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Uniform decay estimates for solutions of a class of retarded integral inequalities ☆

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Abstract

Some uniform decay estimates are established for solutions of the following type of retarded integral inequalities:

$$y(t) \le E(t,\tau)\|y_\tau\| + \int_\tau^t K_1(t,s)\|y_s\|ds + \int_t^\infty K_2(t,s)\|y_s\|ds + \rho, \qquad t \ge \tau \ge 0.$$

As a simple example of application, the retarded scalar functional differential equation $\dot{x} = -a(t)x + B(t, x_t)$ is considered, and the global asymptotic stability of the equation is proved under weaker conditions. Another example is the ODE system $\dot{x} = F_0(t,x) + \sum_{i=1}^m F_i(t, x(t - r_i(t)))$ on $\mathbb{R}^n$ with superlinear nonlinearities F_i $(0 \le i \le m)$. The existence of a global pullback attractor of the system is established under appropriate dissipation conditions.

The third example for application concerns the study of the dynamics of the functional cocycle system $\frac{du}{dt} + Au = F(\theta_t p, u_t)$ in a Banach space X with sublinear nonlinearity. In particular, the existence and uniqueness of a nonautonomous equilibrium solution Γ is obtained under the hyperbolicity assumption on operator A and some additional hypotheses, and the global asymptotic stability of Γ is also addressed.

MSC: 34D45; 34K20; 34K38; 35B35; 35B40; 37L15; 37L30

Keywords: Retarded integral inequality; Delay differential equation; Asymptotic stability; Exponential asymptotic stability; Pullback attractor; Nonautonomous equilibrium solution

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1. Introduction

Decay estimate of solutions is a fundamental problem in the qualitative analysis of evolution equations. In most cases this problem can be reduced to differential or integral inequalities. For non-retarded evolution equations, numerous inequalities are available to make the performance of decay estimate fruitful (see e.g. [1,31,45,46]), among which is the remarkable Gronwall-Bellman inequality which was first proposed in Gronwall [15] and later extended to a more general form in Bellman [2]. In contrast, the situation in the case of retarded equations seems to be more complicated. Although there have appeared many nice retarded differential and integral inequalities in the literature (see e.g. [12,13,16,31,35–37,45,61] and references cited therein), the existing ones are far from being adequate to provide easy-to-handle and efficient tools for studying the dynamics of this type of equations, and it is still a challenging task to derive decay estimates for their solutions, even if for the scalar functional differential equation $\dot{x} = f(t, x, x_t)$. In fact, it is often the case that one has to fall his back on differential/integral inequalities without delay when dealing with retarded differential or integral equations, which makes the calculations in the argument much involved and restrictive.

In this paper we investigate the following type of retarded integral inequalities:

$$\begin{aligned} y(t) \le\ & E(t,\tau)\|y_\tau\| + \textstyle\int_\tau^t K_1(t,s)\|y_s\|ds \\ & + \textstyle\int_t^\infty K_2(t,s)\|y_s\|ds + \rho, \qquad \forall t \ge \tau \ge 0, \end{aligned} \tag{1.1}$$

where E, K_1 and K_2 are nonnegative measurable functions on $Q := (\mathbb{R}^+)^2$, $\rho \ge 0$ is a constant, $\|\cdot\|$ denotes the usual sup-norm of the space $\mathcal{C} := C([-r, 0])$ for some given $r \ge 0$, $y(t)$ is a nonnegative continuous function on $[-r, \infty)$ (called a *solution* of (1.1)), and y_t $(t \ge 0)$ denotes the *lift* of y in $\mathcal{C}$,

$$y_t(s) = y(t+s), \qquad s \in [-r, 0]. \tag{1.2}$$

Our main purpose is to establish some uniform decay estimates for its solutions. Specifically, let E be a function on Q satisfying that

$$\lim\nolimits_{t\to\infty} E(t+s, s) = 0 \text{ uniformly w.r.t. } s \in \mathbb{R}^+, \tag{1.3}$$

and suppose

$$\vartheta(E) := \sup_{t\ge s\ge 0} E(t,s) \le \vartheta < \infty, \tag{1.4}$$

$$I(K_1, K_2) := \sup\nolimits_{t\ge 0} \left(\textstyle\int_0^t K_1(t,s)ds + \int_t^\infty K_2(t,s)ds \right) \le \kappa < \infty. \tag{1.5}$$

Denote $\mathscr{L}_r(E; K_1, K_2; \rho)$ the solution set of (1.1), i.e.,

$$\mathscr{L}_r(E; K_1, K_2; \rho) = \{y \in C([-r,\infty)) : \ \ y \ge 0 \text{ and satisfies (1.1)}\}. \tag{1.6}$$

We show that the following theorem holds true.

Theorem 1.1. *Let ϑ and κ be the positive constants in (1.4) and (1.5).*

(1) *If $\kappa < 1$ then for any $R, \varepsilon > 0$, there exists $T > 0$ such that*

$$\|y_t\| < \mu\rho + \varepsilon, \qquad t > T \tag{1.7}$$

for all bounded functions $y \in \mathscr{L}_r(E; K_1, K_2; \rho)$ with $\|y_0\| \le R$, where

$$\mu = 1/(1-\kappa). \tag{1.8}$$

(2) *If $\kappa < 1/(1+\vartheta)$ then there exist $M, \lambda > 0$ (independent of ρ) such that*

$$\|y_t\| \le M\|y_0\|e^{-\lambda t} + \gamma\rho, \qquad t \ge 0 \tag{1.9}$$

for all bounded functions $y \in \mathscr{L}_r(E; K_1, K_2; \rho)$, where

$$\gamma = (\mu+1)/(1-\kappa c), \qquad c = \max\left(\vartheta/(1-\kappa),\ 1\right). \tag{1.10}$$

Remark 1.2. If $\kappa < 1/(1+\vartheta)$ then one trivially verifies that $\kappa c < 1$.

The particular case where $K_2 = 0$ is of crucial importance in applications. In such a case we show that if $I(K_1, 0) \le \kappa < 1$ then any function $y \in \mathscr{L}_r(E; K_1, 0; \rho)$ is automatically bounded. Hence the boundedness requirement on y in Theorem 1.1 can be removed. Consequently we have

Theorem 1.3. *Let $(K_1, K_2) = (K, 0)$, and let ϑ, κ, μ and γ be the same constants as in Theorem 1.1. Then the following assertions hold.*

(1) *If $\kappa < 1$ then for any $R, \varepsilon > 0$, there exists $T > 0$ such that*

$$\|y_t\| < \mu\rho + \varepsilon, \qquad t > T \tag{1.11}$$

for all $y \in \mathscr{L}_r(E; K, 0; \rho)$ with $\|y_0\| \le R$.

(2) *If $\kappa < 1/(1+\vartheta)$ then there exist $M, \lambda > 0$ such that for all $y \in \mathscr{L}_r(E; K, 0; \rho)$,*

$$\|y_t\| \le M\|y_0\|e^{-\lambda t} + \gamma\rho, \qquad t \ge 0. \tag{1.12}$$

Theorem 1.1 can be seen as an extension of the following result in Hale [18] (see [18, pp. 110, Lemma 6.2]) which plays a fundamental role in constructing invariant manifolds of differential equations.

Proposition 1.4. *[18] Suppose $\alpha > 0$, $\gamma > 0$, K, L, M are nonnegative constants and u is a nonnegative bounded continuous solution of the inequality*

$$u(t) \le Ke^{-\alpha t} + L\int_0^t e^{-\alpha(t-s)}u(s)ds + M\int_0^\infty e^{-\gamma s}u(t+s)ds, \qquad t \ge 0. \tag{1.13}$$

If $\beta := L/\alpha + M/\gamma < 1$ *then*

$$u(t) \le (1-\beta)^{-1} K e^{-[\alpha-(1-\beta)^{-1}L]t}, \qquad t \ge 0. \tag{1.14}$$

Note that there is in fact an additional requirement in (1.14) to guarantee the exponential decay of u, that is, $\alpha - (1-\beta)^{-1}L > 0$, or equivalently,

$$L/\alpha + M/\gamma < 1 - L/\alpha. \tag{1.15}$$

Let us say a little more about the special case $M = 0$, in which (1.15) reads as $L/\alpha < 1/2$. In such a case L/α coincides with the constant κ in Theorem 1.3. Setting $K = u_0$ in (1.13), we see that the upper bound ϑ of the decay functor E in (1.13) (corresponding to (1.1)) equals 1. Consequently the smallness requirement on κ in assertion (2) of Theorem 1.3 reduces to that $\kappa = L/\alpha < 1/2$.

On the other hand, if $1/2 \le \kappa = L/\alpha < 1$ then we can only infer from (1.14) that u has at most an exponential growth. However, Theorem 1.3 still assures that a function satisfying the corresponding integral inequality must approach 0 in a uniform manner with respect to initial data in bounded sets.

We also mention that our proof for Theorem 1.1 is significantly different not only from the one for Proposition 1.4 given in [18], but also from those in the literature for other types of differential or integral inequalities.

Remark 1.5. The smallness requirement $\kappa < 1$ in the above theorems is optimal in some sense. This can be seen from the simple example of scalar equation:

$$\dot{x} = -ax + bx(t-1), \tag{1.16}$$

where $a, b > 0$ are constants, for which the assumption $\kappa < 1$ in Theorem 1.1 on the corresponding integral inequality to guarantee the global asymptotic stability of the 0 solution of the equation amounts to require that $b < a$; see Section 3.1 for details. On the other hand, if $b > a$ then simple calculations show that (1.16) has a positive eigenvalue and hence 0 is unstable; see e.g. Kuang [30, Chap. 3, Sect. 2].

Remark 1.6. It remains open whether the assumption $\kappa < 1/(1+\vartheta)$ in Theorem 1.3 to guarantee global exponential decay for (1.1) can be further relaxed in the full generality of the theorem.

As a simple example of applications, we consider the asymptotic stability of the scalar functional differential equation:

$$\dot{x} = -a(t)x + B(t, x_t), \tag{1.17}$$

where $a \in C(\mathbb{R})$, and B is a continuous function on $\mathbb{R} \times C([-r, 0])$ for some fixed $r \ge 0$ with $|B(t,\phi)| \le b(t)\|\phi\|$. Special cases of the equation were studied in the literature by many authors. For instance, in an earlier work of Winston [59], the author considered the case where $a(t)$ is nonnegative and $b(t) \le \theta a(t)$ for some $\theta < 1$. Using Razumikhin's method the author proved the exponential asymptotic stability and the asymptotic stability of the equation under the assumption $a(t) \ge \alpha > 0$ and that $a(t) \ge 0$ with $\int_0^\infty a(t)dt = \infty$, respectively. Here we revisit this problem and allow $a(t)$ to be a function which may change sign. Assume $\lim_{t\to\infty}\int_s^{s+t} a(\tau)d\tau \to \infty$ uniformly w.r.t $s \in \mathbb{R}$. We show that the null solution of (1.17) is globally asymptotically stable provided that

$$\kappa_\tau := \sup_{t\ge\tau}\int_\tau^t E(t,s)b(s)ds < 1, \qquad \forall\, \tau \in \mathbb{R},$$

where $E(t,s) = \exp\left(-\int_s^t a(\sigma)d\sigma\right)$. Some results on global exponential asymptotic stability will also be presented. It is not difficulty to check that if $a(t)$ is a nonnegative function with $\int_0^\infty a(s)ds = \infty$ and $b(t) \le \theta a(t)$ $(t \in \mathbb{R})$ for some $\theta < 1$, then $\kappa_\tau \le \theta < 1$ for all $\tau \in \mathbb{R}$.

As another example of applications for our integral inequalities, we discuss the existence of pullback attractor for ODE system

$$\dot{x} = F_0(t,x) + \sum_{i=1}^m F_i(t, x(t-r_i)), \qquad x = x(t) \in \mathbb{R}^n, \tag{1.18}$$

where $F_i(t,x)$ $(0 \le i \le m)$ are continuous mappings from $\mathbb{R}\times\mathbb{R}^n$ to $\mathbb{R}^n$ which are locally Lipschitz in x in a uniform manner with respect to t on bounded intervals, and $r_i : \mathbb{R} \to [0, r]$ $(1 \le i \le m)$ are measurable functions. The investigation of the dynamics of delayed differential equations in the framework of pullback attractor theory developed in [11,28,29] etc. was first initiated by Caraballo et al. [4]. In recent years there is an increasing interest on this topic for both retarded ODEs and PDEs; see e.g. [5,6,8,10,27,39,47,56,65]. However, we find that the existing works mainly focus on the case where the terms involving time lags have at most sublinear nonlinearities. Here we allow the nonlinearities $F_i(t,x)$ $(0 \le i \le m)$ in (1.18) to be superlinear in space variable x. Suppose

(F) there exist positive constants $p > q \ge 1$, $\alpha_i > 0$ $(0 \le i \le m)$, and nonnegative measurable functions $\beta_i(t)$ $(0 \le i \le m)$ on $\mathbb{R}$ such that

$$(F_0(t,x), x) \le -\alpha_0|x|^{p+1} + \beta_0(t), \qquad \forall\, x \in \mathbb{R}^n,\ t \in \mathbb{R},$$
$$|F_i(t,x)| \le \alpha_i|x|^q + \beta_i(t), \qquad \forall\, x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

We show under some additional assumptions on $\beta_i(t)$ $(0 \le i \le m)$ that system (1.18) is dissipative and has a global pullback attractor.

As our third example to illustrate applications of Theorems 1.1 and 1.3, we finally consider the dynamics of retarded nonlinear evolution equations with sublinear nonlinearities in the general setting of the cocycle system:

$$\frac{du}{dt} + Au = F(\theta_t p, u_t), \qquad p \in \mathcal{H} \tag{1.19}$$

in a Banach space X, where A is a sectorial operator in X with compact resolvent, $\mathcal{H}$ is a compact metric space, and θ_t is a dynamical system on $\mathcal{H}$. We will show under a hyperbolicity assumption on A and some smallness requirements on the growth rate and the Lipschitz constant of $F(p,u)$ in u that the system has a unique nonautonomous equilibrium solution Γ. The global asymptotic stability and exponential stability of Γ will also be addressed.

This paper is organized as follows. Section 2 is devoted to the proofs of the main results, namely, Theorems 1.1 and 1.3; and Section 3 consists of the two examples of ODE systems mentioned above. Section 4 is concerned with the dynamics of system (1.19). We will also talk about in this section how to put a differential equation with multiple variable delays and external forces into the general setting of (1.19).

2. Proofs of Theorems 1.1 and 1.3

For convenience in statement, let us first introduce several classes of functions.

Denote $\mathscr{E}$ the family of *bounded* nonnegative measurable functions on $Q := (\mathbb{R}^+)^2$ satisfying (1.3), and let

$$\mathscr{K}_1 = \{K \in \mathscr{M}^+(Q) : \int_0^t K(t,s)ds < \infty \text{ for all } t \geq 0\},$$
$$\mathscr{K}_2 = \{K \in \mathscr{M}^+(Q) : \int_t^\infty K(t,s)ds < \infty \text{ for all } t \geq 0\},$$

where $\mathscr{M}^+(Q)$ is the family of nonnegative measurable functions on Q. Denote $I(K_1, K_2)$ the constant defined in (1.5) for any $(K_1, K_2) \in \mathscr{K}_1 \times \mathscr{K}_2$.

Let $\mathcal{C}$ be the space $C([-r, 0])$ equipped with the usual sup-norm

$$\|\phi\| = \sup_{s\in[-r,0]} |\phi(s)|, \qquad \phi \in \mathcal{C}.$$

Given $y \in C([-r, T))$ $(T > 0)$, one can assign a function y_t from $[0, T)$ to $\mathcal{C}$ as follows: for each $t \in [0, T)$, y_t is the element in $\mathcal{C}$ defined by (1.2). For convenience, y_t will be referred to as the *lift* of y in $\mathcal{C}$.

2.1. Proof of Theorem 1.1

We begin with the following lemma:

Lemma 2.1. *Assume that $\kappa < 1$. Then for any bounded function $y \in \mathscr{L}_r(E; K_1, K_2; \rho)$,*

$$\|y_t\| \leq c\|y_0\| + \mu\rho, \qquad t \geq 0, \tag{2.1}$$

where c, μ are the constants defined in Theorem 1.1.

Proof. It can be assumed that there is $t > 0$ such that $y(t) > \|y_0\| + \mu\rho$; otherwise (2.1) readily holds true. Write

$$\sup_{t\in\mathbb{R}^+} \|y_t\| = N_\varepsilon(\|y_0\| + \varepsilon) + \mu\rho$$

for $\varepsilon > 0$. We show that $N_\varepsilon \leq c$ for all $\varepsilon > 0$, and the conclusion follows.

For each $\delta > 0$ sufficiently small, pick an $\eta > 0$ with

$$y(\eta) > \sup_{t\in\mathbb{R}^+} \|y_t\| - \delta.$$

Then by (1.1) we have

$$\begin{aligned} N_\varepsilon(\|y_0\|+\varepsilon)+\mu\rho-\delta &= \sup_{t\in\mathbb{R}^+}\|y_t\|-\delta < y(\eta) \\ &\le E(\eta,0)\|y_0\| + \int_0^\eta K_1(\eta,s)\|y_s\|ds \\ &\qquad + \int_\eta^\infty K_2(\eta,s)\|y_s\|ds+\rho \\ &\le \vartheta(\|y_0\|+\varepsilon)+\kappa\left(N_\varepsilon(\|y_0\|+\varepsilon)+\mu\rho\right)+\rho. \end{aligned}$$

Setting $\delta\to 0$ we obtain that

$$\begin{aligned} N_\varepsilon(\|y_0\|+\varepsilon)+\mu\rho &\le \vartheta(\|y_0\|+\varepsilon)+\kappa\left(N_\varepsilon(\|y_0\|+\varepsilon)+\mu\rho\right)+\rho \\ &= (\vartheta+\kappa N_\varepsilon)(\|y_0\|+\varepsilon)+(\kappa\mu+1)\rho. \end{aligned} \tag{2.2}$$

The choice of μ implies that

$$\kappa\mu+1=\mu. \tag{2.3}$$

Hence (2.2) implies that

$$N_\varepsilon(\|y_0\|+\varepsilon)\le(\vartheta+\kappa N_\varepsilon)(\|y_0\|+\varepsilon).$$

It follows that $N_\varepsilon\le\vartheta/(1-\kappa)\le c$. This completes the proof of (2.1). □

Let $y\in\mathscr{L}_r(E;K_1,K_2;\rho)$. For $\sigma>0$, if we set $\tilde{y}(t)=y(\sigma+t)$ and define

$$\tilde{E}(t,s)=E(t+\sigma,s+\sigma),\qquad \tilde{K}_i(t,s)=K_i(t+\sigma,s+\sigma)\ (i=1,2)$$

for $t,s\ge 0$, then one trivially checks that $\tilde{y}\in\mathscr{L}_r(\tilde{E};\tilde{K}_1,\tilde{K}_2;\rho)$ with

$$I(\tilde{K}_1,\tilde{K}_2)\le I(K_1,K_2)\le\kappa<1.$$

Thus if y is bounded, then by Lemma 2.1 one also concludes that

$$\|y_{t+\sigma}\|\le c\|y_\sigma\|+\mu\rho,\qquad t,\sigma\ge 0. \tag{2.4}$$

Proof of Theorem 1.1. (1) Assume $\kappa<1$. To verify assertion (1), we first show that if $y\in\mathscr{L}_r(E;K_1,K_2;\rho)$ is a bounded function, then

$$\limsup_{t\to\infty}\|y_t\|\le\mu\rho. \tag{2.5}$$

Let us argue by contradiction and suppose

$$\limsup_{t\to\infty}\|y_t\|=\mu\rho+\delta$$

for some $\delta > 0$. Take a monotone sequence $\tau_n \to \infty$ such that $\lim_{n\to\infty} y(\tau_n) = \mu\rho + \delta$. For any $\varepsilon > 0$, take a $\tau > 0$ sufficiently large so that

$$\|y_t\| < \mu\rho + \delta + \varepsilon, \qquad t \geq \tau.$$

Then for $\tau_n > \tau$, by (1.1) we deduce that

$$\begin{aligned} y(\tau_n) &\leq E(\tau_n, \tau)\|y_\tau\| + \int_\tau^{\tau_n} K_1(\tau_n, s)\|y_s\|ds + \int_{\tau_n}^\infty K_2(\tau_n, s)\|y_s\|ds + \rho \\ &\leq E(\tau_n, \tau)\|y_\tau\| + \kappa\,(\mu\rho + \delta + \varepsilon) + \rho. \end{aligned}$$

Setting $n \to \infty$ in the above inequality, it yields

$$\mu\rho + \delta \leq \kappa\,(\mu\rho + \delta + \varepsilon) + \rho.$$

Since ε is arbitrary, we conclude that

$$\mu\rho + \delta \leq (\kappa\mu + 1)\rho + \kappa\delta.$$

Therefore by (2.3) one has $\delta \leq \kappa\delta$, which leads to a contradiction and verifies (2.5).

Now we complete the proof of assertion (1). Let $R > 0$. Denote

$$\mathscr{B}_R = \{y \in \mathscr{L}_r(E; K_1, K_2; \rho) : \ y \text{ is bounded with } \|y_0\| \leq R\}.$$

By (2.1) we see that $\mathscr{B}_R$ is uniformly bounded. Hence the envelope

$$y^*(t) = \sup_{y\in\mathscr{B}_R} y(t)$$

of the family $\mathscr{B}_R$ is a bounded nonnegative measurable function on $[-r, \infty)$. (The measurability of y^* follows from the simple observation that

$$\{t \in (-r, \infty) : \ y^*(t) > a\} = \bigcup_{y\in\mathscr{B}_R}\{t \in (-r, \infty) : \ y(t) > a\}$$

is an open subset of $\mathbb{R}$ for any $a \in \mathbb{R}$.) As in the case of a continuous function, we use the notation y_t^* $(t \geq 0)$ to denote the lift of y^* in the space of measurable functions on $[-r, 0]$ $(y_t^*(\cdot) = y^*(t + \cdot))$ and write $\|y_t^*\| = \sup_{s\in[-r,0]} y_t^*(s)$. (One should distinguish $\|y_t^*\|$ with the L^∞-norm $\|y_t^*\|_{L^\infty(-r,0)}$ of y_t^*, although it can be shown by using the definition of y^* and the continuity of the functions $y \in \mathscr{B}_R$ that the two quantities coincide for y_t^*.) We claim that $\varphi(t) := \|y_t^*\|$ is a measurable function on $[0, \infty)$. Indeed, one trivially verifies that

$$\|y_t^*\| = \sup_{y\in\mathscr{B}_R} \|y_t\|, \qquad t \geq 0.$$

Since $\|y_t\|$ is continuous in t for every y, the conclusion immediately follows.

We infer from (1.1) that

$$\begin{aligned} y(t) \le\ & E(t,\tau)\|y_\tau^*\| + \int_\tau^t K_1(t,s)\|y_s^*\|ds \\ & + \int_t^\infty K_2(t,s)\|y_s^*\|ds + \rho, \qquad \forall t \ge \tau \ge 0 \end{aligned}$$

for any $y \in \mathscr{B}_R$. Further taking supremum in the lefthand side of the above inequality with respect to $y \in \mathscr{B}_R$ it yields

$$\begin{aligned} y^*(t) \le\ & E(t,\tau)\|y_\tau^*\| + \int_\tau^t K_1(t,s)\|y_s^*\|ds \\ & + \int_t^\infty K_2(t,s)\|y_s^*\|ds + \rho, \qquad \forall t \ge \tau \ge 0. \end{aligned} \tag{2.6}$$

The only difference between (1.1) and the above inequality (2.6) is that the function y^* in (2.6) may not be continuous. Note that we do not make use of any continuity requirement on y in the proofs of Lemma 2.1 and (2.5). Therefore all the arguments therein can be directly carried over to y^* without any modifications except that the function y is replaced by y^*. As a result, we deduce that $\limsup_{t\to\infty}\|y_t^*\| \le \mu\rho$. Hence for any $\varepsilon > 0$ there is a $T > 0$ such that

$$\|y_t^*\| < \mu\rho + \varepsilon, \qquad t > T,$$

from which assertion (1) immediately follows.

(2) Now we assume $\kappa < 1/(1+\vartheta)$. To obtain the exponential decay estimate in (1.9), we first prove a temporary result:

There exist $T, \lambda > 0$ such that if $\|y_0\| \le N_0 + \gamma\rho$ with $N_0 > 0$, then

$$\|y_t\| \le N_0 e^{-\lambda t} + \gamma\rho, \qquad t \ge T. \tag{2.7}$$

For this purpose, we take

$$\sigma = (1+\kappa c)/2. \tag{2.8}$$

Since $\kappa c < 1$ (see Remark 1.2), it is clear that $\sigma < 1$. Define

$$\eta = \min\{s \ge 0:\ \|y_t\| \le \sigma N_0 + \gamma\rho \text{ for all } t \ge s\}.$$

The key point is to estimate the upper bound of η.

Because $\gamma > \mu$ and $N_0 > 0$, by (2.5) it is clear that $\eta < \infty$. We may assume $\eta > r$ (otherwise we are done). Then by continuity of y one necessarily has

$$\|y_\eta\| = \sigma N_0 + \gamma\rho.$$

For simplicity, write $E(t,0) := b(t)$. Given $t \in [\eta - r, \eta]$, by (1.1) we have

$$\begin{aligned}
y(t) &\le b(t)\|y_0\| + \textstyle\int_0^t K_1(t,s)\|y_s\|ds + \int_t^\infty K_2(t,s)\|y_s\|ds + \rho \\
&\le (\text{by } (2.1)) \le \|b_\eta\|\|y_0\| + \kappa(c\|y_0\| + \mu\rho) + \rho \\
&\le \left(\|b_\eta\| + \kappa c\right)\|y_0\| + (\kappa\mu + 1)\rho \\
&\le \left(\|b_\eta\| + \kappa c\right)(N_0 + \gamma\rho) + \mu\rho.
\end{aligned}$$

Here we have used the fact that $\kappa\mu + 1 = \mu$ (see (2.3)). Therefore

$$\begin{aligned}
\sigma N_0 + \gamma\rho &= \|y_\eta\| = \max_{t\in[\eta-r,\eta]} y(t) \\
&\le \left(\|b_\eta\| + \kappa c\right) N_0 + \left(\left(\|b_\eta\| + \kappa c\right)\gamma + \mu\right)\rho.
\end{aligned} \tag{2.9}$$

Take a number $t_0 > 0$ such that

$$E(t+s,s)\gamma \le 1, \qquad \forall\, t \ge t_0,\ s \in \mathbb{R}^+. \tag{2.10}$$

If $\eta \le t_0 + r$ then we are done. Thus we assume that $\eta > t_0 + r$. Then by the definition of γ and (2.10) one deduces that

$$\gamma = \kappa c\gamma + \mu + 1 \ge \left(\|b_\eta\| + \kappa c\right)\gamma + \mu.$$

It follows by (2.9) that $\sigma N_0 \le \left(\|b_\eta\| + \kappa c\right) N_0$. Hence

$$\|b_\eta\| \ge \sigma - \kappa c = (1-\kappa c)/2 > 0. \tag{2.11}$$

Take a number $t_1 > 0$ such that

$$E(t+s,s) < (1-\kappa c)/2, \qquad t > t_1,\ s \in \mathbb{R}^+. \tag{2.12}$$

(2.11) then implies that $\eta \le t_1 + r$. Hence we conclude that

$$\eta \le T := \max(t_0, t_1) + r. \tag{2.13}$$

By far we have proved that if $\|y_0\| \le N_0 + \gamma\rho$ ($N_0 > 0$) then

$$\|y_t\| \le \sigma N_0 + \gamma\rho, \qquad t \ge T.$$

Let $\tilde{y}(t) = y(t+T)$, and set

$$\tilde{E}(t,s) = E(t+T, s+T), \qquad \tilde{K}_i(t,s) = K_i(t+T, s+T)$$

for $t, s \ge 0$, $i = 1, 2$. Then $\tilde{y} \in \mathscr{L}_r(\tilde{E}; \tilde{K}_1, \tilde{K}_2; \rho)$ with

$$I(\tilde{K}_1, \tilde{K}_2) \le I(K_1, K_2) \le \kappa < 1/(1+\vartheta).$$

Since $\|\tilde{y}_0\| \le \sigma N_0 + \gamma\rho$, the same argument as above applies to show that

$$\|\tilde{y}_t\| \le \sigma(\sigma N_0) + \gamma\rho, \qquad t \ge T,$$

that is,

$$\|y_t\| \le \sigma^2 N_0 + \gamma\rho, \qquad t \ge 2T.$$

(We emphasize that the numbers t_0 and t_1 in (2.10) and (2.12) can be chosen independent of $s \in \mathbb{R}^+$. This plays a crucial role in the above argument.) Repeating the above procedure we finally obtain that

$$\|y_t\| \le \sigma^n N_0 + \gamma\rho, \qquad t \ge nT, \; n = 1, 2, \cdots . \tag{2.14}$$

Setting $\lambda = -(\ln\sigma)/2T$, one trivially verifies that

$$\sigma^n \le e^{-\lambda t}, \qquad t \in [nT, (n+1)T]$$

for all $n \ge 1$. (2.7) then follows from (2.14).

We are now in a position to complete the proof of the theorem.

Note that (2.1) implies that if $\|y_0\| = 0$ then

$$\|y_t\| \le \mu\rho \le \gamma\rho, \qquad t \ge 0,$$

and hence the conclusion readily holds true. Thus we assume that $\|y_0\| > 0$. Take $N_0 = \|y_0\|$. Clearly $\|y_0\| = N_0 \le N_0 + \gamma\rho$. Therefore by (2.7) we have

$$\|y_t\| \le \|y_0\| e^{-\lambda t} + \gamma\rho, \qquad t \ge T. \tag{2.15}$$

On the other hand, by (2.4) we deduce that

$$\|y_t\| \le c\|y_0\| + \mu\rho \le c\|y_0\| + \gamma\rho, \qquad t \in [0, T].$$

Set $M = ce^{\lambda T}$. Then

$$\|y_t\| \le c\|y_0\| + \gamma\rho \le Me^{-\lambda t}\|y_0\| + \gamma\rho, \qquad t \in [0, T].$$

Combining this with (2.15) we finally arrive at the estimate

$$\|y_t\| \le M\|y_0\| e^{-\lambda t} + \gamma\rho, \qquad t \ge 0.$$

The proof of the theorem is complete. □

Remark 2.2. In many examples from applications, the function $E(t, s)$ in (1.1) takes the form:

$$E(t, s) = M_0 e^{-\lambda_0 (t-s)},$$

where M_0 and λ_0 are positive constants. In such a case one can write out the constants M and λ in (1.9) and (1.12) explicitly.

Indeed, the number t_0 and t_1 in (2.10) and (2.12) can be taken, respectively, as

$$t_0 = \lambda_0^{-1}\ln(M_0\gamma), \qquad t_1 = \lambda_0^{-1}\ln\left(\frac{2M_0}{1-\kappa c}\right).$$

Consequently the number T in (2.13) reads as $T = \lambda_0^{-1}M_1 + r$, where

$$M_1 = \max\left(\ln(M_0\gamma),\ \ln\left(\frac{2M_0}{1-\kappa c}\right)\right).$$

Thus we infer from the proof of Theorem 1.1 that

$$\lambda = -\frac{\ln\sigma}{2T} = \frac{\ln 2 - \ln(1+\kappa c)}{2(M_1 + r\lambda_0)}\lambda_0,$$
$$M = ce^{\lambda T} = c\sqrt{2/(1+\kappa c)}.$$

In particular, if $r = 0$ then we have

$$\lambda = \theta\lambda_0, \qquad \theta = \frac{\ln 2 - \ln(1+\kappa c)}{2M_1}.$$

Remark 2.3. In the general case, (1.3) implies that there is a bounded nonnegative function $e(t)$ on $\mathbb{R}^+$ with $e(t) \to 0$ as $t \to \infty$ such that

$$E(t+s, s) \le e(t), \qquad t, s \ge 0. \tag{2.16}$$

One can easily see that the numbers t_0 and t_1 in (2.10) and (2.12) can be chosen in such a way that they only depend upon the constants γ, κ, c and the function $e(t)$. Consequently the constants M and λ in Theorem 1.1 (2) (which are defined explicitly below (2.14) in the proof of the theorem) only depend upon γ, κ, c, σ and $e(t)$. Since γ, c and σ are completely determined by ϑ and κ (see Theorem 1.1 and (2.8) for the definitions of these constants), we finally conclude that M and λ only depend upon ϑ, κ and $e(t)$.

2.2. *Proof of Theorem 1.3*

Proof. The conclusions of Theorem 1.3 immediately follow from Theorem 1.1 as long as Lemma 2.4 below is proved. □

Lemma 2.4. *Let $E \in \mathscr{E}$, and $K_1 = K \in \mathscr{K}_1$. Suppose $I(K, 0) \le \kappa < 1$. Let $r, \rho \ge 0$, and let y be a nonnegative continuous function on $[-r, T)$ $(0 < T \le \infty)$ satisfying the integral inequality*

$$y(t) \le E(t, 0)\|y_0\| + \int_0^t K(t, s)\|y_s\|ds + \rho, \qquad 0 \le t < T. \tag{2.17}$$

Then

$$y(t) \le (c+1)(\|y_0\|+1)+\mu\rho, \qquad t\in[0,T), \tag{2.18}$$

where μ and c are the constants defined in Theorem 1.1.

Proof. Suppose the contrary. There would exist $0<\tau<T$ such that

$$y(\tau)=c'(\|y_0\|+1)+\mu\rho, \qquad y(t)\le c'(\|y_0\|+1)+\mu\rho \ \ (t\in[0,\tau)),$$

where $c'=c+1$. By (2.17) we see that

$$\begin{aligned} c'(\|y_0\|+1)+\mu\rho &= y(\tau) \\ &\le E(\tau,0)\|y_0\| + \textstyle\int_0^\tau K(\tau,s)\|y_s\|ds+\rho \\ &\le \vartheta(\|y_0\|+1)+\kappa[c'(\|y_0\|+1)+\mu\rho]+\rho \\ &\le (\vartheta+\kappa c')(\|y_0\|+1)+(\kappa\mu+1)\rho. \end{aligned} \tag{2.19}$$

By (2.3) we have $\kappa\mu+1=\mu$. Hence (2.19) implies

$$c'(\|y_0\|+1)\le(\vartheta+\kappa c')(\|y_0\|+1),$$

that is, $c'\le\vartheta+\kappa c'$. Therefore

$$c+1=c'\le\vartheta/(1-\kappa)\le c,$$

a contradiction. □

Remark 2.5. A classical result closely related to Theorem 1.3 is the Halanay's inequality (which is also called by some authors the Gronwall-Halanay inequality): If a nonnegative function y on $[t_0-r,\ \infty)$ satisfies

$$\dot y(t)\le -\alpha y(t)+\beta\|y_t\|, \qquad t\ge t_0, \tag{2.20}$$

where $\alpha>\beta>0$ are constants, then there exist $\gamma>0$ and $k>0$ such that

$$y(t)\le ke^{-\gamma(t-t_0)}, \qquad t\ge t_0;$$

see Halanay [16, pp. 378]. For simplicity we may put $t_0=0$. Using a similar argument as in the proof of Proposition 3.3 below, one can easily show that a function y satisfying (2.20) fulfills the integral inequality (1.1) with $K_2=0$ and

$$E(t,s)=e^{-\alpha(t-s)}, \qquad K_1(t,s)=\beta E(t,s)$$

for $t,s\ge0$. Note that

$$\vartheta = \sup_{t\geq s\geq 0} E(t,s) = 1, \qquad \kappa = \sup_{t\geq 0} \int_0^t K_1(t,s)ds = \beta/\alpha.$$

Thus the assumption that $\kappa < 1$ in Theorem 1.3 amounts to say that $\alpha > \beta$. Hence Theorem 1.3 can be seen as a generalization of the Halanay's inequality.

On the other hand, we emphasize that in the special case of (2.20), Halanay's result is stronger than Theorem 1.3 in the way that it guarantees the exponential convergence of $y(t)$ to 0 under the assumption that $\beta < \alpha$, whereas under this weaker assumption Theorem 1.3 only gives convergence result. This is also one of the reasons why we are interested in the question proposed in Remark 1.6.

An integral version of the Halanay's inequality can be found in a recent work of Chen [9, Lemma 3.2] along with a very simple proof: Let y be a nonnegative continuous function on $[-r,\infty)$. Suppose that for $\alpha > 0$, there exist two positive constants $M, \beta > 0$ such that $y(t) \leq Me^{-\alpha t}$ $(t \in [-r, 0])$ and that

$$y(t) \leq Me^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} \|y_s\| ds, \qquad t \geq 0. \tag{2.21}$$

If $\beta < \alpha$, then $y(t) \leq Me^{-\mu t}$ for $t \geq -r$, where $\mu \in (0,\alpha)$ is a constant satisfying that $\frac{\beta}{\alpha-\mu} e^{\mu r} = 1$. One advantage of this integral inequality is that it significantly reduces the smoothness requirement on the function y. This may greatly enlarge the applicability of the inequality. Other types of extensions of the Halanay's inequality can be found in [21,58] etc. and references therein.

3. Asymptotic behavior of ODE systems

This section consists of two examples of ODE systems illustrating possible applications of the integral inequalities given here. For the general theory of delay differential equations, one may consult the excellent books [19,30,49,60].

3.1. Asymptotic stability of a scalar functional ODE

Our first example concerns the asymptotic stability of the scalar functional differential equation:

$$\dot{x} = -a(t)x + B(t, x_t), \tag{3.1}$$

where x_t is the lift of $x = x(t)$ in $\mathcal{C} := C([-r, 0])$ ($r \geq 0$ is fixed), $a \in C(\mathbb{R})$, and B is a continuous function on $\mathbb{R} \times \mathcal{C}$. We always assume that B satisfies the following local Lipschitz condition in the second variable: For any compact interval $J \subset \mathbb{R}$ and $R > 0$, there exists $L > 0$ such that

$$|B(t,\phi) - B(t,\phi')| \leq L\|\phi - \phi'\|, \qquad \forall \phi, \phi' \in \overline{\mathcal{B}}_R,\ t \in J.$$

Here and below $\mathcal{B}_R$ denotes the ball in $\mathcal{C}$ centered at 0 with radius $R > 0$.

Given $(\tau,\phi)\in\mathbb{R}\times\mathcal{C}$, the above smoothness requirements on a and B are sufficient to guarantee the existence and uniqueness of a local solution $x(t)=x(t;\tau,\phi)$ $(t\ge\tau)$ of (3.1) with initial value $x_\tau=\phi\in\mathcal{C}$; see [19, Chap. 2, Theorems 2.1, 2.3]. We also assume that

$$|B(t,\phi)|\le b(t)\|\phi\|,\qquad (t,\phi)\in\mathbb{R}\times\mathcal{C} \tag{3.2}$$

for some nonnegative function $b\in C(\mathbb{R})$, so that $x(t;\tau,\phi)$ globally exists for each $(\tau,\phi)\in\mathbb{R}\times\mathcal{C}$. Furthermore, (3.2) implies that 0 is a solution of (3.1).

Definition 3.1. The null solution 0 of (3.1) is said to be

(1) globally asymptotically stable (GAS in short), if *(i)* it is stable, i.e., for every $\tau\in\mathbb{R}$ and $\varepsilon>0$, there exists $\delta>0$ such that $x(t;\tau,\phi)\in\mathcal{B}_\varepsilon$ for all $t\ge\tau$ and $\phi\in\mathcal{B}_\delta$, and *(ii)* it is globally attracting, meaning that $x(t;\tau,\phi)\to 0$ as $t\to\infty$ for every $(\tau,\phi)\in\mathbb{R}\times\mathcal{C}$;
(2) globally exponentially asymptotically stable (GEAS in short), if for every $\tau\in\mathbb{R}$, there exist positive constants $M,\lambda>0$ such that

$$|x(t;\tau,\phi)|\le M\|\phi\|e^{-\lambda(t-\tau)},\qquad \forall\, t\ge\tau,\ \ \phi\in\mathcal{C}. \tag{3.3}$$

Remark 3.2. The notions given in the above definition are the global versions of some corresponding local ones for functional differential equations in [19, Chap. 5, Def. 1.1] and [59, Def. 2.1-2.3], etc.

We now assume that a satisfies the following hypothesis:

(A1) $\int_s^{s+t}a(\sigma)d\sigma\to\infty$ as $t\to\infty$ uniformly with respect to $s\in\mathbb{R}$.

Define two functions $E(t,s)$ and $K(t,s)$ on $\mathbb{R}^2$ as below: $\forall\,(t,s)\in\mathbb{R}^2$,

$$E(t,s)=\exp\left(-\int_s^t a(\sigma)d\sigma\right),\qquad K(t,s)=E(t,s)b(s).$$

By (A1) one trivially verifies that

$$\lim_{t\to\infty}E(t+s,s)=0 \text{ uniformly w.r.t. } s\in\mathbb{R}. \tag{3.4}$$

For each $\tau\in\mathbb{R}$, set

$$\vartheta_\tau=\sup_{t\ge s\ge\tau}E(t,s),\qquad \kappa_\tau=\sup_{t\ge\tau}\int_\tau^t K(t,s)ds.$$

Proposition 3.3. *The null solution of (3.1) is GAS if $\kappa_\tau<1$ for all $\tau\in\mathbb{R}$. If we further assume that $\kappa_\tau<1/(1+\vartheta_\tau)$ for $\tau\in\mathbb{R}$, then it is GEAS.*

Proof. Let $\tau\in\mathbb{R}$. Write $x(t)=x(t;\tau,\phi)$. For any $t\ge\eta\ge\tau$, multiplying (3.1) with $E(t,\eta)^{-1}=\exp\left(\int_\eta^t a(\sigma)d\sigma\right)$, we obtain that

$$\frac{d}{dt}\left(E(t,\eta)^{-1}x\right) = E(t,\eta)^{-1}B(t,x_t). \tag{3.5}$$

Integrating (3.5) in t between η and t, it yields

$$x(t) = E(t,\eta)x(\eta) + \int_{\eta}^{t} E(t,s)B(s,x_s)ds. \tag{3.6}$$

(Here we have used the simple observation that $E(t,\eta)E(s,\eta)^{-1} = E(t,s)$.) Hence

$$|x(t)| \le E(t,\eta)\|x_\eta\| + \int_{\eta}^{t} K(t,s)\|x_s\|ds, \qquad \forall t \ge \eta \ge \tau. \tag{3.7}$$

Rewriting t, s and η in (3.7) as $t+\tau$, $s+\tau$ and $\eta+\tau$, respectively, i.e., performing a τ-translation on the variables in (3.7), we obtain that

$$y(t) \le E_\tau(t,\eta)\|y_\eta\| + \int_{\eta}^{t} K_\tau(t,s)\|y_s\|ds, \qquad \forall t \ge \eta \ge 0, \tag{3.8}$$

where $y(t) = |x(t+\tau)|$, and

$$E_\tau(t,s) = E(t+\tau, s+\tau), \qquad K_\tau(t,s) = K(t+\tau, s+\tau) \tag{3.9}$$

for $t, s \ge 0$. Note that

$$\vartheta_\tau = \sup_{t\ge s\ge 0} E_\tau(t,s), \qquad \kappa_\tau = \sup_{t\ge 0}\int_0^t K_\tau(t,s)ds.$$

Assume that $\kappa_\tau < 1$. Then by Theorem 1.3 one deduces that for any $R, \varepsilon > 0$, there exists $T > 0$ such that

$$|x(t;\tau,\phi)| < \varepsilon, \qquad \forall t > \tau + T,\ \phi \in \mathcal{B}_R. \tag{3.10}$$

On the other hand, we infer from Lemma 2.1 that $|x(t;\tau,\phi)| \le c_\tau\|\phi\|$ for all $t \ge \tau$ and $\phi \in \mathcal{C}$, where $c_\tau = \max\left(\vartheta_\tau/(1-\kappa_\tau),\ 1\right)$, from which it follows that the 0 solution is stable at τ. Thus we see that 0 is GAS. (We mention that the stability of the null solution can be also deduced by using (3.10) and the continuity property of $x(t;\tau,\phi)$ in ϕ. We omit the details.)

The second conclusion is a direct consequence of Theorem 1.3 (2). □

Remark 3.4. If a is a bounded function on $\mathbb{R}$ and κ_τ fulfills a stronger uniform smallness requirement:

$$\kappa := \sup_{\tau\in\mathbb{R}} \kappa_\tau < 1/(1+\vartheta), \tag{3.11}$$

where $\vartheta = \sup_{\tau\in\mathbb{R}} \vartheta_\tau$, then it can be shown that there exist positive constants $M, \lambda > 0$ independent of $\tau \in \mathbb{R}$ such that (3.3) holds true. In such a case we simply say that the solution 0 of (3.1) is uniformly GEAS.

To see this, we define for each $(\tau, s) \in \mathbb{R} \times \mathbb{R}^+$ a function $e_{\tau,s}$ on $\mathbb{R}^+$:

$$e_{\tau,s}(t) = E_\tau(s+t, s), \qquad t \in \mathbb{R}^+.$$

By *(A1)* we see that $\lim_{t\to\infty} e_{\tau,s}(t) = 0$ uniformly with respect to $(\tau, s) \in \mathbb{R} \times \mathbb{R}^+$. Using this simple fact and the boundedness of a one easily examines that the family $\{e_{\tau,s}\}_{(\tau,s)\in\mathbb{R}\times\mathbb{R}^+}$ is uniformly bounded on $\mathbb{R}^+$. Define

$$e(t) = \sup_{(\tau,s)\in\mathbb{R}\times\mathbb{R}^+} e_{\tau,s}(t), \qquad t \in \mathbb{R}^+.$$

Then $e(t) \to 0$ as $t \to \infty$. Since for every $\tau \in \mathbb{R}$, we have

$$E_\tau(s+t, s) \le e(t), \qquad t, s \ge 0,$$

invoking Remark 2.3 we deduce by (3.8) and (3.11) that there exist $M, \lambda > 0$ independent of $\tau \in \mathbb{R}$ such that (3.3) holds for all solutions of (3.1).

Remark 3.5. If $a(t) \ge 0$ for $t \in \mathbb{R}$, then $\vartheta_\tau = 1$ for all $\tau \in \mathbb{R}$, and the hypothesis on κ_τ to guarantee GEAS of the null solution reduces to that $\kappa_\tau < 1/2$.

In such a case one can also easily verify that $\kappa_\tau \le \theta < 1$ for all $\tau \in \mathbb{R}$ if the following hypotheses in Winston [59] are fulfilled:

(A2) $b(t) \le \theta a(t)$ $(t \in \mathbb{R})$ for some $\theta < 1$; and (A3) $\int_0^\infty a(t)dt = \infty$.

It follows that the null solution 0 of (3.1) is GAS. If a is bounded and $\theta < 1/2$, then we also infer from Remark 3.4 that 0 is uniformly GEAS.

Example 3.1. Let $a(t)$ be a continuous ω-periodic $(\omega > 0)$ function. Denote $a^+(t)$ $(a^-(t))$ the positive (negative) part of $a(t)$ (hence $a(t) = a^+(t) - a^-(t)$). Let

$$I = \int_0^\omega a(t)dt, \qquad I^\pm = \int_0^\omega a^\pm(t)dt.$$

Clearly $I = I^+ - I^-$. For $s \in \mathbb{R}$ and $t \ge 0$, we observe that

$$\begin{aligned} \int_s^{s+t} a(\sigma)d\sigma &= \int_s^{s+m_t\omega} a(\sigma)d\sigma + \int_{s+m_t\omega}^{s+t} a(\sigma)d\sigma \\ &= m_t I + \int_{s+m_t\omega}^{s+t} a(\sigma)d\sigma \\ &\ge m_t I - \int_{s+m_t\omega}^{s+t} a^-(\sigma)d\sigma \ge m_t I - I^-, \end{aligned} \tag{3.12}$$

where $m_t = [t/\omega]$ is the integer part of t/ω.

Now suppose that $I > 0$. Then by (3.12) we have that

$$\int_s^{s+t} a(\sigma)d\sigma \ge m_t I - I^- \ge \left(\frac{t}{\omega} - 1\right) I - I^- = \Lambda t - I^+, \tag{3.13}$$

where $\Lambda = \frac{I}{\omega}$, and

$$\int_s^{s+t} a(\sigma)d\sigma \ge m_t I - I^- \ge -I^-. \tag{3.14}$$

By (3.13) it is obviously that a fulfills hypothesis (A1).

We infer from (3.14) that for any $\tau \in \mathbb{R}$,

$$E_\tau(t,s) = \exp\left(-\int_s^t a(\sigma+\tau)d\sigma\right) \le e^{I^-} := \vartheta, \qquad t \ge s \ge 0. \tag{3.15}$$

Assume that the function b in (3.2) is bounded. Set $\beta = \sup_{t\ge 0} b(t)$. Then

$$\int_0^t K_\tau(t,s)ds = \int_0^t E_\tau(t,s)b(s+\tau)ds \le (\text{by } (3.13)) \le \beta\omega e^{I^+}/I := \kappa \tag{3.16}$$

for all $t \ge 0$. Thus in the case where a is periodic and b is bounded, we have

Proposition 3.6. *If $\beta < \beta_1 := I/(\omega e^{I^+})$, the null solution of (3.1) is GAS; and if $\beta < \beta_2 := I/(\omega e^{I^+}(1+e^{I^-}))$, then it is GEAS.*

Proof. Assume $\beta < \beta_1$. Then by (3.16) we see that

$$\kappa_\tau := \sup_{t\ge 0} \int_0^t K_\tau(t,s)ds < 1, \qquad \forall \tau \in \mathbb{R}.$$

We infer from (3.15) that $\vartheta_\tau := \sup_{t\ge s\ge 0} E_\tau(t,s) \le e^{I^-} := \vartheta$. Thus if we assume $\beta < \beta_2$, then one trivially verifies that

$$\kappa_\tau \le (\text{by } (3.16)) \le \beta\omega e^{I^+}/I < 1/(1+\vartheta), \qquad \tau \in \mathbb{R}.$$

Now the conclusion directly follows from Propositions 3.3. $\square$

A concrete example as in Example 3.1 is the linear equation:

$$\dot{x} = -(\sin t + \varepsilon)x + \beta x(t-1), \quad t > 0, \tag{3.17}$$

where $0 < \varepsilon, \beta < 1$ are constants. Simple calculations show that

$$I^+ < 2 + 2\pi\varepsilon, \qquad I^- < 2.$$

It is easy to check that if $\beta < \varepsilon e^{-(2+2\pi\varepsilon)}$, then the first hypothesis in Proposition 3.6 is fulfilled, and hence the null solution 0 of the equation is GAS. If we further assume that $\beta < \varepsilon e^{-(2+2\pi\varepsilon)} / \left(1 + e^2\right)$, then it is GEAS.

3.2. Pullback attractors of an ODE system with delays

As a second example, we consider in this part the existence of pullback attractors of the ODE system:

$$\dot{x} = F_0(t, x) + \sum_{i=1}^{m} F_i(t, x(t - r_i)), \qquad x = x(t) \in \mathbb{R}^n \tag{3.18}$$

with superlinear nonlinearities F_i $(0 \le i \le m)$.

Assume that F_i $(0 \le i \le m)$ are continuous mappings from $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which are locally Lipschitz in the space variable x in a uniform manner with respect to t on bounded intervals and satisfy the structure condition **(F)** given in Section 1, and $r_i : \mathbb{R} \to [0, r]$ $(1 \le i \le m)$ are measurable functions.

Denote $\mathcal{C}$ the space $C([-r, 0], \mathbb{R}^n)$ equipped with the usual norm $\|\cdot\|$. By the hypotheses on F_i and the delay functions r_i, it can be easily shown that the initial value problem of (3.18) is well-posed. Specifically, for each $\tau \in \mathbb{R}$ and $\phi \in \mathcal{C}$ the system has a unique solution $x(t; \tau, \phi) := x(t)$ on a maximal existence interval $[\tau - r, T_\phi)$ $(T_\phi > \tau)$ with

$$x(\tau + s) = \phi(s), \qquad s \in [-r, 0].$$

For convenience, we call the lift x_t of $x(t)$ the *solution curve* of (3.18) in $\mathcal{C}$ with initial value $x_\tau = \phi$, denoted hereafter by $x_t(\tau, \phi)$.

Lemma 3.7. *Suppose that there exist $M, N > 0$ such that*

$$\textstyle\sum_{i=0}^{m} \int_s^t \beta_i(\mu) d\mu \le M(t - s) + N, \qquad -\infty < s < t < \infty, \tag{3.19}$$

*where β_i $(0 \le i \le m)$ are the functions in **(F)**. Then each solution $x(t; \tau, \phi)$ of (3.18) is globally defined for $t \ge \tau$. Furthermore, there exist $C, \lambda, \rho > 0$ independent of $\tau \in \mathbb{R}$ such that*

$$|x(t; \tau, \phi)| \le C\|\phi\| e^{-\lambda(t-\tau)} + \rho, \qquad \forall t \ge \tau, \ (\tau, \phi) \in \mathbb{R} \times \mathcal{C}. \tag{3.20}$$

Proof. Let $x = x(t) := x(t; \tau, \phi)$ be a solution of (3.18) with maximal existence interval $[\tau - r, T_\phi)$. Set $\gamma := p(q - 1)/(p - q) + 1$. Taking the inner product of both sides of (3.18) with $|x|^{\gamma-1}x$, we find that

$$\begin{aligned} \tfrac{1}{\gamma+1} \tfrac{d}{dt} |x|^{\gamma+1} &= |x|^{\gamma-1}(F_0(t, x), x) + |x|^{\gamma-1} \textstyle\sum_{i=1}^{m} (F_i(t, x(t - r_i)), x) \\ &\le \left(-\alpha_0 |x|^{\gamma+p} + \beta_0(t)|x|^{\gamma-1}\right) + \textstyle\sum_{i=1}^{m} \left(\alpha_i |x|^\gamma \|x_t\|^q + \beta_i(t)|x|^\gamma\right). \end{aligned}$$

The classical Young's inequality implies that

$$|x|^{\gamma}\|x_t\|^{q}\le \varepsilon\|x_t\|^{\gamma+1}+C_\varepsilon|x|^{\gamma(\gamma+1)/((\gamma+1)-q)}$$

for any $\varepsilon>0$. Here and below C_ε denotes a general constant depending upon ε. By the choice of γ one easily verify that $\gamma(\gamma+1)/((\gamma+1)-q)<\gamma+p$. Hence using the Young's inequality once again we deduce that

$$|x|^{\gamma}\|x_t\|^{q}\le \varepsilon\|x_t\|^{\gamma+1}+\varepsilon|x|^{\gamma+p}+C_\varepsilon.$$

We also have

$$|x|^{\gamma-1},\,|x|^{\gamma}\le \varepsilon|x|^{\gamma+1}+C_\varepsilon.$$

Combining the above estimates together it gives

$$\begin{aligned}\tfrac{1}{\gamma+1}\tfrac{d}{dt}|x|^{\gamma+1}\ &\le -\left(\alpha_0-\varepsilon\alpha\right)|x|^{\gamma+p}+\varepsilon\alpha\|x_t\|^{\gamma+1}\\ &\quad+\varepsilon\beta(t)|x|^{\gamma+1}+C_\varepsilon(\beta(t)+1),\end{aligned}\tag{3.21}$$

where

$$\alpha=\textstyle\sum_{i=1}^{m}\alpha_i,\qquad \beta(t)=\sum_{i=0}^{m}\beta_i(t).$$

It can be assumed that $\varepsilon\alpha<\alpha_0$. Noticing that $s^{\gamma+1}\le s^{\gamma+p}+1$ for all $s\ge 0$, by (3.21) we find that

$$\frac{d}{dt}|x|^{\gamma+1}\le -a_\varepsilon(t)|x|^{\gamma+1}+\varepsilon(\gamma+1)\alpha\|x_t\|^{\gamma+1}+C_\varepsilon(\beta(t)+1),\tag{3.22}$$

where $a_\varepsilon(t)=(\gamma+1)\left(\alpha_0-\varepsilon\alpha-\varepsilon\beta(t)\right)$.

Let $E_\varepsilon(t,s)=\exp\left(-\int_s^t a_\varepsilon(\mu)d\mu\right)$ $(t\ge s\ge\tau)$. In what follows we always assume $\varepsilon<1$ and that $\varepsilon(\gamma+1)(\alpha+M)<c_0/2$, where $c_0=(\gamma+1)\alpha_0$. Then

$$\begin{aligned}-\textstyle\int_s^t a_\varepsilon(\mu)d\mu\ &=-c_0(t-s)+\varepsilon(\gamma+1)\left(\alpha(t-s)+\textstyle\int_s^t\beta(\mu)d\mu\right)\\ &\le (\text{by } (3.19))\le\\ &\le -\left(c_0-\varepsilon(\gamma+1)(\alpha+M)\right)(t-s)+\varepsilon(\gamma+1)N\\ &\le -c_1(t-s)+c_2,\qquad t\ge s\ge\tau,\end{aligned}\tag{3.23}$$

where $c_1=\frac{c_0}{2}$, and $c_2=(\gamma+1)N$. By (3.23) we see that

$$E_\varepsilon(t,s)\le e^{c_2}e^{-c_1(t-s)}:=E(t,s),\qquad t\ge s\ge\tau.\tag{3.24}$$

Clearly

$$\lim_{t\to\infty}E(t+s,s)=0$$

uniformly with respect to $s \in \mathbb{R}$.

Now performing a similar argument as in the proof of Proposition 3.3 on the differential inequality (3.22), one can obtain that

$$\begin{aligned} |x(t)|^{\gamma+1} &\le E_\varepsilon(t,\eta)\|x_\eta\|^{\gamma+1} + \varepsilon \int_\eta^t K_\varepsilon(t,s)\|x_s\|^{\gamma+1} ds \\ &+ C_\varepsilon \int_\eta^t E_\varepsilon(t,s)\tilde{\beta}(s)ds, \qquad \tau \le \eta < t < T_\phi, \end{aligned}$$

where

$$K_\varepsilon(t,s) = \alpha(\gamma+1)E_\varepsilon(t,s), \qquad \tilde{\beta}(t) = \beta(t) + 1.$$

Hence by (3.24) we have

$$\begin{aligned} |x(t)|^{\gamma+1} &\le E(t,\eta)\|x_\eta\|^{\gamma+1} + \varepsilon \int_\eta^t K(t,s)\|x_s\|^{\gamma+1} ds \\ &+ C_\varepsilon \int_\eta^t E(t,s)\tilde{\beta}(s)ds \end{aligned} \tag{3.25}$$

where $K(t,s) = \alpha(\gamma+1)E(t,s)$.

We observe that

$$\begin{aligned} \int_\eta^t E(t,s)\tilde{\beta}(s)ds &= e^{c_2} \int_\eta^t e^{-c_1(t-s)}\tilde{\beta}(s)ds \\ &\le e^{c_2} \int_0^\infty e^{-c_1 s}\tilde{\beta}(t-s)ds. \end{aligned} \tag{3.26}$$

Note that

$$\begin{aligned} \int_0^\infty e^{-c_1 s}\tilde{\beta}(t-s)ds &= \sum_{k=0}^\infty \int_k^{k+1} e^{-c_1 s}\tilde{\beta}(t-s)ds \\ &\le \sum_{k=0}^\infty e^{-c_1 k} \int_k^{k+1} \tilde{\beta}(t-s)ds \\ &\le (\text{by } (3.19)) \le (M+N+1) \sum_{k=0}^\infty e^{-c_1 k}. \end{aligned}$$

Therefore by (3.25) and (3.26) we deduce that

$$|x(t)|^{\gamma+1} \le E(t,\eta)\|x_\eta\|^{\gamma+1} + \varepsilon \int_\eta^t K(t,s)\|x_s\|^{\gamma+1} ds + C'_\varepsilon \tag{3.27}$$

for all $\tau \le \eta < t < T_\phi$. As in the proof of Proposition 3.3, performing a τ-translation on the variables in (3.27), we obtain that

$$y(t) \le E(t,\eta)\|y_\eta\| + \varepsilon \int_\eta^t K(t,s)\|y_s\| ds + C'_\varepsilon, \qquad \forall t \ge \eta \ge 0, \tag{3.28}$$

where $y(t) = |x(t+\tau)|^{\gamma+1} = |x(t+\tau;\tau,\phi)|^{\gamma+1}$. Here we have used the translation invariance property of E and K: for any $t, s \ge 0$,

$$E(t+\tau, s+\tau) = E(t,s), \qquad K(t+\tau, s+\tau) = K(t,s).$$

Simple calculations yield $\sup_{-\infty<\eta<t<\infty} \int_\eta^t E(t,s)ds = e^{c_2}/c_1$, and hence

$$\kappa_0 := \sup_{-\infty<\eta<t<\infty} \int_\eta^t K(t,s)ds = \alpha(\gamma+1)e^{c_2}/c_1.$$

Note also that $\vartheta := \sup_{t\geq s\geq 0} E(t,s) = e^{c_2}$.

We now fix an $\varepsilon > 0$ sufficiently small so that

$$\kappa := \varepsilon\kappa_0 < 1/(1+\vartheta).$$

Then the requirement in Theorem 1.3 is fulfilled. Thus by virtue of Lemma 2.4 we first deduce that $x(t)$ is bounded on $[\tau - r, T_\phi)$. It follows that $T_\phi = \infty$. Further since $y \in \mathscr{L}_r(E; \varepsilon K, 0; C'_\varepsilon)$ for all $\tau \in \mathbb{R}$, where $\mathscr{L}_r(E; \varepsilon K, 0; C'_\varepsilon)$ denotes the family of nonnegative continuous functions on $\mathbb{R}^+$ satisfying (3.28) (see also (1.6)), invoking Theorem 1.3 one immediate concludes that there exist $C, \lambda, \rho > 0$ independent of τ such that (3.20) holds. $\square$

Lemma 3.7 enables us to define a *process* $\Phi(t,\tau)$ on $\mathcal{C}$:

$$\Phi(t,\tau)\phi = x_t(\tau,\phi), \qquad t \geq \tau > -\infty, \ \phi \in \mathcal{C}, \tag{3.29}$$

where $x_t(\tau,\phi)$ is the solution curve of (3.18) in $\mathcal{C}$ with $x_\tau(\tau,\phi) = \phi$ defined as above. Φ possesses the following basic properties:

- $\Phi(t,\tau) : \mathcal{C} \to \mathcal{C}$ is a continuous mapping for each fixed $(t,\tau) \in \mathbb{R}^2$, $t \geq \tau$;
- $\Phi(\tau,\tau) = \mathrm{id}_{\mathcal{C}}$ for all $\tau \in \mathbb{R}$, where $\mathrm{id}_{\mathcal{C}}$ is the identity mapping on $\mathcal{C}$;
- $\Phi(t,\tau) = \Phi(t,s)\Phi(s,\tau)$ for all $t \geq s \geq \tau$.

For system (3.18), the estimate given in Lemma 3.7 is sufficient to guarantee the existence of a global pullback attractor; see [3,4] etc. (The interested reader is referred to [7] etc. for the general theory of pullback attractors.) Hence we have

Theorem 3.8. *Assume the hypotheses in Lemma 3.7. Then Φ has a (unique) global pullback attractor in $\mathcal{C}$. Specifically, there is a unique family $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ of compact sets contained in the ball $\overline{\mathcal{B}}_\rho$ in $\mathcal{C}$ centered at 0 with radius ρ such that*

(1) $\Phi(t,\tau)A(\tau) = A(t)$ *for all* $t \geq \tau$*;*
(2) *for any bounded set* $B \subset \mathcal{C}$*,*

$$\lim_{\tau\to-\infty} d_H\left(\Phi(t,\tau)B,\ A(t)\right) = 0$$

for all $t \in \mathbb{R}$*, where* $d_H(\cdot,\cdot)$ *denotes the Hausdorff semi-distance in* $\mathcal{C}$*,*

$$d_H(M,N) = \sup_{\phi\in M} \inf\{\|\phi - \psi\| : \ \psi \in N\}, \qquad \forall M, N \subset \mathcal{C}.$$

4. On the dynamics of retarded evolution equations with sublinear nonlinearities

As our third example to illustrate the application of Theorems 1.1 and 1.3, we investigate the dynamics of abstract retarded functional differential equations with sublinear nonlinearities in the general setting of cocycle systems.

Let $\mathcal{H}$ be a compact metric space with metric $d(\cdot,\cdot)$. Assume that there has been given a dynamical system θ on $\mathcal{H}$, i.e., a continuous mapping $\theta : \mathbb{R}\times\mathcal{H}\to\mathcal{H}$ satisfying the group property: for all $p\in\mathcal{H}$ and $s,t\in\mathbb{R}$,

$$\theta(0,p)=p, \qquad \theta(s+t,p)=\theta(s,\theta(t,p)).$$

As usual, we will rewrite $\theta(t,p)=\theta_t p$.

In what follows we always assume that $\mathcal{H}$ is *minimal* (with respect to θ). This means that θ has no proper nonempty compact invariant subsets in $\mathcal{H}$.

Let X be a real Banach space with norm $\|\cdot\|_0$, and let A be a sectorial operator on X with compact resolvent. Denote X^s $(s\ge 0)$ the fractional power of X generated by A with norm $\|\cdot\|_s$; see [20, Chap. 1] for details.

Let $0\le r<\infty$, and $\alpha\in[0,1)$. Denote $\mathcal{C}_\alpha=C([-r,0],X^\alpha)$. $\mathcal{C}_\alpha$ is equipped with the norm $\|\cdot\|_{\mathcal{C}_\alpha}$ defined by

$$\|\phi\|_{\mathcal{C}_\alpha}=\max_{[-r,0]}\|\phi(s)\|_\alpha, \qquad \phi\in\mathcal{C}_\alpha.$$

Given a continuous function $u:[t_0-r,T)\to X^\alpha$, denote by u_t the lift of u in $\mathcal{C}_\alpha$,

$$u_t(s)=u(t+s), \qquad s\in[-r,0],\ t\ge t_0.$$

The retarded functional cocycle system we are concerned with is as follows:

$$\frac{du}{dt}+Au=F(\theta_t p,u_t), \qquad t\ge 0,\ p\in\mathcal{H}, \tag{4.1}$$

where F is a continuous mapping from $\mathcal{H}\times\mathcal{C}_\alpha$ to X. Later we will show how to put a nonlinear evolution equation like

$$\frac{du}{dt}+Au=f(u(t-r_1),\cdots,u(t-r_m))+h(t)$$

into the abstract form of (4.1). For convenience in statement, $\mathcal{H}$ and θ are usually called the *base space* and the *driving system* of (4.1), respectively.

Denote by $\mathcal{B}_R$ the ball in $\mathcal{C}_\alpha$ centered at 0 with radius R.

Assume that F satisfies the following conditions:

(F1) $F(p,\phi)$ is *locally Lipschitz* in ϕ uniformly w.r.t $p\in\mathcal{H}$, namely, for any $R>0$, there exists $L_R>0$ such that

$$\|F(p,\phi)-F(p,\phi')\|_0\le L_R\|\phi-\phi'\|_{\mathcal{C}_\alpha}, \qquad \forall\phi,\phi'\in\overline{\mathcal{B}}_R,\ p\in\mathcal{H}.$$

(F2) There exist $C_0, C_1 > 0$ such that

$$\|F(p,\phi)\|_0 \le C_0\|\phi\|_{\mathcal{C}_\alpha} + C_1, \qquad \forall (p,\phi) \in \mathcal{H} \times \mathcal{C}_\alpha.$$

Under the above assumptions, the same argument as in the proof of [51, Proposition 3.1] with minor modifications applies to show the existence and uniqueness of global mild solutions for (4.1): For each initial data $\phi \in \mathcal{C}_\alpha := C([-r,0], X^\alpha)$ and $p \in \mathcal{H}$, there is a unique continuous function $u : [-r,\infty) \to X^\alpha$ with $u(t) = \phi(t)$ $(-r \le t \le 0)$ satisfying the integral equation

$$u(t) = e^{-At}\phi(0) + \int_0^t e^{-A(t-s)} F(\theta_s p, u_s) ds, \qquad t \ge 0.$$

A solution of (4.1) clearly depends on p. For convenience, given $p \in \mathcal{H}$, we call a solution u of (4.1) a *solution pertaining to* p. We will use the notation $u(t; p, \phi)$ to denote the solution of (4.1) on $[-r,\infty)$ pertaining to p with initial value $\phi \in \mathcal{C}_\alpha$. The solutions of (4.1) generate a *cocycle* Φ on $\mathcal{C}_\alpha$,

$$\Phi(t,p)\phi = u_t, \qquad t \ge 0,\ (p,\phi) \in \mathcal{H} \times \mathcal{C}_\alpha,$$

where u_t is the lift of the solution $u(t) = u(t; p, \phi)$ in $\mathcal{C}_\alpha$.

Since $\mathcal{H}$ is compact and A has compact resolvent, using a similar argument as in the proof of [50, Proposition 4.1], it can be shown that for each fixed $t > r$, $\Phi(t,p)\phi$ is compact as a mapping from $\mathcal{H} \times \mathcal{C}_\alpha$ to $\mathcal{C}_\alpha$. Making use of this basic fact one can easily verify that Φ is *asymptotically compact*, that is, Φ enjoys the following property:

(AC) For any sequences $t_n \to \infty$ and $(p_n, \phi_n) \in \mathcal{H} \times \mathcal{C}_\alpha$, if $\bigcup_{n\ge 1} \Phi([0,t_n], p_n)\phi_n$ is bounded in $\mathcal{C}_\alpha$ then the sequence $\Phi(t_n, p_n)\phi_n$ has a convergent subsequence.

4.1. Basic integral formulas on bounded solutions

Suppose A has a spectral decomposition $\sigma(A) = \sigma^- \cup \sigma^+$, where

$$\mathrm{Re}\, z \le -\beta < 0\ (z \in \sigma^-), \qquad \mathrm{Re}\, z \ge \beta > 0\ (z \in \sigma^+) \tag{4.2}$$

for some $\beta > 0$. Let $X = X_1 \oplus X_2$ be the corresponding direct sum decomposition of X with X_1 and X_2 being invariant subspaces of A. Denote $P_i : X \to X_i$ $(i = 1,2)$ the projection from X to X_i, and write $A_i = A|_{X_i}$. By the basic knowledge on sectorial operators (see Henry [20, Chap. 1]), there exists $M \ge 1$ such that

$$\|\Lambda^\alpha e^{-A_1 t}\| \le Me^{\beta t}, \qquad \|e^{-A_1 t}\| \le Me^{\beta t}, \qquad t \le 0, \tag{4.3}$$

$$\|\Lambda^\alpha e^{-A_2 t} P_2 \Lambda^{-\alpha}\| \le Me^{-\beta t}, \qquad \|\Lambda^\alpha e^{-A_2 t}\| \le Mt^{-\alpha} e^{-\beta t}, \qquad t > 0. \tag{4.4}$$

The verification of the following basic integral formulas on bounded solutions are just slight modifications of the corresponding ones for that of equations without delays (see e.g. [17, pp. 180, Lemma A.1] and [24]), and hence is omitted.

Lemma 4.1. *Let $u : [-r, +\infty) \to X^\alpha$ be a bounded continuous function. Then u is a solution of (4.1) on $[-r, \infty)$ pertaining to $p \in \mathcal{H}$ if and only if u solves the integral equation*

$$\begin{aligned} u(t) = \;& e^{-A_2 t} P_2 u(0) + \int_0^t e^{-A_2(t-s)} P_2 F(\theta_s p, u_s) ds \\ & - \int_t^\infty e^{-A_1(t-s)} P_1 F(\theta_s p, u_s) ds, \qquad t \geq 0. \end{aligned}$$

4.2. Existence of bounded complete solutions

For nonlinear evolution equations, bounded complete solutions are of equal importance as equilibrium ones. This is because that the long-term dynamics of an equation is determined not only by the distribution of its equilibrium solutions, but also by that of all its bounded complete trajectories. In fact, for a nonautonomous evolution equation it may be of little sense to talk about equilibrium solutions in the usual terminology.

In this subsection we establish an existence result for bounded complete solutions of equation (4.1). For this purpose we need first to give some a priori estimates.

Let C_0, C_1 be the constants in (F2), and set

$$\kappa_0 = \sup_{t \geq 0} \left(\int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} ds + \int_t^\infty e^{\beta(t-s)} ds \right). \tag{4.5}$$

Lemma 4.2. *Suppose A has a spectral decomposition as in (4.2), and that $C_0 < 1/(\kappa_0 M)$. Then for any $R, \varepsilon > 0$, there exists $T > 0$ such that for all bounded solutions $u(t) = u(t; p, \phi)$ of (4.1) with $\phi \in \overline{\mathcal{B}}_R$,*

$$\|u(t)\|_\alpha < \rho + \varepsilon, \qquad t > T, \tag{4.6}$$

where $\rho = C_1 M (1 - \kappa_0 C_0 M)^{-1} \int_0^\infty (1 + s^{-\alpha}) e^{-\beta s} ds$. Consequently

$$\sup_{t \in \mathbb{R}} \|\gamma(t)\|_\alpha \leq \rho \tag{4.7}$$

for all bounded complete solutions $\gamma(t)$ of (4.1).

Proof. (1) Let $u(t) = u(t; p, \phi)$ be a bounded solution of (4.1) on $[-r, \infty)$. For any $\tau \geq 0$, set $v(t) = u(t + \tau)$ $(t \geq 0)$. Then v is a bounded solution of (4.1) pertaining to $q = \theta_\tau p$. Hence we infer from Lemma 4.1 that

$$\begin{aligned} v(t) = \;& e^{-A_2 t} P_2 v(0) + \int_0^t e^{-A_2(t-s)} P_2 F(\theta_s q, v_s) ds \\ & - \int_t^\infty e^{-A_1(t-s)} P_1 F(\theta_s q, v_s) ds, \qquad t \geq 0. \end{aligned} \tag{4.8}$$

Therefore by (4.3), (4.4) and (F2), we deduce that

$$\begin{aligned} \|v(t)\|_\alpha \leq \;& M e^{-\beta t} \|v_0\|_{\mathcal{C}_\alpha} + \int_0^t K_1(t,s) \|v_s\|_{\mathcal{C}_\alpha} ds \\ & + \int_t^\infty K_2(t,s) \|v_s\|_{\mathcal{C}_\alpha} ds + C_2, \qquad t \geq 0, \end{aligned}$$

where

$$K_1(t,s) = C_0 M (t-s)^{-\alpha} e^{-\beta(t-s)}, \qquad K_2(t,s) = C_0 M e^{\beta(t-s)}, \tag{4.9}$$

and $C_2 = C_1 M \int_0^\infty (1+s^{-\alpha}) e^{-\beta s} ds$. That is, u satisfies

$$\begin{aligned} \|u(t)\|_\alpha \le\ & M e^{-\beta(t-\tau)} \|u_\tau\|_{\mathcal{C}_\alpha} + \int_\tau^t K_1(t,s) \|u_s\|_{\mathcal{C}_\alpha} ds \\ & + \int_t^\infty K_2(t,s) \|u_s\|_{\mathcal{C}_\alpha} ds + C_2, \qquad t \ge \tau \ge 0. \end{aligned} \tag{4.10}$$

Applying Theorem 1.1 one deduces that if $C_0 < 1/(\kappa_0 M)$ then for any $R, \varepsilon > 0$, there exists $T > 0$ such that (4.6) holds true for all $p \in \mathcal{H}$ and $\phi \in \overline{\mathcal{B}}_R$.

(2) Let $\gamma(t)$ be a bounded complete solution of (4.1) pertaining to some $q \in \mathcal{H}$. Pick an $R > 0$ such that $\|\gamma(t)\|_\alpha < R$ for all $t \in \mathbb{R}$. Then for any $\varepsilon > 0$, there is $T > 0$ such that (4.6) holds for all $p \in \mathcal{H}$ and $\phi \in \overline{\mathcal{B}}_R$. Taking $p = \theta_{-T} q$ and $\phi = \gamma(-T)$, one finds that

$$\|\gamma(0)\|_\alpha = \|u(T; p, \phi)\|_\alpha < \rho + \varepsilon.$$

Since ε is arbitrary, we conclude that $\|\gamma(0)\|_\alpha \le \rho$.

In a similar fashion it can be shown that $\|\gamma(t)\|_\alpha \le \rho$ for all $t \in \mathbb{R}$. $\square$

Thanks to Lemma 4.2, one can now show by very standard argument via the Conley index theory that equation (4.1) has a bounded complete solution u. Specifically, we have the following existence result.

Theorem 4.3. *Assume the hypotheses in Lemma 4.2. Then for any $p \in \mathcal{H}$, (4.1) has at least one bounded complete solution u pertaining to p.*

Proof. The estimate (4.7) allows us to prove by using the Conley index theory and some standard argument that (4.1) has at least one bounded complete solution $\gamma = \gamma(t)$ pertaining to some $p_0 \in \mathcal{H}$. The interested reader is referred to [55, Sect. 7] and [34] for details.

To show that for any $p \in \mathcal{H}$, equation (4.1) has at least one bounded complete solution u pertaining to p, we consider the skew-product flow Π on $\mathscr{X} = \mathcal{H} \times \mathcal{C}_\alpha$ defined as below:

$$\Pi(t)(p, \phi) = (\theta_t p, \Phi(t, p)\phi), \qquad (p, \phi) \in \mathscr{X}, \ t \ge 0. \tag{4.11}$$

The asymptotic compactness of Φ implies that Π is asymptotically compact. Let $\varphi(t) = (\theta_t p_0, \gamma_t)$. Then $\varphi = \varphi(t)$ is a bounded complete trajectory of Π.

Let $\mathcal{S} = \omega(\varphi)$ be the ω-limit set of φ,

$$\omega(\varphi) = \bigcap_{\tau \ge 0} \overline{\{\varphi(t) : \ t \ge \tau\}}. \tag{4.12}$$

By the basic knowledge in the dynamical systems theory we know that $\mathcal{S}$ is a nonempty compact invariant set of Π. Set $K = P_{\mathcal{H}} \mathcal{S}$, where $P_{\mathcal{H}} : \mathscr{X} \to \mathcal{H}$ is the projection. One can easily verify that K is a nonempty compact invariant set of the driving system θ. Hence due to the minimality hypothesis on $\mathcal{H}$ we deduce that $K = \mathcal{H}$. Consequently for each $p \in \mathcal{H}$, there is a $\phi \in \mathcal{C}_\alpha$ such that $(p, \phi) \in \mathcal{S}$. Let $(\theta_t p, u_t)$ be a bounded complete trajectory of Π in $\mathcal{S}$ through (p, ϕ). Set

$$u(t) = u_t(0), \qquad t \in \mathbb{R}.$$

Then $u(t)$ is bounded complete solution of (4.1) pertaining to p. $\square$

4.3. Existence of a nonautonomous equilibrium solution

For the sake of simplicity in statement, instead of (F1) and (F2), in this section we assume that $F(p,\phi)$ is *globally Lipschitz* in ϕ uniformly w.r.t $p \in \mathcal{H}$, i.e.,

(F3) there exist $L > 0$ such that

$$\|F(p,\phi) - F(p,\phi')\|_0 \le L\|\phi - \phi'\|_{\mathcal{C}_\alpha}, \qquad \forall \phi, \phi' \in \mathcal{C}_\alpha,\ p \in \mathcal{H}.$$

In such a case, since

$$\|F(p,\phi)\|_0 = \|F(p,\phi) - F(p,0)\|_0 + \|F(p,0)\|_0 \le L\|\phi\|_{\mathcal{C}_\alpha} + \|F(p,0)\|_0,$$

we see that hypothesis (F2) is automatically fulfilled with

$$C_0 = L, \qquad C_1 = \max_{p\in\mathcal{H}} \|F(p,0)\|_0. \tag{4.13}$$

Definition 4.4. A nonautonomous equilibrium solution of (4.1) is a continuous mapping $\Gamma \in C(\mathcal{H}, X^\alpha)$ such that $\gamma_p(t) := \Gamma(\theta_t p)$ is a bounded complete solution of (4.1) pertaining to p for each $p \in \mathcal{H}$.

Theorem 4.5. *Suppose A has a spectral decomposition as in (4.2), and that $L < 1/(\kappa_0 M)$. Then the following assertions hold:*

(1) *Equation (4.1) has a nonautonomous equilibrium solution $\Gamma \in C(\mathcal{H}, X^\alpha)$.*
(2) *For any $R, \varepsilon > 0$, there exists $T > 0$ such that for any bounded solution $u(t) = u(t; p, \phi)$ with $\phi \in \overline{\mathcal{B}}_R$,*

$$\|u(t) - \Gamma(\theta_t p)\|_\alpha < \varepsilon, \qquad t > T.$$

(3) *There exists $c > 0$ such that for any bounded solution $u(t) = u(t; p, \phi)$,*

$$\|u(t) - \Gamma(\theta_t p)\|_\alpha \le c \max_{s\in[-r,0]} \|\phi(s) - \Gamma(\theta_s p)\|_\alpha, \qquad t \ge 0.$$

Proof. (1) We continue the argument in the proof of Theorem 4.3. Set

$$\mathcal{S}[p] = \{\phi :\ (p,\phi) \in \mathcal{S}\}, \qquad p \in \mathcal{H},$$

where $\mathcal{S}$ is the ω-limit set of φ given by (4.12). Using the compactness of $\mathcal{S}$ one easily checks that $\mathcal{S}[p]$ is upper semicontinuous, i.e., given $p \in \mathcal{H}$, for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\mathcal{S}[q]$ is contained in the ε-neighborhood of $\mathcal{S}[p]$ for all q with $d(q,p) < \delta$. In what follows we show that $\mathcal{S}[p]$ is a singleton. Consequently the upper semicontinuity of $\mathcal{S}[p]$ reduces to the continuity of $\mathcal{S}[p]$ in p.

Let $\phi_1, \phi_2 \in \mathcal{S}[p]$. As in the proof of Theorem 4.3 we know that Φ has two bounded complete trajectories γ_t^i $(i = 1, 2)$ in $\mathcal{C}_\alpha$ pertaining to p with $\gamma_0^i = \phi_i$. We check that $\gamma_t := \gamma_t^1 - \gamma_t^2 \equiv 0$ for $t \in \mathbb{R}$, or equivalently,

$$\gamma(t) := \gamma^1(t) - \gamma^2(t) \equiv 0, \qquad \text{where } \gamma^i(t) = \gamma_t^i(0). \tag{4.14}$$

It then follows that $\phi_1 = \phi_2$, hence $\mathcal{S}[p]$ is a singleton.

For $\eta \in \mathbb{R}$, we write $\varphi^i(t) = \gamma^i(t + \eta)$. Then $\varphi^i(t)$ is a solution of (4.1) pertaining to $q = \theta_\eta p$. By Lemma 4.1 we have

$$\begin{aligned}\varphi^i(t) = {} & e^{-A_2 t} P_2 \varphi^i(0) + \textstyle\int_0^t e^{-A_2(t-s)} P_2 F\left(\theta_s q,\, \varphi_s^i\right) ds \\ & - \textstyle\int_t^\infty e^{-A_1(t-s)} P_1 F\left(\theta_s q,\, \varphi_s^i\right) ds, \qquad t \geq 0.\end{aligned}$$

Let $\varphi(t) := \varphi^1(t) - \varphi^2(t)$. Then

$$\begin{aligned}\varphi(t) = {} & e^{-A_2 t} P_2 \varphi(0) + \textstyle\int_0^t e^{-A_2(t-s)} P_2 \left(F\left(\theta_s q,\, \varphi_s^1\right) - F\left(\theta_s q,\, \varphi_s^2\right)\right) ds \\ & - \textstyle\int_t^\infty e^{-A_1(t-s)} P_1 \left(F\left(\theta_s q,\, \varphi_s^1\right) - F\left(\theta_s q,\, \varphi_s^2\right)\right) ds, \qquad t \geq 0.\end{aligned}$$

Thus by (F3) we deduce that

$$\begin{aligned}\|\varphi(t)\|_\alpha \leq {} & M e^{-\beta t} \|\varphi_0\|_{\mathcal{C}_\alpha} + LM \textstyle\int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} \|\varphi_s\|_{\mathcal{C}_\alpha} ds \\ & + LM \textstyle\int_t^\infty e^{-\beta(s-t)} \|\varphi_s\|_{\mathcal{C}_\alpha} ds, \qquad \forall t \geq 0.\end{aligned} \tag{4.15}$$

Since $\varphi(t) = \gamma^1(t + \eta) - \gamma^2(t + \eta)$ and $\eta \in \mathbb{R}$ can be taken arbitrary, it can be easily seen that (4.15) is readily satisfied by all the translations $\varphi(\cdot + \tau)$ of φ, i.e.,

$$\begin{aligned}\|\varphi(t+\tau)\|_\alpha \leq {} & M e^{-\beta t} \|\varphi_\tau\|_{\mathcal{C}_\alpha} + LM \textstyle\int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} \|\varphi_{s+\tau}\|_{\mathcal{C}_\alpha} ds \\ & + LM \textstyle\int_t^\infty e^{-\beta(s-t)} \|\varphi_{s+\tau}\|_{\mathcal{C}_\alpha} ds, \qquad \forall t \geq 0.\end{aligned}$$

Rewriting $t + \tau$ as t, the above inequality can be put into the following one:

$$\begin{aligned}\|\varphi(t)\|_\alpha \leq {} & E(t, \tau) \|\varphi_\tau\|_{\mathcal{C}_\alpha} + \textstyle\int_\tau^t K_1(t, s) \|\varphi_s\|_{\mathcal{C}_\alpha} ds \\ & + \textstyle\int_t^\infty K_2(t, s) \|\varphi_s\|_{\mathcal{C}_\alpha} ds, \qquad \forall t \geq \tau,\end{aligned} \tag{4.16}$$

where $E(t, s) = M e^{-\beta(t-s)}$, and

$$K_1 = LM(t-s)^{-\alpha} e^{-\beta(t-s)}, \qquad K_2 = LM e^{-\beta(s-t)}. \tag{4.17}$$

Applying Theorem 1.1 (1) to $y(t) = \|\varphi(t)\|_\alpha$, we deduce by (4.16) that if $L < 1/(\kappa_0 M)$, then for any $\varepsilon > 0$ there exists $T > 0$ (independent of η) such that $\|\varphi(t)\|_\alpha < \varepsilon$ for all $t > T$, that is,

$$\|\gamma(t+\eta)\|_\alpha < \varepsilon, \qquad t > T,\ \eta \in \mathbb{R}. \tag{4.18}$$

Now for any $\tau \in \mathbb{R}$, setting $t = T + 1$ and $\eta = \tau - (T + 1)$ in (4.18) we obtain that $\|\gamma(\tau)\|_\alpha < \varepsilon$. Since ε is arbitrary, one immediately concludes that $\gamma(\tau) = 0$, which justifies the validity of (4.14).

Now we write $\mathcal{S}[p] = \{\phi_p\}$. Then ϕ_p is continuous in p, and the invariance property of $\mathcal{S}$ implies that $\gamma_t := \phi_{\theta_t p}$ is a complete trajectory of the cocycle Φ in $\mathcal{C}_\alpha$ for each $p \in \mathcal{H}$. Define

$$\Gamma(p) = \phi_p(0), \qquad p \in \mathcal{H}.$$

Clearly $\Gamma \in C(\mathcal{H}, X^\alpha)$. It is easy to see that for each $p \in \mathcal{H}$, $\gamma_p(t) := \Gamma(\theta_t p) = \phi_{\theta_t p}(0)$ is a complete solution of (4.1) pertaining to p. Hence Γ is a nonautonomous equilibrium solution of equation (4.1).

(2)-(3) Let $p \in \mathcal{H}$, and let $u(t) = u(t; p, \phi)$ be a bounded solution of (4.1). Then the same argument as above with minor modifications applies to show that (4.16) is fulfilled by $\varphi(t) := u(t) - \Gamma(\theta_t p)$ for all $t \ge \tau \ge 0$. Assertions (2) and (3) then immediately follow from Theorem 1.1 and Lemma 2.1. □

4.4. *Global asymptotic stability of the equilibrium*

Now we pay some attention to the particular case where $\sigma(A)$ lies in the right half plane. We continue the argument in Section 4.3 and assume that F satisfies the global Lipschitz condition (F3).

Given $(p, \phi) \in \mathcal{H} \times \mathcal{C}_\alpha$, we write $u(t) = u(t; p, \phi)$. Since the spectral set $\sigma^- = \emptyset$, using the constant variation formula it can be shown that

$$\|u(t)\|_\alpha \le Me^{-\beta(t-\tau)}\|u_\tau\|_{\mathcal{C}_\alpha} + \int_\tau^t K_1(t,s)\|u_s\|_{\mathcal{C}_\alpha} ds + \rho_0, \qquad t \ge \tau \ge 0 \tag{4.19}$$

where $\rho_0 = C_1 M \int_0^\infty s^{-\alpha} e^{-\beta s} ds$ (C_1 is the constant given in (4.13)), and $K_1(t, s)$ is the function given in (4.17). The calculations involved here are similar to those as in the proof of Lemma 4.2. We omit the details. Let

$$\kappa_0 = \sup_{t\ge 0}\left(\int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} ds\right), \qquad \rho = (1 - \kappa_0 LM)^{-1}\rho_0,$$

where L is the constant in (F3). Applying Theorem 1.3 (1) one deduces that if $L < 1/(\kappa_0 M)$, then for any $R, \varepsilon > 0$, there exists $T > 0$ such that

$$\|u(t)\|_\alpha < \rho + \varepsilon, \qquad \forall t > T,\ (p, \phi) \in \mathcal{H} \times \overline{\mathcal{B}}_R. \tag{4.20}$$

Let Γ be the nonautonomous equilibrium solution given by Theorem 4.5. As a direct consequence of (4.20) and Theorem 4.5, we have

Theorem 4.6. *Suppose $L < 1/(\kappa_0 M)$, and that*

$$\mathrm{Re}\, z \ge \beta > 0, \qquad \forall z \in \sigma(A). \tag{4.21}$$

Then Γ is uniformly globally asymptotically stable in the following sense:

(1) *Γ is uniformly table, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(p, \phi) \in \mathcal{H} \times \mathcal{C}_\alpha$ with $\max_{s\in[-r,0]} \|\phi(s) - \Gamma(\theta_s p)\|_\alpha < \delta$,*

$$\|u(t) - \Gamma(\theta_t p)\|_\alpha < \varepsilon, \qquad t \geq 0. \tag{4.22}$$

(2) *Γ is uniformly globally attracting, i.e., for any $R, \varepsilon > 0$, there exists $T > 0$ such that for all $p \in \mathcal{H}$ and $\phi \in \overline{\mathcal{B}}_R$,*

$$\|u(t) - \Gamma(\theta_t p)\|_\alpha < \varepsilon, \qquad t > T. \tag{4.23}$$

Proof. The uniform stability of Γ follows from Theorem 4.5 (3), and the uniform global attraction of Γ is a consequence of (4.20) and some general results on the uniform forward attraction properties of pullback attractors; see e.g. [57, Theorem 3.3]. $\square$

If we impose on L a stronger smallness requirement, then it can be shown that Γ is uniformly globally exponentially asymptotically stable.

Theorem 4.7. *Assume that A satisfies (4.21). If $L < 1/(\kappa_0 M(1 + M))$, then there exist $C, \lambda > 0$ such that for all $(p, \phi) \in \mathcal{H} \times \mathcal{C}_\alpha$,*

$$\|u(t) - \Gamma(\theta_t p)\|_\alpha \leq Ce^{-\lambda t} \max_{s\in[-r,0]} \|\phi(s) - \Gamma(\theta_s p)\|_\alpha\,, \qquad t \geq 0.$$

Proof. Let $\varphi(t) = u(t) - \Gamma(\theta_t p)$. Using a parallel argument as in the proof of Lemma 4.2 (1), we can obtain that

$$\|\varphi(t)\|_\alpha \leq Me^{-\beta(t-\tau)}\|\varphi_\tau\|_{\mathcal{C}_\alpha} + \int_\tau^t K_1(t,s)\|\varphi_s\|_{\mathcal{C}_\alpha} ds. \qquad t \geq \tau \geq 0.$$

If $L < 1/(\kappa_0 M(1 + M))$ then the functions $E(t, s) := Me^{-\beta(t-s)}$ and $K_1(t, s)$ fulfill the requirements in Theorem 1.3. Thus there exist constants $C, \lambda > 0$ independent of φ such that

$$\|\varphi(t)\|_\alpha \leq Ce^{-\lambda t}\|\varphi_0\|_{\mathcal{C}_\alpha}, \qquad t \geq 0.$$

The conclusion of the theorem then immediately follows. $\square$

4.5. Nonlinear evolution equations with multiple delays

Let us now consider the nonlinear evolution equation

$$\frac{du}{dt} + Au = f(u(t - r_1), \cdots, u(t - r_m)) + h(t) \tag{4.24}$$

with multiple delays, where X and A are the same as in Subsection 4.1, f is a continuous mapping from $(X^\alpha)^m$ to X for some $\alpha \in [0, 1)$, $h \in C(\mathbb{R}, X)$, $r_i \in C(\mathbb{R}, \mathbb{R}^+)$, and

$$0 \le r_i(t) \le r < \infty, \qquad 1 \le i \le m.$$

It is well known that (4.24) covers a large number of concrete examples from applications. Our main goal here is to demonstrate how to put such an equation into the abstract form of (4.1).

The initial value problem of (4.24) reads as follows:

$$\begin{cases} \frac{du}{dt} + Au = f(u(t-r_1), \cdots, u(t-r_m)) + h(t), & t \ge \tau, \\ u(\tau+s) = \phi(s), & s \in [-r, 0], \end{cases} \tag{4.25}$$

where $\phi \in \mathcal{C}_\alpha = C([-r,0], X^\alpha)$, and $\tau \in \mathbb{R}$ is given arbitrary. Rewriting $t-\tau$ as t, one obtains an equivalent form of (4.25):

$$\begin{cases} \frac{dv}{dt} + Av = f(v(t-\tilde{r}_1), \cdots, v(t-\tilde{r}_m)) + \tilde{h}(t), & t \ge 0, \\ v(s) = \phi(s), & s \in [-r, 0], \end{cases} \tag{4.26}$$

where $v(t) = u(t+\tau)$, and

$$\tilde{r}_i(t) = r_i(t+\tau), \qquad \tilde{h}(t) = h(t+\tau).$$

Denote $\mathcal{Y}$ the space $C(\mathbb{R})^m \times C(\mathbb{R}, X)$ equipped with the *compact-open topology* (under which a sequence $p_n(t)$ in $\mathcal{Y}$ is convergent *iff* it is uniformly convergent on any compact interval $I \subset \mathbb{R}$). Let θ be the translation operator on $\mathcal{Y}$,

$$\theta_\tau p = p(\cdot + \tau), \qquad \forall p \in \mathcal{Y}, \; \tau \in \mathbb{R}.$$

Set

$$p^*(t) = (r_1(t), \cdots, r_m(t), \, h(t)), \tag{4.27}$$

and assume that $p^*(t)$ is *translation compact* in $\mathcal{Y}$, i.e., the hull

$$\mathcal{H} = \mathcal{H}[p^*] := \overline{\{\theta_\tau p^* : \; \tau \in \mathbb{R}\}}$$

of p^* in $\mathcal{Y}$ is a compact subset of $\mathcal{Y}$.

We also assume that $\mathcal{H}$ is minimal w.r.t θ. This requirement is naturally fulfilled when p^* is, say for instance, periodic, pseudo-periodic, or almost periodic.

Define a function $F : \mathcal{H} \times \mathcal{C}_\alpha \to X$ as

$$F(p, \phi) = f(\phi(-p_1(0)), \cdots, \phi(-p_m(0))) + p_{m+1}(0) \tag{4.28}$$

for any $p = (p_1, \cdots, p_{m+1}) \in \mathcal{H}$. Observing that

$$(r_1(t+\tau), \cdots, r_m(t+\tau), \, h(t+\tau)) = p^*(t+\tau) = (\theta_{t+\tau} p^*)(0),$$

we can rewrite the righthand side of the equation in (4.26) as follows:

$$
\begin{aligned}
& f(v(t-\tilde{r}_1),\cdots,v(t-\tilde{r}_m))+\tilde{h}(t) \\
= \; & F(\theta_{t+\tau}p^*,v_t)=F(\theta_t p,v_t), \qquad p=\theta_\tau p^*.
\end{aligned}
$$

Consequently (4.26) can be reformulated as

$$
\begin{cases} \frac{dv}{dt}+Av=F(\theta_t p,v_t), & t\geq 0,\ \ p\in\{\theta_\tau p^*:\ \tau\in\mathbb{R}\}, \\ v_0=\phi. \end{cases} \tag{4.29}
$$

Since $\{p=\theta_\tau p^*:\ \tau\in\mathbb{R}\}$ is dense in $\mathcal{H}$, for theoretical completeness we usually embed (4.29) into the following cocycle system:

$$
\begin{cases} \frac{dv}{dt}+Av=F(\theta_t p,v_t), & t\geq 0,\ \ p\in\mathcal{H}, \\ v_0=\phi. \end{cases} \tag{4.30}
$$

Now assume f satisfies the following conditions:

(f1) f is *locally Lipschitz*, namely, for any $R>0$, there exists $L_f=L_f(R)>0$ such that for all $u_i,u_i'\in X^\alpha$ $(1\leq i\leq m)$ with $\|u_i\|_\alpha,\|u_i'\|_\alpha\leq R$,

$$
\|f(u_1,\cdots,u_m)-f(u_1',\cdots,u_m')\|_0\leq L_f(\|u_1-u_1'\|_\alpha+\cdots+\|u_m-u_m'\|_\alpha).
$$

(f2) There exist $C_0,C_1>0$ such that

$$
\|f(u_1,\cdots,u_m)\|_0\leq C_0(\|u_1\|_\alpha+\cdots+\|u_m\|_\alpha)+C_1, \qquad \forall u_i\in X^\alpha.
$$

Then one can trivially verify that the mapping F defined by (4.28) satisfies hypotheses (F1) and (F2).

Remark 4.8. Note that if the function p^* in (4.27) is periodic (resp. quasi-periodic, almost periodic), then $\theta_t p$ is periodic (resp. quasi-periodic, almost periodic) for any fixed $p\in\mathcal{H}:=\mathcal{H}[p^*]$. Let Γ be the equilibrium solution of (4.30) given in Theorems 4.6 and 4.7. Then since $\Gamma(q)$ is continuous in q, we deduce that $\gamma_p:=\Gamma(\theta_t p)$ is periodic (resp. quasi-periodic, almost periodic) as well. Therefore these two theorems give the existence of asymptotically stable periodic (resp. pseudo periodic, almost periodic) solutions for equation (4.24).

The interested reader is referred to [19,23,25,26,33,38,40,42–44,53,54] etc. for some classical results and new trends on periodic solutions of delay differential equations, and to [22,32,41,48, 60,62–64] and references therein for typical results on almost periodic solutions.

Remark 4.9. In the case where the functions h and r_i $(1\leq i\leq m)$ in the equation (4.24) are not translation compact (or, the righthand side of the equation takes a more general form like $g(t,u(t-r_1),\cdots,u(t-r_m))$), the framework of cocycle systems does not seem to be quite suitable to handle the problem, because the base space $\mathcal{H}$ of the cocycle system corresponding to the equation may not be compact. Instead, the processes one may be more appropriate.

Set $F(t,\phi)=f(\phi(-r_1),\cdots,\phi(-r_m))+h(t)$ $(t\in\mathbb{R},\ \phi\in\mathcal{C}_\alpha)$. Then (4.24) can be put into a functional one:

$$
\frac{du}{dt}+Au=F(t,u_t). \tag{4.31}
$$

Suppose (4.31) has a unique global solution $x(t; \tau, \phi)$ ($t \in [\tau - r, \infty)$) for each initial data $(\tau, \phi) \in \mathbb{R} \times \mathcal{C}_\alpha$. Denote by $x_t(\tau, \phi)$ the lift of $x(t) := x(t; \tau, \phi)$ in $\mathcal{C}_\alpha$. Then as in (3.29), we can define a process $P(t, \tau)$ on $\mathcal{C}_\alpha$. This allows us to take some steps in the investigation of the dynamics of the equation. For instance, under similar hypotheses as in Section 4.4, it is desirable to prove that the equation has a unique bounded complete (entire) solution $\gamma(t)$ ($t \in \mathbb{R}$) which is uniformly globally (exponentially) asymptotically stable by employing the pullback attractor theory for processes.

The situation becomes quite complicated if the operator A fails to be a dissipative one, i.e., the spectral set σ^- in (4.2) is non-void. One drawback is that both the pullback attractor theory and the Conley index theory are not applicable in proving the existence of bounded complete solutions of the equation. If the delay functions $r_i(t)$ are constants, then since the external force $h(t)$ and the nonlinear term in the righthand side of (4.24) are separate, one may try to get a bounded complete solution $\gamma(t)$ of the equation by considering periodic approximations of $h(t)$. However, if the functions $r_i(t)$ also depend on t, we are not sure whether such a strategy still works. There are also many other interesting questions such as the synchronizing property of the bounded complete solution $\gamma(t)$ with the external force (in case $r_i(t)$ are constant functions) and a more detailed description of the dynamics of the equation. (Note that even if in the case where $h(t)$ and $r_i(t)$ are translation compact, Theorem 4.5 only gives us some information on the asymptotic behavior of bounded solutions of the equation. A natural question is to ask: What can we say about those unbounded solutions?) All these questions deserve to be clarified, and a further study on the geometric theory of functional differential equations in a processes fashion may be helpful for us to take some steps, in which the integral inequality (1.1) may once again play a fundamental role.

4.6. Neural networks with multiple delays

As a concrete example, we consider the following reaction diffusion neural network system with multiple delays:

$$\begin{cases} \frac{\partial u_i}{\partial t} = \operatorname{div}(a_i(x)\nabla u_i) + \sum_{j=1}^n b_{ij} u_j + \\ \qquad\qquad + \sum_{j=1}^n T_{ij} g_j(x, u_j, u_j(x, t - r_{ij})) + J_i(x,t), \\ u_i(x,t) = 0, \qquad t \geq 0, \ x \in \partial\Omega, \qquad i = 1, 2, ..., n. \end{cases} \tag{4.32}$$

Here $\Omega \subset \mathbb{R}^m$ is a bounded domain with a smooth boundary $\partial\Omega$, $a_i \in C^1(\overline{\Omega})$ and is positive everywhere on $\overline{\Omega}$, b_{ij} and T_{ij} are constant coefficients,

$$0 \leq r_{ij} \leq r < \infty, \qquad 1 \leq i, j \leq n,$$

and $J_i(x, t)$ are bounded inputs. We refer the interested reader to [14,52] etc. for a physical background of this type of systems.

Let A_i be the elliptic operator given by

$$A_i u = -\sum_{k=1}^m \frac{\partial}{\partial x_k} \left(a_i(x) \frac{\partial u}{\partial x_k} \right)$$

associated with the corresponding boundary condition. It is a basic knowledge (see e.g. Henry [20, Chap.7]) that A_i is a sectorial operator in $L^2(\Omega)$ with compact resolvent.

For notational simplicity, we use the same notation g_j to denote the Nemytskii operator generated by the function $g_j(x, u, v)$, i.e.,

$$g_j(u, v)(x) = g_j(x, u, v) \ (x \in \Omega), \qquad u, v \in L^2(\Omega).$$

Let $J_i(t) = J_i(\cdot, t)$. Then (4.32) takes a slightly abstract form:

$$\frac{du_i}{dt} + A_i u_i = \sum_{j=1}^{n} b_{ij} u_j + \sum_{j=1}^{n} T_{ij} g_j(u_j, u_j(t - r_{ij})) + J_i(t), \qquad 1 \le i \le n. \tag{4.33}$$

Set $H = \left(L^2(\Omega)\right)^n$, and let $u = (u_1, \cdots, u_n)'$. Denote

$$Au = (A_1 u_1, ..., A_n u_n)', \qquad u \in D(A) \subset H.$$

(It is clear that A is a sectorial operator in H.) Let $\mathcal{C}_0 = C([-r, 0], H)$, and define an operator $G : \mathcal{C}_0 \to H^n = (L^2(\Omega))^{n \times n}$ as follows: $\forall \phi = (\phi_1, \cdots, \phi_n)' \in \mathcal{C}_0$,

$$G(\phi) = (\psi_{ji})_{n \times n}, \qquad \text{where } \psi_{ij} = g_j(\phi_j(0), \phi_j(-r_{ij})).$$

Let $T = \left(T_{ij}\right)_{n \times n}$. Write $TG(\phi) = \left([TG(\phi)]_{ij}\right)_{n \times n}$, and define

$$F(\phi) = (F_1(\phi), F_2(\phi), \cdots, F_n(\phi))', \qquad F_i(\phi) = [TG(\phi)]_{ii}.$$

Then (4.33) can be reformulated as

$$\frac{du}{dt} + Au = Bu + F(u_t) + J(t), \tag{4.34}$$

where $B = \left(b_{ij}\right)_{n \times n}$, and $J = (J_1, \cdots, J_n)'$.

Since (4.34) is nonautonomous, generally the initial value problem reads

$$\begin{cases} \frac{dv}{dt} + Av = Bv + F(v_t) + J(t + \tau), & t \ge 0, \\ v_0 = \phi \in \mathcal{C}_0, \end{cases} \tag{4.35}$$

where $v(t) = u(t + \tau)$, and $\tau \in \mathbb{R}$ denotes the initial time. We assume that J is translation compact in $\mathcal{Y}$. Denote $\mathcal{H}$ the hull $\mathcal{H}[J]$ of the function J in $\mathcal{Y}$. Then as in the previous subsection one can embed (4.35) into the cocycle system:

$$\begin{cases} \frac{dv}{dt} + (A - B)v = F(\theta_t p, v_t), & t \ge 0, \ \ p \in \mathcal{H}, \\ v_0 = \phi \in \mathcal{C}_0, \end{cases} \tag{4.36}$$

where

$$F(p, \phi) = F(\phi) + p(0), \qquad p \in \mathcal{H}, \ \phi \in \mathcal{C}_0.$$

For simplicity, we always assume that $g_j(x,u,v)$ are continuous and *globally Lipschitz* in (u,v) uniformly for $x \in \Omega$, that is, there exists $L_j > 0$ such that

$$|g_j(x,t_1,s_1) - g_j(x,t_2,s_2)| \le L_j(|t_1 - t_2| + |s_1 - s_2|)$$

for all $t_i, s_i \in \mathbb{R}$ and $x \in \Omega$. Then for the Nemytskii operator g_j of the function $g_j(x,u,v)$, we have

$$\|g_j(u_1,v_1) - g_j(u_2,v_2)\|_{L^2(\Omega)} \le L_j(\|u_1 - u_2\|_{L^2(\Omega)} + \|v_1 - v_2\|_{L^2(\Omega)}).$$

Further by some simple calculations it can be shown that

$$\|F(p,\phi) - F(p,\phi')\|_H \le L\|\phi - \phi'\|_{\mathcal{C}_0}$$

with $L = 2\left(\sum_{i=1}^n \left(\sum_{j=1}^n |T_{ij}|L_j\right)^2\right)^{1/2}$. This allows us to carry over all the results on the abstract evolution equation (4.1) to system (4.36). In particular, by Remark 4.8 we have the following theorem.

Theorem 4.10. *Suppose Re $(\sigma(A - B)) \ge \beta > 0$, and that $L < 1/(MI)$, where M is the constant appearing in* (4.3) *corresponding to operator $A - B$, and*

$$I = \sup_{t\ge 0} \int_0^t e^{-\beta(t-s)} ds.$$

Let $J(t) = (J_1(t), \cdots, J_n(t))'$ be a periodic (resp. quasi-periodic, almost periodic) function. Then system (4.32) *has a unique periodic (resp. quasi-periodic, almost periodic) solution γ which is globally uniformly asymptotically stable.*

If we further assume $L < 1/(MI(1 + M))$, then γ is globally exponentially asymptotically stable.

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