



Strong convergence rate of splitting schemes for stochastic nonlinear Schrödinger equations [☆]

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Abstract

In this paper, we show that solutions of stochastic nonlinear Schrödinger (NLS) equations can be approximated by solutions of coupled splitting systems. Based on these systems, we propose a new kind of fully discrete splitting schemes which possess algebraic strong convergence rates for stochastic NLS equations. Key ingredients of our approach are using the exponential integrability and stability of the corresponding splitting systems and numerical approximations. In particular, under very mild conditions, we derive the optimal strong convergence rate $\mathcal{O}(N^{-2} + \tau^{\frac{1}{2}})$ of the spectral splitting Crank–Nicolson scheme, where N and τ denote the dimension of the approximate space and the time step size, respectively.

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1. Introduction

For stochastic partial differential equations (SPDEs) with monotone coefficients, there exist fruitful results on strong error analysis of temporal and/or spatial numerical approximations by using semigroup or variational frameworks (see e.g. [1,4,5,13,18,20,22,26]). However, for SPDEs with non-monotone coefficients, so far as we know, it is a long standing open problem to construct temporal numerical approximations and full discretizations which possess algebraic strong convergence rates. This is the main motivation of the present paper.

As a classical type of SPDEs with non-monotone coefficients, stochastic NLS equations model the propagation of nonlinear dispersive waves in inhomogeneous or random media (see e.g. [21] and references therein). Our main purpose in this paper is to construct temporal approximations and fully discrete schemes possessing algebraic strong convergence rates for the one-dimensional stochastic NLS equation

$$\begin{cases} i du + (\Delta u + \lambda |u|^2 u) dt = u \circ dW(t), & \text{in } (0, T] \times \mathcal{O}; \\ u(t) = 0, & \text{on } [0, T] \times \partial\mathcal{O}; \\ u(0) = \xi, & \text{in } \mathcal{O}, \end{cases} \quad (1)$$

where $T > 0$, $\mathcal{O} = (0, 1)$, and $\lambda = 1$ or -1 corresponds to focusing or defocussing cases, respectively. Here $W = \{W(t) : t \in [0, T]\}$ is a $L^2(\mathcal{O}; \mathbb{R})$ -valued Q -Wiener process on a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, i.e., there exists an orthonormal basis $\{e_k\}_{k \in \mathbb{N}_+}$ of $L^2(\mathcal{O}; \mathbb{R})$ and a sequence of mutually independent, real-valued Brownian motions $\{\beta_k\}_{k \in \mathbb{N}_+}$ such that $W(t) = \sum_{k \in \mathbb{N}_+} Q^{\frac{1}{2}} e_k \beta_k(t)$, $t \in [0, T]$.

Eq. (1) has been investigated both theoretically and numerically. The well-posedness of Eq. (1) has been proved by [12] in \mathbb{H}^1 , by [6] and [10] in \mathbb{H}^2 for defocussing and focusing cases, respectively. There are also many authors constructing numerical approximations of Eq. (1) and obtaining convergence rates in certain sense such as pathwise or in probability weaker than in strong sense (see e.g. [6,7,11,13,24] and references therein). A progress has been made by [10] where the authors obtain a strong convergence rate of spatial centered difference method for Eq. (1). Besides the \mathbb{H}^2 -a priori estimations, the key ingredient to derive strong convergence rates is the \mathbb{H}^1 -exponential integrability of both the exact and numerical solutions (see also [16,17]). This type of exponential integrability is also useful to get the strongly continuous dependence on initial data of both the exact and numerical solutions, to derive a large deviation principle of Freidlin–Wentzell type (see [10, Corollaries 3.1 and 3.2]) and to deduce Gaussian tail estimations of these solutions (see Corollary 4.1). We refer to [16,17,19] and references therein for the exponential integrability of a kind of stochastic evolution equations with non-monotone coefficients and of their numerical approximations. So far as we know, there exists no result about this type of exponential integrability for a temporally discrete approximation of Eq. (1). In this work we propose a temporal splitting Crank–Nicolson scheme (see (35)), based on a splitting approach and its corresponding splitting processes which are shown to admit the desired exponential integrability.

Let us loosely describe the achievement of a sequence of splitting processes through the splitting approach. Given an $M \in \mathbb{N}^+$, denote $\mathbb{Z}_M = \{0, 1, \dots, M\}$. Let $\tau = \frac{T}{M}$ and $\{T_m := (t_m, t_{m+1}], t_m = m\tau, m \in \mathbb{Z}_{M-1}\}$ be a uniform partition of the interval $(0, T]$. Our main idea is to split Eq. (1) in $T_m, m \in \mathbb{Z}_{M-1}$, into a deterministic NLS equation with random initial datum and a linear SPDE:

$$du_{\tau,m}^D(t) = \left(\mathbf{i} \Delta u_{\tau,m}^D(t) + \mathbf{i} \lambda |u_{\tau,m}^D(t)|^2 u_{\tau,m}^D(t) \right) dt, \quad u_{\tau,m}^D(t_m) = u_\tau(t_m), \quad (2)$$

$$du_{\tau,m}^S(t) = -\mathbf{i} u_{\tau,m}^S(t) \circ dW(t), \quad u_{\tau,m}^S(t_m) = u_{\tau,m}^D(t_{m+1}). \quad (3)$$

Eq. (2) and (3) are subjected to homogeneous Dirichlet boundary conditions on $T_m \times \partial\mathcal{O}$. Then, in Section 2, we give the definition of a auxiliary splitting process u_τ with the initial datum $u_\tau(0) = \xi$. It is shown that this splitting process $u_\tau = \{u_\tau(t) : t \in [0, T]\}$ is left-continuous with finite right-hand limits and \mathcal{F}_t -adapted (see Proposition 2.1). Since Eq. (2) has no analytic solution, we apply the Crank–Nicolson scheme to temporally discretize Eq. (2). Based on the analytical expression of the solution of Eq. (3), we get the splitting Crank–Nicolson scheme starting from ξ :

$$\begin{cases} u_{m+1}^D = u_m + \mathbf{i} \tau \Delta u_{m+\frac{1}{2}}^D + \mathbf{i} \lambda \tau \frac{|u_m|^2 + |u_{m+1}^D|^2}{2} u_{m+\frac{1}{2}}^D, \\ u_{m+1} = \exp(-\mathbf{i}(W_{t_{m+1}} - W_{t_m})) u_{m+1}^D, \end{cases} \quad m \in \mathbb{Z}_{M-1}, \quad (4)$$

with $u_{m+\frac{1}{2}}^D = \frac{1}{2}(u_m + u_{m+1}^D)$.

Our first goal is to prove that both $u_\tau = \{u_\tau(t) : t \in [0, T]\}$ and $\{u_m\}_{m \in \mathbb{Z}_M}$ converge to the exact solution $u = \{u(t) : t \in [0, T]\}$ of Eq. (1) with strong order 1/2 (see Theorem 2.2 and Theorem 3.1). The key requirement is the exponential integrability properties of u_τ and u_m , which is proved by an exponential integrability lemma established in [8, Corollary 2.4] or [10, Proposition 3.1] (see Lemmas 2.2 and 3.1). To the best of our knowledge, Theorem 3.1 is the first result about strong convergence rates of temporal approximations for Eq. (1) or even for SPDEs with non-monotone coefficients. We also note that there are several results to numerically approximate SPDEs by splitting schemes (see e.g. [2,9,14,23,24,15] and references therein). [15] obtains a strong convergence rate of a splitting scheme for a linear SPDE about stochastic filtering problem; [23] gets a strong convergence rate for a linear stochastic Schrödinger equation. However, the splitting schemes in [15,23] applied to Eq. (1) would fail to satisfy the desired exponential integrability.

Our second goal is to construct a fully discrete scheme possessing optimal algebraic strong convergence rate based on the aforementioned splitting approach. To this end, we apply the splitting Crank–Nicolson scheme (4) to the spatially spectral Galerkin discretization in Section 4 and get the spectral splitting Crank–Nicolson scheme (55). The spectral approximate solution u^N is shown to converge to u with strong convergence rate $\mathcal{O}(N^{-2})$, where N is the dimension of the spectral approximate space. Combining the strong error estimate of the splitting Crank–Nicolson scheme (4), we finally derive the optimal strong convergence rate $\mathcal{O}(N^{-2} + \tau^{\frac{1}{2}})$ of this fully discrete scheme (see Theorem 4.2). The optimality of convergence rate is in the sense that the temporal and strong convergence rate coincides with the optimal temporal Hölder regularity and spatial Sobolev regularity (see e.g. [3]) under the mild regularity assumptions on initial datum ξ and noise. We remark that there exist a lot of alternative choices of spatial discretizations.

For instance, we apply this splitting approach to the spatial centered difference method analyzed in [10, Theorem 4.1] and get the related full discretization with algebraic strong convergence rate.

Our article is organized as follows. We give a detailed analysis for the splitting process u_τ in Section 2. Meanwhile, we study the evolutions about the charge, energy and a Lyapunov functional used to control the \mathbb{H}^2 -norm of u_τ as well as its exponential integrability in \mathbb{H}^1 . Based on the \mathbb{H}^2 -a priori estimation and exponential integrability of u_τ and the exact solution u , we deduce the strong error estimate between u_τ and u . In Section 3, we analyze the temporal splitting Crank–Nicolson scheme and obtain its strong convergence rate. To perform an implementary full discretization, we apply the proposed splitting Crank–Nicolson scheme to the spectral Galerkin approximate equation in Section 4 and get the strong convergence rate of this spectral splitting Crank–Nicolson scheme for Eq. (1). Similar arguments are also applied to spatially discrete equation by centered difference method.

To close this section, we introduce some frequently used notations and assumptions. The norm and inner product of $L^2 := L^2(\mathcal{O}; \mathbb{C})$ is denoted by $\|\cdot\|$ and $\langle u, v \rangle := \Re \left[\int_{\mathcal{O}} \bar{u}(x)v(x)dx \right]$, respectively. $H^s := H^s(\mathcal{O})$, $H_0^s := H_0^s(\mathcal{O})$, $s \in \mathbb{N}$, and $\mathbb{H}^s = \mathbb{H}^s(\mathcal{O})$, $\mathbb{H}_0^s := \mathbb{H}_0^s(\mathcal{O})$, $s \in \mathbb{N}$ denote the real-valued Sobolev space and the complex-valued Sobolev space, respectively. Throughout we assume that T is a fixed positive number, $\xi \in \mathbb{H}_0^1 \cap \mathbb{H}^2$ is a deterministic function and $Q^{\frac{1}{2}} \in \mathcal{L}_2^2 = \mathcal{L}_2(H, H_0^1 \cap H^2)$, i.e.,

$$\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 := \sum_{k \in \mathbb{N}_+} \|Q^{\frac{1}{2}}e_k\|_{H^2}^2 < \infty,$$

where $\{e_k\}_{k \in \mathbb{N}_+}$ is any orthonormal basis of $L^2(\mathcal{O}; \mathbb{R})$. We use C and C' to denote a generic constant, independent of the time step size τ and the dimension N , which differs from one place to another.

2. Splitting process

We first give the definitions of the auxiliary splitting processes $u_\tau(t)$ and $u_\tau^D(t)$, $t \in [0, T]$ with the initial datum ξ . For simplicity, we denote the solution operators of Eq. (2) and (3) in T_m as $\Phi_{m,t-t_m}^D$ and $\Phi_{m,t-t_m}^S$, respectively. Next we set the splitting process u_τ in T_m as

$$u_\tau(t) := u_{\tau,m}^S(t) := (\Phi_{j,t-t_m}^S \Phi_{j,\tau}^D) \prod_{j=0}^{m-1} (\Phi_{j,\tau}^S \Phi_{j,\tau}^D) u_\tau(0), \quad t \in T_m, \quad (5)$$

and

$$u_\tau^D(t) := u_{\tau,m}^D(t) := \Phi_{j,t-t_m}^D \prod_{j=0}^{m-1} (\Phi_{j,\tau}^S \Phi_{j,\tau}^D) u_\tau(0), \quad t \in [t_m, t_{m+1}),$$

with $u_\tau^D(T) := \prod_{j=0}^{M-1} (\Phi_{j,\tau}^S \Phi_{j,\tau}^D) u_\tau(0)$. Our main purpose is to prove that u_τ possesses the exponential integrability and is a nice approximation of the exact solution u of Eq. (1).

We first recall the following known results about the well-posedness and strongly continuous dependence on initial data for Eq. (1) as well as exponential integrability of u . These properties are useful deriving algebraic strong convergence rates for numerical approximations of Eq. (1).

Theorem 2.1. *Let $p \geq 1$. Eq. (1) possesses a unique strong solution $u = \{u(t) : t \in [0, T]\}$ satisfying*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u(t)\|_{\mathbb{H}^2}^p \right] < \infty \quad (6)$$

and depending on the initial data in $L^p(\Omega; \mathcal{C}([0, T]; L^2))$, i.e., if we assume that $\xi^I, \xi^{II} \in \mathbb{H}_0^1 \cap \mathbb{H}^2$ and that u^I and u^{II} are the solutions of Eq. (1) with initial data ξ^I and ξ^{II} , respectively, then there exists a constant $C = C(\xi^I, \xi^{II}, Q, T, p)$ such that

$$\left(\mathbb{E} \left[\sup_{t \in [0, T]} \|u^I(t) - u^{II}(t)\|^p \right] \right)^{\frac{1}{p}} \leq C \|\xi^I - \xi^{II}\|. \quad (7)$$

Moreover, there exist constants C and α depending on ξ, Q and T such that

$$\sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{\|\nabla u(t)\|^2}{e^{\alpha t}} \right) \right] \leq C. \quad (8)$$

Proof. We refer to [10], Theorem 2.1, Corollary 3.1 and Proposition 3.1, respectively, for the well-posedness and \mathbb{H}^2 -a priori estimate (6), strongly continuous dependence estimate (7) and exponential integrability estimate (8). \square

2.1. Stability of splitting process

In this part, we prove that the splitting process $u_\tau = \{u_\tau(t) : t \in [0, T]\}$ defined by (5) is well-defined and uniformly bounded in $L^p(\Omega; L^\infty([0, T]; \mathbb{H}^2))$ for any $p \geq 1$.

We start with the evolution of the charge and the energy of u_τ , i.e., $\|u_\tau\|^2$ and $H(u_\tau) := \frac{1}{2} \|\nabla u_\tau\|^2 - \frac{\lambda}{4} \|u_\tau\|_{L^4}^4$.

Proposition 2.1. *The splitting process $u_\tau = \{u_\tau(t) : t \in [0, T]\}$ is uniquely solvable and \mathcal{F}_t -measurable. Moreover, for any $t \in [0, T]$ there holds a.s. that*

$$\|u_\tau(t)\|^2 = \|\xi\|^2 \quad (9)$$

and that

$$\begin{aligned} H(u_\tau(t)) &= H(\xi) + \int_0^t \langle \nabla u_\tau, \dot{u}_\tau d(\nabla W(r)) \rangle \\ &\quad + \frac{1}{2} \sum_{k \in \mathbb{N}^+} \int_0^t \|u_\tau \nabla(Q^{\frac{1}{2}} e_k)\|^2 dr. \end{aligned} \quad (10)$$

Proof. Let $m \in \mathbb{Z}_{M-1}$ and $t \in T_m$. Since Eq. (2) can be seen as a special equation of (1) with $Q = 0$, $u_\tau^D(t)$ is uniquely solvable and \mathcal{F}_{t_m} -measurable by Theorem 2.1. Moreover,

$$\|u_\tau^D(t)\|^2 = \|u_{\tau,m}^D(t_m)\|^2, \quad H(u_\tau^D(t)) = H(u_{\tau,m}^D(t_m)) \quad \text{a.s.} \quad (11)$$

On the other hand, Eq. (3) has an \mathcal{F}_t -measurable analytic solution given by $u_\tau^S(t) = \exp(-i(W(t) - W(t_m)))u_{\tau,m}^S(t_m)$, and thus $u_\tau^S(t)$ preserves the modular length as well as the charge, i.e.,

$$|u_\tau^S(t)| = |u_{\tau,m}^S(t_m)|, \quad \|u_\tau^S(t)\|^2 = \|u_{\tau,m}^S(t_m)\|^2. \quad (12)$$

This implies that u_τ is uniquely solvable and \mathcal{F}_t -measurable and

$$\|u_\tau(t)\|^2 = \|u_\tau(t_m)\|^2,$$

from which we obtain (9) by iterations on m . By Itô formula, we have

$$\begin{aligned} H(u_\tau(t)) &= H(u_\tau(t_m)) + \int_{t_m}^t \langle \nabla u_\tau, i u_\tau d(\nabla W(r)) \rangle \\ &\quad + \frac{1}{2} \sum_{k \in \mathbb{N}_{t_m}^*} \int_{t_m}^t \|u_\tau \nabla(Q^{\frac{1}{2}} e_k)\|^2 dr. \end{aligned} \quad (13)$$

Combining (11) and (13), we obtain (10) by iterations. \square

The above charge conservation law (9) and energy evolution (10) imply the following boundedness in $L^p(\Omega; L^\infty([0, T]; \mathbb{H}^1))$ for any $p \geq 1$.

Corollary 2.1. *For any $p \geq 1$, there exists a constant $C = C(\xi, Q, T, p)$ such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u_\tau(t)\|_{\mathbb{H}^1}^p \right] \leq C. \quad (14)$$

Proof. Let $t \in T_m$ for $m \in \mathbb{Z}_{M-1}$ and $p \geq 4$. Applying the Itô formula and the energy evolution law (13) of u_τ^S , we obtain

$$\begin{aligned} H^{\frac{p}{2}}(u_{\tau,m}^S(t)) &= H^{\frac{p}{2}}(u_{\tau,m}^S(t_m)) + \frac{p}{4} \sum_{k \in \mathbb{N}_{t_m}^*} \int_{t_m}^t H^{\frac{p}{2}-1}(u_{\tau,m}^S) \|u_{\tau,m}^S \nabla(Q^{\frac{1}{2}} e_k)\|^2 dr \\ &\quad + \frac{p}{2} \int_{t_m}^t H^{\frac{p}{2}-1}(u_{\tau,m}^S) \left\langle \nabla u_{\tau,m}^S, i u_{\tau,m}^S d(\nabla W(r)) \right\rangle \end{aligned}$$

$$+ \frac{p(p-2)}{8} \sum_{k \in \mathbb{N}_{t_m}^*} \int_{t_m}^t H^{\frac{p}{2}-2}(u_{\tau,m}^S) \left\langle \nabla u_{\tau,m}^S, \mathbf{i} u_{\tau,m}^S \nabla (Q^{\frac{1}{2}} e_k) \right\rangle dr.$$

Taking expectations on both sides of the above equality, using the Hölder and Gagliardo–Nirenberg inequalities, we get

$$\mathbb{E} \left[H^{\frac{p}{2}}(u_{\tau,m}^S(t)) \right] \leq \mathbb{E} \left[H^{\frac{p}{2}}(u_{\tau}(t_m)) \right] + C \int_{t_m}^t \left(1 + \mathbb{E} \left[H^{\frac{p}{2}}(u_{\tau,m}^S(s)) \right] \right) ds.$$

The Gronwall inequality yields that

$$\mathbb{E} \left[H^{\frac{p}{2}}(u_{\tau,m}^S(t)) \right] \leq e^{C\tau} \left(\mathbb{E} \left[H^{\frac{p}{2}}(u_{\tau}(t_m)) \right] + C\tau \right). \quad (15)$$

Using the energy conservation law of Eq. (2) and substituting iteratively the estimation (15) for $\mathbb{E} \left[H^{\frac{p}{2}}(u_{\tau}(t_m)) \right]$ in T_{m-1} , we have

$$\begin{aligned} \mathbb{E} \left[H^{\frac{p}{2}}(u_{\tau}(t)) \right] &\leq (e^{C\tau})^2 \mathbb{E} \left[H^{\frac{p}{2}}(u_{\tau}(t_{m-1})) \right] + C\tau(1 + e^{C\tau}) \leq \dots \\ &\leq (e^{C\tau})^{m+1} H^{\frac{p}{2}}(\xi) + C\tau \sum_{k=0}^m (e^{C\tau})^k \\ &\leq e^{CT} H^{\frac{p}{2}}(\xi) + C\tau \frac{e^{CT} - 1}{e^{C\tau} - 1} \leq e^{CT} (1 + H^{\frac{p}{2}}(\xi)). \end{aligned} \quad (16)$$

Therefore,

$$\sup_{t \in [0, T]} \mathbb{E} \left[\|u_{\tau}(t)\|_{\mathbb{H}^1}^p \right] \leq C. \quad (17)$$

By (10), we have that

$$\begin{aligned} \sup_{t \in [0, T]} H(u_{\tau}(t)) &\leq H(\xi) + \sup_{t \in [0, T]} \left| \int_0^t \langle \nabla u_{\tau}(r), \mathbf{i} u_{\tau}(r) dW(r) \rangle \right| \\ &\quad + \frac{1}{2} \sum_{k \in \mathbb{N}} \int_0^T \|u_{\tau}(r) \nabla (Q^{\frac{1}{2}} e_k)\|^2 dr. \end{aligned}$$

By the Burkholder–Davis–Gundy inequality and charge conservation law (9), we get

$$\left\| \sup_{t \in [0, T]} \left| \int_0^t \langle \nabla u_{\tau}(r), \mathbf{i} u_{\tau}(r) dW(r) \rangle \right| \right\|_{L^{\frac{p}{2}}(\Omega)} \leq C \sup_{t \in [0, T]} \|\nabla u_{\tau}(r)\|_{L^{\frac{p}{2}}(\Omega; L^2)},$$

which is bounded due to (17). This in turn implies (14) for $p \geq 4$. The estimation of (14) for $p \in [1, 4)$ follows by the Hölder inequality. \square

Similar arguments yields the boundedness of u_τ^D .

Corollary 2.2. *The auxiliary process u_τ^D is the right continuous and \mathcal{F}_t -adapted. Moreover, for any $p \geq 1$, there exists a constant $C = C(\xi, Q, T, p)$ such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u_\tau^D(t)\|_{\mathbb{H}^1}^p \right] \leq C. \quad (18)$$

Now we can show the \mathbb{H}^2 -a priori estimate of u_τ through the evolution of the Lyapunov functional

$$f(u) := \|\Delta u\|^2 + \lambda \langle \Delta u, |u|^2 u \rangle. \quad (19)$$

Proposition 2.2. *For any $p \geq 1$, there exists a constant $C = C(\xi, Q, T, p)$ such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u_\tau(t)\|_{\mathbb{H}^2}^p \right] \leq C. \quad (20)$$

Proof. Let $t \in T_m$ for some $m \in \mathbb{Z}_{M-1}$ and $p = 2$. By the same arguments in [10, Theorem 2.1] (see (2.5) and (2.8) in [10] with $Q = 0$), we have

$$\begin{aligned} & \mathbb{E} \left[f(u_\tau^D(t)) \right] - \mathbb{E} \left[f(u_\tau^D(t_m)) \right] \\ &= \int_{t_m}^t \mathbb{E} \left[\left\langle \Delta u_{\tau,m}^D(r), \mathbf{i} |u_{\tau,m}^D(r)|^4 u_{\tau,m}^D(r) \right\rangle \right] dr \\ & \quad + \lambda \int_{t_m}^t \mathbb{E} \left[\left\langle \Delta u_{\tau,m}^D(r), 4\mathbf{i} |\nabla u_{\tau,m}^D(r)|^2 u_{\tau,m}^D(r) + 2\mathbf{i} (\nabla u_{\tau,m}^D(r))^2 \overline{u_{\tau,m}^D(r)} \right\rangle \right] dr \\ & \leq C \left(1 + \sup_{t \in [0, T]} \mathbb{E} \left[\|u_\tau^D(t)\|_{\mathbb{H}^1}^{10} \right] \right) \tau + C \int_{t_m}^t \mathbb{E} \left[f(u_{\tau,m}^D(r)) \right] dr. \end{aligned}$$

Combined with the relationship between $H(u_\tau^D)$ and $\|u_\tau^D(t)\|_{\mathbb{H}^1}$, the a priori estimate (14) and the Gronwall inequality imply that

$$\mathbb{E} \left[f(u_{\tau,m}^D(t)) \right] \leq e^{C\tau} \mathbb{E} \left[f(u_{\tau,m}^D(t_m)) \right] + C\tau. \quad (21)$$

On the other hand, applying the Itô formula to $f(u_\tau)$, we obtain

$$\begin{aligned}
& f(u_\tau(t)) - f(u_\tau^D(t_{m+1})) \\
&= 2 \int_{t_m}^t \left\langle \Delta u_{\tau,m}^S, \Delta \left(-\mathbf{i} u_{\tau,m}^S dW(r) - \frac{1}{2} u_{\tau,m}^S F_Q dr \right) \right\rangle \\
&\quad + \lambda \int_{t_m}^t \left\langle \Delta u_{\tau,m}^S, |u_{\tau,m}^S|^2 \left(-\mathbf{i} u_{\tau,m}^S dW(r) - \frac{1}{2} u_{\tau,m}^S F_Q dr \right) \right\rangle \\
&\quad + \lambda \int_{t_m}^t \left\langle \Delta \left(-\mathbf{i} u_{\tau,m}^S dW(r) - \frac{1}{2} u_{\tau,m}^S F_Q dr \right), |u_{\tau,m}^S|^2 u_{\tau,m}^S \right\rangle \\
&\quad + 2\lambda \int_{t_m}^t \left\langle \Delta(-\mathbf{i} u_{\tau,m}^S Q^{\frac{1}{2}} e_k), -\mathbf{i} |u_{\tau,m}^S|^2 u_{\tau,m}^S Q^{\frac{1}{2}} e_k \right\rangle dr \\
&\quad + \lambda \int_{t_m}^t \left\langle \Delta u_{\tau,m}^S, -|u_{\tau,m}^S|^2 u_{\tau,m}^S F_Q \right\rangle dr + \int_{t_m}^t \sum_{k \in \mathbb{N}} \|\Delta(u_{\tau,m}^S Q^{\frac{1}{2}} e_k)\|^2 dr.
\end{aligned}$$

Taking expectation and using the Hölder and Gagliardo–Nirenberg inequalities and the \mathbb{H}^1 -a priori estimate (17), we obtain

$$\mathbb{E}[f(u_\tau(t))] \leq \mathbb{E}\left[f(u_\tau^D(t_{m+1}))\right] + C \int_{t_m}^t \left(1 + \mathbb{E}\left[f(u_{\tau,m}^S)\right]\right) dr. \quad (22)$$

Then the Gronwall inequality implies that

$$\mathbb{E}[f(u_\tau(t))] \leq e^{C\tau} \mathbb{E}\left[f(u_\tau^D(t_{m+1}))\right] + C\tau. \quad (23)$$

Similar iterative arguments to derive (16) applying to $\mathbb{E}[f(u_\tau^S(t))]$, combining with (23), yield that

$$\mathbb{E}[f(u_\tau(t))] \leq e^{CT} (1 + f(\xi)).$$

These estimations in turn show that

$$\sup_{t \in [0, T]} \mathbb{E} \left[\|u_\tau(t)\|_{\mathbb{H}^2}^2 \right] \leq C.$$

To derive (20) for $p \geq 4$, one only need to apply Itô formula to $f^{\frac{p}{2}}(u_\tau(t))$ and the Burkholder–Davis–Gundy inequality to the stochastic integral as in Lemma 2.1. The estimation of (20) for $p \in [1, 4)$ follows from the Hölder inequality. We omit the details here to avoid the tedious calculations. \square

Corollary 2.3. For any $p \geq 1$, there exists a constant $C = C(\xi, Q, T, p)$ such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u_\tau^D(t)\|_{\mathbb{H}^2}^p \right] \leq C. \quad (24)$$

Proof. The estimation (24) follows from (21) and iterative arguments. \square

2.2. Exponential integrability of splitting process

In this part we prove the \mathbb{H}^1 -exponential integrability for $u_\tau = \{u_\tau(t) : t \in [0, T]\}$. This property is the key ingredient to derive the strong error estimate between u_τ and u . We recall a useful exponential integrability lemma.

Lemma 2.1. Let \mathbb{H} be a separable Hilbert space, $U \in C^2(\mathbb{H}; \mathbb{R})$, \bar{U} be a measurable functional in \mathbb{H} and X be an \mathbb{H} -valued, adapted stochastic process with continuous sample paths satisfying $\int_0^T \|\mu(X_s)\| + \|\sigma(X_s)\|^2 dr < \infty$ a.s., and for all $t \in [0, T]$, $X_t = X_0 + \int_0^t \mu(X_s) dr + \int_0^t \sigma(X_s) dW_s$ a.s. Assume that there exists an \mathcal{F}_0 -measurable random variable $\alpha \in [0, \infty)$ such that a.s.

$$\begin{aligned} DU(X)\mu(X) + \frac{\text{tr}[D^2U(X)\sigma(X)\sigma^*(X)]}{2} \\ + \frac{\|\sigma^*(X)DU(X)\|^2}{2e^{\alpha t}} + \bar{U}(X) \leq \alpha U(X), \end{aligned}$$

then

$$\sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{U(X_t)}{e^{\alpha t}} + \int_0^t \frac{\bar{U}(X_r)}{e^{\alpha s}} dr \right) \right] \leq \mathbb{E} \left[e^{U(X_0)} \right].$$

Proof. See [10, Lemma 3.1] or [8, Corollary 2.4]. \square

Based on Lemma 2.1, we present the exponential integrability of u_τ . It should be mentioned that for SPDEs, the existence of the strong solution is not uncommon. However, we can use the finite-dimensional approximation and Fatou lemma to rigorously prove the following lemma for the mild solution, under the assumption that $Q \in \mathcal{L}_2^2$ and $\xi \in \mathbb{H}_0^1 \cap \mathbb{H}^2$.

Lemma 2.2. There exist constants C and α depending on ξ , Q and T such that

$$\sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{H(u_\tau(t))}{e^{\alpha t}} \right) \right] \leq C \quad (25)$$

and

$$\sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{\|\nabla u_\tau(t)\|^2}{e^{\alpha t}} \right) \right] \leq C. \quad (26)$$

Proof. We first prove (25). Since Eq. (2) possesses the energy conservation law, we focus on Eq. (3). For simplicity, we denote $\sigma(u) = -\mathbf{i}uQ^{\frac{1}{2}}$, $\mu(u) = -\frac{1}{2}uF_Q$ and omit the variable t in u_τ^S . Simple calculations yield that in T_m ,

$$\begin{aligned} DH(u_\tau)\mu(u_\tau) + \frac{\operatorname{tr}[\sigma(u_\tau)\sigma^*(u_\tau)D^2H(u_\tau)]}{2} + \frac{\|\sigma^*(u_\tau)DH(u_\tau)\|^2}{2e^{\alpha_\lambda(t-t_m)}} \\ = -\frac{1}{2}\langle \nabla u_\tau, u_\tau \nabla F_Q \rangle + \frac{\sum_{k \in \mathbb{N}} \langle \nabla u_\tau, \mathbf{i}u_\tau Q^{\frac{1}{2}} e_k \rangle^2}{2e^{\alpha_\lambda(t-t_m)}}. \end{aligned}$$

We conclude that

$$\begin{aligned} DH(u_\tau)\mu(u_\tau) + \frac{\operatorname{tr}[\sigma(u_\tau)\sigma^*(u_\tau)D^2H(u_\tau)]}{2} \\ + \frac{\|\sigma^*(u_\tau)DH(u_\tau)\|^2}{2e^{\alpha_\lambda(t-t_m)}} \leq \alpha_\lambda H(u_\tau) + \beta_\lambda \end{aligned}$$

with

$$\alpha_{-1} = 2\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 \|\xi\|^2, \quad \beta_{-1} = \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 \|\xi\|^2$$

and

$$\alpha_1 = 4\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 \|\xi\|^2, \quad \beta_1 = \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 (\|\xi\|^2 + \|\xi\|^8).$$

Applying Lemma 2.1 with $\bar{U} = -\beta_\lambda$ and the energy conservation law of u_τ^D in T_m , we obtain

$$\sup_{t \in T_m} \mathbb{E} \left[\exp \left(\frac{H(u_\tau(t))}{e^{\alpha_\lambda(t-t_m)}} \right) \right] \leq e^{\beta_\lambda(t-t_m)} \mathbb{E} \left[e^{H(u_\tau(t_m))} \right].$$

Similar arguments yield that

$$\mathbb{E} \left[\exp \left(\frac{H(u_\tau(t))e^{-\alpha_\lambda t_m}}{e^{\alpha_\lambda(t-t_m)}} \right) \right] \leq e^{\beta_\lambda(t-t_m)} \mathbb{E} \left[\exp(e^{-\alpha_\lambda t_m} H(u_\tau(t_m))) \right].$$

Repeating the previous procedure, we get

$$\sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{H(u_\tau(t))}{e^{\alpha_\lambda t}} \right) \right] \leq e^{\beta_\lambda t + H(\xi)},$$

which is (26). Using the relation between the energy functional $H(u)$ and $\|\nabla u\|^2$ (more details we refer to [10, Proposition 3.1] for the proof of (8)), we obtain (26). \square

Corollary 2.4. *There exist constants C and α depending on ξ , $Q^{\frac{1}{2}}$ and T such that*

$$\sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{\|\nabla u_{\tau}^D(t)\|^2}{e^{\alpha t}} \right) \right] \leq C \quad (27)$$

Proof. The estimate (27) follows immediately from the proof of Lemma 2.2. \square

2.3. Strong convergence rate of splitting process

Based on the \mathbb{H}^2 -a priori estimate in Proposition 2.2 and the \mathbb{H}^1 -exponential integrability in Lemma 2.2, we can estimate the strong error between u_{τ} and u .

Theorem 2.2. *For any $p \geq 1$, there exist a constant $C = C(\xi, Q, T, p)$ such that*

$$\left(\mathbb{E} \left[\sup_{m \in \mathbb{Z}_{M+1}} \|u(t_m) - u_{\tau}(t_m)\|^p \right] \right)^{\frac{1}{p}} \leq C \tau^{\frac{1}{2}}. \quad (28)$$

Denote by $e_{m+1} := u(t_{m+1}) - u_{\tau}(t_{m+1})$ in T_m , $m \in \mathbb{Z}_M$, the local error. Note that $u_{\tau}(t_{m+1}) = u_{\tau, m}^S(t_{m+1})$, $u_{\tau, m}^S(t_m) = u_{\tau, m}^D(t_{m+1}) = u_{\tau}(t_m) + \int_{t_m}^{t_{m+1}} \mathbf{i} \Delta u_{\tau, m}^D(r) + \mathbf{i} \lambda |u_{\tau, m}^D(r)|^2 u_{\tau, m}^D(r) dr$ and $e_m = u(t_m) - u_{\tau}^D(t_m)$. Then

$$u(t_m) - u_{\tau, m}^S(t_m) = e_m - \mathbf{i} \int_{t_m}^{t_{m+1}} \left[\Delta u_{\tau, m}^D + \lambda |u_{\tau, m}^D|^2 u_{\tau, m}^D \right] dr. \quad (29)$$

We need the following representations of the differences $u - u_{\tau, m}^D$ and $u - u_{\tau, m}^S$ in T_m .

Lemma 2.3. *For any $s \in T_m$ with $m \in \mathbb{Z}_M$, we have*

$$u(s) - u_{\tau, m}^D(s) = e_m + \int_{t_m}^s L_m^D dr - \mathbf{i} \int_{t_m}^s u dW(r) \quad (30)$$

and

$$\begin{aligned} u(s) - u_{\tau, m}^S(s) &= e_m + \int_{t_m}^s L_m^S dr - \mathbf{i} \int_{t_m}^{t_{m+1}} \left[\Delta u_{\tau, m}^D + \lambda |u_{\tau, m}^D|^2 u_{\tau, m}^D \right] dr \\ &\quad - \mathbf{i} \int_{t_m}^s \left[u - u_{\tau, m}^S \right] dW(r), \end{aligned} \quad (31)$$

where

$$L_m^D := i\Delta \left[u - u_{\tau,m}^D \right] + i\lambda(|u|^2 u - |u_{\tau,m}^D|^2 u_{\tau,m}^D) - \frac{1}{2} u F_Q, \quad (32)$$

$$L_m^S := i\Delta u + i\lambda|u|^2 u - \frac{1}{2}(u - u_{\tau,m}^S) F_Q. \quad (33)$$

Proof. Note that $u(s) - u_{\tau,m}^D(s) = [u(s) - u(t_m)] + e_m + [u_{\tau,m}^D(t_m) - u_{\tau,m}^D(s)]$. Combining (1) and (2), we get (30). Similarly, $u(s) - u_{\tau,m}^S(s) = [u(s) - u(t_m)] + e_m + [u_{\tau,m}^D(t_m) - u_{\tau}^D(t_{m+1})] + [u_{\tau,m}^S(t_m) - u_{\tau,m}^S(s)]$, which yields (31) by (1), (2) and (3). \square

We also need to estimate the following stochastic integrals:

$$\begin{aligned} S_1^m &:= \int_{t_m}^{t_{m+1}} \|W(s) - W(t_m)\|_{\mathbb{H}^1}^2 ds, \\ S_2^m &:= \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^s u(r) dW(r) \right\|_{\mathbb{H}^2}^2 ds, \\ S_3^m &:= \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^s \int_{t_m}^r L_m^S dr_1 dW(r) \right\| ds, \\ S_4^m &:= \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^s \int_{t_m}^r [u(r_1) - u^S(r_1)] dW(r_1) dW(r) \right\| ds. \end{aligned}$$

Lemma 2.4. For any $p \geq 1$ and $m \in \mathbb{Z}_M$, there exists a constant $C = C(\xi, Q, T, p)$ such that

$$\|S_j^m\|_{L^p(\Omega)} \leq C\tau^2, \quad j = 1, 2, 4; \quad \|S_3^m\|_{L^p(\Omega)} \leq C\tau^{\frac{5}{2}}.$$

Proof. Let $p \geq 2$. By the Minkowski and Burkholder–Davis–Gundy inequalities and the a priori estimate (6), we have

$$\begin{aligned} \|S_2^m\|_{L^p(\Omega)} &\leq \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^s u(r) dW(r) \right\|_{L^{2p}(\Omega; \mathbb{H}^2)}^2 ds \\ &\leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 \int_{t_m}^{t_{m+1}} \int_{t_m}^s \|u(r)\|_{L^{2p}(\Omega; \mathbb{H}^2)}^2 dr ds \leq C\tau^2. \end{aligned}$$

This in turn shows $\|S_1^m\|_{L^p(\Omega)} \leq C\tau^2$ only by substituting $u \equiv 1$.

Applying the Burkholder–Davis–Gundy inequality twice and the charge conservation law, we obtain

$$\begin{aligned}
& \|S_4^m\|_{L^p(\Omega)} \\
& \leq C \int_{t_m}^{t_{m+1}} \left[\int_{t_m}^s \sum_{k \in \mathbb{N}} \left\| \int_{t_m}^r (u(r_1) - u_\tau^S(r_1)) dW(r_1) Q^{\frac{1}{2}} e_k \right\|_{L^{\frac{p}{2}}(\Omega; L^2)}^2 dr \right]^{\frac{1}{2}} ds \\
& \leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2} \int_{t_m}^{t_{m+1}} \left[\int_{t_m}^s \left\| \int_{t_m}^r (u(r_1) - u_\tau^S(r_1)) dW(r_1) \right\|_{L^p(\Omega; L^2)}^2 dr \right]^{\frac{1}{2}} ds \\
& \leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 \int_{t_m}^{t_{m+1}} \left[\int_{t_m}^s \int_{t_m}^r \|u(r_1) - u_\tau^S(r_1)\|_{L^p(\Omega; L^2)}^2 dr_1 dr \right]^{\frac{1}{2}} ds \leq C \tau^2.
\end{aligned}$$

Similar arguments yield that $\|S_3^m\|_{L^p(\Omega)} \leq C \tau^{\frac{5}{2}}$ and $\|S_4^m\|_{L^p(\Omega)} \leq C \tau^{\frac{5}{2}}$ for $p \geq 2$. We complete the proof for $p \in [1, 2)$ by the Hölder inequality. \square

Lemma 2.5. For any $p \geq 1$, there exists a constant $C = C(\xi, Q, T)$ such that

$$\left\| \exp \left(2 \int_0^T \|u(s)\|_{L^\infty} \|u_\tau^D(s)\|_{L^\infty} ds \right) \right\|_{L^p(\Omega)} \leq C \quad (34)$$

Proof. By the Cauchy–Schwarz, Gagliardo–Nirenberg, Young, Jensen and Minkowski inequalities, we get

$$\begin{aligned}
& \left\| \exp \left(2 \int_0^T \|u(s)\|_{L^\infty} \|u_\tau(s)\|_{L^\infty} ds \right) \right\|_{L^p(\Omega)} \\
& \leq \left\| \exp \left(\int_0^T 2 \|\xi\| \|\nabla u\| ds \right) \right\|_{L^{2p}(\Omega)} \left\| \exp \left(\int_0^T 2 \|\xi\| \|\nabla u_\tau^D\| ds \right) \right\|_{L^{2p}(\Omega)} \\
& \leq \exp \left(\int_0^T 2pT e^{\alpha T} \|\xi\|^2 ds \right) \left\| \exp \left(\int_0^T \frac{\|\nabla u\|^2}{2pT e^{\alpha T}} ds \right) \right\|_{L^{2p}(\Omega)} \\
& \quad \times \exp \left(\int_0^T 2pT e^{\alpha T} \|\xi\|^2 ds \right) \left\| \exp \left(\int_0^T \frac{\|\nabla u_\tau^D\|^2}{2pT e^{\alpha T}} ds \right) \right\|_{L^{2p}(\Omega)} \\
& \leq C \left(\sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{\|\nabla u(t)\|^2}{e^{\alpha t}} \right) \right] \cdot \sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{\|\nabla u_\tau^D(t)\|^2}{e^{\alpha t}} \right) \right] \right)^{\frac{1}{2p}},
\end{aligned}$$

where α is presented in (8) or (25). From the above estimations we obtain (34) combining with Theorem 2.1 and Corollary 2.4. \square

Our last preliminary result is the following discrete Gronwall inequality (see [25, Lemma 1.4.2]).

Lemma 2.6. *Let $m \in \mathbb{N}$ and $\{p_j\}_{j \in \mathbb{N}}$, $\{k_j\}_{j \in \mathbb{N}}$ are nonnegative number sequences. Assume that the sequence $\{\phi_j\}_{j \in \mathbb{N}}$ satisfies*

$$\phi_{m+1} \leq \phi_0 + \sum_{j=0}^m p_j + \sum_{j=0}^m k_j \phi_j,$$

then

$$\phi_{m+1} \leq \left(\phi_0 + \sum_{j=0}^m p_j \right) \exp \left(\sum_{j=0}^m k_j \right).$$

Proof of Theorem 2.2. By Itô formula and (29), we have

$$\begin{aligned} \|e_{m+1}\|^2 &= \left\| e_m - \int_{t_m}^{t_{m+1}} \mathbf{i} \left[\Delta u_{\tau,m}^D + \lambda |u_{\tau,m}^D|^2 u_{\tau,m}^D \right] dr \right\|^2 \\ &\quad + 2 \int_{t_m}^{t_{m+1}} \left\langle u - u_{\tau,m}^S, \mathbf{i} \left[\Delta u + \lambda |u|^2 u \right] \right\rangle ds := I_1^m + I_2^m. \end{aligned}$$

The first item I_1^m has the estimation by the Gagliardo–Nirenberg inequality:

$$\begin{aligned} I_1^m &= \|e_m\|^2 - 2 \left\langle e_m, \int_{t_m}^{t_{m+1}} \mathbf{i} \left[\Delta u_{\tau,m}^D + \lambda |u_{\tau,m}^D|^2 u_{\tau,m}^D \right] ds \right\rangle \\ &\quad + \left\| \int_{t_m}^{t_{m+1}} \left[\Delta u_{\tau,m}^D + \lambda |u_{\tau,m}^D|^2 u_{\tau,m}^D \right] ds \right\|^2 \\ &\leq \|e_m\|^2 - 2 \left\langle e_m, \int_{t_m}^{t_{m+1}} \mathbf{i} \left[\Delta u_{\tau,m}^D + \lambda |u_{\tau,m}^D|^2 u_{\tau,m}^D \right] ds \right\rangle \\ &\quad + C \tau^2 \left(1 + \sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^2 \right). \end{aligned}$$

Substituting (31) into $u - u_{\tau,m}^S$, we divide I_2^m into

$$\begin{aligned}
I_2^m &= 2 \int_{t_m}^{t_{m+1}} \left\langle \int_{t_m}^s L_m^S(r) dr, \mathbf{i}(\Delta u(s) + \lambda |u(s)|^2 u(s)) \right\rangle ds \\
&\quad + 2 \int_{t_m}^{t_{m+1}} \left\langle e_m, \mathbf{i}(\Delta u(s) + \lambda |u(s)|^2 u(s)) \right\rangle ds \\
&\quad - 2 \left\langle \int_{t_m}^{t_{m+1}} \Delta u_{\tau,m}^D(r) + \lambda |u_{\tau,m}^D(r)|^2 u_{\tau,m}^D(r) dr, \right. \\
&\quad \quad \left. \int_{t_m}^{t_{m+1}} \Delta u(s) + \lambda |u(s)|^2 u(s) ds \right\rangle \\
&\quad - 2 \int_{t_m}^{t_{m+1}} \left\langle \int_{t_m}^s (u(r) - u_{\tau,m}^S(r)) dW(r), \Delta u(s) + \lambda |u(s)|^2 u(s) \right\rangle ds \\
&:= I_{21}^m + I_{22}^m + I_{23}^m + I_{24}^m.
\end{aligned}$$

By the Hölder and Gagliardo–Nirenberg inequalities, we get

$$\begin{aligned}
I_{21}^m &\leq C\tau^2 \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 + \sup_{t \in T_m} \|u_\tau(t)\|_{\mathbb{H}^2}^2 \right), \\
I_{23}^m &\leq C\tau^2 \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 + \sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^2 \right).
\end{aligned}$$

Substituting (31) into I_{24}^m , we have

$$\begin{aligned}
I_{24}^m &= -2 \int_{t_m}^{t_{m+1}} \left\langle e_m [W(s) - W(t_m)], \Delta u + \lambda |u|^2 u \right\rangle ds \\
&\quad - 2 \int_{t_m}^{t_{m+1}} \left\langle \int_{t_m}^s \int_{t_m}^r L_m^S(u(r_1)) dr_1 dW(r), \Delta u + \lambda |u|^2 u \right\rangle ds \\
&\quad + 2 \left\langle \mathbf{i} \int_{t_m}^{t_{m+1}} \left[\Delta u_{\tau,m}^D + \lambda |u_{\tau,m}^D|^2 u_{\tau,m}^D \right] dr_1, \right. \\
&\quad \quad \left. \int_{t_m}^{t_{m+1}} \left[\Delta u + \lambda |u|^2 u \right] [W(s) - W(t_m)] ds \right\rangle
\end{aligned}$$

$$\begin{aligned}
 & + 2 \int_{t_m}^{t_{m+1}} \left\langle \int_{t_m}^s \int_{t_m}^r \mathbf{i} [u - u_{\tau,m}^S] dW(r_1) dW(r), \Delta u + \lambda |u|^2 u \right\rangle ds \\
 & := I_{241}^m + I_{242}^m + I_{243}^m + I_{244}^m.
 \end{aligned}$$

The Cauchy–Schwarz and Gagliardo–Nirenberg inequalities and the charge conservation law for Eq. (1) yield that

$$I_{241}^m \leq \tau \|e_m\|^2 + C \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 \right) S_1^m.$$

For other terms, we have

$$I_{243}^m \leq C \tau^2 \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 \right) + C \tau \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 \right) S_1^m$$

and

$$I_{242}^m + I_{244}^m \leq C \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2} \right) (S_3^m + S_4^m).$$

Then

$$I_{24}^m \leq \tau \|e_m\|^2 + C \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 \right) (\tau^2 + S_1^m + S_3^m + S_4^m).$$

Summing up I_1^m and I_2^m and using the integrating by parts, we deduce that

$$\begin{aligned}
 & \|e_{m+1}\|^2 - \|e_m\|^2 \\
 & \leq \tau \|e_m\|^2 + 2 \int_{t_m}^{t_{m+1}} \left\langle \Delta e_m, \mathbf{i} [u - u_{\tau,m}^D] \right\rangle ds \\
 & \quad + 2\lambda \int_{t_m}^{t_{m+1}} \left\langle e_m, \mathbf{i} [|u|^2 u - |u_{\tau,m}^D|^2 u_{\tau,m}^D] \right\rangle ds \\
 & \quad + C \tau^2 \left(1 + \sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^2 + \sup_{t \in T_m} \|u_{\tau,m}(t)\|_{\mathbb{H}^2}^2 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 \right) \\
 & \quad + C \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 \right) (S_1^m + S_3^m + S_4^m).
 \end{aligned}$$

Denote by

$$J_1^m := 2 \int_{t_m}^{t_{m+1}} \left\langle \Delta e_m, \mathbf{i} \left[u - u_{\tau,m}^D \right] \right\rangle ds,$$

$$J_2^m := 2\lambda \int_{t_m}^{t_{m+1}} \left\langle e_m, \mathbf{i} \left[|u|^2 u - |u_{\tau,m}^D|^2 u_{\tau,m}^D \right] \right\rangle ds.$$

Substituting (30) into J_1^m , by the Hölder inequality and the integration by parts, we have

$$\begin{aligned} J_1^m &= 2 \int_{t_m}^{t_{m+1}} \left\langle \Delta e_m, \mathbf{i} \int_{t_m}^s L_m^D(r) dr \right\rangle ds \\ &\quad + 2 \int_{t_m}^{t_{m+1}} \left\langle e_m, \Delta \left(\int_{t_m}^s u(r) dW(r) \right) \right\rangle ds \\ &\leq \tau \|e_m\|^2 + C\tau^2 \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 + \sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^2 \right) + CS_2^m. \end{aligned}$$

For term J_2^m , using the cubic difference formula $|a|^2 a - |b|^2 b = (|a|^2 + |b|^2)(a - b) + ab(\bar{a} - \bar{b})$ and (30) in Lemma 2.3, we obtain

$$\begin{aligned} J_2^m &= 2\lambda \int_{t_m}^{t_{m+1}} \left\langle e_m, \mathbf{i} \left(|u|^2 + |u_{\tau,m}^D|^2 \right) \left[\int_{t_m}^s L^D(r) dr - \mathbf{i} \int_{t_m}^s u(r) dW(r) \right] \right\rangle ds \\ &\quad + 2\lambda \int_{t_m}^{t_{m+1}} \left\langle e_m, \mathbf{i} u u_{\tau,m}^D \left[\bar{e}_m + \int_{t_m}^s \overline{L_m^D}(r) dr + \mathbf{i} \int_{t_m}^s \bar{u}(r) dW(r) \right] \right\rangle ds. \end{aligned}$$

The Cauchy–Schwarz and Gagliardo–Nirenberg inequalities imply

$$\begin{aligned} J_2^m &\leq \left(2\tau + 2 \int_{t_m}^{t_{m+1}} \|u\|_{L^\infty} \|u_{\tau,m}^D\|_{L^\infty} ds \right) \|e_m\|^2 + C\tau^3 \left(\sup_{t \in T_m} \|u(t)\|_{L^\infty}^4 \right. \\ &\quad \left. + \sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{L^\infty}^4 \right) \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^2 + \sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^2 + S_2^m \right). \end{aligned}$$

Therefore, we obtain

$$\|e_{m+1}\|^2 \leq \|e_m\|^2 + \left(4\tau + 2 \int_{t_m}^{t_{m+1}} \|u(s)\|_{L^\infty} \|u_{\tau,m}^D(s)\|_{L^\infty} ds \right) \|e_m\|^2$$

$$\begin{aligned}
& + C \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^6 + \sup_{t \in T_m} \|u_{\tau,m}(t)\|_{\mathbb{H}^2}^6 + \sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^6 \right) \\
& \times \left(\tau^2 + \sum_{j=1}^4 S_j^m \right).
\end{aligned}$$

Set

$$\begin{aligned}
k_m &:= 4\tau + 2 \int_{t_m}^{t_{m+1}} \|u(s)\|_{L_\infty} \|u_{\tau,m}^D(s)\|_{L_\infty} ds, \\
p_m &:= C \left(1 + \sup_{t \in T_m} \|u(t)\|_{\mathbb{H}^2}^6 + \sup_{t \in T_m} \|u_{\tau,m}(t)\|_{\mathbb{H}^2}^6 + \sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^6 \right) \\
& \times \left(\tau^2 + \sum_{j=1}^4 S_j^m \right).
\end{aligned}$$

Then

$$\|e_{m+1}\|^2 \leq \|e_m\|^2 + k_m \|e_m\|^2 + p_m \leq \cdots \leq \|e_0\|^2 + \sum_{n=0}^m k_n \|e_n\|^2 + \sum_{n=0}^m p_n.$$

Applying Lemma 2.6, we have

$$\begin{aligned}
& \|e_{m+1}\|^2 \\
& \leq C \exp \left(4T + 2 \int_0^{t_{m+1}} \|u(t)\|_{L_\infty} \|u_{\tau,m}^D(t)\|_{L_\infty} dt \right) \times \left(\tau + \sum_{n=0}^m \left[\sum_{j=1}^4 S_j^n \right] \right) \\
& \times \left(1 + \sup_{t \in [0,T]} \|u(t)\|_{\mathbb{H}^2}^6 + \sup_{t \in [0,T]} \|u_\tau(t)\|_{\mathbb{H}^2}^6 + \sup_{t \in [0,T]} \|u_\tau^D(t)\|_{\mathbb{H}^2}^6 \right).
\end{aligned}$$

Then taking $\frac{p}{2}$ -moments on both sides and using the Hölder inequality, we obtain

$$\begin{aligned}
& \mathbb{E} \left[\sup_{m \in \mathbb{Z}_{M+1}} \|e_{m+1}\|^p \right] \\
& \leq C \left\| \exp \left(2 \int_0^T \|u(t)\|_{L_\infty} \|u_\tau^D(t)\|_{L_\infty} dt \right) \right\|_{L^p(\Omega)}^{\frac{p}{2}} \left\| \tau + \sum_{n=0}^M \sum_{j=1}^4 S_j^n \right\|_{L^{2p}(\Omega)}^{\frac{p}{2}} \\
& \times \left(1 + \left(\mathbb{E} \left[\sup_{t \in [0,T]} \|u(t)\|_{\mathbb{H}^2}^{12p} \right] \right)^{\frac{1}{4}} + \left(\mathbb{E} \left[\sup_{t \in [0,T]} \|u_\tau^D(t)\|_{\mathbb{H}^2}^{12p} \right] \right)^{\frac{1}{4}} \right).
\end{aligned}$$

Now the exponential moments' estimation (34) in Lemma 2.5 and \mathbb{H}^2 -a priori estimations (6) and (20) imply that

$$\mathbb{E} \left[\sup_{m \in \mathbb{Z}_{M+1}} \|e_{m+1}\|^p \right] \leq C \left(\tau + \sum_{n=0}^M \sum_{j=1}^4 \|S_j^n\|_{L^{2p}(\Omega)} \right)^{\frac{p}{2}}.$$

Taking the p -th root of the above inequality, we obtain

$$\left(\mathbb{E} \left[\sup_{m \in \mathbb{Z}_{M+1}} \|u(t_m) - u_\tau(t_m)\|^p \right] \right)^{\frac{1}{p}} \leq C \tau^{\frac{1}{2}},$$

which completes the proof of (28) by Lemma 2.4.

Remark 2.1. From the proof, we can also obtain the following error estimate between the splitting process u_τ and u ,

$$\left(\mathbb{E} \left[\sup_{t \in [0, T]} \|u(t) - u_\tau(t)\|^p \right] \right)^{\frac{1}{p}} \leq C \tau^{\frac{1}{2}}.$$

3. Splitting Crank–Nicolson scheme

For the nonlinear case, the splitting process u_τ defined by (5) is not a proper numerical method since Eq. (2) does not possess a analytic solution. To obtain a temporal discretization, we use the modified Crank–Nicolson scheme to temporally discretize $u_\tau = \{u_\tau(t) : t \in [0, T]\}$ and get the following splitting Crank–Nicolson scheme starting from ξ :

$$\begin{cases} u_{m+1}^D = u_m + i\tau \Delta u_{m+\frac{1}{2}}^D + i\lambda\tau \frac{|u_m|^2 + |u_{m+1}^D|^2}{2} u_{m+\frac{1}{2}}^D, \\ u_{m+1} = \exp(-i(W_{t_{m+1}} - W_{t_m})) u_{m+1}^D, \quad m \in \mathbb{Z}_{M-1}, \end{cases} \quad (35)$$

where $u_{m+\frac{1}{2}}^D = \frac{1}{2}(u_m + u_{m+1}^D)$. It is not difficult to show that $\|u_m\| = \|u_{m+1}^D\|$ and $H(u_m) = H(u_{m+1}^D)$, $m \in \mathbb{Z}_{M-1}$. Moreover, the splitting Crank–Nicolson scheme (35) preserves the discrete charge a.s., i.e.,

$$\|u_m\|^2 = \|\xi\|^2, \quad m \in \mathbb{Z}_M. \quad (36)$$

Similarly to u_τ , u_m has a local continuous extension u_m^S in T_m , $m \in \mathbb{Z}_{M-1}$, through the stochastic flow of Eq. (3).

Our main goal in this section is to estimate the strong convergence rate of the splitting Crank–Nicolson scheme (35). At first, we show the exponential integrability of $\{u_m\}_{m \in \mathbb{Z}_M}$, which implies the \mathbb{H}^1 -a priori estimate.

Lemma 3.1. *There exist constants C and α depending on ξ , Q and T such that*

$$\sup_{m \in \mathbb{Z}_M} \mathbb{E} \left[\exp \left(\frac{H(u_m)}{e^{\alpha t_m}} \right) \right] \leq C \quad (37)$$

and

$$\sup_{m \in \mathbb{Z}_M} \mathbb{E} \left[\exp \left(\frac{\|\nabla u_m\|^2}{e^{\alpha t_m}} \right) \right] \leq C. \quad (38)$$

Proof. The proof is similar to that of Lemma 2.2 and we omit the details. \square

Corollary 3.1. *For any $p \geq 1$, there exist a constant $C = C(\xi, Q, T, p)$ such that*

$$\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|u_m\|_{\mathbb{H}^1}^p \right] \leq C. \quad (39)$$

Proof. By Lemma 3.1, we have

$$\sup_{m \in \mathbb{Z}_M} \mathbb{E} \left[\|u_m\|_{\mathbb{H}^1}^p \right] \leq C.$$

The above inequality in turn yields (39) by similar arguments in Lemma 2.1. \square

Our next technical requirement is the uniform \mathbb{H}^2 -a priori estimate of u_m , $m \in \mathbb{Z}_M$. We need the following useful result. For convenience, we recall that

$$u_{m+1}^D - u_m = \mathbf{i} \tau \Delta u_{m+\frac{1}{2}}^D + \mathbf{i} \lambda \tau \frac{|u_m|^2 + |u_{m+1}^D|^2}{2} u_{m+\frac{1}{2}}^D, \quad m \in \mathbb{Z}_M. \quad (40)$$

Lemma 3.2. *Assume that $u_m, u_{m+1}^D \in \mathbb{H}_0^1 \cap \mathbb{H}^2$ for some $m \in \mathbb{Z}_{M-1}$. There exists a constant $C = C(\xi, Q)$ such that*

$$\|u_{m+1}^D - u_m\|^2 \leq C \tau \left(\|\nabla u_{m+1}^D\|^2 + \|\nabla u_m\|^2 \right),$$

$$\|u_{m+1}^D - u_m\| \leq C \tau (\|\nabla u_{m+1}^D\| + \|\nabla u_m\|) + \frac{\tau}{2} (\|\Delta u_m\| + \|\Delta u_{m+1}^D\|),$$

$$\|\nabla u_{m+1}^D - \nabla u_m\|^2 \leq C \tau \left(\|\nabla u_{m+1}^D\|^4 + \|\nabla u_m\|^4 \right) + \frac{\tau}{2} (\|\Delta u_m\|^2 + \|\Delta u_{m+1}^D\|^2).$$

Proof. Taking inner product with $u_{m+1}^D - u_m$ on (40) and using the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} \|u_{m+1}^D - u_m\|^2 &= \tau \left\langle u_{m+1}^D - u_m, \mathbf{i} \Delta u_{m+\frac{1}{2}}^D \right\rangle \\ &\quad + \lambda \tau \left\langle u_{m+1}^D - u_m, \mathbf{i} \frac{|u_{m+1}^D|^2 + |u_m|^2}{2} u_{m+\frac{1}{2}}^D \right\rangle \end{aligned}$$

$$\leq C\tau \left(\|\nabla u_{m+1}^D\|^2 + \|\nabla u_m\|^2 \right).$$

Taking L^2 -norm on both sides of (40), we get

$$\|u_{m+1}^D - u_m\| \leq \frac{\tau}{2} (\|\Delta u_m\| + \|\Delta u_{m+1}^D\|) + C\tau (\|\nabla u_{m+1}^D\| + \|\nabla u_m\|).$$

Taking inner product with $\Delta(u_{m+1}^D - u_m)$ on (40), using the integrating by parts and the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} & \|\nabla u_{m+1}^D - \nabla u_m\|^2 \\ &= \left\langle \Delta u_{m+1}^D - \Delta u_m, i\tau \Delta u_{m+\frac{1}{2}}^D + i\tau \frac{|u_{m+1}^D|^2 + |u_m|^2}{2} u_{m+\frac{1}{2}}^D \right\rangle \\ &\leq \frac{\tau}{2} (\|\Delta u_{m+1}^D\|^2 + \|\Delta u_m\|^2) + C\tau (\|\nabla u_{m+1}^D\|^4 + \|\nabla u_m\|^4), \end{aligned}$$

which completes the proof. \square

Lemma 3.3. For any $p \geq 1$, there exists a constant $C = C(\xi, Q, T, p)$ such that

$$\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|u_m\|_{\mathbb{H}^2}^p \right] \leq C. \quad (41)$$

Proof. In terms of the \mathbb{H}^1 -a priori estimate (39) in Corollary 3.1, we focus on the a priori estimate in \mathbb{H}^2 . Let $m \in \mathbb{Z}_M$. Taking complex inner product on both sides of (40) with $\Delta(u_{m+1}^D - u_m)$, and then taking the imaginary part, we obtain

$$\begin{aligned} & \|\Delta u_{m+1}^D\|^2 - \|\Delta u_m\|^2 \\ &= -\frac{\lambda}{2} \left\langle \Delta(u_{m+1}^D - u_m), (|u_m|^2 + |u_{m+1}^D|^2)(u_m + u_{m+1}^D) \right\rangle. \end{aligned}$$

By the identity $(|a|^2 + |b|^2)(a + b) = 2|a|^2a + 2|b|^2b - (|b|^2 - |a|^2)(b - a)$, $a, b \in \mathbb{C}$ we have

$$\begin{aligned} & \left\langle \Delta(u_{m+1}^D - u_m), (|u_m|^2 + |u_{m+1}^D|^2)(u_m + u_{m+1}^D) \right\rangle \\ &= 2 \left\langle \Delta u_{m+1}^D, |u_{m+1}^D|^2 u_{m+1}^D \right\rangle - 2 \left\langle \Delta u_m, |u_m|^2 u_m \right\rangle \\ &\quad + \frac{1}{2} \left\langle \Delta u_{m+1}^D - \Delta u_m, (|u_{m+1}^D|^2 - |u_m|^2)(u_m - u_{m+1}^D) \right\rangle \\ &\quad - \left\langle \Delta u_{m+1}^D, (|u_{m+1}^D|^2 u_{m+1}^D - |u_m|^2 u_m) \right\rangle \\ &\quad + \left\langle \Delta u_{m+1}^D - \Delta u_m, |u_{m+1}^D|^2 u_{m+1}^D \right\rangle. \end{aligned}$$

Recalling the definition (19) of the Lyapunov functional f , we have

$$\begin{aligned}
& f(u_{m+1}^D) - f(u_m) \\
&= \|\Delta u_{m+1}^D\|^2 - \|\Delta u_m\|^2 \\
&\quad + \lambda \left[\langle \Delta u_{m+1}^D, |u_{m+1}^D|^2 u_{m+1}^D \rangle - \langle \Delta u_m, |u_m|^2 u_m \rangle \right] \\
&= \lambda \left\langle \Delta u_{m+1}^D, \left(|u_{m+1}^D|^2 u_{m+1}^D - |u_m|^2 u_m \right) \right\rangle \\
&\quad - \lambda \left\langle \Delta (u_{m+1}^D - u_m), |u_{m+1}^D|^2 u_{m+1}^D \right\rangle \\
&\quad + \frac{\lambda}{2} \left\langle \Delta (u_{m+1}^D - u_m), \left(|u_{m+1}^D|^2 - |u_m|^2 \right) (u_m - u_{m+1}^D) \right\rangle.
\end{aligned}$$

By the cubic difference formula, we obtain

$$\begin{aligned}
& \left\langle \Delta u_{m+1}^D, \left(|u_{m+1}^D|^2 u_{m+1}^D - |u_m|^2 u_m \right) \right\rangle \\
&= \left\langle \Delta u_{m+1}^D, \left(|u_{m+1}^D|^2 + |u_m|^2 \right) (u_{m+1}^D - u_m) \right\rangle \\
&\quad + \left\langle \Delta u_{m+1}^D, u_{m+1}^D u_m (\overline{u_{m+1}^D} - \overline{u_m}) \right\rangle \\
&= \left\langle \Delta u_{m+1}^D, \left(|u_{m+1}^D|^2 + |u_m|^2 \right) (u_{m+1}^D - u_m) \right\rangle \\
&\quad - \left\langle \Delta u_{m+1}^D, u_{m+1}^D |u_{m+1}^D - u_m|^2 \right\rangle \\
&\quad + \left\langle \Delta u_{m+1}^D, (u_{m+1}^D)^2 (\overline{u_{m+1}^D} - \overline{u_m}) \right\rangle.
\end{aligned}$$

On the other hand, the integration by parts and the equality $\Delta(|u|^2 u) = 2|u|^2 \Delta u + 4|\nabla u|^2 u + 2(\nabla u)^2 \bar{u} + (u)^2 \Delta \bar{u}$ yield that

$$\begin{aligned}
& \left\langle \Delta u_{m+1}^D - \Delta u_m, |u_{m+1}^D|^2 u_{m+1}^D \right\rangle \\
&= \left\langle u_{m+1}^D - u_m, \left(2|u_{m+1}^D|^2 \Delta u_{m+1}^D + \Delta \overline{u_{m+1}^D} (u_{m+1}^D)^2 \right) \right\rangle \\
&\quad + \left\langle u_{m+1}^D - u_m, \left(4|\nabla u_{m+1}^D|^2 u_{m+1}^D + 2(\nabla \overline{u_{m+1}^D})^2 u_{m+1}^D \right) \right\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& f(u_{m+1}^D) - f(u_m) \\
&= \lambda \left\langle \Delta u_{m+1}^D, \left(-|u_{m+1}^D|^2 + |u_m|^2 \right) (u_{m+1}^D - u_m) \right\rangle \\
&\quad - \lambda \left\langle \Delta u_{m+1}^D, u_{m+1}^D |u_{m+1}^D - u_m|^2 \right\rangle \\
&\quad - \lambda \left\langle u_{m+1}^D - u_m, 4 \left(|\nabla u_{m+1}^D|^2 u_{m+1}^D + 2(\nabla \overline{u_{m+1}^D})^2 u_{m+1}^D \right) \right\rangle \\
&\quad - \frac{\lambda}{2} \left\langle \Delta (u_{m+1}^D - u_m), \left(|u_{m+1}^D|^2 - |u_m|^2 \right) (u_m - u_{m+1}^D) \right\rangle
\end{aligned}$$

$$:= II_1^m + II_2^m + II_3^m + II_4^m.$$

By the Hölder, Young and Gagliardo–Nirenberg inequalities, we obtain

$$\begin{aligned} II_1^m &\leq \|u_{m+1}^D - u_m\|_{L^4}^2 \|\Delta u_{m+1}^D\| (\|u_{m+1}^D\|_{L^\infty} + \|u_m\|_{L^\infty}) \\ &\leq \frac{\tau}{16} \|\Delta u_{m+1}^D\|^2 + \frac{C}{\tau} \|\nabla u_{m+1}^D - \nabla u_m\| \|u_{m+1}^D - u_m\|^3 \\ &\quad \times (\|u_{m+1}^D\|_{L^\infty}^2 + \|u_m\|_{L^\infty}^2) \\ &\leq \frac{\tau}{16} \|\Delta u_{m+1}^D\|^2 + \frac{1}{4} \|\nabla u_{m+1}^D - \nabla u_m\|^2 \\ &\quad + C\tau \left(\|\nabla u_{m+1}^D\|^4 + \|\nabla u_m\|^4 \right) + \frac{1}{\tau^5} \|u_{m+1}^D - u_m\|^{12}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} II_2^m + II_4^m &\leq \frac{\tau}{16} (\|\Delta u_{m+1}^D\|^2 + \|\Delta u_m\|^2) + \frac{1}{4} \|\nabla u_{m+1}^D - \nabla u_m\|^2 \\ &\quad + C\tau \left(\|\nabla u_{m+1}^D\|^4 + \|\nabla u_m\|^4 \right) + \frac{1}{\tau^5} \|u_{m+1}^D - u_m\|^{12}. \end{aligned}$$

Lemma 3.2 and the charge and energy conservation laws of (40) yield that

$$\begin{aligned} II_3^m &\leq C \|u_{m+1}^D - u_m\| \|\nabla u_{m+1}^D\|^2 \|u_{m+1}^D\|_{L^\infty} \\ &\leq C\tau \|\nabla u_{m+1}^D\|^{\frac{5}{2}} \left(\|\Delta u_{m+\frac{1}{2}}^D\| + \|\nabla u_{m+1}^D\| + \|\nabla u_m\| \right) \\ &\leq \frac{\tau}{2} \|\Delta u_{m+\frac{1}{2}}^D\|^2 + C\tau (\|\nabla u_{m+1}^D\|^5 + \|\nabla u_{m+1}^D\|^{\frac{7}{2}} + \|\nabla u_{m+1}^D\|^{\frac{5}{2}} \|\nabla u_m\|). \end{aligned}$$

Again by Lemma 3.2, we conclude that

$$\begin{aligned} f(u_{m+1}^D) &\leq f(u_m) + \frac{1}{2} \tau (\|\Delta u_m\|^2 + \|\Delta u_{m+1}^D\|^2) \\ &\quad + C\tau \left(1 + \|\nabla u_{m+1}^D\|^{12} + \|\nabla u_m\|^{12} \right). \end{aligned}$$

By (19), the integrating by parts and Sobolev inequality, we achieve

$$f(u_{m+1}^D) \leq \frac{1 + \frac{\tau}{2}}{1 - \frac{\tau}{2}} f(u_m) + \frac{C\tau}{1 - \frac{\tau}{2}} \left(1 + \|\nabla u_{m+1}^D\|^{12} + \|\nabla u_m\|^{12} \right).$$

The fact $\tau < 1$ implies that

$$f(u_{m+1}^D) \leq (1 + 2\tau) f(u_m) + C\tau \left(1 + \|\nabla u_{m+1}^D\|^{12} + \|\nabla u_m\|^{12} \right).$$

Notice that u_m can be extend to a continuous process $u_m^S(t)$ as the solution of Eq. (3) with initial data u_{m+1}^D in T_m . The same arguments to derive (22) lead to

$$\mathbb{E} \left[f(u_m^S(t)) \right] - \mathbb{E} \left[f(u_{m+1}^D) \right] \leq C\tau + C \int_{t_m}^t \mathbb{E} \left[f(u_m^S(r)) \right] dr, \quad t \in T_m.$$

The Gronwall inequality and the above two inequalities imply that

$$\mathbb{E} \left[f(u_{m+1}) \right] \leq e^{C\tau} ((1 + 2\tau)\mathbb{E} [f(u_m)] + C\tau).$$

Then Lemma 2.6 yields that

$$\mathbb{E} \left[f(u_{m+1}) \right] \leq Ce^{CT} (1 + f(\xi)),$$

which in turn leads to

$$\sup_{m \in \mathbb{Z}_M} \mathbb{E} \left[\|\Delta u_m\|^2 \right] \leq C.$$

Similar arguments to derive (20) in Proposition 2.2 complete the proof of (41). \square

Corollary 3.2. *For any $p \geq 1$, there exist a constant $C = C(\xi, Q, T, p)$ such that*

$$\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M \setminus \{0\}} \|u_m^D\|_{\mathbb{H}^2}^p \right] \leq C. \quad (42)$$

Proof. The proof of (42) follows immediately from the proof of (41). \square

Based on Lemmas 3.1 and 3.3, in the rest of this section we show that the splitting Crank–Nicolson scheme (35) possesses strong convergence rate $\mathcal{O}(\tau^{\frac{1}{2}})$. To the best of our knowledge, this is the first result about strong convergence rate of temporal discretization for SPDEs with non-monotone coefficients.

We begin with the following lemma which is a discrete version of Lemma 2.5.

Lemma 3.4. *For any $p \geq 1$, there exist a constant $C = C(\xi, Q, T, p)$ such that*

$$\left\| \exp \left(2\tau \sum_{m \in \mathbb{Z}_M} \|u_\tau(t_m)\|_{L_\infty} \|u_m\|_{L_\infty} \right) \right\|_{L^p(\Omega)} \leq C.$$

Proof. Applying the Hölder, Young and Gagliardo–Nirenberg inequalities and using the charge conservation laws (9) and (36) and the Jensen inequality, we have

$$\begin{aligned}
& \left\| \exp \left(2\tau \sum_{m \in \mathbb{Z}_{M+1}} \|u_\tau(t_m)\|_{L^\infty} \|u_m\|_{L^\infty} \right) \right\|_{L^p(\Omega)} \\
& \leq \exp \left(4pT^2 e^{\alpha T} \|\xi\|^2 \right) \prod_{m \in \mathbb{Z}_{M+1}} \left\| \exp \left(\frac{\|\nabla u_\tau(t_m)\|^2 \tau}{2pT e^{\alpha T}} \right) \right\|_{L^{2(M+2)p}(\Omega)} \\
& \quad \times \prod_{m \in \mathbb{Z}_{M+1}} \left\| \exp \left(\frac{\|\nabla u_m\|^2 \tau}{2pT e^{\alpha T}} \right) \right\|_{L^{2(M+2)p}(\Omega)} \\
& \leq C \left(\sup_{m \in \mathbb{Z}_{M+1}} \mathbb{E} \left[\exp \left(\frac{\|\nabla u_m\|^2}{e^{\alpha T}} \right) \right] \cdot \sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{\|\nabla u_\tau(t)\|^2}{e^{\alpha T}} \right) \right] \right)^{\frac{1}{2p}},
\end{aligned}$$

where α is the parameter appearing in (26) or (38). We complete the proof by the exponential integrability of u_τ and u_m in Lemmas 2.2 and 3.1. \square

For the convenience, we can also define the continuous extension of u_m as

$$\widehat{u}_\tau(t) := \widehat{u}_{\tau,m}^S(t) := (\Phi_{j,t-t_m}^S \widehat{\Phi}_{j,\tau}^D) \prod_{j=1}^{m-1} (\Phi_{j,\tau}^S \widehat{\Phi}_{j,\tau}^D) u_\tau(0), \quad t \in T_m,$$

where $\widehat{\Phi}_{j,\tau}^D$ is the solution operator of the Crank–Nicolson scheme.

Theorem 3.1. *For any $p \geq 1$, there exist a constant $C = C(\xi, Q, T, p)$ such that*

$$\left(\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|u(t_m) - u_m\|^p \right] \right)^{\frac{1}{p}} \leq C \tau^{\frac{1}{2}}. \quad (43)$$

Proof. We only prove the case $p = 2$, since the proof for other cases is similar to the proof of Theorem 2.2. The strong error can be split as

$$\begin{aligned}
& \mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|u(t_m) - u_m\|^2 \right] \\
& \leq 2\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|u(t_m) - u_\tau(t_m)\|^2 \right] + 2\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|u_\tau(t_m) - u_m\|^2 \right].
\end{aligned}$$

It suffices to estimate the last term due to Theorem 2.2. Denote by $\widehat{e}_{m+1} := u(t_{m+1}) - u_{m+1}$, $m \in \mathbb{Z}_{M-1}$. Applying Itô formula to $\|\widehat{e}_{m+1}\|^2$ in T_m , we obtain

$$\begin{aligned}
\|\widehat{e}_{m+1}\|^2 &= \|u_{\tau,m}^S(t_m) - \widehat{u}_{\tau,m}^S(t_m)\|^2 \\
&= \|\widehat{e}_m\|^2 + 2\langle \widehat{e}_m, \mathbf{i} \int_{t_m}^{t_{m+1}} \Delta u_\tau^D(s) - \Delta u_{m+\frac{1}{2}}^D ds \rangle
\end{aligned}$$

$$\begin{aligned}
 & + 2 \langle \widehat{e}_m, \mathbf{i} \lambda \int_{t_m}^{t_{m+1}} |u_{\tau,m}^D|^2 u_{\tau,m}^D - \frac{|u_m|^2 + |u_{m+1}^D|^2}{2} u_{m+\frac{1}{2}}^D ds \rangle \\
 & + \left\| \int_{t_m}^{t_{m+1}} \Delta \left[u_{\tau,m}^D - u_{m+\frac{1}{2}}^D \right] + |u_{\tau}^D|^2 u_{\tau}^D - \frac{|u_m|^2 + |u_{m+1}^D|^2}{2} u_{m+\frac{1}{2}}^D ds \right\|^2 \\
 & := \|\widehat{e}_m\|^2 + III_1^m + III_2^m + III_3^m.
 \end{aligned}$$

The integration by parts, the Hölder and Sobolev inequalities yield that

$$\begin{aligned}
 III_1^m & = \left\langle \Delta \widehat{e}_m, -2 \int_{t_m}^{t_{m+1}} \int_{t_m}^s \Delta u_{\tau,m}^D(r) + \lambda |u_{\tau,m}^D(r)|^2 u_{\tau,m}^D(r) dr ds \right\rangle \\
 & + 2\tau^2 \left\langle \Delta \widehat{e}_m, \Delta u_{m+\frac{1}{2}}^D + \frac{|u_m|^2 + |u_{m+1}^D|^2}{2} u_{m+\frac{1}{2}}^D \right\rangle \\
 & \leq C\tau^2 \left(\sup_{t \in T_M} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^2 + \|u_m\|_{\mathbb{H}^2}^2 + \|u_{m+1}^D\|_{\mathbb{H}^2}^2 \right).
 \end{aligned}$$

For the term III_2^m , using the cubic difference formula, the Gagliardo–Nirenberg inequality, the charge and energy conservation laws of Eq. (2) and Lemma 3.2, we get

$$\begin{aligned}
 III_2^m & = 2 \left\langle \widehat{e}_m, \mathbf{i} \lambda \int_{t_m}^{t_{m+1}} |u_{\tau,m}^D(s)|^2 u_{\tau,m}^D(s) - |u_{\tau,m}^D(t_m)|^2 u_{\tau,m}^D(t_m) ds \right\rangle \\
 & + 2 \left\langle \widehat{e}_m, \mathbf{i} \lambda \int_{t_m}^{t_{m+1}} |u_{\tau,m}^D(t_m)|^2 u_{\tau,m}^D(t_m) - |u_m|^2 u_m ds \right\rangle \\
 & + 2 \left\langle \widehat{e}_m, \mathbf{i} \lambda \int_{t_m}^{t_{m+1}} |u_m|^2 u_m - \frac{|u_m|^2 + |u_{m+1}^D|^2}{2} u_{m+\frac{1}{2}}^D ds \right\rangle \\
 & \leq 2\tau \|\widehat{e}_m\|^2 + C \int_{t_m}^{t_{m+1}} \|u_{\tau,m}^D(s)\|_{\mathbb{H}^1}^2 \|u_{\tau}^D(s) - u_{\tau,m}^D(t_m)\|^2 ds \\
 & + 2\tau \|u_{\tau,m}^D(t_m)\|_{L^\infty} \|u_m\|_{L^\infty} \|\widehat{e}_m\|^2 \\
 & + C\tau \left(\|u_m\|_{\mathbb{H}^1}^2 + \|u_{m+1}^D\|_{\mathbb{H}^1}^2 \right) \|u_{m+1}^D - u_m\|^2 \\
 & \leq \tau(2 + 2\|u_{\tau}(t_m)\|_{L^\infty} \|u_m\|_{L^\infty}) \|\widehat{e}_m\|^2 \\
 & + C\tau^3 \left(1 + \sup_{t \in T_m} \|u_{\tau}^D(r)\|_{\mathbb{H}^2}^4 + \|u_{m+1}^D\|_{\mathbb{H}^2}^6 + \|u_m\|_{\mathbb{H}^2}^6 \right).
 \end{aligned}$$

Analogously,

$$III_3^m \leq C\tau^2 \left(\sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^2 + \|u_m\|_{\mathbb{H}^2}^2 + \|u_{m+1}^D\|_{\mathbb{H}^2}^2 \right).$$

Summing up the estimations of III_1^m - III_3^m , we get

$$\begin{aligned} \|\widehat{e}_{m+1}\|^2 &\leq \|\widehat{e}_m\|^2 + \tau(2 + 2\|u_\tau(t_m)\|_{L_\infty}\|u_m\|_{L_\infty})\|\widehat{e}_m\|^2 \\ &\quad + C\tau^2 \left(1 + \sup_{t \in T_m} \|u_{\tau,m}^D(t)\|_{\mathbb{H}^2}^4 + \|u_{m+1}^D\|_{\mathbb{H}^2}^6 + \|u_m\|_{\mathbb{H}^2}^6 \right). \end{aligned}$$

Applying Lemma 2.6, we have

$$\begin{aligned} \|\widehat{e}_{m+1}\|^2 &\leq C\tau \exp \left(2T + 2\tau \sum_{n=0}^m \|u_\tau(t_n)\|_{L_\infty}\|u_n\|_{L_\infty} \right) \\ &\quad \times \left(1 + \sup_{t \in [0,T]} \|u_\tau^D(t)\|_{\mathbb{H}^2}^4 + \sup_{m \in \mathbb{Z}_M} \|u_{m+1}^D\|_{\mathbb{H}^2}^6 + \sup_{m \in \mathbb{Z}_{M+1}} \|u_m\|_{\mathbb{H}^2}^6 \right). \end{aligned}$$

Then taking expectations on both sides and using the Hölder inequality, we obtain

$$\begin{aligned} &\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|\widehat{e}_{m+1}\|^2 \right] \\ &\leq C\tau \left\| \exp \left(2\tau \sum_{m \in \mathbb{Z}_M} \|u_\tau(t_m)\|_{L_\infty}\|u_m\|_{L_\infty} \right) \right\|_{L^2(\Omega)} \\ &\quad \times \left(1 + \left(\mathbb{E} \left[\sup_{t \in [0,T]} \|u_\tau^D(t)\|_{\mathbb{H}^2}^8 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|u_{m+1}^D\|_{\mathbb{H}^2}^{12} \right] \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|u_m\|_{\mathbb{H}^2}^{12} \right] \right)^{\frac{1}{2}} \right). \end{aligned}$$

We conclude (3.1) by combining Lemmas 3.3 and 3.4 and Corollaries 2.3 and 3.2. \square

4. Full discretizations

In this section, we first discretize Eq. (1) in space by spectral Galerkin method and then apply the splitting Crank–Nicolson scheme in Section 3 to the spatially discrete equation. Our main goal is to derive the strong error estimate of this fully discrete scheme.

4.1. Spatial spectral Galerkin approximations

In this part, we use the spectral Galerkin method to spatially discretize Eq. (1) and analyze its strong convergence rate.

Let V_N be the subspace of L^2 consisting of the first N eigenvectors of the Dirichlet Laplacian operator. Denote by $\mathcal{P}^N : L^2 \rightarrow V_N$ the spectral Galerkin projection defined by $\langle \mathcal{P}^N u, v \rangle = \langle u, v \rangle$ for any $u \in L^2$ and $v \in V_N$. It is clear that \mathcal{P}^N is a self-adjoint and idempotent operator, i.e.,

$$(\mathcal{P}^N)^* = \mathcal{P}^N, \quad (\mathcal{P}^N)^2 = \mathcal{P}^N. \quad (44)$$

In order to inherit the charge conservation law, we directly use the spectral Galerkin method to approximate Eq. (1). The corresponding numerical solution $u^N = \{u^N(t) : t \in [0, T]\}$, $N \in \mathbb{N}$ satisfies

$$du^N = \mathbf{i} \left(\Delta u^N + \lambda \mathcal{P}^N (|u^N|^2 u^N) - \mathcal{P}^N (u^N \circ dW(t)) \right), \quad u^N(0) = \mathcal{P}^N \xi. \quad (45)$$

We remark that the above approximation is different from applying the spectral Galerkin to approximate its equivalent Itô type SNLS equation

$$\begin{aligned} d\tilde{u}^N &= \left(\mathbf{i} \Delta \tilde{u}^N + \mathbf{i} \lambda \mathcal{P}^N (|\tilde{u}^N|^2 \tilde{u}^N) - \frac{1}{2} \mathcal{P}^N (F_Q \tilde{u}^N) \right) dt \\ &\quad - \mathbf{i} \mathcal{P}^N (\tilde{u}^N dW(t)), \quad \tilde{u}^N(0) = \mathcal{P}^N \xi. \end{aligned} \quad (46)$$

The following lemma shows that, contrary to the deterministic NLS equation, i.e., Eq. (1) with $Q = 0$, Eq. (46) does not possess the charge conservation law.

Lemma 4.1. *The charge of u^N is conserved, i.e., $\|u^N(t)\| = \|u^N(0)\|$ a.s. The charge of \tilde{u}^N is not conserved, and satisfies the following evolution:*

$$\|\tilde{u}^N(t)\|^2 = \|\tilde{u}^N(0)\|^2 - \int_0^t \sum_{k \in \mathbb{N}} \|(I - \mathcal{P}^N)(\tilde{u}^N Q^{\frac{1}{2}} e_k)\|^2 dr. \quad (47)$$

In particular, the charge of u^N decreases, i.e.,

$$\|\tilde{u}^N(t)\|^2 \leq \|\tilde{u}^N(0)\|^2, \quad \text{a.s.} \quad (48)$$

Proof. The conservation of $\|u^N\|$ is immediately obtained due to the chain formula. However, the spatial approximation applied to Itô formula is totally different. Applying Itô formula to the

functional $\frac{1}{2} \|\tilde{u}^N\|^2$ and using the fact (44), we have

$$\begin{aligned} & \frac{1}{2} \|\tilde{u}^N(t)\|^2 - \frac{1}{2} \|\tilde{u}^N(0)\|^2 \\ &= \int_0^t \langle \tilde{u}^N, \mathbf{i} \mathcal{P}^N(\Delta \tilde{u}^N) \rangle dr + \int_0^t \langle \tilde{u}^N, \mathbf{i} \lambda \mathcal{P}^N(|\tilde{u}^N|^2 \tilde{u}^N) \rangle dr \\ & \quad - \int_0^t \langle \tilde{u}^N, \mathbf{i} \mathcal{P}^N(\tilde{u}^N dW(r)) \rangle - \frac{1}{2} \int_0^t \langle \tilde{u}^N, \mathcal{P}^N(\tilde{u}^N F_Q) \rangle dr \\ & \quad + \frac{1}{2} \int_0^t \sum_{k \in \mathbb{N}} \|\mathcal{P}^N(\tilde{u}^N Q^{\frac{1}{2}} e_k)\|^2 dr \\ &= -\frac{1}{2} \int_0^t \sum_{k \in \mathbb{N}} \|(I - \mathcal{P}^N)(\tilde{u}^N Q^{\frac{1}{2}} e_k)\|^2 dr \leq 0. \end{aligned}$$

This completes the proof of (47) and (48). \square

Based on the charge evolution of u^N and \tilde{u}^N , we have the following \mathbb{H}^2 -a priori estimates of u^N and \tilde{u}^N , which are essential to deduce the optimal convergence rate.

Lemma 4.2. *For any $p \geq 1$, there exists a constant $C = C(\xi, Q, T, p)$ such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u^N(t)\|_{\mathbb{H}^2}^p \right] \leq C, \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|\tilde{u}^N(t)\|_{\mathbb{H}^2}^p \right] \leq C \quad (49)$$

Proof. We refer to the proof of [10, Theorem 2.1] since the procedures are similar. \square

To deduce the strong convergence rate of the spectral Galerkin approximations (45), we also need the following exponential integrability of u^N and \tilde{u}^N by applying Lemma 2.1 to $H(u^N)$ and $H(\tilde{u}^N)$, whose proof is similar to that of [10, Proposition 3.1] or Lemma (2.2) and we omit the proof.

Proposition 4.1. *There exist some positive constants α and C depending on ξ, Q and T such that*

$$\sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{\|\nabla u^N(t)\|^2}{e^{\alpha t}} \right) \right] \leq C, \quad \sup_{t \in [0, T]} \mathbb{E} \left[\exp \left(\frac{\|\nabla \tilde{u}^N(t)\|^2}{e^{\alpha t}} \right) \right] \leq C. \quad (50)$$

Now we are in position of estimating the strong convergence rate between the exact solution u and the spectral Galerkin solution u^N, \tilde{u}^N . Recall the following frequently used estimate for spectral Galerkin projection:

$$\|(I - \mathcal{P}^N)v\| \leq \lambda_{N+1}^{-1} \|v\|_{\mathbb{H}^2}, \quad v \in \mathbb{H}^2, \quad (51)$$

where λ_N is the N -th eigenvalue of Dirichlet negative Laplacian: $\lambda_N = (N\pi)^2$, $N \in \mathbb{N}_+$.

Theorem 4.1. *For any $p \geq 1$, there exists a constant $C = C(\xi, Q, T, p)$ such that*

$$\left(\mathbb{E} \left[\sup_{t \in [0, T]} \|u(t) - u^N(t)\|^p \right] \right)^{\frac{1}{p}} \leq CN^{-2}, \quad (52)$$

$$\left(\mathbb{E} \left[\sup_{t \in [0, T]} \|u(t) - \tilde{u}^N(t)\|^p \right] \right)^{\frac{1}{p}} \leq CN^{-2}. \quad (53)$$

Proof. For simplicity, we only prove the case $p = 2$ for $\tilde{u}^N(t)$, the proof for the other case and u^N is similar. Denote by $\epsilon^N := u - \tilde{u}^N$. Subtracting Eq. (45) from Eq. (1) yields that

$$\begin{aligned} d\epsilon^N &= \mathbf{i}\Delta\epsilon^N dt + \mathbf{i}\lambda(|u|^2 u - \mathcal{P}^N(|\tilde{u}^N|^2 \tilde{u}^N))dt \\ &\quad - \frac{1}{2} \left[u F_Q - \mathcal{P}^N(\tilde{u}^N F_Q) \right] dt - \mathbf{i} \left[u dW(t) - \mathcal{P}^N(\tilde{u}^N dW(t)) \right]. \end{aligned}$$

Applying Itô formula to the functional $\frac{1}{2} \|\epsilon^N\|^2$, we get

$$\begin{aligned} & \frac{1}{2} \|\epsilon^N(t)\|^2 - \frac{1}{2} \|(I - \mathcal{P}^N)\xi\|^2 \\ &= \int_0^t \langle \epsilon^N, \mathbf{i}(\Delta u^N - \mathcal{P}^N \Delta \tilde{u}^N) \rangle dr + \int_0^t \langle \epsilon^N, \mathbf{i}\lambda(|u|^2 u - \mathcal{P}^N(|\tilde{u}^N|^2 \tilde{u}^N)) \rangle dr \\ &\quad - \frac{1}{2} \int_0^t \langle \epsilon^N, (u F_Q - \mathcal{P}^N(\tilde{u}^N F_Q)) \rangle dr - \int_0^t \langle \epsilon^N, \mathbf{i}(u dW - \mathcal{P}^N(\tilde{u}^N dW(r))) \rangle \\ &\quad + \frac{1}{2} \int_0^t \sum_{k \in \mathbb{N}} \left(\|(I - \mathcal{P}^N)(u^N Q^{\frac{1}{2}} e_k)\|^2 + \|\mathcal{P}^N(\epsilon^N Q^{\frac{1}{2}} e_k)\|^2 \right) dr \\ &:= I_1^N + I_2^N + I_3^N + I_4^N + I_5^N. \end{aligned}$$

Due to the property of the spectral Galerkin projection operator \mathcal{P}^N , $I_1^N = 0$. By the identity $|a|^2 a - |b|^2 b = (|a|^2 + |b|^2)(a - b) + ab(\bar{a} - \bar{b})$ for $a, b \in \mathbb{C}$ and the estimation (51), we have

$$\begin{aligned}
I_2^N &= \int_0^t \langle \epsilon^N, \mathbf{i}\lambda(|u|^2 u - |\tilde{u}^N|^2 \tilde{u}^N) \rangle + \langle \epsilon^N, \mathbf{i}\lambda(I - \mathcal{P}^N)(|\tilde{u}^N|^2 \tilde{u}^N) \rangle dr \\
&\leq \int_0^t \|\epsilon^N\|^2 \|u\|_{L_\infty} \|\tilde{u}^N\|_{L_\infty} dr + \int_0^t \|\epsilon^N\| \|(I - \mathcal{P}^N)(|\tilde{u}^N|^2 \tilde{u}^N)\| dr \\
&\leq \frac{1}{2} \lambda_{N+1}^{-2} \int_0^t \|\tilde{u}^N\|_{\mathbb{H}^2}^6 dr + \int_0^t \left(\frac{1}{2} + \|u\|_{L_\infty} \|\tilde{u}^N\|_{L_\infty} \right) \|\epsilon^N\|^2 dr.
\end{aligned}$$

It follows from (51) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
I_3^N + I_5^N &= -\frac{1}{2} \int_0^t \sum_k \|(I - \mathcal{P}^N)(\epsilon^N Q^{\frac{1}{2}} e_k)\|^2 dr \\
&\quad - \frac{1}{2} \int_0^t \langle \epsilon^N, (I - \mathcal{P}^N)(\tilde{u}^N F_Q) \rangle dr \\
&\quad + \frac{1}{2} \int_0^t \sum_k \|(I - \mathcal{P}^N)(u Q^{\frac{1}{2}} e_k)\|^2 dr \\
&\leq \frac{\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2}{2} \int_0^t \left(\|\epsilon^N\|^2 + \lambda_{N+1}^{-2} \|u\|_{\mathbb{H}^2}^2 + \lambda_{N+1}^{-2} \|\tilde{u}^N\|_{\mathbb{H}^2}^2 \right) dr.
\end{aligned}$$

By the properties of \mathcal{P}^N , we can rewrite the third term I_3^N as

$$I_4^N = - \int_0^t \langle (I - \mathcal{P}^N)u, \mathbf{i}(I - \mathcal{P}^N)(\tilde{u}^N dW(r)) \rangle.$$

Combining the above estimations, we obtain

$$\begin{aligned}
\|\epsilon^N(t)\|^2 &\leq \lambda_{N+1}^{-2} \left(\|\xi\|^2 + \int_0^t \left[\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 (\|\tilde{u}^N\|_{\mathbb{H}^2}^2 + \|u\|_{\mathbb{H}^2}^2) + \|\tilde{u}^N\|_{\mathbb{H}^2}^6 \right] dr \right) \\
&\quad + 2 \left| \int_0^t \langle (I - \mathcal{P}^N)u, \mathbf{i}(I - \mathcal{P}^N)(\tilde{u}^N dW(r)) \rangle \right| \\
&\quad + \int_0^t \left(1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 + 2\|u\|_{L_\infty} \|\tilde{u}^N\|_{L_\infty} \right) \|\epsilon^N\|^2 dr.
\end{aligned}$$

The Gronwall inequality implies that

$$\begin{aligned} \|\epsilon^N(t)\|^2 &\leq \exp \left(\int_0^T 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 + 2\|u\|_{L_\infty} \|\tilde{u}^N\|_{L_\infty} dr \right) \\ &\times \left(\lambda_{N+1}^{-2} \left(\|\xi\|^2 + \int_0^T \left[\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2 (\|\tilde{u}^N\|_{\mathbb{H}^2}^2 + \|u\|_{\mathbb{H}^2}^2) + \|u^N\|_{\mathbb{H}^2}^6 \right] dr \right) \right. \\ &\left. + 2 \sup_{t \in [0, T]} \left| \int_0^t \langle (I - \mathcal{P}^N)u, \mathbf{i}(I - \mathcal{P}^N)(\tilde{u}^N dW(r)) \rangle \right| \right). \end{aligned}$$

Now taking the supreme over t , taking expectation on both sides in the above inequality and using the a priori estimation (49) in Lemma 4.2 as well as the exponential integrability (50) in Lemma 4.1, we obtain

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\epsilon^N(t)\|^2 \right] \leq C \left(\lambda_{N+1}^{-2} + R^N \right),$$

where

$$R^N := \left\| \int_0^t \langle (I - \mathcal{P}^N)u, \mathbf{i}(I - \mathcal{P}^N)(\tilde{u}^N dW(r)) \rangle \right\|_{L^2(\Omega; L^\infty([0, T]))}.$$

It suffices to estimate the stochastic integral term appeared above. The Burkholder–Davis–Gundy inequality and the estimation (51) lead to

$$\begin{aligned} R^N &\leq C \left(\sum_{k=1}^{\infty} \int_0^t \mathbb{E} \left[\|(I - \mathcal{P}^N)u\|^2 \|(I - \mathcal{P}^N)\tilde{u}^N Q^{\frac{1}{2}} e_k\|^2 \right] dr \right)^{\frac{1}{2}} \\ &\leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2} \lambda_{N+1}^{-2} \left(\int_0^t \mathbb{E} \left[\|u\|_{\mathbb{H}^2}^2 \|\tilde{u}^N\|_{\mathbb{H}^2}^2 \right] ds \right)^{\frac{1}{2}} \leq C \lambda_{N+1}^{-2}. \end{aligned}$$

This completes the proof of (52). \square

Remark 4.1. For general finite element methods, the projection operator \mathcal{P}^N does not commute with Δ in the term $\langle \epsilon^N, \mathbf{i}(\Delta u^N - \mathcal{P}^N \Delta u^N) \rangle$. As a result, this term produce $\|(I - \mathcal{P}^N)\Delta u^N\|$ which requires more regularity. Compared to finite element methods, the spectral Galerkin method can achieve the optimal order.

4.2. Spectral splitting Crank–Nicolson scheme

Motivated by Theorem 3.1 and Theorem 4.1, we propose a fully discrete splitting Crank–Nicolson scheme to inherit the charge conservation law, stability and exponential integrability. As in Section 2, we split the spatially semi-discrete spectral approximations (45) into

$$\begin{cases} du^{(N,D)} = (\mathbf{i}\Delta u^{(N,D)} + \mathbf{i}\lambda \mathcal{P}^N(|u^{(N,D)}|^2 u^{(N,D)})) dt, \\ du^{(N,S)} = -\mathbf{i}\mathcal{P}^N(u^{(N,S)} \circ dW(t)). \end{cases} \quad (54)$$

Then we apply the temporal splitting Crank–Nicolson scheme (35) to the above spectral splitting systems (54), which leads to the following fully discrete spectral splitting Crank–Nicolson scheme:

$$\begin{cases} u_{m+1}^{(N,D)} = u_m^N + \mathbf{i}\tau \Delta u_{m+\frac{1}{2}}^{(N,D)} + \frac{\mathbf{i}\lambda\tau}{2} \mathcal{P}^N \left((|u_m^N|^2 + |u_{m+1}^{(N,D)}|^2) u_{m+\frac{1}{2}}^{(N,D)} \right), \\ u_{m+1}^N = \exp(-\mathbf{i}\mathcal{P}^N(W(t_{m+1}) - W(t_m))) \cdot u_{m+1}^{(N,D)}, \end{cases} \quad (55)$$

where $u_{m+\frac{1}{2}}^{(N,D)} = \frac{1}{2}(u_m^N + u_{m+1}^{(N,D)})$.

It is not difficult to see that the charge of Eq. (54) is conserved since both spatial and temporal numerical methods preserves the charge. Similarly to the proof of Lemma (2.2) and Lemma 3.3, we have the \mathbb{H}^1 -exponential integrability and \mathbb{H}^2 -a priori estimate of u_m^N , $m \in \mathbb{Z}_M$.

Lemma 4.3. *Let $p \geq 1$. There exist constants C and α depending on ξ , Q and T and $C' = C'(\xi, Q, T, p)$ such that*

$$\sup_{m \in \mathbb{Z}_M} \mathbb{E} \left[\exp \left(\frac{\|\nabla u_m^N\|^2}{e^{\alpha t_m}} \right) \right] \leq C$$

and

$$\mathbb{E} \left[\sup_{m \in \mathbb{Z}_M} \|u_m^N\|_{\mathbb{H}^2}^p \right] \leq C'.$$

As a consequence of the exponential integrability, we have the following Gaussian tail estimations for $u(t)$, $u_\tau(t)$, u_m , $u_m^N(t)$ and u_m^N , $t \in [0, T]$, $m \in \mathbb{Z}_M$ with $M \in \mathbb{N}$ and $N \in \mathbb{N}_+$, which have their own interest.

Corollary 4.1. *There exist constants C and η depending on ξ , Q and T such that for any $x \in \mathbb{R}_+$,*

$$\begin{aligned} & \sup_{t \in [0, T]} \mathbb{P}(\|u(t)\|_{\mathbb{H}^1} \geq x) + \sup_{t \in [0, T]} \mathbb{P}(\|u_\tau(t)\|_{\mathbb{H}^1} \geq x) + \sup_{m \in \mathbb{Z}_M} \mathbb{P}(\|u_m\|_{\mathbb{H}^1} \geq x) \\ & + \sup_{t \in [0, T]} \mathbb{P}(\|u^N(t)\|_{\mathbb{H}^1} \geq x) + \sup_{m \in \mathbb{Z}_M} \mathbb{P}(\|u_m^N\|_{\mathbb{H}^1} \geq x) \leq C \exp(-\eta x^2). \end{aligned}$$

Proof. We only prove the Gaussian tail estimation of u ; the other cases are similar. By the Chebyshev inequality and the exponential integrability of u in Theorem 2.1, we deduce that for any $y \geq 1$ there holds that

$$\begin{aligned} \mathbb{P}\left(\|u(t)\|_{\mathbb{H}^1} \geq \sqrt{\ln(y)e^{\alpha T}}\right) &= \mathbb{P}\left(\exp\left(e^{-\alpha T}\|u(t)\|_{\mathbb{H}^1}^2\right) \geq y\right) \\ &\leq \frac{\mathbb{E}\left[\exp\left(e^{-\alpha T}\|u(t)\|_{\mathbb{H}^1}^2\right)\right]}{y}. \end{aligned}$$

Let $x = \sqrt{\ln(y)e^{\alpha T}}$. Then we obtain

$$\begin{aligned} \mathbb{P}\left(\|u(t)\|_{\mathbb{H}^1} \geq x\right) &\leq \mathbb{E}\left[\exp\left(e^{-\alpha T}\|u(t)\|_{\mathbb{H}^1}^2\right)\right] \exp\left(-e^{-\alpha T}x^2\right) \\ &\leq C \exp(-\eta x^2) \end{aligned}$$

for C as in (8) and $\eta = e^{-\alpha T}$. \square

Based on Theorems 3.1 and 4.1 and Lemma 4.3, we derive the strong error between u and u_m^N .

Theorem 4.2. For any $p \geq 1$, there exists $C = C(\xi, Q, T, p)$ such that

$$\left(\mathbb{E}\left[\sup_{m \in \mathbb{Z}_m} \|u(t_m) - u_m^N\|^p\right]\right)^{\frac{1}{p}} \leq C(N^{-2} + \tau^{\frac{1}{2}}). \quad (56)$$

Proof. We split the error into a spatial error and a temporal error:

$$\|u(t_m) - u_m^N\|^p \leq C(\|u(t_m) - u_\tau^N(t_m)\|^p + \|u_\tau^N(t_m) - u_m^N\|^p).$$

The spatial error is controlled by (52) in Theorem 4.1, and the temporal error is estimated by (43) in Theorem 3.1 and the a priori estimations in Lemma 4.3. \square

4.3. Finite difference splitting Crank–Nicolson scheme

The proposed temporal splitting approach gives a kind of full discretizations with different spatial approximations, such as the following finite difference splitting Crank–Nicolson scheme. Let $h = \frac{1}{N+1}$ and $\{x_n := nh, n \in \mathbb{Z}_{N+1}\}$ be a partition of the spatial interval \mathcal{O} . We define a grid function f^h in $\{x_n\}_{n \in \mathbb{Z}_{N+1}}$ as $f^h(x_n) = f^h(n)$ with $f^h(0) = 0$ and $f^h(N+1) = 0$, $m \in \mathbb{Z}_{N+1}$. Denote $\delta_+^h f^h(n) := \frac{f^h(n+1) - f^h(n)}{h}$ and $\delta_-^h f^h(n) := \frac{f^h(n) - f^h(n-1)}{h}$. The finite difference splitting

Crank–Nicolson scheme is

$$\begin{cases} u_{m+1}^{(h,D)} = u_m^h + i\tau \left(\delta_+^h \delta_-^h u_{m+\frac{1}{2}}^{(h,D)} + \frac{\lambda}{2} \left(|u_m^h|^2 + |u_{m+1}^{(h,D)}|^2 \right) u_{m+\frac{1}{2}}^{(h,D)} \right), \\ u_{m+1}^h = \exp(-i(W(t_{m+1}) - W(t_m))) u_{m+1}^{(h,D)}, \quad m \in \mathbb{Z}_M, \end{cases} \quad (57)$$

where $u_{m+\frac{1}{2}}^{(h,D)} = \frac{1}{2}(u_m^h + u_{m+1}^{(h,D)})$.

By combining the error estimate of spatial centered difference method in [10, Theorem 4.1] and similar arguments in Theorem 4.2, the strong error order of the above finite difference splitting Crank–Nicolson scheme (57) can be obtained.

Remark 4.2. Combining our temporal splitting Crank–Nicolson scheme with spatially numerical methods such as Galerkin finite element method and certain finite difference methods, we can also obtain strong convergence rates of related fully discrete splitting schemes. However, as noted in Remark 4.1, the strong convergence rate of finite difference method may not be optimal under minimal regularity assumptions on initial datum ξ and noise's covariance operator Q . That's one motivation why we study spatial Galerkin method.

5. Numerical experiments

In this section, we present several numerical tests to verify our theoretic results including the evolution of charge, the exponential moments of energy and the strong convergence rates for the numerical schemes.

We use the spectral splitting Crank–Nicolson scheme (55) to fully discretize the following stochastic NLS equation with a noise intensity $\varepsilon \in \mathbb{R}$:

$$\begin{cases} i du + (\Delta u + |u|^2 u) dt = \varepsilon u \circ dW(t), & (t, x) \in (0, T] \times (0, 1); \\ u(t, 0) = u(t, 1) = 0, & t \in [0, T]; \\ u(0, x) = \sin(\pi x), & x \in (0, 1). \end{cases}$$

Here we take the Q -Wiener process as

$$W(t, x) = \sum_{k=1}^{\infty} \frac{\sqrt{2} \sin(k\pi x)}{1 + k^{2.6}} \beta_k(t), \quad (t, x) \in [0, T] \times (0, 1).$$

To simulate this process, we truncate the series by the first K terms with various $K \in \mathbb{N}_+$ and take $P = 1000$ trajectories.

For the simulations of the evolution of charge and exponential moment of energy, we take $T = 100$, $\varepsilon = 10$, $N = 2^6$, $h = 2^{-6}$ and $\tau = 2^{-10}$. Fig. 1 shows the charge conservation law of the spectral splitting Crank–Nicolson scheme (55). Fig. 2 illustrates the exponential moment of energy for schemes (55) with different parameters $\alpha = 0.7$ and $\alpha = 1$ appeared in Lemma 3.1, which recover the exponential integrability result (37).

Next we turn to the tests about the temporal and spatial strong convergence rates of the schemes (55). Errors of the numerical solutions against τ and h on a log-log scale are displayed in Fig. 3. We also apply two noise intensities $\varepsilon = 1$ and $\varepsilon = 10$ in these tests to check the strong

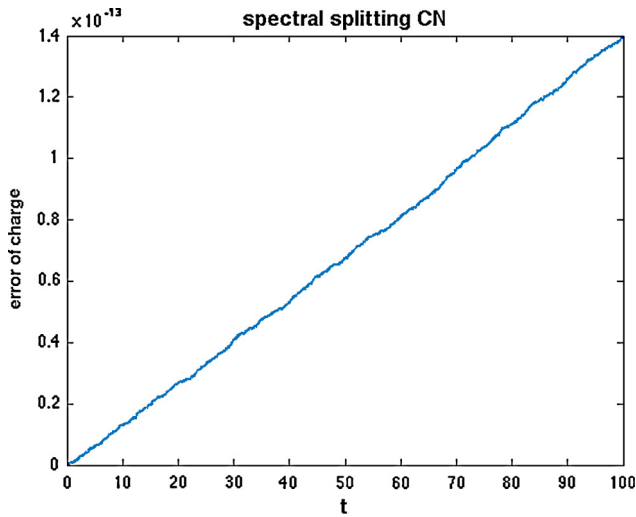


Fig. 1. Evolution of error for charge ($\|u_m^N\|^2 - \|\xi\|^2$) by spectral splitting Crank–Nicolson scheme with $K = N$.

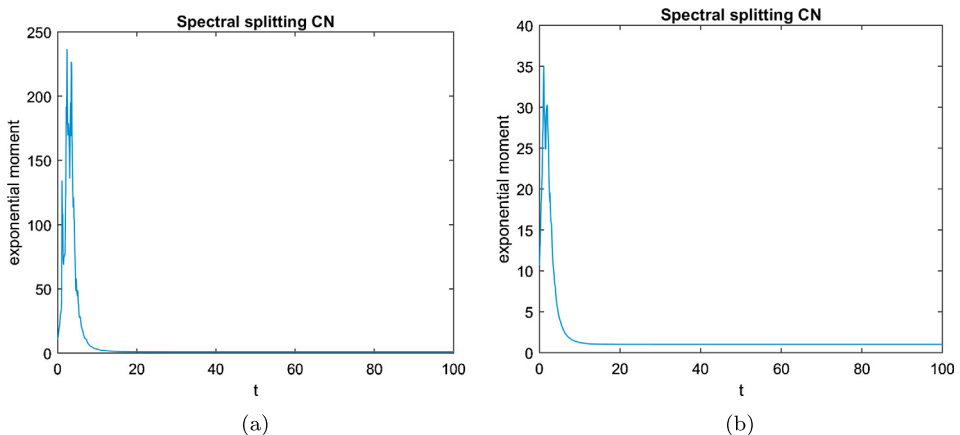


Fig. 2. Exponential integrability for spectral splitting Crank–Nicolson: (a) $\alpha = 0.7$ (b) $\alpha = 1$.

convergence results. More precisely, the left figure (a) in Fig. 3 presents the temporal strong convergence rate for the proposed scheme. Since there is no analytic solution for SPDEs with non-monotone nonlinearity in nearly all cases, we first compute a reference solution u_{ref} on a fine mesh $\tau_{ref} = 2^{-14}$. Compared to five coarser grids by $\tau = 2^p \tau_{ref}$, $p = 1, \dots, 5$, the strong errors at $T = 1$ are plotted with $N = 2^8$. The right figure (b) in Fig. 3 shows the spatial strong convergence rate. For fixed $\tau = 2^{-8}$, the corresponding reference spatial mesh is $h_{ref} = 2^{-10}$ and other five coarser grids are $h = 2^p h_{ref}$, $p = 1, \dots, 5$. The slopes in these figures indicate that the proposed scheme possesses the temporal strong convergence order 1/2 and spatial strong convergence order 2, which coincides with the theoretical result in Theorem 4.2.

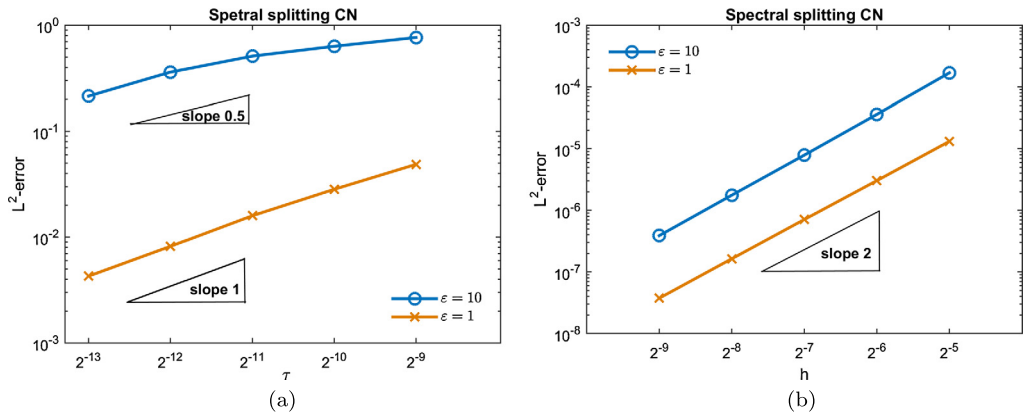


Fig. 3. Temporal and spatial convergence rates for pseudo-spectral splitting Crank–Nicolson: (a) temporal strong convergence rate (b) spatial convergence rate.

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