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Heat kernels of non-symmetric Lévy-type operators ☆

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Abstract

We construct the fundamental solution (the heat kernel) p^κ to the equation $\partial_t = \mathcal{L}^\kappa$, where under certain assumptions the operator $\mathcal{L}^\kappa$ takes one of the following forms,

$$\mathcal{L}^\kappa f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x) - \mathbf{1}_{|z|<1} \langle z, \nabla f(x) \rangle) \kappa(x,z) J(z)\, dz\,,$$

$$\mathcal{L}^\kappa f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x)) \kappa(x,z) J(z)\, dz\,,$$

$$\mathcal{L}^\kappa f(x) := \frac{1}{2} \int_{\mathbb{R}^d} (f(x+z) + f(x-z) - 2f(x)) \kappa(x,z) J(z)\, dz\,.$$

In particular, $J\colon \mathbb{R}^d \to [0,\infty]$ is a Lévy density, i.e., $\int_{\mathbb{R}^d} (1 \wedge |x|^2) J(x) dx < \infty$. The function $\kappa(x,z)$ is assumed to be Borel measurable on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $0 < \kappa_0 \leqslant \kappa(x,z) \leqslant \kappa_1$, and $|\kappa(x,z) - \kappa(y,z)| \leqslant \kappa_2 |x-y|^\beta$ for some $\beta \in (0,1)$. We prove the uniqueness, estimates, regularity and other qualitative properties of p^κ.

MSC: primary 60J35, 47G20; secondary 60J75, 47D03

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Keywords: Heat kernel estimates; Lévy-type operator; Non-symmetric non-local operator; Non-symmetric Markov process; Feller semigroup; Levi's parametrix method

1. Introduction

The goal of this paper is to extend (and improve) the results of [16] and [45] to more general operators than therein considered. These operators will be non-symmetric and not necessarily stable-like. On the occasion we mostly cover (excluding one case which study we postpone) a contemporaneous paper [41] (see also [15] and [49]). Let $d \in \mathbb{N}$ and $\nu : [0, \infty) \to [0, \infty]$ be a non-increasing function satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(|x|)dx < \infty .$$

We consider $J : \mathbb{R}^d \to [0, \infty]$ such that for some $\gamma_0 \in [1, \infty)$ and all $x \in \mathbb{R}^d$,

$$\gamma_0^{-1}\nu(|x|) \leqslant J(x) \leqslant \gamma_0 \nu(|x|) . \tag{1}$$

Further, suppose that $\kappa(x, z)$ is a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$0 < \kappa_0 \leqslant \kappa(x, z) \leqslant \kappa_1 , \tag{2}$$

and for some $\beta \in (0, 1)$,

$$|\kappa(x, z) - \kappa(y, z)| \leqslant \kappa_2 |x - y|^\beta . \tag{3}$$

For $r > 0$ we define

$$h(r) := \int_{\mathbb{R}^d} \left(1 \wedge \frac{|x|^2}{r^2}\right) \nu(|x|)dx , \qquad K(r) := r^{-2} \int_{|x|<r} |x|^2 \nu(|x|)dx .$$

The above functions play a prominent role in the paper. Our main assumption is *the weak scaling condition* at the origin: there exist $\alpha_h \in (0, 2]$ and $C_h \in [1, \infty)$ such that

$$h(r) \leqslant C_h \lambda^{\alpha_h} h(\lambda r) , \quad \lambda \leqslant 1, r \leqslant 1 . \tag{4}$$

In a similar fashion: there exist $\beta_h \in (0, 2]$ and $c_h \in (0, 1]$ such that

$$h(r) \geqslant c_h \lambda^{\beta_h} h(\lambda r) , \quad \lambda \leqslant 1, r \leqslant 1 . \tag{5}$$

Definition 1. We define the following three sets of assumptions,

(P1) (1)–(4) hold and $1 < \alpha_h \leqslant 2$,

(P2) (1)–(5) hold and $0<\alpha_h \leqslant \beta_h < 1$,
(P3) (1)–(4) hold, J is symmetric and $\kappa(x,z)=\kappa(x,-z)$, $x,z\in\mathbb{R}^d$.

We say that (P) holds if (P1) or (P2) or (P3) is satisfied.

In each case (P1), (P2), (P3), respectively, we consider an operator

$$\mathcal{L}^{\kappa} f(x) := \int_{\mathbb{R}^d} (f(x+z)-f(x)-\mathbf{1}_{|z|<1}\langle z, \nabla f(x)\rangle)\kappa(x,z)J(z)\,dz\,, \tag{6}$$

$$\mathcal{L}^{\kappa} f(x) := \int_{\mathbb{R}^d} (f(x+z)-f(x))\kappa(x,z)J(z)\,dz\,, \tag{7}$$

$$\mathcal{L}^{\kappa} f(x) := \frac{1}{2}\int_{\mathbb{R}^d} (f(x+z)+f(x-z)-2f(x))\kappa(x,z)J(z)\,dz\,. \tag{8}$$

We denote by $\mathcal{L}^{\kappa,\varepsilon} f$ the expressions (6), (7), (8) with $J(z)$ replaced by $J_\varepsilon(z) := J(z)\mathbf{1}_{|z|>\varepsilon}$, $\varepsilon\in[0,1]$. We apply the above operators (in a strong or weak sense) only when they are well defined according to the following definition. Let $f\colon\mathbb{R}^d\to\mathbb{R}$ be a Borel measurable function.

Strong operator: The operator $\mathcal{L}^{\kappa} f$ is well defined if the corresponding integral converges absolutely, and in the case (P1) the gradient $\nabla f(x)$ exists for every $x\in\mathbb{R}^d$.
Weak operator: The operator $\mathcal{L}^{\kappa,0^+} f$ is well defined if the limit exists for every $x\in\mathbb{R}^d$,

$$\mathcal{L}^{\kappa,0^+} f(x) := \lim_{\varepsilon\to 0^+} \mathcal{L}^{\kappa,\varepsilon} f(x)\,,$$

where for $\varepsilon\in(0,1]$ the (strong) operators $\mathcal{L}^{\kappa,\varepsilon} f$ are well defined.

The operator $\mathcal{L}^{\kappa,0^+}$ is an extension of $\mathcal{L}^{\kappa,0}=\mathcal{L}^{\kappa}$, meaning that if $\mathcal{L}^{\kappa} f$ is well defined, then so is $\mathcal{L}^{\kappa,0^+} f$ and $\mathcal{L}^{\kappa,0^+} f=\mathcal{L}^{\kappa} f$. Therefore, it is desired to prove the existence of a solution to the equation $\partial_t=\mathcal{L}^{\kappa}$ and the uniqueness of a solution to $\partial_t=\mathcal{L}^{\kappa,0^+}$.

We emphasize that in general we do not assume the symmetry of J. We also point out that whenever J is symmetric and $\kappa(x,z)=\kappa(x,-z)$, $x,z\in\mathbb{R}^d$, then for any bounded function $f\in C^2(\mathbb{R}^d)$ the three operators (6)–(8) coincide and

$$\mathcal{L}^{\kappa} f(x) = \lim_{\varepsilon\to 0^+} \int_{|z|>\varepsilon} (f(x+z)-f(x))\kappa(x,z)J(z)\,dz\,. \tag{9}$$

The above equality may hold for other particular choices of f. The assumptions on f may also be relaxed after replacing the left hand side with $\mathcal{L}^{\kappa,0^+} f(x)$.

Here are our main results.

Theorem 1.1. *Assume* (P) *and let* $T > 0$*. There is a unique function* $p^\kappa(t,x,y)$ *on* $(0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ *such that*

(i) *For all* $t \in (0,T]$*,* $x, y \in \mathbb{R}^d$*,* $x \neq y$*,*

$$\partial_t p^\kappa(t,x,y) = \mathcal{L}_x^{\kappa,0^+} p^\kappa(t,x,y)\,. \tag{10}$$

(ii) *The function* $p^\kappa(t,x,y)$ *is jointly continuous on* $(0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ *and for any* $f \in C_c^\infty(\mathbb{R}^d)$*,*

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\kappa(t,x,y) f(y)\,dy - f(x) \right| = 0\,. \tag{11}$$

(iii) *For every* $t_0 \in (0,T)$ *there are* $c > 0$ *and* $f_0 \in L^1(\mathbb{R}^d)$ *such that for all* $t \in (t_0, T]$*,* $x, y \in \mathbb{R}^d$*,*

$$|p^\kappa(t,x,y)| \le c f_0(x-y)\,, \tag{12}$$

and

$$|\mathcal{L}_x^{\kappa,\varepsilon} p^\kappa(t,x,y)| \leqslant c\,, \qquad \varepsilon \in (0,1]\,. \tag{13}$$

In the case (P1)*, additionally:*

(iv) *For every* $t \in (0,T]$ *there is* $c > 0$ *such that for all* $x, y \in \mathbb{R}^d$*,*

$$|\nabla_x p^\kappa(t,x,y)| \leqslant c\,. \tag{14}$$

In the next theorem we collect more qualitative properties of $p^\kappa(t,x,y)$. To this end, for $t > 0$ and $x \in \mathbb{R}^d$ we define *the bound function*,

$$\Upsilon_t(x) := \left([h^{-1}(1/t)]^{-d} \wedge \frac{t K(|x|)}{|x|^d} \right). \tag{15}$$

Theorem 1.2. *Assume* (P)*. The following hold true.*

(1) *(Non-negativity) The function* $p^\kappa(t,x,y)$ *is non-negative on* $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$*.*
(2) *(Conservativeness) For all* $t > 0$*,* $x \in \mathbb{R}^d$*,*

$$\int_{\mathbb{R}^d} p^\kappa(t,x,y)dy = 1\,.$$

(3) *(Chapman-Kolmogorov equation) For all* $s, t > 0$*,* $x, y \in \mathbb{R}^d$*,*

$$\int_{\mathbb{R}^d} p^\kappa(t,x,z) p^\kappa(s,z,y)\,dz = p^\kappa(t+s,x,y)\,.$$

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(4) *(Upper estimate) For every $T > 0$ there is $c > 0$ such that for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,*

$$p^\kappa(t, x, y) \leqslant c\Upsilon_t(y - x) .$$

(5) *(Factional derivative) For every $T > 0$ there is $c > 0$ such that for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,*

$$|\mathcal{L}_x^{\kappa,\varepsilon} p^\kappa(t, x, y)| \leqslant ct^{-1}\Upsilon_t(y - x) , \qquad \varepsilon \in [0, 1] .$$

(6) *(Gradient) If $1-\alpha_h < \beta \wedge \alpha_h$, then for every $T > 0$ there is $c > 0$ such that for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,*

$$\left|\nabla_x p^\kappa(t, x, y)\right| \leqslant c\Big[h^{-1}(1/t)\Big]^{-1} \Upsilon_t(y - x) .$$

(7) *(Continuity) The function $\mathcal{L}_x^\kappa p^\kappa(t, x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.*

(8) *(Strong operator) For all $t > 0$, $x, y \in \mathbb{R}^d$,*

$$\partial_t p^\kappa(t, x, y) = \mathcal{L}_x^\kappa\, p^\kappa(t, x, y) .$$

(9) *(Hölder continuity) For all $T > 0$, $\gamma \in [0, 1] \cap [0, \alpha_h)$, there is $c > 0$ such that for all $t \in (0, T]$ and $x, x', y \in \mathbb{R}^d$,*

$$\left|p^\kappa(t, x, y) - p^\kappa(t, x', y)\right| \leqslant c(|x - x'|^\gamma \wedge 1) \Big[h^{-1}(1/t)\Big]^{-\gamma} \big(\Upsilon_t(y - x) + \Upsilon_t(y - x')\big).$$

(10) *(Hölder continuity) For all $T > 0$, $\gamma \in [0, \beta) \cap [0, \alpha_h)$, there is $c > 0$ such that for all $t \in (0, T]$ and $x, y, y' \in \mathbb{R}^d$,*

$$\left|p^\kappa(t, x, y) - p^\kappa(t, x, y')\right| \leqslant c(|y - y'|^\gamma \wedge 1) \Big[h^{-1}(1/t)\Big]^{-\gamma} \big(\Upsilon_t(y - x) + \Upsilon_t(y - x')\big).$$

The constants in (4)–(6) may be chosen to depend only on $d, \gamma_0, \kappa_0, \kappa_1, \kappa_2, \beta, \alpha_h, C_h, h, T$ (and β_h, c_h in the case (P2)). The same for (9) and (10) but with additional dependence on γ.

For $t > 0$ we define

$$P_t^\kappa f(x) = \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y)\, dy , \quad x \in \mathbb{R}^d , \tag{16}$$

whenever the integral exists in the Lebesgue sense. We also put P_0^κ to be the identity operator.

Theorem 1.3. *Assume* (P). *The following hold true.*

(1) *$(P_t^\kappa)_{t\geqslant 0}$ is an analytic strongly continuous positive contraction semigroup on $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$.*
(2) *$(P_t^\kappa)_{t\geqslant 0}$ is an analytic strongly continuous semigroup on every $(L^p(\mathbb{R}^d), \|\cdot\|_p)$, $p \in [1, \infty)$.*
(3) *Let $(\mathcal{A}^\kappa, D(\mathcal{A}^\kappa))$ be the generator of $(P_t^\kappa)_{t\geqslant 0}$ on $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$. Then*

(a) $C_0^2(\mathbb{R}^d) \subseteq D(\mathcal{A}^\kappa)$ *and* $\mathcal{A}^\kappa = \mathcal{L}^\kappa$ *on* $C_0^2(\mathbb{R}^d)$,
(b) $(\mathcal{A}^\kappa, D(\mathcal{A}^\kappa))$ *is the closure of* $(\mathcal{L}^\kappa, C_c^\infty(\mathbb{R}^d))$,
(c) *the function* $x \mapsto p^\kappa(t,x,y)$ *belongs to* $D(\mathcal{A}^\kappa)$ *for all* $t>0$, $y \in \mathbb{R}^d$, *and*

$$\mathcal{A}_x^\kappa\, p^\kappa(t,x,y) = \mathcal{L}_x^\kappa\, p^\kappa(t,x,y) = \partial_t p^\kappa(t,x,y)\,, \qquad x \in \mathbb{R}^d\,.$$

(4) *Let* $(\mathcal{A}^\kappa, D(\mathcal{A}^\kappa))$ *be the generator of* $(P_t^\kappa)_{t\geqslant 0}$ *on* $(L^p(\mathbb{R}^d), \|\cdot\|_p)$, $p \in [1,\infty)$. *Then*
(a) $C_c^2(\mathbb{R}^d) \subseteq D(\mathcal{A}^\kappa)$ *and* $\mathcal{A}^\kappa = \mathcal{L}^\kappa$ *on* $C_c^2(\mathbb{R}^d)$,
(b) $(\mathcal{A}^\kappa, D(\mathcal{A}^\kappa))$ *is the closure of* $(\mathcal{L}^\kappa, C_c^\infty(\mathbb{R}^d))$,
(c) *the function* $x \mapsto p^\kappa(t,x,y)$ *belongs to* $D(\mathcal{A}^\kappa)$ *for all* $t>0$, $y \in \mathbb{R}^d$, *and in* $L^p(\mathbb{R}^d)$,

$$\mathcal{A}^\kappa\, p^\kappa(t,\cdot,y) = \mathcal{L}^\kappa\, p^\kappa(t,\cdot,y) = \partial_t p^\kappa(t,\cdot,y)\,.$$

Finally, (by probabilistic methods) we provide a lower bound for the heat kernel $p^\kappa(t,x,y)$. The defnition of σ can be found at the beginning of Section 2.

Theorem 1.4. *Assume* (P). *The following hold true.*

(i) *There are* $T_0 = T_0(d,\nu,\sigma,\kappa_2,\beta) > 0$ *and* $c = c(d,\nu,\sigma,\kappa_2,\beta) > 0$ *such that for all* $t \in (0,T_0]$, $x,y \in \mathbb{R}^d$,

$$p^\kappa(t,x,y) \geqslant c\left([h^{-1}(1/t)]^{-d} \wedge t\nu(|x-y|)\right). \tag{17}$$

(ii) *If additionally* ν *is positive, then for every* $T>0$ *there is* $c = c(d,T,\nu,\sigma,\kappa_2,\beta) > 0$ *such that* (17) *holds for all* $t \in (0,T]$ *and* $x,y \in \mathbb{R}^d$.
(iii) *If additionally there are* $\bar\beta \in [0,2)$ *and* $\bar c > 0$ *such that* $\bar c \lambda^{d+\bar\beta}\nu(\lambda r) \leqslant \nu(r)$, $\lambda \leqslant 1$, $r>0$, *then for every* $T>0$ *there is* $c = c(d,T,\nu,\sigma,\kappa_2,\beta,\bar c,\bar\beta) > 0$ *such that for all* $t \in (0,T]$ *and* $x,y \in \mathbb{R}^d$,

$$p^\kappa(t,x,y) \geqslant c\Upsilon_t(y-x)\,. \tag{18}$$

Remark 1.5. Theorem 1.3 guarantees that $(P_t^\kappa)_{t\geqslant 0}$ is a Feller semigroup and therefore there exists *the canonical Feller process* $X = (X_t)_{t\geqslant 0}$ corresponding to $(P_t^\kappa)_{t\geqslant 0}$ with trajectories that are càdlàg functions (see [43, page 380]). The process X is the unique *solution to the martingale problem* for $(\mathcal{L}^\kappa, C_c^\infty(\mathbb{R}^d))$. The latter follows from part (3a) of Theorem 1.3 and [23, Theorem 4.4.1] (see also [23, Theorem 1.2.12 and Proposition 4.1.7]).

Remark 1.6. The upper estimate of the heat kernel leads to a sufficient condition for a Borel measure to belong to the Kato class with respect to $p^\kappa(t,x,y)$, equivalently, to $X = (X_t)_{t\geqslant 0}$. Similarly, the lower bound provides a necessary condition (cf. [49, Theorem 2.7]). Moreover, if (P) and the assumption of Theorem 1.4(iii) are satisfied, then p^κ is locally in time and globally in space comparable with the heat kernel p of a pure-jump Lévy process $Y = (Y_t)_{t\geqslant 0}$ corresponding to $\nu(|x|)$ (see Section 5, [30, Remark 5.7 and Corollary 5.14]). Thus the Kato class for X and Y is the same. The function Kato classes that consist of absolutely continuous measures are for Lévy processes well studied [31].

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We would like to stress that whenever refering to [30] we mean the first version of the preprint. It is important since the content of [30] (after certain extensions) was divided into two parts: [29] containing results for gerneral Lévy processes and (upcoming version of) [30] focusing on off-diagonal heat kernel estimates.

Remark 1.7. If (2), (3) hold, then $|\kappa(x,z)-\kappa(y,z)| \leqslant (2\kappa_1 \vee \kappa_2)|x-y|^{\beta_1}$ for every $\beta_1 \in [0,\beta]$.

For the purpose of the introduction we give an example right at this moment.

Example 1. Our results apply if (1) holds with $\nu(r) = r^{-d}[\log(1+r^{\alpha/2})]^{-2}$, where $\alpha \in (0,2)$. Indeed, the conditions (4) and (5) are satisfied with $\alpha_h = \beta_h = \alpha$, see [30, Example 2]. Further, Theorem 1.4(iii) also applies. We emphasize that such ν does not have the logarithmic moment at infinity,

$$\int_{\mathbb{R}^d} \ln\left(1+|z|^2\right)\nu(|z|)dz = \infty\,.$$

The non-local integro-differential operators under our considerations belong to the class of operators known as *Lévy-type*. Due to the Courrège-Waldenfels theorem [35, Theorem 4.5.21], [10, Theorem 2.21] those operators are generic for Feller semigroups whose infinitesimal generator has sufficiently rich domain. We refer the reader to [35–37] and [10] for a broad survey on Lévy-type operators. Nevertheless, it is highly non-trivial to construct the semigroup from a given Lévy-type operator with non-constant coefficients, and even more difficult to investigate its heat kernel. The tool used in this paper is the parametrix method, proposed by E. Levi [59] to solve elliptic Cauchy problems. It was successfully applied in the theory of partial differential equations [27], [63], [19], [24], with an overview in the monograph [25], as well as in the theory of pseudo-differential operators [22], [45], [49], [55], [69]. In particular, operators comparable in a sense with the fractional Laplacian were intensively studied [20], [21], [52], [54], [22], also very recently [16], [41], [15], [56]. More detailed historical comments on the development of the method can be found in [25, Bibliographical Remarks] and in the introductions of [49] and [5].

We will now elaborate on our assumptions in view of the literature in terms of two selected aspects: the admissible Lévy measures and the symmetry condition. This will not fully exhaust the relations between all various papers, their assumptions and results.

First we focus on the Lévy measure $J(z)dz$ and we point out three papers [16], [45], [49], two of which are at the opposite poles. In the paper [16] the authors concentrate on a particular isotropic α-stable case $J(z) = |z|^{-d-\alpha}$, $\alpha \in (0,2)$, and, among other things, give explicit estimates of the fundamental solution. In [49] much more general not necessarily absolutely continuous Lévy measures are treated, but the estimates are stated in a rather implicit form of compound kernels. Finally the paper [45] is situated between those extremes. The authors of [45] follow the road-map of [16] and consider $J(z)$ comparable with a Lévy density $j(|z|)$ of a subordinate Brownian motion. In this respect our assumption is given by (1) and stands for the comparability of $J(z)$ with an isotropic unimodal Lévy density $\nu(|z|)$, which allows for much larger class of Lévy measures than in [45]. In particular, we can consider compactly supported Lévy measures. With this in mind it locates us between [45] and [49].

Another assumption on the Lévy measure is the weak scaling (4), which naturally generalizes the scaling property of the isotropic α-stable case [16], and is also present in [45] and [49].

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More precisely, the condition [45, (1.4)] is equivalent to (4) due to (85) and (86), while under (1) the condition [49, A1] is equivalent to (4). The latter is a consequence of the equivalence of conditions (C3) and (C4) in [30, Theorem 3.1], (A1) and (A3) in [30, Lemma 2.3] and (85) below. In other words, here our assumptions coincide with those of [49] restricted to absolutely continuous Lévy measures satisfying (1). In fact, in (P2) we also need one more weak scaling (5), but this case is not in question of any of the papers [16], [45], [49].

Furthermore, in comparison with [45] we avoid two more technical assumptions [45, (1.5) and (1.9)] on the behavior of the Lévy measure at infinity. This is achieved by the choice of the form of the bound function $\Upsilon_t(x)$ supported by outcomes of [30], and the formulation of the maximum principle in Theorem 4.1. We note for instance that the Lévy measure in Example 1, which is admissible by our assumptions, does not satisfy [45, (1.5)], see (85) and Lemma 6.2, so the result of [45] cannot be applied in that case.

The assumptions (2) and (3) on the function $\kappa(x,z)$ are common. In both papers [16] and [45] also the symmetry condition, i.e., the symmetry of J and $\kappa(x,z)=\kappa(x,-z)$ for all $x,z\in\mathbb{R}^d$, is required. We cover such situation in the case (P3). We note in passing that this is a different symmetry than the one used in the theory of Dirichlet forms [26]. In the cases (P1) and (P2) the symmetry condition is absent. As explained before (9) the symmetry enables to represent the operator $\mathcal{L}^\kappa$ in various equivalent forms, which facilitates calculations. In the non-symmetric case the intrinsic drift $\int_{|z|<1} z\kappa(x,z)J(z)dz$ may not be negligible and one has to be more specific in the choice of the operator. In two recent papers [41] and [15] the authors investigate the non-symmetric case for $J(z)=|z|^{-d-\alpha}$ and they consider the operator (a) (6) if $\alpha\in(1,2)$; (b) (6) if $\alpha=1$ and $\int_{r<|z|<R} z\kappa(x,z)J(z)dz=0$; (c) (7) if $\alpha\in(0,1)$. The cases (a) and (b) are covered in the present paper by cases (P1) and (P2). The case (b) with extensions is a subject of our forthcoming paper. In [49], apart from the symmetric case, also (a) and (P1) are included in the discussion (with the presence of a bounded Hölder continuous first order term).

Finally we devote a few words to qualitative improvements that we make even in the cases discussed in [16] and [45]. First of all in Theorem 1.1 we significantly simplify the formulation of the uniqueness of p^κ. In Theorem 1.2 we extend the range of α_h and β for which the gradient ∇p^κ exists, we prove joint continuity of $\mathcal{L}^\kappa p^\kappa$ and Hölder continuity in the second spatial coordinate of p^κ. In Theorem 1.3 we provide more detailed analysis of the semigroup P_t^κ and its generator on various spaces. As a consequence in part (8) of Theorem 1.2 we have that $p^\kappa(t,x,y)$ solves the equation $\partial_t=\mathcal{L}_x^\kappa$ (and $\partial_t=\mathcal{L}_x^{\kappa,0^+}$) for all $t>0$, $x,y\in\mathbb{R}^d$, without the restriction $x\neq y$ (cf. [16, (1.7)], [45, (1.10)]). Up to our knowledge the solvability of the equation with the strong operator $\mathcal{L}_x^\kappa$ is a novelty, and demands many technical reinforcements.

To sum up, we utterly generalize [16] and [45] by restricting very weak assumptions of [49] to Lévy measures satisfying (1) (the case (P2) is not considered in [49]). Moreover, we strengthen certain results even for the isotopic α-stable case [16] and we propose new outcomes. We also extend the core parts of [41] and [15] for the non-symmetric case (excluding non-symmetry with $\alpha=1$, time-dependence and small Kato drift). Other closely related papers treat for instance (symmetric) singular Lévy measures [5], [56] or (symmetric) exponential Lévy measures [44]. Our contribution is that under relatively weak assumptions, and with a satisfactory generality that allows for non-symmetric Lévy measures, we obtain explicit results, which are a proper extension of the α-stable case. To avoid ambiguity we give full proofs of all statements. We also refer the reader to [17] for partial survey and correction of certain gaps of [16].

In order to start the procedure of constructing the solution to the Lévy-type operator one needs suitable knowledge about the solution to the operator with frozen coefficients which leads back to the Lévy case. This initial information usually determines the results accessible by the parametrix method. Therefore we observe pairs of papers like [14,16], [42,45], [46,49], [47,48], [7,5]. In our case we base on the results of [30], which has roots in [42]. Another important ingredient of the preliminaries are the so-called convolution inequalities used to deal with multiply iterated integrals that appear in the construction. For the α-stable case they can be found for instance in [52, Lemma 5]. In Lemma 5.17 we propose a refined version motivated by [45, Lemma 2.6] with more parameters and for function ρ_β^γ defined by means of the bound function.

There exist other methods to associate semigroup and heat kernel to an operator. Some rely on the symbolic calculus [67], [33], [58], [32], [34], [36], [8], [9], other on Dirichlet forms [26], [11], [1], [12], [13] or perturbation series [62], [4], [39], [40], [38], [6]. For probabilistic methods and applications we refer the reader to [18], [53], [61], [50], [57], [56].

The remainder of the paper is organized as follows. In Section 2 we use the results of [30] as a starting point to establish further uniform properties of the heat kernel $p^{\mathfrak{K}}(t,x,y)$ of the Lévy operator $\mathcal{L}^{\mathfrak{K}}$. In Section 3 we carry out the construction of $p^\kappa(t,x,y)$ and we prove its primary properties. According to the parametrix method we anticipate that

$$p^\kappa(t,x,y) = p^{\mathfrak{K}_y}(t,x,y) + \int_0^t \int_{\mathbb{R}^d} p^{\mathfrak{K}_z}(t-s,x,z)q(s,z,y)\,dzds\,,$$

where $q(t,x,y)$ solves the equation

$$q(t,x,y) = q_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)q(s,z,y)\,dzds\,,$$

and $q_0(t,x,y) = \left(\mathcal{L}_x^{\mathfrak{K}_x} - \mathcal{L}_x^{\mathfrak{K}_y}\right)p^{\mathfrak{K}_y}(t,x,y)$. Here $p^{\mathfrak{K}_w}$ is the heat kernel of the Lévy operator $\mathcal{L}^{\mathfrak{K}_w}$ obtained from the operator $\mathcal{L}^\kappa$ by freezing its coefficients: $\mathfrak{K}_w(z) = \kappa(w,z)$. In Section 3.1 we examine $p^{\mathfrak{K}_y}(t,x,y)$. In Section 3.2 we define $q(t,x,y)$ explicitly via the perturbation series and we study its properties. In Section 3.3 we investigate $\phi_y(t,x) = \int_0^t \int_{\mathbb{R}^d} p^{\mathfrak{K}_z}(t-s,x,z)q(s,z,y)\,dzds$, which is the most technical part, and several improvements that we make there affect the eventual results. Finally, in Section 3.4 we collect initial properties of p^κ that follow directly from the construction. In Section 4 we establish a nonlocal maximum principle, analyze the semigroup $(P_t^\kappa)_{t\geq 0}$, complement the fundamental properties of p^κ and prove Theorems 1.1–1.3. In Section 5 and 6 we store auxiliary results such as features of the bound function, 3G-type inequalities, convolution inequalities.

We end this section with comments on the notation. Throughout the article $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface measure of the unit sphere in $\mathbb{R}^d$. By $c(d,\ldots)$ we denote a generic positive constant that depends only on the listed parameters $d,\ldots$. As usual $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$. We use ":=" to denote a definition. In what follows the constants γ_0, κ_0, κ_1, κ_2, β, α_h, C_h, β_h, c_h can be regarded as fixed.

Excluding Section 5 and 6 **we assume in the whole paper that** (P) **holds**. However, in theorems and propositions we explicitly formulate all assumptions. If needed we make a restriction to (P1) or (P2) or (P3).

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2. Analysis of the heat kernel of $\mathcal{L}^{\mathfrak{K}}$

In this section we assume that $\mathfrak{K}\colon \mathbb{R}^d \to [0,\infty)$ is such that

$$0 < \kappa_0 \leqslant \mathfrak{K}(z) \leqslant \kappa_1 \,.$$

In the case (P1), (P2), (P3) we consider an operator $\mathcal{L}^{\mathfrak{K}}$ defined by taking $\kappa(x,z) = \mathfrak{K}(z)$ in (6), (7), (8), respectively. The operator uniquely determines a Lévy process and its density $p^{\mathfrak{K}}(t,x,y) = p^{\mathfrak{K}}(t,y-x)$ (see Section 6. In particular, (96) holds by (4), (86), (85) and (1)). To simplify the notation we introduce

$$\delta_1^{\mathfrak{K}}(t,x,y;z) := p^{\mathfrak{K}}(t,x+z,y) - p^{\mathfrak{K}}(t,x,y) - \mathbf{1}_{|z|<1}\left\langle z, \nabla_x p^{\mathfrak{K}}(t,x,y)\right\rangle, \tag{19}$$

$$\delta_2^{\mathfrak{K}}(t,x,y;z) := p^{\mathfrak{K}}(t,x+z,y) - p^{\mathfrak{K}}(t,x,y)\,, \tag{20}$$

$$\delta_3^{\mathfrak{K}}(t,x,y;z) := \frac{1}{2}(p^{\mathfrak{K}}(t,x+z,y) + p^{\mathfrak{K}}(t,x-z,y) - 2p^{\mathfrak{K}}(t,x,y))\,. \tag{21}$$

Thus we have

$$\mathcal{L}_x^{\mathfrak{K}_1}\, p^{\mathfrak{K}_2}(t,x,y) = \int\limits_{\mathbb{R}^d} \delta^{\mathfrak{K}_2}(t,x,y;z)\,\mathfrak{K}_1(z) J(z)dz\,, \tag{22}$$

where $\delta^{\mathfrak{K}}$ is one of the above functions appropriate to the case under consideration. We also introduce the sets of parameters $\sigma_1 = (\gamma_0, \kappa_0, \kappa_1, \alpha_h, C_h, h)$, $\sigma_2 = (\gamma_0, \kappa_0, \kappa_1, \alpha_h, \beta_h, C_h, c_h, h)$, $\sigma_3 = (\gamma_0, \kappa_0, \kappa_1, \alpha_h, C_h, h)$, and we write shortly σ if the case is clear from the context.

The result below is the initial point of the whole paper.

Proposition 2.1. *Assume* (P). *For every $T > 0$ and $\beta \in \mathbb{N}_0^d$ there exists a constant $c = c(d,T,\beta,\sigma)$ such that for all $t \in (0,T]$, $x,y \in \mathbb{R}^d$,*

$$|\partial_x^{\beta} p^{\mathfrak{K}}(t,x,y)| \leqslant c\left[h^{-1}(1/t)\right]^{-|\beta|} \Upsilon_t(y-x)\,.$$

Proof. In the case (P1), (P2) and (P3) the result follows from [30, Section 5.2]. □

Lemma 2.2. *Assume* (P). *For every $T, \theta > 0$ there exists a constant $\tilde{c} = \tilde{c}(d,T,\theta,\nu,\sigma)$ such that for all $t \in (0,T]$ and $|x-y| \leqslant \theta h^{-1}(1/t)$,*

$$p^{\mathfrak{K}}(t,x,y) \geqslant \tilde{c}\left[h^{-1}(1/t)\right]^{-d}.$$

Proof. In the case (P3) the estimate follows from [30, Corollary 5.11]. In the cases (P1) and (P2) we also use [30, Corollary 5.11] but with $x - y - tb_{[h_0^{-1}(1/t)]}$ in place of x as we have that $|tb_{[h_0^{-1}(1/t)]}| \leqslant a h_0^{-1}(1/t)$ for $a = a(d,T,\sigma)$, see proof of Proposition 5.9 and 5.10 in [30]. □

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2.1. Increments and integrals of $p^{\mathfrak{K}}(t,x,y)$

We simplify the notation by introducing the following expressions. For $t>0$, $x,y,z\in\mathbb{R}^d$,

$$\mathcal{F}_1 := \Upsilon_t(y-x-z)\mathbf{1}_{|z|\geqslant h^{-1}(1/t)} + \left[\left(\frac{|z|}{h^{-1}(1/t)}\right)^2 \wedge \left(\frac{|z|}{h^{-1}(1/t)}\right)\right]\Upsilon_t(y-x),$$

$$\mathcal{F}_2 := \Upsilon_t(y-x-z)\mathbf{1}_{|z|\geqslant h^{-1}(1/t)} + \left[\left(\frac{|z|}{h^{-1}(1/t)}\right) \wedge 1\right]\Upsilon_t(y-x),$$

$$\mathcal{F}_3 := \Upsilon_t(y-x\pm z)\mathbf{1}_{|z|\geqslant h^{-1}(1/t)} + \left[\left(\frac{|z|}{h^{-1}(1/t)}\right)^2 \wedge 1\right]\Upsilon_t(y-x).$$

In the last line we use $f(x\pm z)$ in place of $f(x+z)+f(x-z)$. Hereinafter we add arguments $(t,x,y;z)$ when referring to functions defined above.

Lemma 2.3. *Assume* (P). *For every $T>0$ there exists a constant $c=c(d,T,\sigma)$ such that for all $t\in(0,T]$, $x,y,z\in\mathbb{R}^d$ we have* $\left|p^{\mathfrak{K}}(t,x+z,y)-p^{\mathfrak{K}}(t,x,y)\right|\leqslant c\,\mathcal{F}_2(t,x,y;z)$.

Proof. If $|z|\geq h^{-1}(1/t)$, the result follows from Proposition 2.1. If $|z|<h^{-1}(1/t)$, we use Proposition 2.1 and

$$p^{\mathfrak{K}}(t,x+z,y)-p^{\mathfrak{K}}(t,x,y) = \int_0^1 \left\langle z, \nabla_x p^{\mathfrak{K}}(t,x+\theta z,y)\right\rangle d\theta\,,$$

to obtain $|p^{\mathfrak{K}}(t,x+z,y)-p^{\mathfrak{K}}(t,x,y)|\leq c_1(|z|/h^{-1}(1/t))\int_0^1\Upsilon_t(y-x-\theta z)d\theta$. Since $\theta|z|\leq h^{-1}(1/t)$, we get from Corollary 5.10 that

$$|p^{\mathfrak{K}}(t,x+z,y)-p^{\mathfrak{K}}(t,x,y)|\leqslant c_2\left(\frac{|z|}{h^{-1}(1/t)}\right)\Upsilon_t(y-x)\,. \quad \square$$

Lemma 2.4. *Assume* (P1). *For every $T>0$ there exists a constant $c=c(d,T,\sigma_1)$ such that for all $t\in(0,T]$, $x,y,z\in\mathbb{R}^d$ we have* $|\delta_1^{\mathfrak{K}}(t,x,y;z)|\leqslant c\big(\mathcal{F}_1(t,x,y;z)\mathbf{1}_{|z|<1}+\mathcal{F}_2(t,x,y;z)\mathbf{1}_{|z|\geqslant 1}\big)$.

Proof. For $|z|\geqslant 1$ we apply Lemma 2.3. Let $|z|<1$. If $|z|\geqslant h^{-1}(1/t)$, then by Proposition 2.1,

$$|\delta_1^{\mathfrak{K}}(t,x,y;z)|\leqslant c\left(\Upsilon_t(y-x-z)+\left(\frac{|z|}{h^{-1}(1/t)}\right)\Upsilon_t(y-x)\right).$$

If $|z|<h^{-1}(1/t)$, by Proposition 2.1 and Corollary 5.10 we have

$$|\delta_1^{\mathfrak{K}}(t,x,y;z)|\leqslant |z|^2\sum_{|\beta|=2}\int_0^1\int_0^1 |\partial_x^{\beta}p^{\mathfrak{K}}(t,x+\theta'\theta z,y)|d\theta' d\theta$$

$$\leqslant c\left(\frac{|z|}{h^{-1}(1/t)}\right)^2\Upsilon_t(y-x)\,. \quad \square$$

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Lemma 2.5. *Assume* (P1). *For every $T>0$ there exists a constant $c=c(d,T,\sigma_1)$ such that for all $t\in(0,T]$, $x,x',y,z\in\mathbb{R}^d$ satisfying $|x'-x|<h^{-1}(1/t)$ we have*

$$|\delta_1^{\mathfrak{K}}(t,x',y;z)-\delta_1^{\mathfrak{K}}(t,x,y;z)|\leqslant c\left(\frac{|x'-x|}{h^{-1}(1/t)}\right)\left(\mathcal{F}_1(t,x,y;z)\mathbf{1}_{|z|<1}+\mathcal{F}_2(t,x,y;z)\mathbf{1}_{|z|\geqslant 1}\right).$$

Proof. We denote $w=x'-x$ and we use Proposition 2.1 and Corollary 5.10 repeatedly. Note that $|w|<h^{-1}(1/t)$ and

$$\delta_1^{\mathfrak{K}}(t,x+w,y;z)-\delta_1^{\mathfrak{K}}(t,x,y;z)=\int_0^1\left\langle w,\nabla_x\delta_1^{\mathfrak{K}}(t,x+\theta w,y;z)\right\rangle d\theta\,. \tag{23}$$

For $|z|\geqslant 1$, if $|z|\geqslant h^{-1}(1/t)$, we apply (23) to get

$$|\delta_1^{\mathfrak{K}}(t,x+w,y;z)-\delta_1^{\mathfrak{K}}(t,x,y;z)|\leqslant c\left(\frac{|w|}{h^{-1}(1/t)}\right)\left(\Upsilon_t(y-x-z)+\Upsilon_t(y-x)\right).$$

If $|z|<h^{-1}(1/t)$, we have

$$\begin{aligned}|\delta_2^{\mathfrak{K}}(t,x+w,y;z)-\delta_2^{\mathfrak{K}}(t,x,y;z)|&\leqslant |z||w|\sum_{|\beta|=2}\int_0^1\int_0^1|\partial_x^{\beta}p^{\mathfrak{K}}(t,x+\theta w+\theta' z,y)|\,d\theta' d\theta\\&\leqslant c\left(\frac{|w|}{h^{-1}(1/t)}\right)\left(\frac{|z|}{h^{-1}(1/t)}\right)\Upsilon_t(y-x)\,.\end{aligned}$$

Let $|z|<1$. If $|z|\geqslant h^{-1}(1/t)$, then we use (23) to obtain

$$\begin{aligned}&|\delta_1^{\mathfrak{K}}(t,x+w,y;z)-\delta_1^{\mathfrak{K}}(t,x,y;z)|\\&\leqslant c\left(\frac{|w|}{h^{-1}(1/t)}\right)\left(\Upsilon_t(y-x-z)+\left(\frac{|z|}{h^{-1}(1/t)}\right)\Upsilon_t(y-x)\right).\end{aligned}$$

If $|z|<h^{-1}(1/t)$, then

$$\begin{aligned}&|\delta_1^{\mathfrak{K}}(t,x+w,y;z)-\delta_1^{\mathfrak{K}}(t,x,y;z)|\\&\leqslant |w||z|^2\sum_{|\beta|=3}\int_0^1\int_0^1\int_0^1|\partial_x^{\beta}p^{\mathfrak{K}}(t,x+\theta w+\theta''\theta' z,y)|\,d\theta'' d\theta' d\theta\\&\leqslant c\left(\frac{|w|}{h^{-1}(1/t)}\right)\left(\frac{|z|}{h^{-1}(1/t)}\right)^2\Upsilon_t(y-x)\,.\qquad\square\end{aligned}$$

We note that the estimate for $|\delta_2^{\mathfrak{K}}(t,x,y;z)|\leqslant c\,\mathcal{F}_2(t,x,y;z)$ is given in Lemma 2.3.

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Lemma 2.6. *Assume* (P2). *For every $T>0$ there exists a constant $c=c(d,T,\sigma_2)$ such that for all $t\in(0,T]$, $x,x',y,z\in\mathbb{R}^d$ satisfying $|x'-x|<h^{-1}(1/t)$ we have*

$$|\delta_2^{\mathfrak{K}}(t,x',y;z)-\delta_2^{\mathfrak{K}}(t,x,y;z)|\leqslant c\left(\frac{|x'-x|}{h^{-1}(1/t)}\right)\mathcal{F}_2(t,x,y;z)\,.$$

Proof. The proof is the same as for the case $|z|\geqslant h^{-1}(1/t)$ in the proof of Lemma 2.5. □

Lemma 2.7. *Assume* (P3). *For every $T>0$ there exists a constant $c=c(d,T,\sigma_3)$ such that for all $t\in(0,T]$, $x,y,z\in\mathbb{R}^d$ we have $|\delta_3^{\mathfrak{K}}(t,x,y;z)|\leqslant c\,\mathcal{F}_3(t,x,y;z)$.*

Proof. If $|z|\geqslant h^{-1}(1/t)$ we apply Proposition 2.1. If $|z|<h^{-1}(1/t)$, by Proposition 2.1 and Corollary 5.10 we get

$$\begin{aligned}|\delta_3^{\mathfrak{K}}(t,x,y;z)|&\leqslant |z|^2\sum_{|\beta|=2}\int_0^1\int_{-1}^1|\partial_x^{\beta}p^{\mathfrak{K}}(t,x+\theta'\theta z,y)|\,d\theta' d\theta\\&\leqslant c\left(\frac{|z|}{h^{-1}(1/t)}\right)^2\Upsilon_t(y-x)\,. \quad\square\end{aligned}$$

Lemma 2.8. *Assume* (P3). *For every $T>0$ the exists a constant $c=c(d,T,\sigma_3)$ such that for all $t\in(0,T]$, $x,x',y,z\in\mathbb{R}^d$ satisfying $|x'-x|<h^{-1}(1/t)$ we have*

$$|\delta_3^{\mathfrak{K}}(t,x',y;z)-\delta_3^{\mathfrak{K}}(t,x,y;z)|\leqslant c\left(\frac{|x'-x|}{h^{-1}(1/t)}\right)\mathcal{F}_3(t,x,y;z)\,.$$

Proof. Let $w=x'-x$. We use Proposition 2.1 and Corollary 5.10 repeatedly. We have

$$\delta_3^{\mathfrak{K}}(t,x+w,y;z)-\delta_3^{\mathfrak{K}}(t,x,y;z)=\int_0^1\Big\langle w,\nabla_x\delta_3^{\mathfrak{K}}(t,x+\theta w,y;z)\Big\rangle d\theta\,.$$

If $|z|\geqslant h^{-1}(1/t)$, then

$$|\delta_3^{\mathfrak{K}}(t,x+w,y;z)-\delta_3^{\mathfrak{K}}(t,x,y;z)|\leqslant c\left(\frac{|w|}{h^{-1}(1/t)}\right)\left(\Upsilon_t(y-x\pm z)+\Upsilon_t(y-x)\right).$$

If $|z|<h^{-1}(1/t)$, then

$$\begin{aligned}&|\delta_3^{\mathfrak{K}}(t,x+w,y;z)-\delta_3^{\mathfrak{K}}(t,x,y;z)|\\&\leqslant |w||z|^2\sum_{|\beta|=3}\int_0^1\int_0^1\int_{-1}^1|\partial_x^{\beta}p^{\mathfrak{K}}(t,x+\theta w+\theta''\theta' z,y)|\,d\theta'' d\theta' d\theta\\&\leqslant c\left(\frac{|w|}{h^{-1}(1/t)}\right)\left(\frac{|z|}{h^{-1}(1/t)}\right)^2\Upsilon_t(y-x)\,. \quad\square\end{aligned}$$

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The next result is the counterpart of [16, Theorem 2.4] and [45, Theorem 3.4].

Theorem 2.9. *Assume* (P). *For every $T>0$ there exists a constant $c=c(d,T,\sigma)$ such that for all $t\in(0,T]$, $x,x',y\in\mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} |\delta^{\mathfrak{K}}(t,x,y;z)|\,\nu(|z|)dz \leqslant ct^{-1}\Upsilon_t(y-x)\,, \quad \textit{and}$$

$$\int_{\mathbb{R}^d} |\delta^{\mathfrak{K}}(t,x',y;z)-\delta^{\mathfrak{K}}(t,x,y;z)|\,\nu(|z|)dz$$

$$\leqslant c\left(\frac{|x'-x|}{h^{-1}(1/t)}\wedge 1\right)t^{-1}\big(\Upsilon_t(y-x')+\Upsilon_t(y-x)\big).$$

Proof. The first statement follows immediately from Lemma 2.4, 2.3 and 2.7 supported by Lemma 5.5 and 5.9. We prove the second part. If $|x'-x|\geqslant h^{-1}(1/t)$, then

$$\int_{\mathbb{R}^d} \Big(|\delta^{\mathfrak{K}}(t,x',y;z)|+|\delta^{\mathfrak{K}}(t,x,y;z)|\Big)\,\nu(|z|)dz \leqslant ct^{-1}\left(\Upsilon_t(y-x')+\Upsilon_t(y-x)\right)\,.$$

If $|x'-x|<h^{-1}(1/t)$, we rely on Lemma 2.5, 2.6 and 2.8 as well as Lemma 5.5 and 5.9. $\square$

2.2. *Continuous dependence of heat kernels with respect to $\mathfrak{K}$*

We discuss $\mathfrak{K}$, $\mathfrak{K}_1$, $\mathfrak{K}_2$ as introduced at the beginning of Section 2. In what follows $\|\cdot\|=\|\cdot\|_\infty$.

Lemma 2.10. *Assume* (P). *For all $t>0$, $x,y\in\mathbb{R}^d$ and $s\in(0,t)$,*

$$\frac{d}{ds}\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,x,z)p^{\mathfrak{K}_2}(t-s,z,y)\,dz$$

$$=\int_{\mathbb{R}^d} \mathcal{L}_x^{\mathfrak{K}_1} p^{\mathfrak{K}_1}(s,x,z)\,p^{\mathfrak{K}_2}(t-s,z,y)\,dz-\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,x,z)\,\mathcal{L}_z^{\mathfrak{K}_2}p^{\mathfrak{K}_2}(t-s,z,y)\,dz\,,$$

and

$$\int_{\mathbb{R}^d} \mathcal{L}_x^{\mathfrak{K}} p^{\mathfrak{K}_1}(s,x,z)p^{\mathfrak{K}_2}(t-s,z,y)\,dz=\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,x,z)\,\mathcal{L}_z^{\mathfrak{K}}p^{\mathfrak{K}_2}(t-s,z,y)\,dz\,.$$

Proof. Note that the difference quotient equals

$$\int_{\mathbb{R}^d} \frac{1}{h}\Big[p^{\mathfrak{K}_1}(s+h,x,z)-p^{\mathfrak{K}_1}(s,x,z)\Big]\,p^{\mathfrak{K}_2}(t-s-h,z,y)\,dz$$

$$+\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,x,z)\frac{1}{h}\left[p^{\mathfrak{K}_2}(t-s-h,z,y)-p^{\mathfrak{K}_2}(t-s,z,y)\right]dz\,.$$

If $2|h|<(t-s)\wedge s$, by Lemma 6.1(a), (22), Theorem 2.9, Proposition 2.1 and (94) the first integrand is bounded by $s^{-1}\Upsilon_{s/2}(z-x)\Upsilon_{(t-s)/2}(y-z)$ up to multiplicative constant. Therefore we can use the dominated convergence theorem. Similarly, we deal with the second integral. Next, by (22), Theorem 2.9 and Proposition 2.1 the following integrals converge absolutely and thus the change of the order of integration is justified,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_1}(s,x,z;w)\,\mathfrak{K}(w)J(w)dw\right)p^{\mathfrak{K}_2}(t-s,z,y)dz\\
&\quad=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_1}(s,x,z;w)\,p^{\mathfrak{K}_2}(t-s,z,y)dz\right)\mathfrak{K}(w)J(w)dw\\
&\quad=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}p^{\mathfrak{K}_1}(s,x,z)\delta^{\mathfrak{K}_2}(t-s,z,y;w)dz\right)\mathfrak{K}(w)J(w)dw\\
&\quad=\int_{\mathbb{R}^d}p^{\mathfrak{K}_1}(s,x,z)\left(\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_2}(t-s,z,y;w)\,\mathfrak{K}(w)J(w)dw\right)dz\,.
\end{aligned}$$

In the third equality in the case (P1) we used integration by parts. $\square$

The following result is the counterpart of [16, Theorem 2.5] and [45, Theorem 3.5].

Theorem 2.11. *Assume* (P). *For every $T>0$ there exists a constant $c=c(d,T,\sigma)$ such that for all $t\in(0,T]$, $x,y,z\in\mathbb{R}^d$,*

$$|p^{\mathfrak{K}_1}(t,x,y)-p^{\mathfrak{K}_2}(t,x,y)|\leqslant c\|\mathfrak{K}_1-\mathfrak{K}_2\|\Upsilon_t(y-x)\,,$$

$$|\nabla_x p^{\mathfrak{K}_1}(t,x,y)-\nabla_x p^{\mathfrak{K}_2}(t,x,y)|\leqslant c\|\mathfrak{K}_1-\mathfrak{K}_2\|\left[h^{-1}(1/t)\right]^{-1}\Upsilon_t(y-x)\,,$$

$$\int_{\mathbb{R}^d}|\delta^{\mathfrak{K}_1}(t,x,y;z)-\delta^{\mathfrak{K}_2}(t,x,y;z)|\,\nu(|z|)dz\leqslant c\|\mathfrak{K}_1-\mathfrak{K}_2\|t^{-1}\Upsilon_t(y-x)\,.$$

Proof. (i) The first equality below follows from the strong continuity of the semigroup of a Lévy process and Lemma 6.1(b). Then by Lemma 2.10,

$$p^{\mathfrak{K}_1}(t,x,y)-p^{\mathfrak{K}_2}(t,x,y)=\lim_{\varepsilon_1,\varepsilon_2\to 0^+}\int_{\varepsilon_1}^{t-\varepsilon_2}\frac{d}{ds}\left(\int_{\mathbb{R}^d}p^{\mathfrak{K}_1}(s,x,z)p^{\mathfrak{K}_2}(t-s,z,y)dz\right)ds$$

$$= \lim_{\varepsilon_1\to 0^+} \int_{\varepsilon_1}^{t/2} \int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,x,z) \left(\mathcal{L}_z^{\mathfrak{K}_1} - \mathcal{L}_z^{\mathfrak{K}_2}\right) p^{\mathfrak{K}_2}(t-s,z,y)\, dz ds$$

$$+ \lim_{\varepsilon_2\to 0^+} \int_{t/2}^{t-\varepsilon_2} \int_{\mathbb{R}^d} \left(\mathcal{L}_x^{\mathfrak{K}_1} - \mathcal{L}_x^{\mathfrak{K}_2}\right) p^{\mathfrak{K}_1}(s,x,z) p^{\mathfrak{K}_2}(t-s,z,y)\, dz ds\,.$$

By Proposition 2.1, (22), Theorem 2.9, Corollary 5.14 and Lemma 5.6,

$$\int_{\varepsilon}^{t/2} \int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,x,z) |\left(\mathcal{L}_z^{\mathfrak{K}_1} - \mathcal{L}_z^{\mathfrak{K}_2}\right) p^{\mathfrak{K}_2}(t-s,z,y)|\, dz ds$$

$$\leqslant c\|\mathfrak{K}_1 - \mathfrak{K}_2\| \int_{\varepsilon}^{t/2} \int_{\mathbb{R}^d} \Upsilon_s(z-x) \left(\int_{\mathbb{R}^d} |\delta^{\mathfrak{K}_2}(t-s,z,y;w)| \nu(|w|) dw \right) dz ds$$

$$\leqslant c\|\mathfrak{K}_1 - \mathfrak{K}_2\| \int_{\varepsilon}^{t/2} \int_{\mathbb{R}^d} \Upsilon_s(z-x)\, (t-s)^{-1} \Upsilon_{t-s}(y-z)\, dz ds$$

$$\leqslant c\|\mathfrak{K}_1 - \mathfrak{K}_2\| \Upsilon_t(y-x) \int_{\varepsilon}^{t/2} t^{-1} ds\,.$$

Similarly,

$$\int_{t/2}^{t-\varepsilon} \int_{\mathbb{R}^d} |\left(\mathcal{L}_x^{\mathfrak{K}_1} - \mathcal{L}_x^{\mathfrak{K}_2}\right) p^{\mathfrak{K}_1}(s,x,z)|\, p^{\mathfrak{K}_2}(t-s,z,y)\, dz ds$$

$$\leqslant c\|\mathfrak{K}_1 - \mathfrak{K}_2\| \int_{t/2}^{t-\varepsilon} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\delta^{\mathfrak{K}_1}(s,x,z;w)| \nu(|w|) dw \right) \Upsilon_{t-s}(y-z)\, dz ds$$

$$\leqslant c\|\mathfrak{K}_1 - \mathfrak{K}_2\| \int_{t/2}^{t-\varepsilon} \int_{\mathbb{R}^d} s^{-1} \Upsilon_s(z-x) \Upsilon_{t-s}(y-z)\, dz ds \leqslant c\|\mathfrak{K}_1 - \mathfrak{K}_2\| \Upsilon_t(y-x)\,.$$

(ii) Define $\widehat{\mathfrak{K}}_i(z) = \mathfrak{K}_i(z) - \kappa_0/2$, $i = 1, 2$. By the construction of the Lévy process we have

$$p^{\mathfrak{K}_i}(t,x,y) = \int_{\mathbb{R}^d} p^{\kappa_0/2}(t,x,w) p^{\widehat{\mathfrak{K}}_i}(t,w,y)\, dw\,. \tag{24}$$

By Lemma 2.3 we can differentiate under the integral in (24). Together with Proposition 2.1, Corollary 5.14, Lemma 5.6 and the monotonicity of h^{-1} we obtain

$$
\begin{aligned}
|\nabla_x p^{\mathfrak{K}_1}(t,x,y)-\nabla_x p^{\mathfrak{K}_2}(t,x,y)| &= \left| \int_{\mathbb{R}^d} \nabla_x p^{\kappa_0/2}(t,x,w) \left(p^{\widehat{\mathfrak{K}}_1}(t,w,y) - p^{\widehat{\mathfrak{K}}_2}(t,w,y) \right) dw \right| \\
&\leqslant c\|\mathfrak{K}_1-\mathfrak{K}_2\| \int_{\mathbb{R}^d} \left[h^{-1}(1/t)\right]^{-1} \Upsilon_t(w-x)\Upsilon_t(y-w)\,dw \\
&\leqslant c\|\mathfrak{K}_1-\mathfrak{K}_2\| \left[h^{-1}(1/t)\right]^{-1} \Upsilon_{2t}(y-x) \\
&\leqslant c\|\mathfrak{K}_1-\mathfrak{K}_2\| \left[h^{-1}(1/t)\right]^{-1} 2\Upsilon_t(y-x)\,.
\end{aligned}
$$

(iii) By (24) we have

$$
|\delta^{\mathfrak{K}_1}(t,x,y;z)-\delta^{\mathfrak{K}_2}(t,x,y;z)| = \left| \int_{\mathbb{R}^d} \delta^{\kappa_0/2}(t,x,w;z) \left(p^{\widehat{\mathfrak{K}}_1}(t,w,y) - p^{\widehat{\mathfrak{K}}_2}(t,w,y) \right) dw \right|.
$$

Then by Theorem 2.9, Corollary 5.14 and Lemma 5.6,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\delta^{\mathfrak{K}_1}(t,x,y;z)-\delta^{\mathfrak{K}_2}(t,x,y;z)|\,\nu(|z|)dz \\
&\quad\leqslant \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\delta^{\kappa_0/2}(t,x,w;z)|\,\nu(|z|)dz \right) \left| p^{\widehat{\mathfrak{K}}_1}(t,w,y) - p^{\widehat{\mathfrak{K}}_2}(t,w,y) \right| dw \\
&\quad\leqslant c\|\mathfrak{K}_1-\mathfrak{K}_2\| \int_{\mathbb{R}^d} t^{-1}\Upsilon_t(w-x)\Upsilon_t(y-w)\,dw \leqslant c\|\mathfrak{K}_1-\mathfrak{K}_2\| t^{-1}\Upsilon_{2t}(y-x)\,. \qquad \square
\end{aligned}
$$

3. Levi's construction of heat kernels

For a fixed $w\in\mathbb{R}^d$, define $\mathfrak{K}_w(z)=\kappa(w,z)$ and let $p^{\mathfrak{K}_w}(t,x,y)$ be the heat kernel of the operator $\mathcal{L}^{\mathfrak{K}_w}$ as introduced in Section 2. For all $t>0$, $x,y,w\in\mathbb{R}^d$,

$$
\partial_t p^{\mathfrak{K}_w}(t,x,y) = \mathcal{L}_x^{\mathfrak{K}_w} p^{\mathfrak{K}_w}(t,x,y)\,, \tag{25}
$$

where for every $v\in\mathbb{R}^d$ we have $\mathcal{L}_x^{\mathfrak{K}_v} p^{\mathfrak{K}_w}(t,x,y)=\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_w}(t,x,y;z)\kappa(v,z)J(z)dz$ (see (22)).

3.1. Properties of $p^{\mathfrak{K}_y}(t,x,y)$

Lemma 3.1. *The functions $p^{\mathfrak{K}_w}(t,x,y)$ and $\nabla_x p^{\mathfrak{K}_w}(t,x,y)$ are jointly continuous in $(t,x,y,w)\in(0,\infty)\times(\mathbb{R}^d)^3$.*

Proof. Recall that $p^{\mathfrak{K}_w}(t,x,y)=p^{\mathfrak{K}_w}(t,y-x)$. By the triangle inequality,

$$|p^{\mathfrak{K}_w}(t,x)-p^{\mathfrak{K}_{w_0}}(t_0,x_0)|\leqslant|p^{\mathfrak{K}_w}(t,x)-p^{\mathfrak{K}_{w_0}}(t,x)|+|p^{\mathfrak{K}_{w_0}}(t,x)-p^{\mathfrak{K}_{w_0}}(t_0,x_0)|\,.$$

The first term is small by Theorem 2.11 and (3), and the second by Lemma 6.1. Similar proof is valid for $\nabla_x p^{\mathfrak{K}_w}(t,x,y)$. $\square$

Lemma 3.2. *The function $\mathcal{L}_x^{\mathfrak{K}_v}p^{\mathfrak{K}_w}(t,x,y)$ is jointly continuous in $(t,x,y,w,v)\in(0,\infty)\times(\mathbb{R}^d)^4$.*

Proof. By Lemma 3.1 the function $\delta^{\mathfrak{K}_w}(t,x,y;z)$ is jointly continuous in (t,x,y,w). Recall that $\kappa(v,z)$ is continuous in v and bounded. We let $(t_n,x_n,y_n,w_n,v_n)\to(t,x,y,w,v)$ such that $0<\varepsilon\leqslant t_n\leqslant T$. Further, by Lemma 2.7, 2.3, 2.4 we have respectively,

$$|\delta_3^{\mathfrak{K}_{w_n}}(t_n,x_n,y_n;z)|\leqslant c\Upsilon_\varepsilon(0)\left[\left(\frac{|z|}{h^{-1}(1/\varepsilon)}\right)^2\wedge 1\right],$$

$$|\delta_2^{\mathfrak{K}_{w_n}}(t_n,x_n,y_n;z)|\leqslant c\Upsilon_\varepsilon(0)\left[\left(\frac{|z|}{h^{-1}(1/\varepsilon)}\right)\wedge 1\right],$$

$$|\delta_1^{\mathfrak{K}_{w_n}}(t_n,x_n,y_n;z)|\leqslant c\Upsilon_\varepsilon(0)\left[\mathbf{1}_{|z|\geqslant 1\wedge h^{-1}(1/\varepsilon)}+\left(\frac{|z|}{h^{-1}(1/\varepsilon)}\right)^2\mathbf{1}_{|z|\leqslant 1}\right].$$

Thus the sequence $\delta^{\mathfrak{K}_{w_n}}(t_n,x_n,y_n;z)\kappa(v_n,z)\nu(|z|)$ is bounded by an integrable function and we can use the dominated convergence theorem. $\square$

For $\gamma,\beta\in\mathbb{R}$ we introduce the following function (see Appendix 5.4)

$$\rho_\gamma^\beta(t,x):=\left[h^{-1}(1/t)\right]^\gamma\left(|x|^\beta\wedge 1\right)t^{-1}\Upsilon_t(x)\,. \tag{26}$$

Lemma 3.3. *For every $T>0$ there exists a constant $c=c(d,T,\sigma)$ such that for all $t\in(0,T]$, $x,x',y,w\in\mathbb{R}^d$ and $\gamma\in[0,1]$,*

$$|p^{\mathfrak{K}_w}(t,x,y)-p^{\mathfrak{K}_w}(t,x',y)|\leqslant c(|x-x'|^\gamma\wedge 1)\,t\left(\rho_{-\gamma}^0(t,x-y)+\rho_{-\gamma}^0(t,x'-y)\right).$$

Proof. We use Lemma 2.3 and $(|x-x'|/h^{-1}(1/t)\wedge 1)\leqslant(|x-x'|^\gamma\wedge 1)\left[h^{-1}(1/t)\right]^{-\gamma}\times\left[h^{-1}(1/T)\vee 1\right]$. $\square$

The following result is the counterpart of [16, Lemma 3.2 and 3.3].

Lemma 3.4. *Let $\beta_1\in[0,\beta]\cap[0,\alpha_h)$. For every $T>0$ there exists a constant $c=c(d,T,\sigma,\kappa_2,\beta_1)$ such that for all $t\in(0,T]$, $x,y,w\in\mathbb{R}^d$,*

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$$\int_{\mathbb{R}^d} |\delta^{\mathfrak{K}_y}(t,x,y;z)|\,\kappa(w,z)J(z)dz \leqslant c\rho_0^0(t,x-y)\,, \tag{27}$$

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \delta^{\mathfrak{K}_y}(t,x,y;z)\,dy \right| \kappa(x,z)J(z)dz \leqslant ct^{-1}\left[h^{-1}(1/t)\right]^{\beta_1}, \tag{28}$$

$$\left| \int_{\mathbb{R}^d} \nabla_x p^{\mathfrak{K}_y}(t,x,y)\,dy \right| \leqslant c\left[h^{-1}(1/t)\right]^{-1+\beta_1}. \tag{29}$$

Furthermore,

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^{\mathfrak{K}_y}(t,x,y)\,dy - 1 \right| = 0 \tag{30}$$

Proof. The inequality (27) follows from (2), (1) and Theorem 2.9. Let I be the expression on the left hand side of (28). Since $\int_{\mathbb{R}^d} p^{\mathfrak{K}_w}(t,x,y)dy = 1$ and $\int_{\mathbb{R}^d} \partial_{x_i} p^{\mathfrak{K}_w}(t,x,y)dy = \partial_{x_i} \int_{\mathbb{R}^d} p^{\mathfrak{K}_w}(t,x,y)dy = 0$ (see Lemma 2.3) we have

$$\int_{\mathbb{R}^d} \delta^{\mathfrak{K}_w}(t,x,y;z)\,dy = 0\,, \qquad x,w,z\in\mathbb{R}^d\,. \tag{31}$$

Then by (31), (1), Theorem 2.11 and Remark 1.7,

$$\begin{aligned}
I &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \left(\delta^{\mathfrak{K}_y}(t,x,y;z) - \delta^{\mathfrak{K}_x}(t,x,y;z) \right) dy \right| \kappa(x,z)J(z)dz \\
&\leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left| \delta^{\mathfrak{K}_y}(t,x,y;z) - \delta^{\mathfrak{K}_x}(t,x,y;z) \right| \kappa(x,z)J(z)dzdy \\
&\leqslant c \int_{\mathbb{R}^d} \|\kappa(y,\cdot)-\kappa(x,\cdot)\| t^{-1}\Upsilon_t(y-x)\,dy \\
&\leqslant c \int_{\mathbb{R}^d} (|y-x|^{\beta_1}\wedge 1)t^{-1}\Upsilon_t(y-x)\,dy\,.
\end{aligned}$$

The result follows now from Lemma 5.17(a). For (29) by Theorem 2.11 and Lemma 5.17(a),

$$\left| \int_{\mathbb{R}^d} \nabla_x p^{\mathfrak{K}_y}(t,x,y)\,dy \right| = \left| \int_{\mathbb{R}^d} \left(\nabla_x p^{\mathfrak{K}_y}(t,x,y) - \nabla p^{\mathfrak{K}_x}(t,\cdot,y)(x) \right) dy \right|$$

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$$\leqslant c \int_{\mathbb{R}^d} \|\kappa(y,\cdot)-\kappa(x,\cdot)\| \left[h^{-1}(1/t)\right]^{-1} \Upsilon_t(y-x)\,dy$$

$$\leqslant c \left[h^{-1}(1/t)\right]^{-1} \int_{\mathbb{R}^d} (|y-x|^{\beta_1}\wedge 1)\Upsilon_t(y-x)\,dy \leqslant c\left[h^{-1}(1/t)\right]^{-1+\beta_1}.$$

Eventually, by Theorem 2.11 and Lemma 5.17(a),

$$\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} p^{\mathfrak{K}_y}(t,x,y)\,dy-1\right| \leqslant \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|p^{\mathfrak{K}_y}(t,x,y)-p^{\mathfrak{K}_x}(t,x,y)\right|dy$$

$$\leqslant c \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} \|\kappa(y,\cdot)-\kappa(x,\cdot)\|\Upsilon_t(y-x)dy$$

$$\leqslant c \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} (|y-x|^{\beta_1}\wedge 1)\Upsilon_t(y-x)\,dy \leqslant c\left[h^{-1}(1/t)\right]^{\beta_1} \to 0,$$

as $t\to 0^+$. This ends the proof. □

3.2. *Construction of $q(t,x,y)$*

For $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ define

$$q_0(t,x,y) := \int_{\mathbb{R}^d} \delta^{\mathfrak{K}_y}(t,x,y;z)\,(\kappa(x,z)-\kappa(y,z))\,J(z)dz = \left(\mathcal{L}_x^{\mathfrak{K}_x}-\mathcal{L}_x^{\mathfrak{K}_y}\right)p^{\mathfrak{K}_y}(t,x,y).$$

Directly from Lemma 3.2 we have the following result.

Lemma 3.5. *The function $q_0(t,x,y)$ is jointly continuous in $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.*

In the next lemma we collect estimates on q_0.

Lemma 3.6. *For every $T>0$ there exists a constant $c=c(d,T,\sigma,\kappa_2)\geqslant 1$ such that for all $\beta_1\in[0,\beta]$, $t\in(0,T]$ and $x,x',y,y'\in\mathbb{R}^d$*

$$|q_0(t,x,y)| \leqslant c\rho_0^{\beta_1}(t,y-x), \tag{32}$$

and for every $\gamma\in[0,\beta_1]$,

$$|q_0(t,x,y)-q_0(t,x',y)|$$
$$\leqslant c\left(|x-x'|^{\beta_1-\gamma}\wedge 1\right)\left\{\left(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\right)(t,x-y)+\left(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\right)(t,x'-y)\right\}, \tag{33}$$

and

$$
\begin{aligned}
&|q_0(t,x,y)-q_0(t,x,y')| \\
&\leqslant c\left(|y-y'|^{\beta_1-\gamma}\wedge 1\right)\left\{\left(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\right)(t,x-y)+\left(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\right)(t,x-y')\right\}.
\end{aligned} \tag{34}
$$

Proof. (i) (32) follows from (1), Remark 1.7 and Theorem 2.9.

(ii) For $|x-x'|\geqslant 1$ the inequality holds by (32) and (92):

$$
|q_0(t,x,y)|\leqslant c\rho_0^{\beta_1}(t,y-x)\leqslant c\left[h^{-1}(1/T)\vee 1\right]^{\beta_1-\gamma}\rho_{\gamma-\beta_1}^{\beta}(t,y-x).
$$

For $1\geqslant|x-x'|\geqslant h^{-1}(1/t)$ the result follows from (32) and

$$
\begin{aligned}
|q_0(t,x,y)|\leqslant c\rho_0^{\beta_1}(t,y-x)&=c\left[h^{-1}(1/t)\right]^{\beta_1-\gamma}\rho_{\gamma-\beta_1}^{\beta_1}(t,y-x)\\
&\leqslant c|x-x'|^{\beta_1-\gamma}\rho_{\gamma-\beta_1}^{\beta_1}(t,y-x).
\end{aligned}
$$

Now, (1), Remark 1.7 and Theorem 2.9 provide that

$$
\begin{aligned}
|q_0(t,x,y)-q_0(t,x',y)|&=\Bigg|\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_y}(t,x,y;z)(\kappa(x,z)-\kappa(y,z))J(z)dz\\
&\quad-\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_y}(t,x',y;z)(\kappa(x',z)-\kappa(y,z))J(z)dz\Bigg|\\
&\leqslant\gamma_0\int_{\mathbb{R}^d}|\delta^{\mathfrak{K}_y}(t,x,y;z)-\delta^{\mathfrak{K}_y}(t,x',y;z)|\,|\kappa(x,z)-\kappa(y,z)|\,\nu(|z|)dz\\
&\quad+\gamma_0\int_{\mathbb{R}^d}|\delta^{\mathfrak{K}_y}(t,x',y;z)|\,|\kappa(x,z)-\kappa(x',z)|\,\nu(|z|)dz\\
&\leqslant c\left(|x-y|^{\beta_1}\wedge 1\right)\int_{\mathbb{R}^d}|\delta^{\mathfrak{K}_y}(t,x,y;z)-\delta^{\mathfrak{K}_y}(t,x',y;z)|\,\nu(|z|)dz\\
&\quad+c\left(|x-x'|^{\beta_1}\wedge 1\right)\int_{\mathbb{R}^d}|\delta^{\mathfrak{K}_y}(t,x',y;z)|\,\nu(|z|)dz\\
&\leqslant c\left(|x-y|^{\beta_1}\wedge 1\right)\left(\frac{|x-x'|}{h^{-1}(1/t)}\wedge 1\right)\left(\rho_0^0(t,x-y)+\rho_0^0(t,x'-y)\right)\\
&\quad+c\left(|x-x'|^{\beta}\wedge 1\right)\rho_0^0(t,x'-y).
\end{aligned}
$$

Applying $(|x-y|^{\beta_1}\wedge 1)\leqslant(|x-x'|^{\beta_1}\wedge 1)+(|x'-y|^{\beta_1}\wedge 1)$ we obtain

$$|q_0(t,x,y)-q_0(t,x',y)|\leqslant c\left(\frac{|x-x'|}{h^{-1}(1/t)}\wedge 1\right)\left(\rho_0^{\beta_1}(t,x-y)+\rho_0^{\beta_1}(t,x'-y)\right)\\ +c\left(|x-x'|^{\beta_1}\wedge 1\right)\rho_0^0(t,x'-y).$$

Thus in the last case $|x-x'|\leqslant h^{-1}(1/t)\wedge 1$ we have $|x-x'|/h^{-1}(1/t)\leqslant |x-x'|^{\beta_1-\gamma}\times\left[h^{-1}(1/t)\right]^{\gamma-\beta_1}$ and $|x-x'|^{\beta_1}\leqslant|x-x'|^{\beta_1-\gamma}\left[h^{-1}(1/t)\right]^{\gamma}$.

(iii) We treat the cases $|y-y'|\geqslant 1$ and $1\geqslant|y-y'|\geqslant h^{-1}(1/t)$ like in part (ii). Now note that by (1), Remark 1.7, $\delta^{\mathfrak{K}}(t,x,y;z)=\delta^{\mathfrak{K}}(t,-y,-x;z)$ and Theorem 2.9 and 2.11,

$$\begin{aligned}
&|q_0(t,x,y)-q_0(t,x,y')|\\
&\le\left|\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_y}(t,x,y;z)\left(\kappa(y',z)-\kappa(y,z)\right)J(z)dz\right|\\
&\quad+\left|\int_{\mathbb{R}^d}\left(\delta^{\mathfrak{K}_y}(t,x,y;z)-\delta^{\mathfrak{K}_y}(t,x,y';z)\right)\left(\kappa(x,z)-\kappa(y',z)\right)J(z)dz\right|\\
&\quad+\left|\int_{\mathbb{R}^d}\left(\delta^{\mathfrak{K}_y}(t,x,y';z)-\delta^{\mathfrak{K}_{y'}}(t,x,y';z)\right)\left(\kappa(x,z)-\kappa(y',z)\right)J(z)dz\right|\\
&\leqslant c\left(|y-y'|^{\beta_1}\wedge 1\right)\rho_0^0(t,x-y)\\
&\quad+c\left(|x-y'|^{\beta_1}\wedge 1\right)\left(\frac{|y-y'|}{h^{-1}(1/t)}\wedge 1\right)\left(\rho_0^0(t,x-y)+\rho_0^0(t,x-y')\right)\\
&\quad+c\left(|y-y'|^{\beta_1}\wedge 1\right)\rho_0^0(t,x-y')\,.
\end{aligned}$$

Applying $(|x-y'|^{\beta_1}\wedge 1)\leqslant(|x-y|^{\beta_1}\wedge 1)+(|y-y'|^{\beta_1}\wedge 1)$ we obtain

$$|q_0(t,x,y)-q_0(t,x,y')|\leqslant c\left(\frac{|y-y'|}{h^{-1}(1/t)}\wedge 1\right)\left(\rho_0^{\beta_1}(t,x-y)+\rho_0^{\beta_1}(t,x-y')\right)\\ +c\left(|y-y'|^{\beta_1}\wedge 1\right)\left(\rho_0^0(t,x-y)+\rho_0^0(t,x-y')\right).$$

This proves (34) in the case $|y-y'|\leqslant h^{-1}(1/t)\wedge 1$. $\square$

For $n\in\mathbb{N}$ and $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ we inductively define

$$q_n(t,x,y):=\int_0^t\int_{\mathbb{R}^d}q_0(t-s,x,z)q_{n-1}(s,z,y)\,dzds\,. \tag{35}$$

The following result is the counterpart of [16, Theorem 3.1] and [45, Theorem 4.5].

Theorem 3.7. *Assume* (P). The series $q(t,x,y) := \sum_{n=0}^{\infty} q_n(t,x,y)$ is absolutely and locally uniformly convergent on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and solves the integral equation

$$q(t,x,y) = q_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)q(s,z,y)\,dzds\,. \tag{36}$$

Moreover, for every $T>0$ and $\beta_1 \in (0,\beta]\cap(0,\alpha_h)$ there is a constant $c = c(d,T,\sigma,\kappa_2,\beta_1)$ such that on $(0,T]\times\mathbb{R}^d\times\mathbb{R}^d$,

$$|q(t,x,y)| \leqslant c\big(\rho_0^{\beta_1} + \rho_{\beta_1}^0\big)(t,x-y)\,, \tag{37}$$

and for any $\gamma\in(0,\beta_1]$ there is $c = c(d,T,\sigma,\kappa_2,\beta_1,\gamma)$ such that on $(0,T]\times\mathbb{R}^d\times\mathbb{R}^d$,

$$\begin{aligned}&|q(t,x,y)-q(t,x',y)|\\ &\leqslant c\left(|x-x'|^{\beta_1-\gamma}\wedge 1\right)\left\{\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(t,x-y)+\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(t,x'-y)\right\},\end{aligned} \tag{38}$$

and

$$\begin{aligned}&|q(t,x,y)-q(t,x,y')|\\ &\leqslant c\left(|y-y'|^{\beta_1-\gamma}\wedge 1\right)\left\{\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(t,x-y)+\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(t,x-y')\right\}.\end{aligned} \tag{39}$$

Proof. Let $T>0$ be fixed. In what follows $t\in(0,T]$, $x,y\in\mathbb{R}^d$. We also fix $M>0$ such that $M<\beta_1$ and $\beta_1+M<\alpha_h\wedge 1$. Furthermore, we set $\beta_0=\beta_1+M$ when using Lemma 5.17. For clarity we denote $C_1 = 5cc_2$ and $C_2 = h^{-1}(1/T)\vee 1$, where c and c_2 are taken from Lemma 3.6 and 5.17(b), respectively.

Step 1. First we justify that

$$|q_n(t,x,y)| \leqslant \gamma_n\big(\rho^0_{\beta_1+nM}+\rho^{\beta_1}_{nM}\big)(t,x-y)\,, \tag{40}$$

where

$$\gamma_n := C_1^{n+1}C_2^{(\beta_1-M)n}\prod_{j=1}^n B\left(M/2, jM/2\right) = C_1\Gamma(M/2)\frac{\left(C_1C_2^{\beta_1-M}\Gamma(M/2)\right)^n}{\Gamma\left((n+1)M/2\right)}.$$

For $n=1$ by (32) and Lemma 5.17(c) with $n_1=n_2=\beta_1+M$, $m_1=m_2=M$, we have

$$\begin{aligned}|q_1(t,x,y)| &\leqslant c^2\int_0^t\int_{\mathbb{R}^d}\rho_0^{\beta_1}(t-s,x-z)\rho_0^{\beta_1}(s,z-y)\,dzds\\ &\leqslant 2c^2c_2\,B(M/2,M/2)\left(\rho^0_{\beta_1+M}+\rho^{\beta_1}_M\right)(t,x-y)\,.\end{aligned}$$

Further, assuming (40) for $n\in\mathbb{N}$ we get by (32), Lemma 5.17(c) with $n_1=n_2=m_1=M$, $m_2=0$ and $n_1=n_2=\beta_1+M$, $m_1=m_2=M$, by the monotonicity of Beta function and (92),

$$
\begin{aligned}
|q_{n+1}(t,x,y)| &\leqslant c\,\gamma_n \int_0^t \int_{\mathbb{R}^d} \rho_0^{\beta_1}(t-s,x,z)\big(\rho^0_{\beta_1+nM}+\rho^{\beta_1}_{nM}\big)(s,z,y)\,dz ds \\
&\leqslant c\,\gamma_n c_2\, B(M/2,(\beta_1+nM)/2)\,\big(3\rho^0_{\beta_1+(n+1)M}+\rho^{\beta_1}_{\beta_1+nM}\big)(t,x-y) \\
&\quad + c\,\gamma_n c_2\, B(M/2,(n+1)M/2)\,\big(2\rho^0_{\beta_1+(n+1)M}+2\rho^{\beta_1}_{(n+1)M}\big)(t,x-y) \\
&\leqslant \gamma_n 5cc_2 C_2^{\beta_1-M}\, B(M/2,(n+1)M/2)\,\big(\rho^0_{\beta_1+(n+1)M}+\rho^{\beta_1}_{(n+1)M}\big)(t,x-y) \\
&\leqslant \gamma_{n+1}\big(\rho^0_{\beta_1+(n+1)M}+\rho^{\beta_1}_{(n+1)M}\big)(t,x-y)\,.
\end{aligned}
$$

Thus (40) follows by induction. Then by (92) we have

$$
|q_n(t,x,y)| \leqslant \gamma_n \left[h^{-1}(1/T)\right]^{nM} \big(\rho^0_{\beta_1}+\rho_0^{\beta_1}\big)(t,x-y)\,.
$$

Finally,

$$
\sum_{n=0}^{\infty}|q_n(t,x,y)| \leqslant \left(C_1\Gamma(M/2)\sum_{n=0}^{\infty}\frac{\left(C_1C_2^{\beta_1}\,\Gamma(M/2)\right)^n}{\Gamma((n+1)M/2)}\right)\big(\rho^0_{\beta_1}+\rho_0^{\beta_1}\big)(t,x-y)\,.
$$

Now, the series defining q is absolutely and uniformly convergent on $[\varepsilon,T]\times\mathbb{R}^d\times\mathbb{R}^d$ and has the desired bound (37). The equation (36) follows from the definition of q_n.

Step 2. By (33), (37) and Lemma 5.17(c) with the usual parameters and once with $n_1=n_2=m_1=m_2=\beta_1$,

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d} |q_0(t-s,x,z)-q_0(t-s,x',z)||q(s,z,y)|\,dz ds \leqslant c\left(|x-x'|^{\beta_1-\gamma}\wedge 1\right) \\
&\times \int_0^t\int_{\mathbb{R}^d} \left\{\left(\rho^0_\gamma+\rho^{\beta_1}_{\gamma-\beta_1}\right)(t,x-z)+\left(\rho^0_\gamma+\rho^{\beta_1}_{\gamma-\beta_1}\right)(t,x'-z)\right\}\big(\rho_0^{\beta_1}+\rho^0_{\beta_1}\big)(t,z-y)\,dz ds \\
&\leqslant c\left(|x-x'|^{\beta_1-\gamma}\wedge 1\right)\left\{\left(\rho^0_{\gamma+\beta_1}+\rho^0_\gamma+\rho^{\beta_1}_\gamma\right)(t,x-y)+\left(\rho^0_{\gamma+\beta_1}+\rho^0_\gamma+\rho^{\beta_1}_\gamma\right)(t,x'-y)\right\}.
\end{aligned}
$$

Finally, we use (36), (33) and the above together with (92).

Step 3. In order to prove (39), similarly to *Step 1*, using induction we get by (34), Lemma 5.17(c) and (92),

$$
\begin{aligned}
&|q_n(t,x,y)-q_n(t,x,y')| \\
&\quad\leqslant \gamma_n'\left(|y-y'|^{\beta_1-\gamma}\wedge 1\right)\Big\{\big(\rho^0_{\gamma+nM}+\rho^{\beta_1}_{\gamma-\beta_1+nM}\big)(t,x-y) \\
&\qquad + \big(\rho^0_{\gamma+nM}+\rho^{\beta_1}_{\gamma-\beta_1+nM}\big)(t,x-y')\Big\},
\end{aligned}
$$

where $\gamma_n' = C_1^{n+1}C_2^{(\beta_1-M)n}(C_hC_2^2)^{n(\beta_1-\gamma)/\alpha_h}\prod_{j=1}^n B\left(\beta_1/2,(\gamma+(j-1)M)/2\right)$. $\square$

3.3. Properties of $\phi_y(t,x)$

Let

$$\phi_y(t,x,s) := \int_{\mathbb{R}^d} p^{\mathfrak{K}_z}(t-s,x,z)q(s,z,y)\,dz, \quad x\in\mathbb{R}^d,\ 0\leqslant s<t\,, \tag{41}$$

and

$$\phi_y(t,x) := \int_0^t \phi_y(t,x,s)\,ds = \int_0^t\int_{\mathbb{R}^d} p^{\mathfrak{K}_z}(t-s,x,z)q(s,z,y)\,dzds\,. \tag{42}$$

Lemma 3.8. *Let $\beta_1\in(0,\beta]\cap(0,\alpha_h)$. For every $T>0$ there exists a constant $c=c(d,T,\sigma,\kappa_2,\beta_1)$ such that for all $t\in(0,T]$, $x,y\in\mathbb{R}^d$,*

$$|\phi_y(t,x)|\leqslant ct\big(\rho_0^{\beta_1}+\rho_{\beta_1}^0\big)(t,x-y)\,.$$

For any $T>0$ and $\gamma\in[0,1]\cap[0,\alpha_h)$ there exists a constant $c=c(d,T,\sigma,\kappa_2,\beta_1,\gamma)$ such that for all $t\in(0,T]$, $x,x',y\in\mathbb{R}^d$,

$$\begin{aligned}&|\phi_y(t,x)-\phi_y(t,x')|\\&\leqslant c(|x-x'|^\gamma\wedge 1)\,t\left\{\big(\rho^0_{\beta_1-\gamma}+\rho^{\beta_1}_{-\gamma}\big)(t,x-y)+\big(\rho^0_{\beta_1-\gamma}+\rho^{\beta_1}_{-\gamma}\big)(t,x'-y)\right\}.\end{aligned}$$

For any $T>0$ and $\gamma\in(0,\beta_1]$ there exists a constant $c=c(d,T,\sigma,\kappa_2,\beta_1,\gamma)$ such that for all $t\in(0,T]$, $x,y,y'\in\mathbb{R}^d$,

$$\begin{aligned}&|\phi_y(t,x)-\phi_{y'}(t,x)|\\&\leqslant c(|y-y'|^{\beta_1-\gamma}\wedge 1)\,t\left\{\big(\rho^0_{\gamma}+\rho^{\beta_1}_{\gamma-\beta_1}\big)(t,x-y)+\big(\rho^0_{\gamma}+\rho^{\beta_1}_{\gamma-\beta_1}\big)(t,x-y')\right\}.\end{aligned}$$

Proof. By Proposition 2.1 and (37),

$$|\phi_y(t,x)|\leqslant c\int_0^t\int_{\mathbb{R}^d}(t-s)\rho_0^0(t-s,x-z)\big(\rho_0^{\beta_1}+\rho^0_{\beta_1}\big)(s,z-y)\,.$$

By Lemma 3.3 and (37),

$$|\phi_y(t,x)-\phi_y(t,x')|\leqslant \int_0^t\int_{\mathbb{R}^d}\left|p^{\mathfrak{K}_z}(t-s,x,z)-p^{\mathfrak{K}_z}(t-s,x',z)\right|q(s,z,y)\,dzds$$

$$\leqslant c(|x-x'|^\gamma\wedge 1)\int_0^t\int_{\mathbb{R}^d}(t-s)\left(\rho^0_{-\gamma}(t-s,x-z)+\rho^0_{-\gamma}(t-s,x'-z)\right)$$

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$$\big(\rho_0^{\beta_1}+\rho_{\beta_1}^0\big)(s,z-y)\,dzds\,.$$

By Proposition 2.1 and (39)

$$\begin{aligned}|\phi_y(t,x)-\phi_{y'}(t,x)| &\leqslant \int_0^t\int_{\mathbb{R}^d} p^{\mathfrak{K}_z}(t-s,x,z)|q(s,z,y)-q(s,z,y')|\,dzds\\ &\leqslant c(|y-y'|^{\beta_1-\gamma}\wedge 1)\int_0^t\int_{\mathbb{R}^d}(t-s)\rho_0^0(t-s,x-z)\Big\{\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(s,z-y)\\ &\qquad +\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(s,z-y')\Big\}\,dzds\end{aligned}$$

Finally, the results follow from Lemma 5.17(c). □

Lemma 3.9. *The function $\phi_y(t,x)$ is jointly continuous in $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.*

Proof. First we prove the continuity in t variable for fixed $x,y\in\mathbb{R}^d$. We have for all $\varepsilon\in(0,t)$ and $|h|<\varepsilon/2$,

$$\begin{aligned}\phi_y(t+h,x)-\phi_y(t,x) &= \int_0^{t-\varepsilon}\big(\phi_y(t+h,x,s)-\phi_y(t,x,s)\big)\,ds\\ &\quad+\int_{t-\varepsilon}^{t+h}\phi_y(t+h,x,s)\,ds-\int_{t-\varepsilon}^{t}\phi_y(t,x,s)\,ds\,.\end{aligned}$$

By Proposition 2.1, (37) and (94), for $s\in(0,t-\varepsilon)$ we get

$$p^{\mathfrak{K}_z}(t-s+h,x,z)|q(s,z,y)|\leqslant c\,2t\,\rho_0^0(\varepsilon/2,0)\big(\rho_0^{\beta_1}+\rho_{\beta_1}^0\big)(s,z-y)\,.$$

The right hand side is by Lemma 5.17(a) and 5.15 integrable over $(0,t-\varepsilon)\times\mathbb{R}^d$ in $dzds$. Thus by Lemma 3.1 and the dominated convergence theorem we have $\lim_{h\to 0}\int_0^{t-\varepsilon}\phi_y(t+h,x,s)\,ds=\int_0^{t-\varepsilon}\phi_y(t,x,s)\,ds$ for every $\varepsilon\in(0,t)$. Next, we show that given $\varepsilon_1>0$ there exists $\varepsilon\in(0,t/3)$ such that for all $r\in\mathbb{R}$ satisfying $|r-t|<\varepsilon/2$,

$$\int_{t-\varepsilon}^{r}\big|\phi_y(r,x,s)\big|\,ds<\varepsilon_1\,.$$

Indeed, by (37), the monotonicity of h^{-1} and (94) in the first inequality, and Proposition 2.1 and Lemma 5.6 in the second inequality, we have

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$$\int_{t-\varepsilon}^{r} \left|\phi_y(r,x,s)\right| ds \leqslant c \left(\int_{t-\varepsilon}^{r} \int_{\mathbb{R}^d} p^{\mathfrak{K}_z}(r-s,x,z)\, dz ds \right) \rho_0^0(t/2,0) \leqslant c\, 2\varepsilon\, \rho_0^0(t/2,0)\,.$$

This ends the proof of the continuity in $t > 0$. The joint continuity follows from $|\phi_y(t,x) - \phi_{y_0}(t_0,x_0)| \leqslant |\phi_y(t,x) - \phi_y(t,x_0)| + |\phi_y(t,x_0) - \phi_{y_0}(t,x_0)| + |\phi_{y_0}(t,x_0) - \phi_{y_0}(t_0,x_0)|$ and Lemma 3.8. □

The following result is the counterpart of [16, Lemma 3.6].

Lemma 3.10. *Assume that $1-\alpha_h < \beta \wedge \alpha_h$. For every $T > 0$ there exists a constant $c = c(d,T,\sigma,\kappa_2,\beta_1)$ such that for all $t \in (0,T]$, $x, y \in \mathbb{R}^d$,*

$$\nabla_x \phi_y(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla_x p^{\mathfrak{K}_z}(t-s,x,z) q(s,z,y)\, dz ds\,, \tag{43}$$

$$\left|\nabla_x \phi_y(t,x)\right| \leqslant c \left[h^{-1}(1/t)\right]^{-1} t\, \rho_0^0(t,x-y)\,. \tag{44}$$

Proof. Let $\beta_1 \in (0,\beta]$ such that $1-\alpha_h < \beta_1 < \alpha_h$ and $t \in (0,T]$. We set $\beta_0 = \beta_1$ when using Lemma 5.17. We first note that by Lemma 2.3, (37) and Lemma 5.17(b) for $s \in (0,t)$,

$$\nabla_x \phi_y(t,x,s) = \int_{\mathbb{R}^d} \nabla_x p^{\mathfrak{K}_z}(t-s,x,z) q(s,z,y)\, dz\,. \tag{45}$$

Now, let $|\varepsilon| \leqslant h^{-1}(3/t)$ and $\widetilde{x} = x + \varepsilon\theta e_i$. We have by [64, Theorem 7.21],

$$\mathrm{I}_0 := \left| \frac{1}{\varepsilon}\big(\phi_y(t,x+\varepsilon e_i,s) - \phi_y(t,x,s)\big) \right| = \left| \int_0^1 \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} p^{\mathfrak{K}_z}(t-s,\widetilde{x},z) q(s,z,y)\, dz d\theta \right|.$$

For $s \in (0,t/2]$ we first use Proposition 2.1 and (37), then Lemma 5.17(b) (once with $n_1 = m_1 = 0$, $n_2 = m_2 = \beta_1$) and (93), and finally the monotonicity of h^{-1}, (A2) of Lemma 5.3 and Proposition 5.8 to get

$$\mathrm{I}_0 \leqslant c \int_0^1 \int_{\mathbb{R}^d} (t-s) \rho_{-1}^0(t-s,\widetilde{x}-z) \big(\rho_0^{\beta_1} + \rho_{\beta_1}^0\big)(s,z-y)\, dz d\theta$$

$$\leqslant c \left[h^{-1}(1/(t-s))\right]^{-1} \left(\int_0^1 \rho_0^0(t,\widetilde{x}-y)\, d\theta \right)$$

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$$\times\left(1+\left[h^{-1}(1/s)\right]^{\beta_1}+(t-s)s^{-1}\left[h^{-1}(1/s)\right]^{\beta_1}\right)$$

$$\leqslant c\left[h^{-1}(1/t)\right]^{-1}\rho_0^0(t,x-y)\left(1+t\,s^{-1}\left[h^{-1}(1/s)\right]^{\beta_1}\right).$$

For $s\in(t/2,t)$ we fix $\gamma>0$ such that $\beta_1-\gamma>(1-\alpha_h)\vee 0$. Then by Proposition 2.1, (38), (29) and (37) we have

$$\begin{aligned}
\mathrm{I}_0 &\leqslant \int_0^1\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_i}p^{\mathfrak{K}_z}(t-s,\widetilde{x},z)\right||q(s,z,y)-q(s,\widetilde{x},y)|\,dzd\theta\\
&+\int_0^1\left|\int_{\mathbb{R}^d}\frac{\partial}{\partial x_i}p^{\mathfrak{K}_z}(t-s,\widetilde{x},z)\,dz\right||q(s,\widetilde{x},y)|\,d\theta\\
&\leqslant c\int_0^1\int_{\mathbb{R}^d}(t-s)\rho_{-1}^{\beta_1-\gamma}(t-s,\widetilde{x}-z)\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(s,z-y)\,dzd\theta\\
&+c\int_0^1\left(\int_{\mathbb{R}^d}(t-s)\rho_{-1}^{\beta_1-\gamma}(t-s,\widetilde{x}-z)\,dz\right)\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(s,\widetilde{x}-y)\,d\theta\\
&+c\left[h^{-1}(1/(t-s))\right]^{-1+\beta_1}\int_0^1\big(\rho_0^{\beta_1}+\rho_{\beta_1}^0\big)(s,\widetilde{x}-y)\,d\theta=:\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3\,.
\end{aligned}$$

By Lemma 5.17(b) with $(n_1,n_2)=(\beta_1-\gamma,0)$ and $(n_1,n_2)=(\beta_1-\gamma,\beta_1)$, (93), the monotonicity of h^{-1}, (A2) of Lemma 5.3 and Proposition 5.8 we get

$$\begin{aligned}
\mathrm{I}_1 &\leqslant c\Bigg(\left[h^{-1}(1/(t-s))\right]^{-1+\beta_1-\gamma}\left[h^{-1}(1/s)\right]^{\gamma}+(t-s)\left[h^{-1}(1/(t-s))\right]^{-1}s^{-1}\left[h^{-1}(1/s)\right]^{\gamma}\\
&\quad+\left[h^{-1}(1/(t-s))\right]^{-1+\beta_1-\gamma}\left[h^{-1}(1/s)\right]^{\gamma-\beta_1}\Bigg)\left(\int_0^1\rho_0^0(t,\widetilde{x}-y)\,d\theta\right)\\
&\leqslant c\Bigg(\left[h^{-1}(1/(t-s))\right]^{-1+\beta_1-\gamma}+(t-s)\left[h^{-1}(1/(t-s))\right]^{-1}t^{-1}\\
&\quad+\left[h^{-1}(1/(t-s))\right]^{-1+\beta_1-\gamma}\left[h^{-1}(1/t)\right]^{\gamma-\beta_1}\Bigg)\rho_0^0(t,x-y).
\end{aligned}$$

Next, by Lemma 5.17(a), (93), (94), the monotonicity of h^{-1}, (A2) of Lemma 5.3 and Proposition 5.8,

$$
\begin{aligned}
\mathrm{I}_2 &\leqslant c\left[h^{-1}(1/(t-s))\right]^{-1+\beta_1-\gamma}\left(\left[h^{-1}(1/s)\right]^{\gamma}+\left[h^{-1}(1/s)\right]^{\gamma-\beta_1}\right)\left(\int_0^1 \rho_0^0(t/2,\widetilde{x}-y)\,d\theta\right)\\
&\leqslant c\left[h^{-1}(1/(t-s))\right]^{-1+\beta_1-\gamma}\left(1+\left[h^{-1}(1/t)\right]^{\gamma-\beta_1}\right)\rho_0^0(t,x-y).
\end{aligned}
$$

Similarly, $\mathrm{I}_3 \leqslant c\left[h^{-1}(1/(t-s))\right]^{-1+\beta_1}\rho_0^0(t,x-y)$. Finally, the expression I_0 is bounded by a function independent of ε, which by Lemma 5.15 is integrable over $(0,t)$ in s, since $\alpha_h > 1/2$. Then (43) follows by the dominated convergence theorem and (45). More precisely, Lemma 5.15 assures that uniformly in ε we have for $t \in (0,T]$,

$$
\int_0^t \mathrm{I}_0\,ds \leqslant c\,t\left[h^{-1}(1/t)\right]^{-1}\left(1+\left[h^{-1}(1/t)\right]^{\beta_1-\gamma}+\left[h^{-1}(1/t)\right]^{\beta_1}\right)\rho_0^0(t,x-y)\,,
$$

which proves (44) due to monotonicity of h^{-1}. □

Lemma 3.11. *Let $\beta_1 \in (0,\beta] \cap (0,\alpha_h)$. For all $T>0$, $\gamma \in (0,\beta_1]$ there exist constants $c_1 = c_1(d,T,\sigma,\kappa_2,\beta_1)$ and $c_2 = c_2(d,T,\sigma,\kappa_2,\beta_1,\gamma)$ such that for all $0<s<t\leqslant T$, $x,y\in\mathbb{R}^d$,*

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\delta^{\mathfrak{K}_z}(t-s,x,z;w)||q(s,z,y)|\,dz\right)\kappa(x,w)J(w)dw\\
&\qquad\leqslant c_1\int_{\mathbb{R}^d}\rho_0^0(t-s,x-z)\big(\rho_0^{\beta_1}+\rho_{\beta_1}^0\big)(s,z-y)\,dz\,, \qquad (46)
\end{aligned}
$$

$$
\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_z}(t-s,x,z;w)q(s,z,y)\,dz\right|\kappa(x,w)J(w)dw \leqslant c_2\big(\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3\big), \qquad (47)
$$

where

$$
\begin{aligned}
\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3 := &\int_{\mathbb{R}^d}\rho_0^{\beta_1-\gamma}(t-s,x-z)\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(s,z-y)\,dz\\
&+(t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1-\gamma}\big(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\big)(s,x-y)\\
&+(t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1}\big(\rho_0^{\beta_1}+\rho_{\beta_1}^0\big)(s,x-y)\,.
\end{aligned}
$$

Proof. The inequality (46) follows from (27) and (37). Next, let I_0 be the left hand side of (47). By (38), Lemma 3.4, (37), and Lemma 5.17(a),

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$$
\begin{aligned}
\mathrm{I}_0 &\leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\delta^{\mathfrak{K}_z}(t-s,x,z;w)||q(s,z,y)-q(s,x,y)|\,dz\,\kappa(x,w)J(w)dw \\
&\quad + \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} \delta^{\mathfrak{K}_z}(t-s,x,z;w)\,dz\right|\kappa(x,w)J(w)dw\,|q(s,x,y)| \\
&\leqslant c\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\delta^{\mathfrak{K}_z}(t-s,x,z;w)|\kappa(x,w)J(w)dw\right)\left(|x-z|^{\beta_1-\gamma}\wedge 1\right) \\
&\quad \times \left(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\right)(s,z-y)\,dz \\
&\quad + c\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\delta^{\mathfrak{K}_z}(t-s,x,z;w)|\kappa(x,w)J(w)dw\right)\left(|x-z|^{\beta_1-\gamma}\wedge 1\right)dz \\
&\quad \times \left(\rho_\gamma^0+\rho_{\gamma-\beta_1}^{\beta_1}\right)(s,x-y) \\
&\quad + c\,(t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1}\left(\rho_0^{\beta_1}+\rho_{\beta_1}^0\right)(s,x-y) \leqslant c(\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3)\,. \quad \square
\end{aligned}
$$

Lemma 3.12. *Let $\beta_1 \in (0,\beta] \cap (0,\alpha_h)$. For every $T>0$ there exists a constant $c = c(d,T,\sigma,\kappa_2,\beta_1)$ such that for all $t\in(0,T]$, $x,y\in\mathbb{R}^d$,*

$$
\int_{\mathbb{R}^d}\int_0^t\left|\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_z}(t-s,x,z;w)q(s,z,y)\,dz\right|ds\,\kappa(x,w)J(w)dw \leqslant c\rho_0^0(t,x-y)\,, \tag{48}
$$

$$
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^t\left|\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_z}(t-s,x,z;w)q(s,z,y)\,dz\right|ds\,\kappa(x,w)J(w)dw\,dy \leqslant ct^{-1}\left[h^{-1}(1/t)\right]^{\beta_1}. \tag{49}
$$

Proof. Let I_0 be the left hand side of (47). For $s\in(0,t/2]$ we use (46), Lemma 5.17(b) and the monotonicity of h^{-1} to get

$$
\begin{aligned}
\mathrm{I}_0 &\leqslant c\left((t-s)^{-1}+(t-s)^{-1}\left[h^{-1}(1/s)\right]^{\beta_1}+s^{-1}\left[h^{-1}(1/s)\right]^{\beta_1}\right)\rho_0^0(t,x-y) \\
&\leqslant c\left(t^{-1}+s^{-1}\left[h^{-1}(1/s)\right]^{\beta_1}\right)\rho_0^0(t,x-y)\,.
\end{aligned}
$$

For $s\in(t/2,t)$ we fix $\gamma\in(0,\beta_1)$ and we use (47). Then by Lemma 5.17(b) with $(n_1,n_2)=(\beta_1-\gamma,0)$ and $(n_1,n_2)=(\beta_1-\gamma,\beta_1)$, (93), the monotonicity of h^{-1} and (A2) of Lemma 5.3,

$$
\mathrm{I}_1 \leqslant c\Bigg((t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1-\gamma}\left[h^{-1}(1/s)\right]^{\gamma}+s^{-1}\left[h^{-1}(1/s)\right]^{\gamma}
$$

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$$+(t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1-\gamma}\left[h^{-1}(1/s)\right]^{\gamma-\beta_1}\Big)\,\rho_0^0(t,x-y)$$
$$\leqslant c\left(t^{-1}+(t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1-\gamma}\left[h^{-1}(1/t)\right]^{\gamma-\beta_1}\right)\rho_0^0(t,x-y)\,.$$

Next, like above

$$\mathrm{I}_2\leqslant c\,(t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1-\gamma}\left(\left[h^{-1}(1/s)\right]^{\gamma}+\left[h^{-1}(1/s)\right]^{\gamma-\beta_1}\right)\rho_0^0(s,x-y)$$
$$\leqslant c\,(t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1-\gamma}\left[h^{-1}(1/t)\right]^{\gamma-\beta_1}\rho_0^0(t,x-y)\,.$$

Similarly, $\mathrm{I}_3\leqslant c\,(t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1}\rho_0^0(t,x-y)$. Finally, by Lemma 5.15,

$$\int_0^t \mathrm{I}_0\,ds\leqslant c\left(1+\left[h^{-1}(1/t)\right]^{\beta_1}\right)\rho_0^0(t,x-y)\,,$$

which proves (48). Now, by (47), Lemma 5.17(a) and the monotonicity of h^{-1},

$$\int_{\mathbb{R}^d}\big(\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3\big)\,dy\leqslant c(t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{\beta_1-\gamma}s^{-1}\left[h^{-1}(1/s)\right]^{\gamma}\,.$$

The result follows by integration in s and application of Lemma 5.15. □

Lemma 3.13. *We have for all $t>0$, $x,y\in\mathbb{R}^d$,*

$$\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x)=\int_0^t\int_{\mathbb{R}^d}\mathcal{L}_x^{\mathfrak{K}_x}p^{\mathfrak{K}_z}(t-s,x,z)q(s,z,y)\,dzds\,. \tag{50}$$

Further, $\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x)$ is jointly continuous in $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

Proof. (i) By (41), and (45) in the case (P1),

$$\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x,s)=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_z}(t-s,x,z;w)q(s,z,y)\,dz\right)\kappa(x,w)J(w)dw\,. \tag{51}$$

By finiteness of (46) and Fubini's theorem,

$$\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x,s)=\int_{\mathbb{R}^d}\mathcal{L}_x^{\mathfrak{K}_x}p^{\mathfrak{K}_z}(t-s,x,z)q(s,z,y)\,dz\,. \tag{52}$$

Finally, by (42), and (43) in the case (P1), in the first equality, and (48), (46) in the second (allowing to change the order of integration twice) we prove (50) as follows

$$\begin{aligned}\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x) &= \int_{\mathbb{R}^d}\left(\int_0^t\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_z}(t-s,x,z;w)q(s,z,y)\,dzds\right)\kappa(x,w)J(w)dw\\ &= \int_0^t\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\delta^{\mathfrak{K}_z}(t-s,x,z;w)\,\kappa(x,w)J(w)dw\right)q(s,z,y)\,dzds\,.\end{aligned}$$

(ii) Fix $t_0>0$, $x_0,y_0\in\mathbb{R}^d$. For clarity we define $f(t,x,y):=\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x)$. Note that by (50) and (52) we have $f(t,x,y)=I(t,x,y)+\int_0^{t_0-\varepsilon_0}\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x,s)\,ds$, where $I(t,x,y):=\int_{t_0-\varepsilon_0}^t\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x,s)ds$. First we show that given $\varepsilon>0$ there exists $\varepsilon_0\in(0,t_0/3)$ such that for all $r\in\mathbb{R}$ satisfying $|r-t_0|<\varepsilon_0/2$, and all $x,y\in\mathbb{R}^d$,

$$|I(r,x,y)|<\varepsilon\,. \tag{53}$$

Indeed, by (51), (47), the estimates of I_1, I_2, I_3 from the proof of Lemma 3.12, the monotonicity of h^{-1} and (94),

$$\begin{aligned}|I(r,x,y)| &\leqslant c\int_{t_0-\varepsilon_0}^{r}\left(r^{-1}+(r-s)^{-1}\left[h^{-1}(1/(r-s))\right]^{\beta_1-\gamma}\left[h^{-1}(1/r)\right]^{\gamma-\beta_1}\right)ds\,\rho_0^0(r,x-y)\\ &\leqslant c\int_0^{2\varepsilon_0}\left(2/t_0+u^{-1}\left[h^{-1}(1/u)\right]^{\beta_1-\gamma}\left[h^{-1}(2/t_0)\right]^{\gamma-\beta_1}\right)du\,\rho_0^0(t_0/2,0)<\varepsilon\,.\end{aligned}$$

Using (53), (27), (94), (39) and Lemma 5.17(a) we get for all $t>0$ satisfying $|t-t_0|<\varepsilon_0/2$, and all $x,y\in\mathbb{R}^d$,

$$\begin{aligned}|f(t,x,y)-f(t,x,y_0)| &\leqslant 2\varepsilon+\int_0^{t_0-\varepsilon_0}\int_{\mathbb{R}^d}\left|\mathcal{L}_x^{\mathfrak{K}_x}p^{\mathfrak{K}_z}(t-s,x,z)\right||q(s,z,y)-q(s,z,y_0)|\,dzds\\ &\leqslant 2\varepsilon+c\rho_0^0(\varepsilon_0/2,0)\,(|y-y_0|^{\beta_1-\gamma}\wedge 1)\int_0^{t_0}s^{-1}\left[h^{-1}(1/s)\right]^{\gamma}ds\,.\end{aligned}$$

Again by (53) we have for $t>0$, $|t-t_0|<\varepsilon_0/2$, $x\in\mathbb{R}^d$,

$$|f(t,x,y_0)-f(t_0,x_0,y_0)|\leqslant 2\varepsilon+\left|\int_0^{t_0-\varepsilon_0}\mathcal{L}_x^{\mathfrak{K}_x}\phi_{y_0}(t,x,s)\,ds-\int_0^{t_0-\varepsilon_0}\mathcal{L}_x^{\mathfrak{K}_{x_0}}\phi_{y_0}(t_0,x_0,s)\,ds\right|.$$

By (27), (37) and (94), for $s \in (0, t_0 - \varepsilon_0)$, $x \in \mathbb{R}^d$ we get

$$\left|\mathcal{L}_x^{\mathfrak{K}_x} p^{\mathfrak{K}_z}(t-s,x,z)\right| |q(s,z,y_0)| \leqslant c\, \rho_0^0(\varepsilon_0/2, 0)\big(\rho_0^{\beta_1} + \rho_{\beta_1}^0\big)(s, z-y_0)\,.$$

By Lemma 5.17(a) and 5.15 the right hand side is integrable over $(0, t_0-\varepsilon_0)\times\mathbb{R}^d$ in $dzds$. Thus by (52), Lemma 3.2 and the dominated convergence theorem $\int_0^{t_0-\varepsilon_0} \mathcal{L}_x^{\mathfrak{K}_x}\phi_{y_0}(t,x,s)ds = \int_0^{t_0-\varepsilon_0} \mathcal{L}_x^{\mathfrak{K}_{x_0}}\phi_{y_0}(t_0,x_0,s)ds$ as $(t,x)\to(t_0,x_0)$. Finally, if $(t,x,y)\to(t_0,x_0,y_0)$, then

$$\begin{aligned}\lim|f(t,x,y) - f(t_0,x_0,y_0)| &\leqslant \lim\left(|f(t,x,y)-f(t,x,y_0)| + |f(t,x,y_0)-f(t_0,x_0,y_0)|\right)\\ &\leqslant 4\varepsilon. \quad \square\end{aligned}$$

The following result is the counterpart of [16, Lemma 3.5] and [45, Lemma 4.6].

Lemma 3.14. *For all $t > 0$, $x, y \in \mathbb{R}^d$, $x \neq y$, we have*

$$\phi_y(t,x) = \int_0^t \left(q(r,x,y) + \int_0^r \int_{\mathbb{R}^d} \mathcal{L}_x^{\mathfrak{K}_z} p^{\mathfrak{K}_z}(r-s,x,z) q(s,z,y)\, dzds \right) dr\,. \tag{54}$$

Proof. *Step 1.* Note that for every $s \in (0,t)$ and all $x, y \in \mathbb{R}^d$,

$$\partial_t \phi_y(t,x,s) = \int_{\mathbb{R}^d} \partial_t p^{\mathfrak{K}_z}(t-s,x,z) q(s,z,y)\, dz\,. \tag{55}$$

The above follows from (41), the mean value theorem, (25), (27), (94) and integrability of $|q(s,z,y)|$ in z (see (37)) which justifies the use the dominated convergence theorem.

Step 2. Let $T > 0$. We prove that there exists $c > 0$ such that for all $t \in (0,T]$, $x, y \in \mathbb{R}^d$,

$$\int_0^t \int_0^r |\partial_r \phi_y(r,x,s)|\, ds\, dr \leqslant ct \frac{K(|x-y|)}{|x-y|^d}\,. \tag{56}$$

By (55), (25) and (52) we get $\partial_r \phi_y(r,x,s) = \mathcal{L}_x^{\mathfrak{K}_x}\phi_y(r,x,s) - \int_{\mathbb{R}^d} q_0(r-s,x,z)q(s,z,y)\,dz$. Next, applying (51) and (48) we have $\int_0^t\int_0^r |\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(r,x,s)|dsdr \leqslant ctK(|x-y|)/|x-y|^d$. By (32), (37), Lemma 5.17(c) (once with $n_1 = m_1 = n_2 = m_2 = \beta_1$) and by (92), (93),

$$\begin{aligned}&\int_0^t \left(\int_0^r \int_{\mathbb{R}^d} |q_0(r-s,x,z)q(s,z,y)|\, dzds \right) dr\\ &\leqslant c \int_0^t \rho_0^0(r, x-y)\, dr \leqslant ctK(|x-y|)/|x-y|^d\,.\end{aligned}$$

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Step 3. We claim that for fixed $s>0$, $x,y\in\mathbb{R}^d$,

$$\lim_{t\to s^+}\phi_y(t,x,s)=q(s,x,y)\,. \tag{57}$$

In view of (41) and (30) it suffices to consider the following expression for $\delta>0$ as $t\to s^+$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} p^{\mathfrak{K}_z}(t-s,x,z)\big(q(s,z,y)-q(s,x,y)\big)dz\right| \\
&\leqslant \int_{|x-z|<\delta} p^{\mathfrak{K}_z}(t-s,x,z)|q(s,z,y)-q(s,x,y)|\,dz \\
&+\int_{|x-z|\geqslant\delta} p^{\mathfrak{K}_z}(t-s,x,z)\big(|q(s,z,y)|+|q(s,x,y)|\big)dz =: \mathrm{I}_1+\mathrm{I}_2\,.
\end{aligned}$$

By (38) for any $\varepsilon>0$ there is $\delta>0$ such that $|q(s,z,y)-q(s,x,y)|<\varepsilon$ if $|z-x|<\delta$. Together with Proposition 2.1 and Lemma 5.6 we get $\mathrm{I}_1\leqslant c\varepsilon$. By (37) there is $c>0$ such that $|q(s,z,y)|\leqslant c$ for all $z\in\mathbb{R}^d$. Then by Proposition 2.1 we have

$$\mathrm{I}_2\leqslant c(t-s)\int_{|x-z|\geqslant\delta} K(|x-z|)|x-z|^{-d}dz\leqslant c(t-s)h(\delta)\xrightarrow{t\to s^+}0\,.$$

Step 4. Let $x\neq y$. By (25) and (55) in the first equality, (56) and Fubini's theorem in the second, (56) that allows to put the limit in the third equality, and (42), (57), [64, Theorem 7.21] in the last,

$$\begin{aligned}
&\int_0^t\int_0^r\int_{\mathbb{R}^d}\mathcal{L}_x^{\mathfrak{K}_z}p^{\mathfrak{K}_z}(r-s,x,z)q(s,z,y)\,dzdsdr=\int_0^t\int_0^r\partial_r\phi_y(r,x,s)\,dsdr\\
&=\int_0^t\int_s^t\partial_r\phi_y(r,x,s)\,drds=\int_0^t\lim_{\varepsilon\to0+}\int_{s+\varepsilon}^t\partial_r\phi_y(r,x,s)\,drds=\phi_y(t,x)-\int_0^t q(s,x,y)\,ds\,.
\end{aligned}$$

This ends the proof. □

Corollary 3.15. *For all $x,y\in\mathbb{R}^d$, $x\neq y$, the function $\phi_y(t,x)$ is differentiable in $t>0$ and*

$$\partial_t\phi_y(t,x)=q_0(t,x,y)+\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x)\,. \tag{58}$$

Proof. By (54), (36) and (50) we have

$$\phi_y(t,x)=\int_0^t\left(q_0(r,x,y)+\int_0^r\int_{\mathbb{R}^d}\mathcal{L}_x^{\mathfrak{K}_x}p^{\mathfrak{K}_z}(r-s,x,z)q(s,z,y)\,dzds\right)dr$$

$$= \int_0^t \left(q_0(r,x,y) + \mathcal{L}_x^{\mathfrak{K}_x} \phi_y(r,x) \right) dr \,.$$

Lemma 3.5 and 3.13 assure that the integrand is continuous and the result follows. □

3.4. Properties of $p^\kappa(t,x,y)$

Now we define and study the function

$$\begin{aligned} p^\kappa(t,x,y) &:= p^{\mathfrak{K}_y}(t,x,y) + \phi_y(t,x) \\ &= p^{\mathfrak{K}_y}(t,x,y) + \int_0^t \int_{\mathbb{R}^d} p^{\mathfrak{K}_z}(t-s,x,z) q(s,z,y)\, dz ds \,. \end{aligned} \tag{59}$$

Lemma 3.16. *Let $\beta_1 \in (0,\beta] \cap (0,\alpha_h)$. For every $T>0$ there exists a constant $c = c(d,T,\sigma,\kappa_2,\beta_1)$ such that for all $t \in (0,T]$, $x,y \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} |\delta^\kappa(t,x,y;z)|\, \kappa(x,z) J(z) dz \leqslant c\rho_0^0(t,x-y)\,, \tag{60}$$

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \delta^\kappa(t,x,y;z)\, dy \right| \kappa(x,z) J(z) dz \leqslant ct^{-1} \left[h^{-1}(1/t) \right]^{\beta_1} . \tag{61}$$

Proof. By (59), and (43) in the case (P1),

$$\delta^\kappa(t,x,y;w) = \delta^{\mathfrak{K}_y}(t,x,y;w) + \int_0^t \int_{\mathbb{R}^d} \delta^{\mathfrak{K}_z}(t-s,x,z;w) q(s,z,y)\, dz ds \,.$$

The inequalities result from Lemma 3.4 and 3.12. □

The following result is the counterpart of [16, Lemma 3.7 and 4.2].

Lemma 3.17.

(a) *The function $p^\kappa(t,x,y)$ is jointly continuous on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.*
(b) *For every $T>0$ there is a constant $c = c(d,T,\sigma,\kappa_2,\beta)$ such that for all $t \in (0,T]$ and $x,y \in \mathbb{R}^d$,*

$$|p^\kappa(t,x,y)| \leqslant ct\rho_0^0(t,x-y).$$

(c) *For all $t>0$, $x,y \in \mathbb{R}^d$, $x \neq y$,*

$$\partial_t p^\kappa(t,x,y) = \mathcal{L}_x^\kappa p^\kappa(t,x,y)\,.$$

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(d) *For every* $T>0$ *there is a constant* $c=c(d,T,\sigma,\kappa_2,\beta)$ *such that for all* $t\in(0,T]$, $x,y\in\mathbb{R}^d$ *and* $\varepsilon\in[0,1]$,

$$|\mathcal{L}_x^{\kappa,\varepsilon} p^\kappa(t,x,y)| \leqslant c\rho_0^0(t,x-y)\,, \tag{62}$$

and if $1-\alpha_h<\beta\wedge\alpha_h$, *then*

$$\left|\nabla_x p^\kappa(t,x,y)\right| \leqslant c\Big[h^{-1}(1/t)\Big]^{-1} t\rho_0^0(t,x-y)\,. \tag{63}$$

(e) *For all* $T>0$, $\gamma\in[0,1]\cap[0,\alpha_h)$, *there is a constant* $c=c(d,T,\sigma,\kappa_2,\beta,\gamma)$ *such that for all* $t\in(0,T]$ *and* $x,x',y\in\mathbb{R}^d$,

$$\left|p^\kappa(t,x,y)-p^\kappa(t,x',y)\right| \leqslant c(|x-x'|^\gamma\wedge 1)\,t\left(\rho_{-\gamma}^0(t,x-y)+\rho_{-\gamma}^0(t,x'-y)\right).$$

For all $T>0$, $\gamma\in[0,\beta)\cap[0,\alpha_h)$, *there is a constant* $c=c(d,T,\sigma,\kappa_2,\beta,\gamma)$ *such that for all* $t\in(0,T]$ *and* $x,y,y'\in\mathbb{R}^d$,

$$\left|p^\kappa(t,x,y)-p^\kappa(t,x,y')\right| \leqslant c(|y-y'|^\gamma\wedge 1)\,t\left(\rho_{-\gamma}^0(t,x-y)+\rho_{-\gamma}^0(t,x-y')\right).$$

(f) *The function* $\mathcal{L}_x^\kappa p^\kappa(t,x,y)$ *is jointly continuous on* $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

Proof. The statement of (a) follows from Lemma 3.1 and 3.9. Part (b) is a result of Proposition 2.1 and Lemma 3.8. The equation in (c) is a consequence of (59), (25) and (58): $\partial_t p^\kappa(t,x,y)=\mathcal{L}_x^{\mathfrak{K}_x} p^{\mathfrak{K}_y}(t,x,y)+\mathcal{L}_x^{\mathfrak{K}_x}\phi_y(t,x)=\mathcal{L}_x^{\mathfrak{K}_x} p^\kappa(t,x,y)$. We get (62) by (60). For the proof of (63) we use Proposition 2.1 and (44). The first inequality of part (e) follows from Lemma 3.3 and 3.8, and (92), (93). The same argument suffices for the second inequality when supported by

$$\begin{aligned}|p^{\mathfrak{K}_y}(t,x,y)-p^{\mathfrak{K}_{y'}}(t,x,y')| &\leqslant |p^{\mathfrak{K}_y}(t,-y,-x)-p^{\mathfrak{K}_y}(t,-y',-x)|\\ &\quad+|p^{\mathfrak{K}_y}(t,x,y')-p^{\mathfrak{K}_{y'}}(t,x,y')|\end{aligned}$$

and Theorem 2.11. Part (f) follows from Lemma 3.2 and 3.13. $\square$

4. Main results

4.1. A nonlocal maximum principle

Recall that $\mathcal{L}^{\kappa,0^+}f:=\lim_{\varepsilon\to0^+}\mathcal{L}^{\kappa,\varepsilon}f$ is an extension of $\mathcal{L}^\kappa f:=\mathcal{L}^{\kappa,0}f$. Moreover, in the case (P1), the well-posedness of those operators require the existence of the gradient ∇f.

Theorem 4.1. *Assume* (P). *Let* $T>0$ *and* $u\in C([0,T]\times\mathbb{R}^d)$ *be such that*

$$\|u(t,\cdot)-u(0,\cdot)\|_\infty \xrightarrow{t\to0^+} 0\,, \qquad \sup_{t\in[0,T]}\|u(t,\cdot)\mathbf{1}_{|\cdot|\geqslant r}\|_\infty \xrightarrow{r\to\infty} 0\,. \tag{64}$$

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Assume that $u(t,x)$ satisfies the following equation: for all $(t,x) \in (0,T] \times \mathbb{R}^d$,

$$\partial_t u(t,x) = \mathcal{L}_x^{\kappa,0^+} u(t,x)\,. \tag{65}$$

If $\sup_{x\in\mathbb{R}^d} u(0,x) \geqslant 0$, then for every $t \in (0,T]$,

$$\sup_{x\in\mathbb{R}^d} u(t,x) \leqslant \sup_{x\in\mathbb{R}^d} u(0,x)\,. \tag{66}$$

Proof. For arbitrary $\lambda > 0$ we consider $\widetilde{u}(t,x) = e^{-\lambda t} u(t,x)$. Then for all $(t,x) \in (0,T] \times \mathbb{R}^d$,

$$\partial_t \widetilde{u}(t,x) = \left(-\lambda + \mathcal{L}_x^{\kappa,0^+}\right) \widetilde{u}(t,x)\,.$$

By letting $\lambda \to 0^+$, it suffices to prove that

$$\sup_{x\in\mathbb{R}^d} \widetilde{u}(t,x) \leqslant \sup_{x\in\mathbb{R}^d} \widetilde{u}(0,x) = \sup_{x\in\mathbb{R}^d} u(0,x), \qquad \text{for every } t \in (0,T]\,. \tag{67}$$

Suppose that (67) does not hold. Then $\widetilde{u}(t',x') > \sup_{x\in\mathbb{R}^d} \widetilde{u}(0,x) \geqslant 0$ for some $(t',x') \in (0,T] \times \mathbb{R}^d$. Thus by continuity and (64) the function $\widetilde{u}$ attains a positive maximum at some $(t_0,x_0) \in (0,T] \times \mathbb{R}^d$. Consequently, $\partial_t \widetilde{u}(t_0,x_0) \geqslant 0$, $\mathcal{L}_x^{\kappa,0^+} \widetilde{u}(t_0,x_0) \leqslant 0$ and

$$0 \leqslant \partial_t \widetilde{u}(t_0,x_0) = \left(-\lambda + \mathcal{L}_x^{\kappa,0^+}\right) \widetilde{u}(t_0,x_0) \leqslant -\lambda \widetilde{u}(t_0,x_0)\,,$$

which is a contradiction. □

Corollary 4.2. *If $u_1, u_2 \in C([0,T] \times \mathbb{R}^d)$ satisfy (64), (65) and $u_1(0,x) = u_2(0,x)$, then $u_1 \equiv u_2$ on $[0,T] \times \mathbb{R}^d$.*

4.2. *Properties of the semigroup $(P_t^\kappa)_{t\geq 0}$*

Define

$$P_t^\kappa f(x) = \int_{\mathbb{R}^d} p^\kappa(t,x,y) f(y) dy.$$

We first collect some properties of $\Upsilon_t * f$.

Remark 4.3. We have $\Upsilon_t * f \in C_b(\mathbb{R}^d)$ for any $f \in L^p(\mathbb{R}^d)$, $p \in [1,\infty]$. Moreover, $\Upsilon_t * f \in C_0(\mathbb{R}^d)$ for any $f \in L^p(\mathbb{R}^d) \cup C_0(\mathbb{R}^d)$, $p \in [1,\infty)$. Further, there is $c = c(d)$ such that $\|\Upsilon_t * f\|_p \leqslant c\|f\|_p$ for all $t > 0$, $p \in [1,\infty]]$. The above follows from $\Upsilon_t \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d)$ for every $q \in [1,\infty]$ (see Lemma 5.6), and from properties of the convolution.

Lemma 4.4.

(a) *We have* $P_t^\kappa f \in C_b(\mathbb{R}^d)$ *for any* $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty]$. *Moreover,* $P_t^\kappa f \in C_0(\mathbb{R}^d)$ *for any* $f \in L^p(\mathbb{R}^d) \cup C_0(\mathbb{R}^d)$, $p \in [1, \infty)$. *For every* $T > 0$ *there exists a constant* $c = c(d, T, \sigma, \kappa_2, \beta)$ *such that for all* $t \in (0, T]$ *we get*

$$\|P_t^\kappa f\|_p \leqslant c\|f\|_p \,.$$

(b) $P_t^\kappa : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$, $t > 0$, *and for any bounded uniformly continuous function* f,

$$\lim_{t \to 0^+} \|P_t^\kappa f - f\|_\infty = 0 \,.$$

(c) $P_t^\kappa : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $t > 0$, $p \in [1, \infty)$, *and for any* $f \in L^p(\mathbb{R}^d)$,

$$\lim_{t \to 0^+} \|P_t^\kappa f - f\|_p = 0 \,.$$

Proof. Part (a) follows from Remark 4.3 and Lemma 3.17. It remains to prove the continuity as $t \to 0^+$. We fix $T > 0$ and let $t \in (0, T]$. By Lemma 3.8, Young's inequality and Lemma 5.17(a) we have

$$\Big\| \int_{\mathbb{R}^d} \phi_y(t, \cdot) f(y) dy \Big\|_p \leqslant c \left[h^{-1}(1/t)\right]^{\beta_1} \|f\|_p \xrightarrow{t \to 0^+} 0 \,.$$

Then by (59),

$$\|P_t^\kappa f - f\|_p \leqslant \Big\| \int_{\mathbb{R}^d} p^{\mathfrak{K}_y}(t, \cdot, y)\big[f(y) - f(\cdot)\big] dy \Big\|_p + \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^{\mathfrak{K}_y}(t, x, y) dy - 1 \right| \|f\|_p$$
$$+ c\left[h^{-1}(1/t)\right]^{\beta_1} \|f\|_p \,,$$

and by (30) it suffices to consider the first term. By Proposition 2.1, Minkowski's integral inequality and Lemma 5.6, for $f_z(x) := f(x + z)$, we have

$$\Big\| \int_{\mathbb{R}^d} p^{\mathfrak{K}_y}(t, \cdot, y)\big[f(y) - f(\cdot)\big] dy \Big\|_p \leqslant c \int_{\mathbb{R}^d} \Upsilon_t(z) \|f_z - f\|_p \, dz$$
$$\leqslant c \int_{|z| < \delta} \Upsilon_t(z) \|f_z - f\|_p \, dz + 2c\|f\|_p \int_{|z| \geqslant \delta} tK(|x|)|x|^{-d} \, dz \leqslant c\left(\varepsilon + th(\delta)\|f\|_p\right),$$

where $\delta > 0$ is such that $\|f_z - f\|_p < \varepsilon$ for $|z| < \delta$. This ends the proof. □

Lemma 4.5. *Assume* (P1). *For any $f \in L^p(\mathbb{R}^d)$, $p \in [1,\infty]$, we have for all $t>0$, $x \in \mathbb{R}^d$,*

$$\nabla_x P_t^\kappa f(x) = \int_{\mathbb{R}^d} \nabla_x p^\kappa(t,x,y) f(y) dy\,, \tag{68}$$

and for any $f \in L^\infty(\mathbb{R}^d)$ and all $t>0$, $x \in \mathbb{R}^d$,

$$\nabla_x \left(\int_0^t P_s^\kappa f(x)\, ds \right) = \int_0^t \nabla_x P_s^\kappa f(x)\, ds\,. \tag{69}$$

Proof. By (63) and Corollary 5.10 for $|\varepsilon| < h^{-1}(1/t)$,

$$\left| \frac{1}{\varepsilon}(p^\kappa(t,x+\varepsilon e_i,y) - p^\kappa(t,x,y)) \right| |f(y)| \leqslant c \left[h^{-1}(1/t)\right]^{-1} \Upsilon_t(x-y)|f(y)|\,.$$

The right hand side is integrable by Remark 4.3. We can use the dominated convergence theorem, which gives (68). Now, for $f \in L^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \left| \frac{1}{\varepsilon}(p^\kappa(s,x+\varepsilon e_i,y) - p^\kappa(s,x,y)) \right| |f(y)| dy \leqslant \int_{\mathbb{R}^d} \int_0^1 \left| \partial_{x_i} p^\kappa(s,x+\theta\varepsilon e_i,y) \right| d\theta\, |f(y)| dy$$

$$\leqslant c \left[h^{-1}(1/s)\right]^{-1} \int_0^1 \big(\Upsilon_s * |f|\big)(x+\theta\varepsilon e_i)\, d\theta \leqslant c \left[h^{-1}(1/s)\right]^{-1} \|\Upsilon_s * |f|\|_\infty\,.$$

The right hand side is bounded by $c\left[h^{-1}(1/s)\right]^{-1} \|f\|_\infty$ (Remark 4.3), which is integrable over $(0,t)$ by (A2) of Lemma 5.3 and $\alpha_h > 1$. Finally, (69) follows by dominated convergence theorem. $\square$

Lemma 4.6. *For any function $f \in L^p(\mathbb{R}^d)$, $p \in [1,\infty]$, and all $t>0$, $x \in \mathbb{R}^d$,*

$$\mathcal{L}_x^\kappa P_t^\kappa f(x) = \int_{\mathbb{R}^d} \mathcal{L}_x^\kappa p^\kappa(t,x,y) f(y) dy\,. \tag{70}$$

Further, for every $T>0$ there exists a constant $c>0$ such that for all $f \in L^p(\mathbb{R}^d)$, $t \in (0,T]$,

$$\|\mathcal{L}^\kappa P_t^\kappa f\|_p \leqslant c t^{-1} \|f\|_p\,. \tag{71}$$

Proof. By the definition, and (68) in the case (P1),

$$\mathcal{L}_x^\kappa P_t^\kappa f(x) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \delta^\kappa(t,x,y;z) f(y) dy \right) \kappa(x,z) J(z) dz\,. \tag{72}$$

The equality follows from Fubini's theorem justified by (60) and Remark 4.3. The inequality follows then from (70), (62) and again Remark 4.3. □

Lemma 4.7. *Let $f \in C_0(\mathbb{R}^d)$. For $t > 0$, $x \in \mathbb{R}^d$ we define $u(t,x) = P_t^\kappa f(x)$ and $u(0,x) = f(x)$. Then $u \in C([0,T] \times \mathbb{R}^d)$, (64) holds and $\partial_t u(t,x) = \mathcal{L}_x^\kappa u(t,x)$ for all $t, T > 0$, $x \in \mathbb{R}^d$.*

Proof. First we show that (i), (ii), (iii) (and (iv) in the case (P1)) of Theorem 1.1 hold true. Indeed, it follows from Lemma 3.17, Lemma 4.4(b) and (94). Moreover, part (iii) holds with $f_0 = \Upsilon_{t_0}$. Except the last part (and one use of Lemma 4.4(b)) we base the proof solely on the properties from Theorem 1.1. Note that $u(t,x) = \int_{\mathbb{R}^d} p^\kappa(t,x,x-z) f(x-z)\,dz$ and we have $|p^\kappa(t,x,x-z)f(x-z)| \leqslant c f_0(z)\|f\|_\infty$ for all $t \in [t_0, T]$, $x \in \mathbb{R}^d$. Thus we can use the dominated convergence theorem and the joint continuity to get $u \in C((0,T] \times \mathbb{R}^d)$. The first part of the statement follows by combining the latter with $\|u(t,\cdot) - u(0,\cdot)\|_\infty \to 0$, $t \to 0^+$ (see Lemma 4.4(b) and (11)). Let $\varepsilon > 0$. By previous line there is $t_0 > 0$ such that $|u(t,x)| \leqslant |f(x)| + \varepsilon$ for all $t \in [0, t_0]$, $x \in \mathbb{R}^d$, while for $t \in [t_0, T]$, $x \in \mathbb{R}^d$ we have $|u(t,x)| \leqslant c(f_0 * |f|)(x)$, which is an element of $C_0(\mathbb{R}^d)$. This finishes the proof of (64). Finally, we prove the last part. By the mean value theorem, Lemma 3.17(c), (62), (94) and the dominated convergence theorem $\partial_t u(t,x) = \int_{\mathbb{R}^d} \partial_t p^\kappa(t,x,y) f(y)dy$. Then we apply Lemma 3.17(c) and Lemma 4.6. □

The following result is the counterpart of [16, Lemma 4.3].

Lemma 4.8. *For any bounded (uniformly) Hölder continuous function $f \in C_b^\eta(\mathbb{R}^d)$, $\eta > 0$, and all $t > 0$, $x \in \mathbb{R}^d$, we have $\int_0^t |\mathcal{L}_x^\kappa P_s^\kappa f(x)| ds < \infty$ and*

$$\mathcal{L}_x^\kappa \left(\int_0^t P_s^\kappa f(x)\,ds \right) = \int_0^t \mathcal{L}_x^\kappa P_s^\kappa f(x)\,ds\,. \tag{73}$$

Proof. By the definition, and Lemma 4.5 in the case (P1),

$$\mathcal{L}_x^\kappa \int_0^t P_s^\kappa f(x)\,ds = \int_{\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} \delta^\kappa(s,x,y;z) f(y) dy ds \right) \kappa(x,z) J(z) dz\,.$$

Note that by (72) the poof will be finished if we can change the order of integration from $dsdz$ to $dzds$. To this end we use Fubini's theorem justified by the following. We have $|f(y) - f(x)| \leqslant c(|y-x|^\eta \wedge 1)$ and we can assume that $\eta < \alpha_h$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_0^t \left| \int_{\mathbb{R}^d} \delta^\kappa(s,x,y;z) f(y) dy \right| ds\, \kappa(x,z) J(z) dz \\
&\qquad \leqslant \int_{\mathbb{R}^d} \int_0^t \left| \int_{\mathbb{R}^d} \delta^\kappa(s,x,y;z) \big[f(y) - f(x) \big] dy \right| ds\, \kappa(x,z) J(z) dz
\end{aligned}$$

$$+\int_{\mathbb{R}^d}\int_0^t\left|\int_{\mathbb{R}^d}\delta^\kappa(s,x,y;z)f(x)dy\right|ds\,\kappa(x,z)J(z)dz =: \mathrm{I}_1+\mathrm{I}_2\,.$$

By (60) we have $\mathrm{I}_1 \leqslant c\int_0^t\int_{\mathbb{R}^d}\rho_0^\eta(s,y-x)dyds$, while by (61) $\mathrm{I}_2 \leqslant c\int_0^t s^{-1}\left[h^{-1}(1/s)\right]^{\beta_1}ds$. The integrals are finite by Lemma 5.17(a) and 5.15. $\square$

Proposition 4.9. *Assume* (P). *For any $f\in C_b^2(\mathbb{R}^d)$ and all $t>0$, $x\in\mathbb{R}^d$,*

$$P_t^\kappa f(x)-f(x)=\int_0^t P_s^\kappa\mathcal{L}^\kappa f(x)\,ds\,. \tag{74}$$

Proof. (i) Note that $\mathcal{L}^\kappa f\in C_0(\mathbb{R}^d)$ for any $f\in C_0^2(\mathbb{R}^d)$.

(ii) We will show that if $f\in C_0^{2,\varepsilon}(\mathbb{R}^d)$, then $\mathcal{L}^\kappa f$ is (uniformly) Hölder continuous. To this end we use [2, Theorem 5.1]. For $x,z\in\mathbb{R}^d$ define

$$E_zf(x)=f(x+z)-f(x)\,,\qquad F_zf(x)=f(x+z)-f(x)-\langle z,\nabla f(x)\rangle\,.$$

We only consider the cases (P1) and (P3). The case (P2) is similar (see Lemma 5.5). Then $\mathcal{L}^{\mathfrak{K}_y}f(x)=\int_{|z|<1}F_zf(x)\kappa(y,z)J(z)dz+\int_{|z|\geqslant1}E_zf(x)\kappa(y,z)J(z)dz$. Using (2), (3), (1) and [2, Theorem 5.1(b) and (e)],

$$\begin{aligned}
|\mathcal{L}^\kappa f(x)-\mathcal{L}^\kappa f(y)| &\leqslant |\mathcal{L}^{\mathfrak{K}_x}f(x)-\mathcal{L}^{\mathfrak{K}_y}f(x)|+|\mathcal{L}^{\mathfrak{K}_y}f(x)-\mathcal{L}^{\mathfrak{K}_y}f(y)|\\
&\leqslant c|x-y|^\beta+\int_{|z|<1}|F_zf(x)-F_zf(y)|\,\kappa(y,z)J(z)dz\\
&\quad+\int_{|z|\geqslant1}|E_zf(x)-E_zf(y)|\,\kappa(y,z)J(z)dz\\
&\leqslant c|x-y|^\beta+c|x-y|^\varepsilon\int_{|z|<1}|z|^2\nu(|z|)dz+c|x-y|\int_{|z|\geqslant1}\nu(|z|)dz\,.
\end{aligned}$$

(iii) We will prove that (74) holds if $f\in C_0^{2,\varepsilon}(\mathbb{R}^d)$. Let u_0 and u_1 be defined as in Lemma 4.7 such that $u_0(0,x)=\mathcal{L}^\kappa f(x)$ and $u_1(0,x)=f(x)$. Further, let

$$u_2(t,x):=f(x)+\int_0^t P_s^\kappa\mathcal{L}^\kappa f(x)ds=f(x)+\int_0^t u_0(s,x)ds\,.$$

By Lemma 4.7 for u_0 we get that $u_2\in C([0,T]\times\mathbb{R}^d)$, (64) holds for u_2 and $\partial_tu_2(t,x)=u_0(t,x)$. Using (ii), Lemma 4.8, Lemma 4.7 for u_0 and [64, Theorem 7.21] we have

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$$\mathcal{L}_x^\kappa u_2(t,x) = \mathcal{L}^\kappa f(x) + \int_0^t \mathcal{L}_x^\kappa u_0(s,x)\,ds = \mathcal{L}^\kappa f(x) + \int_0^t \partial_s u_0(s,x)\,ds$$

$$= \mathcal{L}^\kappa f(x) + \lim_{\varepsilon\to 0^+} \int_\varepsilon^t \partial_s u_0(s,x)\,ds = u_0(t,x) = \partial_t u_2(t,x)\,.$$

Thus we can apply Corollary 4.2 to u_1 and u_2, which implies the claim.

(iv) We will extend (74) to $f \in C_b^2(\mathbb{R}^d)$ by approximation. Take $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi(x) = 1$ if $|x| \leqslant 1$, $\varphi(x) = 0$ if $|x| \geqslant 2$ and set $\varphi_n(x) = \varphi(x/n)$. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be standard mollifier such that $\operatorname{supp}(\phi_n) \subset B(0, 1/n)$. Then $f_n = (f * \phi_n)\cdot\varphi_n \in C_c^\infty(\mathbb{R}^d)$ and $f_n \to f$, $\nabla f_n \to \nabla f$ pointwise. Thus $E_z f_n(x) \to E_z f(x)$ and $F_z f_n(x) \to F_z f(x)$. Further, since $\|\partial_x^\beta (f * \phi_n)\|_\infty \leqslant \|\partial_x^\beta f\|_\infty$ for every multi-index $|\beta| \leqslant 2$, there is $c > 0$ such that for all $x, z \in \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$|E_z f_n(x)| \leqslant c\,(|z| \wedge 1)\,, \qquad |F_z f_n(x)| \leqslant c\,|z|^2\,.$$

Therefore, $\mathcal{L}^\kappa f_n(x) \to \mathcal{L}^\kappa f(x)$ and $\|\mathcal{L}^\kappa f_n\|_\infty \leqslant c < \infty$. The result follows from (74) for f_n and the dominated convergence theorem (see Lemma 3.17(b) and 5.6). □

Lemma 4.10. *The function $p^\kappa(t,x,y)$ is non-negative, $\int_{\mathbb{R}^d} p^\kappa(t,x,y)dy = 1$ and $p^\kappa(t+s,x,y) = \int_{\mathbb{R}^d} p^\kappa(t,x,z)p^\kappa(s,z,y)dz$ for all $s,t > 0$, $x,y \in \mathbb{R}^d$.*

Proof. By Lemma 4.7 we can apply Theorem 4.1 to $u_1(t,x) := P_t^\kappa f(x)$, $u_1(0,x) := f(x)$ for any $f \in C_0(\mathbb{R}^d)$ and $T > 0$. The choice of $f \leqslant 0$ results in $u_1(t,x) \leqslant 0$ and proves the non-negativity of $p^\kappa(t,x,y)$. Next, given $s > 0$, $y \in \mathbb{R}^d$, by Lemma 3.17(a)(b) we can take $f(x) = p^\kappa(s,x,y)$. We also consider $u_2(t,x) = p^\kappa(t+s,x,y)$. It is clear from Lemma 3.17(a)(b)(c) and (62) that u_2 satisfies assumptions of Corollary 4.2. Hence $P_t^\kappa p(s,\cdot,y)(x) = p^\kappa(s+t,x,y)$. Finally, putting $f = 1$ in Proposition 4.9 we get $P_t^\kappa 1 - 1 = 0$. □

4.3. Proofs of Theorems 1.1–1.3

Proof of Theorem 1.3. By Lemma 4.4 and 4.10 the family $(P_t^\kappa)_{t\geqslant 0}$ is a strongly continuous positive contraction semigroup on $(L^p(\mathbb{R}^d), \|\cdot\|_p)$, $p \in [1,\infty)$, and (additionally contraction) on $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$. We postpone the proof of the analyticity. Note that for there exists $c > 0$ such that for every $g \in C_0^2(\mathbb{R}^d)$ (resp. $g \in C^2(\mathbb{R}^d)$ and $\partial^\beta g \in L^p(\mathbb{R}^d)$ for every multi-index $|\beta| \leqslant 2$),

$$\left\|\mathcal{L}^\kappa g\right\|_p \leqslant c \sum_{|\beta|\leqslant 2} \left\|\partial^\beta g\right\|_p\,.$$

The inequality follows by recovering increments of function g from its partial derivatives, Minkowski's inequality and integrability properties of the measure $J(z)dz$. We also see that $\mathcal{L}^\kappa f \in C_0(\mathbb{R}^d)$ for $f \in C_0^2(\mathbb{R}^d)$ (resp. $\mathcal{L}^\kappa f \in L^p(\mathbb{R}^d)$ for $f \in C_c^2(\mathbb{R}^d)$). By Proposition 4.9, Minkowski's integral inequality and Lemma 4.4, we have for $f \in C_0^2(\mathbb{R}^d)$ (resp. $f \in C_c^2(\mathbb{R}^d)$),

$$\left\| \frac{P_t^\kappa f - f}{t} - \mathcal{L}^\kappa f \right\|_p \leqslant \int_0^t \| P_s^\kappa \mathcal{L}^\kappa f - \mathcal{L}^\kappa f \|_p \frac{ds}{t},$$

which tends to zero as $t \to 0^+$, and ends the proof of 3(a) and 4(a). In order to prove 3(b) and 4(b) we investigate $(\bar{\mathcal{A}}_c^\kappa, D(\bar{\mathcal{A}}_c^\kappa))$ the closure of $(\mathcal{A}_c^\kappa, D(\mathcal{A}_c^\kappa)) := (\mathcal{L}^\kappa, C_c^\infty(\mathbb{R}^d))$ in $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ (resp. $(L^p(\mathbb{R}^d), \|\cdot\|_p)$).

Step 1. We show that $g \in D(\bar{\mathcal{A}}_c^\kappa)$ and $\bar{\mathcal{A}}_c^\kappa g = \mathcal{L}^\kappa g$ if $g \in C_0^\infty(\mathbb{R}^d)$ (resp. $g \in C_0^\infty(\mathbb{R}^d)$ and $\partial^\beta g \in L^p(\mathbb{R}^d)$ for every multi-index β). Take $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi(x) = 1$ if $|x| \leqslant 1$, $\varphi(x) = 0$ if $|x| \geqslant 2$ and set $\varphi_n(x) = \varphi(x/n)$. Then $g_n = g \cdot \varphi_n \in C_c^\infty(\mathbb{R}^d)$ and for every $|\beta| \leqslant 2$,

$$\|\partial^\beta (g_n - g)\|_p \leqslant \|(\partial^\beta g)(\varphi_n - 1)\|_p + c/n,$$

where c depends only on d, $\|\partial^\beta g\|_p$, $|\beta| \leqslant 1$, and $\|\partial^\beta \varphi\|_\infty$, $|\beta| \leqslant 2$. Then $\|g_n - g\|_p \to 0$ and

$$\begin{aligned} \|\bar{\mathcal{A}}_c^\kappa g_n - \mathcal{L}^\kappa g\|_p = \|\mathcal{L}^\kappa g_n - \mathcal{L}^\kappa g\|_p &\leqslant c \sum_{|\beta| \leqslant 2} \|\partial^\beta (g_n - g)\|_p \\ &\leqslant c \sum_{|\beta| \leqslant 2} \|(\partial^\beta g)(\varphi_n - 1)\|_p + c/n, \end{aligned}$$

which ends the proof of that part.

Step 2. We show that $P_t^\kappa f \in D(\bar{\mathcal{A}}_c^\kappa)$ and $\bar{\mathcal{A}}_c^\kappa P_t^\kappa f = \mathcal{L}^\kappa P_t^\kappa f$ for all $t > 0$ and $f \in C_0(\mathbb{R}^d)$ (resp. $f \in L^p(\mathbb{R}^d)$). Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a standard mollifier such that $\mathrm{supp}(\phi_n) \subset B(0, 1/n)$. Then by Lemma 4.4 $h_n := (P_t^\kappa f) * \phi_n \in C_0^\infty(\mathbb{R}^d)$ (resp. $h_n \in C_0^\infty(\mathbb{R}^d)$ and $\partial^\beta h_n \in L^p(\mathbb{R}^d)$ for every β) and $\|h_n - P_t^\kappa f\|_p \to 0$ as $n \to \infty$. By *Step 1.*, the definition ((68), (63) and Remark 4.3 in the case (P1)),

$$\bar{\mathcal{A}}_c^\kappa h_n(x) = \mathcal{L}^\kappa h_n(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \delta^\kappa(t, x - w, y; z) f(y) dy \right) \phi_n(w) \kappa(x, z) J(z)\, dw dz.$$

Using Fubini's theorem (see (60), Remark 4.3) and (72) we have

$$\begin{aligned} &\bar{\mathcal{A}}_c^\kappa h_n(x) - (\mathcal{L}^\kappa P_t^\kappa f) * \phi_n(x) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \delta^\kappa(t, x - w, y; z) f(y) dy \right) \big(\kappa(x, z) - \kappa(x - w, z)\big) J(z) dz\, \phi_n(w) dw. \end{aligned}$$

Let n be large enough so that the support of ϕ_n is contained in a ball of radius $\varepsilon > 0$. By (3), (2), (60) and Remark 4.3 we get $\|\bar{\mathcal{A}}_c^\kappa h_n - (\mathcal{L}^\kappa P_t^\kappa f) * \phi_n\|_p \leqslant c\, \varepsilon^\beta t^{-1} \|f\|_p$ with c independent of large $n \in \mathbb{N}$. We also have by (70), Lemma 3.17(f) and (d), Corollary 5.10 and Remark 4.3 that $\mathcal{L}^\kappa P_t^\kappa f \in C_0(\mathbb{R}^d)$ (resp. $\mathcal{L}^\kappa P_t^\kappa f \in L^p(\mathbb{R}^d)$ by (71)). Thus $\|\bar{\mathcal{A}}_c^\kappa h_n - \mathcal{L}^\kappa P_t^\kappa f\|_p \to 0$ as $n \to \infty$, which ends the proof.

Step 3. Obviously, $\bar{\mathcal{A}}^\kappa_c \subseteq \mathcal{A}^\kappa$, i.e., $D(\bar{\mathcal{A}}^\kappa_c) \subseteq D(\mathcal{A}^\kappa)$ and $\bar{\mathcal{A}}^\kappa_c = \mathcal{A}^\kappa$ on $D(\bar{\mathcal{A}}^\kappa_c)$. It remains to show the converse inclusion. Let $f \in D(\mathcal{A}^\kappa)$ and define $f_n = P^\kappa_{1/n} f$. We have $\|f_n - f\|_p \to 0$. By *Step 2* also $f_n \in D(\bar{\mathcal{A}}^\kappa_c)$. Since $\mathcal{A}^\kappa$ commutes with P^κ_t on $D(\mathcal{A}^\kappa)$, we get

$$\|\bar{\mathcal{A}}^\kappa_c f_n - \mathcal{A}^\kappa f\|_p = \|\mathcal{A}^\kappa f_n - \mathcal{A}^\kappa f\|_p = \|P^\kappa_{1/n}\mathcal{A}^\kappa f - \mathcal{A}^\kappa f\|_p \to 0\,.$$

This finally gives $\bar{\mathcal{A}}^\kappa_c = \mathcal{A}^\kappa$.

Now, by *Step 2* P^κ_t is differentiable in $C_0(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)$), $t > 0$, and $\partial_t P^\kappa_t = \mathcal{A}^\kappa P^\kappa_t = \mathcal{L}^\kappa P^\kappa_t$ on $C_0(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)$) (see [60, Chapter 1, Theorem 2.4(c)]. Therefore, since for all $s > 0$, $y \in \mathbb{R}^d$, the function $f(x) = p^\kappa(s, x, y)$ belongs to $C_0(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)$), parts 3(c) and 4(c) follow. We prove the analyticity. Take numbers $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that the operator norm $\|P^\kappa_t\| \leqslant Me^{\omega t}$ (see [60, Chapter 1, Theorem 2.2]). Define $T_t := e^{-\lambda t}P^\kappa_t$, $\lambda = \omega + 1$. It suffices to show the analyticity of $(T_t)_{t\geqslant 0}$. Note that $(T_t)_{t\geqslant 0}$ is generated by $A = (-\lambda + \mathcal{A}^\kappa)$ and that T_t is differentiable in $C_0(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)$), $t > 0$, and $AT_t = (-\lambda + \mathcal{L}^\kappa)T_t$ on $C_0(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)$). Then, by (71) for $t \in (0, 2]$,

$$\|AT_t f\|_p \leqslant |\lambda|\|T_t f\|_p + e^{2|\lambda|}\|\mathcal{L}^\kappa P^\kappa_t f\|_p \leqslant (|\lambda|M + ce^{2|\lambda|}t^{-1})\|f\|_p \leqslant c_1 t^{-1}\|f\|_p\,.$$

Next, by [60, Chapter 1, Theorem 2.4(c)] for $t \geqslant 2$,

$$\|AT_t f\|_p = \|T_{t-1}AT_1 f\|_p \leqslant \|T_{t-1}\|\|AT_1 f\|_p \leqslant Me^{-(t-1)}c_1\|f\|_p \leqslant c_2 t^{-1}\|f\|_p.$$

We conclude that $\|AT_t\| \leqslant Ct^{-1}$ for all $t > 0$. The analyticity follows from [60, Chapter 2, Theorem 5.2(d)]. $\square$

Proof of Theorem 1.2. All the properties are collected in Lemma 3.17 and 4.10, except for part (8), which is given in Theorem 1.3 part 3(c). $\square$

Proof of Theorem 1.1. Suppose there is another function $\tilde{p}^\kappa(t, x, y)$ that is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and satisfies (10), (11), (12), (13). In the case (P1) we also assume (14). Let $f \in C^\infty_c(\mathbb{R}^d)$. For $t \in (0, T]$, $x \in \mathbb{R}^d$ define $u_1(t, x) = \int_{\mathbb{R}^d} p^\kappa(t, x, y)f(y)dy$, $u_2(t, x) = \int_{\mathbb{R}^d} \tilde{p}^\kappa(t, x, y)f(y)dy$ and $u_1(0, x) = u_2(0, x) = f(x)$. We will justify that Corollary 4.2 applies to u_1 and u_2. For u_1 it follows directly from Lemma 4.7. For u_2 the proof is the same as the proof of Lemma 4.7, except for the last part. Thus it remains to show (65) for u_2. By the mean value theorem, (10), (13) and the dominated convergence theorem we get $\partial_t u(t, x) = \int_{\mathbb{R}^d} \partial_t \tilde{p}^\kappa(t, x, y)f(y)dy$. Now, it suffices to show

$$\mathcal{L}^{\kappa,0^+}_x u_2(t, x) = \int_{\mathbb{R}^d} \mathcal{L}^{\kappa,0^+}_x \tilde{p}^\kappa(t, x, y)f(y)dy\,. \tag{75}$$

Note that in the case (P1) by (14) we get $\nabla_x u_2(t, x) = \int_{\mathbb{R}^d} \nabla_x \tilde{p}^\kappa(t, x, y)f(y)dy$. By Fubini's theorem, justified by (12), and (14) in the case (P1), for $\varepsilon > 0$ we have $\mathcal{L}^{\kappa,\varepsilon}_x u(t, x) = \int_{\mathbb{R}^d} \mathcal{L}^{\kappa,\varepsilon}_x \tilde{p}^\kappa(t, x, y)f(y)dy$. Using (13) and the dominated convergence theorem we get (75). Finally, by Corollary 4.2 $u_1 \equiv u_2$ on $[0, T] \times \mathbb{R}^d$, hence $\tilde{p}^\kappa(t, x, y) = p^\kappa(t, x, y)$, since $f \in C^\infty_c(\mathbb{R}^d)$ was arbitrary. $\square$

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4.4. Lower bound of $p^\kappa(t,x,y)$

Lemma 4.11. *Assume that there exist $T, R, c > 0$ such that*

$$p^\kappa(t,x,y) \geq c\left[h^{-1}(1/t)\right]^{-d}, \qquad t\in(0,T],\ |x-y|\leqslant Rh^{-1}(1/t). \tag{76}$$

Then there is $C = C(d,\sigma,T,R,c) > 0$ such that

$$p^\kappa(t,x,y) \geq C\left(\left[h^{-1}(1/t)\right]^{-d} \wedge t\nu(|x-y|)\right), \qquad t\in(0,T],\ x,y\in\mathbb{R}^d.$$

Proof. Let $X=(X_t)_{t\geqslant 0}$ the Feller process corresponding to $(P_t^\kappa)_{t\geqslant 0}$. By Remark 1.5 (cf. [10, Theorem 3.21]) for every $f\in C_0^2(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(x) - \int_0^t \mathcal{L}^\kappa f(X_{s-})\,ds\,, \qquad t\geqslant 0\,, \tag{77}$$

is a martingale with respect to the filtration $\sigma(X_s : s\leqslant t)$. Let $A\subseteq\mathbb{R}^d$ be compact and $f\in C_c^\infty(\mathbb{R}^d)$ such that $\mathrm{supp}(f)\cap A=\emptyset$. Define a martingale $N_t^f := \int_0^t \mathbf{1}_A(X_{s-})dM_s^f$. By [66, Theorem 3.5] we get that

$$N_t^f = \sum_{s\leqslant t}\mathbf{1}_A(X_{s-})f(X_s) - \int_0^t \mathbf{1}_A(X_{s-})\int_{\mathbb{R}^d} f(X_{s-}+y)\kappa(X_{s-},y)J(y)dyds.$$

Let $B\subseteq\mathbb{R}^d$ be compact and satisfy $A\cap B=\emptyset$. Taking $f_n\in C_c^\infty(\mathbb{R}^d)$ such that $0\leqslant f_n\leqslant 1$ and $f_n\downarrow \mathbf{1}_B$ we get that

$$\sum_{s\leqslant t}\mathbf{1}_A(X_{s-})\mathbf{1}_B(X_s) - \int_0^t \mathbf{1}_A(X_{s-})\int_{\mathbb{R}^d}\mathbf{1}_B(X_{s-}+y)\kappa(X_{s-},y)J(y)dyds$$

is a martingale. Therefore, by the optional stopping theorem for every bounded stopping time τ and compact sets $A,B\subseteq\mathbb{R}^d$ such that $A\cap B=\emptyset$ we obtain

$$\mathbb{E}^x\sum_{0<s\leq\tau}\mathbf{1}_A(X_{s-})\mathbf{1}_B(X_s) = \mathbb{E}^x\int_0^\tau \mathbf{1}_A(X_s)\int_{\mathbb{R}^d}\mathbf{1}_B(X_s+y)\kappa(X_s,y)J(y)dyds. \tag{78}$$

We can and do assume that $R\leqslant 2$. Fix $M=h^{-1}(1/T)$ and let $r_t=(R/2)h^{-1}(2/t)$. By Remark 5.2 we stretch the range of scaling in (4) (and (5) in the case (P2)) to $(0,M]$. For $D\in\mathcal{B}(\mathbb{R}^d)$ we define $\tau_D := \inf\{t\geqslant 0 : X_t\notin D\}$ the first exit time of X from D. We claim that there exists $\lambda=\lambda(d,\sigma,T,R)\in(0,1/2]$ such that for every $t\in(0,T]$,

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}^x\left(\tau_{B(x,r_t/4)} \leqslant \lambda t\right) \leqslant \frac{1}{2}\,. \tag{79}$$

By [10, Theorem 5.1] there is an absolute constant c_1 such that for all $r, s > 0$ and $x \in \mathbb{R}^d$

$$\mathbb{P}^x\left(\tau_{B(x,r)} \leqslant s\right) \leqslant c_1 s \sup_{|x-z|\leqslant r} \sup_{|\xi|\leqslant 1/r} |q(z,\xi)|\,,$$

where $q(z,\xi)$ is the symbol of the operator $\mathcal{L}^\kappa$ (see [10, Corollary 2.23] for definition). In the case (P1), (P2), (P3) we use, respectively, that for every $\varphi \in \mathbb{R}$ we have $\left|e^{i\varphi} - 1 - i\varphi\mathbf{1}_{|w|\leqslant 1}\right| \leqslant 2(|\varphi|^2 \wedge |\varphi|)\mathbf{1}_{|w|\leqslant 1} + 2\cdot\mathbf{1}_{|w|>1}$, $\left|e^{i\varphi} - 1\right| \leqslant 2(|\varphi| \wedge 1)$ and $|1-\cos(\varphi)| \leqslant 2(|\varphi|^2 \wedge 1)$. Therefore, by (1), (2) and Lemma 5.5 we obtain $\sup_{|\xi|\leqslant 1/r} |q(z,\xi)| \leqslant c_2 h(r)$ for all $z \in \mathbb{R}^d$, $0 < r \leqslant M$ and some $c_2 = c_2(d,\sigma,T,R)$. Hence

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}^x\left(\tau_{B(x,r)} \leqslant s\right) \leqslant c_2 s h(r), \qquad 0 < r \leqslant M,\ s > 0\,.$$

By $r_t/4 \leqslant M$ and $c_2(\lambda t)h(r_t/4) \leqslant 2c_2\lambda(8/R)^2 = c_3^{-1}\lambda$ the inequality (79) holds with $\lambda = (1 \wedge c_3)/2$.

We consider $|y-x| \geqslant Rh^{-1}(1/t)$, which implies that $|x-y| \geqslant 2r_t$. By the strong Markov property and (79) we have for $\zeta := \inf\{s \geqslant 0\colon X_s \in B(y, 3r_t/4)\}$,

$$\begin{aligned}
\mathbb{P}^x\Big(X_{\lambda t} \in B(y,r_t)\Big) &\geqslant \mathbb{P}^x\Big(\zeta \leqslant \lambda t,\ \sup_{s\in[\zeta,\zeta+\lambda t]} |X_s - X_\zeta| < r_t/4\Big) \\
&= \mathbb{E}^x\left(\zeta \leqslant \lambda t;\ \mathbb{P}^{X_\zeta}\Big(\sup_{s\in[0,\lambda t]} |X_s - X_0| < r_t/4\Big)\right) \\
&\geqslant \mathbb{P}^x\big(\zeta \leqslant \lambda t\big) \inf_{z\in\mathbb{R}^d} \mathbb{P}^z\big(\tau_{B(z,r_t/4)} > \lambda t\big) \\
&\geqslant \frac{1}{2}\mathbb{P}^x\big(\zeta \leqslant \lambda t\big) \geqslant \frac{1}{2}\mathbb{P}^x\left(X_{\lambda t\wedge\tau_{B(x,r_t)}} \in \overline{B(y,r_t/2)}\right)\,.
\end{aligned} \tag{80}$$

Noticing that $X_{s-} \in A := \overline{B(x,r_t)}$ for $s \leqslant \tau_{B(x,r_t)}$ and $X_s \notin B := \overline{B(y,r_t/2)}$ for $s < \tau_{B(x,r_t)}$ we have

$$\mathbf{1}_{\overline{B(y,r_t/2)}}\left(X_{\lambda t\wedge\tau_{B(x,r_t)}}\right) = \sum_{s\leq\lambda t\wedge\tau_{B(x,r_t)}} \mathbf{1}_A(X_{s-})\mathbf{1}_B(X_s)\,.$$

Thus, by the Lévy system formula (78) we obtain

$$\begin{aligned}
\mathbb{P}^x\left(X_{\lambda t\wedge\tau_{B(x,r_t)}} \in \overline{B(y,r_t/2)}\right) &= \mathbb{E}^x\left[\int_0^{\lambda t\wedge\tau_{B(x,r_t)}} \mathbf{1}_A(X_{s-}) \int_{\mathbb{R}^d} \mathbf{1}_B(X_s+z)\kappa(X_s,z)J(z)\,dzds\right] \\
&\geqslant \kappa_0\gamma_0^{-1}\mathbb{E}^x\left[\int_0^{\lambda t\wedge\tau_{B(x,r_t/4)}} \int_{B(y,r_t/2)} \nu(|X_s-z|)\,dzds\right].
\end{aligned} \tag{81}$$

Let z_0 be the point on the line segment $[x,y]$ such that $|z_0-y|=3r_t/8$. Then $B(z_0,r_t/8)\subseteq B(y,r_t/2)$ and $|X_s-z|<|x-y|$ if $X_s\in B(x,r_t/4)$, $z\in B(z_0,r_t/8)$. Hence the monotonicity of ν and (79) imply that

$$\begin{aligned}
&\mathbb{E}^x\left[\int_0^{\lambda t\wedge\tau_{B(x,r_t/4)}}\int_{B(y,r_t/2)}\nu(|X_s-z|)\,dz\,ds\right]\\
&\geqslant \mathbb{E}^x\left[\lambda t\wedge\tau_{B(x,r_t/4)}\right]\int_{B(z_0,r_t/8)}\nu(|x-y|)\,dz\\
&\geqslant \lambda t\,\mathbb{P}^x\left(\tau_{B(x,r_t/4)}\geqslant\lambda t\right)|B(z_0,r_t/8)|\,\nu(|x-y|)\\
&\geqslant c(d)\lambda t\left[h^{-1}(2/t)\right]^d\nu(|x-y|). \qquad (82)
\end{aligned}$$

Combining (80), (81) and (82) for $c_4=c_4(d,\sigma,T,R)$ and all $t\in(0,T]$, $|x-y|\geqslant Rh^{-1}(1/t)$,

$$\mathbb{P}^x(X_{\lambda t}\in B(y,r_t))\geqslant c_4\,t\left[h^{-1}(2/t)\right]^d\nu(|x-y|)\,. \qquad (83)$$

Finally, since $\lambda\in(0,1/2]$ we have $r_t\leqslant Rh^{-1}(1/[(1-\lambda)t])$. Therefore, by (76), (83), Lemma 5.3 and the monotonicity of h^{-1} we get for all $t\in(0,T]$, $|x-y|\geqslant Rh^{-1}(1/t)$,

$$\begin{aligned}
p^\kappa(t,x,y)&\geqslant\int_{B(y,r_t)}p^\kappa(\lambda t,x,z)p^\kappa((1-\lambda)t,z,y)\,dz\\
&\geqslant\mathbb{P}^x\left(X_{\lambda t}\in B(y,r_t)\right)\inf_{|z-y|<r_t}p^\kappa((1-\lambda)t,z,y)\\
&\geqslant cc_4\,\lambda t\left[h^{-1}(2/t)\right]^d\nu(|x-y|)\left[h^{-1}(1/t)\right]^{-d}\geqslant C\,t\nu(|x-y|)\,.\qquad\square
\end{aligned}$$

Proof of Theorem 1.4. (i) Let $t\in(0,1]$ and $|x-y|\leqslant h^{-1}(1/t)$. By Lemma 2.2 there is $c_1=c_1(d,\nu,\sigma)$ such that

$$p^{\mathfrak{K}_y}(t,x,y)\geqslant c_1\left[h^{-1}(1/t)\right]^{-d}.$$

By Lemma 3.8 with $c_2=c_2(d,\sigma,\kappa_2,\beta)$,

$$\begin{aligned}
\left|\phi_y(t,x)\right|&\leqslant c_2t\left(\rho_0^\beta+\rho_\beta^0\right)(t,x-y)\leqslant c_2t\left[h^{-1}(1/t)\right]^\beta\rho_0^0(t,x-y)\\
&\leqslant c_2\left[h^{-1}(1/t)\right]^\beta\left[h^{-1}(1/t)\right]^{-d}.
\end{aligned}$$

Thus $|\phi_y(t,x)|\leqslant(c_1/2)[h^{-1}(1/t)]^{-d}$ for all $t\in(0,T_0]$, where $T_0=1\wedge[h((2c_2/c_1)^{-1/\beta})]^{-1}$. By (59) we conclude that for all $t\in(0,T_0]$ and $|x-y|\leqslant h^{-1}(1/t)$ we have

$$p^{\kappa}(t,x,y) \geqslant (c_1/2)\left[h^{-1}(1/t)\right]^{-d}. \tag{84}$$

This ends the proof of this part due to Lemma 4.11.

(ii) It suffices to show that if (76) holds for $T>0$ and $R=1$, then it holds for $3T/2$ and $R=1$ (which allows to obtain (76) from (84) and then apply Lemma 4.11). Let $t \in [T, 3T/2]$ and $|x-y| \leqslant h^{-1}(1/t)$, then for $r = h^{-1}(1/(t-T/2))$,

$$\begin{aligned} p^{\kappa}(t,x,y) &\geqslant \int_{B(y,r)} p^{\kappa}(T/2,x,z)\,p^{\kappa}(t-T/2,z,y)\,dz \\ &\geqslant \inf_{|x-z|\leqslant 2h^{-1}(1/T)} p^{\kappa}(T,x,z)|B(y,r)|c\left[h^{-1}(1/(t-T/2))\right]^{-d} \\ &\geqslant c(\omega_d/d) \inf_{|x-z|\leqslant 2h^{-1}(1/T)} p^{\kappa}(T,x,z) \geqslant c' = c'(d,\nu,\sigma,T,c) > 0\,. \end{aligned}$$

We have used Lemma 4.11 and the positivity of ν in the last inequality. Finally, we use that $[h^{-1}(1/T)]^{-d} \geqslant [h^{-1}(1/t)]^{-d}$.

(iii) The statement follows from part (ii) and Lemma 5.4. □

5. Appendix – unimodal Lévy processes

Let $d \in \mathbb{N}$ and $\nu : [0,\infty) \to [0,\infty]$ be a non-increasing function satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(|x|)dx < \infty\,.$$

For any such ν there exists a unique pure-jump isotropic unimodal Lévy process X (see [3], [68]). The characteristic exponent Φ of X takes the form

$$\Phi(x) = \mathrm{Re}[\Phi(x)] = \int_{\mathbb{R}^d} \big(1-\cos\langle x,z\rangle\big)\nu(|z|)dz\,.$$

For $r>0$ we define $h(r)$ and $K(r)$ as in the introduction, and we let $\Phi^*(r) := \sup_{|z|\leqslant r} \mathrm{Re}[\Phi(z)]$. Then (see [3, Proposition 2]),

$$(1/\pi^2)\Phi^*(|x|) \leqslant \Phi(x) \leqslant \Phi^*(|x|)\,. \tag{85}$$

It is also known that (see [28, Lemma 4]),

$$\frac{1}{8(1+2d)} h(1/r) \leqslant \Phi^*(r) \leqslant 2h(1/r)\,. \tag{86}$$

Note that $h(0^+) < \infty$ (h is bounded) if and only if $\nu(\mathbb{R}^d) < \infty$, i.e., the corresponding Lévy process is a compound Poisson process. In the whole section **we assume that** $h(0^+) = \infty$. We collect and prove general estimates for functions K, h and Υ_t (see [30, Section 2 and 6]).

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5.1. Properties of K and h

The following properties are often used without further comment.

Lemma 5.1. *We have*

(1) K *and* h *are continuous and* $\lim_{r\to\infty} h(r) = \lim_{r\to\infty} K(r) = 0$.
(2) $r^2K(r)$ *and* $r^2h(r)$ *are non-decreasing.*
(3) $r^{-d}K(r)$ *and* $h(r)$ *are strictly decreasing.*
(4) $\lambda^2 K(\lambda r) \leqslant K(r) \leqslant \lambda^{-d} K(\lambda r)$ *and* $\lambda^2 h(\lambda r) \leqslant h(r)$, $\lambda \leqslant 1$, $r > 0$.
(5) $\sqrt{\lambda}h^{-1}(\lambda u) \leqslant h^{-1}(u)$, $\lambda \geqslant 1$, $u > 0$.
(6) $\nu(r) \leqslant (\omega_d/(d+2))^{-1} r^{-d} K(r)$,
(7) *For all* $0 < a < b \leqslant \infty$,

$$h(b) - h(a) = -\int_a^b 2K(r)r^{-1}\, dr\,.$$

(8) *For all* $r > 0$,

$$\int_{|z|\geqslant r} \nu(dz) \leqslant h(r) \quad \textit{and} \quad \int_{|z|<r} |z|^2\nu(dz) \leqslant r^2 h(r)\,.$$

We consider the scaling conditions: there are $\alpha_h \in (0,2]$, $C_h \in [1,\infty)$ and $\theta_h \in (0,\infty]$ such that

$$h(r) \leqslant C_h \lambda^{\alpha_h} h(\lambda r), \qquad \lambda \leqslant 1,\ r < \theta_h. \tag{87}$$

In like manner, there are $\beta_h \in (0,2]$, $c_h \in (0,1]$ and $\theta_h \in (0,\infty]$ such that

$$c_h\, \lambda^{\beta_h}\, h(\lambda r) \leqslant h(r)\,, \quad \lambda \leqslant 1,\ r < \theta_h. \tag{88}$$

Remark 5.2. If $\theta_h < \infty$ in (87), we can stretch the range of scaling to $r < R < \infty$ at the expense of the constant C_h. Indeed, by continuity of h, for $\theta_h \leqslant r < R$,

$$h(r) \leqslant h(\theta_h) \leqslant C_h\lambda^{\alpha_h} h(\lambda\theta_h) \leqslant C_h(r/\theta_h)^2\lambda^{\alpha_h} h(\lambda r) \leqslant [C_h(R/\theta_h)^2]\lambda^{\alpha_h} h(\lambda r).$$

Similarly, if $\theta_h < \infty$ in (88) we extend the scaling to $r < R$ as follows, for $\theta_h \leqslant r < R$,

$$h(r) \geqslant (\theta_h/R)^2 h(\theta_h) \geqslant c_h(\theta_h/R)\lambda^{\beta_h} h(\lambda\theta_h) \geqslant [c_h(\theta_h/R)^2]\lambda^{\beta_h} h(\lambda r)\,.$$

Lemma 5.3. *Let* $\alpha_h \in (0,2]$, $C_h \in [1,\infty)$ *and* $\theta_h \in (0,\infty]$. *The following are equivalent.*

(A1) *For all* $\lambda \leqslant 1$ *and* $r < \theta_h$,

$$h(r) \leqslant C_h\lambda^{\alpha_h} h(\lambda r)\,.$$

(A2) *For all* $\lambda \geqslant 1$ *and* $u > h(\theta_h)$*,*

$$h^{-1}(u) \leqslant (C_h\lambda)^{1/\alpha_h}\, h^{-1}(\lambda u)\,.$$

Further, consider

(A3) *There is* $\underline{c} \in (0,1]$ *such that for all* $\lambda \geqslant 1$ *and* $r > 1/\theta_h$*,*

$$\Phi^*(\lambda r) \geqslant \underline{c}\lambda^{\alpha_h}\,\Phi^*(r)\,.$$

(A4) *There is* $c > 0$ *such that for all* $r < \theta_h$*,*

$$h(r) \leqslant cK(r)\,.$$

Then, (A1) *gives* (A3) *with* $\underline{c} = 1/(c_d C_h)$*,* $c_d = 16(1+2d)$*, while* (A3) *gives* (A1) *with* $C_h = c_d/\underline{c}$*.* (A1) *implies* (A4) *with* $c = c(\alpha_h, C_h)$*.* (A4) *implies* (A1) *with* $\alpha_h = 2/c$ *and* $C_h = 1$*.*

Lemma 5.4. *The following are equivalent.*

(A′1) *There are* $T_1 \in (0,\infty]$*,* $c_1 > 0$ *such that for all* $r < T_1$*,*

$$c_1 r^{-d} K(r) \leqslant \nu(r)\,.$$

(A′2) *There are* $T_2 \in (0,\infty]$*,* $c_2 \in (0,1]$ *and* $\beta_2 \in (0,2)$ *such that for all* $\lambda \leqslant 1$ *and* $r < T_2$*,*

$$c_2\lambda^{\beta_2} K(\lambda r) \leqslant K(r)\,.$$

(A′3) *There are* $T_3 \in (0,\infty]$*,* $c_3 \in (0,1]$ *and* $\beta_3 \in [0,2)$ *such that for all* $\lambda \leqslant 1$ *and* $r < T_3$*,*

$$c_3\lambda^{d+\beta_3}\nu(\lambda r) \leqslant \nu(r)\,.$$

Moreover, (A′1) *implies* (A′2) *with* $T_2 = T_1$*,* $c_2 = 1$ *and* $\beta_2 = \beta_2(d, c_1)$*. From* (A′1) *we get* (A′3) *with* $T_3 = T_1$*,* $c_3 = c_3(d, c_1)$ *and* $\beta_3 = \beta_3(d, c_1)$*. The condition* (A′2) *gives* (A′1) *with* $T_1 = (c_2/2)^{1/(2-\beta_2)} T_2$ *and* $c_1 = c_1(d, c_2, \beta_2)$*. From* (A′3) *we have* (A′1) *with* $T_1 = T_3$ *and* $c_1 = c_1(d, c_3, \beta_3)$*.*

Lemma 5.5. *Let* h *satisfy* (87) *with* $\alpha_h > 1$*, then*

$$\int\limits_{r\leqslant|z|<\theta_h} |z|\nu(|z|)dz \leqslant \frac{(d+2)C_h}{\alpha_h - 1}\, rh(r)\,, \qquad r > 0\,.$$

Let h *satisfy* (88) *with* $\beta_h < 1$*, then*

$$\int\limits_{|z|<r} |z|\nu(|z|)dz \leqslant \frac{d+2}{c_h(1-\beta_h)}\, rh(r)\,, \qquad r < \theta_h\,.$$

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5.2. Properties of the bound function $\Upsilon_t(x)$

We collect properties of the bound function defined in (15).

Lemma 5.6. *We have*

$$(\omega_d/2) \leqslant \int_{\mathbb{R}^d} \Upsilon_t(x)\, dx \leqslant (\omega_d/2)(1+2/d), \qquad t>0\,.$$

Lemma 5.7. *Fix $t>0$. There is a unique solution $r_0>0$ of*

$$tK(r)r^{-d} = [h^{-1}(1/t)]^{-d} = \Upsilon_t(r)\,,$$

and $r_0 \in [h^{-1}(3/t), h^{-1}(1/t)]$.

Proposition 5.8. *Let $a \geqslant 1$. There is $c = c(d,a)$ such that for all $t>0$,*

$$\Upsilon_t(x+z) \leqslant c\,\Upsilon_t(x)\,, \qquad \text{if} \qquad |z| \leqslant \left[a\, h^{-1}(3/t)\right] \vee \frac{|x|}{2}\,.$$

Lemma 5.9. *There exists a constant $c = c(d, \alpha_h, C_h)$ such that for all $t>0$ and $x \in \mathbb{R}^d$,*

$$\int_{|z|\geqslant h^{-1}(1/t)} \Upsilon_t(x-z)\nu(|z|)dz \leqslant ct^{-1}\Upsilon_t(x)\,.$$

Proof. We split the integral into parts. If $|z| \leqslant |x|/2$, then by Proposition 5.8 we get $\Upsilon_t(x-z)\nu(|z|) \leqslant c\Upsilon_t(x)\nu(|z|)$ and we apply Lemma 5.1. If $|x|/2 \leqslant |z|$, then we first simply use $\Upsilon_t(x-z) \leqslant \left[h^{-1}(1/t)\right]^{-d}$ and Lemma 5.1 to find a bound by $t^{-1}\left[h^{-1}(1/t)\right]^{-d}$. At the same time we have $\Upsilon_t(x-z)\nu(|z|) \leqslant \Upsilon_t(x-z)\nu(|x|/2) \leqslant c\Upsilon_t(x-z)K(|x|)|x|^{-d}$, which together with Lemma 5.6 give a bound by $t^{-1}(tK(|x|)|x|^{-d})$. Finally, we take the minimum. $\square$

We collect further properties of the bound function under (87).

Corollary 5.10. *Let h satisfy (87). For every $a \geqslant 1$ there is $c = c(d, a, \alpha_h, C_h)$ such that*

$$\Upsilon_t(x+z) \leqslant c\,\Upsilon_t(x)\,, \qquad \text{if} \qquad |z| \leqslant \left[a\, h^{-1}(1/t)\right] \vee \frac{|x|}{2} \quad \text{and} \quad t < 1/h(\theta_h)\,.$$

Corollary 5.11. *Let h satisfy (87). For $t>0$, $x \in \mathbb{R}^d$ define*

$$\varphi_t(x) = \begin{cases} [h^{-1}(1/t)]^{-d}, & |x| \leqslant h^{-1}(1/t)\,, \\ tK(|x|)|x|^{-d}, & |x| > h^{-1}(1/t)\,. \end{cases}$$

Then $\Upsilon_t(x) \leqslant \varphi_t(x) \leqslant c\,\Upsilon_t(x)$ for all $t < 1/h(\theta_h)$, $x \in \mathbb{R}^d$ and a constant $c = c(\alpha_h, C_h)$.

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Lemma 5.12. *Let h satisfy (87). For all $\beta \in [0, \alpha_h)$ and $t < 1/h(\theta_h)$ we have*

$$\int_{\mathbb{R}^d} (|x|^\beta \wedge 1)\Upsilon_t(x)\,dx \leqslant 2\omega_d \frac{C_h\big(1+1/\theta_h^\beta\big)}{\alpha_h-\beta}\left[h^{-1}(1/t)\right]^\beta .$$

Proof. If $\beta = 0$, the result follows from Lemma 5.6. Assume that $\beta > 0$. We have

$$\int_{|x|<h^{-1}(1/t)} |x|^\beta \Upsilon_t(x)\,dx \leqslant \int_{|x|<h^{-1}(1/t)} \left[h^{-1}(1/t)\right]^\beta \left[h^{-1}(1/t)\right]^{-d} dx = \frac{\omega_d}{d}\left[h^{-1}(1/t)\right]^\beta .$$

The integral over the set $|x| \geqslant h^{-1}(1/t)$ is bounded by the sum

$$\int_{|x|\geqslant\theta_h} \Upsilon_t(x)\,dx + \int_{h^{-1}(1/t)\leqslant|x|<\theta_h} |x|^\beta \Upsilon_t(x)\,dx .$$

Further, we have

$$\int_{|x|\geqslant\theta_h} \Upsilon_t(x)\,dx \leqslant t\omega_d \int_{\theta_h}^{\infty} K(r)r^{-1}\,dr = \frac{t\omega_d}{2}h(\theta_h) \leqslant \frac{\omega_d C_h}{2\theta_h^\beta}\left[h^{-1}(1/t)\right]^\beta ,$$

where the last inequality follows from $r^{1/\beta}h^{-1}(r) \geqslant C_h^{-1/\beta}u^{1/\beta}h^{-1}(u)$ for $r \geqslant u \geqslant h(\theta_h)$, which is a consequence of (A2) of Lemma 5.3, the assumption $0 < \beta < \alpha_h$ and continuity of h^{-1}. Now (87) with $\lambda = h^{-1}(1/t)/r$ gives

$$\begin{aligned}
\int_{h^{-1}(1/t)\leqslant|x|<\theta_h} |x|^\beta \Upsilon_t(x)\,dx &\leqslant t \int_{h^{-1}(1/t)\leqslant|x|<\theta_h} |x|^{\beta-d} K(|x|)\,dx \leqslant t\omega_d \int_{h^{-1}(1/t)}^{\theta_h} r^{\beta-1}h(r)\,dr \\
&\leqslant t\omega_d C_h \int_{h^{-1}(1/t)}^{\theta_h} r^{\beta-1}\left[\frac{h^{-1}(1/t)}{r}\right]^{\alpha_h} (1/t)\,dr \leqslant \omega_d C_h \left[h^{-1}(1/t)\right]^{\alpha_h} \int_{h^{-1}(1/t)}^{\infty} r^{\beta-1-\alpha_h}\,dr \\
&\leqslant \frac{\omega_d C_h}{\alpha_h-\beta}\left[h^{-1}(1/t)\right]^\beta . \qquad \square
\end{aligned}$$

5.3. 3G-type inequalities

Let $\phi\colon [0,\infty) \to (0,\infty)$ be non-increasing and such that

$$\lambda^{\alpha_\phi}\phi(\lambda t) \leqslant c_\phi \phi(t), \qquad \lambda \leqslant 1,\ \ t < \theta_\phi,$$

for some $\alpha_\phi \leqslant 1$, $c_\phi \in [1,\infty)$ and $\theta_\phi \in (0,\infty]$. For $t > 0$ and $x \in \mathbb{R}^d$ we consider

$$\widehat{\Upsilon}_t(x) = \phi(t)\Upsilon_t(x) . \tag{89}$$

Proposition 5.13. *Let h satisfy (87). There exists a constant $c = c(d, \alpha_h, C_h, \alpha_\phi, c_\phi)$ such that for all $s + t < 1/h(\theta_h) \wedge \theta_\phi$, $x, y \in \mathbb{R}^d$,*

$$\widehat{\Upsilon}_s(x) \wedge \widehat{\Upsilon}_t(y) \leqslant c\widehat{\Upsilon}_{s+t}(x+y)\,. \tag{90}$$

Proof. First, note that $t \mapsto \phi(t)\big[h^{-1}(1/t)\big]^{-d}$ and $r \mapsto r^{-d}K(r)$ are non-increasing. Thus

$$\phi(s)\big[h^{-1}(1/s)\big]^{-d} \wedge \phi(t)\big[h^{-1}(1/t)\big]^{-d} \leqslant \phi((s+t)/2)\big[h^{-1}(2/(s+t))\big]^{-d},$$

and

$$\frac{K(|x|)}{|x|^d} \wedge \frac{K(|y|)}{|y|^d} \leqslant \frac{K((|x|+|y|)/2)}{\big[(|x|+|y|)/2\big]^d} \leqslant 2^{d+2}\frac{K(|x+y|)}{|x+y|^d}\,.$$

Since $\alpha_\phi \leqslant 1$ for $\lambda \leqslant 1$ we have $\phi(\lambda t)(\lambda t) \leqslant c_\phi \phi(t)t$ on $(0, \theta_\phi)$. For $s + t \in (0, \theta_\phi)$ we get

$$\big[\phi(s)\,s\big] \vee \big[\phi(t)\,t\big] \leqslant c_\phi\, \phi(s+t)\,(s+t)\,.$$

Finally,

$$\begin{aligned}
\widehat{\Upsilon}_s(x) \wedge \widehat{\Upsilon}_t(y) &\leqslant \phi(s)\big[h^{-1}(1/s)\big]^{-d} \wedge \phi(t)\big[h^{-1}(1/t)\big]^{-d} \\
&\quad \wedge c_\phi\Big(\phi(s+t)(s+t)\Big)\left[\frac{K(|x|)}{|x|^d} \wedge \frac{K(|y|)}{|y|^d}\right] \\
&\leqslant \phi((s+t)/2)\big[h^{-1}(2/(s+t))\big]^{-d} \wedge 2^{d+2}c_\phi\Big(\phi(s+t)(s+t)\Big)\frac{K(|x+y|)}{|x+y|^d}\,.
\end{aligned}$$

The inequality follows by scaling conditions for ϕ and h^{-1} (see Lemma 5.3). $\square$

Since we can take $\phi \equiv 1$ with $(\alpha_\phi, \theta_\phi, c_\phi) = (0, 0, 1)$ we recover the classical 3G inequality.

Corollary 5.14. *Let h satisfy (87). There exists a constant $c = c(d, \alpha_h, C_h)$ such that for all $s + t < 1/h(\theta_h)$, $x, y \in \mathbb{R}^d$,*

$$\Upsilon_s(x) \wedge \Upsilon_t(y) \leqslant c\Upsilon_{s+t}(x+y)\,. \tag{91}$$

5.4. Convolution inequalities

Let $B(a, b)$ be the beta function, i.e., $B(a, b) = \int_0^1 s^{a-1}(1-s)^{b-1}ds$, $a, b > 0$.

Lemma 5.15. *Let $\theta, \eta \in \mathbb{R}$. The inequality*

$$\int_0^t u^{-\eta}[h^{-1}(1/u)]^\gamma (t-u)^{-\theta}[h^{-1}(1/(t-u))]^\beta\, du \leqslant c\, t^{1-\eta-\theta}[h^{-1}(1/t)]^{\gamma+\beta}\,, \quad t > 0\,,$$

holds in the following cases:

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(i) *for all* $\beta, \gamma \geqslant 0$ *such that* $\beta/2 + 1 - \theta > 0$, $\gamma/2 + 1 - \eta > 0$ *with* $c = B(\beta/2 + 1 - \theta, \gamma/2 + 1 - \eta)$,

(ii) *under* (4), *for all* $t \in (0, T]$, $T > 0$, *and all* $\beta, \gamma \in \mathbb{R}$ *such that* $(\beta/2) \wedge (\beta/\alpha_h) + 1 - \theta > 0$, $(\gamma/2) \wedge (\gamma/\alpha_h) + 1 - \eta > 0$ *with*

$$c = (C_h[h^{-1}(1/T) \vee 1]^2)^{-(\beta\wedge 0+\gamma\wedge 0)/\alpha_h} B \\ \times \Big((\beta/2) \wedge (\beta/\alpha_h) + 1 - \theta, (\gamma/2) \wedge (\gamma/\alpha_h) + 1 - \eta\Big).$$

Proof. Let I be the above integral. By the change of variable $s = u/t$ we get that

$$I = t^{1-\eta-\theta} \int_0^1 s^{-\eta} \left[h^{-1}(t^{-1}s^{-1})\right]^{\gamma} (1-s)^{-\theta} \left[h^{-1}(t^{-1}(1-s)^{-1})\right]^{\beta} ds\,.$$

Since $s^{-1} \geqslant 1$ and $(1-s)^{-1} \geqslant 1$ we have $h^{-1}(t^{-1}s^{-1}) \leqslant s^{1/2}h^{-1}(t^{-1})$ and $h^{-1}(t^{-1}(1-s)^{-1}) \leqslant (1-s)^{1/2}h^{-1}(t^{-1})$. Hence,

$$I \leqslant t^{1-\eta-\theta} \left[h^{-1}(1/t)\right]^{\gamma+\beta} \int_0^1 s^{\gamma/2-\eta}(1-s)^{\beta/2-\theta}\, ds \\ = B(\beta/2 + 1 - \theta, \gamma/2 + 1 - \eta) t^{1-\eta-\theta} \left[h^{-1}(1/t)\right]^{\gamma+\beta}\,.$$

This proves (i). The cases (ii) follows from Lemma 5.3 and Remark 5.2 that guarantee

$$\left[h^{-1}(t^{-1}s^{-1})\right]^{\gamma} \leqslant (C_h[h^{-1}(1/T) \vee 1]^2)^{-\gamma/\alpha_h} s^{\gamma/\alpha_h} \left[h^{-1}(1/t)\right]^{\gamma}\,, \\ \left[h^{-1}(t^{-1}(1-s)^{-1})\right]^{\beta} \leqslant (C_h[h^{-1}(1/T) \vee 1]^2)^{-\beta/\alpha_h} (1-s)^{\beta/\alpha_h} \left[h^{-1}(1/t)\right]^{\beta}\,,$$

if $\gamma < 0$ or $\beta < 0$, respectively. □

For $\gamma, \beta \in \mathbb{R}$ we consider the function ρ_γ^β defined in (26).

Remark 5.16. The monotonicity of h^{-1} assures the following,

$$\rho_{\gamma_1}^{\beta}(t, x) \leqslant \left[h^{-1}(1/T)\right]^{\gamma_1-\gamma_2} \rho_{\gamma_2}^{\beta}(t, x)\,, \qquad (t, x) \in (0, T] \times \mathbb{R}^d, \quad \gamma_2 \leqslant \gamma_1, \tag{92}$$

$$\rho_{\gamma}^{\beta_1}(t, x) \leqslant \rho_{\gamma}^{\beta_2}(t, x), \qquad (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad 0 \leqslant \beta_2 \leqslant \beta_1, \tag{93}$$

$$\rho_0^0(\lambda t, x) \leqslant \rho_0^0(t, x)\,, \qquad (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad \lambda \geqslant 1. \tag{94}$$

Lemma 5.17. *Assume* (4) *and let* $\beta_0 \in [0, 1] \cap [0, \alpha_h)$.

(a) *For every $T > 0$ there exists a constant $c_1 = c_1(d, \beta_0, \alpha_h, C_h, h^{-1}(1/T) \vee 1)$ such that for all $t \in (0, T]$ and $\beta \in [0, \beta_0]$,*

$$\int_{\mathbb{R}^d} \rho_0^{\beta}(t,x)\,dx \leqslant c_1 t^{-1} \left[h^{-1}(1/t)\right]^{\beta} .$$

(b) *For every $T > 0$ there exists a constant $c_2 = c_2(d, \beta_0, \alpha_h, C_h, h^{-1}(1/T) \vee 1) \geq 1$ such that for all $\beta_1, \beta_2, n_1, n_2, m_1, m_2 \in [0, \beta_0]$ with $n_1, n_2 \leqslant \beta_1 + \beta_2$, $m_1 \leqslant \beta_1$, $m_2 \leqslant \beta_2$ and all $0 < s < t \leqslant T$, $x \in \mathbb{R}^d$,*

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho_0^{\beta_1}(t-s, x-z)\rho_0^{\beta_2}(s,z)\,dz \\
&\qquad \leqslant c_2\Big[\Big((t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{n_1} + s^{-1}\left[h^{-1}(1/s)\right]^{n_2}\Big)\rho_0^0(t,x) \\
&\qquad\quad + (t-s)^{-1}\left[h^{-1}(1/(t-s))\right]^{m_1}\rho_0^{\beta_2}(t,x) + s^{-1}\left[h^{-1}(1/s)\right]^{m_2}\rho_0^{\beta_1}(t,x)\Big].
\end{aligned}$$

(c) *Let $T > 0$. For all $\gamma_1, \gamma_2 \in \mathbb{R}$, $\beta_1, \beta_2, n_1, n_2, m_1, m_2 \in [0, \beta_0]$ with $n_1, n_2 \leqslant \beta_1 + \beta_2$, $m_1 \leqslant \beta_1$, $m_2 \leqslant \beta_2$ and $\theta, \eta \in [0, 1]$, satisfying*

$$\begin{aligned}
&(\gamma_1 + n_1 \wedge m_1)/2 \wedge (\gamma_1 + n_1 \wedge m_1)/\alpha_h + 1 - \theta > 0\,, \\
&(\gamma_2 + n_2 \wedge m_2)/2 \wedge (\gamma_2 + n_2 \wedge m_2)/\alpha_h + 1 - \eta > 0\,,
\end{aligned}$$

and all $0 < s < t \leqslant T$, $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} (t-s)^{1-\theta}\,\rho_{\gamma_1}^{\beta_1}(t-s, x-z)\, s^{1-\eta}\,\rho_{\gamma_2}^{\beta_2}(s,z)\,dz ds \\
&\qquad \leqslant c_3\, t^{2-\eta-\theta}\Big(\rho^0_{\gamma_1+\gamma_2+n_1} + \rho^0_{\gamma_1+\gamma_2+n_2} + \rho^{\beta_2}_{\gamma_1+\gamma_2+m_1} + \rho^{\beta_1}_{\gamma_1+\gamma_2+m_2}\Big)(t,x)\,, \qquad (95)
\end{aligned}$$

where $c_3 = c_2\,(C_h[h^{-1}(1/T) \vee 1]^2)^{-(\gamma_1\wedge 0+\gamma_2\wedge 0)/\alpha_h} B\,(k+1-\theta,\, l+1-\eta)$ and

$$k = \left(\frac{\gamma_1 + n_1 \wedge m_1}{2}\right) \wedge \left(\frac{\gamma_1 + n_1 \wedge m_1}{\alpha_h}\right), \qquad l = \left(\frac{\gamma_2 + n_2 \wedge m_2}{2}\right) \wedge \left(\frac{\gamma_2 + n_2 \wedge m_2}{\alpha_h}\right).$$

Proof. Part (a) follows immediately from Lemma 5.12 and Remark 5.2. We prove part (b). Proposition 5.13 with $\phi(t) = t^{-1}$ provides with $c = c(d, \alpha_h, C_h, h^{-1}(1/T) \vee 1) > 0$,

$$\frac{\rho_0^0(t-s, x-z)\rho_0^0(s,z)}{c\,\rho_0^0(t,x)} \leqslant \rho_0^0(t-s, x-z) + \rho_0^0(s,z).$$

Combining with (see formulas following [16, (2.5)] or use $(a+b)^{\beta} \leqslant a^{\beta} + b^{\beta}$, $(a+b) \wedge 1 \leqslant (a \wedge 1) + (b \wedge 1)$ and $(a \wedge 1)(b \wedge 1) \leqslant (ab) \wedge 1$ for any $\beta \in [0, 1]$, $a, b \geqslant 0$)

$$\left(|x-z|^{\beta_1}\wedge 1\right)\left(|z|^{\beta_2}\wedge 1\right)\leqslant\left(|x-z|^{\beta_1+\beta_2}\wedge 1\right)+\left(|x-z|^{\beta_1}\wedge 1\right)\left(|x|^{\beta_2}\wedge 1\right),$$
$$\left(|x-z|^{\beta_1}\wedge 1\right)\left(|z|^{\beta_2}\wedge 1\right)\leqslant\left(|z|^{\beta_1+\beta_2}\wedge 1\right)+\left(|x|^{\beta_1}\wedge 1\right)\left(|z|^{\beta_2}\wedge 1\right),$$

we have by (93),

$$\begin{aligned}
&\frac{\rho_0^{\beta_1}(t-s,x-z)\rho_0^{\beta_2}(s,z)}{c\,\rho_0^0(t,x)}\\
&\leqslant\Big(\left(|x-z|^{\beta_1+\beta_2}\wedge 1\right)+\left(|x-z|^{\beta_1}\wedge 1\right)\left(|x|^{\beta_2}\wedge 1\right)\Big)\rho_0^0(t-s,x-z)\\
&\quad+\Big(\left(|z|^{\beta_1+\beta_2}\wedge 1\right)+\left(|x|^{\beta_1}\wedge 1\right)\left(|z|^{\beta_2}\wedge 1\right)\Big)\rho_0^0(s,z)\\
&=\rho_0^{\beta_1+\beta_2}(t-s,x-z)+\left(|x|^{\beta_2}\wedge 1\right)\rho_0^{\beta_1}(t-s,x-z)+\rho_0^{\beta_1+\beta_2}(s,z)+\left(|x|^{\beta_1}\wedge 1\right)\rho_0^{\beta_2}(s,z)\\
&\leqslant\rho_0^{n_1}(t-s,x-z)+\left(|x|^{\beta_2}\wedge 1\right)\rho_0^{m_1}(t-s,x-z)+\rho_0^{n_2}(s,z)+\left(|x|^{\beta_1}\wedge 1\right)\rho_0^{m_2}(s,z)\,.
\end{aligned}$$

Integrating both sides in z and applying (a) we obtain (b). For the proof of (c) we multiply both sides of (b) by

$$(t-s)^{1-\theta}\left[h^{-1}(1/(t-s))\right]^{\gamma_1}s^{1-\eta}\left[h^{-1}(1/s)\right]^{\gamma_2},$$

integrate in s and apply Lemma 5.15 to reach (c) with a constant

$$\begin{aligned}
&c_2\left(C_h\left[h^{-1}(1/T)\vee 1\right]^2\right)^{-(\gamma_1\wedge 0+\gamma_2\wedge 0)/\alpha_h}\\
&\times\max\Bigg\{B\left(k_1+1-\theta,\frac{\gamma_2}{2}\wedge\frac{\gamma_2}{\alpha_h}+2-\eta\right);B\left(\frac{\gamma_1}{2}\wedge\frac{\gamma_1}{\alpha_h}+2-\theta,\,l_1+1-\eta\right);\\
&\qquad B\left(k_2+1-\theta,\frac{\gamma_2}{2}\wedge\frac{\gamma_2}{\alpha_h}+2-\eta\right);B\left(\frac{\gamma_1}{2}\wedge\frac{\gamma_1}{\alpha_h}+2-\theta,\,l_2+1-\eta\right)\Bigg\},
\end{aligned}$$

where $k_1=(\frac{\gamma_1+n_1}{2})\wedge(\frac{\gamma_1+n_1}{\alpha_h})$, $k_2=(\frac{\gamma_1+m_1}{2})\wedge(\frac{\gamma_1+m_1}{\alpha_h})$ and $l_1=(\frac{\gamma_2+n_2}{2})\wedge(\frac{\gamma_2+n_2}{\alpha_h})$ $l_2=(\frac{\gamma_2+m_2}{2})\wedge(\frac{\gamma_2+m_2}{\alpha_h})$, which by monotonicity of Beta function is smaller than c_3. □

Remark 5.18. When using Lemma 5.17 without specifying the parameters we apply the usual case, i.e., $n_1=n_2=\beta_1+\beta_2$ ($\leqslant\beta_0$), $m_1=\beta_1$, $m_2=\beta_2$. Similarly, if only n_1, n_2 are specified, then $m_1=\beta_1$, $m_2=\beta_2$.

6. Appendix – general Lévy process

Let $d\in\mathbb{N}$ and $Y=(Y_t)_{t\geqslant 0}$ be a Lévy process in $\mathbb{R}^d$ ([65]). Recall that there is a well known one-to-one correspondence between Lévy processes in $\mathbb{R}^d$ and the convolution semigroups of probability measures $(P_t)_{t\geqslant 0}$ on $\mathbb{R}^d$. The characteristic exponent Ψ of Y is defined by

$$\mathbb{E}e^{i\langle x,Y_t\rangle} = \int_{\mathbb{R}^d} e^{i\langle x,y\rangle} P_t(dy) = e^{-t\Psi(x)}, \qquad x \in \mathbb{R}^d,$$

and equals

$$\Psi(x) = \langle x, Ax\rangle - i\langle x, b\rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x,z\rangle} - 1 - i\langle x, z\rangle \mathbf{1}_{|z|<1}\right) N(dz).$$

Here A is a symmetric non-negative definite matrix, $b \in \mathbb{R}^d$ and $N(dz)$ is a Lévy measure, i.e., a measure satisfying

$$N(\{0\}) = 0, \qquad \int_{\mathbb{R}^d} (1 \wedge |z|^2) N(dz) < \infty.$$

We have $P_t f(x) = \mathbb{E} f(Y_t + x)$ and $(P_t)_{t\geqslant 0}$ is a strongly continuous positive contraction semigroup on $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ with the infinitesimal generator $(L, D(L))$ such that $C_0^2(\mathbb{R}^d) \subseteq D(L)$ and for $f \in C_0^2(\mathbb{R}^d)$ we have

$$\begin{aligned} Lf &= \mathcal{L}f(x) \\ &:= \sum_{i,j=1}^d A_{ij} \frac{\partial f(x)}{\partial x_i \partial x_j} + \langle b, \nabla f(x)\rangle + \int_{\mathbb{R}^d} (f(x+z) - f(x) - \mathbf{1}_{|z|<1}\langle z, \nabla f(x)\rangle) N(dz). \end{aligned}$$

Note that the above equality on $C_c^\infty(\mathbb{R}^d)$ uniquely determines $(L, D(L))$ and the generating triplet (A, N, b) (see [65, Theorem 31.5 an 8.1]). We make the following assumption on the real part of Ψ,

$$\lim_{|x|\to\infty} \frac{\mathrm{Re}[\Psi(x)]}{\log|x|} = \infty. \tag{96}$$

In particular, $N(\mathbb{R}^d) = \infty$, thus Y is not a compound Poisson process. It follows from [51, Theorem 2.1] (we only use implication which does not require $A = 0$) that Y_t has a density $p(t, x)$ for every $t > 0$ and

$$p(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle x,z\rangle} e^{-t\Psi(z)}\, dz, \qquad p(t, \cdot) \in C_0^\infty(\mathbb{R}^d). \tag{97}$$

We denote $p(t, x, y) = p(t, y - x)$ and observe that $P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y)\, dy$.

Lemma 6.1.

(a) *For all $x, y \in \mathbb{R}^d$, the function $t \mapsto p(t, x, y)$ is differentiable on $(0, \infty)$ and*

$$\frac{\partial p(t, x, y)}{\partial t} = -(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y-x,z\rangle} \Psi(z) e^{-t\Psi(z)}\, dz = \mathcal{L}_x p(t, x, y).$$

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(b) *Let $\varepsilon > 0$. There is a constant $c > 0$ such that for all $s, t \geq \varepsilon$, $x, x', y \in \mathbb{R}^d$,*

$$|p(t,x,y) - p(s,x',y)| \leq c\left(|t-s| + |x-x'|\right) .$$

(c) *Let $\varepsilon > 0$. There is a constant $c > 0$ such that for all $s, t \geq \varepsilon$, $x, x', y \in \mathbb{R}^d$,*

$$|\nabla p(t,x,y) - \nabla p(s,x',y)| \leq c\left(|t-s| + |x-x'|\right) .$$

Proof. (a) Note that for any $t > 0$ and any $h \in \mathbb{R}$ such that $t/2 + h > 0$,

$$\frac{p(t+h,x,y) - p(t,x,y)}{h} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y-x,z\rangle} e^{-t\Psi(z)} \frac{e^{-h\Psi(z)} - 1}{h}\, dz.$$

The absolute value of the integrand is bounded by $|\Psi(z)|e^{-(t/2)\mathrm{Re}[\Psi(z)]}$ which is integrable since $|\Psi(z)| \leqslant c(|z|^2 + 1)$ (see [10, Proposition 2.17]). The claim follows from the dominated convergence theorem. The second equality follows from the semigroup property and (97). Indeed, $P_h p(t,\cdot,y)(x) = \int_{\mathbb{R}^d} p(h,x,z)p(t,z,y)dz = p(t+h,x,y)$. Hence $\mathcal{L}_x p(t,x,y) = \lim_{h\to 0^+}(p(t+h,x,y) - p(t,x,y))/h$.

(b) By (a) we have

$$\sup_{t\geqslant\varepsilon,\, x,y\in\mathbb{R}^d} \left|\frac{\partial p(t,x,y)}{\partial t}\right| \leqslant \int_{\mathbb{R}^d} |\Psi(z)|e^{-\varepsilon\,\mathrm{Re}[\Psi(z)]}\, dz = c_1 < \infty.$$

And

$$\sup_{t\geqslant\varepsilon,\, x,y\in\mathbb{R}^d} \left|\frac{\partial p(t,x,y)}{\partial x_k}\right| \leqslant \max_{k=1,\dots,d} \int_{\mathbb{R}^d} |z_k|e^{-\varepsilon\,\mathrm{Re}[\Psi(z)]}\, dz = c_2 < \infty.$$

These imply the claim with $c = c_1 + dc_2$.

(c) Like above we have $\frac{\partial^2}{\partial t\partial x_k} p(t,x,y) = -(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y-x,z\rangle}\Psi(z)z_k e^{-t\Psi(z)}\, dz$. Then we use $\sup_{t\geqslant\varepsilon,\, x,y\in\mathbb{R}^d} |\frac{\partial^2}{\partial t\partial x_k} p(t,x,y)|$ and $\sup_{t\geqslant\varepsilon,\, x,y\in\mathbb{R}^d} |\frac{\partial^2}{\partial x_j\partial x_k} p(t,x,y)|$. □

We record a general fact which follows from [28, Lemma 4] and Fubini's theorem.

Lemma 6.2. *Let $\Psi^*(r) := \sup_{|z|\leqslant r} \mathrm{Re}[\Psi(z)]$ for $r > 0$. Then*

$$\int_0^1 \frac{\Psi^*(r)}{r} dr < \infty \qquad \Longleftrightarrow \qquad \int_{\mathbb{R}^d} \ln\left(1+|z|^2\right) N(dz) < \infty .$$

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