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Perturbations of planar quasilinear differential systems ☆

Kenta Itakura [a], Masakazu Onitsuka [b], Satoshi Tanaka [c,*]

[a] *Matsue Yamamoto Metal Co., Ltd, Hokuryo-cho 30, Matsue-shi, Shimane 690–0816, Japan*

[b] *Department of Applied Mathematics, Faculty of Science, Okayama University of Science, Ridai-cho 1–1, Okayama 700–0005, Japan*

[c] *Mathematical Institute, Tohoku University, Aoba 6–3, Aramaki, Aoba-ku, Sendai 980–8578, Japan*

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Dedicated to Professor Tetsutaro Shibata on the occasion of his sixtieth birthday

Abstract

The quasilinear differential system

$$x' = ax + b|y|^{p^*-2}y + k(t, x, y), \quad y' = c|x|^{p-2}x + dy + l(t, x, y)$$

is considered, where a, b, c and d are real constants with $b^2 + c^2 > 0$, p and p^* are positive numbers with $(1/p) + (1/p^*) = 1$, and k and l are continuous for $t \geq t_0$ and small $x^2 + y^2$. When $p = 2$, this system is reduced to the linear perturbed system. It is shown that the behavior of solutions near the origin $(0, 0)$ is very similar to the behavior of solutions to the unperturbed system, that is, the system with $k \equiv l \equiv 0$, near $(0, 0)$, provided k and l are small in some sense. It is emphasized that this system can not be linearized at $(0, 0)$ when $p \neq 2$, because the Jacobian matrix can not be defined at $(0, 0)$. Our result will be applicable to study radial solutions of the quasilinear elliptic equation with the differential operator $r^{-(\gamma-1)}(r^\alpha|u'|^{\beta-a}u')'$, which includes p-Laplacian and k-Hessian.

MSC: 34D05; 34D10; 35J92

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* Corresponding author.

E-mail addresses: onitsuka@xmath.ous.ac.jp (M. Onitsuka), satoshi.tanaka.d4@tohoku.ac.jp (S. Tanaka).

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1. Introduction

In this paper, we consider the quasilinear differential system

$$\begin{aligned} x' &= ax + b\phi_{p^*}(y) + k(t, x, y), \\ y' &= c\phi_p(x) + dy + l(t, x, y), \end{aligned} \tag{1.1}$$

where a, b, c and d are real constants with $b^2 + c^2 > 0$, k and l are continuous for $t \geq t_0$ and small $x^2 + y^2$, p and p^* are positive numbers satisfying

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

and, for $q > 1$, ϕ_q is defined by

$$\phi_q(s) = \begin{cases} |s|^{q-2}s, & s \neq 0, \\ 0, & s = 0. \end{cases}$$

We note here that $p > 1$, $p^* > 1$ and ϕ_{p^*} is the inverse function of ϕ_p. Throughout this paper, we assume that

$$k(t, x, y) = o\left((|x|^p + |y|^{p^*})^{\frac{1}{p}}\right), \quad l(t, x, y) = o\left((|x|^p + |y|^{p^*})^{\frac{1}{p^*}}\right) \tag{1.2}$$

as $|x|^p + |y|^{p^*} \to 0$ uniformly in $t \geq t_0$, that is, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|k(t, x, y)| \leq \varepsilon(|x|^p + |y|^{p^*})^{\frac{1}{p}}, \quad |l(t, x, y)| \leq \varepsilon(|x|^p + |y|^{p^*})^{\frac{1}{p^*}}$$

for $|x|^p + |y|^{p^*} \leq \delta$ and $t \geq t_0$. Then $k(t, 0, 0) = l(t, 0, 0) = 0$ for $t \geq t_0$ and system (1.1) has the zero solution $(x(t), y(t)) \equiv (0, 0)$. For the zero solution, we have the following result which will be proven in Section 2.

Proposition 1.1. *The zero solution* $(x(t), y(t)) \equiv (0, 0)$ *of* (1.1) *is unique, that is,* (1.1) *has no solution* (x, y) *such that* $(x(t_1), y(t_1)) = (0, 0)$ *for some* $t_1 \geq t_0$ *and* $(x(t), y(t)) \not\equiv (0, 0)$.

Remark 1.1. Condition (1.2) of Proposition 1.1 is optimal in the following sense. Gazzola, Serrin and Tang [12, Corollary 1 and see also Theorem 1 (i)] prove that

$$(r^{N-1}\phi_p(u'))' + r^{N-1}(-u^{m-1} + u^{q-1}) = 0$$

has a solution u such that $u(r) > 0$ on $[0, R)$ and $u(r) \equiv 0$ on $[R, \infty)$ for some $R > 0$, where $N > p$ and $1 < m < p < q < pN/(N-p)$. Now we set

$$x(t) = r^{-\frac{p}{p-m}}u(r), \quad y(t) = r^{-\frac{m(p-1)}{p-m}}\phi_p(u'(r)), \quad r = e^t.$$

Then $(x(t), y(t))$ is a solution of

$$x' = -\frac{p}{p-m}x + \phi_{p^*}(y), \quad y' = -\Big(\frac{m(p-1)}{p-m} + N - 1\Big)y + L(t,x), \tag{1.3}$$

where

$$L(t,x) = x^{m-1} - e^{\frac{p(q-m)}{p-m}t}x^{q-1}.$$

We see that $(x(t), y(t)) \not\equiv 0$ on $(-\infty, \log R)$ and $(x(t), y(t)) \equiv 0$ on $[\log R, \infty)$. Therefore, (1.3) neither satisfies (1.2) nor has the unique zero solution.

If $b^2 + c^2 = 0$, then (1.1) is a linear perturbed system, and such a case is not treated in this paper. When $p = 2$, system (1.1) is reduced to the linear perturbation system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k(t,x,y) \\ l(t,x,y) \end{pmatrix}, \tag{1.4}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The study of the linear perturbation theory on differential systems has a long history. For example, the following result is well-known. See [3, Theorem 1.1 in Chapter 13].

Theorem A. *Assume that*

$$k(t,x,y) = o\left(\sqrt{x^2+y^2}\right), \quad l(t,x,y) = o\left(\sqrt{x^2+y^2}\right)$$

as $x^2 + y^2 \to 0$ uniformly in $t \geq t_0$ and every eigenvalue of A has a negative real part, that is, $a + d < 0$. Then the zero solution of (1.4) is exponentially stable. Moreover, if $(a - d)^2 + 4bc < 0$, then the origin $(0,0)$ of (1.4) is a stable focus.

It goes without saying that the linear perturbation theory is bringing us great benefits. On the other hand, the perturbation theory on (1.1) will be applicable to study radial solutions of the quasilinear elliptic equation with the differential operator $r^{-(\gamma-1)}(r^\alpha|u'|^{\beta-a}u')'$, which includes Laplacian, p-Laplacian and k-Hessian. For example, Miyamoto and Takahashi [19] transformed the quasilinear elliptic equation

$$(r^\alpha|u'|^{\beta-1}u')' + r^{\gamma-1}|u|^{p-1}u = 0$$

into the system

$$x' = qx - q\phi_{(\beta+1)^*}(y), \quad y' = -by + b|x|^{p-1}x,$$

where $q = (-\alpha+\beta+\gamma)/(p-\beta)$ and $b = \alpha - \beta(q+1)$. Here, $b|x|^{p-1}x$ is the perturbation term. See also, Bidaut-Véron [1], Cîrstea [2], Flores and Franca [7], Franca [8], [9], [10], and Miyamoto [18]. In Section 9, we consider the quasilinear elliptic equation

$$\mathrm{div}(|x|^\alpha|\nabla u|^{p-2}\nabla u) + \frac{c}{|x|^{p-\alpha}}|u|^{p-2}u + |x|^\beta|u|^{q-2}u = 0 \quad \text{in } \mathbf{R}^N - \{0\}, \tag{1.5}$$

where $N > p$, $q > p > 1$ and $\alpha, \beta, c \in \mathbf{R}$. Radial solutions of (1.5) satisfy

$$r^{-(N-1)}(r^{N-1+\alpha}\phi_p(u'))' + \frac{c}{r^{p-\alpha}}\phi_p(u) + r^\beta\phi_q(u) = 0, \quad r > 0. \tag{1.6}$$

We assume that $p > \alpha - \beta$ and $N - p + \alpha > 0$. By setting $a = (p - \alpha + \beta)/(q - p)$, $\eta = (a+1)(p-1)$, $x(t) = r^a u(r)$, $y(t) = -r^\eta\phi_p(u'(r))$ and $r = e^t$, equation (1.6) is transformed into

$$x' = ax - \phi_{p^*}(y), \quad y' = c\phi_p(x) + (\eta - N + 1 - \alpha)y + \phi_q(x). \tag{1.7}$$

Conversely, if $(x(t), y(t))$ is a solution of (1.7), then $u(r) := r^{-a}x(\log r)$ is a solution of (1.6) and $u'(r) = -ar^{-a-1}\phi_{p^*}(y(\log r))$. In section 9, we apply our results to (1.7). Especially, we find a continuum of singular solutions of (1.5) with $c = 0$. Such a result was obtained by Franca [8] when $\alpha = 0$ and Troy and Krisner [30] when $\alpha = \beta = 0$ and $p = 2$.

It is natural to expect that the behavior of solutions of (1.1) near the origin $(0, 0)$ is very similar to the behavior of solutions to the unperturbed system

$$x' = ax + b\phi_{p^*}(y), \quad y' = c\phi_p(x) + dy \tag{1.8}$$

near $(0, 0)$, provided k and l are small in some sense. It is emphasized that this system can not be linearized at $(0, 0)$ when $p \neq 2$, because the Jacobian matrix

$$\begin{pmatrix} a & b(p^*-1)|y|^{p^*-2} \\ c(p-1)|x|^{p-2} & d \end{pmatrix}$$

can not be defined at $(0, 0)$.

There are a lot of studies on quasilinear differential equations. Sometimes, however, it can not be linearized by the nonlinearity of its differential operator. To overcome such a problem, it is worthwhile to investigate the perturbed system (1.1).

The system (1.8) is studied in [6], [22] and [27]. The nonautonomous case is treated in [5], [13], [14], [15], [16], [17], [20], [21], [23], [24], [25], [26], and [29]. These studies succeeded in generalizing results on linear systems. Especially, in [22], the function

$$\begin{aligned} f(\lambda) &= \phi_p(\lambda - a)[(p-1)\lambda - d] - \phi_p(b)c \\ &= \frac{1}{p^*}\Big[p|\lambda - a|^p + (pa - p^*d)\phi_p(\lambda - a) - p^*\phi_p(b)c\Big] \end{aligned} \tag{1.9}$$

is introduced and it is found that the equation

$$f(\lambda) = 0 \tag{1.10}$$

plays the role of the characteristic equation for (1.8). Moreover, (1.10) and its root are regarded as generalizations of the characteristic equation and its eigenvalue, respectively. According to [22], we define the following constants

$$T := a + d, \quad D := \phi_p(a)d - \phi_p(b)c, \quad \Delta := \left| \frac{a}{p^*} - \frac{d}{p} \right|^p + \phi_p(b)c.$$

In the case where $p = 2$, setting $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we find that $T = \mathrm{tr} A$, $D = \det A$ and

$$4\Delta = (a - d)^2 + 4bc = (\mathrm{tr} A)^2 - 4 \det A,$$

which is the discriminant of the characteristic equation

$$(\lambda - a)(\lambda - d) - bc = 0.$$

By the proof of Proposition 1.3 in [22], we find that

$$\min_{\lambda \in \mathbf{R}} f(\lambda) = f(T/p) = -\Delta.$$

Since $f(0) = D$, we note that

$$D > -\Delta. \tag{1.11}$$

The following two results are obtained in [22, Proposition 1.3 and Corollary 1.1].

Proposition B. *The following* (i)–(iii) *hold*:

(i) *if* $\Delta < 0$, *then* (1.10) *has no real root and* $f(\lambda) > 0$ *on* $\mathbf{R}$;
(ii) *if* $\Delta = 0$, *then* (1.10) *has a unique real root* $\lambda = T/p$ *and* $f(\lambda) > 0$ *on* $(-\infty, T/p) \cup (T/p, \infty)$;
(iii) *if* $\Delta > 0$, *then* (1.10) *has two real roots* λ_1 *and* λ_2 *with* $\lambda_1 < \lambda_2$, $f(\lambda) < 0$ *on* (λ_1, λ_2), *and* $f(\lambda) > 0$ *on* $(-\infty, \lambda_1) \cup (\lambda_2, \infty)$.

Moreover, in the case $\Delta > 0$, *the following* (a)–(c) *hold*:

(a) *if* $D < 0$, *then* $\lambda_1 < 0 < \lambda_2$;
(b) *if* $D > 0$ *and* $T < 0$, *then* $\lambda_1 < \lambda_2 < 0$;
(c) *if* $D > 0$ *and* $T > 0$, *then* $0 < \lambda_1 < \lambda_2$.

Theorem C. *The origin* $(0, 0)$ *of system* (1.8) *is classified as follows*:

(i) *if* $D < 0$, *then* $(0, 0)$ *is a saddle*;
(ii) *if* $D > 0$, $\Delta > 0$ *and* $T < 0$, *then* $(0, 0)$ *is a stable node*;
(iii) *if* $D > 0$, $\Delta > 0$ *and* $T > 0$, *then* $(0, 0)$ *is an unstable node*;
(iv) *if* $\Delta < 0$ *and* $T < 0$, *then* $(0, 0)$ *is a stable focus*;
(v) *if* $\Delta < 0$ *and* $T = 0$, *then* $(0, 0)$ *is a center*;
(vi) *if* $\Delta < 0$ *and* $T > 0$, *then* $(0, 0)$ *is an unstable focus*.

The main results of this paper are as follows.

Theorem 1.1. *Assume that $T < 0$ and $D > 0$. Then the zero solution of system (1.1) is exponentially stable.*

Theorem 1.2. *Assume that $T < 0$ and $\Delta < 0$. Then the origin $(0, 0)$ of system (1.1) is a stable focus in the following sense. The zero solution of system (1.1) is exponentially stable and every solution $(x(t), y(t))$ near the origin is rotating infinitely around the origin in a clockwise [respectively counter-clockwise] direction as $t \to \infty$ when $b > 0$ [respectively $b < 0$].*

Theorem 1.3. *Assume that $T < 0$, $D > 0$ and $\Delta > 0$. Then the origin $(0, 0)$ of system (1.1) is a stable node in the following sense. The zero solution of system (1.1) is exponentially stable, (1.10) has two real roots λ_1 and λ_2 with $\lambda_1 < \lambda_2 < 0$, and every solution $(x(t), y(t))$ of (1.1) near the origin satisfies*

$$\lim_{t\to\infty} \frac{\phi_{p^*}(y(t))}{x(t)} = \frac{\lambda_i - a}{b} \tag{1.12}$$

for some $i \in \{1, 2\}$ when $b \neq 0$, and

$$\lim_{t\to\infty} \frac{\phi_p(x(t))}{y(t)} = 0 \quad \text{or} \quad \frac{(p-1)a - d}{c}$$

when $b = 0$. Moreover, for every solution $(x(t), y(t))$ of (1.1) near the origin and each $\varepsilon > 0$, there exist $t_1 \geq t_0$, $i \in \{1, 2\}$ and $C_0 > 0$ such that

$$C_0 e^{p(\lambda_i - \varepsilon)t} \leq |x(t)|^p + |y(t)|^{p^*} \leq C_0 e^{p(\lambda_i + \varepsilon)t}, \quad t \geq t_1, \tag{1.13}$$

and, in particular, if $b \neq 0$, then there exist $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 e^{(\lambda_i - \varepsilon)t} \leq |x(t)| \leq C_2 e^{(\lambda_i + \varepsilon)t}, \quad t \geq t_1. \tag{1.14}$$

Remark 1.2. In Theorems 1.2 and 1.3, a focus and a node may be not standard ones, respectively. Namely, the focus in Theorem 1.2 might not have the topological conjugation property to the unperturbed system, and the stable manifold in Theorem 1.3 might not be 1 dimensional.

Remark 1.3. We can consider the case where $t \to -\infty$ in the following way. If $(x(t), y(t))$ is a solution of (1.1) on $(-\infty, t_0]$, then $(x(-t), y(-t))$ is a solution of

$$x' = -ax - b\phi_{p^*}(y) - k(-t, x, y), \quad y' = -c\phi_p(x) - dy - l(-t, x, y)$$

on $[-t_0, \infty)$.

The existence of a local solution $(x(t), y(t))$ to (1.1) with an initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$ is guaranteed by the Peano existence theorem. By a standard argument on a general theory on ordinary differential equations, the solution $(x(t), y(t))$ is continuable whenever it is bounded.

When the origin $(0, 0)$ is a saddle, we need more assumptions. We consider the autonomous differential system

$$x' = ax + b\phi_{p^*}(y) + k(x, y), \quad y' = c\phi_p(x) + dy + l(x, y), \tag{1.15}$$

where a, b, c and d are real constants with $b^2 + c^2 > 0$, k and l are continuous in the neighborhood of $(0, 0)$, p and p^* are positive numbers satisfying $(1/p) + (1/p^*) = 1$.

Theorem 1.4. *Let $D < 0$. Assume that k and l are continuous in the neighborhood of $(0, 0)$ and*

$$k(x, y) = o\left((|x|^p + |y|^{p^*})^{\frac{2}{p}}\right), \quad l(x, y) = o\left(|x|^p + |y|^{p^*}\right) \tag{1.16}$$

as $|x|^p + |y|^{p^*} \to 0$. Then the origin $(0, 0)$ of system (1.15) is a saddle in the following sense. Equation (1.10) has two real roots λ_1 and λ_2 with $\lambda_1 < 0 < \lambda_2$, and there exists $\delta > 0$ such that every solution $(x(t), y(t))$ of (1.15) with $0 < |x(t_1)|^p + |y(t_1)|^{p^*} < \delta$ for some $t_1 \in \mathbf{R}$ satisfies one of the following (i)–(iii):

(i) *there exist t_0 and t_2 such that $t_0 < t_1 < t_2$ and $|x(t_i)|^p + |y(t_i)|^{p^*} > \delta$ for $i = 0, 2$;*

(ii) $(x(t), y(t)) \to (0, 0)$ *as $t \to \infty$ and*

$$\lim_{t\to\infty} \frac{\phi_{p^*}(y(t))}{x(t)} = \frac{\lambda_1 - a}{b} \quad \text{when } b \neq 0,$$
$$\lim_{t\to\infty} \frac{\phi_p(x(t))}{y(t)} = 0 \quad \text{when } b = 0,\ a > 0,$$
$$\lim_{t\to\infty} \frac{\phi_p(x(t))}{y(t)} = \frac{(p-1)a - d}{c} \quad \text{when } b = 0,\ a < 0,$$

and for each $\varepsilon > 0$, there exist $t_2 \geq t_1$ and $C_0 > 0$ such that

$$C_0 e^{p(\lambda_1 - \varepsilon)t} \leq |x(t)|^p + |y(t)|^{p^*} \leq C_0 e^{p(\lambda_1 + \varepsilon)t}, \quad t \geq t_2,$$

and, in particular, if $b \neq 0$, then there exist $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 e^{(\lambda_1 - \varepsilon)t} \leq |x(t)| \leq C_2 e^{(\lambda_1 + \varepsilon)t}, \quad t \geq t_2;$$

(iii) $(x(t), y(t)) \to (0, 0)$ *as $t \to -\infty$ and*

$$\lim_{t\to-\infty} \frac{\phi_{p^*}(y(t))}{x(t)} = \frac{\lambda_2 - a}{b} \quad \text{when } b \neq 0,$$
$$\lim_{t\to-\infty} \frac{\phi_p(x(t))}{y(t)} = \frac{(p-1)a - d}{c} \quad \text{when } b = 0,\ a > 0,$$
$$\lim_{t\to-\infty} \frac{\phi_p(x(t))}{y(t)} = 0 \quad \text{when } b = 0,\ a < 0,$$

and for each $\varepsilon > 0$, there exist $t_0 \leq t_1$ and $C_0 > 0$ such that

$$C_0 e^{p(\lambda_2 + \varepsilon)t} \leq |x(t)|^p + |y(t)|^{p^*} \leq C_0 e^{p(\lambda_2 - \varepsilon)t}, \quad t \leq t_0,$$

and, in particular, if $b \neq 0$*, then there exist* $C_1 > 0$ *and* $C_2 > 0$ *such that*

$$C_1 e^{(\lambda_2+\varepsilon)t} \le |x(t)| \le C_2 e^{(\lambda_2-\varepsilon)t}, \quad t \le t_0.$$

Moreover, there exist solutions $(x(t), y(t))$ *of* (1.15) *satisfying each one of* (ii) *and* (iii).

Remark 1.4. If $D < 0$ and $b = 0$, then $a \neq 0$.

In Section 2, the generalized Prüfer transformation is introduced and the proof of Proposition 1.1 is given. In Section 3, we give lemmas which play crucial roles in this paper. In Sections 4, 5 and 6, we consider the cases where $\Delta < 0$, $\Delta > 0$ and $\Delta = 0$, respectively. In Section 7, we prove Theorems 1.1, 1.2 and 1.3. In Section 8, we give a proof of Theorem 1.4. In Section 9, we consider radial solutions of (1.5) and apply our results to (1.7).

2. Prüfer transformation

Let $p > 1$. A generalization of the classical trigonometric functions, denoted by $\sin_p$ and $\cos_p$ is well-known. See, for example, [4], [5], [6] and [28]. The function $\sin_p$ is defined as the solution of the initial value problem

$$(\phi_p(S'))' + (p-1)\phi_p(S) = 0, \quad S(0) = 0, \; S'(0) = 1.$$

The function $\sin_p$ is defined on $\mathbf{R}$ and is periodic with period $2\pi_p$, where $\pi_p = 2\pi/(p\sin(\pi/p))$. Further, $\sin_p x$ is an odd function having zeros at $x = j\pi_p$, $j \in \mathbf{Z}$; it is positive for $2j\pi_p < x < (2j+1)\pi_p$, $j \in \mathbf{Z}$, and negative for $(2j+1)\pi_p < x < 2(j+1)\pi_p$, $j \in \mathbf{Z}$. The function $\cos_p$ is defined by $\cos_p x = (\sin_p x)'$. Then

$$\begin{gathered} \sin_p(x+\pi_p) = -\sin_p x, \quad \cos_p(x+\pi_p) = -\cos_p x, \\ (\phi_p(\cos_p x))' = -(p-1)\phi_p(\sin_p x) \end{gathered} \tag{2.1}$$

and

$$(\cos_p x)' = (\phi_{p^*}(\phi_p(\cos_p x)))' = -|\cos_p x|^{2-p}\phi_p(\sin_p x) \tag{2.2}$$

if $\cos_p x \neq 0$ or $1 < p \le 2$. Since

$$(|\sin_p x|^p + |\cos_p x|^p)' = (|\sin_p x|^p + |\phi_p(\cos_p x)|^{p^*})' = 0,$$

the generalized Pythagorean identity holds:

$$|\sin_p x|^p + |\cos_p x|^p = |\sin_p x|^p + |\phi_p(\cos_p x)|^{p^*} = 1, \quad x \in \mathbf{R}. \tag{2.3}$$

Hence,

$$|\sin_p x| \le 1, \; |\cos_p x| \le 1, \quad x \in \mathbf{R}.$$

Now we make use of the generalized Prüfer transformation, which was first introduced by Elbert [5]. Let (x, y) be a non-zero solution of (1.1). We define the functions r and θ by

$$x(t) = r(t)\sin_p \theta(t), \quad y(t) = \phi_p(r(t)\cos_p \theta(t)), \tag{2.4}$$

where

$$r(t) = (|x(t)|^p + |y(t)|^{p^*})^{\frac{1}{p}} > 0.$$

Then

$$\begin{aligned} p(r(t))^{p-1}r'(t) &= [(r(t))^p]' \\ &= (|x(t)|^p + |y(t)|^{p^*})' \\ &= p\phi_p(x(t))x'(t) + p^*\phi_{p^*}(y(t))y'(t) \\ &= pa|x(t)|^p + (pb + p^*c)\phi_p(x(t))\phi_{p^*}(y(t)) + p^*d|y(t)|^{p^*} \\ &\quad + p\phi_p(x(t))k(t, x(t), y(t)) + p^*\phi_{p^*}(y(t))l(t, x(t), y(t)). \end{aligned}$$

Now we use the notation:

$$\begin{aligned} G(\theta) &:= pa|\sin_p \theta|^p + (pb + p^*c)\phi_p(\sin_p \theta)\cos_p \theta + p^*d|\cos_p \theta|^p; \\ F(\theta) &:= pb|\cos_p \theta|^p + (pa - p^*d)\sin_p \theta\phi_p(\cos_p \theta) - p^*c|\sin_p \theta|^p; \\ K(t, r, \theta) &:= k(t, r\sin_p \theta, \phi_p(r\cos_p \theta)); \\ L(t, r, \theta) &:= l(t, r\sin_p \theta, \phi_p(r\cos_p \theta)). \end{aligned}$$

We find that

$$\begin{aligned} r' &= \frac{1}{p}r^{1-p}[pa|x|^p + (pb + p^*c)\phi_p(x)\phi_{p^*}(y) + p^*d|y|^{p^*}] \\ &\quad + r^{1-p}\phi_p(x)k(t, x, y) + \frac{p^*}{p}r^{1-p}\phi_{p^*}(y)l(t, x, y) \\ &= \frac{1}{p}G(\theta)r + \phi_p(\sin_p \theta)K(t, r, \theta) + \frac{1}{p-1}r^{2-p}\cos_p \theta L(t, r, \theta). \end{aligned}$$

From (2.4) it follows that

$$x' = r'\sin_p \theta + r\theta'\cos_p \theta$$

and

$$\begin{aligned} y' &= (r^{p-1}\phi_p(\cos_p \theta))' \\ &= (p-1)r^{p-2}r'\phi_p(\cos_p \theta) - (p-1)r^{p-1}\theta'\phi_p(\sin_p \theta), \end{aligned}$$

which imply

$$r^{-1}\phi_p(\cos_p\theta)x' - \frac{1}{p-1}r^{1-p}\sin_p\theta y' = \theta'.$$

Hence, by (1.1), we have

$$\begin{aligned}\theta' &= r^{-1}\phi_p(\cos_p\theta)[ar\sin_p\theta + br\cos_p\theta + K(t,r,\theta)]\\ &\quad - \frac{1}{p-1}r^{1-p}\sin_p\theta[c\phi_p(r\sin_p\theta) + d\phi_p(r\cos_p\theta) + L(t,r,\theta)]\\ &= \frac{1}{p}F(\theta) + r^{-1}\phi_p(\cos_p\theta)K(t,r,\theta) - \frac{1}{p-1}r^{1-p}\sin_p\theta L(t,r,\theta).\end{aligned}$$

Consequently, a solution (x, y) of (1.1) satisfying $(x(t), y(t)) \neq (0, 0)$ is transformed to a solution of the system

$$r' = \frac{1}{p}G(\theta)r + M(t,r,\theta), \quad \theta' = \frac{1}{p}F(\theta) + N(t,r,\theta), \tag{2.5}$$

where

$$\begin{aligned}M(t,r,\theta) &:= \phi_p(\sin_p\theta)K(t,r,\theta) + (p^*-1)r^{2-p}\cos_p\theta L(t,r,\theta),\\ N(t,r,\theta) &:= r^{-1}\phi_p(\cos_p\theta)K(t,r,\theta) - (p^*-1)r^{1-p}\sin_p\theta L(t,r,\theta).\end{aligned}$$

Lemma 2.1. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that M and N are continuous for $t \geq t_0$, $0 \leq r \leq 2\delta$ and $\theta \in \mathbf{R}$, and*

$$|M(t,r,\theta)| \leq p^*\varepsilon r, \quad |N(t,r,\theta)| \leq p^*\varepsilon \tag{2.6}$$

for $0 \leq r \leq \delta$ and $t \geq t_0$.

Proof. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|K(t,r,\theta)| \leq \varepsilon r, \quad |L(t,r,\theta)| \leq \varepsilon r^{p-1}$$

for $0 \leq r \leq \delta$ and $t \geq t_0$. Hence, if $0 \leq r \leq \delta$ and $t \geq t_0$, then

$$\begin{aligned}|M(t,r,\theta)| &\leq |\phi_p(\sin_p\theta)||K(t,r,\theta)| + (p^*-1)r^{2-p}|\cos_p\theta||L(t,r,\theta)|\\ &\leq \varepsilon r + (p^*-1)r^{2-p}\varepsilon r^{p-1}\\ &= p^*\varepsilon r\end{aligned}$$

and

$$\begin{aligned}|N(t,r,\theta)| &\leq r^{-1}|\phi_p(\cos_p\theta)||K(t,r,\theta)| + (p^*-1)r^{1-p}|\sin_p\theta||L(t,r,\theta)|\\ &\leq r^{-1}\varepsilon r + (p^*-1)r^{1-p}\varepsilon r^{p-1}\\ &= p^*\varepsilon. \quad \square\end{aligned}$$

Lemma 2.2. *Let $(r(t), \theta(t))$ be a solution of (2.5) and let I be the maximal interval of the existence for $(r(t), \theta(t))$. If $r(t_1) > 0$ for some $t_1 \in I$, then $r(t) > 0$ on I.*

Proof. Assume that there exists $t_2 \in I$ such that $r(t_2) = 0$. Let $\varepsilon := 1/p^*$. By Lemma 2.1, there exists $\delta > 0$ such that $M(t, r, \theta) \le r$ for $0 \le r \le \delta$ and $t \ge t_0$. Since $r(t_2) = 0$, there exists an interval $J \subset I$ such that $0 \le r(t) \le \delta$ on J and $0 < r(t_3) \le \delta$ for some $t_3 \in J$. Since $G(\theta)$ is periodic, we can take constants $G_1 > 0$ for which

$$|G(\theta)| \le G_1 \quad \text{for } \theta \in \mathbf{R}.$$

Set $C := (G_1/p) + 1$. From (2.5) it follows that

$$-Cr(t) \le r'(t) \le Cr(t), \quad t \in J,$$

that is,

$$(e^{Ct} r(t))' \ge 0, \quad (e^{-Ct} r(t))' \le 0, \quad t \in J.$$

Hence, if $t \le t_2$ and $t \in J$, then

$$e^{Ct} r(t) \le e^{Ct_2} r(t_2) = 0$$

and if $t \ge t_2$ and $t \in J$, then

$$e^{-Ct} r(t) \le e^{-Ct_2} r(t_2) = 0.$$

Consequently $r(t) \le 0$ on J. This contradicts the fact that $r(t_3) > 0$ and $t_3 \in J$. □

Proof of Proposition 1.1. Proposition 1.1 follows from Lemma 2.2 immediately. □

3. Crucial lemmas

The following lemma has been obtained in [22, Lemma 4.1].

Lemma 3.1. *The functions $F(\theta)$ and $G(\theta)$ satisfy $F'(\theta) = -pG(\theta) + pT$.*

Since $F(\theta)$ and $G(\theta)$ are periodic, we can take constants $F_1 > 0$ and $G_1 > 0$ such that

$$|F(\theta)| \le F_1, \quad |G(\theta)| \le G_1 \quad \text{for } \theta \in \mathbf{R}. \tag{3.1}$$

The following two lemmas play crucial roles in this paper.

Lemma 3.2. *Let F_1 and G_1 be constants as in (3.1). Assume that $b \ne 0$, $T < 0$ and that there exist constants $F_0 > 0$, α, β such that $\alpha < \beta$ and*

$$|F(\theta)| \ge F_0, \quad \theta \in [\alpha, \beta].$$

Let $\varepsilon > 0$ *satisfy* $T + C_0\varepsilon < 0$*, where*

$$C_0 := \frac{pp^*}{F_0}(F_0 + |T| + G_1), \tag{3.2}$$

and let $\delta > 0$ *be as in Lemma 2.1. Then every solution* $(r(t), \theta(t))$ *of* (2.5) *with* $0 < r(t_1) < (F_0/F_1)^{1/p}\delta$ *and* $\alpha < \theta(t_1) < \beta$ *for some* $t_1 \ge t_0$ *satisfies*

$$0 < r(t) \le r(t_1)\left(\frac{F_1}{F_0}\right)^{1/p} e^{(T+C_0\varepsilon)(t-t_1)/p} < \delta e^{(T+C_0\varepsilon)(t-t_1)/p} \tag{3.3}$$

for $t \in [t_1, t_2)$*, where* $t_2 = \sup\{s \ge t_1 : \alpha < \theta(t) < \beta \text{ on } [t_1, s)\}$.

Proof. Let $(r(t), \theta(t))$ be a solution of (2.5) with

$$0 < r(t_1) < (F_0/F_1)^{1/p}\delta, \quad \alpha < \theta(t_1) < \beta.$$

Since $0 < F_0 \le F_1$, we note that

$$(F_0/F_1)^{1/p}\delta \le \delta.$$

Define $t_* \ge t_1$ by

$$t_* := \sup\{s \ge t_1 : 0 < r(t) < \delta,\ \alpha < \theta(t) < \beta \text{ on } [t_1, s)\}.$$

Then $t_1 < t_* \le t_2$. Lemma 3.1 implies that

$$\begin{aligned}\frac{d}{dt}\Big((r(t))^p F(\theta(t))\Big) &= T(r(t))^p F(\theta(t)) + p(r(t))^{p-1}F(\theta(t))M(t, r(t), \theta(t)) \\ &\quad + p(r(t))^p[T - G(\theta(t))]N(t, r(t), \theta(t)).\end{aligned}$$

We note that $F(\theta) > 0$ or $F(\theta) < 0$ on $[\alpha, \beta]$. We define σ by

$$\sigma = \begin{cases} +1, & \text{if } F(\theta) > 0 \text{ on } [\alpha, \beta], \\ -1, & \text{if } F(\theta) < 0 \text{ on } [\alpha, \beta]. \end{cases}$$

Set

$$w(t) := (r(t))^p \sigma F(\theta(t)) = (r(t))^p |F(\theta(t))| > 0, \quad t \in [t_1, t_*).$$

Then

$$w'(t) = Tw(t) + p(r(t))^{p-1}|F(\theta(t))|M(t, r(t), \theta(t)) + \sigma p(r(t))^p[T - G(\theta)]N(t, r(t), \theta(t)).$$

From (2.6) it follows that

$$
\begin{aligned}
&\Big| p(r(t))^{p-1}|F(\theta(t))|M(t,r(t),\theta(t)) \\
&\qquad + \sigma p(r(t))^{p}[T-G(\theta)]N(t,r(t),\theta(t))\Big| \\
&\quad \le p(r(t))^{p-1}|M(t,r(t),\theta(t))||F(\theta(t))| \\
&\qquad + p(r(t))^{p}(|T|+|G(\theta)|)|N(t,r(t),\theta(t))| \\
&\quad \le pp^{*}\varepsilon(r(t))^{p}|F(\theta(t))| + \frac{|T|+G_1}{F_0}pp^{*}\varepsilon(r(t))^{p}|F(\theta(t))| \\
&\quad = C_0\varepsilon w(t), \quad t\in[t_1,t_*).
\end{aligned}
$$

Hence,

$$
w'(t)\le (T+C_0\varepsilon)w(t), \quad t\in[t_1,t_*),
$$

that is,

$$
(e^{-(T+C_0\varepsilon)(t-t_1)}w(t))'\le 0, \quad t\in[t_1,t_*). \tag{3.4}
$$

Integrating (3.4) on $[t_1,t]$, we have

$$
\begin{aligned}
w(t)\le e^{(T+C_0\varepsilon)(t-t_1)}w(t_1) &= e^{(T+C_0\varepsilon)(t-t_1)}(r(t_1))^{p}|F(\theta(t_1))| \\
&\le e^{(T+C_0\varepsilon)(t-t_1)}(r(t_1))^{p}F_1, \quad t\in[t_1,t_*).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(r(t))^{p} \le \frac{(r(t_1))^{p}F_1e^{(T+C_0\varepsilon)(t-t_1)}}{|F(\theta(t))|} &\le (r(t_1))^{p}\frac{F_1}{F_0}e^{(T+C_0\varepsilon)(t-t_1)} \\
&< \delta^{p}e^{(T+C_0\varepsilon)(t-t_1)}, \quad t\in[t_1,t_*).
\end{aligned}
$$

Consequently, recalling Lemma 2.2, we obtain (3.3) for $t\in[t_1,t_*)$. This shows that $t_*=t_2$ and completes the proof. □

Lemma 3.3. *Let F_1 and G_1 be constants as in (3.1). Assume that $b\neq 0$, $T<0$ and $|F(\theta)|\ge F_0$ on* **R** *for some $F_0>0$. Let $\varepsilon>0$ satisfy $T+C_0\varepsilon<0$, where C_0 is a constant defined by (3.2) and let $\delta>0$ be as in Lemma 2.1. Then every solution $(r(t),\theta(t))$ of (2.5) with $0<r(t_1)<(F_0/F_1)^{1/p}\delta$ for some $t_1\ge t_0$ satisfies*

$$
0<r(t)<\delta e^{(T+C_0\varepsilon)(t-t_1)/p}, \quad t\ge t_1.
$$

Proof. By Lemma 3.2, we obtain (3.3) whenever $\theta(t)$ is bounded. We assume that there exists $t_3>t_1$ such that $\theta(t)$ is bounded on $[t_1,s]$ for each $s\in(t_1,t_3)$ and

$$
\limsup_{t\to t_3^-}|\theta(t)|=\infty. \tag{3.5}
$$

Since (3.3) holds on $[t_1, s]$ for each $s \in (t_1, t_3)$, we note that $0 < r(t) < \delta$ for $t \in [t_1, t_3)$. By (2.5), we have

$$|\theta'(t)| \le \frac{F_1}{p} + p^*\varepsilon =: C_1, \quad t \in [t_1, t_3).$$

Hence,

$$\theta(t_1) - C_1(t - t_1) \le \theta(t) \le \theta(t_1) + C_1(t - t_1), \quad t \in [t_1, t_3).$$

This contradicts (3.5). Therefore, (3.3) holds on $[t_1, \infty)$. □

4. Case $\Delta < 0$

In this section we consider the case $\Delta < 0$. We note that $b \neq 0$, provided $\Delta < 0$, by the definition of Δ. The following lemma is obtained in [22, Lemma 5.1].

Lemma 4.1. *Assume that $\Delta < 0$. Then $bF(\theta) > 0$ for $\theta \in \mathbf{R}$.*

Theorem 4.1. *Assume that $T < 0$ and $\Delta < 0$. Then there exist $\delta > 0$, $\rho > 0$ and $\mu > 0$ such that every solution $(r(t), \theta(t))$ of (2.5) with $0 < r(t_1) < \rho\delta$ for some $t_1 \ge t_0$ satisfies*

$$0 < r(t) < \delta e^{-\mu(t-t_1)}, \quad t \ge t_1 \tag{4.1}$$

and $b\theta(t) \to \infty$ as $t \to \infty$.

Proof. Since $F(\theta)$ and $G(\theta)$ are periodic, Lemma 4.1 implies that there exist constants $F_0 > 0$, $F_1 > 0$ and $G_1 > 0$ such that

$$F_0 \le |F(\theta)| \le F_1, \quad |G(\theta)| \le G_1 \quad \text{for } \theta \in \mathbf{R}. \tag{4.2}$$

We take $\varepsilon > 0$ satisfying $T + C_0\varepsilon < 0$ and $pp^*\varepsilon < F_0$, where C_0 is a constant defined by (3.2). Recall $b \neq 0$ by $\Delta < 0$. By Lemma 2.1 and 3.3, there exists $\delta > 0$ such that (2.6) holds for $0 \le r \le \delta$ and $t \ge t_0$ and that every solution $(r(t), \theta(t))$ of (2.5) with $0 < r(t_1) < \rho\delta$ for some $t_1 \ge t_0$ satisfies (4.1), where $\rho := (F_0/F_1)^{1/p}$ and $\mu := -(T + C_0\varepsilon)/p > 0$.

Let $(r(t), \theta(t))$ be a solution of (2.5) with $0 < r(t_1) < \rho\delta$. Lemma 4.1 shows that $bF(\theta) = |bF(\theta)| = |b||F(\theta)|$ for $\theta \in \mathbf{R}$. Hence, from (2.5) and (2.6), it follows that

$$b\theta'(t) = \frac{1}{p}|b||F(\theta(t))| + bN(t, r, \theta) \ge \frac{1}{p}|b|F_0 - |b|p^*\varepsilon =: c_0 > 0$$

for $t \ge t_1$. Integrating these inequalities on $[t_1, t]$, we obtain

$$b\theta(t) \ge b\theta(t_1) + c_0(t - t_1), \quad t \ge t_1,$$

which shows that $b\theta(t) \to \infty$ as $t \to \infty$. □

5. Case $\Delta > 0$

In this section we consider the case where $\Delta > 0$. Assume that $b \neq 0$. Since $\cos_p \theta / \sin_p \theta$ is strictly decreasing on $(0, \pi_p)$, we have

$$\lim_{\theta \to 0^+} \frac{\cos_p \theta}{\sin_p \theta} = \infty \quad \text{and} \quad \lim_{\theta \to \pi_p^-} \frac{\cos_p \theta}{\sin_p \theta} = -\infty.$$

Therefore, for each $\lambda_0 \in \mathbf{R}$, there exists a unique $\theta_0 \in (0, \pi_p)$ such that

$$\frac{b \cos_p \theta_0}{\sin_p \theta_0} + a = \lambda_0. \tag{5.1}$$

Lemma 5.1. *Assume that $b \neq 0$. Let λ_0 be a real root of (1.10) with $\lambda_0 \neq 0$ and let $\theta_0 \in (0, \pi_p)$ satisfy (5.1). Then $G(\theta_0) = p\lambda_0$.*

Proof. Since $\cos_p \theta_0 = b^{-1}(\lambda_0 - a) \sin_p \theta_0$, we find that

$$\begin{aligned} G(\theta_0) &= pa|\sin_p \theta_0|^p + (pb + p^* c)\phi_p(\sin_p \theta_0) b^{-1}(\lambda_0 - a) \sin_p \theta_0 \\ &\quad + p^* d |b^{-1}(\lambda_0 - a) \sin_p \theta_0|^p \\ &= |b|^{-p} |\sin_p \theta_0|^p g(\lambda_0), \end{aligned}$$

where

$$\begin{aligned} g(\lambda) &:= p^* d |\lambda - a|^p + \phi_p(b)(pb + p^* c)(\lambda - a) + pa|b|^p \\ &= p^* d(\lambda - a)\phi_p(\lambda - a) + p|b|^p(\lambda - a) + p^* \phi_p(b) c(\lambda - a) + pa|b|^p \\ &= p^*(\lambda - a)[d\phi_p(\lambda - a) + \phi_p(b) c] + p|b|^p \lambda. \end{aligned}$$

Note that $f(\lambda_0) = 0$ is equivalent to

$$d\phi_p(\lambda_0 - a) + \phi_p(b) c = (p-1)\lambda_0 \phi_p(\lambda_0 - a).$$

Hence,

$$\begin{aligned} g(\lambda_0) &= p^*(\lambda_0 - a)(p-1)\lambda_0 \phi_p(\lambda_0 - a) + p|b|^p \lambda_0 \\ &= p\lambda_0(|\lambda_0 - a|^p + |b|^p). \end{aligned}$$

Consequently,

$$\begin{aligned} G(\theta_0) &= |b|^{-p} |\sin_p \theta_0|^p p\lambda_0(|\lambda_0 - a|^p + |b|^p) \\ &= p\lambda_0(|b^{-1}(\lambda_0 - a) \sin_p \theta_0|^p + |\sin_p \theta_0|^p) \\ &= p\lambda_0(|\cos_p \theta_0|^p + |\sin_p \theta_0|^p) \\ &= p\lambda_0. \quad \square \end{aligned}$$

In this section, hereafter, we assume that $T < 0$, $D > 0$, $\Delta > 0$ and $b > 0$. Proposition B implies that (1.10) has two real roots λ_1 and λ_2 with $\lambda_1 < \lambda_2 < 0$, $f(\lambda) < 0$ on (λ_1, λ_2), and $f(\lambda) > 0$ on $(-\infty, \lambda_1) \cup (\lambda_2, \infty)$. We can take $\theta_1, \theta_2 \in (0, \pi_p)$ for which

$$\frac{b \cos_p \theta_i}{\sin_p \theta_i} + a = \lambda_i, \quad i = 1, 2. \tag{5.2}$$

Then $0 < \theta_2 < \theta_1 < \pi_p$.

In this section we prove the following result.

Theorem 5.1. *Assume that $T < 0$, $D > 0$, $\Delta > 0$ and $b > 0$. Let λ_1 and λ_2 be two real roots of (1.10) with $\lambda_1 < \lambda_2 < 0$, and let $\theta_1, \theta_2 \in (0, \pi_p)$ satisfy (5.2) and $t_1 \geq t_0$. Then, for each $\varepsilon > 0$, there exist $\delta > 0$ and $\rho > 0$ such that every solution $(r(t), \theta(t))$ of (2.5) with $0 < r(t_1) < \rho\delta$ satisfies*

$$0 < r(t_2)e^{(\lambda_i - \varepsilon)(t - t_2)} \leq r(t) \leq r(t_2)e^{(\lambda_i + \varepsilon)(t - t_2)} \leq \delta e^{(\lambda_i + \varepsilon)(t - t_2)}, \quad t \geq t_2,$$
$$\lim_{t \to \infty} \theta(t) = \theta_i + j\pi_p$$

for some $t_2 \geq t_1$, $i \in \{1, 2\}$ and $j \in \mathbf{Z}$.

Recalling (2.1), we note that $F(\theta)$ and $G(\theta)$ are periodic with period π_p by (2.1). Thus $F(j\pi_p) = pb > 0$ for $j \in \mathbf{Z}$. Since

$$F(\theta) = \frac{p^* |\sin_p \theta|^p}{\phi_p(b)} f\left(\frac{b \cos_p \theta}{\sin_p \theta} + a\right), \quad \theta \neq j\pi_p, \ j \in \mathbf{Z}, \tag{5.3}$$

it follows that

$$F(\theta_1 + j\pi_p) = F(\theta_2 + j\pi_p) = 0, \quad j \in \mathbf{Z}, \tag{5.4}$$

$$F(\theta) < 0, \quad \theta \in (\theta_2 + j\pi_p, \theta_1 + j\pi_p), \ j \in \mathbf{Z}, \tag{5.5}$$

$$F(\theta) > 0, \quad \theta \in (\theta_1 + j\pi_p, \theta_2 + (j+1)\pi_p), \ j \in \mathbf{Z}. \tag{5.6}$$

Lemma 5.1 implies that

$$G(\theta_i) < 0, \quad i = 1, 2.$$

Let $\sigma > 0$ be an arbitrarily sufficiently small number. Then

$$[\theta_2 - \sigma, \theta_2 + \sigma] \cap [\theta_1 - \sigma, \theta_1 + \sigma] = \emptyset,$$
$$G(\theta) < 0, \quad \theta \in [\theta_2 - \sigma, \theta_2 + \sigma] \cup [\theta_1 - \sigma, \theta_1 + \sigma].$$

There exist $\mu_0 > 0$, $\mu_1 > 0$ and $F_0 > 0$ such that $\mu_0 > \mu_1$ and

$$-\mu_0 \le G(\theta) \le -\mu_1, \quad \theta \in [\theta_2 - \sigma, \theta_2 + \sigma] \cup [\theta_1 - \sigma, \theta_1 + \sigma],$$
$$F(\theta) \le -F_0, \quad \theta \in [\theta_2 + \sigma, \theta_1 - \sigma],$$
$$F(\theta) \ge F_0, \quad \theta \in [\theta_1 + \sigma - \pi_p, \theta_2 - \sigma].$$

We take constants $F_1 > 0$ and $G_1 > 0$ satisfying (3.1). Let $\varepsilon > 0$ satisfy $T + C_0\varepsilon < 0$, $2pp^*\varepsilon < \mu_1$ and $2pp^*\varepsilon < F_0$, where C_0 is the constant defined by (3.2). Let $\delta > 0$ be as in Lemma 2.1. For $j \in \mathbf{Z}$, we define sets P_j, Q_j, R_j and S_j by

$$P_j := \{(r,\theta) : 0 < r \le \delta,\ \theta_1 + \sigma + (j-1)\pi_p \le \theta \le \theta_2 - \sigma + j\pi_p\},$$
$$Q_j := \{(r,\theta) : 0 < r \le \delta,\ \theta_2 - \sigma + j\pi_p \le \theta \le \theta_2 + \sigma + j\pi_p\},$$
$$R_j := \{(r,\theta) : 0 < r \le \delta,\ \theta_2 + \sigma + j\pi_p \le \theta \le \theta_1 - \sigma + j\pi_p\},$$
$$S_j := \{(r,\theta) : 0 < r \le \delta,\ \theta_1 - \sigma + j\pi_p \le \theta \le \theta_1 + \sigma + j\pi_p\}.$$

Lemma 5.2. *Let $(r(t), \theta(t))$ be a solution of* (2.5) *and let $j \in \mathbf{Z}$. Then the following* (i)–(iii) *hold*:

(i) *If $(r(t), \theta(t)) \in Q_j \cup S_j$, then*

$$r'(t) \le -\frac{\mu_1}{2p} r(t) < 0;$$

(ii) *If $(r(t), \theta(t)) \in P_j$, then $\theta'(t) \ge F_0/(2p) > 0$;*
(iii) *If $(r(t), \theta(t)) \in R_j$, then $\theta'(t) \le -F_0/(2p) < 0$.*

Proof. From (2.5) and (2.6), it follows that if $(r(t), \theta(t)) \in Q_j \cup S_j$, then

$$r'(t) \le -\frac{\mu_1}{p} r(t) + p^*\varepsilon r(t) \le -\frac{\mu_1}{p} r(t) + \frac{\mu_1}{2p} r(t) = -\frac{\mu_1}{2p} r(t) < 0,$$

and that if $(r(t), \theta(t)) \in P_j$, then

$$\theta'(t) \ge \frac{F_0}{p} - p^*\varepsilon \ge \frac{F_0}{2p} > 0,$$

and that if $(r(t), \theta(t)) \in R_j$, then

$$\theta'(t) \le -\frac{F_0}{p} + p^*\varepsilon \le -\frac{F_0}{2p} < 0. \quad \square$$

Lemma 5.3. *Let $(r(t), \theta(t))$ be a solution of* (2.5) *and let $t_1 \ge t_0$ and $j \in \mathbf{Z}$. If $(r(t_1), \theta(t_1)) \in P_j$ and $0 < r(t_1) < (F_0/F_1)^{1/p}\delta$, then $(r(t_2), \theta(t_2)) \in P_j \cap Q_j$ for some $t_2 \in [t_1, \infty)$, and hence $\theta(t_2) = \theta_2 - \sigma + j\pi_p$.*

Proof. It is enough to consider the case where $(r(t_1), \theta(t_1)) \notin P_j \cap Q_j$. Let

$$t_2 := \sup\{s \ge t_1 : (r(t), \theta(t)) \in P_j,\ t \in [t_1, s)\}.$$

From (ii) of Lemma 5.2, it follows that $\theta(t)$ is strictly increasing on $[t_1, t_2)$ and

$$\theta(t) \geq \theta(t_1) + \frac{F_0}{2p}(t - t_1), \quad t \in [t_1, t_2). \tag{5.7}$$

Hence $t_2 < \infty$. Indeed, if $t_2 = \infty$, then $\theta(t) \to \infty$ as $t \to \infty$ by (5.7), which means that $(r(t_3), \theta(t_3)) \notin P_j$ for some $t_3 \in (t_1, \infty)$. This is a contradiction. Then Lemma 3.2 implies that

$$0 < r(t) < \delta e^{(T + C_0\varepsilon)(t - t_1)/p} \leq \delta, \quad t \in [t_1, t_2),$$

which shows that $0 < r(t_2) < \delta$. Consequently $(r(t_2), \theta(t_2)) \in P_j \cap Q_j$. $\square$

By the same argument as in the proof of Lemma 5.3, we obtain the following result.

Lemma 5.4. *Let $(r(t), \theta(t))$ be a solution of (2.5) and let $t_1 \geq t_0$ and $j \in \mathbf{Z}$. If $(r(t_1), \theta(t_1)) \in R_j$ and $0 < r(t_1) < (F_0/F_1)^{1/p}\delta$, then $(r(t_2), \theta(t_2)) \in Q_j \cap R_j$ for some $t_2 \in [t_1, \infty)$, and hence $\theta(t_2) = \theta_2 + \sigma + j\pi_p$.*

Lemma 5.5. *Let $(r(t), \theta(t))$ be a solution of (2.5) and let $t_1 \geq t_0$ and $j \in \mathbf{Z}$. If $(r(t_1), \theta(t_1)) \in Q_j$, then $(r(t), \theta(t)) \in Q_j$ and $0 < r(t) \leq \delta e^{-\mu_1(t - t_1)/(2p)}$ for $t \geq t_1$.*

Proof. Let $t_2 := \sup\{s \geq t_1 : (r(t), \theta(t)) \in Q_j,\ t \in [t_1, s)\}$. We will show that $t_2 = \infty$. Assume that $t_2 < \infty$. From Lemma 5.2 it follows $r(t)$ is decreasing on $[t_1, t_2]$. Thus Lemma 2.2 shows that either $(r(t_2), \theta(t_2)) \in P_j$ or $(r(t_2), \theta(t_2)) \in R_j$. If $(r(t_2), \theta(t_2)) \in P_j$, then it must be that $\theta'(t_2) \leq 0$. However, Lemma 5.2 implies that $\theta'(t_2) > 0$ when $(r(t_2), \theta(t_2)) \in P_j$. This is a contradiction. When $(r(t_2), \theta(t_2)) \in R_j$, we can derive a contradiction in the same way. Therefore, $t_2 = \infty$. We conclude that $(r(t), \theta(t)) \in Q_j$ for $t \geq t_1$ and

$$r'(t) \leq -\frac{\mu_1}{2p} r(t) < 0, \quad t \geq t_1,$$

by Lemma 5.2. Hence, $(e^{\mu_1(t - t_1)/(2p)} r(t))' \leq 0$ for $t \geq t_1$, which shows that

$$e^{\mu_1(t - t_1)/(2p)} r(t) \leq r(t_1) \leq \delta, \quad t \geq t_1.$$

Consequently, $0 < r(t) \leq \delta e^{-\mu_1(t - t_1)/(2p)}$ for $t \geq t_1$. $\square$

Lemma 5.6. *Let $(r(t), \theta(t))$ be a solution of (2.5) and let $t_1 \geq t_0$ and $j \in \mathbf{Z}$. If $(r(t_1), \theta(t_1)) \in S_j$ and $0 < r(t_1) < (F_0/F_1)^{1/p}\delta$, then one of the following* (i)–(iii) *holds*:

(i) *$(r(t), \theta(t)) \in S_j$ and $0 < r(t) \leq \delta e^{-\mu_0(t - t_1)/(2p)}$ for $t \geq t_1$;*
(ii) *$(r(t_2), \theta(t_2)) \in R_j$ and $r(t_2) < (F_0/F_1)^{1/p}\delta$ for some $t_2 \in [t_1, \infty)$;*
(iii) *$(r(t_2), \theta(t_2)) \in P_{j+1}$ and $r(t_2) < (F_0/F_1)^{1/p}\delta$ for some $t_2 \in [t_1, \infty)$.*

Proof. Let $t_2 := \sup\{s \geq t_1 : (r(t), \theta(t)) \in S_j,\ t \in [t_1, s)\}$. If $t_2 = \infty$, then we conclude that $0 < r(t) \leq \delta e^{-\mu_1(t - t_1)/(2p)}$ for $t \geq t_1$, by the same argument as in the proof of Lemma 5.5, and hence (i) holds. Suppose that $t_2 < \infty$. From (i) of Lemma 5.2, it follows that $r(t_2) \leq r(t_1) <$

$(F_0/F_1)^{1/p}\delta \le \delta$. Recalling Lemma 2.2, we have $r(t_2) > 0$. Therefore, $(r(t_2), \theta(t_2)) \in R_j$ or $(r(t_2), \theta(t_2)) \in P_{j+1}$. $\square$

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Set $\rho := (F_0/F_1)^{1/p}$ and $\mu := \mu_1/(2p)$. Let $(r(t), \theta(t))$ be a solution of (2.5) with $0 < r(t_1) < \rho\delta$ for some $t_1 \ge t_0$. Then, by Lemmas 5.3, 5.4, 5.5 and 5.6, there exist $t_2 \ge t_1$ and $j \in \mathbf{Z}$ such that, for $t \ge t_2$, $(r(t), \theta(t))$ satisfies $0 < r(t) \le \delta e^{-\mu(t-t_2)}$ and either $(r(t), \theta(t)) \in Q_j$ or $(r(t), \theta(t)) \in S_j$. Since σ is an arbitrarily sufficiently small number, we find that

$$\lim_{t\to\infty} \theta(t) = \theta_i + j\pi_p,$$

where $i = 1$ or 2.

Recalling Lemma 5.1 and the fact that $G(\theta)$ is periodic with period π_p by (2.1), we see that

$$\lim_{t\to\infty} G(\theta(t)) = p\lambda_i.$$

Let $\varepsilon > 0$ be arbitrary. By Lemma 2.1, there exists $t_3 \ge t_2$ such that

$$0 < r(t_3) < \delta, \quad |G(\theta(t)) - p\lambda_i| \le \frac{p\varepsilon}{2}, \quad |M(t, r(t), \theta(t))| \le \frac{\varepsilon}{2} r(t)$$

for $t \ge t_3$. Therefore, from (2.5) it follows that

$$(\lambda_i - \varepsilon) r(t) \le r'(t) \le (\lambda_i + \varepsilon) r(t), \quad t \ge t_0,$$

which implies that

$$r(t_3) e^{(\lambda_i-\varepsilon)(t-t_3)} \le r(t) \le r(t_3) e^{(\lambda_i+\varepsilon)(t-t_3)} \le \delta e^{(\lambda_i+\varepsilon)(t-t_3)}, \quad t \ge t_3. \quad \square$$

6. Case $\Delta = 0$

In this section, hereafter, we assume that $T < 0$, $D > 0$, $\Delta = 0$ and $b > 0$. Proposition B implies that (1.10) has a unique real root $\lambda_0 := T/p$ and $f(\lambda) > 0$ on $(-\infty, \lambda_0) \cup (\lambda_0, \infty)$. We take $\theta_0 \in (0, \pi_p)$ satisfying

$$\frac{b \cos_p \theta_0}{\sin_p \theta_0} + a = \frac{T}{p}. \tag{6.1}$$

Lemmas 3.1 and 5.1 imply that $F'(\theta_0) = 0$. We note that $F(\theta_0) = 0$. We set $h(\theta) := F(\theta)/(\theta - \theta_0)$. By L'Hospital's rule, we see that

$$\lim_{\theta\to\theta_0} h(\theta) = F'(\theta_0) = 0.$$

Hence we regard $h(\theta)$ as a continuous function on $\mathbf{R}$, and we have

$$F(\theta) = h(\theta)(\theta - \theta_0), \quad \theta \in \mathbf{R}. \tag{6.2}$$

Since $F(j\pi_p) = pb > 0$ for $j \in \mathbf{Z}$, recalling (5.3), we conclude that

$$F(\theta_0 + j\pi_p) = 0, \quad j \in \mathbf{Z},$$
$$F(\theta) > 0, \quad \theta \neq \theta_0 + j\pi_p, \ j \in \mathbf{Z}.$$

By Lemma 5.1, we note that $G(\theta_0) = p\lambda_0 = T < 0$. There exist $\sigma \in (0, \pi_p/2)$ and $F_0 > 0$ such that

$$|h(\theta)| \le p, \quad \theta \in [\theta_0 - \sigma, \theta_0 + \sigma],$$
$$F(\theta) \ge F_0, \quad \theta \in [\theta_0 + \sigma - \pi_p, \theta_0 - \sigma],$$
$$2T \le G(\theta) \le T/2, \quad \theta \in [\theta_0 - \sigma, \theta_0 + \sigma].$$

By (6.2), we have

$$F(\theta) \le p|\theta - \theta_0|, \quad \theta \in [\theta_0 - \sigma, \theta_0 + \sigma]. \tag{6.3}$$

In this section we prove the following result.

Theorem 6.1. *Assume that $T < 0$, $D > 0$, $\Delta = 0$ and $b > 0$. Let $\theta_0 \in (0, \pi_p)$ satisfy (6.1) and let $t_1 \ge t_0$. Then there exist $\delta > 0$, $\rho > 0$ and $\mu > 0$ such that every solution $(r(t), \theta(t))$ of (2.5) with $0 < r(t_1) < \rho\delta$ satisfies*

$$0 < r(t) \le \delta e^{-\mu(t - t_2)}, \quad t \ge t_2$$

for some $t_2 \ge t_1$.

Let $F_1 > 0$ and $G_1 > 0$ satisfy (3.1). We take $\varepsilon > 0$ so small that $T + (4/3)C_0\varepsilon < 0$, $2pp^*\varepsilon < -T/2$, $2pp^*\varepsilon < F_0$, and

$$\left(\frac{p^*\varepsilon}{\sigma}\right)^{-T/(4p)} \le \left(\frac{F_0}{F_1}\right)^{1/p}, \tag{6.4}$$

where C_0 is the constant defined by (3.2). Let $\delta > 0$ be as in Lemma 2.1. For $j \in \mathbf{Z}$, we define sets V_j and W_j by

$$V_j := \{(r, \theta) : 0 < r \le \delta, \ \theta_0 + \sigma + (j-1)\pi_p \le \theta \le \theta_0 - \sigma + j\pi_p\},$$
$$W_j := \{(r, \theta) : 0 < r \le \delta, \ \theta_0 - \sigma + j\pi_p \le \theta \le \theta_0 + \sigma + j\pi_p\}.$$

By the same argument as in the proof of Lemmas 5.2 and 5.3, we obtain the following two lemmas.

Lemma 6.1. *Let $(r(t), \theta(t))$ be a solution of (2.5) and let $j \in \mathbf{Z}$. Then the following* (i) *and* (ii) *hold*:

(i) *If* $(r(t), \theta(t)) \in W_j$, *then*

$$r'(t) \le \frac{T}{4p} r(t) < 0;$$

(ii) *If* $(r(t), \theta(t)) \in V_j$, *then* $\theta'(t) \ge F_0/(2p) > 0$.

Lemma 6.2. *Let* $(r(t), \theta(t))$ *be a solution of* (2.5) *and let* $t_1 \ge t_0$ *and* $j \in \mathbf{Z}$. *If* $(r(t_1), \theta(t_1)) \in V_j$ *and* $0 < r(t_1) < (F_0/F_1)^{1/p}\delta$, *then* $(r(t_2), \theta(t_2)) \in V_j \cap W_j$ *for some* $t_2 \in [t_1, \infty)$.

Lemma 6.3. *Let* $(r(t), \theta(t))$ *be a solution of* (2.5) *and let* $t_1 \ge t_0$ *and* $j \in \mathbf{Z}$. *If* $(r(t_1), \theta(t_1)) \in W_j$, *then the following* (i) *or* (ii) *holds*:

(i) $(r(t), \theta(t)) \in W_j$ *and* $0 < r(t) \le \delta e^{T(t-t_1)/(4p)}$ *for* $t \ge t_1$;
(ii) $(r(t_2), \theta(t_2)) \in V_{j+1}$ *and* $0 < r(t) \le r(t_1) e^{T(t-t_1)/(4p)}$ *on* $[t_1, t_2]$ *for some* $t_2 \in [t_1, \infty)$.

Proof. Let $t_2 := \sup\{s \ge t_1 : (r(t), \theta(t)) \in W_j,\ t \in [t_1, s)\}$. First we assume that $t_2 = \infty$. Then $(r(t), \theta(t)) \in W_j$ for $t \ge t_1$. From (i) of Lemma 6.1, it follows that

$$(e^{-T(t-t_1)/(4p)} r(t))' \le 0, \quad t \ge t_1.$$

Integrating this inequality on $[t_1, t]$, we have

$$r(t) \le r(t_1) e^{T(t-t_1)/(4p)} \le \delta e^{T(t-t_1)/(4p)} \tag{6.5}$$

for $t \ge t_1$. Lemma 2.2 implies that $r(t) > 0$ for $t \ge t_1$. Hence (i) of Lemma 6.3 holds.

Now we suppose that $t_2 < \infty$. By Lemma 2.2, we have $r(t) > 0$ on $[t_1, t_2]$. Lemma 6.1 implies that $r(t)$ is strictly decreasing on $[t_1, t_2]$. Thus, either $(r(t_2), \theta(t_2)) \in V_j \cap W_j$ or $(r(t_2), \theta(t_2)) \in V_{j+1} \cap W_j$. Recalling Lemma 6.1, we see that if $(r(t_2), \theta(t_2)) \in V_j \cap W_j$, then $\theta'(t_2) \ge F_0/(2p) > 0$, which shows that $\theta(t) < \theta(t_2)$ on $(t_2 - \rho, t_2)$ for some small $\rho > 0$, and hence $(r(t), \theta(t)) \notin W_j$ on $(t_2 - \rho, t_2)$. This is a contradiction. Therefore, $(r(t_2), \theta(t_2)) \in V_{j+1}$. By the same argument as in the case where $t_2 = \infty$, we conclude that (6.5) holds on $[t_1, t_2]$. Consequently, (ii) of Lemma 6.3 holds. □

Lemma 6.4. *Let* $(r(t), \theta(t))$ *be a solution of* (2.5). *Assume that there exists* $t_1,\ t_2 \in [t_0, \infty)$ *and* $j \in \mathbf{Z}$ *such that* $(r(t_1), \theta(t_1)) \in V_j \cap W_j$, $(r(t_2), \theta(t_2)) \in V_{j+1} \cap W_{j+1}$, *and* $(r(t), \theta(t)) \in W_j \cup V_{j+1}$ *for* $t \in [t_1, t_2]$. *Then* $0 < r(t) \le r(t_1) e^{T(t-t_1)/(8p)}$ *for* $t \in [t_1, t_2]$.

Proof. There exists $s_1 \in (t_1, t_2)$ such that $(r(s_1), \theta(s_1)) \in W_j \cap V_{j+1}$ and $(r(t), \theta(t)) \in V_{j+1}$ for $t \in [s_1, t_2]$. We note that $\theta(t_1) = \theta_0 - \sigma + j\pi_p$, $\theta(s_1) = \theta_0 + \sigma + j\pi_p$, $\theta(t_2) = \theta_0 - \sigma + (j+1)\pi_p$ and

$$\theta_0 - \sigma + j\pi_p \le \theta(t) \le \theta_0 + \sigma + j\pi_p, \quad t \in [t_1, s_1].$$

By (2.5), (2.6) and (6.3), we have

$$\begin{aligned}\theta'(t) &= \frac{1}{p}F(\theta(t)-j\pi_p)+N(t,r(t),\theta(t)) \\ &\le |\theta(t)-j\pi_p-\theta_0|+p^*\varepsilon, \quad t\in[t_1,s_1],\end{aligned}$$

that is,

$$1 \ge \frac{\theta'(t)}{|\theta(t)-j\pi_p-\theta_0|+p^*\varepsilon}, \quad t\in[t_1,s_1].$$

Integrating this inequality on $[t_1,s_1]$, we obtain

$$\begin{aligned}s_1-t_1 &\ge \int_{t_1}^{s_1}\frac{\theta'(t)}{|\theta(t)-j\pi_p-\theta_0|+p^*\varepsilon}dt \qquad (6.6)\\ &= \int_{\theta(t_1)}^{\theta(s_1)}\frac{du}{|u-j\pi_p-\theta_0|+p^*\varepsilon} \\ &= \int_{\theta_0-\sigma+j\pi_p}^{\theta_0+\sigma+j\pi_p}\frac{du}{|u-j\pi_p-\theta_0|+p^*\varepsilon} \\ &= \int_{-\sigma}^{\sigma}\frac{dv}{|v|+p^*\varepsilon} \\ &= 2\int_0^{\sigma}\frac{dv}{v+p^*\varepsilon} \\ &= 2\log\frac{\sigma+p^*\varepsilon}{p^*\varepsilon}. \\ &\ge 2\log\frac{\sigma}{p^*\varepsilon}.\end{aligned}$$

Lemma 6.1 implies that

$$(e^{-T(t-t_1)/(4p)}r(t))' \le 0, \quad t\in[t_1,s_1].$$

Integrating this inequality on $[t_1,t]$, we obtain

$$r(t) \le r(t_1)e^{T(t-t_1)/(4p)}, \quad t\in[t_1,s_1]. \qquad (6.7)$$

Since

$$\frac{T+C_0\varepsilon}{p} = \frac{T}{4p}+\frac{3}{4p}\left(T+\frac{4}{3}C_0\varepsilon\right) < \frac{T}{4p},$$

from Lemma 3.2 and (6.7), it follows that, for $t \in [s_1, t_2]$,

$$
\begin{aligned}
0 < r(t) &\le r(s_1)\left(\frac{F_1}{F_0}\right)^{1/p} e^{(T+C_0\varepsilon)(t-s_1)/p} \\
&\le r(t_1)e^{T(s_1-t_1)/(4p)}\left(\frac{F_1}{F_0}\right)^{1/p} e^{(T+C_0\varepsilon)(t-s_1)/p} \\
&\le r(t_1)e^{T(s_1-t_1)/(4p)}\left(\frac{F_1}{F_0}\right)^{1/p} e^{T(t-s_1)/(4p)} \\
&= r(t_1)e^{T(t-t_1)/(4p)}\left(\frac{F_1}{F_0}\right)^{1/p} \\
&= r(t_1)e^{T(t-t_1)/(8p)}e^{T(t-t_1)/(8p)}\left(\frac{F_1}{F_0}\right)^{1/p} \\
&\le r(t_1)e^{T(t-t_1)/(8p)}e^{T(s_1-t_1)/(8p)}\left(\frac{F_1}{F_0}\right)^{1/p}.
\end{aligned}
$$

By (6.4) and (6.6), we conclude that

$$
0 < r(t) \le r(t_1)e^{T(t-t_1)/(8p)}, \quad t \in [s_1, t_2]. \tag{6.8}
$$

Since $(r(t), \theta(t)) \in W_j \cup V_{j+1}$ for $t \in [t_1, t_2]$, we note that $r(t) > 0$ for $t \in [t_1, t_2]$. Combining (6.7) and (6.8), we have

$$
0 < r(t) \le r(t_1)e^{T(t-t_1)/(8p)}, \quad t \in [t_1, t_2]. \quad \square
$$

Proof of Theorem 6.1. Set $\rho := (F_0/F_1)^{1/p}$ and $\mu := -T/(8p) > 0$. Let $(r(t), \theta(t))$ be a solution of (2.5) with $0 < r(t_1) < \rho\delta$ for some $t_1 \ge t_0$. Then $(r(t_1), \theta(t_1)) \in V_j$ or $(r(t_1), \theta(t_1)) \in W_j$ for some $j \in \mathbf{Z}$.

When $(r(t_1), \theta(t_1)) \in V_j$, Lemma 6.2 shows that $(r(t_2), \theta(t_2)) \in V_j \cap W_j$ for some $t_2 \in [t_1, \infty)$.

Assume that $(r(t_1), \theta(t_1)) \in W_j$. Then either (i) or (ii) of Lemma 6.3 holds. If (i) holds, then Theorem 6.1 follows. If (ii) holds, then $(r(t_2), \theta(t_2)) \in V_{j+1}$ for some $t_2 \ge t_1$ and

$$
0 < r(t_2) \le r(t_1)e^{T(t_2-t_1)/(4p)} \le r(t) < \rho\delta = (F_0/F_1)^{1/p}\delta,
$$

and hence Lemma 6.2 implies that $(r(t_3), \theta(t_3)) \in V_{j+1} \cap W_{j+1}$ for some $t_3 \in [t_2, \infty)$.

Therefore we conclude that either the following (a) or (b) holds:

(a) $(r(t), \theta(t)) \in W_i$ and $0 < r(t) \le \delta e^{T(t-t_*)/(4p)}$ on $[t_*, \infty)$ for some $i \in \mathbf{Z}$ and $t_* \ge t_1$;
(b) there exists $i_0 \in \mathbf{Z}$ and $\{\tau_i\}_{i=i_0}^{\infty}$ such that $(r(\tau_i), \theta(\tau_i)) \in V_i \cap W_i$ and $(r(t), \theta(t)) \in W_i \cup V_{i+1}$ on $[\tau_i, \tau_{i+1}]$ for $i \ge i_0$.

It is sufficient to consider the case (b). Then Lemma 6.4 implies that

$$
0 < r(t) \le r(\tau_i)e^{-\mu(t-\tau_i)}, \quad t \in [\tau_i, \tau_{i+1}] \tag{6.9}
$$

for $i \geq i_0$. Therefore we have

$$r(\tau_i) \leq r(\tau_{i-1})e^{-\mu(\tau_i-\tau_{i-1})}, \quad i \geq i_0+1.$$

From (6.9) it follows that if $t \in [\tau_i, \tau_{i+1}]$ and $i \geq i_0$, then

$$\begin{aligned} 0 < r(t) &\leq r(\tau_{i-1})e^{-\mu(\tau_i-\tau_{i-1})}e^{-\mu(t-\tau_i)} \\ &= r(\tau_{i-1})e^{-\mu(t-\tau_{i-1})} \\ &\leq r(\tau_{i-2})e^{-\mu(\tau_{i-1}-\tau_{i-2})}e^{-\mu(t-\tau_{i-1})} \\ &= r(\tau_{i-2})e^{-\mu(t-\tau_{i-2})} \\ &\vdots \\ &\leq r(\tau_{i_0})e^{-\mu(t-\tau_{i_0})}. \end{aligned}$$

Since $(r(\tau_{i_0}), \theta(\tau_{i_0})) \in V_{i_0} \cap W_{i_0}$, we note that $r(\tau_{i_0}) \leq \delta$. Consequently,

$$0 < r(t) \leq \delta e^{-\mu(t-\tau_{i_0})}, \quad t \geq \tau_{i_0}. \quad \square$$

7. Proofs of main results

In this section we give proof of Theorems 1.1, 1.2 and 1.3.

Proof of Theorem 1.2. Theorem 1.2 follows immediately from Theorem 4.1. $\square$

To prove Theorem 1.3, we need the following two lemmas.

Lemma 7.1. *Let $(x(t), y(t))$ be a solution of (1.1). Then $(\widetilde{x}(t), \widetilde{y}(t)) := (-x(t), y(t))$ is a solution of*

$$\begin{aligned} \widetilde{x}' &= \widetilde{a}\widetilde{x} + \widetilde{b}\phi_{p^*}(\widetilde{y}) + \widetilde{k}(t, \widetilde{x}, \widetilde{y}), \\ \widetilde{y}' &= \widetilde{c}\phi_p(\widetilde{x}) + \widetilde{d}\widetilde{y} + \widetilde{l}(t, \widetilde{x}, \widetilde{y}), \end{aligned} \tag{7.1}$$

where

$$\begin{gathered} \widetilde{a} = a, \quad \widetilde{b} = -b, \quad \widetilde{c} = -c, \quad \widetilde{d} = d, \\ \widetilde{k}(t, \widetilde{x}, \widetilde{y}) = -k(t, -\widetilde{x}, \widetilde{y}), \quad \widetilde{l}(t, \widetilde{x}, \widetilde{y}) = l(t, -\widetilde{x}, \widetilde{y}). \end{gathered}$$

Moreover,

$$\begin{gathered} \widetilde{f}(\lambda) := \phi_p(\lambda - \widetilde{a})[(p-1)\lambda - \widetilde{d}] - \phi_p(\widetilde{b})\widetilde{c} = f(\lambda), \\ \widetilde{T} := \widetilde{a} + \widetilde{d} = T, \quad \widetilde{D} := \phi_p(\widetilde{a})\widetilde{d} - \phi_p(\widetilde{b})\widetilde{c} = D, \\ \widetilde{\Delta} := \left| \frac{\widetilde{a}}{p^*} - \frac{\widetilde{d}}{p} \right|^p + \phi_p(\widetilde{b})\widetilde{c} = \Delta \end{gathered}$$

and

$$\widetilde{k}(t,\widetilde{x},\widetilde{y}) = o\left((|\widetilde{x}|^p + |\widetilde{y}|^{p^*})^{\frac{1}{p}}\right), \quad \widetilde{l}(t,\widetilde{x},\widetilde{y}) = o\left((|\widetilde{x}|^p + |\widetilde{y}|^{p^*})^{\frac{1}{p^*}}\right)$$

as $|\widetilde{x}|^p + |\widetilde{y}|^{p^*} \to 0$ *uniformly in* $t \geq t_0$.

Lemma 7.2. *Assume that* $b = 0$. *Let* $(x(t), y(t))$ *be a solution of* (1.1). *Then* $(\overline{x}(t), \overline{y}(t)) := (y(t), x(t))$ *is a solution of*

$$\begin{aligned} \overline{x}' &= \overline{a}\,\overline{x} + \overline{b}\phi_{\overline{p}^*}(\overline{y}) + \overline{k}(t,\overline{x},\overline{y}), \\ \overline{y}' &= \overline{c}\phi_{\overline{p}}(\overline{x}) + \overline{d}\,\overline{y} + \overline{l}(t,\overline{x},\overline{y}), \end{aligned} \tag{7.2}$$

where

$$\begin{gathered} \overline{p} = p^*, \quad \overline{a} = d, \quad \overline{b} = c \neq 0, \quad \overline{c} = b = 0, \quad \overline{d} = a, \\ \overline{k}(t,\overline{x},\overline{y}) = l(t,\overline{y},\overline{x}), \quad \overline{l}(t,\overline{x},\overline{y}) = k(t,\overline{y},\overline{x}). \end{gathered}$$

Moreover,

$$\begin{gathered} \overline{f}(\lambda) := \phi_{\overline{p}}(\lambda - \overline{a})[(\overline{p}-1)\lambda - \overline{d}] - \phi_{\overline{p}}(\overline{b})\overline{c} = \phi_{p^*}(\lambda - d)[(p^*-1)\lambda - a], \\ \overline{T} := \overline{a} + \overline{d} = T, \quad \overline{D} := \phi_{\overline{p}}(\overline{a})\overline{d} - \phi_{\overline{p}}(\overline{b})\overline{c} = a\phi_{p^*}(d) = \phi_{p^*}(D), \\ \overline{\Delta} := \left|\frac{\overline{a}}{\overline{p}^*} - \frac{\overline{d}}{\overline{p}}\right|^{\overline{p}} + \phi_{\overline{p}}(\overline{b})\overline{c} = \left|\frac{d}{p} - \frac{a}{p^*}\right|^{p^*} = \Delta^{\frac{p^*}{p}}, \end{gathered}$$

and

$$\overline{k}(t,\overline{x},\overline{y}) = o\left((|\overline{x}|^{\overline{p}} + |\overline{y}|^{\overline{p}^*})^{\frac{1}{\overline{p}}}\right), \quad \overline{l}(t,\overline{x},\overline{y}) = o\left((|\overline{x}|^{\overline{p}} + |\overline{y}|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}}\right)$$

as $|\overline{x}|^p + |\overline{y}|^{p^*} \to 0$ *uniformly in* $t \geq t_0$.

Proof of Theorem 1.3. By Proposition B, we note that (1.10) has two real roots λ_1 and λ_2 with $\lambda_1 < \lambda_2 < 0$. First we assume that $b > 0$. Theorem 5.1 implies that the zero solution of system (1.1) is exponentially stable. Moreover, if $(x(t), y(t))$ is a solution of (1.1) near the origin, then there exist $i \in \{1, 2\}$ and $j \in \mathbf{Z}$ such that

$$\lim_{t\to\infty} \frac{\phi_{p^*}(y(t))}{x(t)} = \lim_{t\to\infty} \frac{\cos_p \theta(t)}{\sin_p \theta(t)} = \frac{\cos_p(\theta_i + j\pi_p)}{\sin_p(\theta_i + j\pi_p)} = \frac{\cos_p \theta_i}{\sin_p \theta_i} = \frac{\lambda_i - a}{b},$$

since $\cos_p \theta / \sin_p \theta$ is periodic with period π_p. By Theorem 5.1 again, for each $\varepsilon > 0$, there exist $t_2 \geq t_1$, $i \in \{1, 2\}$ and $C_0 > 0$ such that

$$C_0 e^{(\lambda_i - \varepsilon)t} \leq r(t) \leq C_0 e^{(\lambda_i + \varepsilon)t}, \quad t \geq t_2,$$

which is equivalent to (1.13). By (1.12), there exist $c_1 > 0$ and t_3 such that

$$\frac{|y(t)|^{p^*}}{|x(t)|^p} \le c_1, \quad t \ge t_3.$$

Hence, from (1.13), it follows that

$$|x(t)|^p \le |x(t)|^p + |y(t)|^{p^*} \le C_0^p e^{p(\lambda_i+\varepsilon)t} = C_2 e^{p(\lambda_i+\varepsilon)t}$$

and

$$(1+c_1)|x(t)|^p \ge |x(t)|^p + |y(t)|^{p^*} \ge C_0^p e^{p(\lambda_i+\varepsilon)t} = (1+c_1)C_1 e^{p(\lambda_i+\varepsilon)t}$$

for $t \ge t_3$, where $C_1 = C_0^p/(1+c_1)$ and $C_2 = C_0^p$. Hence we have shown Theorem 1.3 with $b > 0$.

Next we suppose that $b < 0$. Applying Theorem 1.3 with $b > 0$ to (7.1), we conclude that the zero solution of (7.1) is exponentially stable and that if $(\widetilde{x}(t), \widetilde{y}(t)) := (-x(t), y(t))$ is a solution of (7.1) near the origin, then

$$\lim_{t\to\infty} \frac{\phi_{p^*}(y(t))}{x(t)} = \lim_{t\to\infty} \frac{\phi_{p^*}(\widetilde{y}(t))}{-\widetilde{x}(t)} = \frac{\lambda_i - \widetilde{a}}{-\widetilde{b}} = \frac{\lambda_i - a}{b}$$

for some $i \in \{1, 2\}$, and moreover (1.13) holds. This means that Theorem 1.3 with $b < 0$ follows. So we have obtained Theorem 1.3 with $b \neq 0$.

Finally, we assume that $b = 0$. Recalling $b^2 + c^2 > 0$, we find that $c > 0$. Let $(x(t), y(t))$ be a solution of (1.1). Thus we can apply Theorem 1.3 with $b \neq 0$ to (7.2). Then the zero solution of (7.2) is exponentially stable and if $(\overline{x}(t), \overline{y}(t)) := (y(t), x(t))$ is a solution of (7.2) near the origin, then

$$\lim_{t\to\infty} \frac{\phi_p(x(t))}{y(t)} = \lim_{t\to\infty} \frac{\phi_{\overline{p}^*}(\overline{y}(t))}{\overline{x}(t)} = \frac{\overline{\lambda} - \overline{a}}{\overline{b}} = \frac{\overline{\lambda} - d}{c},$$

where $\overline{\lambda} = d$ or $(p-1)a$. We note that

$$|\overline{x}(t)|^{\overline{p}} + |\overline{y}(t)|^{\overline{p}^*} = |y(t)|^{p^*} + |x(t)|^p.$$

Consequently Theorem 1.3 for the case $b = 0$ follows. □

Proof of Theorem 1.1. Theorem 1.1 with $\Delta \neq 0$ follows from Theorems 1.2 and 1.3. By the same argument as in the proof of Theorem 1.3, Theorem 6.1 implies Theorem 1.1 with $\Delta = 0$. □

8. Saddle points

In this section we will show Theorem 1.4. We assume that $D < 0$. From (1.11) it follows that $\Delta > 0$. Proposition B implies that (1.10) has two real roots λ_1 and λ_2 such that $\lambda_1 < 0 < \lambda_2$, $f(\lambda) < 0$ on (λ_1, λ_2), and $f(\lambda) > 0$ on $(-\infty, \lambda_1) \cup (\lambda_2, \infty)$. By the generalized Prüfer transformation (2.4), system (1.1) is transformed into

$$r' = \frac{1}{p} G(\theta) r + M(r, \theta), \quad \theta' = \frac{1}{p} F(\theta) + N(r, \theta), \tag{8.1}$$

where

$$\begin{aligned} M(r, \theta) &:= \phi_p(\sin_p \theta) K(r, \theta) + (p^* - 1) r^{2-p} \cos_p \theta L(r, \theta), \\ N(r, \theta) &:= r^{-1} \phi_p(\cos_p \theta) K(r, \theta) - (p^* - 1) r^{1-p} \sin_p \theta L(r, \theta), \\ K(r, \theta) &:= k(r \sin_p \theta, \phi_p(r \cos_p \theta)), \\ L(r, \theta) &:= l(r \sin_p \theta, \phi_p(r \cos_p \theta)). \end{aligned}$$

First we assume that $b > 0$. We take $\theta_1, \theta_2 \in (0, \pi_p)$ satisfying (5.2). Then $0 < \theta_2 < \theta_1 < \pi_p$ and (5.4)–(5.6) hold. Lemmas 3.1 and 5.1 show that

$$G(\theta_i) = p\lambda_i, \quad F'(\theta_i) = p(T - p\lambda_i), \quad i = 1, 2. \tag{8.2}$$

Since $F(\theta_i) = 0$, we note that

$$\lim_{\theta \to \theta_i} \frac{F(\theta)}{\theta - \theta_i} = F'(\theta_i) = p(T - p\lambda_i).$$

Since $f(0) = D < 0$ and $f(T/p) = -\Delta < 0$, recalling (iii) of Proposition B, we find that $\lambda_1 < T/p < \lambda_2$, that is, $T - p\lambda_2 < 0 < T - p\lambda_1$. Let $\sigma > 0$ be an arbitrarily sufficiently small number. Then

$$\begin{aligned} &[\theta_2 - \sigma, \theta_2 + \sigma] \cap [\theta_1 - \sigma, \theta_1 + \sigma] = \emptyset, \\ &G(\theta) < 0, \quad \theta \in [\theta_1 - \sigma, \theta_1 + \sigma], \\ &G(\theta) > 0, \quad \theta \in [\theta_2 - \sigma, \theta_2 + \sigma], \\ &\frac{F(\theta)}{\theta - \theta_1} \geq \frac{p(T - p\lambda_1)}{2}, \quad \theta \in [\theta_1 - \sigma, \theta_1 + \sigma]. \end{aligned} \tag{8.3}$$

There exist $\mu_1 > 0$ and $F_0 > 0$ such that

$$\begin{aligned} G(\theta) &\leq -\mu_1, \quad \theta \in [\theta_1 - \sigma, \theta_1 + \sigma], \\ G(\theta) &\geq \mu_1, \quad \theta \in [\theta_2 - \sigma, \theta_2 + \sigma], \\ F(\theta) &\leq -F_0, \quad \theta \in [\theta_2 + \sigma, \theta_1 - \sigma], \\ F(\theta) &\geq F_0, \quad \theta \in [\theta_1 + \sigma - \pi_p, \theta_2 - \sigma]. \end{aligned}$$

Let $\varepsilon > 0$ satisfy

$$2pp^*\varepsilon < \mu_1, \quad 2pp^*\varepsilon < F_0, \quad \frac{T - p\lambda_1}{2} - p^*\varepsilon > 0. \tag{8.4}$$

From (1.16) it follows that

$$|K(r,\theta)| \le \varepsilon r^2, \quad |N(r,\theta)| \le \varepsilon r^p$$

for all sufficiently small $r \ge 0$. By the same argument as in the proof of Lemma 2.1, there exists $\delta \in (0, 1)$ such that M and N are continuous for $0 \le r \le 2\delta$ and $\theta \in \mathbf{R}$, and

$$|M(r,\theta)| \le p^*\varepsilon r^2, \quad |N(r,\theta)| \le p^*\varepsilon r \quad \text{for } 0 < r \le \delta. \tag{8.5}$$

Since $\delta \in (0, 1)$, we note that (8.5) implies

$$|M(r,\theta)| \le p^*\varepsilon r, \quad |N(r,\theta)| \le p^*\varepsilon \quad \text{for } 0 < r \le \delta.$$

For $j \in \mathbf{Z}$, we define sets P_j, Q_j, R_j and S_j as in Section 5.

In the same way as Lemma 5.2, we have the following result.

Lemma 8.1. *Let $(r(t), \theta(t))$ be a solution of* (8.1) *and let $j \in \mathbf{Z}$. Then the following* (i)–(iv) *hold*:

(i) *If $(r(t), \theta(t)) \in Q_j$, then $r'(t) \ge \mu_1 r(t)/(2p) > 0$;*
(ii) *If $(r(t), \theta(t)) \in S_j$, then $r'(t) \le -\mu_1 r(t)/(2p) < 0$;*
(iii) *If $(r(t), \theta(t)) \in P_j$, then $\theta'(t) \ge F_0/(2p) > 0$;*
(iv) *If $(r(t), \theta(t)) \in R_j$, then $\theta'(t) \le -F_0/(2p) < 0$.*

Lemma 8.2. *Let $(r(t), \theta(t))$ be a solution of* (8.1) *and let $j \in \mathbf{Z}$. If $(r(t_1), \theta(t_1)) \in P_j \cup Q_j \cup R_j$ for some $t_1 \in \mathbf{R}$, then $r(t_2) > \delta$ for some $t_2 \ge t_1$.*

Proof. First we assume that $(r(t_1), \theta(t_1)) \in Q_j$. We will prove that $r(t_2) > \delta$ for some $t_2 \ge t_1$. Let

$$t_3 := \sup\{s \ge t_1 : (r(t), \theta(t)) \in Q_j,\ t \in [t_1, s)\}.$$

Then $t_3 < \infty$. Indeed, if $t_3 = \infty$, then Lemma 8.1 implies that $(e^{-(\mu_1/(2p))t} r(t))' \ge 0$ for $t \ge t_1$, and hence $r(t) \ge r(t_1)e^{(\mu_1/(2p))(t-t_1)}$ for $t \ge t_1$. This is a contradiction. Thus $t_3 < \infty$. Then either $(r(t_3), \theta(t_3)) \in P_j$ or $(r(t_3), \theta(t_3)) \in R_j$. If $(r(t_3), \theta(t_3)) \in P_j$, then Lemma 8.1 shows that $\theta'(t_3) \ge F_0/(2p) > 0$, which means that $\theta(t_3 - \rho) < \theta(t_3) = \theta_2 - \sigma + j\pi_p$ for all sufficiently small $\rho > 0$. This is a contradiction. When $(r(t_3), \theta(t_3)) \in R_j$, we can derive a contradiction. Consequently, we find that $t_3 < \infty$ and $(r(t_3), \theta(t_3)) \notin P_j \cup R_j$, that is, $(r(t_3), \theta(t_3)) \in Q_j$ and $r(t_3) = \delta$. Using Lemma 8.1 again, we see that $r'(t_3) > 0$, which shows that $r(t_2) > \delta$ for some $t_2 > t_3$.

Now we suppose that $(r(t_1), \theta(t_1)) \in P_j$. We will show that $r(t_2) > \delta$ for some $t_2 \ge t_1$. Assume that $r(t) \le \delta$ for $t \ge t_1$. Recalling Lemma 2.2, we have $r(t) > 0$ for $t \ge t_1$. From Lemma 8.1 it follows that $(r(s_1), \theta(s_1)) \in Q_j$ for some $s_1 \ge t_1$, which implies that $r(t_2) > \delta$ for some $t_2 \ge t_1$.

This is a contradiction. Hence we find that $r(t_2) > \delta$ for some $t_2 \geq t_1$. In the same way, we conclude that if $(r(t_1), \theta(t_1)) \in R_j$, then $r(t_2) > \delta$ for some $t_2 \geq t_1$. $\square$

Lemma 8.3. *Let $(r(t), \theta(t))$ be a solution of (8.1) and let $j \in \mathbf{Z}$. If $(r(t_1), \theta(t_1)) \in S_j$ for some $t_1 \in \mathbf{R}$, then one of the following (i)–(iii) holds:*

(i) *$(r(t), \theta(t)) \in S_j$ for $t \geq t_1$ and $(r(t), \theta(t)) \to (0, \theta_1 + j\pi_p)$ as $t \to \infty$;*
(ii) *$(r(t_2), \theta(t_2)) \in R_j$ for some $t_2 \in [t_1, \infty)$;*
(iii) *$(r(t_2), \theta(t_2)) \in P_{j+1}$ for some $t_2 \in [t_1, \infty)$.*

Proof. Without loss of generality, we may assume that $j = 0$, since $F(\theta)$ and $G(\theta)$ are periodic with period π_p by (2.1). Lemma 8.1 implies that $r(t)$ is strictly decreasing whenever $(r(t), \theta(t)) \in S_0$. Hence, (ii) or (iii) holds or $(r(t), \theta(t)) \in S_0$ for $t \geq t_1$.

Hereafter, we assume that $(r(t), \theta(t)) \in S_0$ for $t \geq t_1$. Then Lemmas 2.2 and 8.1 show that $r(t)$ is strictly decreasing and $r(t) \to 0$ as $t \to \infty$. Now we consider the sets

$$S_+ := \{(r, \theta) \in S_0 : r \leq \theta - \theta_1\},$$
$$S_- := \{(r, \theta) \in S_0 : r \leq -(\theta - \theta_1)\}.$$

By (8.3) and (8.5), we see that if $(r(t), \theta(t)) \in S_+$, then

$$\begin{aligned} \theta'(t) &= \frac{1}{p} F(\theta) + N(r, \theta) \\ &\geq \frac{T - p\lambda_1}{2}(\theta(t) - \theta_1) - p^* \varepsilon r(t) \\ &\geq \frac{T - p\lambda_1}{2}(\theta(t) - \theta_1) - p^* \varepsilon (\theta(t) - \theta_1) \\ &= C(\theta(t) - \theta_1) > 0, \end{aligned} \tag{8.6}$$

where

$$C := \frac{T - p\lambda_1}{2} - p^* \varepsilon > 0.$$

We claim that $(r(t), \theta(t)) \notin S_+$ for $t \geq t_1$. Suppose that $(r(t_2), \theta(t_2)) \in S_+$. Since we have assumed that $(r(t), \theta(t)) \in S_0$ for $t \geq t_1$ and recall that $r(t)$ is strictly decreasing and $r(t) \to 0$ as $t \to \infty$, we see that $(r(t), \theta(t)) \in S_+$ for $t \geq t_2$. From (8.6) it follows that

$$\left(e^{-Ct}(\theta(t) - \theta_1)\right)' \geq 0, \quad t \geq t_2,$$

which means that

$$\theta(t) - \theta_1 \geq e^{C(t - t_2)}(\theta(t_2) - \theta_1), \quad t \geq t_2.$$

Thus, $\theta(t) \to \infty$ as $t \to \infty$. This contradicts the fact that $(r(t), \theta(t)) \in S_0$. Consequently, $(r(t), \theta(t)) \notin S_+$ for $t \geq t_1$ as claimed. In the same way, we conclude that $(r(t), \theta(t)) \notin S_-$ for $t \geq t_1$. Therefore, $\theta(t)$ must converge to θ_1 as $t \to \infty$, provided $(r(t), \theta(t)) \in S_0$ for $t \geq t_1$. $\square$

The following result is known. See, for example, [3, Theorem 6.1 in Chapter 15].

Theorem D. *Let* λ_1 *and* λ_2 *satisfy* $\lambda_1 < 0 < \lambda_2$. *Assume that* k *and* l *are continuous in the neighborhood of* $(0, 0)$ *and*

$$k(x, y) = o\left(\sqrt{x^2 + y^2}\right), \quad l(x, y) = o\left(\sqrt{x^2 + y^2}\right) \quad \text{as } x^2 + y^2 \to 0.$$

Then the linear perturbed system

$$x' = \lambda_1 x + k(x, y), \quad y' = \lambda_2 y + l(x, y)$$

has at least one orbit tending to $(0, 0)$ *at each of the angle* 0 *and* π.

Lemma 8.4. *For each* $j \in \mathbf{Z}$, *system* (8.1) *has a solution* $(r(t), \theta(t))$ *such that* $(r(t), \theta(t)) \to (0, \theta_1 + j\pi_p)$ *as* $t \to \infty$ *and* $r(t) > 0$ *for all sufficiently large* t. *Moreover, for each such a solution and each* $\varepsilon > 0$, *there exists* t_2 *such that*

$$r(t_2)e^{(\lambda_1-\varepsilon)(t-t_2)} \le r(t) \le r(t_2)e^{(\lambda_1+\varepsilon)(t-t_2)}, \quad t \ge t_2. \tag{8.7}$$

Proof. It is sufficient to give a proof for the case where $j = 0$, because $F(\theta)$ and $G(\theta)$ are periodic with period π_p by (2.1). By Taylor's theorem, (5.4) and (8.2), there exists a function ζ such that

$$F(\theta) = p(T - p\lambda_1)(\theta - \theta_1) + \zeta(\theta)(\theta - \theta_1), \quad \theta \in \mathbf{R},$$
$$\lim_{\theta\to\theta_1} \zeta(\theta) = 0.$$

We set $\psi(\theta) := G(\theta) - G(\theta_1)$. Then, by (8.2), we have

$$G(\theta) = p\lambda_1 + \psi(\theta), \quad \theta \in \mathbf{R},$$
$$\lim_{\theta\to\theta_1} \psi(\theta) = 0.$$

Setting $\eta(t) = \theta(t) - \theta_1$, we conclude that (8.1) is equivalent to

$$r' = \lambda_1 r + P(r, \eta), \quad \eta' = (T - p\lambda_1)\eta + Q(r, \eta), \tag{8.8}$$

where

$$P(r, \eta) = \frac{1}{p}\psi(\eta + \theta_1)r + M(r, \eta + \theta_1),$$
$$Q(r, \eta) = \frac{1}{p}\zeta(\eta + \theta_1)\eta + N(r, \eta + \theta_1).$$

We extend the domain of (r, η) of (8.8) to the case $r < 0$ as follows:

$$r' = \lambda_1 r + P(|r|, \eta), \quad \eta' = (T - p\lambda_1)\eta + Q(|r|, \eta). \tag{8.9}$$

Let $\varepsilon > 0$. There exists $\delta > 0$ satisfying (8.5) and P and Q are continuous for $|r| \le 2\delta$ and $\eta \in \mathbf{R}$ and

$$|\zeta(\eta + \theta_1)| < p\varepsilon, \quad |\psi(\eta + \theta_1)| < p\varepsilon \quad \text{for } |\eta| \le \delta.$$

If $\sqrt{r^2 + \eta^2} \le \delta$, then

$$\begin{aligned} |P(|r|, \eta)| &\le \frac{1}{p}|\psi(\eta + \theta_1)||r| + |M(|r|, \eta + \theta_1)| \\ &\le \varepsilon|r| + p^*\varepsilon|r|^2 \\ &\le \varepsilon\sqrt{r^2 + \eta^2} + p^*\varepsilon(r^2 + \eta^2) \\ &= (1 + p^*\sqrt{r^2 + \eta^2})\varepsilon\sqrt{r^2 + \eta^2} \\ &\le (1 + p^*\delta)\varepsilon\sqrt{r^2 + \eta^2} \end{aligned} \tag{8.10}$$

and

$$\begin{aligned} |Q(|r|, \eta)| &\le \frac{1}{p}|\zeta(\eta + \theta_1)||\eta| + |N(|r|, \eta + \theta_1)| \\ &\le \varepsilon|\eta| + p^*\varepsilon|r| \\ &\le \varepsilon\sqrt{r^2 + \eta^2} + p^*\varepsilon\sqrt{r^2 + \eta^2} \\ &= (1 + p^*)\varepsilon\sqrt{r^2 + \eta^2}. \end{aligned}$$

Hence,

$$P(|r|, \eta) = o\left(\sqrt{r^2 + \eta^2}\right), \quad Q(|r|, \eta) = o\left(\sqrt{r^2 + \eta^2}\right)$$

as $r^2 + \eta^2 \to 0$. We recall that $\lambda_1 < 0 < T - p\lambda_1$. Therefore we can apply Theorem D to (8.9). Then (8.8) has a solution $(r(t), \eta(t))$ for which $(r(t), \eta(t)) \to (0, 0)$ as $t \to \infty$ and $r(t) > 0$ for all sufficiently large t. This means that (8.9) has a solution $(r(t), \theta(t))$ such that $(r(t), \theta(t)) \to (0, \theta_1)$ as $t \to \infty$ and $r(t) > 0$ for all sufficiently large t.

Let $(r(t), \eta(t))$ be a solution of (8.8) such that $(r(t), \eta(t)) \to (0, 0)$ as $t \to \infty$ and $r(t) > 0$ for all sufficiently large t. By (8.9) and (8.10), for each $\varepsilon > 0$, there exists t_2 such that $r(t)$ satisfies

$$(\lambda_1 - \varepsilon)r(t) \le r'(t) \le (\lambda_1 + \varepsilon)r(t), \quad t \ge t_2,$$

which shows (8.7). □

Combining Lemmas 8.2, 8.3 and 8.4, we can obtain the following result immediately.

Theorem 8.1. *Assume that $D < 0$ and $\Delta > 0$. Let λ_1 and λ_2 be two real roots of* (1.10) *with $\lambda_1 < 0 < \lambda_2$, and let $\theta_1 \in (0, \pi_p)$ satisfy* (5.2) *and $t_1 \in \mathbf{R}$. Then there exists $\delta > 0$ such that every solution $(r(t), \theta(t))$ of* (8.1) *with $0 < r(t_1) < \delta$ satisfies the following* (i) *or* (ii):

(i) *$r(t_2) > \delta$ for some $t_2 \geq t_1$;*
(ii) *$(r(t), \theta(t)) \to (0, \theta_1 + j\pi_p)$ as $t \to \infty$ for some $j \in \mathbf{Z}$ and for each $\varepsilon > 0$ there exists $t_2 \geq t_1$ such that* (8.7) *holds.*

Moreover, there exist solutions $(r(t), \theta(t))$ of (8.1) *satisfying each* (i) *and* (ii).

Proof of Theorem 1.4. By Theorem 8.1, there exists $\delta > 0$ such that every solution $(r(t), \theta(t))$ of (8.1) with $0 < r(t_1) < \delta^{1/p}$ satisfies either (i) or (ii) of Theorem 8.1. We see that (i) of Theorem 8.1 implies that $|x(t_2)|^p + |y(t_2)|^{p^*} > \delta$ for some $t_2 > t_1$. Now we assume that (ii) of Theorem 8.1 holds. Let $(x(t), y(t))$ be a solution of (1.15) with $0 < |x(t_1)|^p + |y(t_1)|^{p^*} < \delta$. Then $(x(t), y(t)) \to 0$ as $t \to \infty$ and for each $\varepsilon > 0$, there exist $t_2 \geq t_1$ and $C_0 > 0$ such that (1.13) with $i = 1$ holds for $t \geq t_2$. If $b \neq 0$, then

$$\lim_{t\to\infty} \frac{\phi_{p^*}(y(t))}{x(t)} = \lim_{t\to\infty} \frac{\cos_p \theta(t)}{\sin_p \theta(t)} = \frac{\cos_p \theta_1}{\sin_p \theta_1} = \frac{\lambda_1 - a}{b}.$$

By the same argument as in the proof of Theorem 1.3, we obtain (1.14) with $i = 1$ holds for $t \geq t_2$.

Next we suppose that $b = 0$. Then $c \neq 0$. Set $(\overline{x}(t), \overline{y}(t)) := (y(t), x(t))$. We use Lemma 7.2. Then we note that

$$\overline{f}(\lambda) = \phi_{p^*}(\lambda - d)[(p^* - 1)\lambda - a] = 0$$

has roots $\lambda = d$, $a/(p^* - 1) = (p - 1)a$. Since $\phi_p(a)d = D < 0$, we find that either $d < 0 < (p - 1)a$ or $(p - 1)a < 0 < d$. Using the same argument as in the case $b \neq 0$, we conclude that

$$\lim_{t\to\infty} \frac{\phi_p(x(t))}{y(t)} = \lim_{t\to\infty} \frac{\phi_{\overline{p}^*}(\overline{y}(t))}{\overline{x}(t)} = \frac{\overline{\lambda} - \overline{a}}{\overline{b}} = \frac{\overline{\lambda} - d}{c},$$

where $\overline{\lambda} = d$ if $a > 0$ and $\overline{\lambda} = (p - 1)a$ if $a < 0$. Consequently (ii) of Theorem 1.4 holds. Moreover, by Theorem 8.1, there exists a solution $(x(t), y(t))$ of (1.15) satisfying (ii) of Theorem 1.4.

Now we consider the case $t \to -\infty$. If $(x(t), y(t))$ is a solution of (1.15), then $(x(-t), y(-t))$ is a solution of

$$x' = -ax - b\phi_{p^*}(y) - k(x, y), \quad y' = -c\phi_p(x) - dy - l(x, y). \tag{8.11}$$

We note that the characteristic equation

$$\phi_p(\lambda + a)[(p - 1)\lambda + d] - \phi_p(b)c = 0$$

for (8.11) has two real roots $-\lambda_2$ and $-\lambda_1$ and $-\lambda_2 < 0 < -\lambda_1$. Applying the argument above to (8.11), we conclude that every solution $(x(t), y(t))$ of (1.15) satisfies either $|x(t_0)|^p + |y(t_0)|^{p^*} > \delta$ for some $t_0 < t_1$ or (iii) of Theorem 1.4 and we also find that there exists a solution $(x(t), y(t))$ of (1.15) satisfying (iii) of Theorem 1.4. $\square$

9. Application to elliptic equations

In this section we consider equation (1.6) and assume that

$$q > p > \alpha - \beta, \quad N - p + \alpha > 0. \tag{9.1}$$

We set

$$a := \frac{p - \alpha + \beta}{q - p} > 0, \quad \eta := (a+1)(p-1) > 0 \tag{9.2}$$

and define the exponent E_1 and E_2 by

$$E_1 := \frac{(p-1)(N+\beta)}{N-p+\alpha}, \quad E_2 := \frac{(p-1)(N+\beta) + p - \alpha + \beta}{N-p+\alpha},$$

respectively. If $p = 2$ and $\alpha = \beta = 0$, then $E_1 = N/(N-2)$ is the Serrin exponent and $E_2 = (N+2)/(N-2)$ is the Sobolev exponent. Now we set

$$b := -1, \quad d := \eta - N + 1 - \alpha = -\frac{N-p+\alpha}{q-p}(q - 1 - E_1). \tag{9.3}$$

Then

$$T := a + d = -\frac{N-p+\alpha}{q-p}\Big(q - 1 - E_2\Big),$$

$$D := \phi_p(a)d - \phi_p(b)c = -\frac{N-p+\alpha}{q-p}(q-1-E_1)a^{p-1} + c$$

and

$$\Delta := \left|\frac{a}{p^*} - \frac{d}{p}\right|^p + \phi_p(b)c = \Big(\frac{N-p+\alpha}{p}\Big)^p - c.$$

Since $q > p$, we have

$$\phi_q(x) = o\left((|x|^p + |y|^{p^*})^{\frac{1}{p^*}}\right) \quad \text{as } |x|^p + |y|^{p^*} \to 0.$$

First we consider the case $\Delta < 0$.

Proposition 9.1. *Assume that* (9.1) *holds and*

$$c > \Big(\frac{N-p+\alpha}{p}\Big)^p.$$

Then every nontrivial solution u of (1.6) *on* (r_0, ∞) *for some $r > 0$ changes sign infinitely many times. In particular,* (1.6) *has no positive solution on* $(0, \infty)$.

Proof. Using the generalized Prüfer transformation to (1.7), we have

$$\theta'(t) = \frac{1}{p} F(\theta(t)) - (p^* - 1) r^{q-p} |\sin_p \theta(t)|^q, \quad t > t_0,$$

where $t_0 = \log r_0$. By Lemma 4.1, since $F(\theta)$ is periodic, there exists $F_0 > 0$ such that

$$F(\theta) \le -F_0, \quad \theta \in \mathbf{R}.$$

Hence we have

$$\theta'(t) \le -\frac{F_0}{p} < 0, \quad t > t_0,$$

which implies that $\theta(t) \to -\infty$ as $t \to \infty$. Therefore, every nontrivial solution u of (1.6) on (r_0, ∞) changes sign infinitely many times. □

Next we assume that $D < 0$. Then (1.7) has an equilibrium $(x, y) = ((-D)^{1/(q-p)}, a^{p-1}(-D)^{(p-1)/(q-p)})$. Therefore, (1.6) has an exact positive singular solution $u_*(r) := (-D)^{1/(q-p)} r^{-a}$. If $c > 0$, then (1.6) has another singular solution as follows.

Proposition 9.2. *Assume that* (9.1) *holds and* $q - 1 > p$ *and*

$$0 < c < \frac{N - p + \alpha}{q - p}(q - 1 - E_1)\left(\frac{p - \alpha + \beta}{q - p}\right)^{p-1}.$$

Then (1.6) *has a singular solution* $u(r)$ *on* $(0, r_0)$ *for some* $r_0 > 0$ *such that*

$$\lim_{r \to 0} u(r) = \infty, \quad \lim_{r \to 0} r^a u(r) = 0.$$

Proof. We note that $D < 0$ and $l(x, y) := \phi_q(x)$ satisfies

$$l(x, y) = o(|x|^p + |y|^{p^*}) \tag{9.4}$$

as $|x|^p + |y|^{p^*} \to 0$, because of $q - 1 > p$. Recalling (1.11), we have $\Delta > 0$. Proposition B implies that

$$f(\lambda) = \phi_p(\lambda - a)[(p - 1)\lambda - d] + c = 0$$

has two real roots λ_1 and λ_2 with $\lambda_1 < 0 < \lambda_2$, $f(\lambda) < 0$ on (λ_1, λ_2), and $f(\lambda) > 0$ on $(-\infty, \lambda_1) \cup (\lambda_2, \infty)$. Since $f(a) = c > 0$, we conclude that $\lambda_2 < a$. Let $\varepsilon = (a - \lambda_2)/2$. By Theorem 1.4, there exist t_0, $C_0 > 0$ and a solution $(x(t), y(t))$ of (1.7) such that $(x(t), y(t)) \to (0, 0)$ as $t \to -\infty$ and (1.14) with $i = 2$ holds for $t \le t_0$. Hence (1.6) has a solution $u(r)$ on $(0, e^{t_0}]$ such that

$$C_1 r^{\lambda_2 + \varepsilon} \le r^a u(r) \le C_2 r^{\lambda_2 - \varepsilon}, r \in (0, e^{t_0}],$$

which shows that $\lim_{r\to 0} r^a u(r) = 0$. Moreover, since

$$u(r) \geq C_1 r^{\lambda_2 - a + \varepsilon} = C_1 r^{(\lambda_2 - a)/2}, r \in (0, e^{t_0}].$$

Therefore, $\lim_{r\to 0} u(r) = \infty$. □

Finally we consider the case $c = 0$, that is,

$$r^{-(N-1)}(r^{N-1+\alpha}\phi_p(u'))' + r^\beta \phi_q(u) = 0, \quad r > 0. \tag{9.5}$$

Then (9.5) has an exact positive singular solution Ar^{-a}, where $A = (-da^{p-1})^{1/(q-p)}$, a and d are defined in (9.2) and (9.3), respectively. Franca [8] considered the equation

$$\Delta_p u + K(|x|)|u|^{q-2}u = 0$$

and showed the existence of infinitely many positive singular solutions, where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$. Troy and Krisner [30] also found a continuum of singular solutions to

$$\Delta u + |u|^{q-2}u = 0.$$

Franca and Garrione [11] proved the existence of infinitely many positive singular solutions for

$$\Delta u + c\frac{u}{|x|^2} + K(|x|)|u|^{q-2}u = 0,$$

where $c < (n-2)^2/4$.

Theorem 9.1. *Assume that* (9.1) *holds and $E_1 < q - 1 < E_2$. Then there exists a continuum U of singular solutions such that each $u \in U$ is a positive singular solution of* (9.5) *and satisfies*

$$\lim_{r\to 0}\frac{u(r)}{u_*(r)} = 1, \quad \lim_{r\to\infty}\frac{u(r)}{u_*(r)} = 0,$$

where $u_*(r) := Ar^{-a}$ is the exact positive singular solution of (9.5).

Proof. Since $c = 0$, the characteristic equation

$$f(\lambda) = \phi_p(\lambda - a)[(p-1)\lambda - d] = 0$$

for (1.7) has real roots $\lambda_1 := d/(p-1) < 0$ and $\lambda_2 := a > 0$. Since $D < 0$ and (9.4) holds, by Theorem 1.4, there exists a solution $(\overline{x}(t), \overline{y}(t))$ of (1.7) such that $(\overline{x}(t), \overline{y}(t)) \to (0, 0)$ as $t \to \infty$ and

$$\lim_{t\to\infty}\frac{\phi_{p^*}(\overline{y}(t))}{\overline{x}(t)} = \frac{\lambda_1 - a}{b} = -\frac{d}{p-1} + a > 0.$$

Since $(-\overline{x}(t), -\overline{y}(t))$ is also a solution of (1.7), we can assume that $\overline{x}(t) > 0$ and $\overline{y}(t) > 0$ for all sufficiently large t. Let $I := (\sigma, \infty)$ be the maximal interval of the existence for $(\overline{x}(t), \overline{y}(t))$. Now we define the function $Q(x, y)$ by

$$Q(x, y) = \frac{1}{q}|x|^q - \left(\frac{a}{p^*} - \frac{d}{p}\right)xy - \frac{b}{p^*}|y|^{p^*}.$$

We set

$$P(t) = e^{-Tt}Q(\overline{x}(t), \overline{y}(t)).$$

Then

$$P'(t) = \frac{q-p}{pq}Te^{-Tt}|\overline{x}(t)|^q.$$

From $T > 0$, it follows that $P(t) \to 0$ as $t \to \infty$ and $P(t)$ is nondecreasing, which means that $P(t) \le 0$ for $t \in I$. Now we set

$$C = \frac{a}{p^*} - \frac{d}{p}.$$

Since $a > 0$ and $d < 0$, we note that $C > 0$. We consider the sets Ω_0 and Ω_1 defined by

$$\Omega_0 = \{(x, y) : Q(x, y) \le 0,\ x \ge 0,\ y \ge 0\}$$

and

$$\Omega_1 = \left\{(x, y) : (qC)^{-1}x^{q-1} \le y \le (p^*C)^{p-1}x^{p-1},\ x \ge 0\right\},$$

respectively. Then $(\overline{x}(t), \overline{y}(t)) \in \Omega_0$ for $t \in I$. Now we will show that $\Omega_0 \subset \Omega_1$. Note that $b = -1$. Let $(x, y) \in \Omega_0$. Then

$$\frac{1}{q}x^q + \frac{1}{p^*}y^{p^*} = Q(x, y) + Cxy \le Cxy.$$

Hence we have $x^q/q \le Cxy$ which implies that $(qC)^{-1}x^{q-1} \le y$. Moreover we obtain $y^{p^*}/p^* \le Cxy$, that is, $y \le (p^*C)^{p-1}x^{p-1}$. Therefore $\Omega_0 \subset \Omega_1$. Since Ω_1 is bounded, by a standard argument on a general theory on ordinary differential equations, we conclude that $I = \mathbf{R}$.

System (1.7) has equilibriums $(0, 0)$, (A, y_*) and $(-A, -y_*)$, where $y_* = (-da^{q-1})^{\frac{p-1}{q-p}}$. Since

$$Q(A, y_*) = -\left(\frac{1}{p} - \frac{1}{q}\right)|d|^{\frac{q}{q-p}}a^{\frac{(p-1)q}{q-p}} < 0,$$

we find that $(0, 0), (A, y_*) \in \Omega_0$. By

$$\frac{\partial}{\partial x}(ax - \phi_{p^*}(y)) + \frac{\partial}{\partial y}(c\phi_p(x) + dy + \phi_q(x)) = T > 0,$$

the Bendixson-Dulac theorem shows that (1.7) has no nonconstant periodic solution. Therefore, Poincaré-Bendixson theorem implies that either

$$(\overline{x}(t), \overline{y}(t)) \to (0, 0) \quad \text{as } t \to -\infty \tag{9.6}$$

or

$$(\overline{x}(t), \overline{y}(t)) \to (A, y_*) \quad \text{as } t \to -\infty. \tag{9.7}$$

If (9.6) holds, then the orbit of $(\overline{x}(t), \overline{y}(t))$ is homoclinic. However, by the same argument as in the proof of Bendixson-Dulac theorem, we conclude that (1.7) has no homoclinic orbit. Hence (9.7) holds.

Now we have a solution $(\overline{x}(t), \overline{y}(t))$ of (1.7) such that $\overline{x}(t) > 0$, $\overline{y}(t) > 0$ for $t \in \mathbf{R}$ and

$$\lim_{t\to-\infty}(\overline{x}(t), \overline{y}(t)) = (A, y_*), \qquad \lim_{t\to\infty}(\overline{x}(t), \overline{y}(t)) = (0, 0).$$

We define the set

$$U = \{u : u(r) = r^{-a}\overline{x}(\log r + \gamma),\ \gamma \in \mathbf{R}\}.$$

Then $u \in U$ is a positive solution of (9.5) with

$$u(1) = \overline{x}(\gamma), \quad u'(1) = -a\phi_{p^*}(\overline{y}(\gamma))$$

and satisfies

$$\lim_{r\to 0}\frac{u(r)}{u_*(r)} = 1, \quad \lim_{r\to\infty}\frac{u(r)}{u_*(r)} = 0. \quad \square$$

Remark 9.1. The proof of Theorem 9.1 is based on the method introduced by Miyamoto and Takahashi [19, Lemma 2.2]. On the other hand, we need Theorem 1.4 to ensure the existence of a solution $(\overline{x}(t), \overline{y}(t))$ to (1.7) going to $(0, 0)$ as $t \to \infty$.

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