ARTICLE IN PRESS YJDEQ:9589

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Journal of Differential Equations

www.elsevier.com/locate/jde

Local wellposedness of nonlinear Maxwell equations with perfectly conducting boundary conditions

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Received 4 May 2018; revised 9 October 2018

Abstract

In this article we develop the local wellposedness theory for quasilinear Maxwell equations in H^m for all $m \geq 3$ on domains with perfectly conducting boundary conditions. The macroscopic Maxwell equations with instantaneous material laws for the polarization and the magnetization lead to a quasilinear first order hyperbolic system whose wellposedness in H^3 is not covered by the available results in this case. We prove the existence and uniqueness of local solutions in H^m with $m \geq 3$ of the corresponding initial boundary value problem if the material laws and the data are accordingly regular and compatible. We further characterize finite time blowup in terms of the Lipschitz norm and we show that the solutions depend continuously on their data. Finally, we establish the finite propagation speed of the solutions.

MSC: 35L50; 35L60; 35Q61; 35B30

Keywords: Nonlinear Maxwell equations; Perfectly conducting boundary conditions; Quasilinear initial boundary value problem; Hyperbolic system; Local wellposedness; Continuous dependence

1. Introduction and main result

Describing the theory of electromagnetism, the Maxwell equations are one of the fundamental partial differential equations in physics. Equipped with instantaneous nonlinear material laws, they form a quasilinear hyperbolic system. On the full space $\mathbb{R}^d$, for such systems Kato has established a satisfactory local wellposedness theory in $H^s(\mathbb{R}^d)$ for $s > \frac{d}{2} + 1$, see [20]. However,

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https://doi.org/10.1016/j.jde.2018.10.019

Please cite this article in press as: M. Spitz, Local wellposedness of nonlinear Maxwell equations with perfectly conducting boundary conditions, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.019

on a domain $G \neq \mathbb{R}^3$, the Maxwell equations with the boundary conditions of a perfect conductor do not belong to the classes of hyperbolic systems for which one knows such a wellposedness theory. The available results need much more regularity and exhibit a loss of derivatives, see [15].

In this work we provide a complete local wellposedness theory for quasilinear Maxwell equations on domains with a perfectly conducting boundary. We prove the existence and uniqueness of solutions in $H^m(G)$ for all $m \geq 3$ if the material laws and the data are accordingly regular and compatible, provide a blow-up condition in the Lipschitz norm, and show the continuous dependence of the solutions on the data, see also [34]. These results are based on a detailed regularity theory for the corresponding nonautonomous linear equation which we developed in the companion paper [35]. Here and in [35], we crucially use the special structure of the Maxwell system.

The macroscopic Maxwell equations in a domain G read

$$\begin{aligned} \partial_t \boldsymbol{D} &= \operatorname{curl} \boldsymbol{H} - \boldsymbol{J}, && \text{for } x \in G, \quad t \in (t_0, T), \\ \partial_t \boldsymbol{B} &= -\operatorname{curl} \boldsymbol{E}, && \text{for } x \in G, \quad t \in (t_0, T), \\ \operatorname{div} \boldsymbol{D} = \rho, \quad \operatorname{div} \boldsymbol{B} &= 0, && \text{for } x \in G, \quad t \in (t_0, T), \\ \boldsymbol{E} \times \nu = 0, \quad \boldsymbol{B} \cdot \nu &= 0, && \text{for } x \in \partial G, \quad t \in (t_0, T), \\ \boldsymbol{E}(t_0) = \boldsymbol{E}_0, \quad \boldsymbol{H}(t_0) &= \boldsymbol{H}_0, && \text{for } x \in G, \end{aligned} \tag{1.1}$$

for an initial time $t_0 \in \mathbb{R}$. Here $\boldsymbol{E}(t,x), \boldsymbol{D}(t,x) \in \mathbb{R}^3$ denote the electric and $\boldsymbol{H}(t,x), \boldsymbol{B}(t,x) \in \mathbb{R}^3$ the magnetic fields. The charge density $\rho(t,x)$ is related with the current density $\boldsymbol{J}(t,x) \in \mathbb{R}^3$ via

$$\rho(t) = \rho(t_0) - \int_{t_0}^{t} \operatorname{div} \boldsymbol{J}(s) ds$$

for all $t \geq t_0$. In (1.1) we consider the Maxwell system with the boundary conditions of a perfect conductor, where ν denotes the outer normal unit vector of G. System (1.1) has to be complemented by constitutive relations between the electric fields and the magnetic fields, the so called material laws. We choose $\boldsymbol{E}$ and $\boldsymbol{H}$ as state variables and express $\boldsymbol{D}$ and $\boldsymbol{B}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$. The actual form of these material laws is a question of modeling and different kinds have been considered in the literature. The so called retarded material laws assume that the fields $\boldsymbol{D}$ and $\boldsymbol{B}$ depend also on the past of $\boldsymbol{E}$ and $\boldsymbol{H}$, see [2] and [33] for instance. In dynamical material laws the material response is modeled by additional evolution equations for the polarization or magnetization, see e.g. [1], [11], [16], or [17]. In this work we concentrate on the instantaneous material laws, see [6] and [14]. Here the fields $\boldsymbol{D}$ and $\boldsymbol{B}$ are given as local functions of $\boldsymbol{E}$ and $\boldsymbol{H}$, i.e., we assume that there are functions $\theta_1, \theta_2 \colon G \times \mathbb{R}^6 \to \mathbb{R}^3$ such that $\boldsymbol{D}(t,x) = \theta_1(x, \boldsymbol{E}(t,x), \boldsymbol{H}(t,x))$ and $\boldsymbol{B}(t,x) = \theta_2(x, \boldsymbol{E}(t,x), \boldsymbol{H}(t,x))$. The most prominent example is the so called Kerr nonlinearity, where

$$\boldsymbol{D} = \boldsymbol{E} + \vartheta |\boldsymbol{E}|^2 \boldsymbol{E}, \quad \boldsymbol{B} = \boldsymbol{H}, \tag{1.2}$$

with $\vartheta \colon G \to \mathbb{R}^{3\times 3}$ and the vacuum permittivity and permeability set equal to 1 for convenience. We further assume that the current density decomposes as $\boldsymbol{J} = \boldsymbol{J}_0 + \sigma_1(\boldsymbol{E}, \boldsymbol{H})\boldsymbol{E}$, where $\boldsymbol{J}_0$ is an

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external current density and σ_1 denotes the conductivity. If we insert these material laws into (1.1) and formally differentiate, we obtain

$$(\partial_t \boldsymbol{D}, \partial_t \boldsymbol{B}) = \partial_{(\boldsymbol{E},\boldsymbol{H})}\theta(x, \boldsymbol{E}, \boldsymbol{H})\partial_t(\boldsymbol{E}, \boldsymbol{H}) = (\operatorname{curl} \boldsymbol{H} - \boldsymbol{J}, -\operatorname{curl} \boldsymbol{E})$$

for the evolutionary part of (1.1). The resulting equation is a first order quasilinear hyperbolic system. In order to write (1.1) in the standard form of first order systems, we introduce the matrices

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$A_j^{\mathrm{co}} = \begin{pmatrix} 0 & -J_j \\ J_j & 0 \end{pmatrix} \tag{1.3}$$

for $j = 1, 2, 3$. Observe that $\sum_{j=1}^3 J_j \partial_j = \operatorname{curl}$. Writing χ for $\partial_{(\boldsymbol{E},\boldsymbol{H})}\theta$, $f = (-\boldsymbol{J}_0, 0)$, $\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}$, and using $u = (\boldsymbol{E}, \boldsymbol{H})$ as new variable, we finally obtain

$$\chi(u)\partial_t u + \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j u + \sigma(u)u = f. \tag{1.4}$$

Under mild regularity conditions, e.g., $\boldsymbol{D}, \boldsymbol{B} \in C([t_0, T], H^1(G)) \cap C^1([t_0, T], L^2(G))$ and $\operatorname{div} \boldsymbol{J} \in L^1((t_0, T), L^2(G))$, a solution $u = (\boldsymbol{E}, \boldsymbol{H})$ of (1.4) satisfies the divergence conditions in (1.1) if they hold at the initial time t_0. Similarly, the second part of the boundary conditions, i.e., $\boldsymbol{B} \cdot \nu = 0$ on $(t_0, T) \times \partial G$ is true if $\boldsymbol{E} \times \nu = 0$ on $(t_0, T) \times \partial G$ and $\boldsymbol{B}(t_0) \cdot \nu = 0$ on ∂G. We refer to [34, Lemma 7.25] for details. Defining the matrix

$$B = \begin{pmatrix} 0 & \nu_3 & -\nu_2 & 0 & 0 & 0 \\ -\nu_3 & 0 & \nu_1 & 0 & 0 & 0 \\ \nu_2 & -\nu_1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5}$$

on ∂G, we can then cast system (1.1) into the equivalent first order quasilinear hyperbolic initial boundary value problem

$$\begin{cases} \chi(u)\partial_t u + \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j u + \sigma(u)u = f, & x \in G, \quad t \in J; \\ Bu = g, & x \in \partial G, \ t \in J; \\ u(t_0) = u_0, & x \in G, \end{cases} \tag{1.6}$$

with additional conditions for the initial value. Here $J = (t_0, T)$ is an open interval. We allow for inhomogeneous boundary values which are not only interesting from the mathematical point

of view, but also have physical relevance, see [10]. We further assume that χ is symmetric and at least locally positive definite, which includes (1.2). Such assumptions are quite standard already for linear Maxwell equations.

The initial value problem on the full space (without boundary conditions) corresponding to (1.6) has been solved by Kato in [20] in a more general setting. Kato freezes a function $\hat{u}$ in the nonlinearities and employs a priori estimates for the corresponding linear problem previously obtained in [18] and [19]. Then a fixed point argument yields a solution of the quasilinear problem. However, Kato works in a general functional analytic setting which does not cover the Maxwell equations with perfectly conducting boundary conditions. An alternative approach is to use energy techniques in order to derive the a priori estimates for the linear problem needed to apply a fixed point argument. This strategy was applied successfully to quasilinear hyperbolic initial boundary value problems with noncharacteristic boundary (i.e., where the boundary matrix $A(\nu) = \sum_{j=1}^{d} A_j \nu_j$ is nonsingular), see [5,24,32]. Systems with characteristic boundary pose several additional difficulties. In particular, no general theory for the corresponding linear problems is available and even a loss of regularity there is possible, see [26]. In [29] an additional structural assumption is proposed in order to prevent this loss of regularity and a quasilinear result is derived from it in [30]. However, the Maxwell system does not fulfill this structural assumption. A different approach is taken in [15] for quasilinear hyperbolic initial boundary value problems with characteristic boundary. The results there require high regularity (at least $H^6(G)$), are given in Sobolev-like spaces incorporating weights in the normal direction, and contain a loss of regularity. In [31] the authors focus on Maxwell's equations, but treat different boundary conditions (belonging to the class treated in [26], see also [7]) than the perfectly conducting ones. Moreover, only the existence of a local solution of the quasilinear system is obtained there.

Somehow surprisingly, the quasilinear Maxwell system (1.1) with perfectly conducting boundary has not yet been treated in the natural space $H^3(G)$ and even the basic questions on local existence and uniqueness are still open. We will close this gap by providing a complete local wellposedness theory. We will prove that

(1) the system (1.6) has a unique maximal solution u in the function space $\bigcap_{j=0}^{m} C^j((T_-, T_+), H^{m-j}(G))$ for all $m \in \mathbb{N}$ with $m \geq 3$ provided the data are sufficiently regular and compatible (in the sense of (2.10) below) with the material law,
(2) finite existence time can be characterized by blowup in the Lipschitz norm,
(3) the solution depends continuously on the data.

We refer to Theorem 5.3 for the precise statement. The derivation of global properties for (1.6) is a highly nontrivial task. In particular, it is already known that global existence cannot be expected for all data. Blow-up examples in the Lipschitz norm are given in [25]. For different boundary conditions than we consider, blow-up examples in the $H(\mathrm{curl})$-norm are provided in [9].

In order to prove the local wellposedness theorem, we follow the strategy mentioned above. We freeze a function $\hat{u}$ in the nonlinearities in (1.6) and employ energy estimates to set up a fixed point argument. However, energy techniques work in L^2-based spaces but require Lipschitz coefficients, see [12]. The solutions there are constructed in $C(\overline{J}, L^2(G))$ but in view of our fixed point argument, we need that $\chi(u)$ is contained in $W^{1,\infty}(J \times G)$. This gap in integrability is bridged by Sobolev's embedding. If a solution u belongs to $C(\overline{J}, H^m(G)) \cap C^1(\overline{J}, H^{m-1}(G))$ for a number $m \in \mathbb{N}_0$, then $\chi(u)$ is an element of $W^{1,\infty}(J \times G)$ if $m > \frac{3}{2} + 1$ and χ is reasonable regular. We thus require $m \geq 3$. Reasonable regular here means that χ belongs to $C^m(G \times \mathcal{U}, \mathbb{R}^{6\times 6})$ and that χ and all of its derivatives up to order m are bounded on $G \times \mathcal{U}_1$

for every compact subset $\mathcal{U}_1$ of $\mathcal{U}$, where $\mathcal{U}$ is a domain in $\mathbb{R}^6$. For later reference we introduce the spaces

$$\mathcal{ML}^{m,n}(G,\mathcal{U}) = \{\theta \in C^m(G \times \mathcal{U}, \mathbb{R}^{n\times n}) \colon \text{For all } \alpha \in \mathbb{N}_0^9 \text{ with } |\alpha| \le m \text{ and } \mathcal{U}_1 \Subset \mathcal{U} : \sup_{(x,y)\in G\times\mathcal{U}_1} |\partial^\alpha \theta(x,y)| < \infty\},$$

$$\mathcal{ML}^{m,n}_{\mathrm{pd}}(G,\mathcal{U}) = \{\theta \in \mathcal{ML}^{m,n}(G,\mathcal{U}) \colon \text{There exists } \eta > 0 \text{ with } \theta = \theta^T \ge \eta \text{ on } G \times \mathcal{U}\}$$

for $n \in \mathbb{N}$. Actually, we only need $n = 1$ or $n = 6$. Although $C(\overline{J}, H^m(G)) \cap C^1(\overline{J}, H^{m-1}(G))$ embeds into $W^{1,\infty}(J \times G)$ for $m \ge 3$, the techniques we are going to apply to solve (1.6) require that its solution has the same amount of regularity in time as in space. We thus construct solutions of (1.6) in the function spaces

$$G_m(J \times G) = \bigcap_{j=0}^{m} C^j(\overline{J}, H^{m-j}(G))^6 \tag{1.7}$$

for all $m \in \mathbb{N}_0$, cf. [5,24,32]. (We do not indicate the dimension of u below.) Defining the function $e_{-\gamma} : t \mapsto e^{-\gamma t}$, we equip the space $G_m(J \times G)$ with the family of time-weighted norms

$$\|v\|_{G_{m,\gamma}(J\times G)} = \max_{j=0,\dots,m} \|e_{-\gamma}\partial_t^j v\|_{L^\infty(J, H^{m-j}(G))}$$

for all $\gamma \ge 0$. In the case $\gamma = 0$, we also write $\|v\|_{G_m(J\times G)}$ instead of $\|v\|_{G_{m,0}(J\times G)}$. Analogously, any time–space norm indexed by γ means the usual norm complemented by the time weight $e_{-\gamma}$ in the following. We also need the spaces $\tilde{G}_m(J \times G)$, consisting of those functions v with $\partial^\alpha v \in L^\infty(J, L^2(G))$ for all $\alpha \in \mathbb{N}_0^4$ with $0 \le |\alpha| \le m$. We equip them with the same family of norms as $G_m(J \times G)$.

The paper is organized as follows. In Section 2 we first study the regularity properties of $\theta(u)$ for $\theta \in \mathcal{ML}^{m,1}(G,\mathcal{U})$ and $u \in \tilde{G}_m(J \times G)$. Based on Faá di Bruno's formula we find suitable function spaces and provide corresponding estimates for these compositions. We further discuss the compatibility conditions. These are necessary conditions for the existence of a $G_m(J \times G)$-solution, which arise since the differential equation and the boundary condition in (1.6) both yield information about u and its time derivatives on $\{t_0\} \times \partial G$.

In Section 3 we then follow the strategy described above to deduce existence and uniqueness of a solution of (1.6). We freeze a function $\hat{u}$ from $\tilde{G}_m(J \times G)$ and study the arising linear problem

$$\begin{cases} A_0\partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j u + Du = f, & x \in G, \quad t \in J; \\ Bu = g, & x \in \partial G, \ t \in J; \\ u(t_0) = u_0, & x \in G; \end{cases} \tag{1.8}$$

where $A_0 = \chi(\hat{u})$ and $D = \sigma(\hat{u})$. Already the higher order energy estimates for this linear problem are difficult to obtain since the Maxwell system has a characteristic boundary (i.e.,

$A(\nu) = \sum_{j=1}^{3} A_j^{\mathrm{co}} \nu_j$ is singular). Here we rely on [35], where the structure of the Maxwell equations is heavily exploited in order to derive these estimates. We show how the results from [35] can be employed to set up a fixed point argument which yields unique local solutions of the quasilinear problem (1.6). In order to characterize a finite maximal existence time in terms of the Lipschitz norm of the solution, the a priori estimates from (1.8) are not good enough. We have to use in Section 4 that the coefficient A_0 equals $\chi(u)$ with u being the solution of (1.6). Combining Moser-type inequalities and estimates from [35] relying on the structure of (1.8), we can then control the $H^m(G)$-norm of u by its Lipschitz norm. In Section 5 we study the continuous dependence of the solutions on the data. Once again the estimates from [35] cannot be applied directly as they do not prevent the loss of regularity due to the quasilinearity when the difference of two solutions is considered. We therefore have to combine the techniques from [35] with certain decompositions of u already used in the full space case, cf. [3]. Finally, we also prove that solutions of (1.6) have finite propagation speed. Here, we use a simple and flexible method relying on weighted energy estimates for the linear problem, cf. [3].

Standing notation: Let m be a nonnegative integer. We denote by G a domain in $\mathbb{R}^3$ with compact $C^{\max\{m,3\}+2}$-boundary (the assumption that ∂G is compact can be loosened, see [34] for details) or the half-space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \colon x_3 > 0\}$. The set $\mathcal{U}$ is a domain in $\mathbb{R}^6$. We further write $L(A_0, \dots, A_3, D)$ for the differential operator $A_0\partial_t + \sum_{j=1}^{3} A_j \partial_j + D$ and ∂_0 for ∂_t. By J we mean an open time interval and we set $\Omega = J \times \mathbb{R}^3_+$. Finally, the image of a function v is designated by $\operatorname{im} v$.

2. Calculus and compatibility conditions

In the study of quasilinear problems one often has to control compositions $\theta(v)$ in higher regularity in terms of v. Derivatives of $\theta(v)$ can be expressed by the so called Faá di Bruno's formula, which is therefore widespread in the literature, see e.g. [3,5]. More important than the formula itself are the estimates which follow from it. We provide both in the next lemma. Its proof is an iterative application of the chain and product rule combined with Sobolev's embedding for the estimates and it works as the proof of [34, Lemma 7.1]. We work with functions v taking values in $\mathbb{R}^6$ in the following.

Lemma 2.1. *Let $m \in \mathbb{N}$ and $\tilde{m} = \max\{m, 3\}$. Let $\mathcal{U}_1$ be a compact subset of $\mathcal{U} \subseteq \mathbb{R}^6$.*

(1) *Let $\theta \in \mathcal{ML}^{m,1}(G, \mathcal{U})$ and $v \in \tilde{G}_{\tilde{m}}(J \times G)$ with $\operatorname{im} v \subseteq \mathcal{U}$. The function $\theta(v)$ belongs to the function space $W^{1,\infty}(J \times G)$ and all its derivatives up to order m are contained in $L^\infty(J, L^2(G)) + L^\infty(J \times G)$. For $\gamma_1, \dots, \gamma_j \in \mathbb{N}_0^4$ with $|\gamma_i| \le m$, $1 \le j \le |\alpha|$, and $\alpha \in \mathbb{N}_0^4$ with $1 \le |\alpha| \le m$ there exist constants $C(\alpha, \gamma_1, \dots, \gamma_j)$ such that*

$$\partial^\alpha \theta(v) = \sum_{\substack{\beta,\gamma \in \mathbb{N}_0^4, \beta_0 = 0 \\ \beta+\gamma=\alpha}} \sum_{1 \le j \le |\gamma|} \sum_{\substack{\gamma_1,\dots,\gamma_j \in \mathbb{N}_0^4 \setminus \{0\} \\ \sum \gamma_i = \gamma}} \sum_{l_1,\dots,l_j=1}^{6} C(\alpha, \gamma_1, \dots, \gamma_j) \cdot (\partial_{y_{l_j}} \cdots \partial_{y_{l_1}} \partial_x^{(\beta_1,\beta_2,\beta_3)} \theta)(v) \prod_{i=1}^{j} \partial^{\gamma_i} v_{l_i}. \tag{2.1}$$

Moreover, there exists a constant $C(\theta, m, \mathcal{U}_1)$ such that

$$\|\partial^\alpha \theta(v)\|_{L^\infty(J,L^2(G))+L^\infty(J\times G)} \le C(\theta,m,\mathcal{U}_1)(1+\|v\|_{G_{\tilde m}(J\times G)})^m \tag{2.2}$$

for all $\alpha \in \mathbb{N}_0^4$ *with* $|\alpha| \le m$ *and* $v \in \tilde G_{\tilde m}(J\times G)$ *with* $\operatorname{im} v \subseteq \mathcal{U}_1$.

(2) *Let* $\theta \in \mathcal{ML}^{m-1,1}(G,\mathcal{U})$ *and* $v \in H^{\tilde m-1}(G)$ *with* $\operatorname{im} v \subseteq \mathcal{U}$. *The composition* $\theta(v)$ *belongs to* $L^\infty(G)$ *and all of its derivatives up to order* m *are contained in* $L^2(G)+L^\infty(G)$. *We further have*

$$\partial^\alpha \theta(v) = \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^3\\ \beta+\gamma=\alpha}} \sum_{1\le j\le|\gamma|} \sum_{\substack{\gamma_1,\dots,\gamma_j\in\mathbb{N}_0^3\setminus\{0\}\\ \sum\gamma_i=\gamma}} \sum_{l_1,\dots,l_j=1}^{6} C_0(\alpha,\gamma_1,\dots,\gamma_j)$$
$$\cdot (\partial_{y_{l_j}}\cdots\partial_{y_{l_1}}\partial_x^\beta\theta)(v)\prod_{i=1}^{j}\partial^{\gamma_i}v_{l_i} \tag{2.3}$$

for all $v \in H^{\tilde m-1}(G)$ *and* $\alpha \in \mathbb{N}_0^3$ *with* $0<|\alpha|\le m-1$, *and the constants*

$$C_0(\alpha,\gamma_1,\dots,\gamma_j) = C((0,\alpha_1,\alpha_2,\alpha_3),(0,\gamma_{1,1},\gamma_{1,2},\gamma_{1,3}),\dots,(0,\gamma_{j,1},\gamma_{j,2},\gamma_{j,3}))$$

from assertion (1). There further exists a constant $C_0(\theta,m,\mathcal{U}_1)$ *such that*

$$\|\partial^\alpha\theta(v)\|_{L^2(G)+L^\infty(G)} \le C_0(\theta,m,\mathcal{U}_1)(1+\|v\|_{H^{\tilde m-1}(G)})^{m-1} \tag{2.4}$$

for all $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha|\le m-1$ *and* $v \in H^{\tilde m-1}(G)$ *with* $\operatorname{im} v \subseteq \mathcal{U}_1$.

(3) *Assume additionally that* $m \ge 2$. *Let* $\theta \in \mathcal{ML}^{m,1}(G,\mathcal{U})$ *and* $t_0 \in \overline{J}$. *Then* $\partial_t^j\theta(v)(t_0)$ *belongs to* $H^{m-j-1}(G)$ *and there is a constant* $C(\theta,m,\mathcal{U}_1)$ *such that*

$$\begin{aligned}&\|\partial_t^j\theta(v)(t_0)\|_{H^{m-j-1}(G)}\\ &\quad\le C(\theta,m,\mathcal{U}_1)(1+\max_{0\le l\le j}\|\partial_t^l v(t_0)\|_{H^{\tilde m-l-1}(G)})^{m-2}\max_{0\le l\le j}\|\partial_t^l v(t_0)\|_{H^{\tilde m-l-1}(G)}\end{aligned}$$

for all $j \in \{1,\dots,m-1\}$ *and* $v \in \tilde G_{\tilde m}(J\times G)$ *with* $\operatorname{im} v \subseteq \mathcal{U}$ *and* $\operatorname{im} v(t_0) \subseteq \mathcal{U}_1$.

For the contraction property of the fixed point operator in Section 3 and the derivation of the continuous dependence, we need similar estimates for the differences of such compositions. They are given by the next corollary. Its proof relies on Lemma 2.1 and works in the same way as the one of [34, Corollary 7.2].

Corollary 2.2. *Let* $m \in \mathbb{N}$, $\tilde m = \max\{m,3\}$, *and* $\gamma \ge 0$. *Take* $\theta \in \mathcal{ML}^{m-1,1}(G,\mathcal{U})$, $R>0$, *and pick a compact subset* $\mathcal{U}_1$ *of* $\mathcal{U}$.

(1) *Let* $v_1, v_2 \in \tilde G_{\tilde m-1}(J\times G)$ *with the bounds* $\|v_1\|_{G_{\tilde m-1}(J\times G)}, \|v_2\|_{G_{\tilde m-1}(J\times G)} \le R$ *and* $\operatorname{im} v_1, \operatorname{im} v_2 \subseteq \mathcal{U}_1$. *Then there exists a constant* $C = C(\theta,m,R,\mathcal{U}_1)$ *such that*

$$\|(\partial^\alpha\theta(v_1))(t)-(\partial^\alpha\theta(v_2))(s)\|_{L^2(G)}\le C\sum_{\substack{\beta\in\mathbb{N}_0^4\\0\le|\beta|\le\tilde{m}-1}}\|\partial^\beta v_1(t)-\partial^\beta v_2(s)\|_{L^2(G)} \tag{2.5}$$

for almost all $t\in J$ and almost all $s\in J$ if $\alpha\in\mathbb{N}_0^4$ with $0\le|\alpha|\le m-2$. In the case $|\alpha|=m-1$ and $m>1$ we have the estimate

$$\begin{aligned}\|(\partial^\alpha\theta(v_1))(t)-(\partial^\alpha\theta(v_2))(s)\|_{L^2(G)+L^\infty(G)}\le{}& C\sum_{\substack{\beta\in\mathbb{N}_0^4\\0\le|\beta|\le\tilde{m}-1}}\|\partial^\beta v_1(t)-\partial^\beta v_2(s)\|_{L^2(G)}\\ &+C\sum_{0\le j\le m-1}\sum_{\substack{0\le\gamma\le\alpha,\gamma_0=0\\|\gamma|=m-1-j}}\sum_{l_1,\dots,l_j=1}^{6}\|(\partial_{y_{l_j}}\dots\partial_{y_{l_1}}\partial_x^{(\gamma_1,\gamma_2,\gamma_3)}\theta)(v_1)(t)\\ &\qquad-(\partial_{y_{l_j}}\dots\partial_{y_{l_1}}\partial_x^{(\gamma_1,\gamma_2,\gamma_3)}\theta)(v_2)(s)\|_{L^\infty(G)}\end{aligned} \tag{2.6}$$

for almost all $t\in J$ and almost all $s\in J$. If θ additionally belongs to $\mathcal{ML}^{m,1}(G,\mathcal{U})$, the estimate (2.5) is true for almost all $t\in J$ and almost all $s\in J$ in the case $|\alpha|=m-1$. Finally, if $\alpha_0=0$, it is enough to sum in (2.5) and (2.6) over those multiindices β with $\beta_0=0$.

(2) *Let $v_1,v_2\in\tilde{G}_{\tilde{m}-1}(J\times G)$ with the bounds $\|v_1\|_{G_{\tilde{m}-1}(J\times G)},\|v_2\|_{G_{\tilde{m}-1}(J\times G)}\le R$ and $\operatorname{im}v_1,\operatorname{im}v_2\subseteq\mathcal{U}_1$, and $\theta\in\mathcal{ML}^{m,1}(G,\mathcal{U})$. Then there exists a constant $C=C(\theta,m,R,\mathcal{U}_1)$ such that*

$$\|\theta(v_1)-\theta(v_2)\|_{G_{m-1,\gamma}(J\times G)}\le C\|v_1-v_2\|_{G_{\tilde{m}-1,\gamma}(J\times G)}$$

for all $\gamma\ge 0$.

Assume that $u\in G_m(J\times G)$ is a solution of (1.6) with inhomogeneity $f\in H^m(J\times G)$ and initial value $u_0\in H^m(G)$. Lemma 2.1 implies that we can take $p-1$ time-derivatives of the evolution equation in (1.6), insert $t_0\in\overline{J}$, and solve for $\partial_t^p u(t_0)$. This procedure yields a closed expression for $\partial_t^p u(t_0)$ in terms of the material laws and the data for all $p\in\{0,\dots,m\}$, which will be utterly important in the following.

Definition 2.3. Let $J\subseteq\mathbb{R}$ be an open interval, $m\in\mathbb{N}$, and $\chi\in\mathcal{ML}^{m,6}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma\in\mathcal{ML}^{m,6}(G,\mathcal{U})$. We inductively define the operators

$$S_{\chi,\sigma,G,m,p}\colon\overline{J}\times H^{\max\{m,3\}}(J\times G)\times H^{\max\{m,2\}}(G,\mathcal{U})\to H^{m-p}(G)$$

by $S_{\chi,\sigma,G,m,0}(t_0,f,u_0)=u_0$ and

$$S_{\chi,\sigma,G,m,p}(t_0,f,u_0)=\chi(u_0)^{-1}\Big(\partial_t^{p-1}f(t_0)-\sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j S_{\chi,\sigma,G,m,p-1}(t_0,f,u_0)$$

$$-\sum_{l=1}^{p-1}\binom{p-1}{l}M_1^l(t_0,f,u_0)S_{\chi,\sigma,G,m,p-l}(t_0,f,u_0) \tag{2.7}$$

$$-\sum_{l=0}^{p-1}\binom{p-1}{l}M_2^l(t_0,f,u_0)S_{\chi,\sigma,G,m,p-1-l}(t_0,f,u_0)\Big),$$

$$M_k^p=\sum_{1\le j\le p}\ \sum_{\substack{\gamma_1,\dots,\gamma_j\in\mathbb{N}_0^4\setminus\{0\}\\ \sum\gamma_i=(p,0,0,0)}}\ \sum_{l_1,\dots,l_j=1}^{6} C((p,0,0,0),\gamma_1,\dots,\gamma_j)$$

$$\cdot(\partial_{y_{l_j}}\cdots\partial_{y_{l_1}}\theta_k)(u_0)\prod_{i=1}^{j}S_{\chi,\sigma,G,m,|\gamma_i|}(t_0,f,u_0)_{l_i} \tag{2.8}$$

for $1\le p\le m$, $k\in\{1,2\}$, where $\theta_1=\chi$, $\theta_2=\sigma$, $M_2^0=\sigma(u_0)$, and C is the constant from Lemma 2.1. By $H^{\max\{m,2\}}(G,\mathcal{U})$ we mean those functions $u_0\in H^{\max\{m,2\}}(G)$ with $\overline{\operatorname{im}u_0}\subseteq\mathcal{U}$.

Lemma 2.1 then implies that

$$\partial_t^p u(t_0)=S_{\chi,\sigma,G,m,p}(t_0,f,u_0)\quad\text{for all } p\in\{0,\dots,m\} \tag{2.9}$$

if $u\in G_m(J\times G)$ is a solution of (1.6) with data $f\in H^m(J\times G)$ and $u_0\in H^m(G)$. The next lemma shows that the operators $S_{\chi,\sigma,G,m,p}$ indeed map into $H^{m-p}(G)$ and it provides corresponding estimates. The proof relies on Lemma 2.1, Corollary 2.2 and the product estimates from [35, Lemma 2.1]. We refer to [34, Lemma 7.7] for details.

Lemma 2.4. *Let $J\subseteq\mathbb{R}$ be an open interval, $t_0\in\overline{J}$, $m\in\mathbb{N}$, and $\tilde m=\max\{m,3\}$. Take $\chi\in\mathcal{ML}^{m,6}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma\in\mathcal{ML}^{m,6}(G,\mathcal{U})$ Choose data $f,\tilde f\in H^{\tilde m}(J\times G)$ and $u_0,\tilde u_0\in H^{\tilde m}(G)$ such that $\overline{\operatorname{im}u_0}$ and $\overline{\operatorname{im}\tilde u_0}$ are contained in $\mathcal{U}$. Take $r>0$ such that*

$$\sum_{j=0}^{\tilde m-1}\|\partial_t^j f(t_0)\|_{H^{\tilde m-j-1}(G)}+\|u_0\|_{H^{\tilde m}(G)}\le r,$$

$$\sum_{j=0}^{\tilde m-1}\|\partial_t^j \tilde f(t_0)\|_{H^{\tilde m-j-1}(G)}+\|\tilde u_0\|_{H^{\tilde m}(G)}\le r.$$

Then the function $S_{\chi,\sigma,G,m,p}(t_0,f,u_0)$ belongs to $H^{m-p}(G)$ and for a constant $C_1=C_1(\chi,\sigma,m,r,\mathcal{U}_1)$ it satisfies

$$\|S_{\chi,\sigma,G,m,p}(t_0,f,u_0)\|_{H^{m-p}(G)}\le C_1\Big(\sum_{j=0}^{m-1}\|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)}+\|u_0\|_{H^m(G)}\Big)$$

for all $p\in\{0,\dots,m\}$, where $\mathcal{U}_1$ is a compact subset of $\mathcal{U}$ with $\operatorname{im}u_0\subseteq\mathcal{U}_1$. Moreover, there is a constant $C_2=C_2(\chi,\sigma,m,r,\mathcal{U}_2)$ such that

$$\|S_{\chi,\sigma,G,m,p}(t_0, f, u_0) - S_{\chi,\sigma,G,m,p}(t_0, \tilde{f}, \tilde{u}_0)\|_{H^{m-p}(G)}$$
$$\leq C_2\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0) - \partial_t^j \tilde{f}(t_0)\|_{H^{m-j-1}(G)} + \|u_0 - \tilde{u}_0\|_{H^m(G)}\Big)$$

for all $p \in \{0, \dots, m\}$*, where* $\mathcal{U}_2$ *is a compact subset of* $\mathcal{U}$ *with* $\operatorname{im} u_0, \operatorname{im} \tilde{u}_0 \subseteq \mathcal{U}_2$.

In [35] a solution in $G_m(J \times G)$ of the linear problem (1.8) was constructed for boundary data from the space

$$E_m(J \times \partial G) = \bigcap_{j=0}^{m} H^j(J, H^{m+\frac{1}{2}-j}(\partial G)),$$
$$\|g\|_{E_m(J\times\partial G)} = \max_{0\leq j\leq m} \|\partial_t^j g\|_{L^2(J, H^{m+1/2-j}(\partial G))}.$$

We thus also take boundary data $g \in E_m(J \times \partial G)$ for the nonlinear problem (1.6). But then we can differentiate the boundary condition in (1.6) up to $m-1$-times in time and still evaluate in t_0 if u belongs to $G_m(J \times G)$. In combination with (2.9) we deduce the identities

$$BS_{\chi,\sigma,G,m,p}(t_0, f, u_0) = \partial_t^p g(t_0) \quad \text{on } \partial G \tag{2.10}$$

for all $p \in \{0, \dots, m-1\}$, which are thus necessary conditions for the existence of a $G_m(J \times G)$-solution of (1.6). We say that the data tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the *compatibility conditions* of order m if $\overline{\operatorname{im} u_0} \subseteq \mathcal{U}$ and (2.10) holds for all $p \in \{0, \dots, m-1\}$. In the next lemma we relate the operators $S_{\chi,\sigma,G,m,p}$ with their linear counterparts from [35]. Therefore, we have to recall some notation. In [35] we introduced the function spaces

$$F_{m,k}(J \times G) = \{A \in W^{1,\infty}(J \times G)^{k\times k} \colon \partial^\alpha A \in L^\infty(J, L^2(G))^{k\times k} \text{ for all } \alpha \in \mathbb{N}_0^4 \text{ with } 1 \leq |\alpha| \leq m\},$$
$$\|A\|_{F_m(J\times G)} = \max\{\|A\|_{W^{1,\infty}(J\times G)}, \max_{1\leq|\alpha|\leq m} \|\partial^\alpha A\|_{L^\infty(J,L^2(G))}\},$$
$$F^0_{m,k}(G) = \{A \in L^\infty(G)^{k\times k} \colon \partial^\alpha A \in L^2(G)^{k\times k} \text{ for all } \alpha \in \mathbb{N}_0^3 \text{ with } 1 \leq |\alpha| \leq m\},$$
$$\|A\|_{F^0_m(G)} = \max\{\|A\|_{L^\infty(G)}, \max_{1\leq|\alpha|\leq m} \|\partial^\alpha A\|_{L^2(G)}\}$$

for $k \in \mathbb{N}$. Moreover, by $F_{m,k,\eta}(J \times G)$ we mean those functions A from $F_{m,k}(J \times G)$ with $A(t,x) = A(t,x)^T \geq \eta$ for all $(t,x) \in J \times G$ and by $F^{\mathrm{c}}_{m,k}(J \times G)$ those which have a limit as $|(t,x)| \to \infty$. If it is clear from the context which parameter k we consider, we suppress it in the notation.

Remark 2.5. As noted in Remark 1.2 in [35] the linear theory allows for coefficients in $W^{1,\infty}(J \times G)$ whose derivatives up to order m are contained in $L^\infty(J, L^2(G)) + L^\infty(J \times G)$. In view of Lemma 2.1, we can thus apply the linear theory with coefficients $\chi(\hat{u})$ and $\sigma(\hat{u})$ and $\hat{u} \in \tilde{G}_{\tilde{m}}(J \times G)$. However, the part of the derivatives in $L^\infty(J \times G)$ is easier to treat so that we concentrated on coefficients from $F_m(J \times G)$ in [35]. The same is true for the nonlinear

problem. In the proofs we will therefore assume without loss of generality that χ and σ from $\mathcal{ML}^{m,6}(G,\mathcal{U})$ have decaying space derivatives as $|x| \to \infty$. More precisely, for all multiindices $\alpha \in \mathbb{N}_0^9$ with $\alpha_4 = \ldots = \alpha_9 = 0$ and $1 \le |\alpha| \le m$, $R > 0$, $\mathcal{U}_1 \Subset \mathcal{U}$, and $v \in L^\infty(J, L^2(G))$ with $\operatorname{im} v \subseteq \mathcal{U}_1$ and $\|v\|_{L^\infty(J,L^2(G))} \le R$ we have

$$\begin{aligned} &(\partial^\alpha \chi)(v), (\partial^\alpha \sigma)(v) \in L^\infty(J, L^2(G)), \\ &\|(\partial^\alpha \chi)(v)\|_{L^\infty(J,L^2(G))} + \|(\partial^\alpha \sigma)(v)\|_{L^\infty(J,L^2(G))} \le C, \end{aligned} \tag{2.11}$$

where $C = C(\chi, \sigma, m, R, \mathcal{U}_1)$. With this assumption we obtain from Lemma 2.1 that $\chi(\hat{u})$ and $\sigma(\hat{u})$ belong to $F_m(J \times G)$.

Finally, we point out that if G is bounded, the above considerations are unnecessary since $L^2(G) + L^\infty(G) = L^2(G)$ in this case.

If one has a $G_m(J \times G)$-solution u of the linear problem (1.8) with coefficients $A_0 \in F_{\tilde{m},\eta}(J \times G)$ with $\eta > 0$ and $\tilde{m} = \max\{m, 3\}$, $A_1, A_2, A_3 \in F_{\tilde{m}}(J \times G)$ independent of time, and $D \in F_{\tilde{m}}(J \times G)$ (note that we allow more general spatial coefficients here), the same reasoning as above first gives a closed expression for $\partial_t^p u(t_0)$ in terms of the coefficients and the data, which we denote by $S_{G,m,p}(t_0, A_0, \ldots, A_3, D, f, u_0)$, and compatibility conditions on the boundary. We refer to (2.2) and (2.4) in [35] for the precise notion. We then say that the tuple $(t_0, A_0, \ldots, A_3, D, B, f, g, u_0)$ fulfills the linear compatibility conditions of order m if the equations

$$B S_{G,m,p}(t_0, A_0, \ldots, A_3, D, f, u_0) = \partial_t^p g(t_0) \quad \text{on } \partial G \tag{2.12}$$

are satisfied for all $p \in \{0, \ldots, m-1\}$. Since we want to apply the linear theory with coefficients $\chi(\hat{u})$ and $\sigma(\hat{u})$, we have to know in which way the compatibility conditions (2.10) for the nonlinear problem imply the compatibility conditions (2.12) for the resulting linear problem.

Lemma 2.6. *Let $J \subseteq \mathbb{R}$ be an open interval, $t_0 \in \overline{J}$, and $m \in \mathbb{N}$ with $m \ge 3$. Take $\chi \in \mathcal{ML}^{m,6}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^{m,6}(G,\mathcal{U})$. Choose data $f \in H^m(J \times G)$ and $u_0 \in H^m(G)$ such that $\overline{\operatorname{im} u_0}$ is contained in $\mathcal{U}$. Let $r > 0$. Assume that f and u_0 satisfy*

$$\begin{aligned} &\|u_0\|_{H^m(G)} \le r, \qquad \max_{0 \le j \le m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} \le r, \\ &\|f\|_{G_{m-1}(J \times G)} \le r, \quad \|f\|_{H^m(J \times G)} \le r. \end{aligned}$$

(1) *Let $\hat{u} \in \tilde{G}_m(J \times G)$ with $\partial_t^p \hat{u}(t_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$ for $0 \le p \le m-1$. Then $\hat{u}$ fulfills the equations*

$$S_{G,m,p}(t_0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), f, u_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0) \tag{2.13}$$

for all $p \in \{0, \ldots, m\}$.

(2) *There is a constant $C(\chi, \sigma, m, r, \mathcal{U}_1) > 0$ and a function u in $G_m(J \times G)$ realizing the initial conditions*

$$\partial_t^p u(t_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$$

for all $p \in \{0, \dots, m\}$ *and it is bounded by*

$$\|u\|_{G_m(J\times G)} \le C(\chi, \sigma, m, r, \mathcal{U}_1)\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|u_0\|_{H^m(G)}\Big).$$

Here $\mathcal{U}_1$ *denotes a compact subset of* $\mathcal{U}$ *with* $\operatorname{im} u_0 \subseteq \mathcal{U}_1$.

Proof. (1) Assertion (1) follows by induction from the definition of the operators $S_{G,m,p}$ in [35, (2.2)], Lemma 2.1, and the definition of $S_{\chi,\sigma,G,m,p}$ in (2.7).

(2) Since $S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$ belongs to $H^{m-p}(G)$ for all $p \in \{0, \dots, m\}$, an extension theorem (see e.g. Lemma 2.34 in [34]) yields the existence of a function u with $\partial_t^p u(t_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$ for all $p \in \{0, \dots, m\}$ and

$$\|u\|_{G_m(J\times G)} \le C \sum_{p=0}^{m} \|S_{\chi,\sigma,G,m,p}(t_0, f, u_0)\|_{H^{m-p}(G)}.$$

Lemma 2.4 then implies the assertion. □

3. Local existence

In this section we prove existence and uniqueness of a solution of (1.6) by a fixed point argument based on the a priori estimates and the regularity theory from [35] for the corresponding linear problem. By a solution of the nonlinear problem (1.6) we mean a function u which belongs to $\bigcap_{j=0}^{m} C^j(I, H^{m-j}(G))$ with $\overline{\operatorname{im} u(t)} \subseteq \mathcal{U}$ for all $t \in I$ and which satisfies (1.6). Here I is an interval with $t_0 \in I$. Since the main result from [35] is omnipresent in the following, we recall it in Theorem 3.1 below. Prior to this, we want to stress that in [35] the initial boundary value problem (1.8) on general domains G was reduced to a half-space problem via local charts. The localization procedure and a subsequent transform lead to the study of

$$\begin{cases} A_0\partial_t u + \sum_{j=1}^{3} A_j \partial_j u + Du = f, & x \in \mathbb{R}^3_+, \quad t \in J; \\ Bu = g, & x \in \partial\mathbb{R}^3_+, \ t \in J; \\ u(t_0) = u_0, & x \in \mathbb{R}^3_+; \end{cases} \tag{3.1}$$

with coefficients $A_0 \in F^{\mathrm{c}}_{\tilde{m},6,\eta}(\Omega)$, $D \in F^{\mathrm{c}}_{\tilde{m},6}(\Omega)$, $A_3 = A_3^{\mathrm{co}}$, and $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\mathbb{R}^3_+)$, where

$$F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\mathbb{R}^3_+) := \{A \in F_{m,6}(\Omega) \colon \exists \mu_1, \mu_2, \mu_3 \in F_{m,1}(\Omega) \text{ independent of time,}$$
$$\text{constant outside of a compact set such that } A = \sum_{j=1}^{3} A_j^{\mathrm{co}} \mu_j\}.$$

Moreover, in the boundary condition we have $B = B^{\mathrm{co}}$, where B^{co} is a constant matrix in $\mathbb{R}^{2\times 6}$ with rank 2. There further exists another constant matrix $C^{\mathrm{co}} \in \mathbb{R}^{2\times 6}$ with rank 2 such that

$$A_3^{\mathrm{co}} = \frac{1}{2}\Big(C^{\mathrm{co}\,T} B^{\mathrm{co}} + B^{\mathrm{co}\,T} C^{\mathrm{co}}\Big). \tag{3.2}$$

We refer to [35, Section 2] and [34, Chapter 5] for the details.

The main result from [35] shows that the linear initial boundary value problem (1.8) respectively (3.1) has a unique solution in $G_m(J \times G)$ if the coefficients and data are accordingly regular and compatible. Moreover, the $G_m(J \times G)$-norm of the solution can be estimated by the corresponding norms of the data.

Theorem 3.1. *Let $\eta > 0$, $m \in \mathbb{N}_0$, and $\tilde{m} = \max\{m, 3\}$. Fix radii $r \geq r_0 > 0$. Take a domain G with compact $C^{\tilde{m}+2}$-boundary or $G = \mathbb{R}^3_+$. Choose times $t_0 \in \mathbb{R}$, $T' > 0$ and $T \in (0, T')$ and set $J = (t_0, t_0 + T)$. Take coefficients $A_0 \in F^{\mathrm{c}}_{\tilde{m},\eta}(J \times G)$, $D \in F^{\mathrm{c}}_{\tilde{m}}(J \times G)$, and $A_3 = A_3^{\mathrm{co}}$. If $G = \mathbb{R}^3_+$, pick $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\mathbb{R}^3_+)$. Otherwise, let $A_1 = A_1^{\mathrm{co}}$ and $A_2 = A_2^{\mathrm{co}}$. Assume the bounds*

$$\begin{aligned}
&\|A_i\|_{F_{\tilde{m}}(J\times G)} \leq r, \quad \|D\|_{F_{\tilde{m}}(J\times G)} \leq r,\\
&\max\{\|A_i(t_0)\|_{F^0_{\tilde{m}-1}(G)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j A_0(t_0)\|_{H^{\tilde{m}-j-1}(G)}\} \leq r_0,\\
&\max\{\|D(t_0)\|_{F^0_{\tilde{m}-1}(G)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j D(t_0)\|_{H^{\tilde{m}-j-1}(G)}\} \leq r_0,
\end{aligned}$$

for all $i \in \{0, 1, 2\}$. Set $B = B^{\mathrm{co}}$ if $G = \mathbb{R}^3_+$ and define B as in (1.5) else. Choose data $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$ such that the tuple $(t_0, A_0, \ldots, A_3, D, B, f, g, u_0)$ fulfills the linear compatibility conditions (2.12) of order m.

Then the linear initial boundary value problem (3.1) respectively (1.8) has a unique solution u in $G_m(J \times G)$. Moreover, there is a number $\gamma_m = \gamma_m(\eta, r, T') \geq 1$ such that

$$\begin{aligned}
\|u\|^2_{G_{m,\gamma}(J\times G)} \leq (C_{m,0} + TC_m)e^{mC_1T}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|^2_{H^{m-1-j}(G)} + \|g\|^2_{E_{m,\gamma}(J\times\partial G)}\\
+ \|u_0\|^2_{H^m(G)}\Big) + \frac{C_m}{\gamma}\|f\|^2_{H^m_\gamma(J\times G)}
\end{aligned}$$

for all $\gamma \geq \gamma_m$, where $C_i = C_i(\eta, r, T') \geq 1$ and $C_{i,0} = C_{i,0}(\eta, r_0) \geq 1$ for $i \in \{1, m\}$.

We point out that the scope of [35] was to provide the regularity theory for (1.8), building up on the L^2-theory from [12]. The case $m = 0$ in Theorem 3.1 is already contained in [12]. We note that we need a further assumption on our material laws χ and σ to guarantee that $\chi(\hat{u})$ and $\sigma(\hat{u})$ have a limit at infinity, which is required in Theorem 3.1. We therefore define

$$\begin{aligned}
\mathcal{ML}^{m,n,\mathrm{c}}(G,\mathcal{U}) = \{\theta \in \mathcal{ML}^{m,n}(G,\mathcal{U}) \colon \exists A \in \mathbb{R}^{n\times n} \text{ such that for all}\\
(x_k, y_k)_k \in (G\times\mathcal{U})^{\mathbb{N}} \text{ with } |x_k| \to \infty \text{ and } y_k \to 0:\\
\theta(x_k, y_k) \to A \text{ as } k \to \infty\},\\
\mathcal{ML}^{m,n,\mathrm{c}}_{\mathrm{pd}}(G,\mathcal{U}) = \mathcal{ML}^{m,n}_{\mathrm{pd}}(G,\mathcal{U}) \cap \mathcal{ML}^{m,n,\mathrm{c}}(G,\mathcal{U}).
\end{aligned}$$

We point out that $\mathcal{ML}^{m,n,\mathrm{c}}(G,\mathcal{U})$ coincides with $\mathcal{ML}^{m,n}(G,\mathcal{U})$ if G is bounded. Let $\chi \in \mathcal{ML}^{m,6,\mathrm{c}}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^{m,6,\mathrm{c}}(G,\mathcal{U})$ satisfy (2.11) and take a function $\hat{u} \in \tilde{G}_{\tilde{m}}(J \times G)$. Lemma 2.1 and Sobolev's embedding then imply that $\chi(\hat{u})$ is an element of $F^{\mathrm{c}}_{\tilde{m},\eta}(J \times G)$ for a number $\eta > 0$ and $\sigma(\hat{u})$ is contained in $F^{\mathrm{c}}_{\tilde{m}}(J \times G)$. In the next lemma we prove the uniqueness of solutions of (1.6).

Lemma 3.2. *Let $t_0 \in \mathbb{R}, T > 0$, and $J = (t_0, t_0 + T)$. Let $m \in \mathbb{N}$ with $m \geq 3$. Take $\chi \in \mathcal{ML}^{m,6,\mathrm{c}}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^{m,6,\mathrm{c}}(G,\mathcal{U})$. Choose data $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$. Let u_1 and u_2 be two solutions in $G_m(J \times G)$ of (1.6) with inhomogeneity f, boundary value g, and initial value u_0 at initial time t_0. Then $u_1 = u_2$.*

Proof. As explained in Remark 2.5, we assume without loss of generality that χ and σ possess property (2.11). Set

$$K = \{T_0 \in \overline{J} \colon u_1 = u_2 \text{ on } [t_0, T_0]\}.$$

This set is nonempty since $u_1(t_0) = u_0 = u_2(t_0)$. Let $T_1 = \sup K$. The continuity of u_1 and u_2 implies that the two functions coincide on $[t_0, T_1]$.

Since u_1 and u_2 are solutions of (1.6), there is a compact subset $\mathcal{U}_1 \subseteq \mathcal{U}$ such that $\operatorname{im} u_1, \operatorname{im} u_2 \subseteq \mathcal{U}_1$. We now assume that T_1 is not equal to T. We then take a time $T_u \in (T_1, T]$ to be fixed below and we set $J_u = [T_1, T_u]$. We observe that u_1 and u_2 are both solutions of (1.6) in $G_m(J_u \times G)$ with inhomogeneity f, boundary value g, and initial value $u_1(T_1) = u_2(T_1)$. In particular, both functions solve the linear initial boundary value problem (1.8) with data f, g, and $u_1(T_1)$ and differential operator $L_1 := L(\chi(u_1), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(u_1))$ respectively $L_2 := L(\chi(u_2), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(u_2))$. Lemma 2.1 and Sobolev's embedding theorem yield that $\chi(u_1)$, $\chi(u_2)$, $\sigma(u_1)$, and $\sigma(u_2)$ are elements of $F^{\mathrm{c}}_3(J \times G)$. Take $r > 0$ such that $\|u_1\|_{G_3(J_u \times G)} \leq r$. Lemma 2.1 and Remark 2.5 then provide a radius $R = R(\chi, \sigma, r, \mathcal{U}_1)$ such that the bounds

$$\begin{aligned}
&\max\{\|\chi(u_1)\|_{F_3(J\times G)}, \|\sigma(u_1)\|_{F_3(J\times G)}\} \leq R,\\
&\max\{\|\chi(u_1(T_1))\|_{F^0_2(G)}, \max_{1\leq j\leq 2} \|\partial_t^j \chi(u_1)(T_1)\|_{H^{m-1-j}(G)}\} \leq R,\\
&\max\{\|\sigma(u_1(T_1))\|_{F^0_2(G)}, \max_{1\leq j\leq 2} \|\partial_t^j \sigma(u_1)(T_1)\|_{H^{m-1-j}(G)}\} \leq R
\end{aligned}$$

hold true. We further note that $\chi(u_1)$ is symmetric and uniformly positive definite. Therefore, Theorem 3.1 for the differential operator L_1 can be applied to $u_1 - u_2$. We take $\eta = \eta(\chi) > 0$ such that $\chi \geq \eta$ and set $\gamma = \gamma_{3.1,0}(\eta, R)$, where $\gamma_{3.1,0}$ denotes the corresponding constant from Theorem 3.1. Theorem 3.1 and Corollary 2.2 (2) then show that

$$\begin{aligned}
&\|u_1 - u_2\|^2_{G_{0,\gamma}(J_u\times G)}\\
&\quad \leq C_{3.1}(\eta, R, T)\|L_1 u_1 - L_1 u_2\|^2_{L^2_\gamma(J_u\times G)}\\
&\quad = C(\chi, \sigma, r, T, \mathcal{U}_1)\|f - \chi(u_1)\partial_t u_2 - \sigma(u_1)u_2 + \chi(u_2)\partial_t u_2 + \sigma(u_2)u_2 - f\|^2_{L^2_\gamma(J_u\times G)}\\
&\quad \leq C(\chi, \sigma, r, T, \mathcal{U}_1)(T_u - T_1)\|\partial_t u_2\|^2_{L^\infty(J_u\times G)}\|\chi(u_1) - \chi(u_2)\|^2_{G_{0,\gamma}(J_u\times G)}
\end{aligned}$$

$$+ C(\chi,\sigma,r,T,\mathcal{U}_1)(T_u - T_1)\|u_2\|^2_{L^\infty(J_u\times G)}\|\sigma(u_1)-\sigma(u_2)\|^2_{G_{0,\gamma}(J_u\times G)}$$
$$\le C(\chi,\sigma,r,T,\mathcal{U}_1)(\|\partial_t u_2\|^2_{G_2(J_u\times G)} + \|u_2\|^2_{G_2(J_u\times G)})(T_u - T_1)\|u_1-u_2\|^2_{G_{0,\gamma}(J_u\times G)},$$

where $C_{3.1}$ is the corresponding constant from Theorem 3.1. Fixing the generic constant in the last line of the above estimate, we choose $T_u > T_1$ so small that

$$C(\chi,\sigma,r,T,\mathcal{U}_1)(\|\partial_t u_2\|^2_{G_2(J_u\times G)} + \|u_2\|^2_{G_2(J_u\times G)})(T_u - T_1) \le \frac{1}{2}.$$

Hence,

$$\|u_1 - u_2\|_{G_{0,\gamma}(J_u\times G)} = 0,$$

implying $u_1 = u_2$ on $[T_1, T_u]$ and thus on $[t_0, T_u]$. This result contradicts the definition of T_1. We conclude that $T_1 = T$, i.e., $u_1 = u_2$ on J. □

Finally, we can combine all the preparations in order to prove the local existence of solutions of (1.6) using Banach's fixed point theorem. For the self-mapping and the contraction property we heavily rely on Theorem 3.1. Special care in the treatment of the constants is needed to close the argument. In particular, the structure of the constants in Theorem 3.1 is crucial here.

Theorem 3.3. *Let $t_0 \in \mathbb{R}$, $T > 0$, $J = (t_0, t_0 + T)$, and $m \in \mathbb{N}$ with $m \ge 3$. Take $\chi \in \mathcal{ML}^{m,6,c}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^{m,6,c}(G,\mathcal{U})$. Let*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix}$$

for all $x \in \partial G$, where ν denotes the unit outer normal vector of ∂G. Choose an inhomogeneity $f \in H^m(J\times G)$, boundary value $g \in E_m(J\times\partial G)$, and initial value $u_0 \in H^m(G)$ with $\overline{\operatorname{im} u_0} \subseteq \mathcal{U}$ such that the tuple $(\chi,\sigma,t_0,B,f,g,u_0)$ fulfills the nonlinear compatibility conditions (2.10) of order m. Choose a radius $r > 0$ satisfying

$$\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|^2_{H^{m-1-j}(G)} + \|g\|^2_{E_m(J\times\partial G)} + \|u_0\|^2_{H^m(G)} + \|f\|^2_{H^m(J\times G)} \le r^2. \tag{3.3}$$

Take a number $\kappa > 0$ with

$$\operatorname{dist}(\{u_0(x) \colon x \in G\}, \partial\mathcal{U}) > \kappa.$$

Then there exists a time $\tau = \tau(\chi,\sigma,m,T,r,\kappa) > 0$ such that the nonlinear initial boundary value problem (1.6) with data f, g, and u_0 has a unique solution u on $[t_0, t_0+\tau]$ which belongs to $G_m(J_\tau\times G)$, where $J_\tau = (t_0, t_0+\tau)$.

Proof. Without loss of generality we assume $t_0 = 0$ and that (2.11) holds true for χ and σ, cf. Remark 2.5. If $f = 0$, $g = 0$, and $u_0 = 0$, then $u = 0$ is a $G_m(J \times G)$-solution of (1.6) and it is unique by Lemma 3.2. So in the following we assume $\|f\|_{H^m(J\times G)} + \|g\|_{E_m(J\times\partial G)} + \|u_0\|_{H^m(G)} > 0$. Recall that the map $S_{\chi,\sigma,G,m,p}$ was defined in (2.7) for $0 \le p \le m$. Let $\tau \in (0, T]$. We set $J_\tau = (0, \tau)$ and

$$\mathcal{U}_\kappa = \{y \in \mathcal{U} \colon \operatorname{dist}(y, \partial\mathcal{U}) \ge \kappa\} \cap \overline{B}_{2C_{\mathrm{Sob}}r}(0), \tag{3.4}$$

where C_{Sob} denotes the constant for the Sobolev embedding from $H^2(G)$ into $L^\infty(G)$. Then $\mathcal{U}_\kappa$ is compact and $\overline{\operatorname{im} u_0}$ is contained in $\mathcal{U}_\kappa$.

I) Let $R > 0$. We set

$$\begin{aligned} B_R(J_\tau) := \{v \in \tilde{G}_m(J_\tau \times G) \colon \|v\|_{G_m(J_\tau\times G)} \le R,\ \|v - u_0\|_{L^\infty(J_\tau\times G)} \le \kappa/2, \\ \partial_t^j v(0) = S_{\chi,\sigma,G,m,j}(0, f, u_0) \text{ for } 0 \le j \le m-1\} \end{aligned}$$

and equip it with the metric $d(v_1, v_2) = \|v_1 - v_2\|_{G_{m-1}(J_\tau\times G)}$. We first show that $B_R(J_\tau)$ is a complete metric space. Recall that $\tilde{G}_m(J_\tau \times G)$ is continuously embedded in $G_{m-1}(J_\tau \times G)$ so that $B_R(J_\tau)$ is well defined. Moreover, Lemma 2.6 (2) shows that $B_R(J_\tau)$ is nonempty for all $R > C_{2.6(2)}(\chi, \sigma, m, r, \mathcal{U}_\kappa) \cdot (m+1)r$.

Let $(v_n)_n$ be a Cauchy sequence in $(B_R(J_\tau), d)$. The functions v_n then tend to v in $G_{m-1}(J_\tau \times G)$ as $n \to \infty$, and hence v satisfies $\partial_t^j v(0) = S_{\chi,\sigma,G,m,j}(0, f, u_0)$ for $0 \le j \le m-1$ and $\|v\|_{G_{m-1}(J_\tau\times G)} \le R$. Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$. The sequence $(\partial^\alpha v_n)_n$ is bounded in $L^\infty(J_\tau, L^2(G)) = (L^1(J_\tau, L^2(G)))^*$. The Banach–Alaoglu Theorem thus gives a subsequence (again denoted by $(\partial^\alpha v_n)_n$) with σ^*-limit v_α in $L^\infty(J_\tau, L^2(G))$. It is straightforward to check that $\partial^\alpha v = v_\alpha$. In particular, v belongs to $\tilde{G}_m(J_\tau \times G)$ with norm less or equal R. Finally, as $m \ge 3$, we infer

$$\begin{aligned} \|v - u_0\|_{L^\infty(J_\tau\times G)} &\le \|v - v_n\|_{L^\infty(J_\tau\times G)} + \|v_n - u_0\|_{L^\infty(J_\tau\times G)} \\ &\le C_{\mathrm{Sob}}\|v - v_n\|_{G_2(J_\tau\times G)} + \kappa/2 \longrightarrow \kappa/2 \end{aligned}$$

as $n \to \infty$. We conclude that v again belongs to $B_R(J_\tau)$.

II) Let $\hat{u} \in B_R(J_\tau)$. Take $\eta = \eta(\chi) > 0$ such that $\chi \ge \eta$. Then $\chi(\hat{u})$ is contained in $F^{\mathrm{c}}_{m,\eta}(J_\tau \times G)$ and $\sigma(\hat{u})$ is an element of $F^{\mathrm{c}}_m(J_\tau \times G)$ by Lemma 2.1, Remark 2.5, and Sobolev's embedding. Lemma 2.6 (1) and the compatibility of $(\chi, \sigma, t_0, B, f, g, u_0)$ imply that the tuple $(t_0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), B, f, g, u_0)$ fulfills the linear compatibility conditions (2.12). Theorem 3.1 yields a solution $u \in G_m(J_\tau \times G)$ of the linear initial boundary value problem (1.8) with differential operator $L(\chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}))$, inhomogeneity f, boundary value g, and initial value u_0. One thus defines a mapping $\Phi \colon \hat{u} \mapsto u$ from $B_R(J_\tau)$ to $G_m(J_\tau \times G)$. We want to prove that Φ also maps $B_R(J_\tau)$ into $B_R(J_\tau)$ for a suitable radius R and a sufficiently small time interval J_τ.

For this purpose take numbers $\tau \in (0, T]$ and $R > C_{2.6(2)}(\chi, \sigma, m, r, \mathcal{U}_\kappa)(m+1)r$ which will be fixed below. Let $\hat{u} \in B_R(J_\tau)$. We first note that there is a constant $C_{2.4}(\chi, \sigma, m, r, \mathcal{U}_\kappa)$ such that

$$\|S_{\chi,\sigma,G,m,p}(0, f, u_0)\|_{H^{m-p}(G)} \le C_{2.4}(\chi, \sigma, m, r, \mathcal{U}_\kappa) \tag{3.5}$$

for all $p \in \{0, \dots, m\}$ due to Lemma 2.4. Lemma 2.1 (2) further provides a constant $C_{2.1(2)}$ such that

$$\|\chi(\hat{u})(0)\|_{F^0_{m-1}(G)} = \|\chi(u_0)\|_{F^0_{m-1}(G)} \le C_{2.1(2)}(\chi, m, r, \mathcal{U}_\kappa),$$
$$\|\sigma(\hat{u})(0)\|_{F^0_{m-1}(G)} = \|\sigma(u_0)\|_{F^0_{m-1}(G)} \le C_{2.1(2)}(\sigma, m, r, \mathcal{U}_\kappa).$$

Note that $\overline{\operatorname{im} \hat{u}}$ is contained in the compact set

$$\tilde{\mathcal{U}}_\kappa = \mathcal{U}_\kappa + \overline{B}(0, \kappa/2) \subseteq \mathcal{U} \tag{3.6}$$

as $\hat{u} \in B_R(J_\tau)$. From Lemma 2.1 (3) and (3.5) we deduce the bound

$$\begin{aligned}
&\|\partial_t^l \chi(\hat{u})(0)\|_{H^{m-l-1}(G)} \\
&\quad \le C_{2.1(3)}(\chi, m, \mathcal{U}_\kappa)(1 + \max_{0\le k\le l} \|\partial_t^k \hat{u}(0)\|_{H^{m-k-1}(G)})^{m-1} \\
&\quad = C_{2.1(3)}(\chi, m, \mathcal{U}_\kappa)(1 + \max_{0\le k\le l} \|S_{\chi,\sigma,G,m,k}(0, f, u_0)\|_{H^{m-k-1}(G)})^{m-1} \\
&\quad \le C_{2.1(3)}(\chi, m, \mathcal{U}_\kappa)(1 + C_{2.4}(\chi, \sigma, m, r, \mathcal{U}_\kappa))^{m-1}, \\
&\|\partial_t^l \sigma(\hat{u})(0)\|_{H^{m-l-1}(G)} \le C_{2.1(3)}(\sigma, m, \mathcal{U}_\kappa)(1 + C_{2.4}(\chi, \sigma, m, r, \mathcal{U}_\kappa))^{m-1}
\end{aligned}$$

for all $l \in \{1, \dots, m-1\}$. We thus find a radius $r_0 = r_0(\chi, \sigma, m, r, \kappa)$ such that

$$\begin{aligned}
&\max\{\|\chi(\hat{u})(0)\|_{F^0_{m-1}(G)}, \max_{1\le l\le m-1} \|\partial_t^l \chi(\hat{u})(0)\|_{H^{m-l-1}(G)}\} \le r_0, \\
&\max\{\|\sigma(\hat{u})(0)\|_{F^0_{m-1}(G)}, \max_{1\le l\le m-1} \|\partial_t^l \sigma(\hat{u})(0)\|_{H^{m-l-1}(G)}\} \le r_0.
\end{aligned} \tag{3.7}$$

As $\hat{u}$ belongs to $B_R(J_\tau)$, Lemma 2.1 (1) gives

$$\|\chi(\hat{u})\|_{F_m(J\times G)}, \|\sigma(\hat{u})\|_{F_m(J\times G)} \le C_{2.1(1)}(\chi, \sigma, m, \tilde{\mathcal{U}}_\kappa)(1 + R)^m.$$

We thus obtain a radius $R_1 = R_1(\chi, \sigma, m, R, \kappa)$ with

$$\|\chi(\hat{u})\|_{F_m(J\times G)} \le R_1 \qquad \text{and} \qquad \|\sigma(\hat{u})\|_{F_m(J\times G)} \le R_1. \tag{3.8}$$

We next define the constant $C_{m,0} = C_{m,0}(\chi, \sigma, r, \kappa)$ by

$$C_{m,0}(\chi, \sigma, r, \kappa) = C_{3.1,m,0}(\eta(\chi), r_0(\chi, \sigma, m, r, \kappa)), \tag{3.9}$$

where $C_{3.1,m,0}$ denotes the constant $C_{m,0}$ from Theorem 3.1. We set the radius $R = R(\chi, \sigma, m, r, \kappa)$ for $B_R(J_\tau)$ to be

$$R(\chi, \sigma, m, r, \kappa) = \max\left\{\sqrt{6\, C_{m,0}(\chi, \sigma, r, \kappa)}\, r,\ C_{2.6(2)}(\chi, \sigma, m, r, \mathcal{U}_\kappa)(m+1)r + 1\right\}. \tag{3.10}$$

We further introduce the constants

$$\gamma_m = \gamma_m(\chi, \sigma, T, r, \kappa) := \gamma_{3.1,m}(\eta(\chi), R_1(\chi, \sigma, m, R(\chi, \sigma, m, r, \kappa), \kappa), T), \tag{3.11}$$

$$C_m = C_m(\chi, \sigma, T, r) := C_{3.1,m}(\eta(\chi), R_1(\chi, \sigma, m, R(\chi, \sigma, m, r, \kappa), \kappa), T), \tag{3.12}$$

where $\gamma_{3.1,m}$ and $C_{3.1,m}$ denote the corresponding constants from Theorem 3.1. Let

$$C_{2.2(2)}(\theta, m, R, \tilde{\mathcal{U}}_\kappa)$$

denote the constant arising from the application of Corollary 2.2 (2) to the components of $\theta \in \mathcal{ML}^{m,6}(G, \mathcal{U})$.

With these constants at hand we define the parameter $\gamma = \gamma(\chi, \sigma, m, T, r, \kappa)$ and the time step $\tau = \tau(\chi, \sigma, m, T, r, \kappa)$ by

$$\gamma = \max\left\{\gamma_m,\ C_{m,0}^{-1} C_m\right\}, \tag{3.13}$$

$$\tau = \min\Big\{T,\ (2\gamma + mC_{3.1,1})^{-1}\log 2,\ C_m^{-1} C_{m,0},\ (2C_{\mathrm{Sob}}R)^{-1}\kappa, \\ [32R^2 C_{m,0} C_P^2 (C_{2.2(2)}^2(\chi, m, R, \tilde{\mathcal{U}}_\kappa) + C_{2.2(2)}^2(\sigma, m, R, \tilde{\mathcal{U}}_\kappa))]^{-1}\Big\}, \tag{3.14}$$

where C_P and $C_{3.1,1}$ denote the corresponding constants from [35, Lemma 2.1] and Theorem 3.1 respectively. Observe that γ and τ actually only depend on χ, σ, m, T, r, and κ as $C_{m,0}$, C_m, $C_{3.1,1}$, and R only depend on these quantities (see (3.9) to (3.12)). For later reference we note that the choice of γ and τ implies

$$\gamma \geq \gamma_m, \tag{3.15}$$

$$\frac{C_m}{\gamma} \leq C_{m,0}, \tag{3.16}$$

$$\tau \leq T, \tag{3.17}$$

$$(2\gamma + mC_{3.1,1})\,\tau \leq \log 2, \tag{3.18}$$

$$C_m \tau \leq C_{m,0}, \tag{3.19}$$

$$C_{\mathrm{Sob}} R \tau \leq \kappa/2, \tag{3.20}$$

$$4C_{m,0} C_P^2 C_{2.2(2)}^2(\theta, m, 6, R, \tilde{\mathcal{U}}_\kappa) R^2 \tau \leq \frac{1}{8}, \qquad \theta \in \{\chi, \sigma\}. \tag{3.21}$$

III) We want to bound the function $\Phi(\hat{u})$ from step II) by means of Theorem 3.1. In view of the estimates (3.7) and (3.8), the definitions of $C_{m,0}$, γ_m, and C_m in (3.9), (3.11), and (3.12), respectively, fit to the assertion of Theorem 3.1. Using also (3.15) and (3.17), we arrive at the inequality

$$\|\Phi(\hat{u})\|^2_{G_m(J_\tau \times G)} \leq e^{2\gamma\tau} \|\Phi(\hat{u})\|^2_{G_{m,\gamma}(J_\tau \times G)} \\ \leq (C_{m,0} + \tau C_m) e^{(2\gamma + mC_{3.1,1})\tau} \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)}$$

$$+ \|g\|^2_{E_{m,\gamma}(J_\tau\times\partial G)} + \|u_0\|^2_{H^m(G)}\Big) + \frac{C_m}{\gamma} e^{2\gamma\tau} \|f\|^2_{H^m_\gamma(J_\tau\times G)}.$$

Observe that (3.3) yields

$$\|f\|^2_{H^m_\gamma(J_\tau\times G)} \le \|f\|^2_{H^m(J_\tau\times G)} \le \|f\|^2_{H^m(J\times G)} \le r^2$$

and analogously $\|g\|^2_{E_{m,\gamma}(J_\tau\times\partial G)} \le \|g\|^2_{E_m(J\times\partial G)}$. Employing (3.19), (3.16), (3.18), (3.3), and (3.10), we then deduce the inequalities

$$\|\Phi(\hat{u})\|^2_{G_m(J_\tau\times G)} \le (C_{m,0} + C_{m,0})e^{\log 2} r^2 + C_{m,0} e^{\log 2} r^2 = 6C_{m,0} r^2 \le R^2,$$
$$\|\Phi(\hat{u})\|_{G_m(J_\tau\times G)} \le R.$$

Since $\Phi(\hat{u})$ belongs to $G_m(J_\tau \times G)$, identity (2.1) in [35] (which is the linear counterpart to (2.9)) shows that

$$\partial_t^p \Phi(\hat{u})(0) = S_{G,m,p}(0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), f, u_0)$$

for all $p \in \{0, \dots, m\}$. On the other hand, as an element of $B_R(J_\tau)$, the function $\hat{u}$ satisfies $\partial_t^p \hat{u}(0) = S_{\chi,\sigma,G,m,p}(0, f, u_0)$ for all $p \in \{0, \dots, m-1\}$. Lemma 2.6 (1) thus yields

$$\partial_t^p \Phi(\hat{u})(0) = S_{G,m,p}(0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), f, u_0) = S_{\chi,\sigma,G,m,p}(0, f, u_0)$$

for all $p \in \{0, \dots, m-1\}$. We further estimate

$$\begin{aligned}
\|\Phi(\hat{u}) - u_0\|_{L^\infty(J_\tau\times G)} &= \Big\| \Phi(\hat{u})(0) + \int_0^t \partial_t \Phi(\hat{u})(s)ds - u_0 \Big\|_{L^\infty(J_\tau\times G)} \\
&= \Big\| \int_0^t \partial_t \Phi(\hat{u})(s)ds \Big\|_{L^\infty(J_\tau\times G)} \le C_{\mathrm{Sob}} \sup_{t\in(0,\tau)} \int_0^t \|\partial_t \Phi(\hat{u})(s)\|_{H^2(G)} ds \\
&\le C_{\mathrm{Sob}} \tau \|\partial_t \Phi(\hat{u})\|_{G_2(J_\tau\times G)} \le C_{\mathrm{Sob}} \tau R \le \kappa/2
\end{aligned}$$

for all $\hat{u} \in B_R(J_\tau)$, where we used that $\Phi(\hat{u})(0) = u_0$ for $\hat{u} \in B_R(J_\tau)$ and (3.20). We conclude that $\Phi(\hat{u})$ belongs to $B_R(J_\tau)$, i.e., Φ maps $B_R(J_\tau)$ into itself.

IV) Let $\hat{u}_1, \hat{u}_2 \in B_R(J_\tau)$. Since $\chi(\hat{u}_i)$ and $\sigma(\hat{u}_i)$ belong to $F_m(J_\tau \times G)$ for $i \in \{1, 2\}$, Lemma 2.1 of [35] implies that $\chi(\hat{u}_i)\partial_t \Phi(\hat{u}_2)$ and $\sigma(\hat{u}_i)\Phi(\hat{u}_2)$ are elements of the space $\tilde{G}_{m-1}(J_\tau \times G) \hookrightarrow H^{m-1}(J_\tau \times G)$ for $i \in \{1, 2\}$. The function $\Phi(\hat{u}_2)$ thus fulfills

$$\begin{aligned}
&L(\chi(\hat{u}_1), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}_1))\Phi(\hat{u}_2) \\
&\quad = \chi(\hat{u}_1)\partial_t \Phi(\hat{u}_2) + \sigma(\hat{u}_1)\Phi(\hat{u}_2) - \chi(\hat{u}_2)\partial_t \Phi(\hat{u}_2) - \sigma(\hat{u}_2)\Phi(\hat{u}_2) \\
&\qquad + L(\chi(\hat{u}_2), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}_2))\Phi(\hat{u}_2) \\
&\quad = (\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t \Phi(\hat{u}_2) + (\sigma(\hat{u}_1) - \sigma(\hat{u}_2))\Phi(\hat{u}_2) + f
\end{aligned}$$

and this function belongs to $\tilde{G}_{m-1}(J_\tau \times G) \hookrightarrow H^{m-1}(J_\tau \times G)$. We further stress that $\Phi(\hat{u}_1)(0) = u_0 = \Phi(\hat{u}_2)(0)$.

As in step III), properties (3.7), (3.8), (3.9), (3.11), (3.12), (3.15), and (3.17) allow us to apply Theorem 3.1 with differential operator $L(\chi(\hat{u}_1), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}_1))$ and parameter γ on $J_\tau \times G$. We thus obtain the inequality

$$\begin{aligned}
&\|\Phi(\hat{u}_1) - \Phi(\hat{u}_2)\|^2_{G_{m-1}(J_\tau \times G)} \\
&\quad \le e^{2\gamma\tau} \|\Phi(\hat{u}_1) - \Phi(\hat{u}_2)\|^2_{G_{m-1,\gamma}(J_\tau \times G)} \\
&\quad \le (C_{m,0} + \tau C_m) e^{(2\gamma + mC_{3.1,1})\tau} \sum_{j=0}^{m-2} \|\partial_t^j (f - L\Phi(\hat{u}_2))(0)\|^2_{H^{m-2-j}(G)} \\
&\qquad + \frac{C_m}{\gamma} e^{(2\gamma + mC_{3.1,1})\tau} \|f - L\Phi(\hat{u}_2)\|^2_{H^{m-1}_\gamma(J_\tau \times G)} \\
&\quad = (C_{m,0} + \tau C_m) e^{(2\gamma + mC_{3.1,1})\tau} \sum_{j=0}^{m-2} \|\partial_t^j ((\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t \Phi(\hat{u}_2))(0) \\
&\qquad\qquad + \partial_t^j ((\sigma(\hat{u}_1) - \sigma(\hat{u}_2))\Phi(\hat{u}_2))(0)\|^2_{H^{m-2-j}(G)} \\
&\qquad + \frac{C_m}{\gamma} e^{(2\gamma + mC_{3.1,1})\tau} \|(\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t \Phi(\hat{u}_2) \\
&\qquad\qquad + (\sigma(\hat{u}_1) - \sigma(\hat{u}_2))\Phi(\hat{u}_2)\|^2_{H^{m-1}_\gamma(J_\tau \times G)}.
\end{aligned}$$

Lemma 2.1 and the equalities

$$\partial_t^l \hat{u}_1(0) = S_{\chi,\sigma,G,m,l}(0, f, u_0) = \partial_t^l \hat{u}_2(0)$$

for all $l \in \{0, \dots, m-1\}$ imply that the terms in the sum vanish. Employing (3.18), we then deduce

$$\begin{aligned}
\|\Phi(\hat{u}_1) - \Phi(\hat{u}_2)\|^2_{G_{m-1}(J_\tau \times G)} &\le 4C_m \frac{1}{\gamma} \|(\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t \Phi(\hat{u}_2)\|^2_{H^{m-1}_\gamma(J_\tau \times G)} \\
&\quad + 4C_m \frac{1}{\gamma} \|(\sigma(\hat{u}_1) - \sigma(\hat{u}_2))\Phi(\hat{u}_2)\|^2_{H^{m-1}_\gamma(J_\tau \times G)} \\
&=: I_1 + I_2. \qquad (3.22)
\end{aligned}$$

Before going on, we point out that we know from step II) that $\Phi(\hat{u}_2)$ is an element of $B_R(J_\tau)$ and hence

$$\|\partial_t \Phi(\hat{u}_2)\|_{G_{m-1}(J_\tau \times G)} \le \|\Phi(\hat{u}_2)\|_{G_m(J_\tau \times G)} \le R. \qquad (3.23)$$

We now treat the first summand. Lemma 2.1 of [35], estimate (3.23), and Corollary 2.2 (2) show that

$$
\begin{aligned}
I_1 &\le 4C_m \frac{1}{\gamma}\tau \|(\chi(\hat{u}_1)-\chi(\hat{u}_2))\partial_t \Phi(\hat{u}_2)\|^2_{G_{m-1,\gamma}(J_\tau\times G)} \\
&\le 4C_m \frac{1}{\gamma}\tau C_P^2 \|\chi(\hat{u}_1)-\chi(\hat{u}_2)\|^2_{G_{m-1,\gamma}(J_\tau\times G)} \|\partial_t \Phi(\hat{u}_2)\|^2_{G_{m-1}(J_\tau\times G)} \\
&\le 4C_m \frac{1}{\gamma} C_P^2 C^2_{2.2(2)}(\chi, m, R, \tilde{\mathcal{U}}_\kappa) R^2 \tau \|\hat{u}_1-\hat{u}_2\|^2_{G_{m-1,\gamma}(J_\tau\times G)}.
\end{aligned}
$$

Exploiting (3.16) and (3.21), we finally arrive at

$$
I_1 \le \frac{1}{8}\|\hat{u}_1-\hat{u}_2\|^2_{G_{m-1,\gamma}(J_\tau\times G)} \le \frac{1}{8}\|\hat{u}_1-\hat{u}_2\|^2_{G_{m-1}(J_\tau\times G)}. \tag{3.24}
$$

Analogously, we obtain

$$
I_2 \le \frac{1}{8}\|\hat{u}_1-\hat{u}_2\|^2_{G_{m-1}(J_\tau\times G)}. \tag{3.25}
$$

Estimates (3.22), (3.24), and (3.25) imply

$$
\|\Phi(\hat{u}_1)-\Phi(\hat{u}_2)\|_{G_{m-1}(J_\tau\times G)} \le \frac{1}{2}\|\hat{u}_1-\hat{u}_2\|_{G_{m-1}(J_\tau\times G)}.
$$

We conclude that Φ is a strict contraction on $B_R(J_\tau)$.

V) Banach's fixed point theorem thus gives a fixed point $u \in B_R(J_\tau)$, i.e., $\Phi(u) = u$. By definition of Φ, this means that $u \in G_m(J_\tau \times G)$ is a solution of (1.6). Lemma 3.2 shows that u is the only one on $[0, \tau]$. □

Remark 3.4.

(1) Using time reversion and adapting coefficients and data accordingly, we can transfer the result of Theorem 3.3 to the negative time direction, see [34, Remark 3.3] for details.
(2) Standard techniques show that the restriction and the concatenation of solutions of (1.8) are again solutions of (1.8). For the precise statements and the proofs we refer to Lemma 7.13 and Lemma 7.14 in [34].

Theorem 3.3 and Remark 3.4 show that the definition of a maximal solution makes sense.

Definition 3.5. Let $t_0 \in \mathbb{R}$ and $m \in \mathbb{N}$ with $m \ge 3$. Take $\chi \in \mathcal{ML}^{m,6,c}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^{m,6,c}(G,\mathcal{U})$. Choose data $f \in H^m((-T,T)\times G)$, $g \in E_m((-T,T)\times \partial G)$, and $u_0 \in H^m(G)$ for all $T > 0$ and define B as in Theorem 3.3. Assume that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the compatibility conditions (2.10) of order m. We introduce

$$
\begin{aligned}
T_+(m, t_0, f, g, u_0) &= \sup\{\tau \ge t_0 \colon \exists\, G_m\text{-solution of (1.6) on } [t_0, \tau]\}, \\
T_-(m, t_0, f, g, u_0) &= \inf\{\tau \le t_0 \colon \exists\, G_m\text{-solution of (1.6) on } [\tau, t_0]\}.
\end{aligned}
$$

The interval $(T_-(m, t_0, f, g, u_0), T_+(m, t_0, f, g, u_0)) =: I_{max}(m, t_0, f, g, u_0)$ is called the maximal interval of existence.

The name maximal interval of existence is justified by the next proposition. It states that there is a unique solution of (1.6) on the maximal interval of existence which cannot be extended beyond this interval. This solution is also called the maximal solution in the following. The proof works with standard techniques, see [34, Proposition 7.16] for details.

Proposition 3.6. *Let $t_0 \in \mathbb{R}$ and $m \in \mathbb{N}$ with $m \geq 3$. Take $\chi \in \mathcal{ML}^{m,6,\mathrm{c}}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^{m,6,\mathrm{c}}(G,\mathcal{U})$. Choose data $f \in H^m((-T,T) \times G)$, $g \in E_m((-T,T) \times \partial G)$, and $u_0 \in H^m(G)$ for all $T > 0$ and define B as in Theorem 3.3. Assume that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the compatibility conditions (2.10) of order m. Then there exists a unique maximal solution $u \in \bigcap_{j=0}^{m} C^j(I_{max}, H^{m-j}(G))$ of (1.6) on I_{max} which cannot be extended beyond this interval.*

4. Blow-up criteria

We next want to characterize finite maximal existence times, i.e., the situation when $T_+ < \infty$, by a blow-up condition. As it is usually the case when the solution is constructed via Banach's fixed point theorem, the construction allows to derive such a blow-up condition in the norm which controls the initial value. In our case this is the $H^m(G)$-norm.

Lemma 4.1. *Let $t_0 \in \mathbb{R}$ and $m \in \mathbb{N}$ with $m \geq 3$. Take $\chi \in \mathcal{ML}^{m,6,\mathrm{c}}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^{m,6,\mathrm{c}}(G,\mathcal{U})$. Choose data $f \in H^m((-T,T) \times G)$, $g \in E_m((-T,T) \times \partial G)$, and $u_0 \in H^m(G)$ for all $T > 0$ and define B as in Theorem 3.3. Assume that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the compatibility conditions (2.10) of order m. Let u be the maximal solution of (1.6) on I_{max} provided by Proposition 3.6. If $T_+ = T_+(m, t_0, f, g, u_0) < \infty$, then one of the following blow-up properties*

(1) $\liminf_{t \nearrow T_+} \operatorname{dist}(\{u(t,x) \colon x \in G\}, \partial\mathcal{U}) = 0$,
(2) $\lim_{t \nearrow T_+} \|u(t)\|_{H^m(G)} = \infty$

occurs. The analogous result is true for $T_-(m, t_0, f, g, u_0)$.

Proof. Let $T_+ < \infty$ and assume that condition (1) does not hold. This means that there exists $\kappa > 0$ such that

$$\operatorname{dist}(\{u(t,x) \colon x \in G\}, \partial\mathcal{U}) > \kappa$$

for all $t \in (t_0, T_+)$. Assume that there exists a sequence $(t_n)_n$ converging from below to the maximal existence time T_+ such that $\rho := \sup_{n\in\mathbb{N}} \|u(t_n)\|_{H^m(G)}$ is finite. Fix a time $T' > T_+$ and take a radius $r > \rho$ with

$$m\|f\|^2_{G_{m-1}((t_0,T')\times G)} + \|g\|^2_{E_m((t_0,T')\times\partial G)} + \rho^2 + \|f\|^2_{H^m((t_0,T')\times G)} < r^2.$$

Then pick an index $N \in \mathbb{N}$ such that

$$t_N + \tau(\chi, \sigma, m, T' - t_0, r, \kappa) > T_+,$$

for the time step $\tau = \tau(\chi, \sigma, m, T' - t_0, r, \kappa)$ from Theorem 3.3. Identity (2.9) shows that the tuple $(\chi, \sigma, t_N, B, f, g, u(t_N))$ fulfills the compatibility conditions (2.10) of order m. Since the

distance between $\operatorname{im} u(t_N)$ and $\partial\mathcal{U}$ is larger than κ, Theorem 3.3 thus gives a G_m-solution v of (1.6) with inhomogeneity f, boundary value g, and initial value $u(t_N)$ at t_N on $[t_N, t_N+\tau]$. Setting $w(t) := u(t)$ if $t \in [t_0, t_N]$ and $w(t) := v(t)$ if $t \in [t_N, t_N+\tau]$, we obtain a G_m-solution of (1.6) with data f, g, and u_0 on $[t_0, t_N+\tau]$ by Remark 3.4. This contradicts the definition of T_+ since $t_N + \tau > T_+$. The assertion for T_- is proven analogously. □

The blow-up criterion above can be improved. In fact we will show that if $T_+ < \infty$ (and the solution does not come arbitrarily close to $\partial\mathcal{U}$), then the spatial Lipschitz norm of the solution has to blow up when one approaches T_+, see Theorem 5.3 (2) below. There are several examples of quasilinear systems, both on the full space and on domains, where the blow-up condition is given in terms of the Lipschitz norm of the solution, see e.g. [3–5,21,23–25]. This improvement (in comparison with the $H^m(G)$-norm) is possible as one can exploit that a solution u of the nonlinear problem (1.6) solves the linear problem (1.8) with coefficients $\chi(u)$ and $\sigma(u)$. Deriving estimates for the derivatives of u, one can then use so-called Moser-type estimates, see the proof of Proposition 4.4 below. These estimates, introduced in [27] and based on the Gagliardo–Nirenberg estimates from [28], are an efficient tool to estimate products of derivatives of u. However, as our material laws χ and σ do also depend on the space variable x, we cannot use them in their standard form (see e.g. [22,25] and also [34] for domains, where we treated a slightly simpler case). But the proof of the version below still follows the standard ideas already used in [27].

Lemma 4.2. *Let $T > 0$, $J = (0, T)$, and $m \in \mathbb{N}$ with $m \geq 3$. Let $\theta \in \mathcal{ML}^{m,1}(G, \mathcal{U})$ and $v \in G_m(J \times G)$. Assume that there is a number $\zeta_0 > 0$ and a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ such that $\|v\|_{W^{1,\infty}(J\times G)} \leq \zeta_0$ and $\operatorname{im} v \subseteq \mathcal{U}_1$. Then there is a constant $C = C(\theta, \zeta_0, \mathcal{U}_1)$ such that*

$$\|\partial^\beta \theta(v) \partial_t \partial^{\alpha-\beta} v\|_{L^2(J\times G)} + \|\partial^\beta \theta(v) \partial^{\alpha-\beta} v\|_{L^2(J\times G)} \leq C \|v\|_{H^{|\alpha|}(J\times G)}$$

for all $0 < \beta \leq \alpha$ and $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$.

We will employ this lemma in the proof of the next proposition. There it has to be combined with a technique developed in [35] to control the derivatives in normal direction of solutions of (1.6) although this system has a characteristic boundary. For later reference, we recall the key result in this direction. It is a simplified version of [35, Proposition 3.3 and Remark 4.11] and relies heavily on the structure of the Maxwell equations.

Lemma 4.3. *Let $T' > 0$, $\eta > 0$, $\gamma \geq 1$, and $r \geq r_0 > 0$. Pick $T \in (0, T')$ and set $J = (0, T)$. Take $A_0 \in F^{\mathrm{c}}_{0,\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, and $D \in F^{\mathrm{c}}_0(\Omega)$ with*

$$\|A_i\|_{W^{1,\infty}(\Omega)} \leq r, \quad \|D\|_{W^{1,\infty}(\Omega)} \leq r,$$
$$\|A_i(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0, \quad \|D(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0$$

for all $i \in \{0, 1, 2\}$. Choose $f \in H^1(\Omega)$ and $u_0 \in H^1(\mathbb{R}^3_+)$. Let $u \in G_1(\Omega)$ solve the initial value problem

$$\begin{cases} A_0\partial_t u + \sum_{j=1}^{3} A_j\partial_j u + Du = f, & x \in \mathbb{R}^3_+, \quad t \in J; \\ u(0) = u_0, & x \in \mathbb{R}^3_+. \end{cases} \tag{4.1}$$

Then there are constants $C_{1,0} = C_{1,0}(\eta, r_0) \geq 1$ and $C_1 = C_1(\eta, r, T') \geq 1$ such that

$$\|\nabla u\|^2_{G_{0,\gamma}(\Omega)} \leq e^{C_1 T}\Big((C_{1,0} + TC_1)\Big(\sum_{j=0}^{2} \|\partial_j u\|^2_{G_{0,\gamma}(\Omega)} + \|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big) + \frac{C_1}{\gamma}\|f\|^2_{H^1_\gamma(\Omega)}\Big). \tag{4.2}$$

We can now prove the main result of this section, showing that we control the $H^m(G)$-norm of a solution as soon as we control its spatial Lipschitz norm. For the proof we differentiate (1.6) and employ the basic L^2-estimate from Theorem 3.1 to the derivative of u. The Moser-type estimates from Lemma 4.2 allow us to treat the arising inhomogeneities in such a way that the Gronwall lemma yields the desired estimate. However, this approach only works in tangential directions. In order to bound the derivatives of u containing a derivative in normal direction, we have to combine the above approach with Lemma 4.3.

Proposition 4.4. *Let $m \in \mathbb{N}$ with $m \geq 3$ and $t_0 \in \mathbb{R}$. Take functions $\chi \in \mathcal{ML}^{m,6,c}_{\mathrm{pd}}(G, \mathcal{U})$ and $\sigma \in \mathcal{ML}^{m,6,c}(G, \mathcal{U})$. Let*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix}$$

for all $x \in \partial G$, where ν denotes the unit outer normal vector of ∂G. Choose data $u_0 \in H^m(G)$, $g \in E_m((-T, T) \times \partial G)$, and $f \in H^m((-T, T) \times G)$ for all $T > 0$ such that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the compatibility conditions (2.10) of order m. Let u denote the maximal solution of (1.6) provided by Proposition 3.6 on (T_-, T_+). We introduce the quantity

$$\omega(T) = \sup_{t \in (t_0, T)} \|u(t)\|_{W^{1,\infty}(G)}$$

for every $T \in (t_0, T_+)$. We further take $r > 0$ with

$$\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|g\|_{E_m((t_0,T_+)\times\partial G)} + \|u_0\|_{H^m(G)} + \|f\|_{H^m((t_0,T_+)\times G)} \leq r.$$

We set $T^* = T_+$ if $T_+ < \infty$ and take any $T^* > t_0$ if $T_+ = \infty$. Let $\omega_0 > 0$ and let $\mathcal{U}_1$ be a compact subset of $\mathcal{U}$.

Then there exists a constant $C = C(\chi, \sigma, m, r, \omega_0, \mathcal{U}_1, T^* - t_0)$ such that

$$\|u\|^2_{G_m((t_0,T)\times G)} \le C\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|^2_{H^{m-1-j}(G)} + \|u_0\|^2_{H^m(G)} + \|g\|^2_{E_m((t_0,T)\times\partial G)} + \|f\|^2_{H^m((t_0,T)\times G)}\Big)$$

for all $T \in (t_0, T^*)$ which have the property that $\omega(T) \le \omega_0$ and $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T]$. The analogous result is true on (T_-, t_0).

Proof. Without loss of generality we assume $t_0 = 0$ and that χ and σ have property (2.11), cf. Remark 2.5. Let $\omega_0 > 0$ and $\mathcal{U}_1$ be a compact subset of $\mathcal{U}$. If $\omega(T) > \omega_0$ or if the set $\{u(t,x)\colon (t,x) \in [t_0, T] \times G\}$ is not contained in $\mathcal{U}_1$ for all $T \in (0, T^*)$, there is nothing to prove. Otherwise we fix $T' \in (0, T^*)$ with $\omega(T') \le \omega_0$ and $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T']$. Let $T \in (0, T']$ be arbitrary and denote $(0, T) \times \mathbb{R}^3_+$ by Ω. Note that $\omega(T) \le \omega(T') \le \omega_0$ and $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T]$.

To derive the improved estimates, we have to study the problem on the half-space. To that purpose, we apply the localization procedure and the subsequent transform already used in the proof of Theorem 3.1, see [35, Section 2] and [34, Chapter 5]. To streamline the argument, we do not perform the localization here but assume that $G = \mathbb{R}^3_+$ and that we have spatial coefficients $A_1, A_2 \in F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\mathbb{R}^3_+)$ and $A_3 = A_3^{\mathrm{co}}$. The full space case is easier and treated similarly. We refer to [35, page 9] and [34, Proposition 7.20] for the details.

We pick a number $\eta = \eta(\chi) > 0$ such that $\chi \ge \eta$. Consequently, there is a constant C with $|\chi(\xi)^{-1}| \le C\eta^{-1}$ for all $\xi \in \mathbb{R}^6$. Since the function u solves (1.6), we infer

$$\begin{aligned}
\|\partial_t u\|_{L^\infty(\Omega)} &\le \|\chi(u)^{-1} f\|_{L^\infty(\Omega)} + \sum_{j=1}^{3} \|\chi(u)^{-1} A_j \partial_j u\|_{L^\infty(\Omega)} + \|\chi(u)^{-1}\sigma(u)u\|_{L^\infty(\Omega)} \\
&\le C(\eta, \sigma, \mathcal{U}_1)\Big(\|f\|_{H^m(\Omega)} + 3\,\omega(T) + \omega(T)\Big), \\
\|u\|_{W^{1,\infty}(\Omega)} &\le \|\partial_t u\|_{L^\infty(\Omega)} + \omega(T) \le C_{4.3}(\chi, \sigma, r, \omega_0, \mathcal{U}_1).
\end{aligned} \tag{4.3}$$

In the following we will frequently apply (4.3) without further reference.

I) We set

$$\begin{aligned}
f_\alpha = \partial^\alpha f &- \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta \chi(u) \partial_t \partial^{\alpha-\beta} u - \sum_{j=1}^{2} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_j \partial_j \partial^{\alpha-\beta} u \\
&- \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta \sigma(u) \partial^{\alpha-\beta} u
\end{aligned} \tag{4.4}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$. As u is a solution of (1.6), the function $\partial^\alpha u$ solves the linear initial value problem

$$\begin{cases} L(\chi(u), A_1, A_2, A_3, \sigma(u))v = f_\alpha, & x \in \mathbb{R}^3_+, \quad t \in (0,T); \\ v(0) = \partial^{(0,\alpha_1,\alpha_2,\alpha_3)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f, u_0), & x \in \mathbb{R}^3_+; \end{cases} \tag{4.5}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$. Moreover, if additionally $\alpha_3 = 0$, it solves

$$\begin{cases} L(\chi(u), A_1, A_2, A_3, \sigma(u))v = f_\alpha, & x \in \mathbb{R}^3_+, \quad t \in (0,T); \\ Bv = \partial^\alpha g, & x \in \partial\mathbb{R}^3_+, \quad t \in (0,T); \\ v(0) = \partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f, u_0), & x \in \mathbb{R}^3_+. \end{cases} \tag{4.6}$$

Here we used that $\partial_t^j u(0) = S_{\chi,\sigma,\mathbb{R}^3_+,m,j}(0, f, u_0)$ for all $j \in \{0, \dots, m\}$ by (2.9).

In view of (4.6) respectively (4.5), we want to apply Theorem 3.1 respectively Lemma 4.3 to $\partial^\alpha u$. We thus need estimates for $\|f_\alpha\|_{L^2(\Omega)}$ for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$, $\|f_\alpha\|_{H^1(\Omega)}$ for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m-1$, and $\|f_\alpha(0)\|_{L^2(\mathbb{R}^3_+)}$ for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m-1$. We start with the estimate for $\|f_\alpha\|_{L^2(\Omega)}$. Take $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$. Let $\beta \in \mathbb{N}_0^4$ with $0 < \beta \le \alpha$. Lemma 4.2 then implies that

$$\|f_\alpha\|_{L^2(\Omega)} \le \|f\|_{H^{|\alpha|}(\Omega)} + C(\chi, \sigma, r, \omega_0, \mathcal{U}_1)\|u\|_{H^{|\alpha|}(\Omega)}. \tag{4.7}$$

Next, we want to estimate $\|f_\alpha\|_{H^1(\Omega)}$ for $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m-1$. So fix such a multi-index. Let $k \in \{0, \dots, 3\}$ and set $\alpha^k = \alpha + e_k$. A straightforward computation, see e.g. (3.6) in [34], shows the formula

$$f_{\alpha^k} = \partial_k f_\alpha - \partial_k \chi(u)\partial_t \partial^\alpha u - \sum_{j=1}^{2} \partial_k A_j \partial_j \partial^\alpha u - \partial_k \sigma(u)\partial^\alpha u. \tag{4.8}$$

Combined with (4.7), Lemma 4.2 now yields the inequality

$$\|f_\alpha\|_{H^1(\Omega)} \le C(\chi, \sigma, r, \omega_0, \mathcal{U}_1)\Big(\|f\|_{H^{|\alpha|+1}(\Omega)} + \|u\|_{H^{|\alpha|+1}(\Omega)}\Big). \tag{4.9}$$

It remains to estimate $\|f_\alpha(0)\|_{L^2(\mathbb{R}^3_+)}$ for $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m-1$. To that purpose we first insert $t = 0$ in the definition of f_α in (4.4). The product estimates from Lemma 2.1 in [35], the fact that $\partial_t^j u(0) = S_{\chi,\sigma,\mathbb{R}^3_+,m,j}(0, f, u_0)$ for all $j \in \{0, \dots, m\}$ by (2.9), and Lemma 2.4 then lead to the bound

$$\|f_\alpha(0)\|_{L^2(\mathbb{R}^3_+)} \le C(\chi, \sigma, m, r, \omega_0, \mathcal{U}_1)\Big(\sum_{l=0}^{|\alpha|} \|\partial_t^l f(0)\|_{H^{|\alpha|-l}(\mathbb{R}^3_+)} + \|u_0\|_{H^{|\alpha|+1}(\mathbb{R}^3_+)}\Big), \tag{4.10}$$

see Proposition 7.20 in [34] for the details.

II) We will show inductively that there are constants $C_k = C_k(\chi, \sigma, m, r, \omega_0, \mathcal{U}_1, T^*)$ such that

$$\|\partial^\alpha u\|^2_{G_0(\Omega)} \le C_k\Big(\sum_{j=0}^{k-1} \|\partial_t^j f(0)\|^2_{H^{k-1-j}(\mathbb{R}^3_+)} + \|g\|^2_{E_k(J\times\partial\mathbb{R}^3_+)} + \|u_0\|^2_{H^k(\mathbb{R}^3_+)} + \|f\|^2_{H^k(\Omega)}\Big) \tag{4.11}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$ and $k \in \{0, \dots, m\}$.

We first apply Lemma 2.1 (1) and (3) to obtain a radius $R_1 = R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1)$ with

$$\|\chi(u)\|_{W^{1,\infty}(\Omega)} + \|\sigma(u)\|_{W^{1,\infty}(\Omega)} \le R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1),$$
$$\|\chi(u(0))\|_{L^\infty(\mathbb{R}^3_+)} + \|\sigma(u(0))\|_{L^\infty(\mathbb{R}^3_+)} \le R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1).$$

Set $\gamma_0 = \gamma_0(\chi, \sigma, r, \omega_0, \mathcal{U}_1, T^*) = \gamma_{3.1,0}(\eta(\chi), R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1), T^*) \ge 1$, where $\gamma_{3.1,0}$ is the corresponding constant from Theorem 3.1. As u solves (4.6) with $\alpha = 0$, Theorem 3.1 yields

$$\begin{aligned}
\|u\|^2_{G_0(\Omega)} &\le e^{2\gamma_0 T} \sup_{t\in(0,T)} \|e^{-\gamma_0 t} u(t)\|^2_{L^2(\mathbb{R}^3_+)} \\
&\le C_{3.1,0,0}(\eta, R_1) e^{2\gamma_0 T^*} \Big(\|u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g\|^2_{E_{0,\gamma_0}(J\times\partial\mathbb{R}^3_+)}\Big) \\
&\quad + C_{3.1,0}(\eta, R_1, T^*) e^{2\gamma_0 T^*} \frac{1}{\gamma_0} \|f\|^2_{L^2_{\gamma_0}(\Omega)} \\
&\le C_0\Big(\|u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g\|^2_{E_0(J\times\partial\mathbb{R}^3_+)} + \|f\|^2_{L^2(\Omega)}\Big),
\end{aligned}$$

where $C_0 = C_0(\chi, \sigma, r, \omega_0, \mathcal{U}_1, T^*)$ and $C_{3.1,0,0}$ respectively $C_{3.1,0}$ denote the corresponding constants from Theorem 3.1. This inequality shows the claim (4.11) for $k = 0$.

Let $k \in \{1, \dots, m\}$ and assume that (4.11) has been shown for all $0 \le j \le k-1$. We first claim that there are constants $C_{k,\alpha} = C_{k,\alpha}(\chi, \sigma, r, \omega_0, \mathcal{U}_1, T^*)$ such that

$$\|\partial^\alpha u\|^2_{G_0(\Omega)} \le C_{k,\alpha}\Big(\sum_{j=0}^{k-1} \|\partial_t^j f(0)\|^2_{H^{k-1-j}(\mathbb{R}^3_+)} + \|g\|^2_{E_k(J\times\partial\mathbb{R}^3_+)} + \|u_0\|^2_{H^k(\mathbb{R}^3_+)} + \|f\|^2_{H^k(\Omega)} + \int_0^T \sum_{\beta\in\mathbb{N}_0^4, |\beta|=k} \|\partial^\beta u(s)\|^2_{L^2(\mathbb{R}^3_+)} ds\Big) \tag{4.12}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$. We show (4.12) by another induction, this time with respect to α_3.

Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$ and $\alpha_3 = 0$. In step I) we have seen that $\partial^\alpha u$ solves the initial boundary value problem (4.6). Hence, Theorem 3.1 yields

$$\begin{aligned}
&\|\partial^\alpha u\|^2_{G_0(\Omega)} \\
&\quad \le e^{2\gamma_0 T} \sup_{t\in(0,T)} \|e^{-\gamma_0 t} \partial^\alpha u(t)\|^2_{L^2(\mathbb{R}^3_+)}
\end{aligned}$$

$$\leq C_{3.1,0,0}(\eta, R_1)e^{2\gamma_0 T^*}\Big(\|\partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f, u_0)\|^2_{L^2(\mathbb{R}^3_+)}$$
$$+ \|\partial^\alpha g\|^2_{E_{0,\gamma_0}(J\times\partial\mathbb{R}^3_+)}\Big) + C_{3.1,0}(\eta, R_1, T^*)e^{2\gamma_0 T^*}\frac{1}{\gamma_0}\|f_\alpha\|^2_{L^2_{\gamma_0}(\Omega)}$$
$$\leq C(\chi, \sigma, k, r, \omega_0, \mathcal{U}_1, T^*)\Big(\sum_{j=0}^{k-1}\|\partial_t^j f(0)\|^2_{H^{k-1-j}(\mathbb{R}^3_+)} + \|g\|^2_{E_k(J\times\partial\mathbb{R}^3_+)} + \|u_0\|^2_{H^k(\mathbb{R}^3_+)}\Big)$$
$$+ \|f\|^2_{H^k(\Omega)} + \|u\|^2_{H^k(\Omega)}\Big),$$

where we employed Lemma 2.4 and (4.7). Using the induction hypothesis (4.11) for the derivatives of u of order smaller or equal than $k-1$, we arrive at

$$\|\partial^\alpha u\|^2_{G_0(\Omega)} \leq C_{k,\alpha}(\chi, \sigma, r, \omega_0, \mathcal{U}_1, T^*)\Big(\sum_{j=0}^{k-1}\|\partial_t^j f(0)\|^2_{H^{k-1-j}(\mathbb{R}^3_+)} + \|g\|^2_{E_k(J\times\partial\mathbb{R}^3_+)}$$
$$+ \|u_0\|^2_{H^k(\mathbb{R}^3_+)} + \|f\|^2_{H^k(\Omega)} + \int_0^T \sum_{\beta\in\mathbb{N}_0^4, |\beta|=k}\|\partial^\beta u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds\Big),$$

which is (4.12) for all multiindices α with $|\alpha| = k$ and $\alpha_3 = 0$.

Now, assume that there is a number $l \in \{1, \dots, k\}$ such that (4.12) is true for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$ and $\alpha_3 \in \{0, \dots, l-1\}$.

Take $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$ and $\alpha_3 = l$. The multi-index $\alpha' = \alpha - e_3$ belongs to $\mathbb{N}_0^4$ and satisfies $|\alpha'| = k-1 \leq m-1$. Due to step I), we know that $\partial^{\alpha'} u$ solves the initial value problem (4.5) with right-hand side $f_{\alpha'}$ and initial value

$$\partial^{(0,\alpha_1,\alpha_2,\alpha_3-1)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f, u_0).$$

As $|\alpha'| \leq m-1$, the function $f_{\alpha'}$ belongs to $H^1(\Omega)$ by (4.9), the derivative of the higher order initial value $\partial^{(0,\alpha_1,\alpha_2,\alpha_3-1)} S_{\chi,\sigma,G,m,\alpha_0}(0, f, u_0)$ to $H^1(\mathbb{R}^3_+)$ by Lemma 2.4, and $\partial^{\alpha'} u$ to $G_1(\Omega)$. Moreover, $\chi(u)$ and $\sigma(u)$ are elements of $F^{\mathrm{c}}_{0,\eta}(\Omega)$ respectively $F^{\mathrm{c}}_0(\Omega)$, A_1 and A_2 belong to $F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\mathbb{R}^3_+)$ and $A_3 = A_3^{\mathrm{co}}$. We can therefore apply Lemma 4.3. We choose $\gamma = 1$ to infer

$$\|\partial^\alpha u\|^2_{G_0(\Omega)}$$
$$= \|\partial_3\partial^{\alpha'} u\|^2_{G_0(\Omega)} \leq e^{2T}\|\nabla\partial^{\alpha'} u\|^2_{G_{0,1}(\Omega)}$$
$$\leq e^{(2+C_1)T}\Big((C_{1,0} + TC_1)\Big(\sum_{j=0}^{2}\|\partial_j\partial^{\alpha'} u\|^2_{G_{0,1}(\Omega)} + \|f_{\alpha'}(0)\|^2_{L^2(\mathbb{R}^3_+)}\Big)$$
$$+ (C_{1,0} + TC_1)\|\partial^{(0,\alpha_1,\alpha_2,\alpha_3-1)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f, u_0)\|^2_{H^1(\mathbb{R}^3_+)} + C_1\|f_{\alpha'}\|^2_{H^1(\Omega)}\Big)$$

$$\leq C(\chi,\sigma,k,r,\omega_0,\mathcal{U}_1,T^*)\Big(\sum_{j=0}^{2}\|\partial_j\partial^{\alpha'}u\|^2_{G_0(\Omega)}+\sum_{j=0}^{k-1}\|\partial_t^j f(0)\|^2_{H^{k-1-j}(\mathbb{R}^3_+)}$$
$$+\|u_0\|^2_{H^k(\mathbb{R}^3_+)}+\|f\|^2_{H^k(\Omega)}+\|u\|^2_{H^k(\Omega)}\Big), \tag{4.13}$$

where we used (4.10), Lemma 2.4, and (4.9) in the last estimate and where

$$C_{1,0}=C_{1,0}(\chi,\sigma,r,\omega_0,\mathcal{U}_1)=C_{4.3,1,0}(\eta(\chi),R_1(\chi,\sigma,r,\omega_0,\mathcal{U}_1)),$$
$$C_1=C_1(\chi,\sigma,r,\omega_0,\mathcal{U}_1,T^*)=C_{4.3,1}(\eta(\chi),R_1(\chi,\sigma,r,\omega_0,\mathcal{U}_1),T^*).$$

Inserting the induction hypothesis for $\|\partial_j\partial^{\alpha'}u\|^2_{G_0(\Omega)}$ and the induction hypothesis (4.11) for the derivatives of u of order smaller or equal than $k-1$, we obtain (4.12) for all $\alpha\in\mathbb{N}_0^4$ with $|\alpha|=k$ and $\alpha_3=l$. By induction, we thus infer that the estimate in (4.12) is valid for all multiindices $\alpha\in\mathbb{N}_0^4$ with $|\alpha|=k$.

We now sum in (4.12) over all multiindices with $|\alpha|=k$, which yields

$$\sum_{\alpha\in\mathbb{N}_0^4,|\alpha|=k}\|\partial^\alpha u(T)\|^2_{L^2(\mathbb{R}^3_+)}$$
$$\leq\sum_{\alpha\in\mathbb{N}_0^4,|\alpha|=k}\|\partial^\alpha u\|^2_{G_0(\Omega)}$$
$$\leq\Big(\sum_{\alpha\in\mathbb{N}_0^4,|\alpha|=k}C_{k,\alpha}\Big)\Big(\sum_{j=0}^{k-1}\|\partial_t^j f(0)\|^2_{H^{k-1-j}(\mathbb{R}^3_+)}+\|g\|^2_{E_k((0,T)\times\partial\mathbb{R}^3_+)}+\|u_0\|^2_{H^k(\mathbb{R}^3_+)}$$
$$+\|f\|^2_{H^k((0,T)\times\mathbb{R}^3_+)}+\int_0^T\sum_{\beta\in\mathbb{N}_0^4,|\beta|=k}\|\partial^\beta u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds\Big).$$

Recall that the time $T\in(0,T']$ was arbitrary. Since $T\mapsto\|f\|^2_{H^k((0,T)\times\mathbb{R}^3_+)}$ and $T\mapsto\|g\|^2_{E_k((0,T)\times\partial\mathbb{R}^3_+)}$ are monotonically increasing, Gronwall's inequality leads to

$$\sum_{\alpha\in\mathbb{N}_0^4,|\alpha|=k}\|\partial^\alpha u(t)\|^2_{L^2(\mathbb{R}^3_+)}\leq C_k'e^{C_k't}\Big(\sum_{j=0}^{k-1}\|\partial_t^j f(0)\|^2_{H^{k-1-j}(\mathbb{R}^3_+)}+\|g\|^2_{E_k((0,t)\times\mathbb{R}^3_+)}$$
$$+\|u_0\|^2_{H^k(\mathbb{R}^3_+)}+\|f\|^2_{H^k((0,t)\times\mathbb{R}^3_+)}\Big) \tag{4.14}$$

for all $t\in[0,T']$, where we defined $C_k'=C_k'(\chi,\sigma,r,\omega_0,\mathcal{U}_1,T^*)$ by $\sum_{\alpha\in\mathbb{N}_0^4,|\alpha|=k}C_{k,\alpha}$. Defining $C_k=C_k(\chi,\sigma,r,\omega_0,\mathcal{U}_1,T^*)$ by $C_k'e^{C_k'T^*}$ and taking again a fixed time $T\in(0,T']$, we particularly obtain

$$\sum_{\alpha\in\mathbb{N}_0^4,|\alpha|=k} \|\partial^\alpha u(t)\|^2_{L^2(\mathbb{R}^3_+)} \le C_k\Big(\sum_{j=0}^{k-1}\|\partial_t^j f(0)\|^2_{H^{k-1-j}(\mathbb{R}^3_+)} + \|g\|^2_{E_k((0,T)\times\partial\mathbb{R}^3_+)} + \|u_0\|^2_{H^k(\mathbb{R}^3_+)} + \|f\|^2_{H^k((0,T)\times\mathbb{R}^3_+)}\Big)$$

for all $t\in[0,T]$.

We conclude that (4.11) is true for all $\alpha\in\mathbb{N}_0^4$ with $|\alpha|=k$. Again by induction, we infer that (4.11) is true for all $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\in\{0,\dots,m\}$. Summing over all multiindices with absolute value between 0 and m, the assertion of the proposition finally follows. □

This proposition now easily implies the improved blow-up condition. We postpone the statement and its proof to the full local wellposedness theorem below.

5. Continuous dependence and local wellposedness theorem

The investigation of continuous dependence for quasilinear problems is challenging because of a loss of derivatives. It occurs since the difference of two solutions satisfies an equation with a less regular right-hand side. For the same reason one can only hope for continuous (and not Lipschitz-continuous) dependence on the data. We start with an approximation lemma in low regularity. It is the first step to overcome the loss of derivatives.

Lemma 5.1. *Let $J\subset\mathbb{R}$ be an open interval, $t_0\in\overline{J}$, and $\eta>0$. Take coefficients $A_{0,n},A_0\in F^{\mathrm{c}}_{3,\eta}(\Omega)$, $A_1,A_2\in F^{\mathrm{cp}}_{3,\mathrm{coeff}}(\mathbb{R}^3_+)$, $A_3=A_3^{\mathrm{co}}$, and $D_n,D\in F^{\mathrm{c}}_3(\Omega)$ for all $n\in\mathbb{N}$ such that $(A_{0,n})_n$ and $(D_n)_n$ are bounded in $W^{1,\infty}(\Omega)$ and converge to A_0 respectively D in $L^\infty(\Omega)$. Set $B=B^{\mathrm{co}}$. Choose data $u_0\in L^2(\mathbb{R}^3_+)$, $g\in E_0(J\times\partial\mathbb{R}^3_+)$, and $f\in L^2(\Omega)$. Let u_n denote the weak solution of the linear initial boundary value problem (3.1) with differential operator $L(A_{0,n},A_1,A_2,A_3,D_n)$ and these data for all $n\in\mathbb{N}$ and u be the weak solution of (3.1) with differential operator $L(A_0,\dots,A_3,D)$ and the same data. Then $(u_n)_n$ converges to u in $G_0(\Omega)$.*

Proof. Without loss of generality we assume that $J=(0,T)$ for some $T>0$ and $t_0=0$. Set $A_{0,0}=A_0$ and $D_0=D$. Take $r>0$ with $\|A_{0,n}\|_{W^{1,\infty}(\Omega)}\le r$ and $\|D_n\|_{W^{1,\infty}(\Omega)}\le r$ for all $n\in\mathbb{N}_0$.

I) We first assume that u_0 belongs to $H^1(\mathbb{R}^3_+)$, g to $E_1(J\times\partial\mathbb{R}^3_+)$, f to $H^1(\Omega)$, and that $Bu_0=g(0)$ on $\partial\mathbb{R}^3_+$. Then the tuples $(0,A_{0,n},A_1,A_2,A_3,B,D_n,f,g,u_0)$ fulfill the linear compatibility conditions (2.12) of first order for each $n\in\mathbb{N}_0$. The solutions u_n and u are thus contained in $G_1(\Omega)$ by Theorem 3.1. The difference u_n-u further solves the linear initial boundary value problem

$$\begin{cases} L(A_{0,n},A_1,A_2,A_3,D_n)(u_n-u)=f_n, & x\in\mathbb{R}^3_+,\quad t\in J;\\ B(u_n-u)=0, & x\in\partial\mathbb{R}^3_+,\ t\in J;\\ (u_n-u)(0)=0, & x\in\mathbb{R}^3_+;\end{cases}$$

where $f_n=(A_0-A_{0,n})\partial_t u+(D-D_n)u$ for all $n\in\mathbb{N}$. As u is an element of $G_1(\Omega)$, the right-hand side of the differential equation above belongs to $L^2(\Omega)$. Theorem 3.1 thus provides constants $\gamma=\gamma_{3.1,0}(\eta,r)$ and $C_0=\max\{C_{3.1,0,0}(\eta,r),C_{3.1,0}(\eta,r,T)\}$ such that

$$\|u_n - u\|^2_{G_0(\Omega)} \le e^{2\gamma T}\|u_n - u\|^2_{G_{0,\gamma}(\Omega)} \le C_0 e^{2\gamma T}\|(A_0 - A_{0,n})\partial_t u + (D - D_n)u\|^2_{L^2_\gamma(\Omega)}$$
$$\le 2C_0 e^{2\gamma T}(\|A_{0,n} - A_0\|^2_{L^\infty(\Omega)}\|\partial_t u\|^2_{L^2_\gamma(\Omega)} + \|D_n - D\|^2_{L^\infty(\Omega)}\|u\|^2_{L^2_\gamma(\Omega)})$$

for all $n \in \mathbb{N}$. Since $A_{0,n} \to A_0$ and $D_n \to D$ in $L^\infty(\Omega)$, we conclude that the functions u_n tend to u in $G_0(\Omega)$ as $n \to \infty$.

II) We now come to the general case. Take sequences $(f_j)_j$ in $H^1(\Omega)$, $(g_j)_j$ in $E_2(J \times \partial\mathbb{R}^3_+)$, and $(\tilde{u}_{0,j})_j$ in $C^\infty_c(\mathbb{R}^3_+)$ converging to f, g and u_0 in $L^2(\Omega)$, $E_0(J \times \partial\mathbb{R}^3_+)$, and $L^2(\mathbb{R}^3_+)$ respectively. As B is constant and has rank 2, there is a sequence $(\tilde{h}_j)_j$ in $E_2(J \times \partial\mathbb{R}^3_+)$ with $B\tilde{h}_j = g_j$ for all $j \in \mathbb{N}$. Extending $\tilde{h}_j$ to $J \times \mathbb{R}^3_+$ via a suitable sequence of smooth cut-off functions in x_3-direction, we obtain functions h_j in $C(\overline{J}, H^1(\mathbb{R}^3_+))$ such that $u_{0,j} = \tilde{u}_{0,j} + h_j(0)$ converges to u_0 in $L^2(\mathbb{R}^3_+)$ as $j \to \infty$ and $Bu_{0,j} = Bh_j(0) = B\tilde{h}_j(0) = g_j(0)$ on $\partial\mathbb{R}^3_+$ for all $j \in \mathbb{N}$. We refer to step I) of the proof of Theorem 4.13 in [34] for the details of this construction. Note that the tuples $(0, A_{0,n}, A_1, A_2, A_3, D_n, B, f_j, g_j, u_{0,j})$ consequently fulfill the linear compatibility conditions (2.12) of order 1 for all $n, j \in \mathbb{N}$.

Let the function u^j_n denote the weak solution of (3.1) with differential operator $L(A_{0,n}, A_1, A_2, A_3, D_n)$ and data f_j, g_j, and $u_{0,j}$ as well as u^j the weak solution of (3.1) with differential operator $L(A_0, \dots, A_3, D)$, and the same data for all $n, j \in \mathbb{N}$. These solutions belong to $G_1(\Omega)$ by Theorem 3.1. Observe that the difference $u^j_n - u_n$ solves (3.1) with differential operator $L(A_{0,n}, A_1, A_2, A_3, D_n)$, inhomogeneity $f_j - f$, boundary value $g_j - g$, and initial value $u_{0,j} - u_0$, and the function $u^j - u$ solves (3.1) with differential operator $L(A_0, A_1, A_2, A_3, D)$ and the same data. The a priori estimate in Theorem 3.1 thus shows

$$\|u^j_n - u_n\|^2_{G_0(\Omega)} \le e^{2\gamma T}\|u^j_n - u_n\|^2_{G_{0,\gamma}(\Omega)} \tag{5.1}$$
$$\le C_0\, e^{2\gamma T}\Big(\|u_{0,j} - u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g_j - g\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|f_j - f\|^2_{L^2_\gamma(\Omega)}\Big),$$
$$\|u^j - u\|^2_{G_0(\Omega)} \le e^{2\gamma T}\|u^j - u\|^2_{G_{0,\gamma}(\Omega)} \tag{5.2}$$
$$\le C_0\, e^{2\gamma T}\Big(\|u_{0,j} - u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g_j - g\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|f_j - f\|^2_{L^2_\gamma(\Omega)}\Big)$$

for all $n, j \in \mathbb{N}$, where γ and C_0 were introduced in step I).

Let $\varepsilon > 0$. Because of the convergence of the data, we find an index j_0 such that

$$C_0\, e^{2\gamma T}\Big(\|u_{0,j_0} - u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g_{j_0} - g\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|f_{j_0} - f\|^2_{L^2_\gamma(\Omega)}\Big) \le \frac{\varepsilon^2}{9}. \tag{5.3}$$

On the other hand, the tuple $(f_{j_0}, g_{j_0}, u_{0,j_0})$ fulfills the assumptions of step I), which therefore implies $u^{j_0}_n \to u^{j_0}$ in $G_0(\Omega)$ as $n \to \infty$. Hence, there is an index $n_0 \in \mathbb{N}$ such that

$$\|u^{j_0}_n - u^{j_0}\|_{G_0(\Omega)} \le \frac{\varepsilon}{3} \tag{5.4}$$

for all $n \ge n_0$. Combining (5.1) to (5.4), we arrive at

$$\|u_n - u\|_{G_0(\Omega)} \le \|u_n - u^{j_0}_n\|_{G_0(\Omega)} + \|u^{j_0}_n - u^{j_0}\|_{G_0(\Omega)} + \|u^{j_0} - u\|_{G_0(\Omega)} \le \varepsilon$$

for all $n \ge n_0$. □

Please cite this article in press as: M. Spitz, Local wellposedness of nonlinear Maxwell equations with perfectly conducting boundary conditions, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.019

The next lemma contains the core of the proof for the continuous dependence. It states that given a sequence of data converging in H^m respectively E_m and assuming that the sequence of corresponding solutions of (1.6) converges in G_{m-1}, then the solutions also converge in G_m. The proof involves techniques developed for the full space (see e.g. [3]) which prevents to lose regularity because of the quasilinearity. Using methods from [35], we again exploit the structure of Maxwell's equations to avoid the loss of a derivative due to the characteristic boundary.

Lemma 5.2. *Let $J' \subseteq \mathbb{R}$ be an open and bounded interval, $t_0 \in \overline{J'}$, and $m \in \mathbb{N}$ with $m \geq 3$. Take functions $\chi \in \mathcal{ML}^{m,6,\mathrm{c}}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^{m,6,\mathrm{c}}(G,\mathcal{U})$. Set*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $x \in \partial G$, where ν denotes the unit outer normal vector of ∂G. Choose $f_n, f \in H^m(J' \times G)$, $g_n, g \in E_m(J' \times \partial G)$, and $u_{0,n}, u_0 \in H^m(G)$ for all $n \in \mathbb{N}$ with

$$\|u_{0,n} - u_0\|_{H^m(G)} \longrightarrow 0, \quad \|g_n - g\|_{E_m(J' \times \partial G)} \longrightarrow 0, \quad \|f_n - f\|_{H^m(J' \times G)} \longrightarrow 0,$$

as $n \to \infty$. We further assume that (1.6) with data $(t_0, f_n, g_n, u_{0,n})$ and (t_0, f, g, u_0) have $G_m(J' \times G)$-solutions u_n and u for all $n \in \mathbb{N}$, that there is a compact subset $\tilde{\mathcal{U}}_1$ of $\mathcal{U}$ with $\operatorname{im} u(t) \subseteq \tilde{\mathcal{U}}_1$ for all $t \in J'$, that $(u_n)_n$ is bounded in $G_m(J' \times G)$, and that $(u_n)_n$ converges to u in $G_{m-1}(J' \times G)$.

Then the functions u_n converge to u in $G_m(J' \times G)$.

Proof. Without loss of generality we assume that $t_0 = 0$, that $J' = (0, T')$ for a number $T' > 0$, and that χ and σ fulfill (2.11), cf. Remark 2.5. The proof is again reduced to the half-space case $G = \mathbb{R}^3_+$ via local charts. We do not give the details of the localization procedure here but assume as in the proof of Proposition 4.4 that $G = \mathbb{R}^3_+$ and that we have spatial coefficients $A_1, A_2 \in F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\mathbb{R}^3_+)$ $A_3 = A_3^{\mathrm{co}}$, and $B = B^{\mathrm{co}}$. We refer to [35, Section 2] and [34, Chapter 5] for the details.

Let $T \in (0, T']$, $J = (0, T)$, and $\Omega = J \times \mathbb{R}^3_+$. Sobolev's embedding yields a constant C_S depending on the length of the interval J' such that

$$\sum_{j=0}^{m-1} \|\partial_t^j f_n(0) - \partial_t^j f(0)\|_{H^{m-j-1}(\mathbb{R}^3_+)} \leq m C_S \|f_n - f\|_{H^m(J' \times \mathbb{R}^3_+)} \longrightarrow 0 \tag{5.5}$$

as $n \to \infty$. We set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, $u_\infty = u$, $f_\infty = f$, $g_\infty = g$, and $u_{0,\infty} = u_0$. Throughout, let $n \in \overline{\mathbb{N}}$ and $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$. By assumption, (5.5), and Sobolev's embedding there is a radius $r > 0$ such that

$$\|u_n\|_{G_m(J' \times \mathbb{R}^3_+)} + \|u_n\|_{L^\infty(J' \times \mathbb{R}^3_+)} + \sum_{j=1}^{2} \|A_j\|_{F_m(J' \times \mathbb{R}^3_+)} \leq r, \tag{5.6}$$

$$\sum_{j=0}^{m-1} \|\partial_t^j f_n(0)\|_{H^{m-j-1}(\mathbb{R}^3_+)} + \|u_{0,n}\|_{H^m(\mathbb{R}^3_+)} + \|g_n\|_{E_m(J'\times\partial\mathbb{R}^3_+)} + \|f_n\|_{H^m(J'\times\mathbb{R}^3_+)} \le r. \quad (5.7)$$

Moreover, $(u_n)_n$ converges to u in $L^\infty(J'\times G)$ so that there is a compact and connected set $\mathcal{U}_1 \subseteq \mathcal{U}$ and an index n_0 such that $\operatorname{im} u_n(t) \subseteq \mathcal{U}_1$ for all $t \in J'$ and $n \ge n_0$. Without loss of generality we assume $n_0 = 1$. Lemma 2.1 (1) then shows that $\chi(u_n)$ and $\sigma(u_n)$ belong to $F^c_m(J'\times\mathbb{R}^3_+)$ and that there is a radius $R = R(\chi, \sigma, m, r, \mathcal{U}_1)$ with

$$\|\chi(u_n)\|_{F_m(J'\times\mathbb{R}^3_+)} + \|\sigma(u_n)\|_{F_m(J'\times\mathbb{R}^3_+)} \le R. \quad (5.8)$$

Furthermore, $\chi(u_n)$ is symmetric and positive definite with $\chi(u_n) \ge \eta$. We use the operators and right-hand sides

$$\begin{aligned} L_n &= L(\chi(u_n), A_1, A_2, A_3, \sigma(u_n)), \\ f_{\alpha,n} &= \partial^\alpha f_n - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta \chi(u_n) \partial^{\alpha-\beta} \partial_t u_n - \sum_{j=1}^{2} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_j \partial^{\alpha-\beta} \partial_j u_n \\ &\quad - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta \sigma(u_n) \partial^{\alpha-\beta} u_n. \end{aligned} \quad (5.9)$$

As in [35], the function $\partial^\alpha u_n$ then solves the linear initial value problem

$$\begin{cases} L_n v = f_{\alpha,n}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ v(0) = \partial^{(0,\alpha_1,\alpha_2,\alpha_3)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n}), & x \in \mathbb{R}^3_+; \end{cases} \quad (5.10)$$

and it is the solution of the linear initial boundary value problem

$$\begin{cases} L_n v = f_{\alpha,n}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ B v = \partial^\alpha g_n, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ v(0) = \partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n}), & x \in \mathbb{R}^3_+; \end{cases} \quad (5.11)$$

if also $\alpha_3 = 0$. Here we exploited that A_3 and B are constant.

I) To estimate $f_{\alpha,n}$ and $f_{\alpha,n} - f_{\alpha,\infty}$, we introduce the quantity

$$\begin{aligned} h_n(t) = \sum_{i=1}^{3} \sum_{0\le j\le m} \sum_{\substack{0\le\gamma\le\alpha,\gamma_0=0 \\ |\gamma|=m-j}} \sum_{l_1,\dots,l_j=1}^{6} &\|(\partial_{y_{l_j}} \dots \partial_{y_{l_1}} \partial_x^{(\gamma_1,\gamma_2,\gamma_3)} \theta_i)(u_n(t)) \\ &- (\partial_{y_{l_j}} \dots \partial_{y_{l_1}} \partial_x^{(\gamma_1,\gamma_2,\gamma_3)} \theta_i)(u(t))\|_{L^\infty(\mathbb{R}^3_+)} \end{aligned}$$

for all $t \in \overline{J'}$ and $n \in \mathbb{N}$, where $\theta_1 = \chi$, $\theta_2 = \sigma$, and θ_3 is the matrix inverse of χ, cf. Corollary 2.2. Recall that $(u_n)_n$ tends to u uniformly as $n \to \infty$ and that these functions map in the compact set $\mathcal{U}_1$. It follows

$$h_n(t) \longrightarrow 0 \quad \text{for all } t \in \overline{J'} \qquad \text{and} \qquad \int_0^{T'} h_n^2(t)dt \longrightarrow 0 \tag{5.12}$$

as $n \to \infty$. Using Lemma 2.1 of [35] and Corollary 2.2 we derive the bounds

$$\begin{aligned}
&\|f_{\alpha,n}\|_{L^2(\Omega)} \le C(\chi,\sigma,m,r,\mathcal{U}_1,T'),\\
&\|f_{\alpha,n} - f_{\alpha,\infty}\|^2_{L^2(\Omega)}\\
&\quad = \int_0^T \|f_{\alpha,n}(s) - f_{\alpha,\infty}(s)\|^2_{L^2(\mathbb{R}^3_+)} ds\\
&\quad \le C(\chi,\sigma,m,r,\mathcal{U}_1,T')\Big(\|f_n - f\|^2_{H^m(\Omega)} + \|u_n - u\|^2_{G_{m-1}(\Omega)} + \delta_{|\alpha|m}\int_0^T h_n^2(s)ds\\
&\qquad\qquad + \int_0^T \sum_{\tilde\alpha\in\mathbb{N}_0^4, |\tilde\alpha|=m} \|\partial^{\tilde\alpha}u_n(s) - \partial^{\tilde\alpha}u(s)\|^2_{L^2(\mathbb{R}^3_+)} ds\Big).
\end{aligned} \tag{5.13}$$

Let $|\alpha| \le m-1$. Using also (4.8), we then obtain

$$\begin{aligned}
&\|f_{\alpha,n}\|_{G_0(\Omega)} \le C(\chi,\sigma,m,r,\mathcal{U}_1),\\
&\|f_{\alpha,n} - f_{\alpha,\infty}\|_{G_0(\Omega)} \le \|f_n - f\|_{G_{m-1}(\Omega)} + C(\chi,\sigma,m,r,\mathcal{U}_1)\|u_n - u\|_{G_{m-1}(\Omega)},
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&\|f_{\alpha,n}\|_{H^1(\Omega)} \le C(\chi,\sigma,m,r,\mathcal{U}_1,T'),\\
&\|f_{\alpha,n} - f_{\alpha,\infty}\|^2_{H^1(\Omega)}\\
&\quad \le C(\chi,\sigma,m,r,\mathcal{U}_1,T')\Big(\|f_n - f\|^2_{H^m(\Omega)} + \|u_n - u\|^2_{G_{m-1}(\Omega)}\\
&\qquad + \delta_{|\alpha|m-1}\int_0^T h_n^2(s)ds + \int_0^T \sum_{\tilde\alpha\in\mathbb{N}_0^4, |\tilde\alpha|=m} \|\partial^{\tilde\alpha}u_n(s) - \partial^{\tilde\alpha}u(s)\|^2_{L^2(\mathbb{R}^3_+)} ds\Big).
\end{aligned} \tag{5.15}$$

(See [34] for further details.)

II) We now look at the tangential case $\alpha_3 = 0$. To split $\partial^\alpha u_n$, we define the functions

$$w_{0,n} = \partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n}),$$

which belong to $L^2(\mathbb{R}^3_+)$ by Lemma 2.4. Consider the linear initial boundary value problems

$$\begin{cases} L_n v = f_{\alpha,\infty}, & x \in \mathbb{R}^3_+, \quad t \in J;\\ Bv = \partial^\alpha g_\infty, & x \in \partial\mathbb{R}^3_+, \ t \in J;\\ v(0) = w_{0,\infty}, & x \in \mathbb{R}^3_+; \end{cases} \tag{5.16}$$

and

$$\begin{cases} L_n v = f_{\alpha,n} - f_{\alpha,\infty}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ B v = \partial^\alpha g_n - \partial^\alpha g_\infty, & x \in \partial\mathbb{R}^3_+, \ t \in J; \\ v(0) = w_{0,n} - w_{0,\infty}, & x \in \mathbb{R}^3_+. \end{cases} \tag{5.17}$$

Because of the above regularity statements, Theorem 3.1 implies that the problem (5.16) has a unique solution w_n in $G_0(\Omega)$, the problem (5.17) has a unique solution z_n in $G_0(\Omega)$, and that the sum $w_n + z_n$ uniquely solves (5.11). We thus obtain

$$w_n + z_n = \partial^\alpha u_n. \tag{5.18}$$

We point out that in the case $n = \infty$ the initial boundary value problems (5.16) and (5.11) coincide. Since the latter is solved by $\partial^\alpha u_n$ and solutions of that problem are unique by Theorem 3.1, we conclude that

$$w_\infty = \partial^\alpha u_\infty = \partial^\alpha u. \tag{5.19}$$

Since $(u_n)_n$ tends to u uniformly and these functions map into the compact set $\mathcal{U}_1$, we obtain the limits

$$\|\chi(u_n) - \chi(u)\|_{L^\infty(\Omega)} + \|\sigma(u_n) - \sigma(u)\|_{L^\infty(\Omega)} \le C(\chi, \sigma, \mathcal{U}_1)\|u_n - u\|_{G_{m-1}(\Omega)} \longrightarrow 0$$

as $n \to \infty$. In view of (5.8), Lemma 5.1 therefore tells us that

$$\|w_n - \partial^\alpha u\|_{G_0(\Omega)} = \|w_n - w_\infty\|_{G_0(\Omega)} \longrightarrow 0 \tag{5.20}$$

as $n \to \infty$. Define $\gamma = \gamma(\chi, \sigma, m, r, \mathcal{U}_1, T') \ge 1$ by

$$\gamma = \gamma_{3.1,0}(\eta(\chi), R(\chi, \sigma, m, r, \mathcal{U}_1), T'),$$

where $\gamma_{3.1,0}$ is the corresponding constant from Theorem 3.1. The estimate from this theorem applied to (5.17) then yields

$$\begin{aligned} &\|z_n\|^2_{G_0(\Omega)} \le e^{2\gamma T}\|z_n\|^2_{G_{0,\gamma}(\Omega)} \\ &\le C_0 e^{2\gamma T'}\Big(\|w_{0,n} - w_{0,\infty}\|^2_{L^2(\mathbb{R}^3_+)} + \|\partial^\alpha g_n - \partial^\alpha g_\infty\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|f_{\alpha,n} - f_{\alpha,\infty}\|^2_{L^2_\gamma(\Omega)}\Big) \end{aligned} \tag{5.21}$$

where $C_0(\chi, \sigma, m, r, \mathcal{U}_1, T')$ is the maximum of the constants C_0 and $C_{0,0}$ appearing in Theorem 3.1. Because of (5.7), Lemma 2.4 provides a constant $C_{2.4} = C_{2.4}(\chi, \sigma, m, r, \mathcal{U}_1)$ such that

$$\begin{aligned} &\|w_{0,n} - w_{0,\infty}\|_{L^2(\mathbb{R}^3_+)} \\ &\quad = \|\partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n}) - \partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f, u_0)\|_{L^2(\mathbb{R}^3_+)} \end{aligned}$$

$$\leq C_{2.4}\Big(\sum_{j=0}^{m-1}\|\partial_t^j f_n(0)-\partial_t^j f(0)\|_{H^{m-j-1}(\mathbb{R}^3_+)}+\|u_{0,n}-u_0\|_{H^m(\mathbb{R}^3_+)}\Big)$$

for all $n\in\mathbb{N}$. Inserting this estimate together with (5.13) into (5.21), we derive

$$\begin{aligned}\|z_n\|^2_{G_0(\Omega)}\leq C_{5.22}\Big(&\sum_{j=0}^{m-1}\|\partial_t^j f_n(0)-\partial_t^j f(0)\|^2_{H^{m-j-1}(\mathbb{R}^3_+)}+\|u_{0,n}-u_0\|^2_{H^m(\mathbb{R}^3_+)}\\&+\|g_n-g\|^2_{E_m(J\times\partial\mathbb{R}^3_+)}+\|f_n-f\|^2_{H^m(\Omega)}+\|u_n-u\|^2_{G_{m-1}(\Omega)}\Big)\\&+C_{5.22}\int_0^T h_n^2(s)ds+C_{5.22}\int_0^T\sum_{\tilde\alpha\in\mathbb{N}_0^4,|\tilde\alpha|=m}\|\partial^{\tilde\alpha}u_n(s)-\partial^{\tilde\alpha}u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds,\end{aligned}$$

for all $n\in\mathbb{N}$, where we introduce a constant $C_{5.22}=C_{5.22}(\chi,\sigma,m,r,\mathcal{U}_1,T')$. We write a_n' for the first part of the above right-hand side. It follows

$$\|z_n\|^2_{G_0(\Omega)}\leq a_n'+C_{5.22}\int_0^T\sum_{\tilde\alpha\in\mathbb{N}_0^4,|\tilde\alpha|=m}\|\partial^{\tilde\alpha}u_n(s)-\partial^{\tilde\alpha}u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds,\tag{5.22}$$

for all $n\in\mathbb{N}$. Observe that a_n' converges to 0 as $n\to\infty$ by our assumptions, (5.5), and (5.12). Formula (5.18) and inequality (5.22) imply that

$$\begin{aligned}&\|\partial^\alpha u_n-\partial^\alpha u\|^2_{G_0(\Omega)}\\&\quad=\|w_n+z_n-\partial^\alpha u\|^2_{G_0(\Omega)}\leq 2\|w_n-\partial^\alpha u\|^2_{G_0(\Omega)}+2\|z_n\|^2_{G_0(\Omega)}\\&\quad\leq 2\|w_n-\partial^\alpha u\|^2_{G_0(\Omega)}+2a_n'+2C_{5.22}\int_0^T\sum_{\tilde\alpha\in\mathbb{N}_0^4,|\tilde\alpha|=m}\|\partial^{\tilde\alpha}u_n(s)-\partial^{\tilde\alpha}u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds\\&\quad=a_{\alpha,n}+C_{5.23}\int_0^T\sum_{\tilde\alpha\in\mathbb{N}_0^4,|\tilde\alpha|=m}\|\partial^{\tilde\alpha}u_n(s)-\partial^{\tilde\alpha}u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds,\end{aligned}\tag{5.23}$$

for all $n\in\mathbb{N}$. Here we set $C_{5.23}=C_{5.23}(\chi,\sigma,m,r,\mathcal{U}_1,T')$ and note that

$$a_{\alpha,n}:=2\|w_n-\partial^\alpha u\|^2_{G_0(\Omega)}+2a_n'\longrightarrow 0$$

as $n\to\infty$ by (5.20).

III) We claim that for all multiindices $\alpha\in\mathbb{N}_0^4$ with $|\alpha|=m$ there is a sequence $(a_{\alpha,n})_n$ and a constant $C_\alpha=C_\alpha(\chi,\sigma,m,r,\mathcal{U}_1,T')$ such that

$$\|\partial^\alpha u_n-\partial^\alpha u\|^2_{G_0(\Omega)}\leq a_{\alpha,n}+C_\alpha\int_0^T\sum_{\tilde\alpha\in\mathbb{N}_0^4,|\tilde\alpha|=m}\|\partial^{\tilde\alpha}u_n(s)-\partial^{\tilde\alpha}u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds\tag{5.24}$$

for all $n \in \mathbb{N}$ and

$$a_{\alpha,n} \longrightarrow 0 \tag{5.25}$$

as $n \to \infty$. One proves this assertion by induction with respect to α_3. Observe that step II) shows the claim for the case $\alpha_3 = 0$. In the induction step one assumes that there is an index $l \in \{1, \dots, m\}$ such that the assertion is true for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_3 = l - 1$. Take $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_3 = l$. We set $\alpha' = \alpha - e_3$.

Unfortunately we cannot directly apply Lemma 4.3 here, since it was derived for a fixed differential operator. If we apply only one such operator to a difference of solutions we experience the typical loss of derivatives. Therefore, one has to repeat the key step of the proof of Lemma 4.3 for the operators L_n and the difference $\partial^{\alpha'} u_n - \partial^{\alpha'} u$. In this calculation we use results from step II) such as estimate (5.15). Since in this very lengthy reasoning essentially the same arguments are employed as in [35], we decided to omit these calculations here. The details can be found in steps III) to V) of the proof of Lemma 7.22 in [34].

We define a_n and $C_m = C_m(\chi, \sigma, r, \mathcal{U}_1, T')$ by

$$a_n = \sum_{\tilde{\alpha} \in \mathbb{N}_0^4, |\tilde{\alpha}| = m} a_{\tilde{\alpha},n}, \qquad C_m = \sum_{\tilde{\alpha} \in \mathbb{N}_0^4, |\tilde{\alpha}| = m} C_{\tilde{\alpha}},$$

for all $n \in \mathbb{N}$. Summing (5.24) over all multiindices $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$, we then derive

$$\begin{aligned}\sum_{\tilde{\alpha} \in \mathbb{N}_0^4, |\tilde{\alpha}| = m} \|\partial^{\tilde{\alpha}} u_n(T) - \partial^{\tilde{\alpha}} u(T)\|^2_{L^2(\mathbb{R}^3_+)} &\le \sum_{\tilde{\alpha} \in \mathbb{N}_0^4, |\tilde{\alpha}| = m} \|\partial^{\tilde{\alpha}} u_n - \partial^{\tilde{\alpha}} u\|^2_{G_0(\Omega)} \\ &\le a_n + C_m \int_0^T \sum_{\tilde{\alpha} \in \mathbb{N}_0^4, |\tilde{\alpha}| = m} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|^2_{L^2(\mathbb{R}^3_+)} ds\end{aligned}$$

for all $n \in \mathbb{N}$. Since $T \in (0, T']$ was arbitrary, Gronwall's lemma shows that

$$\sum_{\tilde{\alpha} \in \mathbb{N}_0^4, |\tilde{\alpha}| = m} \|\partial^{\tilde{\alpha}} u_n(T) - \partial^{\tilde{\alpha}} u(T)\|^2_{L^2(\mathbb{R}^3_+)} \le a_n e^{C_m T}$$

for all $T \in [0, T']$ and $n \in \mathbb{N}$. As $(a_n)_n$ converges to 0 due to (5.25), we finally arrive at

$$\sum_{\tilde{\alpha} \in \mathbb{N}_0^4, |\tilde{\alpha}| = m} \|\partial^{\tilde{\alpha}} u_n - \partial^{\tilde{\alpha}} u\|^2_{G_0(J' \times \mathbb{R}^3_+)} \le a_n e^{C_m T'} \longrightarrow 0$$

as $n \to \infty$. Since $\|u_n - u\|_{G_{m-1}(J' \times \mathbb{R}^3_+)}$ tends to zero as $n \to \infty$ by assumption, we conclude that $(u_n)_n$ converges to u in $G_m(J' \times \mathbb{R}^3_+)$. $\square$

Finally, we can prove the full local wellposedness theorem. In the following we will write $B_M(x, r)$ for the ball of radius r around a point x from a metric space M. For times $t_0 < T$ we further set

$$M_{\chi,\sigma,m}(t_0,T)=\{(\tilde f,\tilde g,\tilde u_0)\in H^m((t_0,T)\times G)\times E_m((t_0,T)\times\partial G)\times H^m(G):$$
$$(\chi,\sigma,t_0,B,\tilde f,\tilde g,\tilde u_0)\text{ is compatible of order }m\},$$
$$d((\tilde f_1,\tilde g_1,\tilde u_{0,1}),(\tilde f_2,\tilde g_2,\tilde u_{0,2}))$$
$$=\max\{\|\tilde f_1-\tilde f_2\|_{H^m((t_0,T)\times G)},\|\tilde g_1-\tilde g_2\|_{E_m((t_0,T)\times\partial G)},\|\tilde u_{0,1}-\tilde u_{0,2}\|_{H^m(G)}\}.$$

Theorem 5.3. *Let $m\in\mathbb{N}$ with $m\geq 3$ and fix $t_0\in\mathbb{R}$. Take functions $\chi\in\mathcal{ML}^{m,6,c}_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma\in\mathcal{ML}^{m,6,c}(G,\mathcal{U})$ and set*

$$B(x)=\begin{pmatrix}0&\nu_3(x)&-\nu_2(x)&0&0&0\\-\nu_3(x)&0&\nu_1(x)&0&0&0\\\nu_2(x)&-\nu_1(x)&0&0&0&0\end{pmatrix}$$

for all $x\in\partial G$, where ν denotes the unit outer normal vector of ∂G. Choose data $u_0\in H^m(G)$, $g\in E_m((-T,T)\times\partial G)$, and $f\in H^m((-T,T)\times G)$ for all $T>0$ such that $\overline{\operatorname{im}u_0}\subseteq\mathcal{U}$ and the tuple $(\chi,\sigma,t_0,B,f,g,u_0)$ fulfills the compatibility conditions (2.10) of order m. For the maximal existence times from Definition 3.5 we then have

$$T_+=T_+(m,t_0,f,g,u_0)=T_+(k,t_0,f,g,u_0),$$
$$T_-=T_-(m,t_0,f,g,u_0)=T_-(k,t_0,f,g,u_0)$$

for all $k\in\{3,\dots,m\}$. The following assertions are true.

(1) *There exists a unique maximal solution u of (1.6) which belongs to the function space $\bigcap_{j=0}^m C^j((T_-,T_+),H^{m-j}(G))$.*
(2) *If $T_+<\infty$, then*
   (a) *the solution u leaves every compact subset of $\mathcal{U}$, or*
   (b) $\limsup_{t\nearrow T_+}\|\nabla u(t)\|_{L^\infty(G)}=\infty$.
   The analogous result holds for T_-.
(3) *Let $T'\in(t_0,T_+)$. Then there is a number $\delta>0$ such that for all data $\tilde f\in H^m((t_0,T_+)\times G)$, $\tilde g\in E_m((t_0,T_+)\times\partial G)$, and $\tilde u_0\in H^m(G)$ fulfilling*

$$\|\tilde f-f\|_{H^m((t_0,T_+)\times G)}<\delta,\quad\|\tilde g-g\|_{E_m((t_0,T_+)\times\partial G)}<\delta,\quad\|\tilde u_0-u_0\|_{H^m(G)}<\delta$$

and the compatibility conditions (2.10) of order m, the maximal existence time satisfies $T_+(m,t_0,\tilde f,\tilde g,\tilde u_0)>T'$. We denote by $u(\cdot;\tilde f,\tilde g,\tilde u_0)$ the corresponding maximal solution of (1.6). The flow map

$$\Psi\colon B_{M_{\chi,\sigma,m}(t_0,T_+)}((f,g,u_0),\delta)\to G_m((t_0,T')\times G),$$
$$(\tilde f,\tilde g,\tilde u_0)\mapsto u(\cdot;\tilde f,\tilde g,\tilde u_0),$$

is continuous. Moreover, there is a constant $C=C(\chi,\sigma,m,r,T_+-t_0,\kappa_0)$ such that

$$\begin{aligned}
&\|\Psi(\tilde f_1,\tilde g_1,\tilde u_{0,1})-\Psi(\tilde f_2,\tilde g_2,\tilde u_{0,2})\|_{G_{m-1}((t_0,T')\times G)}\\
&\quad\le C\sum_{j=0}^{m-1}\|\partial_t^j\tilde f_1(t_0)-\partial_t^j\tilde f_2(t_0)\|_{H^{m-j-1}(G)}+C\|\tilde g_1-\tilde g_2\|_{E_{m-1}((t_0,T')\times\partial G)}\\
&\qquad+C\|\tilde u_{0,1}-\tilde u_{0,2}\|_{H^m(G)}+C\|\tilde f_1-\tilde f_2\|_{H^{m-1}((t_0,T')\times G)}
\end{aligned}\tag{5.26}$$

for all $(\tilde f_1,\tilde g_1,\tilde u_{0,1}),(\tilde f_2,\tilde g_2,\tilde u_{0,2})\in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f,g,u_0),\delta)$, *where* $\kappa_0=\operatorname{dist}(\operatorname{im}u_0,\partial\mathcal{U})$. *The analogous result is true for* T_-.

Proof. We show the assertion for T_+, the proofs for T_- are analogous. Let $k\in\{3,\dots,m-1\}$. We have $T_+=T_+(m,t_0,f,g,u_0)\le T_+(k,t_0,f,g,u_0)$ by definition. Assume now that $T_+<T_+(k,t_0,f,g,u_0)$. Then $T_+<\infty$ and the maximal $H^m(G)$-solution u of (1.6), which exists on (t_0,T_+), can be extended to a $H^k(G)$-solution on $(t_0,T_+(k,t_0,f,g,u_0))$ by the definition of the maximal existence time and Lemma 3.2. It follows that

$$\sup_{t\in(t_0,T_+)}\|u(t)\|_{H^k(G)}<\infty\quad\text{and}\quad\liminf_{t\nearrow T_+}\operatorname{dist}(\{u(t,x):x\in G\},\partial\mathcal{U})>0.\tag{5.27}$$

Sobolev's embedding thus implies

$$\omega_0:=\sup_{t\in(t_0,T_+)}\|u(t)\|_{W^{1,\infty}(G)}<\infty.$$

Pick a radius $\rho>0$ such that

$$\sum_{j=0}^{m-1}\|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)}+\|g\|_{E_m((t_0,T_+)\times\partial G)}+\|u_0\|_{H^m(G)}+\|f\|_{H^m((t_0,T_+)\times G)}<\rho.$$

Due to (5.27) and the boundedness of u there is a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ such that $\operatorname{im}u(t)\subseteq\mathcal{U}_1$ for all $t\in[t_0,T_+]$. Proposition 4.4 then yields the bound

$$\sup_{t\in(t_0,T_+)}\|u(t)\|^2_{H^m(G)}\le C_{4.4}(\chi,\sigma,m,\rho,\omega_0,\mathcal{U}_1,T_+-t_0)\cdot C\rho^2.$$

But by Lemma 4.1 and (5.27) we have $\lim_{t\nearrow T_+}\|u(t)\|_{H^m(G)}=\infty$ and thus a contradiction. We conclude that $T_+(k,t_0,f,g,u_0)=T_+$.

Assertion (1) is just Proposition 3.6. To show (2), assume that $T_+<\infty$ and that Properties (2)a and (2)b do not hold. We then have

$$\omega_0:=\sup_{t\in(t_0,T_+)}\|u(t)\|_{W^{1,\infty}(G)}<\infty$$

and there is a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ such that $\operatorname{im}u(t)\subseteq\mathcal{U}_1$ for all $t\in[t_0,T_+]$. We apply Proposition 4.4 with $T^*=T_+$ again to deduce

$$\|u(t)\|^2_{H^m(G)}\le C_{4.4}(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,T_+-t_0)\cdot Cr^2$$

for all $t \in (t_0, T_+)$ and thus $\sup_{t\in(t_0,T_+)} \|u(t)\|_{H^m(G)} < \infty$. Lemma 4.1 however shows that $\lim_{t\nearrow T_+} \|u(t)\|_{H^3(G)} = \infty$. We thus obtain a contradiction.

(3) Let $T' \in (t_0, T_+)$. Without loss of generality we assume that χ and σ satisfy (2.11), cf. Remark 2.5. The difficulty in assertion (3) is to make sure that the solutions to the data in the neighborhood we have to construct exist at least till T'. To that purpose we use an iterative scheme that allows us to apply Theorem 3.3 with the same minimal time step size in each iteration.

Recall that by Sobolev's embedding there is a constant depending only on the length of the interval $[t_0, T_+)$ such that

$$\|\tilde{f}\|_{G_{m-1}((t_0,T_+)\times G)} \le C_S \|\tilde{f}\|_{H^m((t_0,T_+)\times G)} \tag{5.28}$$

for all $\tilde{f} \in H^m((t_0, T_+) \times G)$. Fix a time $T^* \in (T', T_+)$. We pick two radii $0 < r_0 < r < \infty$ such that

$$\begin{aligned} &\|u_0\|_{H^m(G)} + \|f\|_{G_{m-1}((t_0,T_+)\times G)} + \|f\|_{H^m((t_0,T_+)\times G)} < r_0, \\ &C_S m r_0 < r, \quad \text{and} \quad \|u\|_{G_m((t_0,T^*)\times G)} < r. \end{aligned} \tag{5.29}$$

Moreover, there is a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ such that $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T^*]$. Lemma 2.1 thus provides a number $\tilde{r} = \tilde{r}(\chi, \sigma, m, r, \mathcal{U}_1)$ with

$$\begin{aligned} &\|\chi(u)\|_{F_m((t_0,T^*)\times G)} + \|\sigma(u)\|_{F_m((t_0,T^*)\times G)} \le \tilde{r}, \\ &\max\{\|\chi(u)(t_0)\|_{F^0_{m-1}(G)}, \max_{1\le j\le m-1} \|\partial_t^j \chi(u)(t_0)\|_{H^{m-j-1}(G)}\} \le \tilde{r}, \\ &\max\{\|\sigma(u)(t_0)\|_{F^0_{m-1}(G)}, \max_{1\le j\le m-1} \|\partial_t^j \sigma(u)(t_0)\|_{H^{m-j-1}(G)}\} \le \tilde{r}. \end{aligned} \tag{5.30}$$

I) Let $t' \in (t_0, T^*)$ and $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in M_{\chi,\sigma,m}(t_0, T_+)$. Assume that the solution $\tilde{u}$ of (1.6) with data $\tilde{f}, \tilde{g}, \tilde{u}_0$ exists on $[t_0, t']$ and thus belongs to $G_m((t_0, t') \times G)$. Pick a radius R' and a compact subset $\tilde{\mathcal{U}}_1$ of $\mathcal{U}$ such that $\|\tilde{u}\|_{G_m((t_0,t')\times G)} \le R'$ and $\operatorname{im} u(t), \operatorname{im} \tilde{u}(t) \subseteq \tilde{\mathcal{U}}_1$ for all $t \in [t_0, t']$. Set $\tilde{T} = T_+ - t_0$. We first show the inequality

$$\begin{aligned} \|\tilde{u} - u\|^2_{G_{m-1}((t_0,t')\times G)} &\le C\|\tilde{f} - f\|^2_{H^{m-1}((t_0,t')\times G)} + C\|\tilde{g} - g\|_{E_{m-1}((t_0,t')\times\partial G)} \\ &\quad + C\Big(\sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}(t_0) - \partial_t^j f(t_0)\|^2_{H^{m-j-1}(G)} + \|\tilde{u}_0 - u_0\|^2_{H^m(G)}\Big), \end{aligned} \tag{5.31}$$

for a constant $C = C(\chi, \sigma, m, r, R', \tilde{\mathcal{U}}_1, \tilde{T})$. To this aim, we apply the linear differential operator $L = L(\chi(u), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(u))$ to $\tilde{u} - u$. We obtain

$$L(\tilde{u} - u) = \tilde{f} + (\chi(u) - \chi(\tilde{u}))\partial_t \tilde{u} + (\sigma(u) - \sigma(\tilde{u}))\tilde{u} - f =: F.$$

Lemma 2.1 and [35, Lemma 2.1] show that F is an element of $H^{m-1}((t_0, t') \times G)$. Set

$$\gamma_0 = \gamma_0(\chi, \sigma, m, r, \mathcal{U}_1, \tilde{T}) = \gamma_{3.1;0}(\eta(\chi), \tilde{r}, \tilde{T}) \ge 1,$$

where $\chi \geq \eta(\chi) > 0$ and $\gamma_{3.1;0}$ is the corresponding constant from Theorem 3.1. This theorem then yields

$$\begin{aligned}
&\|\tilde{u} - u\|^2_{G_{m-1,\gamma}((t_0,t')\times G)} \\
&\quad \leq (C_{3.1;m,0} + \tilde{T} C_{3.1;m}) e^{m C_{3.1;1} \tilde{T}} \Big(\sum_{j=0}^{m-2} \|\partial_t^j F(t_0)\|^2_{H^{m-2-j}(G)} + \|\tilde{u}_0 - u_0\|^2_{H^{m-1}(G)} \\
&\qquad + \|\tilde{g} - g\|^2_{E_{m-1,\gamma}((t_0,t')\times\partial G)} \Big) + \frac{C_{3.1;m}}{\gamma} \|F\|^2_{H^{m-1}_\gamma((t_0,t')\times G)}
\end{aligned} \tag{5.32}$$

for all $\gamma \geq \gamma_0$, where $C_{3.1;m,0} = C_{3.1;m,0}(\eta(\chi), \tilde{r})$, $C_{3.1;m} = C_{3.1;m}(\eta(\chi), \tilde{r}, \tilde{T})$, and $C_{3.1;1} = C_{3.1;1}(\eta(\chi), \tilde{r}, \tilde{T})$ are the corresponding constants from Theorem 3.1. We next apply part (2) of Lemma 2.1 from [35] and then Corollary 2.2 to obtain

$$\begin{aligned}
&\|F\|^2_{H^{m-1}_\gamma((t_0,t')\times G)} \\
&\quad \leq C\|\tilde{f} - f\|^2_{H^{m-1}_\gamma((t_0,t')\times G)} + C\tilde{T}\|\chi(\tilde{u}) - \chi(u)\|^2_{G_{m-1,\gamma}((t_0,t')\times G)} \|\partial_t \tilde{u}\|^2_{G_{m-1}((t_0,t')\times G)} \\
&\qquad + C\tilde{T}\|\sigma(\tilde{u}) - \sigma(u)\|^2_{G_{m-1,\gamma}((t_0,t')\times G)} \|\tilde{u}\|^2_{G_{m-1}((t_0,t')\times G)} \\
&\quad \leq C\|\tilde{f} - f\|^2_{H^{m-1}_\gamma((t_0,t')\times G)} + C(\chi, \sigma, m, r, R', \tilde{\mathcal{U}}_1, \tilde{T}) \|\tilde{u} - u\|^2_{G_{m-1,\gamma}((t_0,t')\times G)}.
\end{aligned} \tag{5.33}$$

Let $j \in \{0, \dots, m-2\}$. To treat $\partial_t^j F(t_0)$, we employ Lemma 2.1 and the definition of the M_k^l in (2.8). The same arguments as in the proof of Lemma 2.4 then show that

$$\begin{aligned}
&\|\partial_t^j F(t_0)\|_{H^{m-2-j}(G)} \\
&\quad \leq \|\partial_t^j F(t_0)\|_{H^{m-1-j}(G)} \\
&\quad \leq C(\chi, \sigma, m, r, R', \tilde{\mathcal{U}}_1) \Big(\sum_{l=0}^{m-1} \|\partial_t^l f(t_0) - \partial_t^l \tilde{f}(t_0)\|_{H^{m-l-1}(G)} + \|u_0 - \tilde{u}_0\|_{H^m(G)} \Big).
\end{aligned} \tag{5.34}$$

We obtain a constant $C_{5.35} = C_{5.35}(\chi, \sigma, m, r, R', \tilde{\mathcal{U}}_1, \tilde{T})$ and the bound

$$\begin{aligned}
&\|\tilde{u} - u\|^2_{G_{m-1,\gamma}((t_0,t')\times G)} \\
&\quad \leq C_{5.35} \Big(\frac{1}{\gamma} \|\tilde{u} - u\|^2_{G_{m-1,\gamma}((t_0,t')\times G)} + \sum_{l=0}^{m-1} \|\partial_t^l \tilde{f}(t_0) - \partial_t^l f(t_0)\|^2_{H^{m-l-1}(G)} \\
&\qquad + \|\tilde{g} - g\|^2_{E_{m-1,\gamma}((t_0,t')\times\partial G)} + \|\tilde{u}_0 - u_0\|^2_{H^m(G)} + \|\tilde{f} - f\|^2_{H^{m-1}_\gamma((t_0,t')\times G)} \Big)
\end{aligned} \tag{5.35}$$

for all $\gamma \geq \gamma_0$ by inserting (5.33) and (5.34) into (5.32). We next fix a number $\gamma = \gamma(\chi, \sigma, m, r, R', \tilde{\mathcal{U}}_1, \tilde{T})$ with $\gamma \geq \gamma_0$ and $C_{5.35}\frac{1}{\gamma} \leq \frac{1}{2}$ to infer that (5.31) is true.

II) Recall that $\mathcal{U}_1$ is a compact subset of $\mathcal{U}$ such that $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T^*]$. Pick a number κ such that

$$2\kappa < \operatorname{dist}(\mathcal{U}_1, \partial\mathcal{U}). \tag{5.36}$$

Take the time step $\tau = \tau(\chi, \sigma, m, \tilde{T}, 4r, \kappa)$ from Theorem 3.3. Choose an index $N \in \mathbb{N}$ with

$$t_0 + (N-1)\tau < T' \le t_0 + N\tau.$$

We set $t_k = t_0 + k\tau$ for $k \in \{1, \dots, N-1\}$. If $t_0 + N\tau < T^*$, we put $t_N = t_0 + N\tau$; else we take any t_N from (T', T^*).

Let $0 < \delta_0 < r_0$ be so small that $C_{\mathrm{Sob}}\delta_0 < \kappa$, where C_{Sob} is the norm of the embedding from $H^2(G) \hookrightarrow L^\infty(G)$. As in (3.4) and (3.6) we define the compact sets

$$\mathcal{U}_\kappa = \{y \in \mathcal{U}\colon \operatorname{dist}(y, \partial\mathcal{U}) \ge \kappa\} \cap B_{2C_{\mathrm{Sob}}4r}(0) \quad \text{and} \quad \tilde{\mathcal{U}}_\kappa = \mathcal{U}_\kappa + \overline{B}(0, \kappa/2).$$

Take $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_0)$. Using the choice of r and r_0 in (5.29), we deduce that

$$\begin{aligned}
&\|\tilde{u}_0\|_{H^m(G)} \le \|u_0\|_{H^m(G)} + \|\tilde{u}_0 - u_0\|_{H^m(G)} \le r_0 + \delta_0 < 2r_0 < 2r,\\
&\|\tilde{g}\|_{E_m((t_0,T')\times\partial G)} \le r_0 + \delta_0 < 2r, \quad \|\tilde{f}\|_{H^m((t_0,T')\times G)} \le r_0 + \delta_0 < 2r,
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
\sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}(t_0)\|_{H^{m-1-j}(G)} &\le m\|\tilde{f}\|_{G_{m-1}((t_0,T')\times G)} \le C_S m\|\tilde{f}\|_{H^m((t_0,T')\times G)}\\
&< 2C_S m r_0 < 2r.
\end{aligned} \tag{5.38}$$

Moreover, $\|\tilde{u}_0 - u_0\|_{L^\infty(G)} \le C_{\mathrm{Sob}}\delta_0 < \kappa$ so that $\operatorname{im}\tilde{u}_0$ is contained in $\mathcal{U}_\kappa$. So Theorem 3.3 shows that the solution $\tilde{u}$ of (1.6) with data $\tilde{f}$, $\tilde{g}$, and $\tilde{u}_0$ at t_0 exists on $[t_0, t_1]$ and belongs to $G_m((t_0, t_1) \times G)$. Moreover, the proof of this theorem yields a radius $R = R_{3.3}(\chi, \sigma, m, 4r, \kappa) > 4r$, see (3.10), such that $\|\tilde{u}\|_{G_m((t_0,t_1)\times G)} \le R$. This proof also shows that $\operatorname{im}\tilde{u}(t) \subseteq \tilde{\mathcal{U}}_\kappa$ for all $t \in [t_0, t_1]$, cf. (3.6). We conclude that Ψ maps $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_0)$ into $B_{G_m((t_0,t_1)\times G)}(0, R)$. We further deduce from (5.31) that there is a constant

$$C_{5.39} = C_{5.39}(\chi, \sigma, m, r, \tilde{T}, \kappa) = C_{5.31}(\chi, \sigma, m, r, R(\chi, \sigma, m, r, \kappa), \tilde{U}_\kappa, \tilde{T})$$

such that

$$\begin{aligned}
&\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0) - \Psi(f, g, u_0)\|^2_{G_{m-1}((t_0,t_1)\times G)}\\
&\quad \le C_{5.39}\|\tilde{f} - f\|^2_{H^{m-1}((t_0,t_1)\times G)} + C_{5.39}\|\tilde{g} - g\|^2_{E_{m-1}((t_0,t')\times\partial G)}\\
&\qquad + C_{5.39}\Big(\sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}(t_0) - \partial_t^j f(t_0)\|^2_{H^{m-j-1}(G)} + \|\tilde{u}_0 - u_0\|^2_{H^m(G)}\Big)
\end{aligned} \tag{5.39}$$

for all $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_0)$.

Next take a sequence $(f_n, g_n, u_{0,n})_n$ in $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_0)$ which converges to (f, g, u_0) in this space. Since

$$\sum_{j=0}^{m-1} \|\partial_t^j f_n(t_0) - \partial_t^j f(t_0)\|^2_{H^{m-j-1}(G)} \le mC_S \|f_n - f\|_{H^m((t_0,T_+)\times G)} \longrightarrow 0 \tag{5.40}$$

as $n \to \infty$, estimate (5.39) yields the limit

$$\|\Psi(f_n, g_n, u_{0,n}) - \Psi(f, g, u_0)\|_{G_{m-1}((t_0,t_1)\times G)} \longrightarrow 0$$

as $n \to \infty$. Lemma 5.2 thus shows that $(\Psi(f_n, g_n, u_{0,n}))_n$ converges to $\Psi(f, g, u_0)$ in $G_m((t_0, t_1) \times G)$. We conclude that the map

$$\Psi\colon B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_0) \to G_m((t_0, t_1) \times G)$$

is continuous at (f, g, u_0). Using also (5.29) and (5.36), we find a number $\delta_1 \in (0, \delta_0]$ such that for all data $(\tilde f, \tilde g, \tilde u_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_1)$ the function $\Psi(\tilde f, \tilde g, \tilde u_0)$ exists on $[t_0, t_1]$ and satisfies (5.39) and

$$\begin{aligned}
&\|\Psi(\tilde f, \tilde g, \tilde u_0)\|_{G_m((t_0,t_1)\times G)} \\
&\quad \le \|\Psi(\tilde f, \tilde g, \tilde u_0) - \Psi(f, g, u_0)\|_{G_m((t_0,t_1)\times G)} + \|\Psi(f, g, u_0)\|_{G_m((t_0,t_1)\times G)} < 2r, \\
&\operatorname{dist}(\operatorname{im}\Psi(\tilde f, \tilde g, \tilde u_0)(t), \partial\mathcal{U}) > \kappa,
\end{aligned}$$

for all $t \in [t_0, t_1]$.

Now assume that there is an index $j \in \{1, \dots, N-1\}$ and a number $\delta_j > 0$ such that $\Psi(\tilde f, \tilde g, \tilde u_0)$ exists on $[t_0, t_j]$ and satisfies

$$\|\Psi(\tilde f, \tilde g, \tilde u_0)\|_{G_m((t_0,t_j)\times G)} < 2r \quad \text{and} \quad \operatorname{dist}(\operatorname{im}\Psi(\tilde f, \tilde g, \tilde u_0)(t), \partial\mathcal{U}) > \kappa$$

for all $t \in [t_0, t_j]$ and $(\tilde f, \tilde g, \tilde u_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_j)$.

Fix such a tuple $(\tilde f, \tilde g, \tilde u_0)$. Then the tuple $(\chi, \sigma, t_j, B, \tilde f, \tilde g, \Psi(\tilde f, \tilde g, \tilde u_0)(t_j))$ fulfills the nonlinear compatibility conditions (2.10) of order m by (2.9) and

$$\begin{aligned}
&\|\Psi(\tilde f, \tilde g, \tilde u_0)(t_j)\|_{H^m(G)} \le \|\Psi(\tilde f, \tilde g, \tilde u_0)\|_{G_m((t_0,t_j)\times G)} < 2r, \\
&\operatorname{dist}(\operatorname{im}\Psi(\tilde f, \tilde g, \tilde u_0)(t_j), \partial\mathcal{U}) > \kappa.
\end{aligned}$$

In view of (5.37) and (5.38), Theorem 3.3 shows that the problem (1.6) with inhomogeneity $\tilde f$, boundary value $\tilde g$, and initial value $\Psi(\tilde f, \tilde g, \tilde u_0)(t_j)$ at initial time t_j has a unique solution $\tilde u^j$ on $[t_j, t_{j+1}]$, which is bounded by R in $G_m((t_j, t_{j+1}) \times G)$ and whose image is contained in $\tilde U_\kappa$. Concatenating $\Psi(\tilde f, \tilde g, \tilde u_0)$ and $\tilde u^j$, we obtain a solution of (1.6) with data $\tilde f$, $\tilde g$, and $\tilde u_0$ at initial time t_0, cf. Remark 3.4. This means that $\Psi(\tilde f, \tilde g, \tilde u_0)$ exists on $[t_0, t_{j+1}]$. Uniqueness of solutions of (1.6), i.e. Lemma 3.2, further yields $\Psi(\tilde f, \tilde g, \tilde u_0)_{|[t_j,t_{j+1}]} = \tilde u^j$ so that

$$\begin{aligned}
\|\Psi(\tilde f, \tilde g, \tilde u_0)\|_{G_m((t_0,t_{j+1})\times G)} &\le \max\{\|\Psi(\tilde f, \tilde g, \tilde u_0)\|_{G_m((t_0,t_j)\times G)}, \|\tilde u^j\|_{G_m((t_j,t_{j+1})\times G)}\} \\
&\le \max\{2r, R\} \le R.
\end{aligned}$$

As for the interval $[t_0, t_1]$, we obtain a number $\delta_{j+1} \in (0, \delta_j]$ such that

$$\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)\|_{G_m((t_0,t_{j+1})\times G)} < 2r \quad \text{and} \quad \operatorname{dist}(\operatorname{im}\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)(t), \partial\mathcal{U}) > \kappa$$

for all $t \in [t_0, t_{j+1}]$ and $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_{j+1})$.

By induction, the above property holds for $j + 1 = N$, so that

$$T_+(m, t_0, \tilde{f}, \tilde{g}, \tilde{u}_0) > t_N \geq T'$$

for all $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N)$.

Next fix two tuples $(\tilde{f}_1, \tilde{g}_1, \tilde{u}_{0,1})$ and $(\tilde{f}_2, \tilde{g}_2, \tilde{u}_{0,2})$ in $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N)$. Replacing u by $\Psi(\tilde{f}_2, \tilde{g}_2, \tilde{u}_{0,2})$ in step I), we deduce from (5.31) that

$$\begin{aligned}
&\|\Psi(\tilde{f}_1, \tilde{g}_1, \tilde{u}_{0,1}) - \Psi(\tilde{f}_2, \tilde{g}_2, \tilde{u}_{0,2})\|^2_{G_{m-1}((t_0,T')\times G)} \\
&\quad \leq C\|\tilde{f}_1 - \tilde{f}_2\|^2_{H^{m-1}((t_0,T')\times G)} + C\|\tilde{g}_1 - \tilde{g}_2\|^2_{E_{m-1}((t_0,T')\times G)} \\
&\qquad + C\Big(\sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}_1(t_0) - \partial_t^j \tilde{f}_2(t_0)\|^2_{H^{m-j-1}(G)} + \|\tilde{u}_{0,1} - \tilde{u}_{0,2}\|^2_{H^m(G)}\Big),
\end{aligned} \tag{5.41}$$

where $C = C(\chi, \sigma, m, r, \tilde{T}, \kappa) = C_{5.31}(\chi, \sigma, m, 2r, 2r, \mathcal{U}_\kappa, \tilde{T})$ and the constant $C_{5.31}$ from (5.31). This estimate implies (5.26).

Finally, we take a sequence $(\tilde{f}_n, \tilde{g}_n, \tilde{u}_{0,n})_n$ in $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N)$ which converges to $(\tilde{f}_1, \tilde{g}_1, \tilde{u}_{0,1})$ in $M_{\chi,\sigma,m}(t_0, T_+)$. Employing (5.28), we deduce from (5.41) that $\Psi(\tilde{f}_n, \tilde{g}_n, \tilde{u}_{0,n})$ tends to $\Psi(\tilde{f}_1, \tilde{g}_1, \tilde{u}_{0,1})$ in the space $G_{m-1}((t_0, T') \times G)$ as $n \to \infty$. Lemma 5.2 therefore implies that

$$\|\Psi(\tilde{f}_n, \tilde{g}_n, \tilde{u}_{0,n}) - \Psi(\tilde{f}_1, \tilde{g}_1, \tilde{u}_{0,1})\|_{G_m((t_0,T')\times G)} \longrightarrow 0$$

as $n \to \infty$. Consequently, the flow map

$$\Psi\colon B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N) \to G_m((t_0, T') \times G)$$

is continuous at $(\tilde{f}_1, \tilde{g}_1, \tilde{u}_{0,1})$ and thus on $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N)$. □

6. Finite propagation speed

We finally prove that solutions of (1.6) have finite propagation speed, i.e., that initial disturbances travel with finite speed. Several techniques to establish this property have been developed in the literature, see e.g. [3,5,13]. While these references work on the full space, finite speed of propagation is proven for an initial boundary value problem in [8], making however several structural assumptions on the problem which are not fulfilled by Maxwell's equations. We thus follow a different approach and show that the technique of weighted energy estimates from [3] is well adaptable to our setting.

We first prove the finite propagation speed property for the corresponding linear problem (3.1). This result can be transferred to domains via localization as in the previous sections, see [34, Chapter 6] for details. We concentrate on the half-space case here since in this case one sees much better how the maximal propagation speed depends on the coefficients. We however note that the coefficients on the half-space arising after localization depend on the charts of ∂G so that the maximal propagation speed of the solution on domains depends on the shape of ∂G. We refer to Theorem 6.1 of [34] for a result on G itself.

In this section we set $J = (0, T)$ for a time $T > 0$.

Theorem 6.1. *Let $\eta > 0$, $A_0 \in F^{\mathrm{c}}_{3,\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{3,\mathrm{coeff}}(\mathbb{R}^3_+)$, $A_3 = A^{\mathrm{co}}_3$, $D \in F^{\mathrm{c}}_3(\Omega)$, and $B = B^{\mathrm{co}}$. We set*

$$C_0 = \frac{1}{\eta} \sum_{j=1}^{3} \|A_j\|_{L^\infty(\Omega)}.$$

Let $R > 0$ and $x_0 \in \overline{\mathbb{R}^3_+}$. We define the backward cone $\mathcal{C}$ by

$$\mathcal{C} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^3 \colon |x - x_0| < R - C_0 t\}.$$

Let $f \in L^2(\Omega)$, $g \in L^2(J, H^{1/2}(\partial\mathbb{R}^3_+))$, and $u_0 \in L^2(\mathbb{R}^3_+)$ satisfy

$$\begin{aligned} f &= 0 \quad \text{on } \mathcal{C} \cap \Omega, \\ g &= 0 \quad \text{on } \mathcal{C} \cap (J \times \partial\mathbb{R}^3_+), \\ u_0 &= 0 \quad \text{on } \mathcal{C}_{t=0} \cap \mathbb{R}^3_+, \end{aligned}$$

where $\mathcal{C}_{t=0} = \{x \in \mathbb{R}^3 \colon (0, x) \in \mathcal{C}\}$. Then the unique solution $u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ of the linear initial boundary value problem (3.1) with inhomogeneity f, boundary value g, and initial value u_0 vanishes on the cone $\mathcal{C}$, i.e.,

$$u(t, x) = 0 \quad \text{for almost all } (t, x) \in \mathcal{C} \cap \Omega.$$

Proof. I) Let $\varepsilon > 0$ and set $K = C_0^{-1}$. We fix a function $\psi \in C^\infty(\mathbb{R}^3)$ with

$$-2\varepsilon + K(R - |x - x_0|) \le \psi(x) \le -\varepsilon + K(R - |x - x_0|) \quad \text{for all } x \in \mathbb{R}^3, \tag{6.1}$$

$$\|\nabla\psi\|_{L^\infty(\mathbb{R}^3)} \le K, \tag{6.2}$$

see step I) of the proof of Theorem 6.1 in [34]. We first assume that f belongs to $H^1(\Omega)$, g to $E_1(J \times \partial\mathbb{R}^3_+)$, and u_0 to $H^1(\mathbb{R}^3_+)$ and that $(t_0, A_0, \dots, A_3, D, B, f, g, u_0)$ fulfills the linear compatibility conditions of first order. Fix $\varepsilon > 0$. We introduce the cone

$$\mathcal{C}_\varepsilon = \{(t, x) \in \mathbb{R} \times \mathbb{R}^3 \colon |x - x_0| < R - C_0 t - C_0 \varepsilon\}.$$

The cone $\mathcal{C}_{t=0,\varepsilon}$ is defined analogously. We set

$$\Phi(t,x) = -t + \psi(x)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^3$. Note that Φ belongs to $C^\infty(\overline{J \times \mathbb{R}^3_+})$. We next want to derive a weighted energy inequality for u. To that purpose, we define

$$u_\tau = e^{\tau\Phi} u, \quad f_\tau = e^{\tau\Phi} f, \quad g_\tau = e^{\tau\Phi(\cdot,0)} g, \quad u_{0,\tau} = e^{\tau\Phi(0,\cdot)} u_0$$

for all $\tau > 0$. Observe that there is a constant $C = C(\tau,\varepsilon)$ such that

$$e^{\tau\Phi(t,x)} + \sum_{j=0}^{3} |\partial_j e^{\tau\Phi(t,x)}| \le C e^{-\tau K|x-x_0|}$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}^3$ and $\tau > 0$. We further observe that u belongs to $G_1(\Omega)$ by Theorem 3.1. Consequently, we infer that u_τ is contained in $G_1(\Omega)$, f_τ in $H^1(\Omega)$, g_τ in $E_1(J \times \partial\mathbb{R}^3_+)$, and $u_{0,\tau}$ in $H^1(\mathbb{R}^3_+)$ for all $\tau > 0$. With this amount of regularity we can compute

$$A_0 \partial_t u_\tau + \sum_{j=1}^{3} A_j \partial_j u_\tau + D u_\tau = f_\tau - \tau\Big(A_0 - \sum_{j=1}^{3} \partial_j \psi A_j\Big) u_\tau \tag{6.3}$$

for all $\tau > 0$. These functions also satisfy $u_\tau(0) = u_{0,\tau}$ on $\mathbb{R}^3_+$ and $B u_\tau = g_\tau$ on $J \times \partial\mathbb{R}^3_+$ for all $\tau > 0$.

Moreover, $A_0 - \sum_{j=1}^{3} \partial_j \psi A_j$ is uniformly positive semidefinite since

$$\begin{aligned}\Big(\big(A_0 - \sum_{j=1}^{3} \partial_j \psi A_j\big)\xi, \xi\Big)_{\mathbb{R}^6\times\mathbb{R}^6} &\ge \eta|\xi|^2 - \sum_{j=1}^{3} \|\partial_j \psi\|_{L^\infty(\mathbb{R}^3)} \|A_j\|_{L^\infty(\Omega)} |\xi|^2 \\ &\ge \eta|\xi|^2 - \eta K C_0 |\xi|^2 = 0\end{aligned}$$

on Ω for $\xi \in \mathbb{R}^6$, where we used the definition of $C_0 = K^{-1}$. Identity (6.3) in combination with this estimate then yields

$$\begin{aligned}&\partial_t \langle A_0 u_\tau, u_\tau\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\ &\quad = \langle \partial_t A_0 u_\tau, u_\tau\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} + 2\Big\langle f_\tau - \sum_{j=1}^{3} A_j \partial_j u_\tau - D u_\tau, u_\tau\Big\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\ &\qquad - 2\tau\Big\langle \Big(A_0 - \sum_{j=1}^{3} \partial_j \psi A_j\Big) u_\tau, u_\tau\Big\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\ &\quad \le \langle \partial_t A_0 u_\tau, u_\tau\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} - 2\sum_{j=1}^{3} \langle A_j \partial_j u_\tau, u_\tau\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\ &\qquad - 2\langle D u_\tau, u_\tau\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} + 2\langle f_\tau, u_\tau\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}\end{aligned}$$

for almost all $t \in J$ and for all $\tau > 0$. Hence,

$$\begin{aligned}
&\eta \|u_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)} \\
&\quad \le \langle A_0 u_\tau, u_\tau \rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\
&\quad = \langle A_0(0) u_\tau(0), u_\tau(0) \rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} + \int_0^t \partial_t \langle A_0 u_\tau, u_\tau \rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}(s) ds \\
&\quad \le \|A_0\|_{L^\infty(\Omega)} \|u_{0,\tau}\|^2_{L^2(\mathbb{R}^3_+)} + (\|\partial_t A_0\|_{L^\infty(\Omega)} + 2\|D\|_{L^\infty(\Omega)}) \int_0^t \|u_\tau(s)\|^2_{L^2(\mathbb{R}^3_+)} ds \\
&\qquad - 2\sum_{j=1}^3 \int_0^t \langle A_j(s) \partial_j u_\tau(s), u_\tau(s) \rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} ds \\
&\qquad + 2\int_0^t \|f_\tau(s)\|_{L^2(\mathbb{R}^3_+)} \|u_\tau(s)\|_{L^2(\mathbb{R}^3_+)} ds
\end{aligned} \tag{6.4}$$

for all $t \in J$ and $\tau > 0$. Since $u_\tau \in G_1(\Omega)$, the symmetry of the matrices A_j further implies

$$\langle A_j \partial_j u_\tau, u_\tau \rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} = -\frac{1}{2} \langle \partial_j A_j u_\tau, u_\tau \rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}, \tag{6.5}$$

$$\begin{aligned}
\langle A_3 \partial_3 u_\tau, u_\tau \rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} &= -\frac{1}{2} \langle \partial_3 A_3 u_\tau, u_\tau \rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\
&\quad - \frac{1}{2} \int_{\partial\mathbb{R}^3_+} \operatorname{tr}(A_3 u_\tau)(\sigma) \operatorname{tr}(u_\tau)(\sigma) d\sigma
\end{aligned} \tag{6.6}$$

on J for $j \in \{1, 2\}$ by integration by parts. We set

$$C_1 = \frac{1}{\eta} \Big(\sum_{j=0}^3 \|A_j\|_{W^{1,\infty}(\Omega)} + 2\|D\|_{L^\infty(\Omega)} + \frac{1}{\eta} \Big).$$

Inserting (6.5) and (6.6) into (6.4), we derive

$$\begin{aligned}
&\eta \|u_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)} \\
&\quad \le \eta C_1 \|u_{0,\tau}\|^2_{L^2(\mathbb{R}^3_+)} + \eta \|f_\tau\|^2_{L^2(\Omega)} + \eta C_1 \int_0^t \|u_\tau(s)\|^2_{L^2(\mathbb{R}^3_+)} ds \\
&\qquad + \langle \operatorname{tr}(A_3 u_\tau), \operatorname{tr} u_\tau \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)}
\end{aligned} \tag{6.7}$$

for all $t \in J$ and $\tau > 0$, where we denote $(0,t) \times \partial\mathbb{R}^3_+$ by Γ_t. In order to estimate the last term in (6.7), we recall that the boundary matrix $A_3 = A_3^{\mathrm{co}}$ decomposes as

$$A_3^{\mathrm{co}} = \frac{1}{2} C^{\mathrm{co}\,T} B^{\mathrm{co}} + \frac{1}{2} B^{\mathrm{co}\,T} C^{\mathrm{co}},$$

see (3.2). Employing $Bu_\tau = g_\tau$, $B = B^{\mathrm{co}}$, and $u \in G_1(\Omega)$, we thus infer

$$\begin{aligned}
\langle \operatorname{tr}(A_3 u_\tau), \operatorname{tr} u_\tau \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} &= \langle C^{\mathrm{co}} \operatorname{tr} u_\tau, B^{\mathrm{co}} \operatorname{tr} u_\tau \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} \\
&= \langle C^{\mathrm{co}} \operatorname{tr} u_\tau, g_\tau \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} = \langle C^{\mathrm{co}} \operatorname{tr} u, g_{2\tau} \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} \\
&\le \|C^{\mathrm{co}} \operatorname{tr} u\|_{L^2(\Gamma_t)} \|g_{2\tau}\|_{L^2(\Gamma_t)} \le \|\operatorname{tr} u\|_{L^2(\Gamma_t)} \|g_{2\tau}\|_{L^2(\Gamma_t)} \\
&\le \|u\|_{H^1(\Omega)} \|g_{2\tau}\|_{L^2(\Gamma)}
\end{aligned} \tag{6.8}$$

for all $t \in J$ and $\tau > 0$, where Γ denotes $J \times \partial\mathbb{R}^3_+$ as usual. We point out that $\|u\|_{H^1(\Omega)}$ is finite as $u \in G_1(\Omega)$. Estimate (6.7) and (6.8) finally lead to

$$\begin{aligned}
\|u_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)} \le C_1 \|u_{0,\tau}\|^2_{L^2(\mathbb{R}^3_+)} + \|f_\tau\|^2_{L^2(\Omega)} + \frac{1}{\eta} \|u\|_{H^1(\Omega)} \|g_{2\tau}\|_{L^2(\Gamma)} \\
+ C_1 \int_0^t \|u_\tau(s)\|^2_{L^2(\mathbb{R}^3_+)} ds
\end{aligned}$$

for all $t \in J$ and $\tau > 0$ so that Gronwall's lemma implies

$$\sup_{t\in J} \|u_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)} \le \Big(C_1 \|u_{0,\tau}\|^2_{L^2(\mathbb{R}^3_+)} + \|f_\tau\|^2_{L^2(\Omega)} + \frac{1}{\eta} \|u\|_{H^1(\Omega)} \|g_{2\tau}\|_{L^2(\Gamma)} \Big) e^{C_1 T} \tag{6.9}$$

for all $\tau > 0$.

III) To exploit the weighted energy estimate (6.9), we take (s,x) from $(J \times \mathbb{R}^3_+) \setminus \mathcal{C}$, i.e., $|x - x_0| \ge R - C_0 s$ and hence

$$-s + K(R - |x - x_0|) = -s + \frac{1}{C_0}(R - |x - x_0|) \le 0.$$

It follows

$$e^{\tau \Phi(s,x)} = e^{\tau(-s + \psi(x))} \le e^{-\tau\varepsilon}$$

for all $\tau > 0$. On the other hand, $f(s,x) = 0$ for almost all $(s,x) \in \mathcal{C}$, $g(s,x) = 0$ for almost all $(s,x) \in \mathcal{C} \cap (J \times \partial\mathbb{R}^3_+)$, and $u_0(x) = 0$ for almost all $x \in \mathcal{C}_{t=0} \cap \mathbb{R}^3_+$. We conclude that

$$|f_\tau(s,x)| \le |f(s,x)| \text{ for all } \tau > 0 \quad \text{and} \quad |f_\tau(s,x)| \longrightarrow 0 \text{ as } \tau \to \infty$$

for almost all $(s,x) \in J \times \mathbb{R}^3_+$ so that

$$\|f_\tau\|_{L^2(\Omega)} \longrightarrow 0$$

as $\tau \to \infty$. Analogously, we deduce

$$\|g_{2\tau}\|_{L^2(\Gamma)} \longrightarrow 0 \quad \text{and} \quad \|u_{0,\tau}\|_{L^2(\mathbb{R}^3_+)} \longrightarrow 0$$

as $\tau \to \infty$. By (6.9) the functions u_τ thus tend to 0 in $G_0(\Omega)$ as $\tau \to \infty$, so that

$$C_2 := \sup_{t\in J, \tau>0} \|u_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)} < \infty. \tag{6.10}$$

Now take a point (t, x) from $\mathcal{C}_{3\varepsilon}$. We then calculate

$$\begin{aligned} 3\varepsilon &< K(R - |x - x_0|) - t \le -t + \psi(x) + 2\varepsilon = \Phi(t, x) + 2\varepsilon, \\ \varepsilon &< \Phi(t, x). \end{aligned}$$

Estimate (6.10) now implies

$$\begin{aligned} \int_{\mathcal{C}_{3\varepsilon}\cap\Omega} |u(t, x)|^2 dx dt &\le e^{-2\varepsilon\tau} \int_{\mathcal{C}_{3\varepsilon}\cap\Omega} e^{2\tau\Phi(t,x)} |u(t, x)|^2 dx dt \\ &\le e^{-2\varepsilon\tau} T \sup_{t\in J} \|u_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)} \le C_2 T e^{-2\varepsilon\tau} \end{aligned}$$

for all $\tau > 0$. Letting $\tau \to \infty$, we obtain $|u(t, x)| = 0$ for almost all $(t, x) \in \mathcal{C}_{3\varepsilon}$.

Finally, we take a sequence $(\varepsilon_n)_n$ in $(0, 1)$ with $\varepsilon_n \to 0$ as $n \to \infty$. Since $u(t, x) = 0$ for almost all $(t, x) \in \mathcal{C}_{3\varepsilon_n}$ for all $n \in \mathbb{N}$, we conclude that

$$u(t, x) = 0 \quad \text{for almost all } (t, x) \in \bigcup_{n\in\mathbb{N}} \mathcal{C}_{3\varepsilon_n} = \mathcal{C}.$$

IV) Now let f, g, and u_0 be as in the assertion. We take a family of functions $(f_\varepsilon, g_\varepsilon, u_{0,\varepsilon})$ in $H^1(\Omega) \times E_1(J \times \partial\mathbb{R}^3_+) \times H^1(\mathbb{R}^3_+)$ for $0 < \varepsilon < 1$ such that f_ε converges to f in $L^2(\Omega)$, g_ε to g in $E_0(J \times \partial\mathbb{R}^3_+)$, and $u_{0,\varepsilon}$ to u_0 in $L^2(\mathbb{R}^3_+)$ as $\varepsilon \to 0$ and the tuples $(0, A_0, \dots, A_3, D, B, f_\varepsilon, g_\varepsilon, u_{0,\varepsilon})$ are compatible of order 1 for all $\varepsilon \in (0, 1)$. Such a family can be constructed as explained at the beginning of step II) of Lemma 5.1. Using a standard mollifier for the regularization as in step I) of the proof of Theorem 4.13 in [34], we can construct this family in such a way that

$$\begin{aligned} &\operatorname{supp} f_\varepsilon \subseteq \operatorname{supp} f + B(0, \varepsilon), \quad \operatorname{supp} g_\varepsilon \subseteq \operatorname{supp} g + B(0, \varepsilon) \\ &\operatorname{supp} u_{0,\varepsilon} \subseteq (\operatorname{supp} u_0 \cup (\operatorname{supp} g(0) \times \{0\})) + B(0, \varepsilon) \end{aligned}$$

for all $\varepsilon \in (0, 1)$.

Let $\varepsilon \in (0, 1)$ and $(s, y) \in \mathbb{R} \times \mathbb{R}^3$ with $(s, y) \in \mathcal{C}^C + B(0, \varepsilon)$, where $\mathcal{C}^C = \mathbb{R}^4 \setminus \mathcal{C}$. We then find $(t, x) \in \mathcal{C}^C$ such that $|(t, x) - (s, y)| < \varepsilon$. Assume that (s, y) belongs to $\mathcal{C}_{(2+\frac{1}{C_0})\varepsilon}$. We then obtain

$$\begin{aligned} R &\le |x-x_0| + C_0 t \le |x-y| + C_0(t-s) + |y-x_0| + C_0 s \\ &\le \varepsilon + C_0\varepsilon + R - C_0\Big(2+\frac{1}{C_0}\Big)\varepsilon = R - C_0\varepsilon, \end{aligned}$$

which is a contradiction. This means that $\mathcal{C}^C + B(0,\varepsilon) \subseteq \mathcal{C}^C_{(2+\frac{1}{C_0})\varepsilon}$ for all $\varepsilon > 0$. We thus arrive at

$$\operatorname{supp} f_\varepsilon \cap \Omega \subseteq (\operatorname{supp} f + B(0,\varepsilon)) \cap \Omega \subseteq \mathcal{C}^C_{(2+\frac{1}{C_0})\varepsilon} \cap \Omega$$

for all $\varepsilon > 0$. Analogously, we derive that

$$\begin{aligned} \operatorname{supp} g_\varepsilon \cap \Gamma &\subseteq (\operatorname{supp} g + B(0,\varepsilon)) \cap \Gamma \subseteq \mathcal{C}^C_{(2+\frac{1}{C_0})\varepsilon} \cap \Gamma, \\ \operatorname{supp} u_{0,\varepsilon} \cap \mathbb{R}^3_+ &\subseteq ((\operatorname{supp} u_0 \cup (\operatorname{supp} g(0) \times \{0\})) + B(0,\varepsilon)) \cap \mathbb{R}^3_+ \\ &\subseteq \mathcal{C}^C_{t=0,(2+\frac{1}{C_0})\varepsilon} \cap \mathbb{R}^3_+, \end{aligned}$$

for all $\varepsilon > 0$, where $\mathcal{C}^C_{t=0,(2+\frac{1}{C_0})\varepsilon} = \mathbb{R}^3 \setminus \mathcal{C}_{t=0,(2+\frac{1}{C_0})\varepsilon}$. Steps II) and III) now show that the unique solution $u_\varepsilon \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ of (3.1) with inhomogeneity f_ε, boundary value g_ε, and initial value $u_{0,\varepsilon}$ vanishes on $\mathcal{C}_{(2+\frac{1}{C_0})\varepsilon}$, i.e., $u_\varepsilon(t,x) = 0$ for almost all $(t,x) \in \mathcal{C}_{(2+\frac{1}{C_0})\varepsilon}$ for each $\varepsilon > 0$.

Take a monotonically decreasing sequence $(\varepsilon_n)_n$ in $(0,1)$ with $\varepsilon_n \to 0$ as $n \to \infty$. By Theorem 3.1 there is a constant C_3 and a number $\gamma > 0$ such that

$$\|u_{\varepsilon_n} - u\|^2_{G_{0,\gamma}(\Omega)} \le C_3\Big(\|u_{0,\varepsilon_n} - u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g_{\varepsilon_n} - g\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \frac{1}{\gamma}\|f_{\varepsilon_n} - f\|^2_{L^2_\gamma(\Omega)}\Big) \to 0$$

as $n \to \infty$, in particular $(u_{\varepsilon_n})_n$ tends to u in $L^2(\Omega)$ as $n \to \infty$. Consequently, there is a subsequence, which we again denote by $(u_{\varepsilon_n})_n$, which converges pointwise almost everywhere to u. Since $\mathcal{C}_{(2+\frac{1}{C_0})\varepsilon_n} \subseteq \mathcal{C}_{(2+\frac{1}{C_0})\varepsilon_m}$ for all $m > n$, we infer that $u(t,x) = 0$ for almost all $(t,x) \in \mathcal{C}_{(2+\frac{1}{C_0})\varepsilon_n}$ for all $n \in \mathbb{N}$. Hence,

$$u(t,x) = 0 \quad \text{for almost all } (t,x) \in \bigcup_{n\in\mathbb{N}} \mathcal{C}_{(2+\frac{1}{C_0})\varepsilon_n} = \mathcal{C}. \quad \square$$

We also formulate the finite propagation speed property using the forward light cone, cf. [3]. This version shows that if the data is supported on a forward light cone, then also the solution is supported on this cone.

Corollary 6.2. *Let $\eta > 0$, $A_0 \in F^{\mathrm{c}}_{3,\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{3,\mathrm{coeff}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, $D \in F^{\mathrm{c}}_3(\Omega)$, and $B = B^{\mathrm{co}}$. We set*

$$C_0 = \frac{1}{\eta}\sum_{j=1}^{3} \|A_j\|_{L^\infty(\Omega)}.$$

Let $R > 0$ and $x_0 \in \overline{\mathbb{R}^3_+}$. We define the forward cone $\mathcal{K}$ by

$$\mathcal{K} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^3 : |x - x_0| \leq R + C_0 t\}.$$

Let $f \in L^2(\Omega)$, $g \in L^2(J, H^{1/2}(\partial\mathbb{R}^3_+))$, and $u_0 \in L^2(\mathbb{R}^3_+)$ such that

$$\begin{aligned} f &= 0 \quad \text{on } \Omega \setminus \mathcal{K}, \\ g &= 0 \quad \text{on } (J \times \partial\mathbb{R}^3_+) \setminus \mathcal{K}, \\ u_0 &= 0 \quad \text{on } \mathbb{R}^3_+ \setminus \mathcal{K}_{t=0}, \end{aligned}$$

where $\mathcal{K}_{t=0} = \{x \in \mathbb{R}^3 : (0, x) \in \mathcal{K}\}$. Then the unique solution $u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ of the linear initial boundary value problem (3.1) with inhomogeneity f, boundary value g, and initial value u_0 is supported in the cone $\mathcal{K}$, i.e.,

$$u(t, x) = 0 \quad \text{for almost all } (t, x) \in \Omega \setminus \mathcal{K}.$$

The assertion can be reduced to Theorem 6.1, see [34, Corollary 6.2] for details.

Remark 6.3. In the framework of Theorem 5.3 assume that the data vanish on a backward light cone or outside of a forward light cone, see Theorem 6.1 respectively Corollary 6.2 for the precise statement. Then also the solution of the nonlinear problem (1.6) vanishes on the backward respectively forward light cone. This assertion follows from the simple observation that the solution u of (1.6) also solves the linear problem (1.8) respectively (3.1) with coefficients $\chi(u)$ and $\sigma(u)$. Theorem 6.1 respectively Corollary 6.2 then yield the assertion.

Acknowledgments

I want to thank my advisor Roland Schnaubelt for helpful feedback on this research and valuable advice regarding this article. I also thank the anonymous referee for useful comments. Moreover, I gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

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Please cite this article in press as: M. Spitz, Local wellposedness of nonlinear Maxwell equations with perfectly conducting boundary conditions, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.019