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Diffusive wave in the low Mach limit for non-viscous and heat-conductive gas ☆

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Abstract

The low Mach number limit for one-dimensional non-isentropic compressible Navier–Stokes system without viscosity is investigated, where the density and temperature have different asymptotic states at far fields. It is proved that the solution of the system converges to a nonlinear diffusion wave globally in time as Mach number goes to zero. It is remarked that the velocity of diffusion wave is proportional with the variation of temperature. Furthermore, it is shown that the solution of compressible Navier–Stokes system also has the same phenomenon when Mach number is suitably small.
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Keywords: Compressible Navier–Stokes equations; Low Mach limit; Nonlinear diffusion wave

1. Introduction

The one-dimensional compressible Navier–Stokes system in Lagrangian coordinates reads

$$\begin{aligned} &v_t - u_x = 0,\\ &u_t + P_x = (\mu \frac{u_x}{v})_x,\\ &(e + \frac{u^2}{2})_t + (Pu)_x = (\kappa \frac{\theta_x}{v} + \mu \frac{u u_x}{v})_x, \end{aligned} \tag{1.1}$$

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where the unknown functions v, u and θ represent the specific volume, velocity and temperature, respectively, while $\mu > 0$ and $\kappa > 0$ denote the viscosity and heat conductivity coefficients respectively. Here, we consider the perfect gas, so that the pressure function P and the internal function e are given by

$$P = R\frac{\theta}{v}, \quad \text{and} \quad e = c_v\theta + const.,$$

where the parameters $R > 0$ and $c_v > 0$ are the gas constant and heat capacity at the constant volume respectively. For simplicity, we assume μ and κ are constants, and normalize $R = 1$ and $c_v = 1$.

The low Mach limit is an important and interesting problem in fluid dynamics. The first result is due to Klainerman and Majda [16,17], in which they proved the incompressible limit of the isentropic Euler equations to the incompressible Euler equations for local smooth solutions. In [1], Alazard showed the incompressible limit for Navier–Stokes equations in the whole space. Note that in [1], the solutions have the same states at the far fields. For related results, see [5,18, 20,21,24,25,30] for Euler equations, and [6,9,12,15,22,28,29] for N–S equations.

Recently, Huang et al. [13] began to study the case that the solutions have different end states and found that the solutions of compressible Navier–Stokes equations converge to a nonlinear diffusion wave solution globally in time as Mach number goes to zero, which is related to the thermal creep flow [8]. That is, the flow in diffusion wave is only driven by the variation of temperature. This phenomenon is quite different from the constant case. Since the diffusion wave is independent of the viscosity $\mu = 0$, we conjecture that the result of [13] is still valid without viscosity, that is $\mu = 0$ in the system (1.1). Precisely speaking, we consider the non-viscous and heat-conductive gas in the following system

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + P_x = 0, \\ &(\theta + \frac{u^2}{2})_t + (Pu)_x = (\kappa\frac{\theta_x}{v})_x, \end{aligned} \tag{1.2}$$

where the only difference with the system (1.1) is that $\mu = 0$. We will prove that the solution of system (1.2) converges to a nonlinear diffusion wave solution globally in time as Mach number tends to zero. Moreover, as the Mach number is suitably small, the flow is only driven by the variation of temperature.

Let ε be the compressibility parameter, which represents the maximum Mach number of the fluid. As in [26], we set

$$t \to \varepsilon t, \quad x \to x, \quad u \to \varepsilon u, \quad \mu \to \varepsilon\mu, \quad \kappa \to \varepsilon\kappa.$$

By the above changes of variables, system (1.2) is written as

$$\begin{aligned} &v^\varepsilon_t - u^\varepsilon_x = 0, \\ &u^\varepsilon_t + \frac{1}{\varepsilon^2}P^\varepsilon_x = 0, \\ &[\theta^\varepsilon + \frac{1}{2}(\varepsilon u^\varepsilon)^2]_t + (P^\varepsilon u^\varepsilon)_x = \kappa(\frac{\theta^\varepsilon_x}{v^\varepsilon})_x. \end{aligned} \tag{1.3}$$

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As $\varepsilon \to 0$, the limit of solutions of (1.3) is called the low Mach limit [16,17]. Similar to [1,13,26], we assume that the pressure is a small perturbation of a given constant state $\bar{P} > 0$, i.e.

$$P = \bar{P} + O(\varepsilon), \tag{1.4}$$

and without loss of generality, we further assume $\bar{P}$ to be 1. Formally, as the Mach number goes to zero, the limit system of (1.3) is

$$\begin{aligned} &(2u - \kappa T_x)_x = 0, \ \ \rho = T^{-1}, \\ &u_t + \pi_x = 0, \\ &\theta_t + u_x = \kappa(\frac{\theta_x}{v})_x, \end{aligned} \tag{1.5}$$

and

$$v_t - u_x = 0,$$

in which π_x is the limit of $\frac{1}{\varepsilon^2} P_x^\varepsilon$.

We will study the low Mach limit and what happens in the limiting process when the background is not constant state, i.e.

$$(v^\varepsilon, \theta^\varepsilon)(x,t) \to (v_\pm, \theta_\pm), \quad \text{as } x \to \pm\infty, \quad \text{with } \frac{\theta_-}{v_-} = \frac{\theta_+}{v_+}, \tag{1.6}$$

where θ_- may not be equal to θ_+. Following [13], we shall construct a spacial diffusive wave $(\bar{v}, \bar{u}, \bar{\theta}, \bar{\pi})$ of (1.5) by choosing

$$(\bar{v}, \bar{u}, \bar{\theta}) \triangleq (T, \frac{\kappa T_x}{2T}, T), \tag{1.7}$$

where $T(\eta)$, $\eta = \frac{x}{\sqrt{1+t}}$ is the unique self-similar solution of the following diffusion equation

$$T_t = (\frac{\kappa T_x}{2T})_x, \qquad \lim_{x \to \pm\infty} T = \theta_\pm, \tag{1.8}$$

and $\bar{\pi}$ is a solution of $(1.5)_2$. For the existence and uniqueness of the self-similar solution of (1.8), see [2], [3] and [27]. Set $\delta = |\theta_+ - \theta_-|$, then the $T(x,t)$ has the following asymptotic expression

$$T_x(x,t) = O(1)\delta(1+t)^{-\frac{1}{2}} e^{-\frac{x^2}{4d(\theta_\pm)(1+t)}}, \quad d(\theta_\pm) = \frac{\kappa}{2\theta_\pm}, \quad \text{as } x \to \pm\infty. \tag{1.9}$$

(We will prove this approximation, which is used in [10,11,14,23], by the idea and method from [7] in **Appendix**.) Because $(\bar{v}, \bar{u}, \bar{\theta})$ is not the solution of (1.3), there will be some non-integrated error terms in the system. It is necessary to introduce a new profile to approximate towards system (1.3). That is

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$$(\tilde{v}, \tilde{u}, \tilde{\theta}) \triangleq (\bar{v}, \bar{u}, \bar{\theta} - \frac{1}{2}(\varepsilon\bar{u})^2). \tag{1.10}$$

Then, we have

$$\|(\bar{v} - \tilde{v}, \bar{u} - \tilde{u}, \bar{\theta} - \tilde{\theta})\|_{L^2_x} \leqslant C\varepsilon^2(1+t)^{-1},$$

which implies that $(\tilde{v}, \tilde{u}, \tilde{\theta})$ approximates the diffusive wave solution $(\bar{v}, \bar{u}, \bar{\theta})$ in L^2-norm. A direct calculation implies

$$\begin{aligned} &\tilde{v}_t - \tilde{u}_x = 0, \\ &\varepsilon^2\tilde{u}_t + \tilde{P}_x = R_{1x}, \\ &[\tilde{\theta} + \frac{1}{2}(\varepsilon\tilde{u})^2]_t + (\tilde{P}\tilde{u})_x = \kappa(\frac{\tilde{\theta}_x}{\tilde{v}})_x + R_{2x}, \end{aligned} \tag{1.11}$$

in which

$$R_1 = \varepsilon^2\frac{\kappa T_t}{2T} - \varepsilon^2\frac{\tilde{u}^2}{2T} = O(1)\delta\varepsilon^2(1+t)^{-1}e^{-\frac{c\pm x^2}{1+t}}, \tag{1.12}$$

and

$$R_2 = \varepsilon^2\kappa\frac{\tilde{u}\tilde{u}_x}{T} - \varepsilon^2\frac{\tilde{u}^3}{2T} = O(1)\delta\varepsilon^2(1+t)^{-\frac{3}{2}}e^{-\frac{c\pm x^2}{1+t}}, \tag{1.13}$$

as $x \to \pm\infty$. The initial data of (1.3) is given by

$$(v^\varepsilon, u^\varepsilon, \theta^\varepsilon)|_{t=0} = (\tilde{v}, \tilde{u}, \tilde{\theta})(x, 0). \tag{1.14}$$

Then we obtain the following global existence and uniform estimates.

Theorem 1 *(Uniform estimates). Let $(\tilde{v}, \tilde{u}, \tilde{\theta})$ be the diffusive wave defined by (1.10) and set $\delta = |\theta_+ - \theta_-|$ to be the wave strength. Then there exist constants $\varepsilon_0 > 0$ and $\delta_0 > 0$, such that if $\varepsilon \leqslant \varepsilon_0$ and $\delta \leqslant \delta_0$, the Cauchy problem (1.3) with the initial values (1.14) has a unique global smooth solution $(v^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ satisfying*

$$\begin{aligned} &\|(v^\varepsilon - \tilde{v}, \varepsilon u^\varepsilon - \varepsilon\tilde{u}, \theta^\varepsilon - \tilde{\theta})\|^2_{L^2_x} \leqslant C\varepsilon^3\sqrt{\delta}(1+t)^{-1+C_0\sqrt{\delta}}, \\ &\|(v^\varepsilon - \tilde{v}, \varepsilon u^\varepsilon - \varepsilon\tilde{u}, \theta^\varepsilon - \tilde{\theta})_x\|^2_{L^2_x} \leqslant C\varepsilon^{\frac{8}{3}}\sqrt{\delta}(1+t)^{-\frac{3}{2}+C_0\sqrt{\delta}}, \\ &\text{and} \qquad \|(\theta^\varepsilon - \tilde{\theta})_{xx}\|^2_{L^2_x} \leqslant C\varepsilon^{\frac{2}{3}}\sqrt{\delta}(1+t)^{-\frac{3}{2}+C_0\sqrt{\delta}}, \end{aligned} \tag{1.15}$$

where C and C_0 are constants independent of ε and δ, and $\|\cdot\|_{L^2_x}$ denotes the L^2-norm with respect to x.

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Note that from (1.6), (1.10), (1.14) and (1.15), it is straightforward to see that $(v^\varepsilon - \tilde{v}, \varepsilon u^\varepsilon - \varepsilon\tilde{u}) \in (L_t^\infty(H_x^1(\mathbb{R})))^2$, $\theta^\varepsilon - \tilde{\theta} \in L_t^\infty(H_x^2(\mathbb{R}))$. Hence the Gagliardo–Nirenberg's inequalities

$$\begin{aligned}
&\|(v^\varepsilon - \tilde{v}, \varepsilon u^\varepsilon - \varepsilon\tilde{u}, \theta^\varepsilon - \tilde{\theta})\|_{L_x^\infty}^2 \\
&\quad \leqslant C\|(v^\varepsilon - \tilde{v}, \varepsilon u^\varepsilon - \varepsilon\tilde{u}, \theta^\varepsilon - \tilde{\theta})\|_{L_x^2}\|(v^\varepsilon - \tilde{v}, \varepsilon u^\varepsilon - \varepsilon\tilde{u}, \theta^\varepsilon - \tilde{\theta})_x\|_{L_x^2} \\
&\text{and} \quad \|(\theta^\varepsilon - \tilde{\theta})_x\|_{L_x^\infty}^2 \leqslant C\|(\theta^\varepsilon - \tilde{\theta})_x\|_{L_x^2}\|(\theta^\varepsilon - \tilde{\theta})_{xx}\|_{L_x^2},
\end{aligned}$$

immediately imply that

Corollary 2 *(Low Mach limit). Under the assumptions of* ***Theorem 1****, in addition, suppose* $\delta_0 \leqslant \frac{1}{16C_0}$*, we have, as* $\varepsilon \to 0$*,*

$$\begin{aligned}
\|(v^\varepsilon - \tilde{v}, \theta^\varepsilon - \tilde{\theta})\|_{L_x^\infty} &\leqslant C\varepsilon^{\frac{17}{12}}\delta^{\frac{1}{4}}(1+t)^{-\frac{1}{2}} \to 0, \\
\|u^\varepsilon - \tilde{u}\|_{L_x^\infty} &\leqslant C\varepsilon^{\frac{5}{12}}\delta^{\frac{1}{4}}(1+t)^{-\frac{1}{2}} \to 0, \\
\text{and} \quad \|(\theta^\varepsilon - \tilde{\theta})_x\|_{L_x^\infty} &\leqslant C\varepsilon^{\frac{5}{6}}\delta^{\frac{1}{4}}(1+t)^{-\frac{1}{2}} \to 0.
\end{aligned} \tag{1.16}$$

From **Corollary 2**, we shall show the behaviors of solutions of (1.3) when the Mach number ε is small.

Without lose of generality, we assume that $\theta_+ > \theta_-$. Then for any given constant $\eta_0 > 0$, by (1.9), there exists $C_{\eta_0} > 0$ such that

$$T'(\eta) > C_{\eta_0}\delta, \quad \text{for } |\eta| \leqslant \eta_0.$$

From (1.7), (1.10) and (1.16), it follows that

Corollary 3 *(Driven by the variation of temperature). For the solution obtained by* ***Theorem 1****, there exists constants* $\varepsilon_1 = \varepsilon_1(\eta_0) \leqslant \varepsilon_0$ *and* $\tilde{C}_0 > 0$*, such that for* $\varepsilon \leqslant \varepsilon_1$*,*

$$\begin{aligned}
0 < \frac{C_{\eta_0}\delta}{\tilde{C}_0\sqrt{1+t}} < \frac{\tilde{\theta}_x}{\tilde{C}_0} \leqslant u^\varepsilon \leqslant \tilde{C}_0\tilde{\theta}_x, \\
0 < \frac{1}{2}\tilde{\theta}_x \leqslant \theta_x^\varepsilon \leqslant \frac{3}{2}\tilde{\theta}_x,
\end{aligned}$$

and

$$\bar{C}_0^{-1}\theta_x^\varepsilon \leqslant u^\varepsilon \leqslant \bar{C}_0\theta_x^\varepsilon, \qquad \text{for } |x| < \eta_0(1+t)^{\frac{1}{2}}, t \geqslant 0, \tag{1.17}$$

where C_1 *is a positive constant depending only on* $\theta_\pm$*.*

Remark 4. Due to $\tilde{u} = \frac{\kappa T_x}{2T}$, the velocity $\tilde{u}$ of system (1.1) is proportional with the variation of temperature $\tilde{\theta}$. Thus the estimate (1.17) shows that, as ε is suitably small, the velocity u^ε of system (1.3) is also proportional with the variation of temperature.

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Finally let us outline the proof of **Theorem 1**. Since there is no viscosity in $(1.3)_2$, the system (1.3) is less dissipative compared with the system (1.1). It is not trivial to obtain the higher order estimates. Indeed, even for the basic energy estimate, the derivative estimate for ψ_y^2 is missing, see (2.13) below. To prove **Theorem 1**, we first obtain the estimates for $\|\zeta_y\|_{H^1}$ though the basic energy. Then, by the compensation method (cf. [19]), we control simultaneously the estimates for $\|(\phi_y, \psi_y)\|_{H^1}$ by $\|\zeta_y\|_{H^1}$. Motivated by [13], a new type differential inequality is essentially used to obtain the desired a priori estimates.

The rest of this paper will be arranged in the following way. Section 2 is devoted to the details of the proof, in which the energy estimates will be used, while the last section is Appendix A to prove an approximation of a self-similar solution, which will be used in the proof of the main results.

2. Proof of Theorem 1

For simplicity, we omit the superscript ε of the variables in this section.

2.1. Reformulation of the system

Set the scaling

$$y = \frac{x}{\varepsilon}, \qquad \tau = \frac{t}{\varepsilon^2},$$

then, from (1.12) and (1.13), we have

$$R_1 = O(1)\delta\varepsilon^2(1+\varepsilon^2\tau)^{-1}e^{-\varepsilon^2\frac{c_\pm y^2}{1+\varepsilon^2\tau}}, \tag{2.1}$$

and

$$R_2 = O(1)\delta\varepsilon^2(1+\varepsilon^2\tau)^{-\frac{3}{2}}e^{-\varepsilon^2\frac{c_\pm y^2}{1+\varepsilon^2\tau}}, \tag{2.2}$$

as $y \to \pm\infty$. Define the perturbation around the profile $(\tilde{v}, \tilde{u}, \tilde{\theta})(\tau, y)$ by

$$(\phi, \psi, \omega, \zeta)(\tau, y) \triangleq (v - \tilde{v}, \varepsilon u - \varepsilon\tilde{u}, \theta + \frac{1}{2}(\varepsilon u)^2 - \tilde{\theta} - \frac{1}{2}(\varepsilon\tilde{u})^2, \theta - \tilde{\theta})(\tau, y),$$

and set

$$(\Phi, \Psi, \bar{W}) \triangleq \int_{-\infty}^{y} (\phi, \psi, \omega)(\tau, y)\mathrm{d}z.$$

It is easy to see from (1.14) that $(\Phi, \Psi, \bar{W})(0, \pm\infty) = 0$, which enables that $(\Phi, \Psi, \bar{W})$ can be defined in Sobolev Space.

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Subtracting (1.11) from (1.3) and integrating the resulting system with respect of y yields

$$\begin{aligned}&\Phi_\tau - \Psi_y = 0,\\ &\Psi_\tau + P - \tilde{P} = -R_1,\\ &\bar{W}_\tau + \varepsilon P u - \varepsilon \tilde{P}\tilde{u} = \kappa\left(\frac{\theta_y}{v} - \frac{\tilde{\theta}_y}{\tilde{v}}\right) - \varepsilon R_2.\end{aligned} \tag{2.3}$$

For the variable $\bar{W}$ is the anti-derivative of the total energy, it is more convenient to introduce another variable which is related to the temperature, that is

$$W \triangleq \bar{W} - \varepsilon\tilde{u}\Psi.$$

By a direct calculation, it holds that

$$\zeta = W_y - Y, \qquad \text{where } Y \triangleq \frac{1}{2}\Psi_y^2 - \varepsilon\tilde{u}_y\Psi. \tag{2.4}$$

Using the new variable W, (2.3) becomes

$$\begin{aligned}&\Phi_\tau - \Psi_y = 0,\\ &\Psi_\tau - \frac{1}{\tilde{v}}\Phi_y + \frac{1}{\tilde{v}}W_y = Q_1,\\ &W_\tau + \Psi_y = \frac{\kappa}{\tilde{v}}W_{yy} + Q_2,\end{aligned} \tag{2.5}$$

where

$$Q_1 = J_1 + \frac{Y}{\tilde{v}} - R_1, \tag{2.6}$$

$$Q_2 = \kappa\left(\frac{1}{v} - \frac{1}{\tilde{v}}\right)\theta_y + J_2 - \varepsilon\tilde{u}_\tau\Psi - \frac{\kappa}{\tilde{v}}Y_y - \varepsilon R_2 + \varepsilon\tilde{u}R_1 \tag{2.7}$$

and

$$J_1 = \frac{\tilde{P}-1}{\tilde{v}}\Phi_y - \left[P - \tilde{P} + \frac{\tilde{P}}{\tilde{v}}(v-\tilde{v}) - \frac{1}{\tilde{v}}(\theta - \tilde{\theta})\right] = O(1)[|(\Phi_y, W_y)|^2 + Y^2 + (\varepsilon\tilde{u})^4], \tag{2.8}$$

$$J_2 = (1-P)\Psi_y = O(1)[|(\Phi_y, \Psi_y, W_y)|^2 + Y^2 + (\varepsilon\tilde{u})^4]. \tag{2.9}$$

Since the local existence of solution of compressible Navier–Stokes system is known, we only need the following a priori estimates to complete the proof of **Theorem 1**.

Proposition 5 *(A priori estimates). Assume that* (Φ, Ψ, W) *is a smooth solution of* (2.5) *with zero initial data in the time interval* $[0, T]$. *Then there exist constants* $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *such that if* $\varepsilon \leqslant \varepsilon_1$ *and* $\delta \leqslant \delta_1$, *then*

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$$\begin{aligned}
&\|(\Phi, \Psi, W)(\tau)\|^2_{L^\infty_y} \leqslant C\varepsilon\sqrt{\delta},\\
&\|(\phi, \psi, \zeta)(\tau)\|^2_{L^2_y} \leqslant C\varepsilon^2\sqrt{\delta}(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}},\\
&\|(\phi_y, \psi_y, \zeta_y)(\tau)\|^2_{L^2_y} \leqslant C\varepsilon^{\frac{11}{3}}\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{3}{2}+C_0\sqrt{\delta}},\\
&\|\zeta_{yy}(\tau)\|^2_{L^2_y} \leqslant C\varepsilon^{\frac{11}{3}}\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{3}{2}+C_0\sqrt{\delta}}.
\end{aligned} \tag{2.10}$$

To obtain the above a priori estimates, we need the following a priori assumption.

$$\begin{aligned}
&\sup_{0\leqslant\tau\leqslant T}\{\|(\Phi, \Psi, W)(\tau)\|^2_{L^\infty_y} + \|(\phi, \psi, \zeta)(\tau)\|^2_{L^2_y} + \frac{1}{\varepsilon^2}\|(\phi_y, \psi_y, \zeta_y)(\tau)\|^2_{L^2_y}\\
&\quad + \frac{1}{\varepsilon^2}\|(\phi_{yy}, \psi_{yy}, \zeta_{yy})(\tau)\|^2_{L^2_y}\} \leqslant \bar{\delta}^2,
\end{aligned} \tag{2.11}$$

where $\bar{\delta}$ is a constant depending only on δ. Immediately, from the a priori assumption and the Galiardo–Nirenberg's inequality, we have

$$\|(\phi, \psi, \zeta)\|^2_{L^\infty_y} \leqslant C\bar{\delta}. \tag{2.12}$$

In the sequel, we will use the energy estimates to prove **Proposition 5**. We will use $\|\cdot\|$ to denote the norm of L^2 with respect to y in the rest of this paper.

2.2. *Basic estimates*

Multiplying $(2.5)_1$ by Φ, $(2.5)_2$ by $\tilde{v}\Psi$ and $(2.5)_3$ by W, respectively, and adding all the resultant equations, we have

$$\begin{aligned}
&(\frac{1}{2}\Phi^2 + \frac{\tilde{v}}{2}\Psi^2 + \frac{1}{2}W^2)_\tau + \frac{\kappa}{\tilde{v}}W_y^2\\
&\quad = \frac{1}{2}\tilde{v}_\tau\Psi^2 - (\frac{\kappa}{\tilde{v}})_y W_y W + \tilde{v}\Psi Q_1 + WQ_2 + (\Phi\Psi - \Psi W + \frac{\kappa}{\tilde{v}}W_y W)_y.
\end{aligned} \tag{2.13}$$

Since the order of the term $\tilde{v}\Psi Q_1$ with respect to $(1+t)$ is not enough if we estimate the right-hand side of (2.13) directly, we need to use a weighted energy method.

Similar to [13], define

$$m = (\Phi, \Psi, W)^t, \tag{2.14}$$

then (2.5) can be written as

$$m_\tau + A_1 m_y = A_2 m_{yy} + A_3, \tag{2.15}$$

where

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{\bar{v}} & 0 & \frac{1}{\bar{v}} \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\kappa}{\bar{v}} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 \\ Q_1 \\ Q_2 \end{pmatrix}.$$

It is easy to see that $\lambda_1 = -\sqrt{\frac{2}{\bar{v}}}, \lambda_2 = 0, \lambda_3 = -\lambda_1$ are the eigenvalues of A_1.

Set

$$L = \frac{1}{2}(l_1, l_2, l_3)^t, \quad R = \frac{1}{2}(r_1, r_2, r_3), \tag{2.16}$$

where $l_i, r_i, i = 1, 2, 3$ are the left and right corresponding eigenvectors and can be chosen as

$$\begin{aligned} &l_1 = (-1, -\frac{2}{\lambda_3}, 1),\ l_2 = (\sqrt{2}, 0, \sqrt{2})\ , l_3 = (-1, \frac{2}{\lambda_3}, 1), \\ &r_1 = (-1, -\lambda_3, 1)^t, r_2 = (\sqrt{2}, 0, \sqrt{2})^t\ , r_3 = (-1, \lambda_3, 1)^t. \end{aligned} \tag{2.17}$$

Then, we have

$$l_i r_j = 4\delta_{ij}, \quad i, j = 1, 2, 3, \qquad \Lambda \triangleq LA_1R = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

and from (2.15), we can get

$$B_\tau + \Lambda B_y = LA_2RB_{yy} + 2LA_2R_yB_y + [(L_\tau + \Lambda L_y)R + LA_2R_{yy}]B + LA_3 \tag{2.18}$$

by multiplying (2.15) by L, where

$$B \triangleq Lm \triangleq (b_1, b_2, b_3)^t,$$

and then

$$\|(b_1, b_3)\| \sim \|(\Phi, \Psi, W)\|, \quad \|b_2\| \sim \|(\Phi, W)\|. \tag{2.19}$$

As what will be mentioned in **Appendix**, we could assume $T_y > 0$, where T is the self-similar solution of (1.8), and set

$$T_1 \triangleq \frac{T}{\theta_+}, \quad \text{then } |T_1 - 1| \leqslant C\delta. \tag{2.20}$$

Let $N > 0$ be a large integer which will be chosen later, and multiply (2.18) by $\tilde{B} \triangleq (T_1^N b_1, b_2, T_1^{-N} b_3)$, then we have

$$\begin{aligned} &(\frac{T_1^N}{2}b_1^2 + \frac{1}{2}b_2^2 + \frac{T_1^{-N}}{2}b_3^2)_\tau + B_yA_4B_y + (\frac{\lambda_1}{2}T_1^N b_1^2 + \frac{\lambda_3}{2}T_1^{-N}b_3^2 - \tilde{B}A_4B_y)_y \\ &-\frac{T_1^{N-1}}{2}(N\lambda_1T_{1y} + T_1\lambda_{1y})b_1^2 + \frac{T_1^{-N-1}}{2}(N\lambda_3T_{1y} + T_1\lambda_{3y})b_3^2 \end{aligned}$$

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$$-\tilde{B}\Lambda L_y RB - (\frac{T_1^N}{2})_\tau b_1^2 - (\frac{T_1^{-N}}{2})_\tau b_3^2$$
$$= (B-\tilde{B})_y A_4 B_y - \tilde{B}A_{4y}B_y + 2\tilde{B}LA_2R_yB_y + \tilde{B}(L_\tau R + LA_2R_{yy})B + \tilde{B}LA_3, \quad (2.21)$$

where

$$A_4 \triangleq LA_2R = \frac{\kappa}{4\tilde{v}}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

is a nonnegative symmetric matrix.

Set

$$E_1 \triangleq \int (\frac{1}{2}\Phi^2 + \frac{\tilde{v}}{2}\Psi^2 + \frac{1}{2}W^2)\mathrm{d}y + \int (\frac{T_1^N}{2}b_1^2 + \frac{1}{2}b_2^2 + \frac{T_1^{-N}}{2}b_3^2)\mathrm{d}y,$$
$$\text{and} \quad K_1 \triangleq \int \frac{\kappa}{\tilde{v}}W_y^2\mathrm{d}y + \int B_yA_4B_y\mathrm{d}y, \quad (2.22)$$

then we have $E_1 \sim \|(\Phi, \Psi, W)\|^2$ and $\|W_y\|^2 \leqslant CK_1$.

Since

$$|\tilde{B}\Lambda L_y RB| \leqslant CT_{1y}(T_1^N\lambda_1 b_1^2 + T_1^{-N}\lambda_3 b_3^2),$$

we can choose N large enough so that

$$-\frac{T_1^{N-1}}{2}(N\lambda_1 T_{1y} + T_1\lambda_{1y})b_1^2 + \frac{T_1^{-N-1}}{2}(N\lambda_3 T_{1y} - T_1\lambda_{3y})b_3^2 - \tilde{B}\Lambda L_y RB$$
$$\geqslant 2\int |T_y|(b_1^2 + b_3^2). \quad (2.23)$$

Next we need to estimate the integral of the right hand side of (2.13) and (2.21) term by term. Using the Cauchy's inequality, we obtain

$$\left|\int \frac{1}{2}\tilde{v}_\tau\Psi^2\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}\|\Psi\|^2,$$

and

$$\left|\int (\frac{\kappa}{\tilde{v}})_y W_y W\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}\|W\|^2 + C\delta\|W_y\|^2.$$

Note that (2.19), then

$$\left|\int (\tilde{v}\Psi Q_1 + WQ_2)\mathrm{d}y\right| \leqslant CI, \quad \text{where } I = \int [|(b_1, b_3)| \cdot |(Q_1, Q_2)| + |b_2Q_2|]\mathrm{d}y. \quad (2.24)$$

For the right-hand side of (2.21), from (2.19), we have

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$$\left|\int [(\frac{T_1^N}{2})_\tau b_1^2 + (\frac{T_1^{-N}}{2})_\tau b_3^2]\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}E_1.$$

Since A_4 is nonnegative and (2.20) holds, by Cauchy's inequality, we can obtain

$$\begin{aligned}\left|\int (\tilde{B}-B)_y A_4 B_y \mathrm{d}y\right| &= \left|\int [(T_1^N - 1, 0, T_1^{-N} - 1)B]_y A_4 B_y \mathrm{d}y\right| \\ &\leqslant \frac{C}{\delta}\int |T_y|^2|B|^2\mathrm{d}y + C\delta\int |B_y|^2\mathrm{d}y \\ &\leqslant C\varepsilon^2\delta(1+t)^{-1}E_1 + C\delta\|(\Phi_y, \Psi_y, W_y)\|^2.\end{aligned}$$

It is easy to check that the terms $\tilde{B}A_{4y}B_y$, $\tilde{B}LA_2R_yB_y$ and $\tilde{B}(L_\tau R + LA_2R_{yy})B$ satisfy the same estimate. For the last term, it is obvious from (2.16) and (2.17) that

$$\left|\int \tilde{B}LA_3\mathrm{d}y\right| \leqslant CI,$$

where I is defined by (2.24).

Substituting all the above estimates into equations (2.13) and (2.21), using (2.22) and (2.23), we have

$$E_{1\tau} + K_1 + 2\int |T_y|(b_1^2 + b_3^2)\mathrm{d}y \leqslant C\varepsilon^2\delta(1+t)^{-1}(E_1+1) + C\delta\|(\Phi_y, \Psi_y, W_y)\|^2 + CI.$$

Next, we only need to calculate I to complete the basic estimate. Since the term εR_2 and $\varepsilon\tilde{u}R_1$ in Q_2 have better decay rate than the term R_1 in Q_1, it is naturally that we only need to estimate $\left|\int Q_1 b_1\mathrm{d}y\right|$ and $\left|\int Q_2 b_2\mathrm{d}y\right|$.

(2.4), (2.8) and (2.12) yield that

$$\int |J_1 b_1|\mathrm{d}y \leqslant C\varepsilon^2\delta(1+t)^{-1}(E_1+1) + C\bar{\delta}\|(\Phi_y, \Psi_y, W_y)\|^2$$

and

$$\int |\frac{Y}{\tilde{v}} b_1|\mathrm{d}y \leqslant C\varepsilon^2\delta(1+t)^{-1}E_1 + C\bar{\delta}\|\Psi_y\|^2.$$

From (2.1), we have

$$\int |R_1 b_1|\mathrm{d}y \leqslant \delta\int |T_y| b_1^2\mathrm{d}y + C\varepsilon^2\delta(1+t)^{-1}.$$

Then, it follows from (2.6) that

$$\int |Q_1 b_1|\mathrm{d}y \leqslant \delta\int |T_y| b_1^2\mathrm{d}y + C\varepsilon^2\delta(1+t)^{-1}(E_1+1) + C\bar{\delta}\|(\Phi_y, \Psi_y, W_y)\|^2.$$

Similarly, from (2.1), (2.2) and (2.9), we obtain

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$$\int (|\Phi_y\tilde{\theta}_y b_1| + |J_2 b_2| + |\varepsilon\tilde{u}_\tau \Phi b_2| + |\varepsilon R_2 b_2| + |\varepsilon\tilde{u} R_1 b_2|)\mathrm{d}y$$
$$\leqslant C\varepsilon^2\delta(1+t)^{-1}(E_1+1) + C(\delta+\bar{\delta})\|(\Phi_y, W_y)\|^2.$$

Using Cauchy's inequality, we obtain from (2.12) that

$$\int |\Phi_y\zeta_y b_2|\mathrm{d}y \leqslant C\bar{\delta}(\|\Phi_y\|^2 + \|\zeta_y\|^2)$$

and

$$\int |Y_y b_2|\mathrm{d}y \leqslant C\varepsilon^2\delta(1+t)^{-1}E_1 + C(\delta+\bar{\delta})\|\Psi_y\|^2 + C\bar{\delta}\|\psi_y\|^2.$$

Thus, it follows from (2.7) that

$$\left|\int Q_2 b_2 \mathrm{d}y\right| \leqslant C\int (|\phi\zeta_y b_2| + |\phi\tilde{\theta}_y b_2| + |J_2 b_2| + |\varepsilon\tilde{u}_\tau\Phi b_2| + |Y_y b_2|$$
$$+ |\varepsilon R_2 b_2| + |\varepsilon\tilde{u} R_1 b_2|)\mathrm{d}y$$
$$\leqslant C\varepsilon^2\delta(1+t)^{-1}(E_1+1) + C(\delta+\bar{\delta})\|(\Phi_y, W_y)\|^2 + C\bar{\delta}\|(\psi_y, \zeta_y)\|^2.$$

Using the above estimates, we have

$$E_{1\tau} + \frac{1}{2}K_1 + \int |T_y|(b_1^2 + b_3^2)\mathrm{d}y$$
$$\leqslant C\varepsilon^2\delta(1+t)^{-1}(E_1+1) + C(\delta+\bar{\delta})\|(\Phi_y, \Psi_y)\|^2 + C\bar{\delta}\|(\psi_y, \zeta_y)\|^2$$

by assuming δ and $\bar{\delta}$ are suitably small.

Since the norm $\|(\Phi_y, \Psi_y)\|$ cannot be controlled by K_1, we need to use the compensation matrix technique. Multiplying $(2.5)_2$ by $-\frac{1}{2}\Phi_y$ and $(2.5)_3$ by Ψ_y, respectively, and adding both the resultant equations, we obtain that

$$\frac{1}{2\tilde{v}}\Phi_y^2 + \frac{1}{2}\Psi_y^2 + (W\Psi_y - \frac{1}{2}\Phi_y\Psi)_\tau$$
$$= (W\Psi_\tau - \frac{1}{2}\Phi_\tau\Psi)_y + \frac{1}{2\tilde{v}}W_y\Phi_y - W_y(Q_1 + \frac{1}{\tilde{v}}\Phi_y - \frac{1}{\tilde{v}}W_y)$$
$$- \frac{1}{2}\Phi_y Q_1 + \frac{\kappa}{\tilde{v}}W_{yy}\Psi_y + \Psi_y Q_2, \tag{2.25}$$

by using $(2.5)_1$ and $(2.5)_2$. Integrating (2.25) with respect of y, using (2.4) and the Cauchy's inequality, and suppose δ and $\bar{\delta}$ are suitably small, we can get

$$\int (\frac{1}{4\tilde{v}}\Phi_y^2 + \frac{1}{4}\Psi_y^2)\mathrm{d}y + \left[\int (W\Psi_y - \frac{1}{2}\Phi_y\Psi)\mathrm{d}y\right]_\tau \leqslant C_1 K_1 + C_1\|\zeta_y\|^2 + C\varepsilon^3\delta(1+t)^{-\frac{3}{2}}.$$

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Choose $\tilde{C}_1$ large enough so that

$$0 \leqslant \frac{1}{4}\int(\Phi_y^2+\Psi_y^2)\mathrm{d}y+\frac{1}{2}\tilde{C}_1E_1 \leqslant \int(W\Psi_y-\frac{1}{2}\Phi_y\Psi)\mathrm{d}y+\tilde{C}_1E_1 \triangleq \tilde{E}_1$$

$$\text{and} \qquad \frac{1}{4}\tilde{C}_1K_1 \leqslant \frac{1}{2}\tilde{C}_1K_1-C_1K_1.$$

Set

$$\tilde{K}_1=\frac{1}{4}\tilde{C}_1K_1+\int(\frac{1}{4\tilde{v}}\Phi_y^2+\frac{1}{4}\Psi_y^2)\mathrm{d}y, \qquad \text{then} \quad \|(\Phi_y,\Psi_y,W_y)\|^2\sim\tilde{K}_1.$$

By a direct calculation, we have

Lemma 6 *(Basic estimate). If δ and $\bar{\delta}$ are suitably small, it holds*

$$\tilde{E}_{1\tau}+\frac{1}{2}\tilde{K}_1+\int|T_y|(b_1^2+b_3^2)\mathrm{d}y \leqslant C\varepsilon^2\delta(1+t)^{-1}(\tilde{E}_1+1)+C\bar{\delta}\|(\psi_y,\zeta_y)\|^2+C_1\|\zeta_y\|^2. \tag{2.26}$$

2.3. First-order derivative estimates

In order to estimate $\|(\phi,\psi,\zeta)\|$, we need to use the convex entropy. By applying ∂y to (2.3), we have

$$\begin{aligned}&\phi_\tau-\psi_y=0,\\&\psi_\tau+(P-\tilde{P})_y=-R_{1y},\\&\zeta_\tau+\varepsilon Pu_y-\varepsilon\tilde{P}\tilde{u}_y=\kappa(\frac{\theta_y}{v}-\frac{\tilde{\theta}_y}{\tilde{v}})_y+Q_3,\end{aligned} \tag{2.27}$$

where

$$Q_3=\varepsilon\tilde{P}_y\tilde{u}+\frac{1}{2}\varepsilon^2(\tilde{u}^2)_\tau-\varepsilon R_{2y} \tag{2.28}$$

and we used the fact that $\varepsilon u_\tau+P_y=0$ which comes from $(1.3)_2$.

Set

$$F(s)=s-1-\ln s,$$

then, it is obvious that $F'(1)=0$, and $F(s)$ is strictly convex around $s=1$. Moreover, using Taylor's formula, we obtain that

$$C_1\phi^2 \leqslant F(\frac{v}{\tilde{v}}) \leqslant C_2\phi^2, \qquad C_1\zeta^2 \leqslant F(\frac{\theta}{\tilde{\theta}}),F(\frac{\tilde{\theta}}{\theta}) \leqslant C_2\zeta^2, \tag{2.29}$$

for some positive constants $C_1, C_2>0$. By a direct calculation, we get

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$$[\tilde{\theta}F(\frac{v}{\tilde{v}})]_\tau = \tilde{\theta}_\tau F(\frac{v}{\tilde{v}}) - \tilde{\theta}(\frac{1}{v} - \frac{1}{\tilde{v}})\phi_\tau - \frac{\tilde{P}\phi^2}{v\tilde{v}}\tilde{v}_\tau, \tag{2.30}$$

and

$$[\tilde{\theta}F(\frac{\theta}{\tilde{\theta}})]_\tau = \frac{\zeta}{\theta}\zeta_\tau - \tilde{\theta}_\tau F(\frac{\tilde{\theta}}{\theta}). \tag{2.31}$$

Multiplying $(2.27)_2$ by ψ and $(2.27)_3$ by $\frac{\zeta}{\theta}$, respectively, it holds that

$$\psi_\tau\psi - \frac{\zeta}{v}\psi_y - \tilde{\theta}(\frac{1}{v} - \frac{1}{\tilde{v}})\phi_y + [(P - \tilde{P})\psi]_y = -\psi R_{1y}, \tag{2.32}$$

and

$$\frac{\zeta}{\theta}\zeta_\tau + \frac{\zeta}{v}\psi_y - \varepsilon\frac{\zeta}{\theta}(P - \tilde{P})\tilde{u}_y = \kappa(\frac{\theta_y}{v} - \frac{\tilde{\theta}_y}{\tilde{v}})_y\frac{\zeta}{\theta} + \frac{\zeta}{\theta}Q_3. \tag{2.33}$$

Therefore, it follows from (2.30)–(2.33) that

$$\begin{aligned}[\tilde{\theta}F(\frac{v}{\tilde{v}}) + \frac{1}{2}\psi^2 + \tilde{\theta}F(\frac{\theta}{\tilde{\theta}})]_\tau = {}& \tilde{\theta}_\tau F(\frac{v}{\tilde{v}}) - \frac{\tilde{P}\tilde{v}_\tau}{v\tilde{v}}\phi^2 - \psi R_{1y} - \tilde{\theta}_\tau F(\frac{\tilde{\theta}}{\theta}) + \varepsilon\frac{\zeta}{\theta}(P - \tilde{P})\tilde{u}_y \\ & + [\kappa(\frac{\theta_y}{v} - \frac{\tilde{\theta}_y}{\tilde{v}})\frac{\zeta}{\theta} - (P - \tilde{P})\psi]_y - \kappa(\frac{\theta_y}{v} - \frac{\tilde{\theta}_y}{\tilde{v}})(\frac{\zeta}{\theta})_y + \frac{\zeta}{\theta}Q_3.\end{aligned} \tag{2.34}$$

Set

$$E_2 = \int [\tilde{\theta}F(\frac{v}{\tilde{v}}) + \frac{1}{2}\psi^2 + \tilde{\theta}F(\frac{\theta}{\tilde{\theta}})]\mathrm{d}y, \qquad \text{and } K_2 = \int \frac{\kappa}{v\theta}\zeta_y^2\mathrm{d}y, \tag{2.35}$$

then by (2.29), we have $E_2 \sim \|(\phi, \psi, \zeta)\|^2$ and there exist a constant $\bar{C}_1 > 0$ such that

$$\|\zeta_y\|^2 \leqslant \bar{C}_1 K_2. \tag{2.36}$$

Next we shall estimate the integral of the terms of the right hand side of (2.34) as in section 2.2. It is easy to see from the definition of the function F that

$$\left|\int \tilde{\theta}_\tau F(\frac{v}{\tilde{v}})\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}\|\phi\|^2,$$

and

$$\left|\int \tilde{\theta}_\tau F(\frac{\tilde{\theta}}{\theta})\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}\|\zeta\|^2.$$

Note that $\Psi = \frac{1}{2}\lambda_3(b_3 - b_1)$, which together with (2.1), yield that

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$$\left|\int \psi R_{1y}\mathrm{d}y\right| = \left|\int \Psi R_{1yy}\mathrm{d}y\right| \leqslant \varepsilon^2(1+t)^{-1}\int |T_y|(b_1^2+b_3^2)\mathrm{d}y + C\varepsilon^4\delta(1+t)^{-2},$$

and from (2.28), (1.10) and (2.2), we could obtain

$$\left|\int \frac{\zeta}{\theta}Q_3\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}\|\zeta\|^2 + C\varepsilon^5\delta(1+t)^{-\frac{5}{2}}.$$

With direct calculation, we have

$$\left|\int \frac{\tilde{P}\tilde{v}_\tau}{v\tilde{v}}\phi^2\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}\|\phi\|^2,$$

and

$$\left|\int \varepsilon\frac{\zeta}{\theta}(P-\tilde{P})\tilde{u}_y\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}\|(\phi,\zeta)\|^2.$$

For the term containing κ, we need to reform it by

$$\kappa(\frac{\theta_y}{v}-\frac{\tilde{\theta}_y}{\tilde{v}})(\frac{\zeta}{\theta})_y = \frac{\kappa}{v\theta}\zeta_y^2 + \frac{\kappa}{\theta}\tilde{\theta}_y(\frac{1}{v}-\frac{1}{\tilde{v}})\zeta_y + \kappa(\frac{1}{\theta})_y[\frac{\zeta_y}{v}+\tilde{\theta}_y(\frac{1}{v}-\frac{1}{\tilde{v}})]\zeta.$$

Then, since

$$\left|\int \frac{\kappa}{\theta}\tilde{\theta}_y\zeta_y(\frac{1}{v}-\frac{1}{\tilde{v}})\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}\|\phi\|^2 + C\delta\|\zeta_y\|^2$$

and

$$\left|\int \kappa(\frac{1}{\theta})_y\zeta[\frac{\zeta_y}{v}+\tilde{\theta}_y(\frac{1}{v}-\frac{1}{\tilde{v}})]\mathrm{d}y\right| \leqslant C(\delta+\bar{\delta}+\sqrt{\bar{\delta}})\|\zeta_y\|^2 + C\varepsilon^2\delta(1+t)^{-1}\|(\phi,\zeta)\|^2$$

from the a priori assumption, the following lemma holds.

Lemma 7 *(First-order derivative estimate). If δ and $\bar{\delta}$ are suitably small, it holds*

$$E_{2\tau} + \frac{1}{2}K_2 \leqslant \varepsilon^2(1+t)^{-1}\int |T_y|(b_1^2+b_3^2)\mathrm{d}y + C\varepsilon^2\delta(1+t)^{-1}E_2 + C\varepsilon^4\delta(1+t)^{-2}. \quad (2.37)$$

Note that there is the norm of ψ_y in the right hand side of (2.26), we need to control it as what we did to the norm of Φ_y, Ψ_y in section 2.2. Rewrite (2.27) in the following form

$$\begin{aligned} &\phi_\tau - \psi_y = 0,\\ &\psi_\tau - \frac{\theta}{v^2}\phi_y + \frac{1}{v}\zeta_y = J_3,\\ &\zeta_\tau + \frac{\theta}{v}\psi_y = J_4, \end{aligned} \quad (2.38)$$

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where

$$J_3=-(\frac{1}{v}-\frac{1}{\tilde{v}})\tilde{\theta}_y+(\frac{\theta}{v^2}-\frac{\tilde{\theta}}{\tilde{v}^2})\tilde{v}_y-R_{1y},\quad J_4=-\varepsilon(P-\tilde{P})\tilde{u}_y+\kappa(\frac{\theta_y}{v}-\frac{\tilde{\theta}_y}{\tilde{v}})_y-Q_3.$$

Multiplying $(2.38)_2$ by $-\frac{1}{2}\phi_y$ and $(2.38)_3$ by $\frac{v}{\theta}\psi_y$, respectively, and adding both the resultant equations, we can obtain that

$$\begin{aligned}\frac{\theta}{2v^2}\phi_y^2+\frac{1}{2}\psi_y^2+(\frac{v}{\theta}\psi_y\zeta-\frac{1}{2}\phi_y\psi)_\tau=&-\frac{1}{2}(\psi_y\psi)_y+\frac{1}{2}\psi_y^2+\frac{1}{2v}\phi_y\zeta_y+(\frac{v}{\theta})_\tau\psi_y\zeta+\frac{v}{\theta}(\psi_\tau\zeta)_y\\&-(\frac{1}{v}\phi_y-\frac{1}{\theta}\zeta_y+\frac{v}{\theta}J_3)\zeta_y-\frac{1}{2}\phi_yJ_3+\frac{v}{\theta}\psi_yJ_4.\end{aligned}\tag{2.39}$$

Integrating (2.39) with respect of y, using the a priori assumption and the Cauchy's inequality, and suppose δ and $\bar{\delta}$ are suitably small, we can get

$$\int(\frac{\theta}{4v^2}\phi_y^2+\frac{1}{4}\psi_y^2)\mathrm{d}y+\left[\int(\frac{v}{\theta}\zeta\psi_y-\frac{1}{2}\phi_y\psi)\mathrm{d}y\right]_\tau\leqslant C_2K_2+C_2\|\zeta_{yy}\|^2+C\varepsilon^5\delta(1+t)^{-\frac{5}{2}}.\tag{2.40}$$

2.4. Higher-order derivative estimates

For the second-order derivative estimate, we need to rewrite (2.27) in the following form

$$\begin{aligned}&\phi_\tau-\psi_y=0,\\&\psi_\tau-\frac{P}{v}\phi_y+\frac{1}{\tilde{v}}\zeta_y=(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}})\tilde{v}_y-(\frac{1}{v}-\frac{1}{\tilde{v}})\theta_y-R_{1y},\\&\zeta_\tau+\psi_y=\kappa(\frac{\theta_y}{v}-\frac{\tilde{\theta}_y}{\tilde{v}})_y+Q_3+\frac{\varepsilon^2\tilde{u}^2}{2\tilde{v}}\psi_y-\varepsilon(P-\tilde{P})u_y,\end{aligned}\tag{2.41}$$

according to (1.10). Then applying ∂y to (2.41) yields that

$$\begin{aligned}&\phi_{y\tau}-\psi_{yy}=0,\\&\psi_{y\tau}-\frac{1}{\tilde{v}}\phi_{yy}+\frac{1}{\tilde{v}}\zeta_{yy}=Q_4-R_{1yy},\\&\zeta_{y\tau}+\psi_{yy}=\kappa(\frac{\theta_y}{v}-\frac{\tilde{\theta}_y}{\tilde{v}})_{yy}+Q_5,\end{aligned}\tag{2.42}$$

where

$$\begin{aligned}Q_4=&\frac{\tilde{P}-1}{\tilde{v}}\phi_{yy}+(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}})\phi_{yy}+2(\frac{\theta_yv_y}{v^2}-\frac{\tilde{\theta}_y\tilde{v}_y}{\tilde{v}^2})-2(\frac{\theta v_y^2}{v^3}-\frac{\tilde{\theta}\tilde{v}_y^2}{\tilde{v}^3})\\&-(\frac{1}{v}-\frac{1}{\tilde{v}})\theta_{yy}+(\frac{\theta}{v^2}-\frac{\tilde{\theta}}{\tilde{v}^2})\tilde{v}_{yy},\end{aligned}$$

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and

$$Q_5 = Q_{3y} + \frac{\varepsilon^2\tilde{u}^2}{2\tilde{v}}\psi_{yy} + \frac{\varepsilon^2}{2}(\frac{\tilde{u}^2}{\tilde{v}})_y\psi_y - \varepsilon(P-\tilde{P})_y u_y - \varepsilon(P-\tilde{P})u_{yy}.$$

Multiplying $(2.42)_1$ by ϕ_y, $(2.42)_2$ by $\tilde{v}\psi_y$ and $(2.42)_3$ by ζ_y, respectively, and adding all the resultant equations, we can obtain that

$$\begin{aligned}(\frac{1}{2}\phi_y^2 + \frac{\tilde{v}}{2}\psi_y^2 + \frac{1}{2}\zeta_y^2)_\tau = (\frac{\tilde{v}}{2})_\tau\psi_y^2 + (\phi_y\psi_y - \psi_y\zeta_y)_y + \kappa\zeta_y(\frac{\theta_y}{v} - \frac{\tilde{\theta}_y}{\tilde{v}})_{yy}\\ + \zeta_y Q_5 + \tilde{v}\psi_y(Q_4 + R_{1yy}).\end{aligned} \tag{2.43}$$

Set

$$E_3 = \int(\frac{1}{2}\phi_y^2 + \frac{\tilde{v}}{2}\psi_y^2 + \frac{1}{2}\zeta_y^2)\mathrm{d}y, \qquad \text{and } K_3 = \int\frac{\kappa}{v}\zeta_{yy}^2\mathrm{d}y,$$

then we have $E_3 \sim \|(\phi_y, \psi_y, \zeta_y)\|^2$ and there exist a constant $\bar{C}_2 > 0$ such that

$$\|\zeta_{yy}\|^2 \leqslant \bar{C}_2 K_3.$$

Similar to what we did in section 2.2 and 2.3, the integral of the first term in the right hand side of (2.43) can be estimated by

$$\left|\int(\frac{\tilde{v}}{2})_\tau\psi_y^2\mathrm{d}y\right| \leqslant C\varepsilon^2\delta(1+t)^{-1}\|\psi_y\|^2.$$

Dealing with the term containing κ, we need to reform it by

$$\kappa\zeta_y(\frac{\theta_y}{v} - \frac{\tilde{\theta}_y}{\tilde{v}})_{yy} = -\frac{\kappa}{v}\zeta_{yy}^2 + \kappa[\zeta_y(\frac{\theta_y}{v} - \frac{\tilde{\theta}_y}{\tilde{v}})_y]_y - (\frac{\kappa}{v})_y\zeta_y\zeta_{yy} - \kappa[(\frac{1}{v} - \frac{1}{\tilde{v}})\tilde{\theta}_y]_y\zeta_{yy}.$$

Then, we have

$$\begin{aligned}\left|\int(\frac{\kappa}{v})_y\zeta_y\zeta_{yy}\mathrm{d}y\right| &\leqslant C\left(\int|\phi_y\zeta_y\zeta_{yy}|\mathrm{d}y + \int|\tilde{v}_y\zeta_y\zeta_{yy}|\mathrm{d}y\right)\\ &\leqslant (\delta + \sqrt{\bar{\delta}})\|\zeta_{yy}\|^2 + C\varepsilon^2\delta(1+t)^{-1}\|\zeta_y\|^2 + \|(\phi_y, \zeta_y)\|^{\frac{9}{2}}\end{aligned}$$

by using Gagliardo–Nirenberg's and Young's inequalities and the a priori assumption (2.11), and

$$\begin{aligned}\left|\int\kappa[(\frac{1}{v} - \frac{1}{\tilde{v}})\tilde{\theta}_y]_y\zeta_{yy}\mathrm{d}y\right| &\leqslant \int|\phi_y\tilde{\theta}_y\zeta_{yy}|\mathrm{d}y + C\int|\phi\tilde{\theta}_{yy}\zeta_{yy}|\mathrm{d}y\\ &\leqslant \delta\|\zeta_{yy}\|^2 + C\varepsilon^2\delta(1+t)^{-1}\|\phi_y\|^2 + C\varepsilon^4\delta(1+t)^{-2}\|\phi\|^2.\end{aligned}$$

Note that we only have $\|\zeta_{yy}\|^2$ in the left hand side, it needs to be careful to estimate the rest terms, not to bring the norms of ϕ_{yy} and ψ_{yy}.

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Since

$$\left|\int \zeta_y Q_{3y}\mathrm{d}y\right| \leqslant C\varepsilon\delta(1+t)^{-\frac{1}{2}}\|\zeta_y\|^2 + C\varepsilon^8\delta(1+t)^{-4},$$

$$\left|\int (\frac{\varepsilon^2\tilde{u}^2}{2\tilde{v}}\psi_y)_y\zeta_y\mathrm{d}y\right| = \left|\int \frac{\varepsilon^2\tilde{u}^2}{2\tilde{v}}\psi_y\zeta_{yy}\mathrm{d}y\right| \leqslant \delta\|\zeta_{yy}\|^2 + C\varepsilon^4\delta(1+t)^{-1}\|\psi_y\|^2$$

and

$$\begin{aligned}\left|\int [\varepsilon(P-\tilde{P})u_y]_y\zeta_y\mathrm{d}y\right| &= \left|\int (P-\tilde{P})(\psi_y+\varepsilon\tilde{u}_y)\zeta_{yy}\mathrm{d}y\right| \\ &\leqslant \bar{\delta}^{\frac{1}{4}}\|\zeta_{yy}\|^2 + C\varepsilon^4\delta(1+t)^{-2}\|(\phi,\zeta)\|^2 + C\|(\phi_y,\phi_y,\zeta_y)\|^{\frac{7}{2}},\end{aligned}$$

in which we used the Gagliardo–Nirenberg's and Poincare's inequalities, we have

$$\begin{aligned}\left|\int \zeta_y Q_5\mathrm{d}y\right| &\leqslant (\delta+\bar{\delta}^{\frac{1}{4}})\|\zeta_{yy}\|^2 + C\varepsilon\delta(1+t)^{-\frac{1}{2}}\|(\psi_y,\zeta_y)\|^2 + C\varepsilon^4\delta(1+t)^{-2}\|(\phi,\zeta)\|^2 \\ &\quad + C\|(\phi_y,\phi_y,\zeta_y)\|^{\frac{7}{2}} + C\varepsilon^8\delta(1+t)^{-4},\end{aligned}$$

while

$$\left|\int \tilde{v}\psi_y R_{1yy}\mathrm{d}y\right| \leqslant C\varepsilon\delta(1+t)^{-\frac{1}{2}}\|\psi_y\|^2 + C\varepsilon^6\delta(1+t)^{-3}$$

from (2.1). Furthermore, it holds

$$\begin{aligned}\left|\int \tilde{v}\psi_y[(\frac{1}{v}-\frac{1}{\tilde{v}})\theta_{yy} + (\frac{\theta}{v^2}-\frac{\tilde{\theta}}{\tilde{v}^2})\tilde{v}_{yy}]\mathrm{d}y\right| &\leqslant C\int |\psi_y\phi\zeta_{yy}| + |\psi_y\phi\tilde{\theta}_{yy}| + |\psi_y(\phi+\zeta)\tilde{v}_{yy}|\mathrm{d}y \\ &\leqslant \bar{\delta}^{\frac{1}{4}}\|\zeta_{yy}\|^2 + C\|(\phi_y,\psi_y)\|^{\frac{7}{2}} + C\varepsilon\delta(1+t)^{-\frac{1}{2}}E_3 \\ &\quad + C\varepsilon^3\delta(1+t)^{-\frac{3}{2}}E_2\end{aligned}$$

by using the Poincare's inequality.

For the term $\tilde{v}\psi_y Q_4$, by a direct calculation, we obtain

$$\begin{aligned}\left|\int 2\tilde{v}\psi_y(\frac{\theta_y v_y}{v^2}-\frac{\tilde{\theta}_y\tilde{v}_y}{\tilde{v}^2})\mathrm{d}y\right| &\leqslant C\int (|\psi_y\zeta_y\phi_y| + |\phi_y\zeta_y\tilde{v}_y| + |\psi_y\tilde{\theta}_y\phi_y| + |\psi_y\tilde{\theta}_y\tilde{v}_y\phi|)\mathrm{d}y \\ &\leqslant \sqrt{\bar{\delta}}\|\zeta_{yy}\|^2 + C\varepsilon\delta(1+t)^{-\frac{1}{2}}E_3 + C\varepsilon^3\delta(1+t)^{-\frac{3}{2}}\|\phi\|^2 \\ &\quad + C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{19}{6}}\end{aligned}$$

and

$$\left|\int 2\tilde{v}\psi_y\left(\frac{\theta v_y^2}{v^3}-\frac{\tilde{\theta}\tilde{v}_y^2}{\tilde{v}^3}\right)\mathrm{d}y\right| \leqslant \sqrt{\bar{\delta}}\|\phi_{yy}\|^2 + C\varepsilon\delta(1+t)^{-\frac{1}{2}}E_3 + C\varepsilon^3\delta(1+t)^{-\frac{3}{2}}E_2 + C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{7}{2}}$$

where we assumed that $\bar{\delta}$ suitably small. On the other hand, from (1.10) and $(2.41)_1$, it follows that

$$\tilde{v}\psi_y\frac{\tilde{P}-1}{\tilde{v}}\phi_{yy} = \frac{1}{2}\varepsilon^2\tilde{u}^2\phi_{yy}\psi_y = \left(\frac{1}{2}\varepsilon^2\tilde{u}^2\phi_y\psi_y\right)_y - \varepsilon^2\tilde{u}\tilde{u}_y\phi_y\psi_y - \frac{1}{4}\varepsilon^2\tilde{u}^2(\phi_y^2)_\tau.$$

Similarly,

$$\tilde{v}\psi_y\left(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\right)\phi_{yy} = \left(\tilde{v}\psi_y\left(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\right)\phi_y\right)_y - \left[\tilde{v}\left(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\right)\right]_y\phi_y\psi_y - \frac{\tilde{v}}{2}\left(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\right)(\phi_y^2)_\tau.$$

Since

$$\left|\int \varepsilon^2\tilde{u}\tilde{u}_y\phi_y\psi_y\mathrm{d}y\right| + \left|\int\left[\tilde{v}\left(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\right)\right]_y\phi_y\psi_y\mathrm{d}y\right| \leqslant C\varepsilon\delta(1+t)^{-\frac{1}{2}}\|(\phi_y,\psi_y)\|^2,$$

and by assuming δ and $\bar{\delta}$ suitably small, from (1.10), (1.7), (1.9) and (2.12), the integral of the terms $-\frac{1}{4}\varepsilon^2\tilde{u}^2(\phi_y^2)_\tau$ and $-\frac{\tilde{v}}{2}\left(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\right)(\phi_y^2)_\tau$ can be absorbed by $E_{3\tau}$. In fact, set

$$\bar{E}_3 = E_3 + \int\left[\frac{1}{4}\varepsilon^2\tilde{u}^2\phi_y^2 + \frac{\tilde{v}}{2}\left(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\right)\phi_y^2\right]\mathrm{d}y,$$

then, $E_3 \sim \bar{E}_3$ from (2.11) as $\bar{\delta}$ suitably small. From (2.41), (1.4) and (1.10), we have

$$\left|\int \frac{1}{4}\varepsilon^2(\tilde{u}^2)_\tau\phi_y^2\mathrm{d}y\right| + \left|\int\left[\frac{\tilde{v}}{2}\left(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\right)\right]_\tau\phi_y^2\mathrm{d}y\right|$$
$$\leqslant C\varepsilon^2\delta(1+t)^{-1}E_3 + C\varepsilon^3\delta(1+t)^{-\frac{3}{2}}E_2 + C\varepsilon^6\delta(1+t)^{-3} + C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{7}{2}}.$$

Thus, it holds that

$$\frac{1}{2}\bar{E}_{3\tau} + \frac{1}{4}K_3 \leqslant C\varepsilon\delta(1+t)^{-\frac{1}{2}}E_3 + C\varepsilon^3\delta(1+t)^{-\frac{3}{2}}E_2 + C\varepsilon^6\delta(1+t)^{-3} + C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{19}{6}} + \sqrt{\bar{\delta}}\|\phi_{yy}\|^2 \tag{2.44}$$

by assuming δ and $\bar{\delta}$ suitably small.

Since the term $\sqrt{\bar{\delta}}\|\phi_{yy}\|^2$ can not be absorbed by the left hand side, we need more calculation. Multiplying $(2.42)_2$ by $-\frac{1}{2}\phi_{yy}$ and $(2.42)_3$ by ψ_{yy}, respectively, and adding both the resultant equations, we obtain that

$$\begin{aligned}&\frac{1}{2\tilde{v}}\phi_{yy}^2+\frac{1}{2}\psi_{yy}^2+(\zeta_y\psi_{yy}-\frac{1}{2}\phi_{yy}\psi_y)_\tau\\&=(\zeta_y\psi_{y\tau}-\frac{1}{2}\phi_{y\tau}\psi_y)_y-\zeta_{yy}(Q_4-R_{1yy}+\frac{1}{\tilde{v}}\phi_{yy}-\frac{1}{\tilde{v}}\zeta_{yy})\\&\quad+\frac{1}{2\tilde{v}}\zeta_{yy}\phi_{yy}-\frac{1}{2}\phi_{yy}(Q_4-R_{1yy})+[\kappa(\frac{\theta_y}{v}-\frac{\tilde{\theta}_y}{\tilde{v}})_{yy}+Q_5]\psi_{yy},\end{aligned}\tag{2.45}$$

by using $(2.42)_1$ and $(2.42)_2$. Integrating (2.45) with respect of y, using the Cauchy's inequality, and suppose δ and $\bar{\delta}$ are suitably small, we can get

$$\int(\frac{1}{4\tilde{v}}\phi_{yy}^2+\frac{1}{4}\psi_{yy}^2)\mathrm{d}y+\left[\int(\zeta_y\psi_{yy}-\frac{1}{2}\phi_{yy}\psi_y)\mathrm{d}y\right]_\tau\leqslant C_3K_3+C_3\|\zeta_{yyy}\|^2+C\varepsilon^7\delta(1+t)^{-\frac{7}{2}}.\tag{2.46}$$

Applying ∂y to (2.42) yields that

$$\begin{aligned}&\phi_{yy\tau}-\psi_{yyy}=0,\\&\psi_{yy\tau}-\frac{1}{\tilde{v}}\phi_{yyy}+\frac{1}{\tilde{v}}\zeta_{yyy}=(\frac{1}{\tilde{v}})_y\phi_{yy}-(\frac{1}{\tilde{v}})_y\psi_{yy}+Q_{4y}-R_{1yyy},\\&\zeta_{yy\tau}+\psi_{yyy}=\kappa(\frac{\theta_y}{v}-\frac{\tilde{\theta}_y}{\tilde{v}})_{yyy}+Q_{5y}.\end{aligned}\tag{2.47}$$

Multiplying $(2.47)_1$ by ϕ_{yy}, $(2.47)_2$ by $\tilde{v}\psi_{yy}$ and $(2.47)_3$ by ζ_{yy}, respectively, and adding all the resultant equations, we can obtain that

$$\begin{aligned}(\frac{1}{2}\phi_{yy}^2+\frac{\tilde{v}}{2}\psi_{yy}^2+\frac{1}{2}\zeta_{yy}^2)_\tau&=(\frac{\tilde{v}}{2})_\tau\psi_{yy}^2+(\phi_{yy}\psi_{yy}-\psi_{yy}\zeta_{yy})_y-(\frac{\tilde{v}_y}{\tilde{v}})_y\phi_{yy}\psi_{yy}+(\frac{\tilde{v}_y}{\tilde{v}})_y\psi_{yy}^2\\&\quad+\kappa\zeta_{yy}(\frac{\theta_y}{v}-\frac{\tilde{\theta}_y}{\tilde{v}})_{yyy}+\zeta_{yy}Q_5+\tilde{v}\psi_{yy}(Q_{4y}+R_{1yyy}).\end{aligned}\tag{2.48}$$

Set

$$E_4=\int(\frac{1}{2}\phi_{yy}^2+\frac{\tilde{v}}{2}\psi_{yy}^2+\frac{1}{2}\zeta_{yy}^2)\mathrm{d}y,\qquad\text{and } K_4=\int\frac{\kappa}{v}\zeta_{yyy}^2\mathrm{d}y,$$

and

$$\bar{E}_4=E_4+\int[\frac{1}{4}\varepsilon^2\tilde{u}^2\phi_{yy}^2+\frac{\tilde{v}}{2}(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}})\phi_{yy}^2]\mathrm{d}y.$$

Then, we have $E_4\sim\|(\phi_{yy},\psi_{yy},\zeta_{yy})\|^2$, $E_4\sim\bar{E}_4$ from (2.11) as $\bar{\delta}$ suitably small, and there exists a constant $\bar{C}_3>0$ such that

$$\|\zeta_{yyy}\|^2\leqslant\bar{C}_3K_4.$$

Just as what we did in second-order derivative estimate, it holds from (2.47) that

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$$\bar{E}_{4\tau}+\frac{1}{2}K_4\leqslant C\varepsilon^2\delta(1+t)^{-1}E_4+C\varepsilon^3\delta(1+t)^{-\frac{3}{2}}E_3+(\delta+\sqrt{\bar{\delta}})E_4+C\varepsilon^8\delta(1+t)^{-4} \quad (2.49)$$

by the aim of the proof of Lemma 4.3 in [4], which made a good estimate on the term containing Q_{4y}.

Similar to what we did in section 2.2, choose $\tilde{C}_3$ large enough so that

$$0\leqslant\frac{1}{4}\int(\phi_{yy}^2+\psi_{yy}^2)\mathrm{d}y+\frac{1}{2}\tilde{C}_3\bar{E}_3\leqslant\int(\zeta_y\psi_{yy}-\frac{1}{2}\phi_{yy}\psi_y)\mathrm{d}y+\tilde{C}_3\bar{E}_3$$

$$\text{and}\qquad \frac{1}{4}\tilde{C}_3K_3\leqslant\frac{1}{2}\tilde{C}_3K_3-C_3K_3.$$

Set

$$\hat{E}_4=\frac{1}{2}\tilde{C}_3\bar{E}_3+\frac{1}{2}\int(\zeta_y\psi_{yy}-\frac{1}{2}\phi_{yy}\psi_y)\mathrm{d}y+\bar{E}_4,$$

$$\hat{K}_4=\frac{1}{8}\tilde{C}_3K_3+\frac{1}{2}\int(\frac{1}{4\tilde{v}}\phi_{yy}^2+\frac{1}{4}\psi_{yy}^2)\mathrm{d}y+\frac{1}{2}K_4,$$

then we have $\hat{E}_4\sim\bar{E}_3+\bar{E}_4$, $\hat{K}_4\sim\bar{E}_4+K_4$, and

$$\hat{E}_{4\tau}+\frac{1}{2}\hat{K}_4\leqslant C\varepsilon\delta(1+t)^{-\frac{1}{2}}\hat{E}_4+C\varepsilon^3\delta(1+t)^{-\frac{3}{2}}E_2+C\varepsilon^6\delta(1+t)^{-3}$$
$$+C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{19}{6}}+C\|(\phi_{yy},\psi_{yy},\zeta_{yy})\|^{\frac{19}{6}}, \quad (2.50)$$

from (2.44) and (2.49) by assuming δ and $\bar{\delta}$ suitably small.

Note that from Young's inequality, we have

$$\varepsilon\delta(1+t)^{-\frac{1}{2}}\hat{E}_4\leqslant C\varepsilon^2\delta(1+t)^{-1}\hat{E}_4+\varepsilon^{\frac{2}{3}}\delta\hat{E}_4,$$

and by using (2.11), we obtain

$$\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{19}{6}}+\|(\phi_{yy},\psi_{yy},\zeta_{yy})\|^{\frac{19}{6}}\leqslant\varepsilon\sqrt{\bar{\delta}}\hat{E}_4.$$

Then, (2.50) can be written as

$$\hat{E}_{4\tau}+\frac{1}{2}\hat{K}_4\leqslant C\varepsilon^2\delta(1+t)^{-1}\hat{E}_4+C\varepsilon^3\delta(1+t)^{-\frac{3}{2}}E_2+C\varepsilon^6\delta(1+t)^{-3}+C\varepsilon^{\frac{2}{3}}(\delta+\sqrt{\bar{\delta}})\hat{E}_4. \quad (2.51)$$

Choose $\tilde{C}_2$ large enough so that

$$0\leqslant\frac{1}{4}\int(\phi_y^2+\psi_y^2)\mathrm{d}y+\frac{1}{2}\tilde{C}_2E_2\leqslant\int(\frac{v}{\theta}\zeta\psi_y-\frac{1}{2}\phi_y\psi)\mathrm{d}y+\tilde{C}_2E_2,$$

$$\frac{1}{4}\tilde{C}_2K_2\leqslant\frac{1}{2}\tilde{C}_2-C_2K_2,$$

$$\text{and}\qquad \frac{1}{4}\tilde{C}_2\hat{K}_4\leqslant\frac{1}{2}\tilde{C}_2\hat{K}_4-C_2\bar{C}_2K_3-C_3\bar{C}_3K_4.$$

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Set

$$\tilde{E}_4 = E_2 + \int (\frac{v}{\theta}\zeta\psi_y - \frac{1}{2}\phi_y\psi)\mathrm{d}y + \frac{1}{2}\tilde{C}_2\varepsilon^{-\frac{2}{3}}\hat{E}_4$$
$$\tilde{K}_4 = \frac{1}{4}\tilde{C}_2K_2 + \int (\frac{\theta}{4v^2}\phi_y^2 + \frac{1}{4}\psi_y^2)\mathrm{d}y + \frac{1}{4}\tilde{C}_2\varepsilon^{-\frac{2}{3}}\hat{K}_4, \quad \text{then } \hat{E}_4 \leqslant C\tilde{K}_4. \tag{2.52}$$

By a direct calculation, we have

Lemma 8 *(Higher-order derivative estimate). If δ, $\bar{\delta}$ and ε are suitably small, it holds*

$$\tilde{E}_{4\tau} + \frac{1}{2}\tilde{K}_4 \leqslant C\varepsilon^2(1+t)^{-1}\int |T_y|(b_1^2 + b_3^2)\mathrm{d}y + C\varepsilon^2\delta(1+t)^{-1}\tilde{E}_4 + C\varepsilon^4\delta(1+t)^{-2}. \tag{2.53}$$

2.5. Proof of Proposition 5

Choose $\tilde{C}_4$ large enough so that

$$\frac{1}{4}\tilde{C}_4\tilde{K}_3 \leqslant \frac{1}{2}\tilde{C}_4\tilde{K}_3 - C_1\tilde{C}_1K_2.$$

Set

$$E_5 = \tilde{E}_1 + \tilde{C}_3\varepsilon^{-2}\tilde{E}_4, \quad \text{and } K_5 = \frac{1}{2}\tilde{K}_1 + \frac{1}{2}\tilde{C}_3\varepsilon^{-2}\tilde{K}_4. \tag{2.54}$$

Then, from (2.26), (2.36) and (2.53), by assuming $\bar{\delta}$ and ε suitably small, we get

$$E_{5\tau} + K_5 \leqslant C_0\varepsilon^2\sqrt{\delta}(1+\varepsilon^2\tau)^{-1}E_5 + C_0\varepsilon^2\delta(1+\varepsilon^2\tau)^{-1}. \tag{2.55}$$

Multiplying (2.55) by $(1+\varepsilon^2\tau)^{-C_0\sqrt{\delta}}$, integrating the resultant inequality over $(0,\tau)$, we obtain that

$$(1+\varepsilon^2\tau)^{-C_0\sqrt{\delta}}E_5 + \int_0^\tau (1+\varepsilon^2 s)^{-C_0\sqrt{\delta}}K_5\mathrm{d}s \leqslant \sqrt{\delta}. \tag{2.56}$$

Thus

$$E_5 \leqslant \sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}} \quad \text{and} \int_0^\tau K_5\mathrm{d}s \leqslant \sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}.$$

Then

$$\|(\Phi, \Psi, W)\|^2 \leqslant \sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}.$$

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In order to get $(2.10)_2$, we need a better decay rate. Multiplying (2.53) by $1+\varepsilon^2\tau$, we have

$$[(1+\varepsilon^2\tau)\tilde{E}_4]_\tau + (1+\varepsilon^2\tau)\tilde{K}_4 \leqslant C\varepsilon^2\int |T_y|(b_1^2+b_3^2)\mathrm{d}y + C\varepsilon^2\delta\tilde{E}_4 + C\varepsilon^4\delta(1+\varepsilon^2\tau)^{-1}$$
$$\leqslant C\varepsilon^2 K_5 + C\varepsilon^4\delta(1+\varepsilon^2\tau)^{-1},$$

by using (2.19), (2.52) and (2.54). Then,

$$\tilde{E}_4 \leqslant C\varepsilon^2\sqrt{\delta}(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}, \quad \text{and} \int_0^\tau (1+\varepsilon^2 s)\tilde{K}_4\mathrm{d}s \leqslant C\varepsilon^2\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}, \tag{2.57}$$

which immediately implies

$$\begin{aligned}
\|(\Phi_y, \Psi_y, W_y, \zeta)\|^2 &\leqslant C\varepsilon^2\sqrt{\delta}(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}\\
\|(\phi_y, \psi_y, \zeta_y)\|^2 &\leqslant C\varepsilon^{\frac{8}{3}}\sqrt{\delta}(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}\\
\|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^2 &\leqslant C\varepsilon^{\frac{8}{3}}\sqrt{\delta}(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}.
\end{aligned} \tag{2.58}$$

Since the time-decay rate of $\|(\phi_y, \psi_y, \zeta_y)\|^2$ and $\|\zeta_{yy}\|^2$ in (2.58) is less than the one in (2.10), we need a better estimate. Multiplying (2.44) by $(1+\varepsilon^2\tau)^{\frac{3}{2}}$ and integrating the resultant inequality over $(0,\tau)$, we obtain from (2.57) and (2.58) that

$$\begin{aligned}
(1+\varepsilon^2\tau)^{\frac{3}{2}}\hat{E}_{4\tau} &\leqslant C\int_0^\tau [\varepsilon\delta(1+\varepsilon^2 s)\hat{E}_4 + \varepsilon^3\delta E_2 + \varepsilon^6\delta(1+\varepsilon^2 s)^{-\frac{3}{2}}\\
&\quad + (1+\varepsilon^2 s)^{\frac{3}{2}}(\|(\phi_y, \psi_y, \zeta_y)\|^{\frac{19}{6}} + \|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^{\frac{19}{6}})]\mathrm{d}s\\
&\leqslant C\varepsilon^{\frac{11}{3}}\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}},
\end{aligned}$$

where we used

$$\int_0^\tau (1+\varepsilon^2 s)^{\frac{3}{2}}(\|(\phi_y, \psi_y, \zeta_y)\|^{\frac{19}{6}} + \|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^{\frac{19}{6}})\mathrm{d}s$$
$$\leqslant \varepsilon^2\int_0^\tau (1+\varepsilon^2 s)(\|(\phi_y, \psi_y, \zeta_y)\|^2 + \|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^2)\mathrm{d}s \leqslant C\varepsilon^4\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}},$$

which comes from (2.58), (2.52) and (2.57). Thus

$$\|(\phi_y, \psi_y, \zeta_y)\|^2 + \|\zeta_{yy}\|^2 \leqslant C\hat{E}_4 \leqslant C\varepsilon^{\frac{11}{3}}\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{3}{2}+C_0\sqrt{\delta}}. \tag{2.59}$$

By using Gagliardo–Nirenberg's inequality, we obtain

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$$\|(\Phi, \Psi, W)\|^2_{L^\infty_y} \leqslant C\|(\Phi, \Psi, W)\|\|(\Phi_y, \Psi_y, W_y)\| \leqslant C\varepsilon\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{1}{4}+\frac{1}{2}C_0\sqrt{\delta}}.$$

Then the a priori assumption (2.11) is closed and the proof of **Proposition 5** is completed.

Appendix A

In this section, we will prove the approximation of $T_x(x,t)$, where T is the self-similar solution $T(\eta)$, $\eta = \frac{x}{\sqrt{1+t}}$ of the equation

$$T_t = (\frac{\kappa T_x}{2T})_x, \qquad \lim_{x\to\pm\infty} T = \theta_\pm, \tag{A.1}$$

when x tends to $\pm\infty$.

Lemma 9. *If $T(\eta)$, $\eta = \frac{x}{\sqrt{1+t}}$ satisfies (A.1), then*

$$T_x(x,t) = O(1)\delta(1+t)^{-\frac{1}{2}}e^{-\frac{x^2}{4d(\theta_\pm)(1+t)}}, \quad d(T) = \frac{\kappa}{2T}, \quad \text{as } x \to \pm\infty. \tag{A.2}$$

From equation (A.1), by direct calculation, we have

$$T_t = -\frac{x}{2}(1+t)^{-\frac{3}{2}}T', \tag{A.3}$$

and

$$T_x = (1+t)^{-\frac{1}{2}}T', \qquad T_{xx} = (1+t)^{-1}T''. \tag{A.4}$$

Plugging them into (A.1), we obtain

$$\frac{\eta}{\kappa}T' + (\frac{T'}{T})' = 0. \tag{A.5}$$

Multiplying (A.5) by $\frac{T}{T'}$, it follows

$$\frac{\eta}{\kappa}T + [ln(\frac{T'}{T})]' = 0. \tag{A.6}$$

Then, we have

$$T'(\eta) = \left(\frac{T'}{T}\bigg|_{\eta=\eta_0}\right) T(\eta)e^{-\int_{\eta_0}^{\eta}\frac{s}{\kappa}T(s)\mathrm{d}s}, \tag{A.7}$$

for any given $\eta_0 \in \mathbb{R}$. Without loss of generality, we assume $\theta_+ > \theta_-$, then $T'(\eta) > 0$.

Set $M > 0$, then for any $\eta, \eta_0 \in (-M, M)$, there exist constants C_1, C_2, c_1, c_2 such that

$$C_1 e^{-c_1(\eta^2-\eta_0^2)} \leqslant \frac{T'(\eta)}{T'(\eta_0)} \leqslant C_2 e^{-c_2(\eta^2-\eta_0^2)}. \tag{A.8}$$

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Integrate (A.8) on $(-M, M)$ with respect of η_0, and suppose M is large enough, then there exist constants C_3, C_4 independent of M such that

$$\frac{C_1 C_3}{2} \leqslant \frac{T(M) - T(-M)}{T'(\eta_0)} \leqslant C_2 C_4. \tag{A.9}$$

Let $M \to \infty$, we can obtain

$$\frac{1}{C_2 C_4} \leqslant \frac{T'(\eta_0)}{\delta} \leqslant \frac{2}{C_1 C_3}, \tag{A.10}$$

which means $T'(\eta_0) \leqslant C\delta$.

On the other hand, we have

$$|T'| \leqslant C e^{-\frac{\theta_\pm}{2\kappa}\eta^2} \tag{A.11}$$

from (A.7). Then it follows $T' = O(1)\delta e^{-\frac{\theta_\pm}{2\kappa}\eta^2}$ as $\eta \to \pm\infty$, which means

$$T_x = O(1)\delta(1+t)^{-\frac{1}{2}} e^{-\frac{\theta_\pm x^2}{2\kappa(1+t)}}, \qquad \text{as } x \to \pm\infty. \tag{A.12}$$

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