearities f.

Allowing any number of unstable equilibria, we confirm the existence of all possible configurations of large-amplitude periodic solutions by constructing the suitable feedback functions and

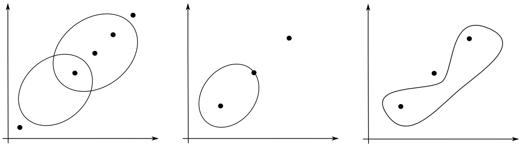

Fig. 1.1. Three examples excluded by Mallet-Paret and Sell. Here we show the images of periodic orbits and equilibria under π_2.

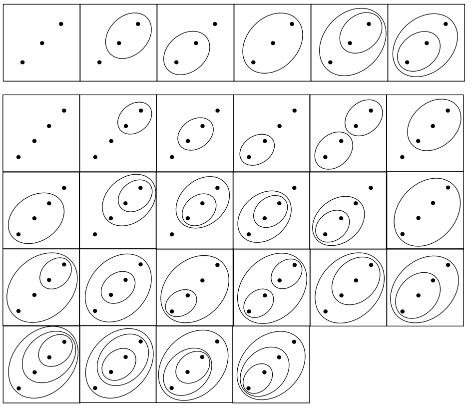

Fig. 1.2. Possible configurations for three or four unstable equilibria: The images of the large-amplitude periodic orbits and the unstable equilibria under π_2.

periodic solutions explicitly. The oscillation frequency of these periodic solutions is the lowest possible. The corresponding periodic orbits are hyperbolic, unstable, and they have exactly one Floquet multiplier outside the unit circle. We do not state uniqueness; there may exist more periodic solutions that cannot be obtained from each other by translation of time and oscillate about the same fixed points of f_μ.

Proving the nonexistence of periodic solutions is a challenging problem in general, see for example the papers [2,12,19] for some well-known results. We can verify that unrequired large-amplitude periodic solutions do not appear for the feedback functions constructed in the paper. So for any configuration in Fig. 1.2, there is a nonlinearity f such that equation (1.1) admits the marked large-amplitude periodic solutions (maybe even more of the same type), but it has none of those that are not indicated.

In the negative feedback case, i.e., when f is nonincreasing, there is at most one equilibrium. Still, it is possible to prove the coexistence of an arbitrary number of slowly oscillatory periodic orbits, see paper [22] for an explicit construction. If f is continuously differentiable, then these periodic orbits are hyperbolic and stable.

2. The main result

Before formulating the main result precisely, we give an introduction to the theoretical background and to the notation used in the paper. Consider equation (1.1) under $(H0)$–$(H2)$.

The phase space for (1.1) is the Banach space $C = C([-1,0],\mathbb{R})$ with the maximum norm. If J is an interval, $u : J \to \mathbb{R}$ is continuous and $[t-1,t] \subseteq J$, then the segment $u_t \in C$ is defined by $u_t(s) = u(t+s)$, $-1 \le s \le 0$.

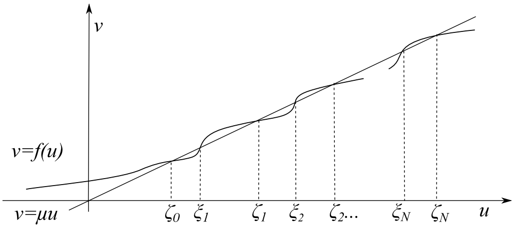

Fig. 2.1. A nonlinearity f giving N unstable and $N+1$ stable equilibria.

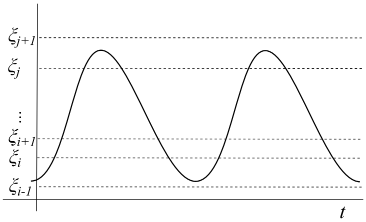

Fig. 2.2. An $\left[i, j\right]$ periodic function.

A solution of equation (1.1) is either a continuous function $x\colon\ [t_0 - 1, \infty) \to \mathbb{R}$, $t_0 \in \mathbb{R}$, that is differentiable for $t > t_0$ and satisfies equation (1.1) on (t_0, ∞), or a continuously differentiable function $x\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation for all $t \in \mathbb{R}$. To all $\varphi \in C$, there corresponds a unique solution $x^{\varphi} : [-1, \infty) \to \mathbb{R}$ with $x_0^{\varphi} = \varphi$.

Let $\Phi\colon\ [0, \infty) \times C \ni (t, \varphi) \mapsto x_t^{\varphi} \in C$ denote the solution semiflow. The global attractor $\mathcal{A}$, if exists, is a nonempty, compact set in C with the following two properties: $\mathcal{A}$ is invariant in the sense that $\Phi(t, \mathcal{A}) = \mathcal{A}$ for all $t \geq 0$. $\mathcal{A}$ attracts bounded sets in the sense that for every bounded set $B \subset C$ and for every open set $U \supset \mathcal{A}$, there exists $t \geq 0$ with $\Phi([t, \infty) \times B) \subset U$. Global attractors are uniquely determined [5].

In this paper the number of unstable equilibria is an arbitrary integer $N \geq 2$. We use the notation $\xi_1 < \xi_2 < \ldots < \xi_N$ for those fixed points of f_μ that give the unstable equilibria. Typically we will consider feedback functions for which f_μ admits $N+1$ further fixed points ζ_j, $j \in \{0, 1, \ldots, N\}$, inducing stable equilibria. Then

$$\zeta_0 < \xi_1 < \zeta_1 < \xi_2 < \zeta_2 < \ldots < \xi_N < \zeta_N.$$

See Fig. 2.1 for an example.

As usual, an arbitrary solution x is called oscillatory about a fixed point χ of f_μ if the set $x^{-1}(\chi) \subset \mathbb{R}$ is not bounded from above. A solution x is slowly oscillatory if for any fixed point χ in $x(\mathbb{R})$ and for any $t \in \mathbb{R}$ such that $[t-1, t]$ is in the domain of x, the function $[t-1, t] \ni s \mapsto x(s) - \chi \in \mathbb{R}$ has one or two sign changes.

As it has been mentioned before, we say that a periodic solution has large amplitude if it oscillates about at least two elements of $\{\xi_1, \xi_2, \ldots, \xi_N\}$. This definition is the straightforward generalization of the one used in [10]. By an $\left[i, j\right]$ periodic solution with $1 \leq i < j \leq N$, we mean a large-amplitude periodic solution that oscillates about the elements of $\left\{\xi_i, \xi_{i+1}, \ldots, \xi_j\right\}$ but not about the elements of $\{\xi_1, \xi_2, \ldots, \xi_{i-1}\} \cup \left\{\xi_{j+1}, \ldots, \xi_N\right\}$, see Fig. 2.2.

If $p : \mathbb{R} \to \mathbb{R}$ is a periodic solution with minimal period $\omega > 1$, one can consider the period map $\Phi(\omega, \cdot)$ and its derivative $M = D_2\Phi(\omega, p_0)$. M is called the monodromy operator. It is a compact operator, and 0 belongs to its spectrum $\sigma = \sigma(M)$. Eigenvalues of finite multiplicity – the so-called Floquet multipliers of the periodic orbit $\mathcal{O}_p = \{p_t : t \in [0, \omega)\}$ – form $\sigma(M) \setminus \{0\}$. It is known that 1 is a Floquet multiplier with eigenfunction $\dot{p}_0$. The periodic orbit $\mathcal{O}_p$ is said to be hyperbolic if the generalized eigenspace of M corresponding to the eigenvalue 1 is one-dimensional, furthermore there are no Floquet multipliers on the unit circle besides 1.

We know a lot about the dynamics from previous works of Krisztin, Walther and Wu in the case when $f'(u) > 0$ for all $u \in \mathbb{R}$. With the notation introduced above, consider the subset

$$C_i = \{\varphi \in C : \zeta_{i-1} \le \varphi(s) \le \zeta_i \text{ for all } s \in [-1, 0]\}, \qquad i \in \{1, \dots, N\}, \tag{2.1}$$

of the phase space C. Clearly, the equilibria $\hat{\zeta}_{i-1}, \hat{\xi}_i, \hat{\zeta}_i$ belong to C_i. The monotonicity of f implies that the set (2.1) is positively invariant under the solution semiflow Φ, see Proposition 3.1 of this paper. Krisztin, Walther and Wu have characterized the closure of the unstable set

$$\left\{\varphi \in C : x^\varphi \text{ exists on } \mathbb{R} \text{ and } x_t^\varphi \to \hat{\xi}_i \text{ as } t \to -\infty\right\}.$$

It has a so-called spindle-like structure: it contains $\hat{\zeta}_{i-1}, \hat{\xi}_i, \hat{\zeta}_i$, periodic orbits oscillating about ξ_i, and heteroclinic connections among them. In the simplest situation the periodic orbit is unique, and it oscillates slowly [12,13]. In other cases, the closure of the unstable set has a more complicated structure. For example, more periodic orbits appear via a series of Hopf-bifurcations in a small neighborhood of $\hat{\xi}_i$ as $f'(\xi_i)$ increases, see [14]. Under certain technical conditions, the closure of the unstable set of $\hat{\xi}_i$ is the global attractor of the restriction $\Phi|_{[0,\infty)\times C_i}$ [7,12]. For further details, see the paper [9], and the references therein.

The monograph [13] of Krisztin, Walther and Wu raised originally the question, whether the global attractor is the union of the global attractors $\mathcal{A}_i$ of the restrictions $\Phi|_{[0,\infty)\times C_i}$, $i \in \{1, \dots, N\}$. We already know from the previous paper [10] of Krisztin and Vas that this is not necessarily the case. In the $N = 2$ case there exists a strictly increasing feedback function f such that equation (1.1) has exactly two periodic orbits outside $\mathcal{A}_1 \cup \mathcal{A}_2$, and the unstable sets of them constitute the global attractor besides $\mathcal{A}_1 \cup \mathcal{A}_2$. These two periodic solutions have large amplitude; they oscillate slowly about ξ_1 and ξ_2. See paper [11] of Krisztin and Vas for the geometrical description of the unstable sets of these large-amplitude periodic orbits.

The purpose of this paper is to develop the result of [10] by investigating what type of large-amplitude periodic solutions may exist for the same nonlinearity f if the number of unstable equilibria is an arbitrary integer greater than 1.

Our main result can be formulated using parenthetical expressions. A pair of parentheses consists of a left parenthesis "(" and a right parenthesis ")", furthermore, "(" precedes ")" if read from left to right. A parenthetical expression of N numbers consists of the integers $1, 2, \dots, N$ and a finite (possibly zero) number of pairs of parentheses such that

- the integers $1, 2, \dots, N$ are used exactly once in increasing order,
- a pair of parenthesis encloses at least two numbers out of $1, 2, \dots, N$, e.g., the expressions (1) 23 or 1()23 are not allowed,
- multiple enclosing of the same sublist of numbers is not allowed, e.g., ((12))3 is not allowed,
- for any two pairs of parentheses, if the left parenthesis "(" of the first pair precedes the left parenthesis "(" of the second one, then the right parenthesis ")" of the second pair precedes the right parenthesis ")" of the first one.

For example, the parenthetical expressions of 3 numbers are

$$123,\ (12)3,\ 1(23),\ (123),\ ((12)3),\ (1(23)). \tag{2.2}$$

We emphasize that parentheses appear in pairs in a correct parenthetical expression, and it is definite which right parenthesis ")" belongs to a given left parenthesis "(".

By the result of Mallet-Paret and Sell, if the derivative of f is positive, $p^1 : \mathbb{R} \to \mathbb{R}$ and $p^2 : \mathbb{R} \to \mathbb{R}$ are periodic solutions of (1.1), and E_i is the set of fixed points of f_μ about which p^i oscillates for both $i \in \{1, 2\}$, then either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$ or $E_1 \cap E_2 = \emptyset$. This assertion is already true under hypotheses $(H0)$–$(H1)$. See Proposition 3.4 in Section 3 for a proof in the $\mu = 1$ case.

This property guarantees that we can assign a correct parenthetical expression of N numbers to each μ and f satisfying $(H0)$ and $(H1)$ if we use the following rule: for all $i < j$, the numbers $i, i+1, \ldots, j$ are enclosed by a pair of parentheses (not containing further numbers) if and only if (1.1) with this parameter μ and nonlinearity f admits at least one $[i, j]$ periodic solution.

The monotonicity of f is important here. In general we cannot guarantee that we can assign a correct parenthetical expression in the above explained way to each $\mu > 0$ and $f \in C^1(\mathbb{R}, \mathbb{R})$. For example, in case of four unstable equilibria, we cannot exclude that the equation has $[1, 3]$ and $[2, 4]$ periodic solutions for the same nonmonotone $f \in C^1(\mathbb{R}, \mathbb{R})$. Then we would get the incorrect expression $(_1 1 (_2 23)_1 4)_2$, where $(_1 123)_1$ corresponds to the $[1, 3]$ periodic solution, and $(_2 234)_2$ corresponds to the $[2, 4]$ periodic solution.

Tibor Krisztin has conjectured that the converse statement is true, that is, we can assign a configuration of large-amplitude periodic solutions to each parenthetical expression. The main result of the paper is the following.

Theorem 2.1. *Fix a parenthetical expression of N numbers, where $N \geq 2$. Then there exists μ and f satisfying $(H0)$–$(H2)$ such that the following assertions hold.*

(i) For this μ and f, there exist exactly N unstable equilibria

$$\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_N \qquad \text{with } \xi_1 < \xi_2 < \ldots < \xi_N.$$

For all $i, j \in \{1, \ldots, N\}$ with $i < j$, the equation (1.1) has an $[i, j]$ periodic solution if and only if there exists a pair of parentheses in the expression that contains only the numbers $i, i+1, \ldots, j$.

(ii) For any $i, j \in \{1, \ldots, N\}$ such that the numbers $i, i+1, \ldots, j$ are enclosed by a pair of parentheses (not containing further integers), at least one of the $[i, j]$ periodic solutions is slowly oscillatory. The corresponding periodic orbit is hyperbolic, with exactly one Floquet multiplier outside the unit circle, which is real, greater than 1 and simple.

Fig. 2.3 shows the configurations corresponding to $((1\,(23))\,(45))\,6$ and $((((12)\,3)\,4)\,5)\,6$: the images of the large-amplitude periodic orbits and unstable equilibria under the projection $\pi_2 : C \ni \varphi \mapsto (\varphi(0), \varphi(-1)) \in \mathbb{R}^2$.

In the proof of assertion (i) of Theorem 2.1, we explicitly construct a nondecreasing C^1-function f. This nonlinearity is close to a step function in the sense that it is constant on certain subintervals of the real line. Roughly speaking, we can control whether certain types of large-amplitude periodic orbits appear or not by setting the heights of the steps properly.

In general, determining the Floquet multipliers is an infinite dimensional problem. Our construction allows us to reduce this problem to a finite dimensional one. This is why we can prove Theorem 2.1.(ii).

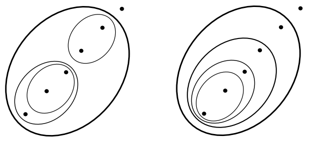

Fig. 2.3. Configurations corresponding to the expressions ((1 (23)) (45)) 6 and ((((12) 3) 4) 5) 6.

The hyperbolicity of the periodic orbits guarantees that Theorem 2.1 remains true for non-decreasing perturbations of the feedback function, see Theorem 8.2. In consequence, we can require f in Theorem 2.1 to be even strictly increasing.

This correspondence between the configurations of large-amplitude periodic solutions and the parenthetical expressions implies the following under hypotheses $(H0)$–$(H2)$. If we ignore the exact number of large-amplitude periodic solutions oscillating about the same given subsets of $\{\xi_1, \xi_2, \ldots, \xi_N\}$, the number of possible configurations for N unstable equilibria equals the number $\mathcal{C}_N$ of ways in which N numbers can be correctly parenthesized. One can check that C_N, $N \geq 2$, are the so so-called large Schröder numbers [1]. Applying a well-known combinatorial tool, generating functions, it can be calculated that

$$\mathcal{C}_N = -\frac{1}{2}\sum_{i=0}^{N}\binom{\frac{1}{2}}{i}\binom{\frac{1}{2}}{N-i}\left(-3-2\sqrt{2}\right)^i\left(-3+2\sqrt{2}\right)^{N-i}, \qquad N \geq 2. \tag{2.3}$$

By this formula, $C_2 = 2$, $C_3 = 6$, $C_4 = 22$, $C_5 = 90$, $C_6 = 394$ and $C_7 = 1806$. Numerical simulation shows that C_N grows geometrically.

It is an interesting problem to show the existence of unstable periodic orbits for delay equations by computer assisted proofs. Using a technique from [21], Szczelina has recently found numerical approximations of apparently unstable orbits in [20] for an equation of the form (1.1). Lessard and Kiss, applying a different approach developed in [16], have rigorously proven the coexistence of three periodic orbits for Wright's equation with two delays in [6], and at least one of them is presumed to be unstable. Although method of Lessard and Kiss can be applied to determine both stable and unstable periodic solutions, it is not suitable for the stability analysis of the obtained solutions.

The paper is organized as follows. For the sake of notational simplicity, we fix μ to be 1. In Section 3 we prove some simple results. The proof of Theorem 2.1.(i) is found in Sections 4–6. In Section 4 we consider feedback functions f for which $f(u) = K\,\mathrm{sgn}(u)$ if $|u| \geq 1$ and $f(u) \in [-K, K]$ if $u \in (-1, 1)$. We explicitly construct periodic solutions for such nonlinearities. Then we use these feedback functions as building blocks in Sections 5 and 6 to determine a nonlinearity satisfying assertion (i) of Theorem 2.1. For a first reading one may skip Section 4, only read Corollary 4.10 without proof, and then look at the construction in Sections 5–6. We give a brief introduction to Floquet theory and then verify Theorem 2.1.(ii) in Section 7. The proof of Theorem 2.1.(ii) cannot be read without knowing the details of Section 4. In Section 8 we explain why the statements of Theorem 2.1 remain true for small perturbations of the nonlinearity. We close the paper with discussing open questions in Section 9.

3. Preliminaries

We fix μ to be 1 in the rest of the paper and consider the equation

$$\dot{x}(t) = -x(t) + f(x(t-1)). \tag{3.1}$$

The results of the paper can be easily modified for other choices of μ as well.

It is natural to use the pointwise ordering on C. For $\varphi, \psi \in C$, we say that

- $\varphi \le \psi$ if $\varphi(s) \le \psi(s)$ for all $s \in [-1, 0]$,
- $\varphi \prec \psi$ if $\varphi \le \psi$ and $\varphi(0) < \psi(0)$.

Relations "$\ge$" and "$\succ$" are defined analogously. The semiflow induced by equation (3.1) is monotone if f is nondecreasing.

Proposition 3.1. *Assume $(H1)$. Let φ and ψ be elements of C with $\varphi \le \psi$ ($\varphi \prec \psi$). Then $x^{\varphi}(t) \le x^{\psi}(t)$ $\left(x^{\varphi}(t) < x^{\psi}(t)\right)$ for all $t \ge 0$.*

Proof. If $x^{\varphi} : [-1, \infty) \to \mathbb{R}$ is a solution of equation (3.1) with $x_0^{\varphi} = \varphi$, then x^{φ} can computed recursively on $[0, \infty)$ using the variation-of-constants formula:

$$x^{\varphi}(t) = x^{\varphi}(n) e^{-(t-n)} + \int_n^t e^{-(t-s)} f\left(x^{\varphi}(s-1)\right) \mathrm{d}s$$

for all nonnegative integers n and $t \in [n, n+1]$. The proposition follows from this formula. $\square$

The next two propositions have appeared in the paper [17] of Mallet-Paret and Sell for the case $f'(u) > 0$, $u \in \mathbb{R}$.

Proposition 3.2. *Assume that $(H1)$ holds, and $p : \mathbb{R} \to \mathbb{R}$ is a periodic solution of (3.1) with minimal period $\omega > 0$. Fix $t_0 < t_1 < t_0 + \omega$ so that $p(t_0) = \min_{t \in \mathbb{R}} p(t)$ and $p(t_1) = \max_{t \in \mathbb{R}} p(t)$. Then*

(i) p is of monotone type in the sense that p is nondecreasing on $[t_0, t_1]$ and nonincreasing on $[t_1, t_0 + \omega]$;

(ii) if p oscillates about a fixed point χ of f, then $p(t_0) < \chi < p(t_1)$.

Proof. Statement (i) is proven in [17] only if $f' > 0$. For the proof of statement (i) under hypothesis $(H1)$, see Proposition 5.1 in [10].

The proof of statement (ii) under $(H1)$. Note that as $\mu = 1$, $\hat{\chi}$ is an equilibrium. It is clear that $p(t_0) \le \chi \le p(t_1)$. If $p(t_0) = \chi$, then with $\varphi = \hat{\chi}$ and $\psi = p_{t_1}$ we have $\varphi \prec \psi$, and

$$\chi = x^{\varphi}(t) < x^{\psi}(t) = p(t + t_1) \qquad \text{for all } t \ge 0$$

by Proposition 3.1. This is impossible as p oscillates about χ. Similarly, $p(t_1) > \chi$. $\square$

Remark 3.3. It follows immediately that if $(H1)$ holds, and $p\colon \mathbb{R} \to \mathbb{R}$ is a periodic solution of (3.1) with minimal period $\omega \in (1, 2)$, then p is slowly oscillatory: On the one hand, Proposition 3.1 easily gives that for all fixed points χ of f in $p(\mathbb{R})$, the map $t \mapsto p(t) - \chi$ has at least one sign change on each interval of length 1. On the other hand, Proposition 3.2 implies that $t \mapsto p(t) - \chi$ has at most two sign changes on each interval of length ω, hence also on each interval of length 1.

For a simple closed curve $c : [a, b] \to \mathbb{R}^2$, let $\mathrm{int}\,(c\,[a, b])$ denote the interior, i.e., the bounded component of $\mathbb{R}^2 \setminus c\,([a, b])$.

Proposition 3.4. *Assume* $(H1)$.

(i) $\pi_2 : C \ni \varphi \mapsto (\varphi(0), \varphi(-1)) \in \mathbb{R}^2$ *maps nonconstant periodic orbits and equilibria of* (3.1) *into simple closed curves and points in* $\mathbb{R}^2$*, respectively. The images of different (nonconstant and constant) periodic orbits are disjoint in* $\mathbb{R}^2$.

(ii) A periodic solution $p : \mathbb{R} \to \mathbb{R}$ *of* (3.1) *with minimal period* $\omega > 0$ *oscillates about a fixed point* χ *of* f *if and only if* $\pi_2\hat{\chi} \in \mathrm{int}\left(\pi_2\mathcal{O}_p\right)$*, where* $\mathcal{O}_p = \{p_t : t \in [0, \omega]\}$.

(iii) In consequence, if $p^1 : \mathbb{R} \to \mathbb{R}$ *and* $p^2 : \mathbb{R} \to \mathbb{R}$ *are periodic solutions of equation* (3.1)*, and* E_i *is the set of fixed points of* f *about which* p^i *oscillates for both* $i \in \{1, 2\}$*, then either*

$$E_1 \subseteq E_2 \quad \textit{and} \quad p_1(\mathbb{R}) \subseteq p_2(\mathbb{R}),$$

or

$$E_2 \subseteq E_1 \quad \textit{and} \quad p_2(\mathbb{R}) \subseteq p_1(\mathbb{R}),$$

or $E_1 \cap E_2 = \emptyset$.

Proof. The paper [17] verifies (i) in the case $f' > 0$, while [10] gives a proof in the slightly more general case $f' \geq 0$. See Proposition 2.4 of [10].

In order to prove (ii), first assume that p oscillates about a fixed point χ of f. Let ω denote the minimal period of p. Set points $t_0 < t_1 < t_0 + \omega$ such that $p(t_0) = \min_{t\in\mathbb{R}} p(t)$ and $p(t_1) = \max_{t\in\mathbb{R}} p(t)$. Then $p(t_0) < \chi < p(t_1)$ by Proposition 3.2.(ii).

According to Proposition 3.2.(i), the set of zeros of $t \mapsto p(t) - \chi$ in (t_0, t_1) is an interval:

$$\{t \in (t_0, t_1) : p(t) = \chi\} = \left[z_{0,}, z_1\right]$$

with $t_0 < z_0 \leq z_1 < t_1$. One may also set z_2 and z_3 so that $[z_2, z_3] \subset (t_1, t_0 + \omega)$, $p(t) = \chi$ for $t \in \left[z_{2,}, z_3\right]$ and $p(t) \neq \chi$ for $t \in (t_1, t_0 + \omega) \setminus \left[z_{2,}, z_3\right]$. Of course, $z_0 = z_1$ or $z_2 = z_3$ is possible.

Consider the curve $\Gamma : [t_0, t_0 + \omega] \ni t \mapsto \pi_2 p_t \in \mathbb{R}^2$. By property (i), Γ is a simple closed curve, and $\Gamma(t) \neq \pi_2\hat{\chi} = (\chi, \chi)$ for $t \in [t_0, t_0 + \omega]$.

For $t \in (z_1, t_1]$, $p(t) > \chi$, $\dot{p}(t) \geq 0$, hence $f(p(t-1)) = \dot{p}(t) + p(t) > \chi$ and necessarily $p(t-1) > \chi$. We claim that $p(t-1) > \chi$ holds also for $t \in \left[z_{0,}, z_1\right]$. If not, then there exists $z^* \in \left[z_{0,}, z_1\right]$ so that $p(z^* - 1) = \chi$, which contradicts $\Gamma(z^*) \neq \pi_2\hat{\chi}$. Therefore

$$\Gamma(t) \in \left\{(u, v) \in \mathbb{R}^2 : u \geq \chi,\ v > \chi\right\} \text{ for } t \in [z_0, t_1].$$

It can be verified in a similar manner that $p(t-1) < \chi$ holds for $t \in [z_2, t_0 + \omega]$ and thus

$$\Gamma(t) \in \left\{(u, v) \in \mathbb{R}^2 : u \leq \chi,\ v < \chi\right\} \text{ for } t \in [z_2, t_0 + \omega].$$

Since Γ is a simple closed curve and there exists no $t \in [t_0, t_0 + \omega] \setminus ([z_0, z_1] \cup [z_2, z_3])$ such that $\Gamma(t)$ is in $\left\{(\chi, v) \in \mathbb{R}^2 : v \in \mathbb{R}\right\}$, we obtain that $\pi_2\hat{\chi} = (\chi, \chi) \in \mathrm{int}\,(\Gamma\,[t_0, t_0 + \omega])$.

The reverse statement is easy. If p does not oscillate about a fixed point χ of f, then $p(t) > \chi$ or $p(t) < \chi$ for all $t \in \mathbb{R}$, and

$$\Gamma(t) \in \left\{(u, v) \in \mathbb{R}^2 : u > \chi,\ v > \chi\right\} \text{ for all } t \in \mathbb{R}$$

or

$$\Gamma(t) \in \left\{(u, v) \in \mathbb{R}^2 : u < \chi,\ v < \chi\right\} \text{ for all } t \in \mathbb{R},$$

respectively. This means that $(\chi, \chi) \notin \operatorname{int}(\Gamma[t_0, t_0 + \omega])$.

Statement (iii) follows at once from (i) and (ii). □

4. Construction of a single periodic solution

Let $K > 1$. We define $\mathcal{F}(K)$ as the class of functions $f \in C^1(\mathbb{R}, \mathbb{R})$ with

- $f(u) \in [-K, K]$ for $u \in (-1, 1)$,
- $f(u) = K\operatorname{sgn}(u)$ for $|u| \geq 1$.

The elements of $\mathcal{F}(K)$ are not required to satisfy $(H1)$ or $(H2)$.

Proposition 4.1. *There exists a threshold number $K_0 > 1$ such that for all $K > K_0$ and $f \in \mathcal{F}(K)$, the equation*

$$\dot{x}(t) = -x(t) + f(x(t-1)) \tag{3.1}$$

has a periodic solution $p : \mathbb{R} \to \mathbb{R}$ with the following properties: The minimal period of p is in $(1, 2)$, $\max_{t\in\mathbb{R}} p(t) \in (1, K)$ and $\min_{t\in\mathbb{R}} p(t) \in (-K, -1)$.

We prove Proposition 4.1 by determining a suitable periodic solution explicitly. The paper [10] has already described two significantly different periodic solutions in the special case when $f \in \mathcal{F}(K)$ and $f(x) = 0$ for all $x \in [-1 + \varepsilon, 1 - \varepsilon]$ with some small $\varepsilon > 0$. Section 3.1 of [10] has determined the first periodic solution that we now denote by p_1. Section 3.2 of [10] has given the second one p_2. The construction below is a generalization of the one that has been published for p_2 in Section 3.2.

In paper [10], the initial functions of p_1 and p_2 were determined as fixed points of three-dimensional maps. Here we not only generalize but also simplify the calculations regarding p_2 because now we obtain the initial function of the periodic solution as the fixed point of a one-dimensional map. The construction of p_1 is indeed three-dimensional, and at this point we cannot extend it to all $f \in \mathcal{F}(K)$.

In the following we assume that $f \in \mathcal{F}(K)$, where $K > 1$.

Step 0. Preliminary observations

For both $i \in \{-K, K\}$, consider the map

$$\Phi_i : \mathbb{R} \times \mathbb{R} \ni \left(s, x^*\right) \mapsto i + \left(x^* - i\right) e^{-s} \in \mathbb{R}.$$

If $t_0 < t_1$, and x is a solution of equation (3.1) on $[t_0 - 1, \infty)$ with $x(t-1) \geq 1$ for all $t \in (t_0, t_1)$, then equation (3.1) reduces to the ordinary differential equation

$$\dot{x}(t) = -x(t) + K$$

on the interval (t_0, t_1), and thus

$$x(t) = \Phi_K(t - t_0, x(t_0)) \qquad \text{for all } t \in [t_0, t_1]. \tag{4.1}$$

Similarly, if $t_0 < t_1$, x is a solution of equation (3.1) on $[t_0 - 1, \infty)$, and $x(t-1) \leq -1$ for all $t \in (t_0, t_1)$, then

$$x(t) = \Phi_{-K}(t - t_0, x(t_0)) \qquad \text{for all } t \in [t_0, t_1]. \tag{4.2}$$

We say that a function $x : [t_0, t_1] \to \mathbb{R}$ is of type (K) (or $(-K)$) on $[t_0, t_1]$, if (4.1) (or (4.2)) holds.

If $x : [t_0 - 1, \infty) \to \mathbb{R}$ is a solution of equation (3.1), and x is type of (i) on $[t_0 - 1, t_1 - 1]$ with some $i \in \{-K, K\}$, then the equality

$$x(t) = x(t_0)e^{t_0 - t} + e^{-t} \int_{t_0}^{t} e^s f(\Phi_i(s - t_0, j))\, \mathrm{d}s \tag{4.3}$$

holds for all $t \in [t_0, t_1]$ with $j = x(t_0 - 1)$. This observation motivates the next definition. A function $x : [t_0, t_1] \to \mathbb{R}$ is of type (i, j) on $[t_0, t_1]$ with $i \in \{-K, K\}$ and $j \in \mathbb{R}$ if (4.3) holds for all $t \in [t_0, t_1]$.

Let T_1 denote the time needed by a function of type $(-K)$ to decrease from 1 to -1. As $K > 1$, T_1 is well-defined, and

$$T_1 = \ln \frac{K+1}{K-1}.$$

Then T_1 is the time needed by a function of type (K) to increase from -1 to 1. Set T_2 to be the time needed by a function of type (K) to increase from -1 to 0:

$$T_2 = \ln \frac{K+1}{K}.$$

As the reader will see from the rest of the section, we search for a periodic solution p that is of type (K) when it increases from -1 to 1, and of type $(-K)$ when it decreases from 1 to -1. Hence, if J is a subinterval of $\mathbb{R}$ mapped by p onto $[-1, 1]$, then the length of J is T_1, furthermore p is of type $(K, -1)$ or of type $(-K, 1)$ on $J + 1 = \{t + 1 : t \in J\}$.

Step 1. A C^1-submanifold of initial functions

We introduce a one-dimensional C^1-submanifold of the phase space C. This manifold will contain the initial segment of the periodic solution.

If K is large enough, then $U^1 = (0, 1 - T_1 - T_2)$ is a nontrivial open interval. For given $a \in U^1$, set $s_i = s_i(a)$, $i \in \{0, 1, 2\}$, and s_3 as

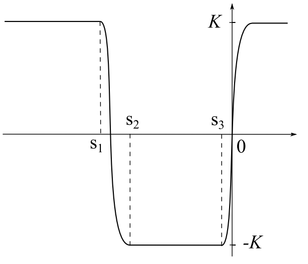

Fig. 4.1. The plot of $h(a)$.

$$\begin{aligned} s_0 &= -1, \\ s_1 &= s_0 + a = -1 + a, \\ s_2 &= s_1 + T_1 = -1 + a + T_1, \\ s_3 &= -T_2. \end{aligned}$$

The definitions of U^1, T_1 and T_2 imply that

$$-1 = s_0 < s_1 < s_2 < s_3 < 0.$$

For all $a \in U^1$, define the function $h(a) \in C^1(\mathbb{R}, \mathbb{R})$ by

$$h(a)(t) = \begin{cases} K, & \text{if } t < s_1, \\ f(\Phi_{-K}(t - s_1, 1)), & \text{if } s_1 \leq t < s_2, \\ -K, & \text{if } s_2 \leq t < s_3, \\ f(\Phi_K(t - s_3, -1)), & \text{if } s_3 \leq t. \end{cases}$$

See Fig. 4.1 for the plot of $h(a)$. Then define the map $\Sigma : U^1 \to C$ by

$$\Sigma(a)(t) = e^{-t} \int_{-1}^{t} e^s h(a)(s)\, ds \qquad \text{for all } -1 \leq t \leq 0. \tag{4.4}$$

It is clear that Σ is continuous on U^1 because $U^1 \ni a \mapsto h(a) \in C(\mathbb{R}, \mathbb{R})$ is continuous. Notice that $\Sigma(a)$ is the unique solution of the initial value problem

$$\begin{cases} \dot{y}(t) = -y(t) + h(a)(t), & -1 \leq t \leq 0, \\ y(-1) = 0. \end{cases} \tag{4.5}$$

The next characterization of $\Sigma\left(U^1\right)$ reveals the idea behind the above definitions. See also Fig. 4.2 for the plot of a typical element of $\Sigma\left(U^1\right)$.

Remark 4.2. A function $\varphi \in C$ belongs to $\Sigma\left(U^1\right)$ if and only if there exists $s_1 \in (-1, -T_1 - T_2)$ so that with $s_2 = s_1 + T_1$ and $s_3 = -T_2$,

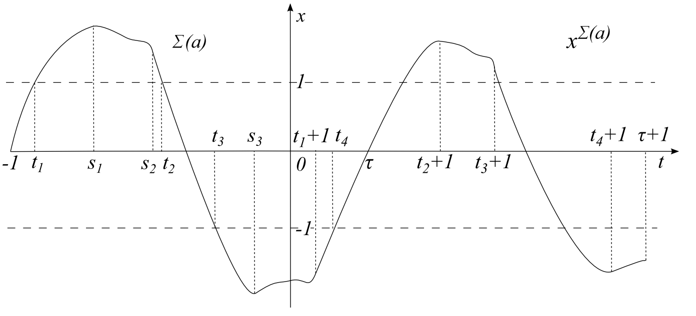

Fig. 4.2. The plot of an element of $\Sigma\left(U^1\right)$ and of the corresponding solution.

(i) $\varphi(-1) = 0$,
(ii) φ is of type (K) on $[-1, s_1]$,
(iii) φ is of type $(-K, 1)$ on $[s_1, s_2]$,
(iv) φ is of type $(-K)$ on $[s_2, s_3]$,
(v) φ is of type $(K, -1)$ on $[s_3, 0]$.

We need to examine the smoothness of Σ. For each fixed $a \in U^1$, the map $\mathbb{R} \ni t \mapsto h(a)(t) \in \mathbb{R}$ is C^1-smooth with derivative $h'(a)$. Fix $t^* \in (s_2, s_3)$. If $a \in U^1$ and $|\delta|$ is small enough, then

$$h(a+\delta)(t) = \begin{cases} h(a)(t-\delta) & \text{if } t \leq t^*, \\ h(a)(t) & \text{if } t > t^*. \end{cases}$$

It follows that

$$\frac{\partial}{\partial a} h(a)(t) = \begin{cases} -h'(a)(t) & \text{if } t \in \left[-1, t^*\right], \\ 0 & \text{if } t \in \left(t^*, 0\right]. \end{cases}$$

Define the nontrivial element $\psi = \psi(a) \in C$ by

$$\psi(t) = e^{-t} \int_{-1}^{t} e^s \frac{\partial}{\partial a} h(a)(s)\, \mathrm{d}s \qquad \text{for all } t \in [-1, 0].$$

Proposition 4.3. *The map $U^1 \ni a \mapsto \Sigma(a) \in C$ is C^1-smooth with $D\Sigma(a)1 = \psi$ for all $a \in U^1$.*

Proof. $\Sigma(a)$ is the unique solution of the initial value problem (4.5). Hence the proposition follows from the differentiability of the solutions of ordinary differential equations with respect to the parameters. □

It follows that $\Sigma\left(U^1\right)$ is a one-dimensional C^1-submanifold of C. We look for a periodic solution with initial segment in $\Sigma\left(U^1\right)$.

We are going to need the exact values of $\Sigma(a)$ at $s_i = s_i(a)$, $i \in \{1, 2, 3\}$, and at 0 for all $a \in U^1$. Let

$$c_1 = \int_0^{T_1} e^u f\left(\Phi_{-K}(u,1)\right) \mathrm{d}u.$$

Note that c_1 is independent of a. Then using the definitions of Σ and h, we deduce that

$$\Sigma(a)(s_1) = e^{-s_1} \int_{-1}^{s_1} K e^s \mathrm{d}s = K\left(1 - e^{-a}\right), \tag{4.6}$$

$$\begin{aligned}
\Sigma(a)(s_2) &= e^{-s_2} \int_{-1}^{s_2} e^s h(a)(s)\, \mathrm{d}s \\
&= e^{s_1 - s_2} \Sigma(a)(s_1) + e^{-s_2} \int_{s_1}^{s_2} e^s f\left(\Phi_{-K}(s - s_1, 1)\right) \mathrm{d}s \\
&= e^{-T_1}\left(\Sigma(a)(s_1) + c_1\right) \\
&= \frac{K-1}{K+1}\left(K\left(1 - e^{-a}\right) + c_1\right),
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\Sigma(a)(s_3) &= e^{-s_3} \int_{-1}^{s_3} e^s h(a)(s)\, \mathrm{d}s \\
&= e^{s_2 - s_3} \Sigma(a)(s_2) + e^{-s_3} \int_{s_2}^{s_3} (-K) e^s \mathrm{d}s \\
&= e^{-1+a+T_1+T_2}\left(\Sigma(a)(s_2) + K\right) - K \\
&= e^{-1+a}(K+1)\left(1 + \frac{c_1}{K} + \frac{K+1}{K-1}\right) - e^{-1}(K+1) - K
\end{aligned} \tag{4.8}$$

and

$$\Sigma(a)(0) = \int_{-1}^{0} e^s h(a)(s)\, \mathrm{d}s = e^{s_3} \Sigma(a)(s_3) + \int_{s_3}^{0} e^s f\left(\Phi_K(s - s_3, -1)\right) \mathrm{d}s. \tag{4.9}$$

We see that $\Sigma(a)(s_i)$, $i \in \{1, 2, 3\}$, and $\Sigma(a)(0)$ are continuously differentiable functions of $a \in U^1$.

Step 2. Construction of a one-dimensional return map

Let

$$U^2 = \left\{ a \in U^1 : \Sigma(a)(s) > 1 \text{ for } s \in [s_1, s_2] \text{ and } \Sigma(a)(s) < -1 \text{ for } s \in [s_3, 0] \right\}.$$

It is easy to see from Proposition 4.3 that U^2 is an open subset of U^1. Later we shall see that U^2 is nonempty if K is large enough.

For $a \in U^2$, there exist

$$-1 < t_1 < s_1 < s_2 < t_2 < t_3 < s_3$$

such that

$$\Sigma(a)(t_1) = \Sigma(a)(t_2) = 1 \qquad \text{and} \qquad \Sigma(a)(t_3) = -1,$$

see Fig. 4.2. As $\Sigma(a)$ is of type (K) on $[-1, s_1]$ and of type $(-K)$ on $[s_2, s_3]$, it is strictly monotone on these intervals. Hence t_1, t_2 and t_3 are unique. For t_1 we have

$$e^{-t_1} \int_{-1}^{t_1} K e^s \mathrm{d}s = 1, \quad \text{and thus } t_1 = -1 + \ln \frac{K}{K-1}. \tag{4.10}$$

Similarly,

$$1 = e^{-t_2} \int_{-1}^{t_2} e^s h(a)(s)\,\mathrm{d}s = e^{s_2 - t_2} \Sigma(a)(s_2) - K e^{-t_2} \int_{s_2}^{t_2} e^s \mathrm{d}s$$

and

$$-1 = e^{-t_3} \int_{-1}^{t_3} e^s h(a)(s)\,\mathrm{d}s = e^{s_2 - t_3} \Sigma(a)(s_2) - K e^{-t_3} \int_{s_2}^{t_3} e^s \mathrm{d}s,$$

from which

$$t_2 = s_2 + \ln \frac{K + \Sigma(a)(s_2)}{K+1} \qquad \text{and} \qquad t_3 = s_2 + \ln \frac{K + \Sigma(a)(s_2)}{K-1} \tag{4.11}$$

follows. Note that $t_3 - t_2 = T_1$ and t_2, t_3 are C^1-smooth functions of a.

Let us introduce the notation

$$c_2 = \int_0^{T_1} e^u f(\Phi_K(u, -1))\,\mathrm{d}u.$$

For $a \in U^2$, consider the solution $x = x^{\Sigma(a)}\colon [-1, \infty) \to \mathbb{R}$ of equation (3.1). We need the following result before defining a further open subset of U^1.

Proposition 4.4. *(i) The maps*

$$U^2 \ni a \mapsto x^{\Sigma(a)}(t_1+1) = e^{-T_1}\Sigma(a)(s_3) + e^{-T_1}c_2 \in \mathbb{R}$$

and

$$U^2 \ni a \mapsto x^{\Sigma(a)}(t_2+1) = K + \frac{K}{K+\Sigma(a)(s_2)}\left(x^{\Sigma(a)}(t_1+1) - K\right)e^{-a} \in \mathbb{R}$$

are continuously differentiable.

(ii) The map

$$U^2 \ni a \mapsto x^{\Sigma(a)}|_{[0,t_1+1]} \in C([0,t_1+1],\mathbb{R})$$

is continuous.

Proof. Statement (i). As T_1 and c_2 are independent of a, $K+\Sigma(a)(s_2) > 0$ and $\Sigma(a)(s_2)$ and $\Sigma(a)(s_3)$ are C^1-smooth functions on U^2, one has to show only that the stated equalities indeed hold. As $\Sigma(a)(-1) = 0$ and $\Sigma(a)$ is of type (K) on $[-1,t_1]$ (see Remark 4.2), $x = x^{\Sigma(a)}$ is of type $(K,0)$ on $[0,t_1+1]$. By (4.3) and (4.9),

$$\begin{aligned} x(t) &= x(0)e^{-t} + e^{-t}\int_0^t e^s f(\Phi_K(s,0))\,\mathrm{d}s \\ &= e^{s_3-t}\Sigma(a)(s_3) + e^{-t}\int_{s_3}^0 e^s f(\Phi_K(s-s_3,-1))\,\mathrm{d}s + e^{-t}\int_0^t e^s f(\Phi_K(s,0))\,\mathrm{d}s \end{aligned}$$

for all $t \in [0,t_1+1]$. It follows immediately from the definition of Φ_K and from $s_3 = -T_2 = \ln(K/(K+1))$ that

$$\Phi_K(s-s_3,-1) = \Phi_K(s,0) \qquad \text{for all } s \in \mathbb{R}.$$

Therefore

$$\begin{aligned} x(t) &= e^{s_3-t}\Sigma(a)(s_3) + e^{-t}\int_{s_3}^t e^s f(\Phi_K(s-s_3,-1))\,\mathrm{d}s \\ &= e^{s_3-t}\Sigma(a)(s_3) + e^{s_3-t}\int_0^{t-s_3} e^u f(\Phi_K(u,-1))\,\mathrm{d}u, \qquad t \in [0,t_1+1]. \end{aligned} \tag{4.12}$$

We see from the definition of s_3 and (4.10) that

$$t_1 + 1 - s_3 = \ln\frac{K}{K-1} - \ln\frac{K}{K+1} = T_1.$$

Hence (4.12) with $t = t_1+1$ gives the formula for $x(t_1+1)$.

By Remark 4.2 and the definition of U^2, $\Sigma(a)$ strictly increases on $[-1, s_1]$, $\Sigma(a)(t) > 1$ for all $t \in [s_1, s_2]$, and $\Sigma(a)$ strictly decreases on $[s_2, s_3]$. It follows that $\Sigma(a)(t) > 1$ for all $t \in (t_1, t_2)$, hence x is of type (K) on the interval $[t_1 + 1, t_2 + 1]$, and thus

$$x(t_2 + 1) = K + (x(t_1 + 1) - K)\, e^{t_1 - t_2}.$$

By (4.10) and (4.11) and the definition of s_2,

$$t_1 - t_2 = \ln \frac{K}{K + \Sigma(a)(s_2)} - a.$$

We obtain that the formula for $x(t_2 + 1)$ indeed holds.

Statement (ii). We see from (4.12) that for all $a_1 \in U^2$ and $a_2 \in U^2$,

$$\begin{aligned}\max_{t\in[0,t_1+1]} \left| x^{\Sigma(a_1)}(t) - x^{\Sigma(a_2)}(t) \right| &= \max_{t\in[0,t_1+1]} e^{s_3 - t} \left|\Sigma(a_1)(s_3) - \Sigma(a_2)(s_3)\right| \\ &\le e^{s_3} \left|\Sigma(a_1)(s_3) - \Sigma(a_2)(s_3)\right|.\end{aligned}$$

Statement (ii) hence follows from the continuity of $U^2 \ni a \mapsto \Sigma(a)(s_3) \in \mathbb{R}$. $\square$

Now let

$$U^3 = \left\{ a \in U^2 : \ x^{\Sigma(a)}(t) < -1 \text{ for all } t \in [0, t_1 + 1] \text{ and } x^{\Sigma(a)}(t_2 + 1) > 0 \right\}.$$

From Proposition 4.4 it is clear that U^3 is an open subset of $\mathbb{R}$. Later we shall see that U^3 is nonempty.

Fig. 4.2 shows an element of $\Sigma\left(U^3\right)$.

Remark 4.5. Observe that the elements of $\Sigma\left(U^3\right)$ can be characterized as follows. A function $\varphi \in C$ belongs to $\Sigma\left(U^3\right)$ if and only if there exists $s_1 \in (-1, -T_1 - T_2)$ so that with $s_2 = s_1 + T_1$ and $s_3 = -T_2$, properties (i)–(v) of Remark 4.2 hold, furthermore

(vi) $\varphi(t) > 1$ for all $t \in [s_1, s_2]$,
(vii) if $-1 < t_1 < s_1$ with $\varphi(t_1) = 1$, then $x^{\varphi}(t) < -1$ for all $t \in [s_3, t_1 + 1]$,
(viii) if $s_2 < t_2 < s_3$ with $\varphi(t_2) = 1$, then $x^{\varphi}(t_2 + 1) > 0$.

For $a \in U^3$, $x = x^{\Sigma(a)}$ is of type (K) on $[t_1 + 1, t_2 + 1]$, hence it is strictly increasing on $[t_1 + 1, t_2 + 1]$. So there exists unique t_4 and τ with $t_1 + 1 < t_4 < \tau < t_2 + 1$ such that $x(t_4) = -1$ and $x(\tau) = 0$, see Fig. 4.2.

As $\Sigma(a)$ strictly decreases on $[t_3, s_3]$, $x(t) < -1$ for all $t \in [s_3, t_1 + 1]$ by Remark 4.5, and x strictly increases on $[t_1 + 1, t_2 + 1]$, we deduce that

$$x(t) < -1 \text{ for } t \in (t_3, t_4) \quad \text{and} \quad x(t) \in (-1, 0) \text{ for } t \in (t_4, \tau). \tag{4.13}$$

Proposition 4.6. *The map*

$$U^3 \ni a \mapsto \tau = \ln \frac{K - x(t_1 + 1)}{K - 1} \in (0, 1) \tag{4.14}$$

is continuously differentiable.

Proof. As x is of type (K) on the interval $[t_1+1, t_2+1]$, we have

$$0 = x(\tau) = K + (x(t_1+1) - K)\, e^{t_1+1-\tau},$$

from which the formula easily follows with the aid of (4.10). It is clear that $\tau \in (0,1)$ because $\tau \in (t_1+1, t_2+1) \subset (0,1)$. The smoothness of τ is a consequence of the smoothness of $x(t_1+1)$. □

Similarly,

$$-1 = x(t_4) = K + (x(t_1+1) - K)\, e^{t_1+1-t_4}$$

and (4.10) together yield that

$$t_4 = \ln \frac{K(K - x(t_1+1))}{K^2-1}. \tag{4.15}$$

As the next result shows, solutions with initial functions in $\Sigma\left(U^3\right)$ return to $\Sigma\left(U^1\right)$.

Proposition 4.7. *Suppose $a \in U^3$ and define t_2 and τ as above. Then $x_{\tau+1} \in \Sigma\left(U^1\right)$ and $x_{\tau+1} = \Sigma(t_2+1-\tau)$.*

Proof. It is clear from the above construction (to be more precise, from the definitions of τ, t_2, t_3, t_4, the fact that x is of type (K) on $[t_1+1, t_2+1]$, property (iv) of Remark 4.2 and the observation (4.13)) that

(i) $x(\tau) = 0$,
(ii) x is of type (K) on $[\tau, t_2+1]$,
(iii) x is of type $(-K, 1)$ on $[t_2+1, t_3+1]$,
(iv) x is of type $(-K)$ on $[t_3+1, t_4+1]$,
(v) x is of type $(K, -1)$ on $[t_4+1, \tau+1]$.

So by Remark 4.2, it suffices to show that

(a) $\hat{s}_1 := (t_2+1) - (\tau+1) = t_2 - \tau$ is in $(-1, -T_1 - T_2)$,
(b) $\hat{s}_2 := (t_3+1) - (\tau+1) = t_3 - \tau$ equals $\hat{s}_1 + T_1$,
(c) $\hat{s}_3 := (t_4+1) - (\tau+1) = t_4 - \tau$ equals $-T_2$.

Property (c) comes from (4.14) and (4.15). By the definition of $\hat{s}_1$, property (b) is equivalent to $t_3 = t_2 + T_1$, which follows from (4.11). It is clear that $\hat{s}_1 > -1$. Hence (a) comes from $\hat{s}_1 = \hat{s}_2 - T_1 < \hat{s}_3 - T_1 = -T_2 - T_1$. □

The above results motivate us to define the map

$$F : U^3 \to \mathbb{R} \text{ by } F(a) = t_2 + 1 - \tau.$$

The next proposition is an immediate consequence of the smoothness of t_2 and τ as functions of a.

Proposition 4.8. *F is C^1-smooth.*

Note that if $a \in U^3$ and $F(a) = a$, then $x_{\tau+1}^{\Sigma(a)} = \Sigma(a)$, and $x^{\Sigma(a)}$ is a periodic solution of equation (3.1) with minimal period $\tau + 1$.

Step 3. The map F has a unique fixed point

A trivial upper bound for the absolute values of c_1 and c_2 is the following:

$$|c_1|, |c_2| \leq K \int_0^{T_1} e^u \mathrm{d}u = \frac{2K}{K-1}. \tag{4.16}$$

If $K > 1$ is fixed, then c_1 and c_2 are uniformly bounded for all $f \in \mathcal{F}(K)$.

We will use a further technical result which holds for more general feedback functions.

Proposition 4.9. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, $K_1 \in \mathbb{R}$, $K_2 \in \mathbb{R}$, $f(u) \in [K_1, K_2]$ for all $u \in \mathbb{R}$, $t_0 \in \mathbb{R}$, and $x : [t_0 - 1, \infty) \to \mathbb{R}$ is a solution of (3.1) with $x(t_0) \in (K_1, K_2)$. Then $x(t) \in (K_1, K_2)$ for all $t \geq t_0$.*

Proof. We prove the upper bound for x. Let $y : \mathbb{R} \to \mathbb{R}$ be the solution of the initial value problem

$$\begin{cases} \dot{y}(t) = -y(t) + K_2, & t \in \mathbb{R}, \\ y(t_0) = x(t_0). \end{cases}$$

Then $y(t) = K_2 + (x(t_0) - K_2) e^{t_0 - t} < K_2$ for $t \in \mathbb{R}$. We know that $\dot{x}(t) \leq -x(t) + K_2$ for all $t \in \mathbb{R}$. Theorem 6.1 of Chapter I.6 in [4] hence implies that for $t \geq t_0$, $x(t) \leq y(t) < K_2$.

The lower bound can be verified analogously. □

Proof of Proposition 4.1. We show that if $K > 1$ is large enough and $f \in \mathcal{F}(K)$, then the map F has a unique fixed point in U^3, namely there exists a unique $a \in U^3$ such that

$$t_2 + 1 - \tau = a. \tag{4.17}$$

Substituting (4.11), (4.14) and then the definitions of s_2 and T_1 into equation (4.17), we obtain that (4.17) is equivalent to $\Sigma(a)(s_2) = -x(t_1 + 1)$. Then using (4.7), (4.8), the formula for $x(t_1 + 1)$ in Proposition 4.4 and again the definition of T_1, we see that $\Sigma(a)(s_2) = -x(t_1 + 1)$ is an equation of second order in e^a: it can be written in the form

$$\alpha z^2 + \beta z + \gamma = 0, \tag{4.18}$$

where $z = e^a$, the coefficients α, β, γ are independent of a, and they are defined as

$$\alpha = e^{-1}\left(\frac{2K}{K-1} + \frac{c_1}{K}\right),$$
$$\beta = \frac{c_1 + c_2}{K+1} - e^{-1},$$
$$\gamma = \frac{-K}{K+1}.$$

Observe that $\alpha > 0$ for all $K > 1$ because of (4.16). As $\gamma < 0$, it is clear that $\sqrt{\beta^2 - 4\alpha\gamma} > |\beta|$. This means that

$$z = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

is a negative solution of (4.18). We conclude that for all $K > 1$ and $f \in \mathcal{F}(K)$, the map F has at most one fixed point a^* in U^3, and it is given by

$$a^* = \ln \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{4.19}$$

It remains to show that if K is chosen sufficiently large, then a^* determined by (4.19) is indeed in U^3 for all $f \in \mathcal{F}(K)$, that is, with the notation used before,

(i) $a^* \in (0, 1 - T_1 - T_2)$,

(ii) $\Sigma\left(a^*\right)(t) > 1$ for $t \in [s_1, s_2]$,

(iii) $x^{\Sigma(a^*)}(t) < -1$ for all $t \in [s_3, t_1 + 1]$,

(iv) $x^{\Sigma(a^*)}(t_2 + 1) > 0$.

Property (i). Applying the bound (4.16) for $|c_1|$ and $|c_1|$, we see that

$$\lim_{K\to\infty} \sup_{f\in\mathcal{F}(K)} \left|\alpha - 2e^{-1}\right| = 0, \quad \lim_{K\to\infty} \sup_{f\in\mathcal{F}(K)} \left|\beta + e^{-1}\right| = 0, \quad \lim_{K\to\infty} \sup_{f\in\mathcal{F}(K)} |\gamma + 1| = 0,$$

and thus

$$\lim_{K\to\infty} \sup_{f\in\mathcal{F}(K)} \left|a^* - \ln \frac{1 + \sqrt{1+8e}}{4}\right| = 0. \tag{4.20}$$

As $\lim_{K\to\infty}(1 - T_1 - T_2) = 1$, property (i) immediately follows for all large K and for all $f \in \mathcal{F}(K)$.

Property (ii). By the definition of Σ and formula (4.6),

$$\begin{aligned}
\Sigma\left(a^*\right)(t) &= e^{-t} \int_{-1}^{t} e^s h(a)(s)\,\mathrm{d}s \\
&= e^{s_1 - t}\Sigma\left(a^*\right)(s_1) + e^{-t}\int_{s_1}^{t} e^s f\left(\Phi_{-K}(s - s_1, 1)\right)\mathrm{d}s \\
&= e^{s_1 - t} K\left(1 - e^{-a^*}\right) + e^{s_1 - t}\int_0^{t - s_1} e^u f\left(\Phi_{-K}(u, 1)\right)\mathrm{d}u
\end{aligned}$$

for all $t \in [s_1, s_2]$. Hence

$$\Sigma\left(a^*\right)(t) \geq e^{s_1 - s_2} K\left(1 - e^{-a^*}\right) - e^{s_1 - s_1}|c_1| = \frac{K-1}{K+1} K\left(1 - e^{-a^*}\right) - |c_1|$$

for all $t \in [s_1, s_2]$. Here we used that $s_2 - s_1 = T_1$. As $|c_1|$ is bounded for $K > 2$, and $1 - e^{-a^*}$ has a positive limit as $K \to \infty$, we see that (ii) is satisfied for all $f \in \mathcal{F}(K)$ if K is large enough.

Property (iii). The definition of Σ gives that for $t \in [s_3, 0]$,

$$\Sigma\left(a^*\right)(t) = e^{s_3 - t}\Sigma\left(a^*\right)(s_3) + e^{s_3 - t}\int_0^{t - s_3} e^u f\left(\Phi_K(u, -1)\right) \mathrm{d}u. \tag{4.21}$$

We see from (4.12) that (4.21) actually holds for all $t \in [s_3, t_1 + 1]$. Regarding the value $\Sigma(a^*)(s_3)$, observe that (4.8), the limit of a^* in (4.20), and the bound for $|c_1|$ together yield that

$$\lim_{K\to\infty} \frac{\Sigma(a^*)(s_3)}{\left(\frac{\sqrt{1+8e}-1}{2e} - 1\right)K} = 1$$

uniformly for $f \in \mathcal{F}(K)$. As the denominator in the above fraction is negative, $\Sigma(a^*)(s_3) < 0$ if K is large enough, and it tends to $-\infty$ as $K \to \infty$.

By using formula (4.21), $\Sigma(a^*)(s_3) < 0$ and $t_1 + 1 - s_3 = T_1$, we now obtain the upper bound

$$\Sigma\left(a^*\right)(t) \le \frac{K-1}{K+1}\Sigma\left(a^*\right)(s_3) + |c_2| \quad \text{for all } t \in [s_3, t_1 + 1].$$

As $\Sigma(a^*)(s_3)$ tends to $-\infty$, and c_2 is bounded if $K > 2$, property (iii) also holds for all $\mathcal{F}(K)$ if K is chosen sufficiently large.

Property (iv). Recall the formula given by Proposition 4.4 for $x^{\Sigma(a^*)}(t_2 + 1)$. With the equality $\Sigma(a^*)(s_2) = -x^{\Sigma(a^*)}(t_1 + 1)$ confirmed at the beginning of this proof, we derive that

$$x^{\Sigma(a^*)}(t_2 + 1) = K - Ke^{-a} = \Sigma\left(a^*\right)(s_1),$$

and hence (iv) follows from (ii).

Define $p : \mathbb{R} \to \mathbb{R}$ as the $(\tau + 1)$-periodic extension of $x^{\Sigma(a^*)}|_{[-1,\tau]}$ to $\mathbb{R}$. Then it is clear from the construction that p is a solution of (3.1), the minimal period of p is $\tau + 1 \in (1, 2)$, $\max_{t\in\mathbb{R}} p(t) > 1$ and $\min_{t\in\mathbb{R}} p(t) < -1$. It follows from Proposition 4.9 that $p(t) \in (-K, K)$ for all real t. $\square$

Extension of the result

An analogous result holds for a wider class of feedback functions. Consider any C^1-nonlinearity defined on a finite closed subinterval of the real line. Next we prove that we can extend this function to the real line such that a new periodic solution appears. The range of this periodic solution contains the original finite interval.

Corollary 4.10. *Let $\eta > 0$, $A_1, A_2, B_1, B_2 \in \mathbb{R}$ with $A_1 < A_2$, $B_1 < B_2$ and $2\eta < A_2 - A_1$. Assume that $\hat{f} \in C^1\left(\left[A_1 + \eta, A_2 - \eta\right], \mathbb{R}\right)$ is given, and*

$$B_1 \le \hat{f}(u) \le B_2 \qquad \text{for all } A_1 + \eta \le u \le A_2 - \eta.$$

Consider the threshold number $K_0 > 1$ from Proposition 4.1. Let

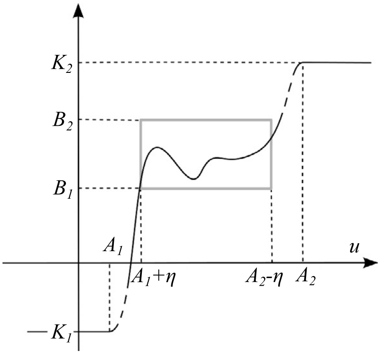

Fig. 4.3. The plot of f in Corollary 4.10.

$$K_1 < \min\left\{\frac{-(A_2 - A_1)K_0 + A_1 + A_2}{2},\ A_1,\ B_1,\ A_1 + A_2 - B_2\right\} \tag{4.22}$$

and $K_2 = A_1 + A_2 - K_1$. Let f be a C^1-extension of $\hat{f}$ to the real line with

$$f(u) = K_1 \quad \text{for } u \le A_1, \qquad f(u) = K_2 \quad \text{for } u \ge A_2,$$

and

$$f(u) \in [K_1, K_2] \quad \text{for } u \in [A_1, A_2].$$

Then equation (3.1) with nonlinearity f has a periodic solution $p : \mathbb{R} \to \mathbb{R}$ such that
(i) the minimal period of p is in $(1, 2)$,
(ii) $\max_{t\in\mathbb{R}} p(t) \in (A_2, K_2)$ and $\min_{t\in\mathbb{R}} p(t) \in (K_1, A_1)$.

See Fig. 4.3 for a plot of f in the corollary.

Proof. First note that if (4.22) holds, then

$$K_2 = A_1 + A_2 - K_1 > \max\{A_2, B_2\}.$$

This observation with (4.22) means that the intervals (K_1, A_1) and (A_2, K_2) in assertion (ii) are indeed both nontrivial, furthermore, $K_1 < B_1 < B_2 < K_2$, so it is possible to choose C^1-extensions f of $\hat{f}$ such that $f(u) \in [K_1, K_2]$ for all $u \in [A_1, A_2]$.

Assume that f is a C^1-function given as in the proposition.

Consider the linear transformation L of $\mathbb{R}$ that maps A_1 to -1 and A_2 to 1:

$$L(u) = \frac{2u - A_1 - A_2}{A_2 - A_1} \qquad \text{for } u \in \mathbb{R}.$$

Let L^{-1} denote the inverse linear transformation. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(u) = Lf\left(L^{-1}u\right)$ for $u \in \mathbb{R}$. Then $g \in C^1(\mathbb{R}, \mathbb{R})$. As the above introduced linear transformations are order preserving, we calculate that

$$g(u) = \frac{2K_1 - A_1 - A_2}{A_2 - A_1} \quad \text{for all } u \le L(A_1) = -1,$$

$$g(u) = \frac{2K_2 - A_1 - A_2}{A_2 - A_1} = -\frac{2K_1 - A_1 - A_2}{A_2 - A_1} \quad \text{for all } u \ge L(A_2) = 1,$$

and

$$\frac{2K_1 - A_1 - A_2}{A_2 - A_1} \le g(u) \le \frac{2K_2 - A_1 - A_2}{A_2 - A_1} \quad \text{for all } u \in (-1, 1).$$

We conclude that $g \in \mathcal{F}(K)$ with

$$K = \frac{A_1 + A_2 - 2K_1}{A_2 - A_1}. \tag{4.23}$$

The assumption (4.22) guarantees that $K > K_0$. By Proposition 4.1, the equation $\dot{y}(t) = -y(t) + g(y(t-1))$ has a periodic solution $q : \mathbb{R} \to \mathbb{R}$. The minimal period of q is in $(1, 2)$, furthermore,

$$\max_{t\in\mathbb{R}} q(t) \in (1, K) \qquad \text{and} \qquad \min_{t\in\mathbb{R}} q(t) \in (-K, -1).$$

Define the periodic function $p : \mathbb{R} \to \mathbb{R}$ by $p(t) = L^{-1}q(t)$ for all $t \in \mathbb{R}$. Substituting p into equation (3.1), one can see that p is a solution of (3.1) with the above chosen nonlinearity f. It is clear that p has the desired properties. $\square$

Note that the bounds B_1 and B_2 in the previous corollary are not necessarily strict bounds for $\hat{f}$.

5. Further auxiliary results

5.1. Two technical results

Set $\mu = 1$ as before, and consider equation (3.1). The first proposition in this section studies the ranges of the large-amplitude periodic solutions.

Proposition 5.1. *Suppose that $(H1)$ holds, and f has exactly $2N + 1$ fixed points*

$$\zeta_0 < \xi_1 < \zeta_1 < \xi_2 < \zeta_2 \ldots < \xi_N < \zeta_N$$

with $N \ge 2$, $f'(\zeta_i) < 1$ for all $i \in \{0, 1, \ldots, N\}$ and $f'(\xi_i) > 1$ for all $i \in \{1, \ldots, N\}$. Assume that $p : \mathbb{R} \to \mathbb{R}$ is an $[i, j]$ periodic solution of equation (3.1) with some integers $1 \le i < j \le N$, namely p oscillates about the elements of $\{\xi_i, \xi_{i+1}, \ldots, \xi_j\}$ but not about the elements of $\{\xi_1, \xi_2, \ldots, \xi_{i-1}\} \cup \{\xi_{j+1}, \ldots, \xi_N\}$. Then

$$\zeta_{i-1} < p(t) < \zeta_j \qquad \text{for all } t \in \mathbb{R}.$$

Proof. 1. Set $t_{min} \in \mathbb{R}$ and $t_{max} \in \mathbb{R}$ such that $p(t_{min}) = \min_{t\in\mathbb{R}} p(t)$ and $p(t_{max}) = \max_{t\in\mathbb{R}} p(t)$. The proof is based on the observation that

$$f(p(t_{min})) < p(t_{min}) \qquad \text{and} \qquad f(p(t_{max})) > p(t_{max}). \tag{5.1}$$

The weaker inequalities $f(p(t_{min})) \le p(t_{min})$ and $f(p(t_{max})) \ge p(t_{max})$ can be seen from

$$0 = \dot{p}(t_{min}) = -p(t_{min}) + f(p(t_{min}-1)) \ge -p(t_{min}) + f(p(t_{min}))$$

and

$$0 = \dot{p}(t_{max}) = -p(t_{max}) + f(p(t_{max}-1)) \le -p(t_{max}) + f(p(t_{max})).$$

Proposition 3.2.(ii) in addition implies that there exist no equilibria $\hat{\chi} \in C$ such that $\chi \in \{p(t_{min}), p(t_{max})\}$, i.e.,

$$f(p(t_{min})) \ne p(t_{min}) \qquad \text{and} \qquad f(p(t_{max})) \ne p(t_{max}).$$

2. We show that $p(t) > \zeta_{i-1}$ for all real t. First suppose that $i = 1$ and $\zeta_{i-1} = \zeta_0$. Note that by the assumptions of the proposition, $f(u) \ge u$ for all $u \in (-\infty, \zeta_0]$, and thus (5.1) excludes the possibility that $p(t_{min}) \le \zeta_0$. Now assume that $i > 1$. Then $p(t) > \xi_{i-1}$ for all real t, otherwise p oscillates about ξ_{i-1}. As $f(u) > u$ for $u \in (\xi_{i-1}, \zeta_{i-1})$ and $f(\zeta_{i-1}) = \zeta_{i-1}$, (5.1) shows that it is impossible that $p(t_{min}) \in (\xi_{i-1}, \zeta_{i-1}]$. Thus $p(t) > \zeta_{i-1}$ for all real t in any case $i \ge 1$.

It is similar to verify that $p(t) < \zeta_j$ for all $t \in \mathbb{R}$. □

The following simple result will be used to exclude the existence of the unwanted large-amplitude periodic solutions.

Proposition 5.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $\zeta_-, \xi_-, \xi_+, \zeta_+ \in \mathbb{R}$ are fixed points of f with $\zeta_- < \xi_- < \xi_+ < \zeta_+$. Suppose that equation (3.1) admits a periodic solution p such that $(\xi_-, \xi_+) \subset p(\mathbb{R}) \subset (\zeta_-, \zeta_+)$. Then*

$$\log \frac{\zeta_+ - \xi_-}{\zeta_+ - \xi_+} \le 1 \qquad \textit{and} \qquad \log \frac{\xi_+ - \zeta_-}{\xi_- - \zeta_-} \le 1. \tag{5.2}$$

Proof. Let $y : \mathbb{R} \to \mathbb{R}$ be the solution of the initial value problem

$$\begin{cases} \dot{y}(t) = -y(t) + \zeta_+, & t \in \mathbb{R}, \\ y(0) = \xi_-. \end{cases}$$

Then $y(t) = \zeta_+ + (\xi_- - \zeta_+) e^{-t}$ for $t \in \mathbb{R}$. It is a straightforward calculation to show that the unique solution of $y(T) = \xi_+$ is

$$T = \log \frac{\zeta_+ - \xi_-}{\zeta_+ - \xi_+},$$

that is, y needs T time to increase from ξ_- to ξ_+.

We may assume (by considering a time shift of p if necessary) that $p(0) = \xi_-$. It is clear that

$$\dot{p}(t) \leq -p(t) + f(\zeta_+) = -p(t) + \zeta_+$$

for all $t \in \mathbb{R}$. In consequence, Theorem 6.1 of Chapter I.6 in [4] implies that for $t \geq 0$, $p(t) \leq y(t)$.

Let $t_* > 0$ be minimal with $p(t_*) = \xi_+$. Necessarily $t_* \leq 1$, otherwise $p_1 \prec \hat{\xi}_+$ and thus $p(t) < \xi_+$ for all $t \geq 1$ by Proposition 3.1. On the other hand, the inequality $p(t) \leq y(t)$ for $t \geq 0$ yields that $t_* \geq T$. Summing up, $T \leq t_* \leq 1$, i.e., the first estimate in (5.2) is true.

The second estimate can be verified in an analogous manner. □

The stability of the equilibria given by the fixed points $\zeta_- < \xi_- < \xi_+ < \zeta_+$ is irrelevant in the above proposition. However, in accordance with our previously introduced conventions in notation, ζ_- and ζ_+ will always denote stable fixed points of f in the forthcoming applications, while ξ_- and ξ_+ will always denote unstable fixed points.

We will also need the next technical condition for continuously differentiable functions defined on $\mathbb{R}$ or on a subinterval of $\mathbb{R}$. Let f'_- and f'_+ denote the left hand and right hand derivatives of f, respectively.

(C) If ζ_- and ζ_+ are the smallest and largest fixed points of f, respectively, then $f'_+(\zeta_-) = f'_-(\zeta_+) = 0$. In addition, f has at least one unstable fixed point in both intervals

$$\left(\zeta_-, \frac{\zeta_- + \zeta_+}{2}\right) \quad \text{and} \quad \left(\frac{\zeta_- + \zeta_+}{2}, \zeta_+\right).$$

5.2. *Nonlinearities generating the simplest configurations of large-amplitude periodic orbits*

Let $\rho\colon [0,2] \to \mathbb{R}$ be a nondecreasing C^1-function with fixed points $0, 1, 2$ such that $\rho(u) < u$ for $u \in (0,1)$, $\rho(u) > u$ for $u \in (1,2)$, $\rho'(0) = \rho'(2) = 0$ and $\rho'(1) > 1$. The function

$$\rho(u) = \sin\left(\frac{\pi}{2}(u-1)\right) + 1$$

is a suitable choice.

For all $M \geq 1$, define

$$f_M : \mathbb{R} \ni u \mapsto \begin{cases} 0 & \text{if } u < 0, \\ \rho(u - 2k) + 2k & \text{if } u \in [2k, 2k+2) \text{ and } k \in \{0, 1, \ldots, M-1\}, \\ 2M & \text{if } u \geq 2M. \end{cases} \tag{5.3}$$

Then f_M satisfies $(H1)$–$(H2)$. It has exactly M unstable fixed points

$$\xi_k = 2k - 1, \quad k \in \{1, 2, \ldots, M\},$$

and it has $M+1$ stable fixed points $\zeta_k = 2k$, $k \in \{0, 1, \ldots, M\}$, with $f'_M(\zeta_k) = 0$ for all $k \in \{0, 1, \ldots, M\}$. See Fig. 5.1 for the plot of f_3.

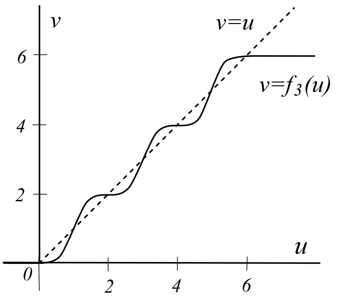

Fig. 5.1. The plot of f_3.

Proposition 5.3. *Let $M \geq 1$. Equation (3.1) with nonlinearity $f = f_M$ admits no large-amplitude periodic solutions.*

Proof. It is clear that if $f = f_1$, then equation (3.1) cannot have large-amplitude periodic solutions. Suppose for contradiction that $M > 1$, $1 \leq i < j \leq M$, and equation (3.1) with nonlinearity $f = f_M$ has an $\left[i, j\right]$ periodic solution p. Then it follows from Proposition 5.1 and the location of the equilibria that $p(t) \in (2i-2, 2j)$ for all real t. We can apply Proposition 5.2 with $\xi_- = \xi_i = 2i - 1$, $\xi_+ = \xi_j = 2j - 1$, $\zeta_- = 2i - 2$ and $\zeta_+ = 2j$. The first inequality in (5.2) already gives that

$$1 \geq \log \frac{\zeta_+ - \xi_-}{\zeta_+ - \xi_+} = \log\left(2\left(j-i\right)+1\right) \underset{j-i\geq 1}{\geq} \log 3,$$

which is impossible as $\log 3 > 1$. □

Observe that if $\eta \in (0,1)$, then $f_M(u) < u$ for all $u \in (0, \eta)$ and $f_M(u) > u$ for all $u \in (2M - \eta, 2M)$.

Let us introduce a new nonlinearity f_M^* for all $M \geq 2$ by modifying f_M on $(-\infty, \eta) \cup (2M - \eta, \infty)$. So let $\eta \in (0,1)$, and choose $K_1, K_2 \in \mathbb{R}$ with

$$K_1 < M\left(1 - K_0\right) < 0 \quad \text{and} \quad K_2 = 2M - K_1 > 2M, \tag{5.4}$$

where $K_0 > 1$ is the threshold number from Proposition 4.1. Set

$$f_M^* : \mathbb{R} \ni u \mapsto \begin{cases} K_1 & \text{if } u \leq 0, \\ \rho_1\left(f_M(u)\right) & \text{if } u \in (0, \eta), \\ f_M(u) & \text{if } u \in \left[\eta, 2M - \eta\right], \\ \rho_2\left(f_M(u)\right) & \text{if } u \in (2M - \eta, 2M), \\ K_2 & \text{if } u \geq 2M, \end{cases} \tag{5.5}$$

where ρ_1 and ρ_2 are defined so that $f_M^* : \mathbb{R} \to \mathbb{R}$ fulfills $(H1)$, furthermore

$$\rho_1\left(f_M(u)\right) < f_M(u) < u \quad \text{for all } u \in (0, \eta) \tag{5.6}$$

and

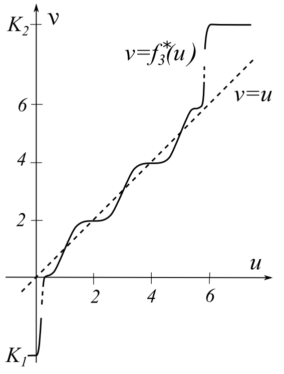

Fig. 5.2. The plot of f_3^*. Observe that f_3 has to be modified for $u < \eta$ and for $u > 6 - \eta$.

$$\rho_2\left(f_M(u)\right) > f_M(u) > u \quad \text{for all } u \in (2M - \eta, 2M). \tag{5.7}$$

See Fig. 5.2 for the plot of f_3^*. One can easily check that the second-order polynomials

$$\rho_1 \colon (0, f_M(\eta)) \ni v \mapsto \frac{K_1}{f_M^2(\eta)} v^2 + \left(1 - \frac{2K_1}{f_M(\eta)}\right) v + K_1 \in \mathbb{R}$$

and

$$\begin{aligned} \rho_2 \colon (f_M(2M - \eta), 2M) \ni v \mapsto & \frac{-K_1}{(2M - f_M(2M - \eta))^2} v^2 \\ & + \left(1 + \frac{2K_1 f_M(2M - \eta)}{(2M - f_M(2M - \eta))^2}\right) v \\ & - \frac{K_1 f_M^2(2M - \eta)}{(2M - f_M(2M - \eta))^2} \in \mathbb{R} \end{aligned}$$

are suitable choices.

Two remarks regarding the above definition: We need condition (5.4) because we intend to apply Corollary 4.10. Conditions (5.6) and (5.7) will be used to guarantee that f_M^* has no fixed points in $(0, \eta) \cup (2M - \eta, 2M)$.

Proposition 5.4. *For all $M \geq 2$, f_M^* satisfies $(H1)$, $(H2)$ and (C). The unstable fixed points of f_M^* are*

$$\xi_k = 2k - 1, \quad k \in \{1, 2, \ldots, M\}.$$

In addition, f_M^* has $M + 1$ stable fixed points. The smallest stable fixed point of f_M^* is $K_1 < 0$, the largest one is $K_2 > 2M$, and the others are

$$\zeta_k = 2k, \quad k \in \{1, 2, \ldots, M - 1\}.$$

The derivative of f_M^* vanishes at its stable fixed points.

Proof. As $K_1 < 0$ and $f_M^*(u) = K_1$ for all $u \leq 0$, K_1 is a stable fixed point of f_M^*, and $\left(f_M^*\right)'(K_1) = 0$. It is clear that f_M^* has no other fixed point in $(-\infty, 0]$. Similarly, $K_2 > 2M$ is the unique fixed point of f_M^* in $[2M, \infty)$, and $\left(f_M^*\right)'(K_2) = 0$. By the choices of ρ_1 and ρ_2, f_M^* has no fixed points in $(0, \eta) \cup (2M - \eta, 2M)$, see (5.6) and (5.7). The assertions regarding the rest of the fixed points follow from the fact that $f_M^*(u) = f_M(u)$ for all $u \in \left[\eta, 2M - \eta\right]$.

The definition of f_M^* and the fact that f_M satisfies $(H1)$–$(H2)$ implies that $(H1)$–$(H2)$ also hold for f_M^*.

Note that $(K_1 + K_2)/2 = M$. So condition (C) holds with $\zeta_- = K_1$, $\zeta_+ = K_2$, $\xi_- = \xi_1 = 1 \in (0, M)$ and $\xi_+ = \xi_M = 2M - 1 \in (M, 2M)$. $\square$

Proposition 5.5. *Let $M \geq 2$. Equation (3.1) with nonlinearity $f = f_M^*$ has a slowly oscillatory $[1, M]$ periodic solution. It admits no $\left[i, j\right]$ periodic solutions for indices $1 \leq i < j \leq M$ with $i > 1$ or $j < M$.*

Proof. It is clear from Proposition 5.4 that the number of unstable fixed points of f_M^* is M, and all of them are found in $(0, 2M)$.

Consider equation (3.1) with nonlinearity $f = f_M^*$, $M \geq 2$. We can apply Corollary 4.10 with K_1, K_2, η chosen as in the definition of f_M^*, $A_1 = 0$, $A_2 = 2M$, $B_1 = 0$ and $B_2 = 2M$. Corollary 4.10 yields that there is a periodic solution $p : \mathbb{R} \to \mathbb{R}$ such that $(0, 2M) \subset p(\mathbb{R}) \subset (K_1, K_2)$, that is, p is a $[1, M]$ periodic solution. As the minimal period of p is in $(1, 2)$, it is necessarily slowly oscillatory, see Remark 3.3.

Now consider any indices i, j with $1 \leq i < j \leq M$ so that $i > 1$ or $j < M$. It remains to exclude the existence of an $\left[i, j\right]$ periodic solution $q : \mathbb{R} \to \mathbb{R}$. First suppose that $1 \leq i < j < M$. Then $q(\mathbb{R}) \subseteq (K_1, 2j)$ by Proposition 5.1, and we can use Proposition 5.2 with $\zeta_- = K_1$, $\zeta_+ = 2j$, $\xi_- = \xi_i = 2i - 1$ and $\xi_+ = \xi_j = 2j - 1$. The first inequality in (5.2) implies that

$$1 \geq \log \frac{\zeta_+ - \xi_-}{\zeta_+ - \xi_+} = \log(2j - (2i - 1)) \geq \log 3,$$

which contradicts $\log 3 > 1$. Using the second inequality in (5.2) with $\zeta_- = 2i - 2$, $\xi_- = 2i - 1$ and $\xi_+ = 2M - 1$, the reader can see in an analogous way that there exist no $[i, M]$ periodic solutions for $1 < i < M$. $\square$

6. The proof of Theorem 2.1.(i)

This section is the proof of Theorem 2.1.(i).

We introduce the following partial order. A pair of parentheses in a parenthetical expression is of 1st level, if it is not nested in any other pair of parentheses. For $n \geq 2$, a pair of parentheses is of nth level, if it is nested in an $(n-1)$th level pair, and not in any mth level pair for $m \geq n$.

An nth level subexpression is an nth level pair of parentheses, together with all the numbers and parentheses enclosed by it.

Example. Consider the expression

$$\Big(1\Big(((23)\,4)\,5\,(67)\,89\Big)\Big)10 \tag{6.1}$$

of 10 numbers for example. Then (1 (((23) 4) 5 (67) 89)) is a 1st level subexpression, while (((23) 4) 5 (67) 89) is of 2nd level, ((23) 4) and (67) are of 3rd level, and (23) is of 4th level. This example shows that a 1st level subexpression is not necessarily the whole expression itself. We also see that not all 3rd level subexpressions contain a 4th level subexpression.

Now suppose that a subexpression contains exactly the numbers $i, i+1, \ldots, j$. Suppose that f is a continuously differentiable, nondecreasing function that is defined on $\mathbb{R}$ or on a subinterval of $\mathbb{R}$, and it satisfies $(H2)$. We say that f generates the subexpression if

- f has exactly $j-i+1$ unstable fixed points $\xi_i < \xi_{i+1} < \ldots < \xi_j$ giving the unstable equilibria $\hat{\xi}_i, \hat{\xi}_{i+1}, \ldots, \hat{\xi}_j$,
- for all $i', j' \in \{i, \ldots, j\}$ with $i' < j'$, equation (3.1) with nonlinearity f admits an $[i', j']$ periodic solution if and only if there exists a pair of parentheses in the subexpression that encloses $i', i'+1, \ldots, j'$ and no other numbers,
- these periodic solutions can be chosen to be slowly oscillatory.

Functions generating the original parenthetical expression are defined in an analogous way.

Outline of the proof.

Set $N \geq 2$, and consider a parenthetical expression of N numbers. The proof of Theorem 2.1.(i) is already complete when this expression contains no parentheses: we know from Proposition 5.3 that f_N generates the trivial expression $12 \ldots N$. Otherwise fix $m \geq 1$ such that the expression contains at least one pair of parentheses of mth level, but none of $(m+1)$th level. The proof proceeds by mathematical induction on the levels of the subexpressions from the mth level to the 1st one: For all n decreasing from m to 1, and for each nth level subexpression, we construct a nonlinear function that satisfies $(H1)$, $(H2)$, (C) and generates the given subexpression. Then as last step of the proof, we obtain a nonlinearity that satisfies $(H1)$, $(H2)$ and generates the original parenthetical expression.

Initial step.

Suppose that $n = m$. Any mth level subexpression has the form $(i \ldots j)$, where $1 \leq i < j \leq N$. By Propositions 5.4 and 5.5, nonlinearity f^*_{j-i+1} defined in (5.5) generates $(i \ldots j)$ and fulfills $(H1)$, $(H2)$ and (C).

Inductive step.

Now suppose that $1 \leq n < m$, and there are functions that not only generate the $(n+1)$th level subexpressions, but also satisfy $(H1)$, $(H2)$ and (C). Fix a subexpression of nth level. Let $i, i+1, \ldots, j$ denote the integers contained by it.

If there exists no $(n+1)$th level subexpression within the subexpression under consideration (i.e., it has the form $(i \ldots j)$), we are ready by Propositions 5.4 and 5.5.

Otherwise we use the nonlinearities generating the $(n+1)$th level subexpressions and the functions f_M defined in (5.3) as “building blocks” to determine a nonlinearity f that generates the fixed nth level subexpression. The procedure is the following.

Step 1. We divide the real line into intervals.

Step 2. We introduce a C^1-function $\hat{f}$ defined piecewise on these intervals such that $\hat{f}$ generates the “inner part” of the considered nth level subexpression (that is, the whole

nth level subexpression except for that pair of parentheses that encloses all numbers $i, i+1, \dots, j$). Roughly speaking, the restriction of $\hat{f}$ to any of these intervals will be either a transformation of f_M, $M \geq 1$, or of a nonlinearity generating an $(n+1)$th level subexpression.

Step 3. At last we modify $\hat{f}$ by using Corollary 4.10 in order to get f generating the given nth level subexpression.

Step 1. (Partition of the real line.)

Let $k \geq 1$ denote the number of $(n+1)$th level subexpressions nested in the considered nth level pair of parentheses. Reading from the left, there is a natural order among these subexpressions. We use this order and distinguish 1st, 2nd, ..., kth subexpression of $(n+1)$th level.

We need to handle that there may exist integers among $i, i+1, \dots, j$ that are not contained in any $(n+1)$th level subexpression. Let $r_1 \geq 0$ be the number of integers among $i, i+1, \dots, j$ that are smaller than any integer in the 1st subexpression of $(n+1)$th level. For all $l \in \{2, \dots, k\}$, let $r_l \geq 0$ denote the number of integers that are greater than any integer contained in the $(l-1)$th subexpression and smaller than any integer in the lth subexpression of $(n+1)$th level. At last, r_{k+1} is the number of integers among $i, i+1, \dots, j$ that are greater than any integer in the kth subexpression of $(n+1)$th level.

Example. Let us return back to our previous example (6.1). Assume that $n = 2$, that is, we look for a nonlinearity f generating the 2nd level subexpression (((23) 4) 5 (67) 89). Then $i = 2$ and $j = 9$. As ((23) 4) and (67) are the 3rd level subexpressions within this subexpression, $k = 2$. In addition, $r_1 = 0$, $r_2 = 1$ and $r_3 = 2$.

Using $k \geq 1$ and r_l, $l \in \{1, \dots, k+1\}$, defined as above, we introduce $k+2$ subintervals of the real line spaced at distances $2r_l$. The first interval is $I_0 = (-\infty, \beta_0]$ with an arbitrary right end point $\beta_0 \in \mathbb{R}$. The endpoints of the next k intervals $I_l = [\alpha_l, \beta_l]$, $l \in \{1, \dots, k\}$, are defined as follows:

$$\alpha_l = \beta_{l-1} + 2r_l \qquad \text{and} \qquad \beta_l = \alpha_l + 2.$$

The length of I_l is 2 for each $l \in \{1, 2, \dots, k\}$. At last, define the left end point of the last interval $I_{k+1} = [\alpha_{k+1}, \infty)$ as

$$\alpha_{k+1} = \beta_k + 2r_{k+1}.$$

With this procedure, we also obtain intervals $J_l = [\beta_{l-1}, \alpha_l]$, $l \in \{1, \dots, k+1\}$, of length $2r_l$. J_l may be trivial as $r_l = 0$ is allowed.

The idea behind this definition is simple. We will set the auxiliary function $\hat{f}$ so that $\hat{f}|_{I_l}$ will generate the lth subexpression of $(n+1)$th level for all $l \in \{1, \dots, k\}$. If $r_l > 0$ for some $l \in \{1, \dots, k+1\}$, then $\hat{f}|_{J_l}$ will generate the trivial parenthetical expression (i.e., the expression containing no parentheses) of r_l numbers. The restrictions $\hat{f}|_{I_0}$ and $\hat{f}|_{I_{k+1}}$ will be constant functions.

The length of the intervals I_l and J_l will play a key role later (in the proof of Proposition 6.4).

Example. Consider example (6.1) again, and suppose that we look for an f generating the 2nd level subexpression (((23) 4) 5 (67) 89). Then $I_0 = (-\infty, \beta_0] = (-\infty, 0]$, $I_1 = [\alpha_1, \beta_1] = [0, 2]$,

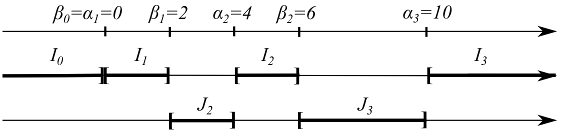

Fig. 6.1. The partition of the real line in our example.

$I_2=\left[\alpha_2,\beta_2\right]=[4,6]$, $I_3=[\alpha_3,\infty)=[10,\infty)$ are good choices. Interval J_1 is trivial, $J_2=\left[\beta_1,\alpha_2\right]=[2,4]$ and $J_3=\left[\beta_2,\alpha_3\right]=[6,10]$, see Fig. 6.1.

Step 2. (The auxiliary function $\hat{f}$.)

We need the subsequent transformations. For $a,b,c,d\in\mathbb{R}$ with $a\neq c$ and $b\neq d$, let $L_{a\to b,c\to d}\colon\mathbb{R}\to\mathbb{R}$ denote the linear map with $L(a)=b$ and $L(c)=d$:

$$L_{a\to b,c\to d}(u)=\frac{u-c}{a-c}b+\frac{u-a}{c-a}d\qquad\text{for } u\in\mathbb{R}.$$

Then $L^{-1}_{a\to b,c\to d}=L_{b\to a,d\to c}$ is the inverse of $L_{a\to b,c\to d}$.

If x is a solution of equation (3.1), then y, defined by $y(t)=L_{a\to b,c\to d}x(t)$ for all t in the domain of x, is a solution of

$$\dot{y}(t)=-y(t)+g(y(t-1)),\tag{6.2}$$

where

$$g\colon\mathbb{R}\ni u\mapsto L_{a\to b,c\to d}f\left(L_{b\to a,d\to c}u\right)\in\mathbb{R}.\tag{6.3}$$

In particular, $L_{a\to b,c\to d}$ creates a bijection between the periodic solutions and the equilibria of (3.1) and (6.2). It is easy to see that for $\chi^*=L_{a\to b,c\to d}\chi$, $g'(\chi^*)=f'(\chi)$, and therefore the transformation preserves the stability of the equilibria. It is also clear that a periodic function x oscillates (slowly) about $\xi_i,\xi_{i+1},\ldots,\xi_j$ if and only if $y=L_{a\to b,c\to d}x$ oscillates (slowly) about $L_{a\to b,c\to d}\xi_i, L_{a\to b,c\to d}\xi_{i+1},\ldots,L_{a\to b,c\to d}\xi_j$. The parenthetical expression generated by g is the same as the one generated by f.

Emphasizing the dependence of g on a,b,c,d, in the following we use the notation

$$T_{a\to b,c\to d}f\colon\mathbb{R}\ni u\mapsto L_{a\to b,c\to d}f\left(L_{b\to a,d\to c}u\right)\in\mathbb{R}.$$

We are ready to introduce the auxiliary function $\hat{f}$.

Let g_l, $l\in\{1,\ldots,k\}$, denote the nonlinearity that generates the lth subexpression of $(n+1)$th level, furthermore satisfies $(H1)$, $(H2)$ and (C). By the induction hypothesis, such g_l exists. Let $a_l\in\mathbb{R}$ and $b_l\in\mathbb{R}$ denote the smallest and largest fixed points of g_l for each $l\in\{1,\ldots,k\}$.

We define $\hat{f}\colon\mathbb{R}\to\mathbb{R}$ using the following three rules.

$(R1)$ Let

$$\hat{f}(u)=T_{a_l\to\alpha_l,b_l\to\beta_l}g_l(u)\quad\text{for all } u\in I_l=\left[\alpha_l,\beta_l\right] \text{ and } l\in\{1,\ldots,k\}.$$

$(R2)$ For all $u \in I_0 = (-\infty, \beta_0]$, set $\hat{f}(u) = \beta_0$. For all $u \in I_{k+1} = [\alpha_{k+1}, \infty)$, set $\hat{f}(u) = \alpha_{k+1}$.

$(R3)$ Whenever $r_l > 0$, i.e., $J_l = [\beta_{l-1}, \alpha_l]$ is nontrivial for some $l \in \{1, 2, \dots, k+1\}$, let

$$\hat{f}(u) = T_{0 \to \beta_{l-1}, 2r_l \to \alpha_l} f_{r_l}(u) \quad \text{for all } u \in J_l = [\beta_{l-1}, \alpha_l],$$

where f_{r_l}, defined by (5.3), denotes a nonlinearity that generates the trivial parenthetical expression of r_l numbers.

It is easy to see the following proposition.

Proposition 6.1. *Function $\hat{f}$ satisfies $(H1)$ and $(H2)$.*

Proof. As the above used functions g_l and f_{r_l} satisfy $(H1)$ and $(H2)$, it suffices to prove that $\hat{f}$ is differentiable at β_0, at α_l and at β_l for all $l \in \{1, \dots, k\}$, furthermore at α_{k+1}. The next three observations guarantee the differentiability of $\hat{f}$ at these points.

1. Recall that $a_l \in \mathbb{R}$ and $b_l \in \mathbb{R}$ denote the smallest and largest fixed points of g_l for each $l \in \{1, \dots, k\}$. By condition (C), $(g_l)'_+(a_l) = (g_l)'_-(b_l) = 0$. Thus by $(R1)$, α_l and β_l are fixed points of $\hat{f}|_{I_l}$ with $\hat{f}'_+(\alpha_l) = \hat{f}'_-(\beta_l) = 0$ for all $l \in \{1, \dots, k\}$. (This is the first place where condition (C) is used.)

2. By $(R2)$, the points β_0 and α_{k+1} are fixed points of $\hat{f}|_{I_0}$ and $\hat{f}|_{I_{k+1}}$, receptively, furthermore $\hat{f}'_-(\beta_0) = \hat{f}'_+(\alpha_{k+1}) = 0$.

3. Regarding rule $(R3)$, recall that 0 and $2r_l$ are the smallest and largest fixed points of f_{r_l}, respectively. We also know that $f'_{r_l}(0) = f'_{r_l}(2r_l) = 0$. Hence if $J_l = [\beta_{l-1}, \alpha_l]$ is nontrivial for some $l \in \{1, 2, \dots, k+1\}$, then β_{l-1} and α_l are fixed points of $\hat{f}|_{J_l}$ with $\hat{f}'_+(\beta_{l-1}) = \hat{f}'_-(\alpha_l) = 0$. $\square$

Example. We return back to our previous example. Suppose g_1 generates $((23)\,4)$ and g_2 generates (67). Actually, our procedure gives that $g_2 = f_2^*$, where f_2^* is defined by (5.5). Now the auxiliary function $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ is given by

$$\hat{f}(u) = \begin{cases} 0 & \text{if } u \le \beta_0 = 0, \\ T_{a_1 \to 0, b_1 \to 2} g_1(u) & \text{if } u \in I_1 = [\alpha_1, \beta_1] = [0, 2], \\ T_{0 \to 2, 2 \to 4} f_1(u) & \text{if } u \in J_2 = [\beta_1, \alpha_2] = [2, 4], \\ T_{a_2 \to 4, b_2 \to 6} g_2(u) & \text{if } u \in I_2 = [\alpha_2, \beta_2] = [4, 6], \\ T_{0 \to 6, 4 \to 10} f_2(u) & \text{if } u \in J_3 = [\beta_2, \alpha_3] = [6, 10], \\ 10 & \text{if } u \ge \alpha_3 = 10, \end{cases}$$

see Fig. 6.2.

It is clear that for any $l \in \{1, \dots, k\}$, there exists a pair of parentheses enclosing all the numbers within the lth subexpression of $(n+1)$th level. In other words, the equation

$$\dot{x}(t) = -x(t) + g_l(x(t-1)) \tag{6.4}$$

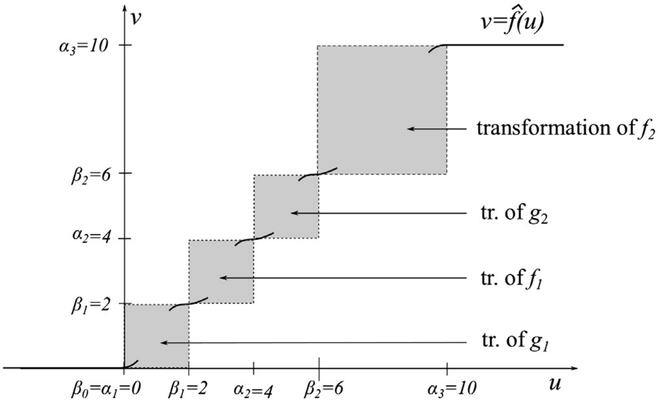

Fig. 6.2. The plot of $\hat{f}$ in our example. On the interval [0, 10], the graph of $\hat{f}$ lies in the gray squares.

admits at least one large-amplitude periodic solution $q^l : \mathbb{R} \to \mathbb{R}$ that oscillates about all the unstable fixed points of g_l. We know from Proposition 5.1 that $q^l(\mathbb{R}) \subset (a_l, b_l)$.

Note that already $g_l|_{q^l(\mathbb{R})}$ generates the lth parenthetical subexpression. This comes from the fact that if r is a periodic solution of (6.4), and r does not oscillate about all unstable fixed points of g_l, then $r(\mathbb{R}) \subsetneq q^l(\mathbb{R})$ by Proposition 3.4.

Define $p^l : \mathbb{R} \to \mathbb{R}$ by

$$p^l(t) = L_{a_l \to \alpha_l, b_l \to \beta_l} q^l(t) \qquad \text{for all real } t,$$

where $a_l, \alpha_l, b_l, \beta_l$ are defined as above. The following result is immediate.

Proposition 6.2. *For all $l \in \{1, \dots, k\}$, $p^l(\mathbb{R}) \subset \operatorname{int} I_l = (\alpha_l, \beta_l)$. Function $\hat{f}|_{p^l(\mathbb{R})}$ generates the lth parenthetical subexpression of $(n+1)$th level.*

At last, let us collect what we know about the fixed points of $\hat{f}$.

$(P1)$ It is obvious that β_0 and α_{k+1} are the smallest and largest fixed points of $\hat{f}$, respectively, and $\hat{f}'(\beta_0) = \hat{f}'(\alpha_{k+1}) = 0$. By construction, $\hat{f}$ has $j - i + 1$ unstable fixed points in (β_0, α_{k+1}).

$(P2)$ Consider any $l \in \{1, \dots, k+1\}$ for which $r_l > 0$. We know the exact location of the fixed points of $\hat{f}$ in the interval $J_l = [\beta_{l-1}, \alpha_l]$ because they arise in the form $L_{0 \to \beta_{l-1}, 2r_l \to \alpha_l} \chi$, where χ is a fixed point of f_{r_l}. As the length of J_l is $\alpha_l - \beta_{l-1} = 2r_l$, the transformation $L_{0 \to \beta_{l-1}, 2r_l \to \alpha_l}$ is only a shift of the real line: it maps all $u \in \mathbb{R}$ to $\beta_{l-1} + u$. Since $2s - 1$ is an unstable fixed point of f_{r_l} for all $s \in \{1, \dots, r_l\}$, we deduce that

$$\beta_{l-1} + 2s - 1, \quad s \in \{1, \dots, r_l\},$$

are the unstable fixed points of $\hat{f}$ in J_l. Similarly,

$$\beta_{l-1} + 2s, \quad s \in \{0, 1, \dots, r_l\},$$

are the stable fixed point of $\hat{f}$ in J_l with zero derivative.

$(P3)$ Regarding the fixed points of $\hat{f}$ in I_l, $l \in \{1, \ldots, k\}$, it is important to note that as g_l satisfies condition (C), $\hat{f}|_{I_l}$ satisfies (C) too.

Step 3. (The function f generating the considered nth level subexpression.)

Now, by modifying $\hat{f}$, we can define a nonlinearity f that generates the fixed nth level subexpression. In the following we apply Corollary 4.10 with $A_1 = B_1 = \beta_0$ and $A_2 = B_2 = \alpha_{k+1}$. The constants η, K_1 and K_2 have to be chosen as given below.

By $(P1)$, β_0 and α_{k+1} are the smallest and largest fixed points of $\hat{f}$, respectively. Set $\eta > 0$ so small that all the other fixed points of $\hat{f}$ belong to $(\beta_0 + \eta, \alpha_{k+1} - \eta)$. By Proposition 6.2, $p^l(\mathbb{R}) \subset \operatorname{int} I_l \subset (\beta_0, \alpha_{k+1})$ for all l. So by decreasing $\eta > 0$ if necessary, we can achieve that the range $p^l(\mathbb{R})$ of the periodic solution p^l is a subset of $(\beta_0 + \eta, \alpha_{k+1} - \eta)$ for all $l \in \{1, \ldots, k\}$.

Choose

$$K_1 < \min\left\{\frac{-(\alpha_{k+1} - \beta_0) K_0 + \beta_0 + \alpha_{k+1}}{2}, \beta_0\right\},$$

where K_0 is the threshold number from Proposition 4.1. Let $K_2 = \beta_0 + \alpha_{k+1} - K_1 > \alpha_{k+1}$.

Set

$$f : \mathbb{R} \ni u \mapsto \begin{cases} K_1, & u \le \beta_0 \\ \rho_1\left(\hat{f}(u)\right), & u \in (\beta_0, \beta_0 + \eta) \\ \hat{f}(u), & u \in \left[\beta_0 + \eta, \alpha_{k+1} - \eta\right] \\ \rho_2\left(\hat{f}(u)\right), & u \in (\alpha_{k+1} - \eta, \alpha_{k+1}) \\ K_2, & u \ge \alpha_{k+1}, \end{cases} \tag{6.5}$$

where ρ_1 and ρ_2 are defined so that $f : \mathbb{R} \to \mathbb{R}$ fulfills $(H1)$, furthermore

$$\rho_1\left(\hat{f}(u)\right) < \hat{f}(u) \quad \text{for all } u \in (\beta_0, \beta_0 + \eta) \tag{6.6}$$

and

$$\rho_2\left(\hat{f}(u)\right) > \hat{f}(u) \quad \text{for all } u \in (\alpha_{k+1} - \eta, \alpha_{k+1}). \tag{6.7}$$

This choice of f is possible. The functions ρ_1 and ρ_2 can be selected as in the definition of f_M^*.

Example. Fig. 6.3 demonstrates that in our example $\hat{f}$ has to be modified on the interval $(-\infty, \eta) \cup (10 - \eta, \infty)$ with some $\eta > 0$ to get a function f generating $(((23)\,4)\,5\,(67)\,89)$.

As $\hat{f}'(\beta_0) = \hat{f}'(\alpha_{k+1}) = 0$, and as $\hat{f}$ has no fixed points in $(\beta_0, \beta_0 + \eta) \cup (\alpha_{k+1} - \eta, \alpha_{k+1})$, it is true that

$$\hat{f}(u) < u \quad \text{for all } u \in (\beta_0, \beta_0 + \eta) \quad \text{and} \quad \hat{f}(u) > u \quad \text{for all } u \in (\alpha_{k+1} - \eta, \alpha_{k+1}).$$

This observation, (6.6) and (6.7) together imply that f possesses no fixed points in $(\beta_0, \beta_0 + \eta) \cup (\alpha_{k+1} - \eta, \alpha_{k+1})$. Next we summarize what else we know about the fixed points of f.

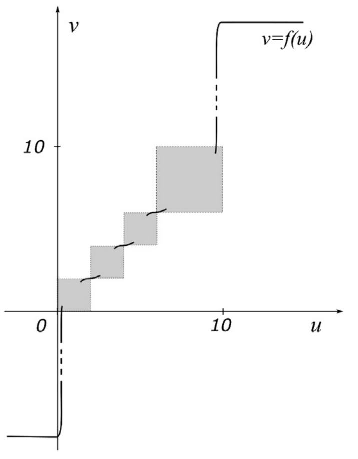

Fig. 6.3. The plot of f in our example.

Proposition 6.3. *Function f satisfies not only $(H1)$ but also $(H2)$ and (C). K_1 and K_2 are the smallest and largest fixed points of f, respectively, with $f'(K_1) = f'(K_2) = 0$. Function f inherits all fixed points χ of $\hat{f}$ in the interval $[\beta_0 + \eta, \alpha_{k+1} - \eta]$ (that is, all fixed points of $\hat{f}$ besides β_0 and α_{k+1}) with $f'(\chi) = \hat{f}'(\chi)$. It has no other fixed points. It follows that f has exactly $j - i + 1$ unstable fixed points.*

Let $\xi_i < \xi_{i+1} < \ldots < \xi_j$ denote the unstable fixed points of $\hat{f}$ and f.

Proof. We omit most of the proof as it is analogous to the proof of Proposition 5.4. We only verify that f satisfies (C) with $\zeta_- = K_1$ and $\zeta_+ = K_2$.

If $r_1 > 0$, then the smallest unstable fixed point of both f and $\hat{f}$ is $\xi_i = \beta_0 + 1$ (see property $(P2)$). If $r_1 = 0$, then the smallest unstable fixed point of f and $\hat{f}$ is the one of $\hat{f}|_{I_1}$. By property $(P3)$, $\hat{f}|_{I_1}$ satisfies (C), so it has an unstable fixed point smaller than $(\alpha_1 + \beta_1)/2 = \alpha_1 + 1 = \beta_0 + 1$. Summing up, $\xi_i \le \beta_0 + 1$. Similarly, $\xi_j \ge \alpha_{k+1} - 1$.

Next we show that $\alpha_{k+1} - \beta_0 \ge 4$. Let $|I|$ denote the length of an interval $I \subset \mathbb{R}$. If $k \ge 2$, then $\alpha_{k+1} - \beta_0 \ge |I_1| + |I_2| = 4$. If $k = 1$, then (as no multiple enclosing of the same sublist of numbers allowed in a correct parenthetical expression) either $r_1 > 0$ or $r_2 > 0$. Suppose $r_1 > 0$ for example. Then $\alpha_{k+1} - \beta_0 \ge |J_1| + |I_1| \ge 4$.

In order to verify (C), we need to confirm that $\xi_i < (K_1 + K_1)/2 < \xi_j$. It is enough to show that $\beta_0 + 1 < (K_1 + K_1)/2 < \alpha_{k+1} - 1$. Actually, using the equality $K_2 = \beta_0 + \alpha_{k+1} - K_1$ and the inequality $\alpha_{k+1} - \beta_0 \ge 4$, we obtain that

$$\frac{K_1 + K_2}{2} = \frac{\beta_0 + \alpha_{k+1}}{2} \in [\beta_0 + 2, \alpha_{k+1} - 2]. \qquad \square$$

By Corollary 4.10, the equation with this nonlinearity f has a periodic solution $p : \mathbb{R} \to \mathbb{R}$ such that $(\beta_0, \alpha_{k+1}) \subset p(\mathbb{R}) \subset (K_1, K_2)$. Necessarily p is an $[i, j]$ solution. Let us see what we know about the other large-amplitude periodic solutions. The length of the intervals I_l and J_l becomes essential in the proof of the following proposition.

Proposition 6.4. *Consider $f: \mathbb{R} \to \mathbb{R}$ obtained above. Assume that equation (3.1) has an $[i', j']$ periodic solution $q: \mathbb{R} \to \mathbb{R}$ so that $i \le i' < j' \le j$, and either $i \ne i'$ or $j \ne j'$. Then an index $l \in \{1, \dots, k\}$ can be given such that $\xi_{i'}, \xi_{i'+1}, \dots, \xi_{j'} \in I_l$.*

Proof. We need to exclude the following cases:

(i) $\xi_{i'} \in J_{l_1}$ and $\xi_{j'} \in I_{l_2}$ with $l_1, l_2 \in \{1, \dots, k\}$, $r_{l_1} > 0$ and $l_1 \le l_2$,
(ii) $\xi_{i'} \in J_{l_1}$ and $\xi_{j'} \in J_{l_2}$ with $l_1, l_2 \in \{1, \dots, k+1\}$, $r_{l_1} > 0$, $r_{l_2} > 0$ and $l_1 \le l_2$,
(iii) $\xi_{i'} \in I_{l_1}$ and $\xi_{j'} \in J_{l_2}$ with $l_1 \in \{1, \dots, k\}$, $l_2 \in \{2, \dots, k+1\}$, $r_{l_2} > 0$ and $l_1 < l_2$,
(iv) $\xi_{i'} \in I_{l_1}$ and $\xi_{j'} \in I_{l_2}$ with $l_1, l_2 \in \{1, \dots, k\}$ and $l_1 < l_2$.

Suppose for contradiction that we are in case (i).

1. We claim that $\xi_{j'} - \xi_{i'} \ge 2$.

On the one hand, we show that $\xi_{j'} \ge \alpha_{l_2} + 1$. Let E_1 and $E_2 = \{\xi_{i'}, \xi_{i'+1}, \dots, \xi_{j'}\}$ be the sets of those unstable fixed points of f about which p^{l_2} and q oscillate, respectively. It is clear that $\xi_{i'} \in E_2 \setminus E_1$ and $\xi_{j'} \in E_1 \cap E_2$. It follows from Proposition 3.4 that $E_1 \subset E_2$, that is, the $[i', j']$ periodic solution q oscillates about all unstable fixed points of f in I_{l_2}. In other words, $\xi_{j'}$ is the largest unstable fixed point of f in I_{l_2}. Since f and $\hat{f}$ have the same unstable fixed points by Proposition 6.3, $\xi_{j'}$ is the largest unstable fixed point of $\hat{f}$ in I_{l_2}. As the restriction of $\hat{f}$ to $I_{l_2} = [\alpha_{l_2}, \beta_{l_2}]$ satisfies (C) by property $(P3)$, we deduce that

$$\xi_{j'} \ge \frac{\alpha_{l_2} + \beta_{l_2}}{2} = \alpha_{l_2} + 1. \tag{6.8}$$

(Note that this is the second place, where condition (C) is crucial.)

On the other hand, we prove that $\xi_{i'} \le \alpha_{l_1} - 1$. Since $\beta_{l_1 - 1} + 2s - 1$, $s \in \{1, \dots, r_{l_1}\}$, are the unstable fixed points of f in $J_{l_1} = [\beta_{l_1 - 1}, \alpha_{l_1}]$ by $(P2)$ and Proposition 6.3, a trivial upper bound for $\xi_{i'}$ is $\beta_{l_1 - 1} + 2r_{l_1} - 1 = \alpha_{l_1} - 1$.

As $l_1 \le l_2$ and thus $\alpha_{l_1} \le \alpha_{l_2}$, we obtain that

$$\xi_{j'} - \xi_{i'} \ge \alpha_{l_2} + 1 - (\alpha_{l_1} - 1) = 2. \tag{6.9}$$

2. We divide case (i) into two subcases.

(a) First suppose that $l_2 < k$ or $l_2 = k$ and $r_{k+1} > 0$. In either case $\beta_{l_2} = \sup I_{l_2}$ is smaller than α_{k+1}, hence β_{l_2} is a stable fixed point not only of $\hat{f}$ but also of f. As K_1 is also a fixed point of f, and $K_1 < \xi_{i'} < \xi_{j'} < \beta_{l_2}$, Proposition 5.1 guarantees that the range $q(\mathbb{R})$ of the $[i', j']$ periodic solution q is a subset of (K_1, β_{l_2}). So we can apply Proposition 5.2 with $\zeta_- = K_1$, $\zeta_+ = \beta_{l_2}$, $\xi_- = \xi_{i'}$ and $\xi_+ = \xi_{j'}$. The first inequality in Proposition 5.2 gives that

$$1 \ge \log \frac{\beta_{l_2} - \xi_{i'}}{\beta_{l_2} - \xi_{j'}} = \log \left(1 + \frac{\xi_{j'} - \xi_{i'}}{\beta_{l_2} - \xi_{j'}}\right). \tag{6.10}$$

As $\beta_{l_2} - \alpha_{l_2} = 2$, estimate (6.8) gives that $\beta_{l_2} - \xi_{j'} < 1$. This observation together with (6.9) implies that the right hand side of inequality (6.10) is not smaller than $\log 3$, which is a contradiction.

(b) Now suppose that $l_2 = k$ and $r_{k+1} = 0$. Recall from the beginning of this proof that $\xi_{j'}$ is the largest unstable fixed point of f in $I_k = [\alpha_k, \beta_k]$. As $r_{k+1} = 0$, we have $\alpha_{k+1} = \beta_k$, which means that f has no unstable fixed points greater than β_k. We conclude that $\xi_{j'}$ is the largest unstable fixed point of f, i.e., $j' = j$. Then necessarily $i' > i$ by our initial assumption.

We claim that $\xi_{i'} - 1$ is a stable fixed point of f. This follows simply from property $(P2)$ and Proposition 6.3 if $l_1 \geq 2$. If $l_1 = 1$, the claim is the consequence of $(P2)$, Proposition 6.3 and the fact that $\xi_{i'}$ is not the smallest unstable fixed point f.

We can apply Proposition 5.2 with $\zeta_- = \xi_{i'} - 1$, $\zeta_+ = K_2$, $\xi_- = \xi_{i'}$ and $\xi_+ = \xi_j$. The second inequality in Proposition 5.2 with (6.9) implies that

$$1 \geq \log \frac{\xi_j - (\xi_{i'} - 1)}{\xi_{i'} - (\xi_{i'} - 1)} \geq \log 3,$$

which is a contradiction again.

Handling the cases (ii)–(iv) is analogous. In each case we can prove that $\xi_{j'} - \xi_{i'} \geq 2$. In each case we can apply Proposition 5.2 with $\xi_- = \xi_{i'}$, $\xi_+ = \xi_{j'}$ and with ζ_-, ζ_+ chosen so that $\zeta_+ - \xi_{j'} \leq 1$ if $j' < j$, and $\xi_i - \zeta_- \leq 1$ if $i' > i$. We omit the details. $\square$

Now it is easy to see the following.

Corollary 6.5. *Function f generates the nth level subexpression under consideration.*

Proof. 1. First of all, by Proposition 6.3, f has $j - i + 1$ unstable fixed points $\xi_i < \xi_{i+1} < \ldots < \xi_j$ in (β_0, α_{k+1}).

2. Consider the pair of parentheses in the subexpression that encloses all the integers $i, \ldots, j$ (that is, the nth level pair of parentheses). It has been already mentioned that the equation with the above constructed nonlinearity f has a periodic solution $p : \mathbb{R} \to \mathbb{R}$ such that $(\beta_0, \alpha_{k+1}) \subset p(\mathbb{R}) \subset (K_1, K_2)$. This comes from Corollary 4.10. Necessarily p is an $[i, j]$ solution. As the minimal period of p is in $(1, 2)$, it is slowly oscillatory, see Remark 3.3.

3. Assume that a given pair of parentheses in our nth level subexpression encloses exactly the numbers $i', i'+1, \ldots j'$, where $i \leq i' < j' \leq j$, and either $i \neq i'$ or $j \neq j'$. Then there is $l \in \{1, \ldots, k\}$ such that this pair of parentheses is included the lth subexpression of $(n+1)$th level. Recall from the definition of f that

$$f|_{[\beta_0+\eta, \alpha_{k+1}-\eta]} = \hat{f}|_{[\beta_0+\eta, \alpha_{k+1}-\eta]},$$

and hence $f|_{p^l(\mathbb{R})} = \hat{f}|_{p^l(\mathbb{R})}$. So by Proposition 6.2, $f|_{p^l(\mathbb{R})}$ generates the lth subexpression of $(n+1)$th level. This means that (3.1) with feedback function f admits a periodic solution $q : \mathbb{R} \to \mathbb{R}$ oscillating slowly about $\xi_{i'}, \xi_{i'+1}, \ldots, \xi_{j'}$.

Conversely, suppose $q : \mathbb{R} \to \mathbb{R}$ is an $[i', j']$ periodic solution of (3.1) so that $i \leq i' < j' \leq j$, and $i \neq i'$ or $j \neq j'$. By Proposition 6.4, an index $l \in \{1, \ldots, k\}$ can be given such that $\xi_{i'}, \xi_{i'+1}, \ldots, \xi_{j'} \in I_l$. Then either q oscillates about all unstable fixed points of f in I_l, or $q(\mathbb{R}) \subsetneq p^l(\mathbb{R})$ by Proposition 3.4. As $f|_{p_l(\mathbb{R})} = \hat{f}|_{p_l(\mathbb{R})}$ generates the lth subexpression of $(n+1)$th level, we see in both cases that there exists a pair of parentheses that encloses only the numbers $\xi_{i'}, \xi_{i'+1}, \ldots, \xi_{j'}$.

Summing up, f generates the considered nth level subexpression. $\square$

Final step.

Assume that there are functions that generate the 1st level subexpressions, furthermore satisfy $(H1)$, $(H2)$ and (C). It remains to show that the original parenthetical expression can be generated. Repeat Steps 1 and 2 with $n = 0$. Let $f = \hat{f}$, where $\hat{f}$ is obtained in Step 2. It is clear that f

fulfills $(H1)$, $(H2)$ and admits N unstable fixed points. One needs to repeat the argument in the proof of Proposition 6.4 with $i=1$ and $j=N$ to show that if $q\colon \mathbb{R}\to\mathbb{R}$ is an $\left[i', j'\right]$ periodic solution with $1\le i'<j'<N$ or with $1<i'<j'\le N$, then an index $l\in\{1,\ldots,k\}$ can be given such that $\xi_{i'},\xi_{i'+1},\ldots,\xi_{j'}\in I_l$. Then it is easy to see – as in the proof of Corollary 6.5 – that f generates the original parenthetical expression of N numbers. We omit the details of this part.

The proof of Theorem 2.1.(i) is complete. Note that all the periodic solutions we constructed are slowly oscillatory. This property is needed to verify Theorem 2.1.(ii).

7. On the Floquet multipliers (the proof of Theorem 2.1.(ii))

Let us recall some facts from Floquet theory. Let $\mu=1$ and suppose $f\colon \mathbb{R}\to\mathbb{R}$ satisfies $(H1)$. Suppose $p:\mathbb{R}\to\mathbb{R}$ is a nonconstant periodic solution of equation

$$\dot{x}\left(t\right)=-x\left(t\right)+f\left(x\left(t-1\right)\right) \tag{3.1}$$

with minimal period $\omega\in(1,2)$.

Consider the monodromy operator $M=D_2\Phi\left(\omega,p_0\right)$. It is well known that $M\varphi=z^{\varphi}_{\omega}$ for all $\varphi\in C$, where $z^{\varphi}:[-1,\infty)\to\mathbb{R}$ is the solution of the linear variational equation

$$\dot{z}(t)=-z(t)+f'\left(p\left(t-1\right)\right)z\left(t-1\right) \tag{7.1}$$

with $z^{\varphi}_0=\varphi$. The solutions of (7.1) are given by the variation-of-constants formula:

$$z^{\varphi}\left(t\right)=e^{n-t}z^{\varphi}\left(n\right)+\int_n^t e^{s-t}f'\left(p\left(s-1\right)\right)z^{\varphi}\left(s-1\right)\mathrm{d}s \tag{7.2}$$

for all nonnegative integers n and $t\in[n,n+1]$.

As mentioned in the introduction, M is a compact operator, and 0 belongs to its spectrum $\sigma=\sigma\left(M\right)$. Eigenvalues of finite multiplicity form $\sigma\left(M\right)\setminus\{0\}$. These eigenvalues are called Floquet multipliers. As $\dot{p}$ is a nonzero solution of the variational equation (7.1), 1 is a Floquet multiplier with eigenfunction $\dot{p}_0$. The periodic orbit $\mathcal{O}_p=\{p_t\colon\ t\in[0,\omega)\}$ is said to be hyperbolic if the generalized eigenspace of M corresponding to the eigenvalue 1 is one-dimensional, furthermore there are no Floquet multipliers on the unit circle besides 1.

The Floquet multipliers are invariant under the time shifts of p. If $a\neq c$ and $b\neq d$, then the Floquet multipliers are also invariant under the linear transformation $L_{a\to b,c\to d}$ mapping a to c and b to d: Consider the periodic function $q\colon\mathbb{R}\to\mathbb{R}$ defined by $q\left(t\right)=L_{a\to b,c\to d}\,p\left(t\right)$, $t\in\mathbb{R}$. Then q is a periodic solution of $\dot{y}\left(t\right)=-y\left(t\right)+g\left(y\left(t-1\right)\right)$, where

$$g\colon\mathbb{R}\ni u\mapsto L_{a\to b,c\to d}\,f\left(L_{b\to a,d\to c}u\right)\in\mathbb{R}.$$

As $g'\left(q\left(t-1\right)\right)=f'\left(p\left(t-1\right)\right)$ for all $t\in\mathbb{R}$, we see that the monodromy operator corresponding to q and g is also determined by the linear variational equation (7.1), i.e., it is the same as the monodromy operator corresponding to p and f.

Let

$$D:=\{\varphi\in C:\varphi\left(s\right)\geq 0\text{ for all }s\in[0,1]\}\qquad\text{and}\qquad \widetilde{D}:=\{\varphi\in D:\varphi\left(0\right)>0\}.$$

The interior of D is

$$\mathring{D} = \{\varphi \in C : \varphi(s) > 0 \text{ for all } s \in [0,1]\}.$$

The formula (7.2) shows that $M(D) \subset D$ and $M\left(\widetilde{D}\right) \subset \mathring{D}$. Furthermore, we see from (7.1) that for each $\varphi \in D$, the function $[0,\infty) \ni t \mapsto e^t z^\varphi(t) \in \mathbb{R}$ is nondecreasing. In particular, $z^\varphi(t) \geq e^{-t}\varphi(0)$ for all $\varphi \in D$ and $t \geq 0$.

We know from paper [18] of Mallet-Paret and Sell or from Appendix VII of monograph [13] of Krisztin, Walther and Wu that $\mathcal{O}_p$ has a real Floquet multiplier $\lambda_1 > 1$ with a strictly positive eigenvector v_1 if $f'(u) > 0$ for all $u \in \mathbb{R}$. Modifying the argument shown in [13], one can prove the same assertion under the weaker assumption $f'(u) \geq 0$, $u \in \mathbb{R}$. Here we give the proof only for the sake of completeness.

Proposition 7.1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $(H1)$, and $p : \mathbb{R} \to \mathbb{R}$ is a periodic solution of equation (3.1) with minimal period $\omega \in (1,2)$. Then there exists $\lambda > 1$ and $\varphi \in \mathring{D}$ such that $M\varphi = \lambda\varphi$.*

Proof. The first step of the proof is to show that $\lambda > 0$ and $\varphi \in \mathring{D}$ can be given with $M\varphi = \lambda\varphi$. Consider the closed, convex and bounded set

$$A = \left\{\varphi \in D : \ \varphi(0) = 1, \ [-1,0] \ni t \mapsto e^t\varphi(t) \in \mathbb{R} \text{ is nondecreasing}\right\} \subset \widetilde{D}.$$

If $\varphi \in A$, then $M\varphi = z^\varphi_\omega \in \mathring{D}$, and $[-1,0] \ni t \mapsto e^t z^\varphi_\omega(t) \in \mathbb{R}$ is nondecreasing. So the map

$$T : \ A \ni \varphi \mapsto \frac{1}{z^\varphi(\omega)} M\varphi \in C$$

is continuous and has range in A. Using the variation-of-constants formula (7.2), one can derive a uniform bound for all $|z^\varphi(t)|$, $t \in [-1,\omega]$, $\varphi \in A$. Then equation (3.1) yields a uniform bound for $|\dot{z}^\varphi(t)|$, $t \in [0,\omega]$, $\varphi \in A$. Also note that $z^\varphi(\omega) \geq e^{-\omega}$. Hence the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t} T(\varphi)(t) = \frac{1}{z^\varphi(\omega)} \dot{z}^\varphi(t+\omega), \qquad \varphi \in A, \ t \in [-1,0],$$

are also uniformly bounded. By the Arzelà–Ascoli theorem, $T(A) \subset A$ is precompact. The Schauder fixed point theorem yields that T has a fixed point, that is, there exists $\varphi \in A$ so that $M\varphi = z^\varphi(\omega)\varphi$. Set $\lambda = z^\varphi(\omega)$. We have already pointed out that $\lambda > 0$. In addition,

$$\varphi = \frac{1}{\lambda} M\varphi \in \mathring{D}.$$

The next step is to verify that $\lambda > 1$. First assume that $\lambda \in (0,1)$. Then $\varphi + \varepsilon\dot{p}_0 \in D$ for some $\varepsilon > 0$ and $M^n(\varphi + \varepsilon\dot{p}_0) = \lambda^n\varphi + \varepsilon\dot{p}_0 \to \varepsilon\dot{p}_0 \notin D$ as $n \to \infty$, which contradicts the fact that $M(D) \subset D$. Next assume that $\lambda = 1$. We may suppose (by shifting p if necessary) that $\dot{p}(0) > 0$. Choose $r > 0$ such that $\varphi + r\dot{p}_0 \in D \backslash \mathring{D}$, i.e., $\varphi(s) + r\dot{p}_0(s) \geq 0$ for all $s \in [-1,0]$ and there exists $s^* \in [-1,0]$ with $\varphi(s^*) + r\dot{p}_0(s^*) = 0$. Then, on the one hand, $M(\varphi + r\dot{p}_0) = \varphi + r\dot{p}_0 \in D \backslash \mathring{D}$. On the other hand, $\varphi(0) + r\dot{p}_0(0) > 0$, hence $\varphi + r\dot{p}_0 \in \widetilde{D}$ and $M(\varphi + r\dot{p}_0) \in \mathring{D}$. We have obtained a contradiction. Therefore $\lambda > 1$. $\square$

Regarding the location of the Floquet multipliers, Theorem 2.1.(ii) states more than Proposition 7.1.

We need Poincaré return maps. Let a closed linear subspace $H \subset C$ of codimension 1 be given so that $p_0 \in H$ and $\dot{p}_0 \notin H$. As before, let Φ denote the solution semiflow corresponding to (3.1), and let x^{φ} denote the solution of (3.1) with initial segment φ. An application of the implicit function theorem yields a convex bounded open neighborhood N of p_0 in H, $\nu \in (0,\omega)$ and a C^1-map $\gamma : N \to (\omega-\nu, \omega+\nu)$ with $\gamma(p_0) = \omega$ so that for each $(t,\varphi) \in (\omega-\nu,\omega+\nu) \times N$, segment x_t^{φ} belongs to H if and only if $t = \gamma(\varphi)$ (see [3,15] and Appendix I in [13]). The Poincaré map P is given by

$$P : N \ni \varphi \mapsto \Phi(\gamma(\varphi), \varphi) \in H.$$

Then P is continuously differentiable, and p_0 is a fixed point of P. In addition, P depends smoothly on the right hand side of (3.1) [15].

Let $\sigma(DP(p_0))$ denote the spectrum of $DP(p_0) : H \to H$. We obtain from Theorem XIV.4.5 in [3] that $\sigma(DP(p_0)) \setminus \{0,1\} = \sigma(M) \setminus \{0,1\}$. For every $\lambda \in \sigma(M) \setminus \{0,1\}$, the projection along $\mathbb{R}\dot{p}_0$ onto H defines an isomorphism from the realified generalized eigenspace of λ and M onto the realified generalized eigenspace of λ and $DP(p_0)$. This means that $\lambda \neq 1$ is a simple Floquet multiplier if and only if λ is a simple eigenvalue of $DP(p_0)$. By Theorem XIV.4.5, $1 \notin \sigma(DP(p_0))$ if and only if the generalized eigenspace associated with 1 and M is one-dimensional. It follows that $\mathcal{O}_p$ is hyperbolic if and only if $DP(p_0)$ has no eigenvalues on the unit circle.

The periodic solutions in the proof of Theorem 2.1 all arise in the form $L_{a\to b, c\to d}\, p$, where $a \neq c$, $b \neq d$, and $p : \mathbb{R} \to \mathbb{R}$ is given by Proposition 4.1. For this reason let us consider a nonlinearity $f \in \mathcal{F}(K)$ with $K > K_0$ and the periodic solution $p : \mathbb{R} \to \mathbb{R}$ of Proposition 4.1. The initial function of p is $p_0 = \Sigma(a^*)$, where a^* is defined by (4.19). In the following let us use any other notation introduced in Section 4. Recall that the minimal period of p is $\omega = \tau + 1 \in (1,2)$. By construction, $p_0 \in H = \{\varphi \in C : \varphi(-1) = 0\}$ and $\dot{p}_0 \notin H$. Consider the corresponding Poincaré map P.

Since P is C^1-smooth and has fixed point $\Sigma(a^*)$, there exists a convex open neighborhood $\hat{N} \subset N$ of p_0 in H so that $P^2 = P \circ P$ is defined on $\hat{N}$. We will use the following observation regarding the range of P^2.

Proposition 7.2. *Consider the periodic solution $p : \mathbb{R} \to \mathbb{R}$ of Proposition 4.1. There exists an open neighborhood $V \subseteq \hat{N}$ of p_0 in H so that if $\varphi \in V$, then $P^2(\varphi) \in \Sigma\left(U^3\right)$.*

Proof. If $\varphi \to p_0$ in C-norm, then $x_1^{\varphi} \to p_1$ in C^1-norm. Hence if $\varphi \in V$, where V is an appropriate open ball in H centered at p_0, then $\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4 \in \mathbb{R}$ can be given close to t_1, t_2, t_3, t_4, respectively, such that

$$-1 < \bar{t}_1 < \bar{t}_2 < \bar{t}_3 < 0 < \bar{t}_1 + 1 < \bar{t}_4,$$
$$\varphi\left(\bar{t}_1\right) = \varphi\left(\bar{t}_2\right) = 1, \quad \varphi\left(\bar{t}_3\right) = x^{\varphi}\left(\bar{t}_4\right) = -1,$$
$$\varphi(s) > 1 \text{ for all } s \in \left(\bar{t}_1, \bar{t}_2\right) \qquad \text{and} \qquad x^{\varphi}(s) < -1 \text{ for all } s \in \left(\bar{t}_3, \bar{t}_4\right).$$

It follows that x^{φ} is of type (K) on $\left[\bar{t}_1 + 1, \bar{t}_2 + 1\right]$ and of type $(-K)$ on $\left[\bar{t}_3 + 1, \bar{t}_4 + 1\right]$.

If V is small enough, then x^φ has a smallest positive zero $\overline{\tau}$ close to $\tau \in (t_4, t_2+1)$ in the interval $\left(\overline{t}_4, \overline{t}_2+1\right)$. Moreover, since x^φ is of type (K) on $\left[\overline{t}_4, \overline{\tau}\right] \subset \left[\overline{t}_1+1, \overline{t}_2+1\right]$ and $x^\varphi\left(\overline{t}_4\right) = -1$, it is of type $(K, -1)$ on $\left[\overline{t}_4+1, \overline{\tau}+1\right]$.

Observe that $P(\varphi) = x^\varphi_{\overline{\tau}+1}$, and we have already verified that

(a) $P(\varphi)(-1) = 0$,

(b) $P(\varphi)$ is of type (K) on $\left[-1, \overline{t}_2 - \overline{\tau}\right]$,

(c) $P(\varphi)$ is of type $(-K)$ on $\left[\overline{t}_3 - \overline{\tau}, \overline{t}_4 - \overline{\tau}\right]$,

(d) $P(\varphi)$ is of type $(K, -1)$ on $\left[\overline{t}_4 - \overline{\tau}, 0\right]$.

If we set $s_1 = \overline{t}_2 - \overline{\tau}$, $s_2 = \overline{t}_3 - \overline{\tau}$ and $s_3 = \overline{t}_4 - \overline{\tau}$, properties (a)–(d) resemble properties (i), (ii), (iv) and (v) of Remark 4.2. However, for any small neighborhood V of p_0 in H, one can find $\varphi \in V$ so that the equality $s_2 = s_1 + T_1$ is not satisfied. Regarding condition (iii) in Remark 4.2, we also cannot guarantee that $P(\varphi)$ is of type $(-K, 1)$ on $[s_1, s_2]$. Hence it may happen that $P(\varphi) \notin \Sigma\left(U^1\right)$ and thus $P(\varphi) \notin \Sigma\left(U^3\right)$.

By construction, $p(t_3+1) > 1$ and $p(t_4+1) < -1$. Therefore we may achieve, by shrinking the radius of V, that $x^\varphi\left(\overline{t}_3+1\right) > 1$ and $x^\varphi\left(\overline{t}_4+1\right) < -1$. In other words, we may achieve that $P(\varphi)\left(\overline{t}_3 - \overline{\tau}\right) > 1$ and $P(\varphi)\left(\overline{t}_4 - \overline{\tau}\right) < -1$. For such initial function φ, let $J \subset \left[\overline{t}_3 - \overline{\tau}, \overline{t}_4 - \overline{\tau}\right]$ denote the subinterval mapped by $P(\varphi)$ onto $[-1, 1]$. By property (c), $P(\varphi)$ is of type $(-K)$ on J. It follows that the length of J is T_1, and $x^{P(\varphi)}$ is of type $(-K, 1)$ on $J + 1 = \{t+1 : t \in J\}$. Repeating the argument above, now it is easy to see that if we take the neighborhood V small enough, then $P^2(\varphi)$ satisfies all conditions (i)–(v) of Remark 4.2.

Using the smooth dependence of solutions on initial data and decreasing the radius of V further, we can achieve that $P^2(\varphi)$ satisfies conditions (vi)–(vii) of Remark 4.5, and thus $P^2(\varphi) \in \Sigma\left(U^3\right)$. $\square$

Let us recall Proposition 4.3 from [10].

Proposition 7.3. *Suppose that $\mathcal{U}_0$ and $\mathcal{U}_1$ are open subsets of $\mathbb{R}^m$, $\mathcal{U}_1 \subset \mathcal{U}_0$ and $u_0 \in \mathcal{U}_1$. Let X be a real Banach space, $\mathcal{V}_0, \mathcal{V}_1$ be open subsets of X with $\mathcal{V}_1 \subset \mathcal{V}_0$, and let $x_0 \in \mathcal{V}_1$. Assume that the maps*

$$Q : \mathcal{U}_0 \to \mathbb{R}^m,\ R : \mathcal{U}_0 \to X,\ S : \mathcal{V}_0 \to X$$

are C^1-smooth, $Q(u_0) = u_0$, $R(u_0) = x_0$, $S(x_0) = x_0$, $Q(\mathcal{U}_1) \subset \mathcal{U}_0$, $S(\mathcal{V}_1) \subset R(\mathcal{U}_1) \subset \mathcal{V}_0$, moreover, $DR(u_0) \in \mathcal{L}(\mathbb{R}^m, X)$ is injective and $S(R(u)) = R(Q(u))$ for all $u \in \mathcal{U}_1$. Then

$$\sigma(DS(x_0)) = \{0\} \cup \sigma(DQ(u_0)),$$

and for each $\lambda \in \sigma(DS(x_0)) \setminus \{0\}$, the corresponding generalized eigenspaces of $DS(x_0)$ and $DQ(u_0)$ have the same dimension.

Now we are in position to complete the proof of Theorem 2.1.

Proof of Theorem 2.1.(ii). Recall that all the periodic solutions determined in the proof of Theorem 2.1.(i) are slowly oscillatory. They can be written in the form $L_{a\to b,c\to d}\,p$, where $a \neq c$, $b \neq d$, and $p : \mathbb{R} \to \mathbb{R}$ is given by Proposition 4.1. As the Floquet multipliers are invariant under such linear transformations, it suffices to prove that the periodic orbits given by Proposition 4.1

are hyperbolic and have exactly one real Floquet multiplier outside the unit circle. We show that this Floquet multiplier is greater than 1 and simple.

Set $X = H$ and $m = 1$. Choose u_0 to be the fixed point a^* of F in U^3 given by (4.19), and let $x_0 = p_0 = \Sigma(a^*)$. Let $\mathcal{U}_0$ be the open set on which $F^2 = F \circ F$ is defined:

$$\mathcal{U}_0 = \left\{a \in U^3 : \ F(a) \in U^3\right\}.$$

Choose $\mathcal{V}_0 = V$, where V is the open neighborhood of $x_0 = p_0$ in H given by Proposition 7.2. Set

$$\mathcal{U}_1 = \left\{a \in \mathcal{U}_0 : \ F^2(a) \in \mathcal{U}_0 \text{ and } \Sigma(a) \in \mathcal{V}_0\right\}.$$

Then $\mathcal{U}_1 \subset \mathcal{U}_0$ is open and $u_0 \in \mathcal{U}_1$. Let $\mathcal{V}_1 \subset \mathcal{V}_0$ be an open ball with $x_0 \in \mathcal{V}_1$ and $P^2(\mathcal{V}_1) \subset \Sigma(\mathcal{U}_1)$. This set exists because $P^2(x_0) = x_0 \in \Sigma(\mathcal{U}_1)$, P^2 is continuous, P^2 maps $\mathcal{V}_0$ into $\Sigma\left(U^3\right)$ by Proposition 7.2, and $\Sigma(\mathcal{U}_1)$ is an open subset of $\Sigma\left(U^3\right)$.

Define

$$Q = F^2 : \mathcal{U}_0 \to \mathbb{R}, \quad R = \Sigma : \mathcal{U}_0 \to H, \quad S = P^2 : \mathcal{V}_0 \to H.$$

Proposition 4.8 shows that Q is C^1-smooth, Proposition 4.3 gives that R is C^1-smooth and $DR(u_0)$ is injective. The map S is also smooth. Clearly $Q(u_0) = u_0$, $R(u_0) = x_0$ and $S(x_0) = x_0$, moreover, $Q(\mathcal{U}_1) \subset \mathcal{U}_0$, $R(\mathcal{U}_1) \subset \mathcal{V}_0$ and $S(\mathcal{V}_1) \subset R(\mathcal{U}_1)$ hold. It is also clear that $S(R(u)) = R(Q(u))$ for all $u \in \mathcal{U}_1$.

As Q is a one-dimensional map, Proposition 7.3 yields that $DS(x_0)$ has at most one nontrivial eigenvalue which is simple. It follows that $DP(x_0) = DP(p_0)$ also has at most one nontrivial eigenvalue which is simple. (Indeed, if μ is an eigenvalue of $DP(x_0)$, then μ^2 is an eigenvalue of $DP(x_0) \circ DP(x_0) = DP^2(x_0) = DS(x_0)$, and the generalized eigenspace of $DP(x_0)$ associated to μ is a subset of the generalized eigenspace of $DS(x_0)$ associated to μ^2.) On the other hand, from $\sigma(DP(p_0)) \setminus \{0, 1\} = \sigma(M) \setminus \{0, 1\}$ and from Proposition 7.1 it follows that $DP(p_0)$ has at least one real eigenvalue that is greater than 1. Summing up, $DP(p_0)$ has exactly one nontrivial eigenvalue λ, which is simple, real and greater than 1. □

Notice that, although we used Proposition 7.3 with $Q = F^2$, we could avoid calculating $DF(a^*)$ with the aid of Proposition 7.1.

8. Perturbations of the feedback function

For $U \subseteq \mathbb{R}$ open, let $C_b^1(U, \mathbb{R})$ denote the space of bounded continuously differentiable functions $g : U \to \mathbb{R}$ with bounded first derivative. We consider the usual C^1-norm on $C_b^1(U, \mathbb{R})$. The nonlinearity constructed in the proof of Theorem 2.1 belongs to $C_b^1(\mathbb{R}, \mathbb{R})$.

The following proposition is a particular case of a more general theorem of Lani-Wayda [15]. This result is the key to our second main theorem considering perturbed nonlinearities.

Proposition 8.1. *Assume that $\mu > 0$, $f \in C_b^1(\mathbb{R}, \mathbb{R})$ and p is a periodic solution of equation (1.1) with minimal period $\omega > 1$ such that $\mathcal{O}_p = \{p_t : t \in [0, \omega)\}$ is hyperbolic. Let a closed linear subspace $H \subset C$ of codimension 1 be given so that $p_0 \in H$ and $\dot{p}_0 \notin H$. Let $U \subset \mathbb{R}$ be* *open with $\{p(t) : t \in [0, \omega)\} \subset U$. Then there exists an open ball $B \subset C_b^1(U, \mathbb{R})$ centered at f, an open neighborhood V of p_0 in H and a C^1-function $\chi : B \to V \subset H$ with $\chi(f) = p_0$ such that for $g \in B$, the solution $x^{\chi(g)}$ of*

$$\dot{x}(t) = -\mu x(t) + g(x(t-1)) \tag{8.1}$$

with initial value $\chi(g)$ is periodic (and therefore can be defined on $\mathbb{R}$). The minimal period of $x^{\chi(g)}$ is in $(\omega - \nu, \omega + \nu)$ with some $\nu > 0$. If $g \in B$, and $\varphi \in V$ is the initial segment of any periodic solution of (8.1) with minimal period in $(\omega - \nu, \omega + \nu)$, then $\varphi = \chi(g)$. If $\|g - f\|_{C_b^1(U,\mathbb{R})} \to 0$, then $\chi(g) \to \chi(f) = p_0$ in C.

The hyperbolicity of the periodic orbits implies that Theorem 2.1 remains true for certain perturbations of the feedback function. The second main result of the paper is the following.

Theorem 8.2. *Fix a parenthetical expression of N numbers, where $N \geq 2$. Set μ and f so that $(H0)$, $(H1)$ and $(H2)$ are satisfied, and Theorem 2.1 holds. Then there exists an open subset $U \subset \mathbb{R}$ and an open ball $B \subset C_b^1(U, \mathbb{R})$ centered at f such that Theorem 2.1 remains true for all nondecreasing $g \in B$.*

Proof. Consider a parenthetical expression of $N \geq 2$ numbers, μ and f as given in the theorem.

Even if we do not distinguish those periodic solutions that can be obtained from each other by translation of time, we cannot exclude that equation (1.1) has an infinite number of large-amplitude slowly oscillatory periodic solutions. First we select a finite number of them. Choose $r > 0$ and slowly oscillatory periodic solutions $p^1, p^2, \dots, p^r : \mathbb{R} \to \mathbb{R}$ so that whenever the numbers $i, i+1, \dots, j$ are enclosed by a pair of parentheses (not containing further numbers) in the expression under consideration, then an index $k \in \{1, 2, \dots, r\}$ can be given such that p^k is an $[i, j]$ periodic solution. By our initial assumption, these solutions can be chosen such that the corresponding orbits are hyperbolic and have one real Floquet multiplier outside the unit circle, which is simple and greater than 1.

Fix an open subset $U \subset \mathbb{R}$ containing all the fixed points of f and including the ranges of $p^1, p^2, \dots, p^r$.

It is clear that if $g \in C_b^1(U, \mathbb{R})$ is close to f in C_b^1-norm, then (8.1) has the same amount of equilibria with the same stability properties. Moreover, if

$$\hat{\xi}_1 \leq \hat{\xi}_2 \leq \ldots \leq \hat{\xi}_N \quad \text{and} \quad \hat{\xi}_1^g \leq \hat{\xi}_2^g \leq \ldots \leq \hat{\xi}_N^g$$

denote the unstable fixed points of f and g, respectively, then

$$\left\| \hat{\xi}_i^g - \hat{\xi}_i \right\| \to 0 \quad \text{for all } i \in \{1, \dots, N\} \text{ as } \|g - f\|_{C_b^1(U,\mathbb{R})} \to 0. \tag{8.2}$$

Let $k \in \{1, 2, \dots, r\}$ be arbitrary. Set $1 \leq i < j \leq N$ such that p^k is an $[i, j]$ periodic solution. As the minimal period of p^k is greater than 1, and the corresponding orbit is hyperbolic, it comes from Proposition 8.1 and (8.2) that a ball $B_k \subset C_b^1(U, \mathbb{R})$ centered at f can be given such that for all $g \in B_k$, equation (8.1) also has a periodic solution $p^{k,g} : \mathbb{R} \to \mathbb{R}$ oscillating about $\xi_i^g, \xi_{i+1}^g, \dots, \xi_j^g$ and no other unstable fixed points of g. By Proposition 8.1, we may assume that the minimal period of $p^{k,g}$ is in $(1, 2)$. Remark 3.3 shows that if $g \in B_k$ is nondecreasing, then $p^{k,g}$ is slowly oscillatory. As the Floquet multipliers depend continuously on the feedback function, we may also assume that $\mathcal{O}_{p^{k,g}} = \left\{ p_t^{k,g} : \ t \in \mathbb{R} \right\}$ has exactly one Floquet multiplier outside the unit circle, which is real, greater than 1, and simple.

It remains to exclude the existence of unrequested large-amplitude periodic solutions. Suppose for contradiction that for some $1 \le i < j \le N$, the numbers $i, i+1, \dots, j$ are not enclosed by a pair of parentheses, and there exists a sequence $(g^n)_{n=1}^{\infty}$ of nondecreasing functions in $\cap_{k=1}^{r} B_k$ such that for all $n \ge 1$, $\|g^n - f\|_{C_b^1} < 1/n$ holds, and equation

$$\dot{x}(t) = -\mu x(t) + g^n(x(t-1)) \tag{8.3}$$

has a large-amplitude periodic solution $q^n : \mathbb{R} \to \mathbb{R}$ oscillating about $\xi_i^{g^n}, \xi_{i+1}^{g^n}, \dots, \xi_j^{g^n}$ and no other unstable fixed point of g^n.

We can easily confirm that the minimal period $\omega^n > 0$ of q^n is smaller than 2 for each $n \ge 1$. Consider Proposition 3.2.(i)–(ii) with $p = q^n$ and $\chi = \xi_i^{g^n}$. We may suppose, by considering a suitable time translate of q^n, that $q^n(t) \ge \xi_i^{g^n}$ for $t \in \left[0, \nu^n\right]$ and $q^n(t) < \xi_i^{g^n}$ for $t \in (\nu^n, \omega^n)$ with some $\nu^n \in (0, \omega^n)$. If $\nu^n \ge 1$ for some n, then Proposition 3.1 would imply that $q^n(t) \ge \xi_i^{g^n}$ for all $t > \nu^n$, which is impossible. So $\nu^n < 1$. Similarly, $\omega^n - \nu^n < 1$. Summing up, $\omega^n < 2$.

Since

$$\sup_{x \in \mathbb{R}} \left| g^n(x) \right| \le \left\| g^n \right\|_{C_b^1} \le \|f\|_{C_b^1} + 1, \qquad n \ge 1,$$

Proposition 4.9 yields that $\left\| q_t^n \right\| \le \|f\|_{C_b^1} + 1$ for all $n \ge 1$ and $t \in \mathbb{R}$. Then (8.3) gives a uniform upper bound for $\left\| \dot{q}_t^n \right\|$, $n \ge 1$, $t \in \mathbb{R}$. The Arzelà–Ascoli theorem hence implies the existence of a subsequence $(q^{n_k})_{k=1}^{\infty}$ that converges to a continuous function $q : \mathbb{R} \to \mathbb{R}$ as $k \to \infty$ uniformly on each compact subset of $\mathbb{R}$. As $(\omega^n)_{n=1}^{\infty}$ is bounded, we may suppose that $\omega^{n_k} \to \omega \ge 0$ as $k \to \infty$. It is easy to see (e.g., by using the variation-of-constant formula) that q is a periodic solution of (1.1) with minimal period ω. It is also clear that q is an $\left[i, j\right]$ periodic solution of (1.1). As f generates the parenthetical expression, we arrived at a contradiction.

It follows that exists an open ball $B \subset \cap_{k=1}^{r} B_k$ centered at f such that equation (8.1) admits exactly the required large-amplitude periodic solutions for all nondecreasing $g \in B$, i.e., Theorem 2.1 remains true for all nondecreasing $g \in B$. $\square$

9. Closing remarks

9.1. The unstable sets of the periodic orbits

Consider a strictly increasing nonlinear function $g \in B$ and any large-amplitude slowly oscillatory (LSOP) solution $p : \mathbb{R} \to \mathbb{R}$ given by Theorem 8.2. As the orbit $\mathcal{O}_p = \{p_t : \ t \in \mathbb{R}\}$ is hyperbolic, and it has exactly one Floquet multiplier outside the unit circle, we expect the unstable set

$$\mathcal{W}^u\left(\mathcal{O}_p\right) = \left\{ \varphi \in C : \ x^{\varphi} \text{ exists on } \mathbb{R} \text{ and } x_t^{\varphi} \to \mathcal{O}_p \text{ as } t \to -\infty \right\}$$

to be a two-dimensional C^1-submanifold of C. Let $\hat{\zeta}_-$ and $\hat{\zeta}_+$ denote the stable equilibria with the property that ζ_- is the maximal fixed point of g with $\zeta_- < \min_{t\in\mathbb{R}} p(t)$ and ζ_+ is the minimal fixed point of g with $\zeta_+ > \max_{t\in\mathbb{R}} p(t)$. We claim that $\mathcal{W}^u(\mathcal{O}_p)\setminus\mathcal{O}_p$ is the union of the two-dimensional heteroclinic sets

$$C_-^p = \left\{\varphi \in \mathcal{W}^u(\mathcal{O}_p) : x_t^\varphi \to \hat{\zeta}_- \text{ as } t \to \infty\right\}$$

and

$$C_+^p = \left\{\varphi \in \mathcal{W}^u(\mathcal{O}_p) : x_t^\varphi \to \hat{\zeta}_+ \text{ as } t \to \infty\right\}.$$

9.2. *The exact number of LSOP solutions*

Let us call two periodic solutions $p: \mathbb{R} \to \mathbb{R}$ and $q: \mathbb{R} \to \mathbb{R}$ significantly different if no constant $T \in \mathbb{R}$ can be given such that $p(t+T) = q(t)$ for all $t \in \mathbb{R}$.

Our main results (Theorem 2.1 or Theorem 8.2) have not discussed the exact number of significantly different slowly oscillatory $[i, j]$ periodic solutions in the case when we do have $[i, j]$ periodic solutions. In general we cannot expect uniqueness. For $N = 2$, the paper [10] has given two slowly oscillatory $[1, 2]$ periodic solutions, and the periodic orbit corresponding to the first solution has three Floquet multipliers outside the unit circle, while the second one has only one. It is an open question whether there exist slowly oscillatory $[1, j]$ periodic solutions for $j \geq 3$ such that the corresponding orbit have more than one Floquet multiplier outside the unit circle.

Although we cannot guarantee uniqueness, we can guarantee the existence of an arbitrary number of $[i, j]$ solutions. This statement can be formulated precisely as follows. Fix $N \geq 2$ and a parenthetical expression of N numbers. Assign an arbitrary positive integer $k_{i,j}$ to all numbers i and j such that $1 \leq i < j \leq N$ and the integers $i, i+1, \ldots, j$ are enclosed by a pair of parenthesis not containing further numbers. Then there exists μ and f satisfying $(H0)$–$(H2)$ such that Theorem 2.1 holds with the addition that if there is a pair of parentheses in the expression that contains only the numbers $i, i+1, \ldots, j$, then equation (1.1) has at least $k_{i,j}$ significantly different $[i, j]$ periodic solutions $p_1, p_2, \ldots, p_{k_{i,j}}$.

We do not intend to give a rigorous proof. We indicate the idea by giving a nonlinearity f^k for all $k \geq 1$ such that f^k satisfies $(H1)$ and $(H2)$, f^k generates $(12\ldots N)$, and equation (3.1) has LSOP solutions $p_1, p_2, \ldots, p_k$ with $(\xi_1, \xi_N) \subset p_1(\mathbb{R}) \subsetneq p_2(\mathbb{R}) \subsetneq \ldots \subsetneq p_k(\mathbb{R})$. This construction goes by induction on k. If $k = 1$, then we are ready by Propositions 5.4 and 5.5. Suppose we have already obtained the nonlinearity f^k for some $k \geq 1$. Then define f^{k+1} as

$$f^{k+1}: \mathbb{R} \ni x \mapsto \begin{cases} K_1, & x \leq \min p_k(\mathbb{R}) - \eta \\ \rho_1\left(f^k(x)\right), & x \in (\min p_k(\mathbb{R}) - \eta, \min p_k(\mathbb{R})) \\ f^k(x), & x \in p_k(\mathbb{R}) \\ \rho_2\left(f^k(x)\right), & x \in (\max p_k(\mathbb{R}), \max p_k(\mathbb{R}) + \eta) \\ K_2, & x \geq \max p_k(\mathbb{R}) + \eta, \end{cases}$$

where $\eta > 0$ is small, and K_1, K_2, ρ_1 and ρ_2 are defined so that $f^{k+1}: \mathbb{R} \to \mathbb{R}$ satisfies $(H1)$ and $(H2)$, furthermore

$$\rho_1\left(f^k(x)\right) < f^k(x) < x \text{ for all } x \in (\min p_k(\mathbb{R}) - \eta, \min p_k(\mathbb{R}))$$

and

$$\rho_2\left(f^k(x)\right) > f^k(x) > x \text{ for all } x \in (\max p_k(\mathbb{R}), \max p_k(\mathbb{R}) + \eta).$$

Using the techniques of this paper, it is easy to see that – with suitably chosen η, K_1, K_2, ρ_1 and ρ_2 – function f^{k+1} possesses the required properties.

9.3. *Further periodic solutions*

A periodic solution of (1.1) is said to have small amplitude if it oscillates only about one unstable fixed point of $f_\mu : \mathbb{R} \ni u \mapsto f(u)/\mu \in \mathbb{R}$. It is easy to guarantee the existence of such solutions: We know that as $f'(\xi_i)$ increases for some $i \in \{1, \ldots, N\}$, small-amplitude periodic solutions oscillating about ξ_i appear via a series of Hopf bifurcations [9,13,14]. However, it is an open problem whether we can ensure their nonexistence for the nonlinearities discussed in the paper. A related result on the nonexistence of small-amplitude periodic solutions is found in [12].

This paper has not studied the existence of large-amplitude rapidly oscillatory periodic (LROP) solutions either. We call a solution $x : [-1, \infty) \to \mathbb{R}$ rapidly oscillatory if for any fixed point χ of f_μ in the range $x(\mathbb{R})$ of x, the function $[-1, \infty) \ni t \mapsto x(t) - \chi \in \mathbb{R}$ has at least three sign changes on each interval of length 1. We conjecture that the existence of LROP solutions can be excluded for the nondecreasing feedback functions in Theorems 2.1 and 8.2 by refining our construction. It would suffice to show that if $K > K_0$ is not too large in Proposition 4.10 and $f \in \mathcal{F}(K)$ is nondecreasing, then equation (3.1) has no periodic solutions with minimal period smaller than 1.

Acknowledgments

Gabriella Vas was supported by the Postdoctoral Fellowship Programme of the Hungarian Academy of Sciences and by the Hungarian Scientific Research Fund, Grant No. K109782. The author would like to thank Tibor Krisztin, for calling her attention to the problem discussed in the paper. The author also thanks Judit Nagy-György and Szabolcs Iván, for determining formula (2.3) for $\mathcal{C}_N$.

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