



# Higher integrability for parabolic systems with Orlicz growth

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## Abstract

We prove higher integrability of the spatial gradient of weak solutions to parabolic systems with  $\varphi$ -growth, where  $\varphi = \varphi(t)$  is a general Orlicz function. The parabolic systems need be neither degenerate nor singular. Our result is a generalized version of the one of Kinnunen and Lewis (2000) [34] for the parabolic  $p$ -Laplace systems.

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## 1. Introduction

Higher integrability results for elliptic problems with Orlicz growth can be easily obtained from the ones for  $p$ -growth problems, hence they are well-known. On the other hand, higher integrability for parabolic problems with Orlicz growth is not simple and, as far as we know, no related result has been reported. The main difficulty is that the lower- and the upper-bound of exponent of the Orlicz function  $\varphi$ , which are denoted by  $p$  and  $q$  in this paper, may be too far

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away from each other to apply techniques used in the standard  $p$ -growth case. Specifically, in  $p$ -growth problems, known proofs use different techniques in the degenerate case ( $p > 2$ ) and the singular case ( $p < 2$ ). However, in the general Orlicz setting, neither of these cases apply when  $p < 2 < q$ , since the problem has characteristics of both the singular and degenerate cases. On the other hand, a unified technique for parabolic problems with  $p(x, t)$ -growth has been recently reported in [2].

We study regularity theory for second-order parabolic systems satisfying a general growth condition of Orlicz type. Precisely, we consider the following parabolic system:

$$\partial_t u_i - \operatorname{div}(A_i(z, u, Du)) = 0 \quad \text{in } \Omega_I = \Omega \times I \subset \mathbb{R}^n \times \mathbb{R}, \quad i = 1, \dots, N, \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) is an open set,  $I \subset \mathbb{R}$  is an interval,  $z = (x, t) \in \Omega \times I$ ,  $u = (u_1, \dots, u_N) \in \mathbb{R}^N$  and  $Du$  is the spatial gradient of  $u$ , i.e.,  $Du = D_x u$ . Here  $A_i : \Omega_I \times \mathbb{R}^N \times \mathbb{R}^{Nn} \rightarrow \mathbb{R}^n$ ,  $i = 1, \dots, N$ , satisfies

$$|A_i(z, u, \xi)| \leq \Lambda \frac{\varphi(|\xi|)}{|\xi|} \quad \text{and} \quad \sum_{i=1}^N A_i(z, u, \xi) \cdot \xi \geq \nu \frac{\varphi(|\xi|)}{|\xi|} \quad (1.2)$$

for all  $z \in \Omega_I$ ,  $u \in \mathbb{R}^N$  and  $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$  and for some  $0 < \nu \leq \Lambda$ , and  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is a weak  $\Phi$ -function which satisfies  $(\text{aInc})_p$  and  $(\text{aDec})_q$  for some

$$\frac{2n}{n+2} < p \leq q. \quad (1.3)$$

We will introduce the definitions of weak  $\Phi$ -function,  $(\text{aInc})$  and  $(\text{aDec})$  in the next section. We note that  $(\text{aInc})_p$  and  $(\text{aDec})_q$  for some  $1 < p \leq q$  are equivalent to the  $\nabla_2$  and  $\Delta_2$  conditions, respectively, see [29, Proposition 2.2.6]. The lower bound  $\frac{2n}{n+2}$  in (1.3) is generally assumed in parabolic regularity theory, see [18] and also [34].

The prototype of (1.1) is the so-called parabolic  $g$ -Laplace system

$$\partial_t u_i - \operatorname{div} \left( \frac{g(|Du|)}{|Du|} Du_i \right) = 0, \quad i = 1, \dots, N.$$

More generally, we may also consider coefficients:

$$\partial_t u_i - \operatorname{div} \left( a(z, u) \frac{g(|Du|)}{|Du|} Du_i \right) = 0, \quad i = 1, \dots, N, \quad \text{where } 0 < \nu \leq a(\cdot, \cdot) \leq \Lambda.$$

Here, we may take  $\varphi(t) = \int_0^t g(s) ds$ . In particular, when  $g(t) = t^{p-1}$ , this system becomes the parabolic  $p$ -Laplace system.

The main result of this paper is to prove higher integrability of the gradient of a weak solution to the system (1.1) together with a reverse Hölder type estimate. The *weak solution to (1.1)* with structure conditions (1.2)–(1.3) is defined as a function  $u \in L^\infty(I, L^2(\Omega, \mathbb{R}^N)) \cap L^1(I, W^{1,1}(\Omega))$  with  $\varphi(|Du|) \in L^1([0, T], L^1(\Omega))$  satisfying

$$-\int_{\Omega_I} u_i \partial_t \zeta_i dz + \int_{\Omega_I} A_i(z, u, Du) \cdot D\zeta_i dz = 0, \quad i = 1, \dots, N, \quad (1.4)$$

for all  $\zeta = (\zeta_1, \dots, \zeta_N) \in C_0^\infty(\Omega_I, \mathbb{R}^N)$ . We show that there exists a universal constant  $\varepsilon > 0$  such that

$$\varphi(|Du|) \in L_{loc}^{1+\varepsilon}(\Omega_I).$$

Regularity theory for the parabolic  $p$ -Laplace systems,  $p \neq 2$ , was first systematically studied by DiBenedetto and Friedman, see [19,20] and also the monographs [18,21]. Later,  $L^q$ -regularity theory was established in [1,34]. In particular, in [34], Kinnunen and Lewis first proved higher integrability for parabolic  $p$ -Laplace systems. We further refer to [4,8,10,35,36,41] and related references for regularity results for parabolic  $p$ -Laplace systems.

In the calculus of variations, partial differential equations with  $p$ -growth can be obtained as Euler-Lagrange equations of functionals with a  $p$ -growth condition that is related to the power function  $t^p$ . Hence we can naturally generalize  $t^p$  to an Orlicz function, and a growth condition related to an Orlicz function is called the Orlicz-growth condition. Regularity results for elliptic equations with Orlicz growth, specifically  $C^\alpha$ - and  $C^{1,\alpha}$ -regularity, were first obtained by Lieberman [37]. Later, he generalized these results to parabolic systems with Orlicz growth [38]. We also refer to regularity results [5,14,16,17,22,24,49] and [6,15,23,32,33,47,50] for the elliptic and parabolic case with Orlicz growth, respectively.

As mentioned above, we shall prove a higher integrability result for parabolic systems with Orlicz growth. The higher integrability is the most basic regularity property of weak solutions for elliptic/parabolic problems in divergence form, and is a crucial ingredient in studying regularity theory, see for instance [28]. It has been obtained first by Elcrat and Meyers [39] for elliptic systems with  $p$ -growth (see also [26,46]) and by Giaquinta and Struwe [27] for parabolic systems with 2-growth (i.e.,  $p = 2$ ). But it was an open problem for about 20 years for parabolic problems with  $p$ -growth ( $p \neq 2$ ), and then Kinnunen and Lewis obtained the result [34]. We also refer to [43–45] for global higher integrability for parabolic problems with  $p$ -growth and [13], [7], [3,9], [11] and [12,25] for higher integrability results for obstacle problems, higher order parabolic systems, parabolic systems with  $p(x, t)$ -growth, doubly nonlinear parabolic systems and porous medium systems, respectively.

Now let us state our higher integrability result for parabolic systems with Orlicz growth.

**Theorem 1.5.** *Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be a weak  $\Phi$ -function satisfying  $(aInc)_p$  and  $(aDec)_q$  with constant  $L \geq 1$  and let  $u$  be a local weak solution to (1.1) with structure conditions (1.2)–(1.3). There exists  $\varepsilon = \varepsilon(n, N, p, q, L, v, \Lambda) > 0$  such that  $\varphi(|Du|) \in L_{loc}^{1+\varepsilon}(\Omega_I)$  with the following estimate: for any  $Q_{4\rho} \Subset \Omega_I$ ,*

$$\int_{Q_\rho} \varphi(|Du|)^{1+\varepsilon} dz \leq c \left[ (\varphi \circ \mathcal{D}^{-1}) \left( \int_{Q_{2\rho}} \varphi(|Du|) dz \right) \right]^\varepsilon \int_{Q_{2\rho}} \varphi(|Du|) dz$$

for some  $c = c(n, N, p, q, L, v, \Lambda) > 0$ , where

$$\mathcal{D}(t) := \min\{t^2, \varphi(t)^{\frac{n+2}{2}} t^{-n}\} \tag{1.6}$$

and  $\mathcal{D}^{-1}$  is the left-inverse of  $\mathcal{D}$ .

We remark that when  $\varphi(t) = t^p$ , we have  $\mathcal{D}(t) = \min\{t^2, t^{\frac{p(n+2)}{2}-n}\}$  and so

$$(\varphi \circ \mathcal{D}^{-1})(t) = \max\{t^{\frac{p}{2}}, t^{\frac{2p}{p(n+2)-2n}}\}$$

Therefore, our result exactly implies the known results for the  $p$ -growth case, see for instance [9].

We would like to introduce the novelties of our approach used in this paper. The main step is to obtain a reverse Hölder type inequality. In this step we cannot take advantage of the approach used in the  $p$ -growth case, which is why the higher integrability for parabolic problems with Orlicz growth has remained unsolved. The first issue is that techniques for  $p > 2$  and  $p < 2$  in the  $p$ -growth case are different which is problematic in the Orlicz case. In this paper, we present a universal approach that is independent of whether the system (1.1) is degenerate, singular or neither. The second problem is that the classical Gagliardo–Nirenberg interpolation inequality, which is an important ingredient in the  $p$ -growth case, is not applicable to the Orlicz setting. In order to overcome this problem, we derive an interpolation inequality for the Orlicz case, see Lemma 2.13. The remaining part follows the approach used in [1,34,40] with modifications for the Orlicz setting using recent tools from [29].

Our paper is organized as follows. In the next section, Section 2, we introduce notation, Orlicz functions and derive an interpolation inequality. In Section 3, we obtain a reverse Hölder inequality. Finally in the last section, Section 4, we prove the main result, Theorem 1.5.

## 2. Preliminaries

### 2.1. Notation

For  $w = (y, \tau) \in \mathbb{R}^n \times \mathbb{R}$  we denote the usual parabolic cylinder by

$$Q_r(w) := B_r(y) \times (\tau - r^2, \tau + r^2),$$

where  $B_r(y)$  is the open ball in  $\mathbb{R}^n$  with center  $y$  and radius  $r$ , and the intrinsic parabolic cylinder (with function  $\varphi$ ) by

$$Q_r^\lambda(w) := B_r(y) \times I_r^\lambda(\tau) \quad \text{where} \quad I_r^\lambda(\tau) := (\tau - \frac{r^2}{\varphi_2(\lambda)}, \tau + \frac{r^2}{\varphi_2(\lambda)}),$$

and, for the function  $\varphi : [0, \infty) \rightarrow [0, \infty)$ , we define

$$\varphi_1(t) := \frac{\varphi(t)}{t} \quad \text{and} \quad \varphi_2(t) := \frac{\varphi(t)}{t^2}.$$

Let  $f, g : [0, \infty) \rightarrow [0, \infty)$ . The function  $f$  is said to be *almost increasing* if there exists  $L \geq 1$  such that  $f(t) \leq Lf(s)$  for all  $0 < t < s < \infty$ . If  $L = 1$  we say  $f$  is increasing. *Almost decreasing* and decreasing are defined analogously. We say that  $f$  and  $g$  are equivalent,  $f \approx g$  if there exists  $L \geq 1$  such that  $L^{-1}f(t) \leq g(t) \leq Lf(t)$  for all  $t \geq 0$ .

We define  $(f)_U := \int_U f \, dz := \frac{1}{|U|} \int_U f \, dz$ .

### 2.2. Orlicz functions

Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  and  $p, q > 0$ . We introduce some conditions.

(aInc) $_p$  The map  $(0, \infty) \ni t \mapsto \varphi(t)/t^p$  is almost increasing with constant  $L \geq 1$ .

(aDec) $_q$  The map  $(0, \infty) \ni t \mapsto \varphi(t)/t^q$  is almost decreasing with constant  $L \geq 1$ .

Note that (aInc) $_p$  implies (aInc) $_{p'}$  for all  $p' < p$  and (aDec) $_q$  implies (aDec) $_{q'}$  for all  $q' > q$ . If  $\varphi$  satisfies (aInc) $_p$  and (aDec) $_q$ , then  $p \leq q$  and for any  $t \in (0, \infty)$  and  $0 < c < 1 < C$ ,

$$c^q L^{-1} \varphi(t) \leq \varphi(ct) \leq c^p L \varphi(t) \quad \text{and} \quad C^p L^{-1} \varphi(t) \leq \varphi(Ct) \leq C^q L \varphi(t).$$

We shall use these inequalities numerous times later without explicit mention.

These conditions allow us to work easily with weak  $\Phi$ -functions, without resorting to tricks to ensure convexity.

**Definition 2.1.** The function  $\varphi : [0, \infty) \rightarrow [0, \infty]$  is said to be a *weak  $\Phi$ -function* if it is increasing with  $\varphi(0) = 0$ ,  $\lim_{t \rightarrow 0^+} \varphi(t) = 0$ ,  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$  and it satisfies (aInc) $_1$ .

As an example of the robustness of this definition, we note that  $\sqrt{\varphi(t^2)}$  need not be convex if  $\varphi$  is, but the (aInc) $_1$  property is conserved. Moreover, the condition (aInc) $_1$  captures some essential features of convexity, as it allows us to use the following Jensen-type inequality (cf. Lemma 4.3.2, [29]).

**Lemma 2.2 (Jensen inequality).** If  $\varphi : [0, \infty) \rightarrow [0, \infty]$  is increasing with  $\varphi(0) = 0$  and satisfies (aInc) $_1$  with constant  $L \geq 1$ , then

$$\varphi\left(\frac{1}{L^2} \int_U |f| dz\right) \leq \int_U \varphi(|f|) dz.$$

Also, we can use the conditions effectively with Young-type inequalities and inverse functions (see the proof of Lemma 2.9). We recall the definition of the conjugate weak  $\Phi$ -function:

$$\varphi^*(s) := \sup_{t \geq 0} (st - \varphi(t)).$$

This definition directly implies Young’s inequality:

$$st \leq \varphi(s) + \varphi^*(t), \quad s, t \geq 0. \tag{2.3}$$

The exact value of  $\varphi^*$  can usually not be determined, but we have the following useful estimate which can be found in the proof of [29, Theorem 2.4.10]:

$$\varphi^*\left(\frac{\varphi(t)}{t}\right) \approx \varphi(t). \tag{2.4}$$

This will be used multiple times in what follows. Moreover, if  $\varphi$  is differentiable with  $\varphi'$  satisfying (aDec) $_q$ , then, by [31, Lemma 3.6(2)],

$$\frac{\varphi(t)}{t} \approx \varphi'(t). \tag{2.5}$$

**Remark 2.6.** Suppose that  $\varphi$  is a weak  $\Phi$ -function which satisfies  $(\text{aInc})_p$  and  $(\text{aDec})_q$  with  $1 \leq p \leq q$ . We can define

$$\tilde{\varphi}(t) := \int_0^t s^{p-1} \sup_{\sigma \in (0,s]} \frac{\varphi(\sigma)}{\sigma^p} ds.$$

Then  $\tilde{\varphi} \approx \varphi$  is differentiable and convex (since its derivative is increasing) and also satisfies  $(\text{aInc})_p$  with  $L = 1$  and  $(\text{aDec})_q$ . Since  $\varphi(\sigma) > 0$  when  $\sigma > 0$ ,  $\tilde{\varphi}$  is also strictly increasing. Since the claims that we are proving are invariant under equivalence of weak  $\Phi$ -functions, we may thus assume when necessary that  $\varphi$  is differentiable, strictly increasing and a bijection.

We next introduce the left-inverse of a weak  $\Phi$ -function:

$$\varphi^{-1}(s) := \inf\{t \geq 0 : \varphi(t) \geq s\}.$$

Clearly,  $\varphi^{-1}(\varphi(t)) \leq t$  and, if  $\varphi$  is continuous,  $\varphi(\varphi^{-1}(s)) \geq s$ . Note that in view of Remark 2.6, we have  $(\varphi \circ \varphi^{-1})(t) \approx (\varphi^{-1} \circ \varphi)(t) \approx t$  if  $\varphi$  satisfies  $(\text{aDec})_q$  with  $q \geq 1$ . By [29, Proposition 2.3.7],  $\varphi^{-1}$  satisfies  $(\text{aInc})_{\frac{1}{q}}$  or  $(\text{aDec})_{\frac{1}{p}}$  if and only if  $\varphi$  satisfies  $(\text{aDec})_q$  or  $(\text{aInc})_p$ , respectively. From these facts and Lemma 2.2 we conclude the Jensen-inequality

$$\int_U \varphi(|f|) dz \leq c\varphi\left(\int_U |f| dz\right) \tag{2.7}$$

when  $\varphi$  satisfies  $(\text{aDec})_1$ .

Let us quote for later use a version of the standard iteration lemma which is particularly adapted to the Orlicz case [30, Lemma 4.2]. Recall that doubling means that  $X(2s) \leq CX(s)$ , which is equivalent to  $(\text{aDec})_q$  for some  $q < \infty$ .

**Lemma 2.8.** *Let  $Z$  be a bounded non-negative function in the interval  $[r, R] \subset \mathbb{R}$  and let  $X$  be a doubling function in  $[0, \infty)$ . Assume that there exists  $\theta \in [0, 1)$  such that*

$$Z(t) \leq X\left(\frac{1}{s-t}\right) + \theta Z(s)$$

for all  $r \leq t < s \leq R$ . Then

$$Z(r) \lesssim X\left(\frac{1}{R-r}\right),$$

where the implicit constant depends only on the doubling constant and  $\theta$ .

We end this subsection with the following lemma and notation which will be used often.

**Lemma 2.9.** Assume that  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is a weak  $\Phi$ -function satisfying  $(aInc)_p$  and  $(aDec)_q$  with  $1 < p \leq q$  and that

$$\int_U \varphi(|f|) dz \leq \varphi(\lambda),$$

where  $f : U \rightarrow \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , and  $\lambda > 0$ . Then, for any  $\theta_0 \in [1 - \frac{1}{q}, 1]$  and  $\delta \in (0, 1)$ ,

$$A_0 \leq \begin{cases} \delta\lambda + c_\delta\varphi^{-1}(\Theta_0) \\ c\lambda \end{cases}$$

for some  $c_\delta = c(p, q, L, \delta) > 0$ , where

$$A_0 := \frac{1}{\varphi_2(\lambda)} \int_U \varphi_1(|f|) dz \quad \text{and} \quad \Theta_0 := \left( \int_U \varphi(|f|)^{\theta_0} dz \right)^{\frac{1}{\theta_0}}. \tag{2.10}$$

**Proof.** Since  $\varphi_1(\varphi^{-1}(t)) \approx \frac{t}{\varphi^{-1}(t)}$  satisfies  $(aDec)_{1-\frac{1}{q}}$ , we find that  $t \mapsto (\varphi_1 \circ \varphi^{-1})(t^{1/\theta_0})$  satisfies  $(aDec)_1$ . Then we obtain by (2.7) and Hölder’s inequality that

$$\int_U \varphi_1(|f|) dz \leq c(\varphi_1 \circ \varphi^{-1})(\Theta_0) \leq c(\varphi_1 \circ \varphi^{-1})\left(\int_U \varphi(|f|) dz\right) \leq c\varphi_1(\lambda).$$

We use only the first inequality to estimate  $\frac{1}{q}$ -part of the integral, and the whole inequality for the remaining  $(1 - \frac{1}{q})$  of the integral. Thus

$$\frac{1}{\varphi_2(\lambda)} \int_U \varphi_1(|f|) dz \leq \frac{c[(\varphi_1 \circ \varphi^{-1})(\Theta_0)]^{\frac{1}{q}}}{\varphi_2(\lambda)} \left( \int_U \varphi_1(|f|) dz \right)^{1-\frac{1}{q}} \leq \frac{c\lambda}{\varphi_1(\lambda)^{\frac{1}{q}}} [(\varphi_1 \circ \varphi^{-1})(\Theta_0)]^{\frac{1}{q}}.$$

Let us define  $H$  by

$$H(t) := G^{-1}(t^q), \quad \text{where} \quad G(t) := \frac{t^{q+1}}{\varphi(t)}.$$

Observe that, since  $G$  satisfies  $(aInc)_1$  and  $(aDec)_{q+1-p}$ ,  $H$  satisfies  $(aInc)_{\frac{q}{q+1-p}}$  and  $(aDec)_q$  and

$$H^{-1}(t) \approx \left( \frac{t^{q+1}}{\varphi(t)} \right)^{\frac{1}{q}} = \frac{t}{\varphi_1(t)^{1/q}}.$$

From  $H^{-1}(t)(H^*)^{-1}(t) \approx t$ , we see that  $(H^*)^{-1}(t) \approx \frac{t}{H^{-1}(t)} \approx \varphi_1(t)^{\frac{1}{q}}$ . Then by Young’s inequality we have that for any  $\delta > 0$ ,

$$\frac{1}{\varphi_2(\lambda)} \int_U \varphi_1(|f|) dz \leq \delta H\left(\frac{\lambda}{\varphi_1(\lambda)^{1/q}}\right) + c_\delta H^*\left([\varphi_1 \circ \varphi^{-1}](\Theta_0)\right)^{\frac{1}{q}} \approx \delta \lambda + c_\delta \varphi^{-1}(\Theta_0).$$

For the upper bound  $c\lambda$ , we simply fix  $\delta = 1$  and use Hölder’s inequality for  $\Theta_0$  as before.  $\square$

### 2.3. Gagliardo–Nirenberg type interpolation inequality for Orlicz functions

In this subsection, we obtain a Gagliardo–Nirenberg type interpolation inequality involving Orlicz functions which will be a crucial ingredient in the proof of a reverse Hölder type estimate. Let us recall a usual scaling invariant version of the Gagliardo–Nirenberg interpolation inequality in balls (see [42]): for  $\gamma > 0$ ,  $p \in [1, n)$ ,  $q \geq 1$  and  $\theta \in [0, 1]$ ,

$$\left(\int_{B_r} \left|\frac{f}{r}\right|^\gamma dx\right)^{\frac{1}{\gamma}} \leq c \left(\int_{B_r} [ |Df|^p + \left|\frac{f}{r}\right|^p ] dx\right)^{\frac{\theta}{p}} \left(\int_{B_r} \left|\frac{f}{r}\right|^q dx\right)^{\frac{1-\theta}{q}} \tag{2.11}$$

for some  $c = c(n, p) > 0$ , provided that

$$\frac{1}{\gamma} \geq \frac{\theta}{p^*} + \frac{1-\theta}{q},$$

where  $p^* = \frac{np}{n-p}$ . Let us prove that this inequality also holds for  $q \in (0, 1)$ . This may be known, but we have not found a proof (e.g. it is not mentioned in [48] which deals with extensions of the parameter ranges).

**Lemma 2.12** (Gagliardo–Nirenberg inequality). *Let  $p \in [1, n)$ ,  $\gamma, q > 0$  and  $\theta \in [0, 1]$ . Then*

$$\left(\int_{B_r} \left|\frac{f}{r}\right|^\gamma dx\right)^{\frac{1}{\gamma}} \leq c \left(\int_{B_r} [ |Df|^p + \left|\frac{f}{r}\right|^p ] dx\right)^{\frac{\theta}{p}} \left(\int_{B_r} \left|\frac{f}{r}\right|^q dx\right)^{\frac{1-\theta}{q}}$$

for some  $c = c(n, p) > 0$ , provided that

$$\frac{1}{\gamma} \geq \frac{\theta}{p^*} + \frac{1-\theta}{q}.$$

**Proof.** For  $q \geq 1$ , the claim is just the Gagliardo–Nirenberg inequality (2.11) quoted above. For  $q \in (0, 1)$ , choose  $\tilde{\theta} \in (0, 1)$  such that

$$\tilde{\theta} p^* + (1 - \tilde{\theta})q = 1.$$

Then by Hölder’s inequality we have

$$\int_{B_r} \left|\frac{f}{r}\right| dx \leq \left(\int_{B_r} \left|\frac{f}{r}\right|^{p^*} dx\right)^{\tilde{\theta}} \left(\int_{B_r} \left|\frac{f}{r}\right|^q dx\right)^{1-\tilde{\theta}}$$

and by the Sobolev–Poincaré inequality

$$\int_{B_r} \left| \frac{f}{r} \right|^{p^*} dx \leq c \left( \int_{B_r} \left[ |Df|^p + \left| \frac{f}{r} \right|^p \right] dx \right)^{\frac{p^*}{p}}.$$

We use the Gagliardo–Nirenberg inequality (2.11) for  $(\gamma, p, q, \theta) = (\gamma, p, 1, \theta_0)$  and the estimates from the previous paragraph to conclude that

$$\begin{aligned} \left( \int_{B_r} \left| \frac{f}{r} \right|^\gamma dx \right)^{\frac{1}{\gamma}} &\leq c \left( \int_{B_r} \left[ |Df|^p + \left| \frac{f}{r} \right|^p \right] dx \right)^{\frac{\theta_0}{p}} \left( \int_{B_r} \left| \frac{f}{r} \right| dx \right)^{1-\theta_0} \\ &\leq c \left( \int_{B_r} \left[ |Df|^p + \left| \frac{f}{r} \right|^p \right] dx \right)^{\frac{\theta_0 + (1-\theta_0)\tilde{\theta}p^*}{p}} \left( \int_{B_r} \left| \frac{f}{r} \right|^q dx \right)^{(1-\tilde{\theta})(1-\theta_0)} \end{aligned}$$

provided that

$$\frac{1}{\gamma} \geq \frac{\theta_0}{p^*} + 1 - \theta_0.$$

Let  $\theta := \theta_0 + (1 - \theta_0)\tilde{\theta}p^* = 1 - (1 - \theta_0)(1 - \tilde{\theta})q$ . Then the exponent of the first term on the right-hand side is  $\frac{\theta}{p}$  and the exponent of the second term is  $\frac{1-\theta}{q}$ . Thus we have the Gagliardo–Nirenberg inequality for parameter value  $\theta$ . It remains to check that the bound for  $\gamma$  is correct. For this we calculate

$$\frac{\theta}{p^*} + \frac{1-\theta}{q} = \frac{\theta_0}{p^*} + \tilde{\theta}(1-\theta_0) + (1-\tilde{\theta})(1-\theta_0) = \frac{\theta_0}{p^*} + 1 - \theta_0. \quad \square$$

We conclude this subsection by deriving a Gagliardo–Nirenberg type inequality for Orlicz functions.

**Lemma 2.13.** *Suppose that  $\psi : [0, \infty) \rightarrow [0, \infty)$  is a weak  $\Phi$ -function and satisfies  $(aDec)_{q_1}$  for some  $q_1 \geq 1$ . For  $p \in [1, n)$ ,  $\gamma, q_2 > 0$  and  $\theta \in [0, 1]$  we have*

$$\left( \int_{B_r} \psi \left( \left| \frac{f}{r} \right| \right)^\gamma dx \right)^{\frac{1}{\gamma}} \leq c \left( \int_{B_r} \left[ \psi(|Df|)^p + \psi \left( \left| \frac{f}{r} \right|^p \right) \right] dx \right)^{\frac{\theta}{p}} \psi \left( \left( \int_{B_r} \left| \frac{f}{r} \right|^{q_2} dx \right)^{\frac{1}{q_2}} \right)^{1-\theta}$$

for some  $c = c(n, L, q_1, q_2) > 0$ , provided that

$$\frac{1}{\gamma} \geq \frac{\theta}{p^*} + \frac{(1-\theta)q_1}{q_2}.$$

**Proof.** In view of Remark 2.6, it suffices to consider  $\psi$  that is strictly increasing and differentiable with  $\psi'(t) \geq 0$ . Then using Young’s inequality (2.3), (2.5) and (2.4) we have that

$$|D(r\psi(|\frac{f}{r}|))| = \psi'(|\frac{f}{r}|)|Df| \leq \psi(|Df|) + \psi^*(\psi'(|\frac{f}{r}|)) \leq \psi(|Df|) + c\psi(|\frac{f}{r}|).$$

Hence applying Lemma 2.12 with  $q = \frac{q_2}{q_1}$  to the function  $r\psi\left(\left|\frac{f}{r}\right|\right)$ , we conclude that

$$\begin{aligned} \left(\int_{B_r} \psi\left(\left|\frac{f}{r}\right|\right)^{\gamma} dx\right)^{\frac{1}{\gamma}} &\leq c\left(\int_{B_r} \left[\psi(|Df|)^p + \psi\left(\left|\frac{f}{r}\right|\right)^p\right] dx\right)^{\frac{\theta}{p}} \left(\int_{B_r} \psi\left(\left|\frac{f}{r}\right|\right)^q dx\right)^{\frac{1-\theta}{q}} \\ &\leq c\left(\int_{B_r} \left[\psi(|Df|)^p + \psi\left(\left|\frac{f}{r}\right|\right)^p\right] dx\right)^{\frac{\theta}{p}} \psi\left(\left(\int_{B_r} \left|\frac{f}{r}\right|^{q_2} dx\right)^{\frac{1}{q_2}}\right)^{1-\theta}; \end{aligned}$$

here the last inequality follows from Lemma 2.2 since  $t \mapsto \psi^{-1}(t^{1/q})^{q_2}$  satisfies (aInc)<sub>1</sub>. □

### 3. Poincare and reverse Hölder type inequalities

In this section, we derive a reverse Hölder type inequality for the gradients of weak solutions to (1.1) on regions satisfying a balancing condition, (3.13). We suppose that  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is a weak  $\Phi$ -function and satisfies (aInc)<sub>p</sub> and (aDec)<sub>q</sub> with constant  $L \geq 1$  and that  $p$  and  $q$  satisfy (1.3). We start with a Caccioppoli type inequality.

**Lemma 3.1** (Caccioppoli inequality). *Let  $u$  be a weak solution to (1.1) with (1.2) and  $Q_R^\lambda \Subset \Omega_I$  with  $\lambda, R > 0$ . For  $r \in [\frac{R}{4}, R)$  and  $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ , we have*

$$\begin{aligned} \varphi_2(\lambda) \sup_{s \in I_r^\lambda} \int_{B_r} \left|\frac{u(s) - a}{r}\right|^2 dx + \int_{Q_r^\lambda} \varphi(|Du|) dz \\ \leq c \int_{Q_R^\lambda} \left[\varphi_2(\lambda) \left|\frac{u - a}{R - r}\right|^2 + \varphi\left(\left|\frac{u - a}{R - r}\right|\right)\right] dz \end{aligned}$$

for some  $c = c(n, N, p, q, L, v, \Lambda) > 0$ , where  $u(s) = u(x, s)$ .

**Proof.** We assume without loss of generality that  $Q_R^\lambda$  is centered at the origin. Let  $\eta \in C_0^1(B_R)$  with  $\eta \equiv 1$  in  $B_r$  and  $|D\eta| \leq c/(R - r)$  and  $\tau \in C^1(\mathbb{R})$  with  $\tau \equiv 0$  in  $(-\infty, -R^2/\varphi_2(\lambda)]$ ,  $\tau \equiv 1$  in  $[-r^2/\varphi_2(\lambda), \infty)$  and  $0 \leq \tau' \leq \varphi_2(\lambda)/(R - r)^2$ . Define

$$\zeta(x, t) := \eta(x)^q \tau(t)^2 (u(x, t) - a)$$

and  $\sigma := -R^2/\varphi_2(\lambda)$ . Using  $\zeta$  as a test function, we have that for  $s \in I_R^\lambda$ ,

$$\int_{\sigma}^s \int_{B_R} [\partial_t u_i \zeta_i + A_i(x, t, u, Du) \cdot D\zeta_i] dx dt = 0, \quad i = 1, \dots, N, \tag{3.2}$$

see Remark 3.3, below. Since  $\partial_t a = 0$ , we note that

$$\begin{aligned} \int_{\sigma}^s \int_{B_R} \partial_t u_i \zeta_i \, dx \, dt &= \int_{\sigma}^s \int_{B_R} \frac{1}{2} \partial_t [\eta^q \tau^2 (u_i - a_i)^2] - \eta_i^q \tau \tau' (u_i - a_i)^2 \, dx \, dt \\ &= \frac{1}{2} \int_{B_R} \eta^q \tau(s)^2 (u_i(s) - a_i)^2 \, dx - \int_{\sigma}^s \int_{B_R} \eta^q \tau \tau' (u_i - a_i)^2 \, dx \, dt, \end{aligned}$$

where  $u_i(s) = u_i(x, s)$ . Then, summing (3.2) for  $i = 1, \dots, N$  and using (1.2), we have

$$\begin{aligned} &\frac{1}{2} \int_{B_R} \eta^q \tau(s)^2 |u(s) - a|^2 \, dx + \nu \int_{\sigma}^s \int_{B_R} \eta^q \tau^2 \varphi(|Du|) \, dx \, dt \\ &\leq -q \int_{\sigma}^s \int_{B_R} \eta^{q-1} \tau^2 \sum_{i=1}^N (u_i - a_i) A_i(x, u, Du) \cdot D\eta \, dx \, dt + \int_{\sigma}^s \int_{B_R} \eta^q \tau \tau' |u - a|^2 \, dx \, dt \\ &\leq c \int_{\sigma}^s \int_{B_R} \tau^2 \eta^{q-1} \frac{\varphi(|Du|)}{|Du|} \left| \frac{u - a}{R - r} \right| \, dx \, dt + \int_{\sigma}^s \int_{B_R} \varphi_2(\lambda) \left| \frac{u - a}{R - r} \right|^2 \, dx \, dt. \end{aligned}$$

Moreover, by Young’s inequality (2.3) and (2.4) and since  $\varphi^*$  satisfies (aInc) $_{\frac{q}{q-1}}$  [29, Proposition 2.4.13], we have that for any  $\varepsilon > 0$

$$\begin{aligned} \eta^{q-1} \frac{\varphi(|Du|)}{|Du|} \left| \frac{u - a}{R - r} \right| &\leq \varepsilon \varphi^* \left( \eta^{q-1} \frac{\varphi(|Du|)}{|Du|} \right) + c_{\varepsilon} \varphi \left( \left| \frac{u - a}{R - r} \right| \right) \\ &\leq c \varepsilon \eta^q \varphi(|Du|) + c_{\varepsilon} \varphi \left( \left| \frac{u - a}{R - r} \right| \right). \end{aligned}$$

We choose  $\varepsilon$  so small that the first term can be absorbed in the left-hand side. Therefore, combining the above inequalities and using the fact that  $\tau \equiv 1$  in  $I_r^\lambda$  and  $\eta \equiv 1$  in  $B_r$ , we have

$$\int_{B_r} |u(s) - a|^2 \, dx + \int_{-\frac{r^2}{\varphi_2(\lambda)}}^s \int_{B_r} \varphi(|Du|) \, dx \, dt \leq c \int_{Q_R^\lambda} \left[ \varphi_2(\lambda) \left| \frac{u - a}{R - r} \right|^2 + \varphi \left( \left| \frac{u - a}{R - r} \right| \right) \right] \, dz$$

for all  $s \in I_r^\lambda$ . Finally, we obtain the claim by dividing both sides by  $r^{n+2} \varphi_2(\lambda)$  and taking into consideration that  $|B_r| \approx r^n$  while  $|Q_R^\lambda| \approx r^{n+2} \varphi_2(\lambda)$ .  $\square$

**Remark 3.3.** When we consider parabolic problems and want to obtain useful estimates such as Caccioppoli inequalities, we have to use test functions depending on the weak solution  $u$  in the weak formulation (1.4). However, the weak solution  $u$  to the parabolic system (1.1) may not be differentiable in the time variable. In fact, we do not need differentiability with respect to the time variable when we prove the existence of weak solution to parabolic problems. In order overcome this difficulty, one way is to consider Steklov averages, see [18] and also [6]. However, since this argument is now quite standard, we shall abuse the notation  $\partial_t u$  without further explanation.

Let  $\eta \in C_0^\infty(B_\rho)$  be a cut-off function such that  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  in  $B_{\rho/2}$ ,  $|D\eta| \leq \frac{4}{\rho}$ . We note that  $\|\eta\|_1 \approx |B_\rho|$ . Define

$$(u)_\rho^\lambda := \frac{1}{\|\eta\|_1} \int_{I_\rho^\lambda} \int_{B_\rho} u \eta \, dx \, dt \quad \text{and} \quad \langle u \rangle_\eta(s) := \frac{1}{\|\eta\|_1} \int_{B_\rho} u(x, s) \eta \, dx \quad \text{for } s \in I_\rho^\lambda.$$

Let us start with a complicated ‘‘Sobolev–Poincaré’’ inequality.

**Lemma 3.4.** *Let  $u$  be a weak solution to (1.1) with (1.2) and  $Q_{4\rho}^\lambda \Subset \Omega_I$  with  $\lambda > 0$  and  $\rho \leq r < R \leq 4\rho$ . For a weak  $\Phi$ -function  $\psi$  satisfying  $(aInc)_{p_1}$  and  $(aDec)_{q_1}$ ,  $1 \leq p_1 \leq q_1$ , we have*

$$\int_{Q_r^\lambda} \psi \left( \left| \frac{u - (u)_\rho^\lambda}{r} \right| \right) dz \leq c\psi(A_0) + c\psi(T(r, R)^{\frac{1}{2}})^{(1-\theta_0)} \int_{Q_r^\lambda} \psi(|Du|)^{\theta_0} dz,$$

for some  $c = c(n, N, p, q, p_1, q_1, \theta_0, L, v, \Lambda) > 0$  provided that

$$\theta_0 p_1 \in [1, n) \quad \text{and} \quad \frac{nq_1}{nq_1 + 2p_1} \leq \theta_0 \leq 1.$$

Here,  $A_0$  is from (2.10) with  $U := Q_r^\lambda$  and  $f = Du$ , and

$$T(r, R) := \int_{Q_R^\lambda} \left[ \left| \frac{u - (u)_\rho^\lambda}{R - r} \right|^2 + \frac{1}{\varphi_2(\lambda)} \varphi \left( \left| \frac{u - (u)_\rho^\lambda}{R - r} \right| \right) \right] dz + A_0^2.$$

**Proof.** By the triangle inequality,

$$\begin{aligned} \int_{Q_r^\lambda} \psi \left( \left| \frac{u - (u)_\rho^\lambda}{r} \right| \right) dz &= \int_{Q_r^\lambda} \psi \left( \left| \frac{u(z) - \langle u \rangle_\eta(t) + \langle u \rangle_\eta(t) - (u)_\rho^\lambda}{r} \right| \right) dz \\ &\leq c \int_{I_r^\lambda} \psi \left( \left| \frac{\langle u \rangle_\eta(t) - (u)_\rho^\lambda}{r} \right| \right) dt + c \int_{Q_r^\lambda} \psi \left( \left| \frac{u(z) - \langle u \rangle_\eta(t)}{r} \right| \right) dz. \end{aligned} \tag{3.5}$$

We first take care of the first term. By the definition of  $\langle u \rangle_\eta$  and using the weak formulation (1.4) with test-function  $\zeta(x, t) := \eta(x)$ , we find that for each  $i = 1, \dots, N$  and  $t \in I_r^\lambda$ ,

$$\begin{aligned}
 |\langle u_i \rangle_\eta(t) - (u_i)_\rho^\lambda| &\leq \sup_{\sigma \in I_r^\lambda} |\langle u_i \rangle_\eta(t) - \langle u_i \rangle_\eta(\sigma)| = \sup_{\sigma \in I_r^\lambda} \left| \int_\sigma^t \partial_t \langle u_i \rangle_\eta(s) ds \right| \\
 &= \sup_{\sigma \in I_r^\lambda} \left| \int_\sigma^t \frac{1}{\|\eta\|_1} \int_{B_r} \partial_t u_i(x, s) \eta(x) dx ds \right| \\
 &\approx \sup_{\sigma \in I_r^\lambda} \left| \int_\tau^t \int_{B_r} A_i(x, s, u, Du) \cdot D\eta dx ds \right| \\
 &\leq \frac{cr}{\varphi_2(\lambda)} \int_{Q_r^\lambda} \varphi_1(|Du|) dz = rA_0.
 \end{aligned}
 \tag{3.6}$$

This gives the first term on the right-hand side of the claim.

We next use the Gagliardo–Nirenberg inequality (Lemma 2.13) with  $(\psi, \gamma, p, q_1, q_2)$  given by  $(\psi^{1/p_1}, p_1, \theta_0 p_1, \frac{q_1}{p_1}, 2)$  to conclude that

$$\int_{B_r} \psi\left(\left|\frac{f}{r}\right|\right) dx \leq c \int_{B_r} [\psi(|Df|)^{\theta_1} + \psi\left(\left|\frac{f}{r}\right|\right)^{\theta_0}] dx \psi\left(\left[\int_{B_r} \left|\frac{f}{r}\right|^2 dx\right]^{\frac{1}{2}}\right)^{1-\theta_0}$$

provided  $\theta_0 p_1 \in [1, n)$  and

$$\frac{1}{p_1} \geq \frac{\theta_0}{(\theta_0 p_1)^*} + \frac{1-\theta_0}{2} \frac{q_1}{p_1} = \frac{1}{p_1} - \frac{\theta_0}{n} + \frac{1-\theta_0}{2} \frac{q_1}{p_1}.$$

This can be written as  $\theta_0 \geq \frac{nq_1}{nq_1+2p_1}$ .

The previous inequality for  $f := u - \langle u \rangle_\eta$  on each time slice gives

$$\begin{aligned}
 \int_{Q_r^\lambda} \psi\left(\left|\frac{u(z) - \langle u \rangle_\eta(t)}{r}\right|\right) dz &= \int_{I_r^\lambda} \int_{B_r} \psi\left(\left|\frac{f(x,t)}{r}\right|\right) dx dt \\
 &\leq c \int_{Q_r^\lambda} [\psi(|Df|)^{\theta_0} + \psi\left(\frac{f}{r}\right)^{\theta_0}] dz \psi\left(\left(\sup_{t \in I_r^\lambda} \int_{B_r} \left|\frac{f(x,t)}{r}\right|^2 dx\right)^{\frac{1}{2}}\right)^{1-\theta_0}.
 \end{aligned}
 \tag{3.7}$$

In the first term we then use the following weighted Poincaré inequality for each time slice:

$$\begin{aligned}
 \int_{B_r} \psi\left(\left|\frac{f(x,t)}{r}\right|\right)^{\theta_0} dx &= \int_{B_r} \psi\left(\left|\frac{u(x,t) - \langle u \rangle_\eta(t)}{r}\right|\right)^{\theta_0} dx \\
 &\leq c \int_{B_r} \psi(|Du(x,t)|)^{\theta_0} dx = c \int_{B_r} \psi(|Df(x,t)|)^{\theta_0} dx,
 \end{aligned}$$

see [29, Lemma 6.2.5 and Corollary 7.4.1(b)] (here we need  $\theta_0 p_1 \geq 1$ ). Finally, from the Caccioppoli inequality (Lemma 3.1) and (3.6) we conclude that

$$\begin{aligned} \sup_{t \in I_r^\lambda} \int_{B_r} \left| \frac{f(x,t)}{r} \right|^2 dx &\leq c \sup_{t \in I_r^\lambda} \int_{B_r} \left| \frac{u(x,t) - (u)_\rho^\lambda}{r} \right|^2 dx + c \sup_{t \in I_r^\lambda} \int_{B_r} \left| \frac{(u)_\rho^\lambda - \langle u \rangle_\eta(t)}{r} \right|^2 dx \\ &\leq c \int_{Q_R^\lambda} \left[ \left| \frac{u - (u)_\rho^\lambda}{R-r} \right|^2 + \frac{1}{\varphi_2(\lambda)} \varphi \left( \left| \frac{u - (u)_\rho^\lambda}{R-r} \right| \right) \right] dz + c A_0^2. \end{aligned}$$

Combining (3.5), (3.6) and (3.7), we obtain the claim.  $\square$

If we choose  $\theta_0 = p_1 = 1$  in the previous lemma, we obtain the following result, since the complicated term involving  $T$  vanishes as its exponent is zero.

**Corollary 3.8 (Poincaré inequality).** *Let  $u$  be a weak solution to (1.1) with (1.2) and  $Q_r^\lambda \Subset \Omega_I$  with  $\lambda > 0$ . For a weak  $\Phi$ -function  $\psi$  satisfying  $(aDec)_{q_1}$  we have*

$$\int_{Q_r^\lambda} \psi \left( \left| \frac{u - (u)_\rho^\lambda}{r} \right| \right) dz \leq c \int_{Q_r^\lambda} \psi(|Du|) dz + c \psi(A_0),$$

where  $A_0$  is from (2.10) with  $U := Q_{2\rho}^\lambda$  and  $f = Du$ .

Over the course of the next two results we will show how the extra terms in the previous lemma can be estimated by suitable quantities when we are in suitable intrinsic cylinders.

**Lemma 3.9.** *We assume the assumptions of Lemma 3.4, and additionally that*

$$\int_{Q_{4\rho}^\lambda} \varphi(|Du|) dz \leq \varphi(\lambda).$$

Then, for some  $c = c(n, N, p, q, p_1, q_1, \theta_0, L, v, \Lambda) > 0$ ,

$$\int_{Q_{2\rho}^\lambda} \psi \left( \left| \frac{u - (u)_\rho^\lambda}{\rho} \right| \right) dz \leq c \psi(A_0) + c \psi(\lambda)^{1-\theta_0} \int_{Q_{2\rho}^\lambda} \psi(|Du|^{\theta_0}) dz,$$

where  $A_0$  is from (2.10) with  $U := Q_{2\rho}^\lambda$  and  $f = Du$ .

**Proof.** The claim follows once we show that  $T$  from Lemma 3.4 satisfies  $T(2\rho, 3\rho) \leq c\lambda$ . We first note from Lemma 2.9 with  $U = Q_r^\lambda$  that

$$\frac{1}{\varphi_2(\lambda)} \int_{Q_r^\lambda} \varphi_1(|Du|) dz \leq c\varphi(\lambda) \quad \text{for all } r \in [\rho, 4\rho]. \tag{3.10}$$

Using this and Corollary 3.8 with  $\psi := \varphi$ , we find that for any  $\rho \leq r < R \leq 4\rho$ ,

$$\int_{Q_R^\lambda} \varphi\left(\left|\frac{u - (u)_\rho^\lambda}{R}\right|\right) dz \leq c \int_{Q_R^\lambda} \varphi(|Du|) dz + c\varphi\left(\frac{1}{\varphi_2(\lambda)} \int_{Q_R^\lambda} \varphi_1(|Du|) dz\right) \leq c\varphi(\lambda)$$

and hence

$$\frac{1}{\varphi_2(\lambda)} \int_{Q_R^\lambda} \varphi\left(\left|\frac{u - (u)_\rho^\lambda}{R - r}\right|\right) dz \leq \left(\frac{R}{R-r}\right)^q \frac{1}{\varphi_2(\lambda)} \int_{Q_R^\lambda} \varphi\left(\left|\frac{u - (u)_\rho^\lambda}{R}\right|\right) dz \leq c\left(\frac{R}{R-r}\right)^q \lambda^2.$$

With the previous inequalities, we have the following estimate for  $T$  from Lemma 3.4: for any  $\rho \leq r < R \leq 4\rho$ ,

$$T(r, R) \leq \left(\frac{R}{R-r}\right)^2 \int_{Q_R^\lambda} \left|\frac{u - (u)_\rho^\lambda}{R}\right|^2 dz + c\left(\frac{R}{R-r}\right)^q \lambda^2. \tag{3.11}$$

Let  $p_0 := \frac{2n}{n+2}$ ,  $\psi(t) := t^2$  and  $\theta_0 := \frac{p_0}{2}$ . Then by Lemma 2.2 for the map  $t \mapsto \varphi(t^{1/p_0})$ ,

$$\int_{Q_r^\lambda} \psi(|Du|)^{\theta_0} dz = \int_{Q_r^\lambda} |Du|^{p_0} dz \leq c\varphi^{-1}\left(\int_{Q_r^\lambda} \varphi(|Du|) dz\right)^{p_0} \leq c\lambda^{p_0}$$

for all  $r \in [\rho, 4\rho]$ . With the last two estimates, (3.10) and Young’s inequality, Lemma 3.4 gives us in this case that

$$\begin{aligned} \int_{Q_r^\lambda} \left|\frac{u - (u)_\rho^\lambda}{r}\right|^2 dz &\leq c\lambda^2 + c\lambda^{p_0} \left(\left(\frac{R}{R-r}\right)^2 \int_{Q_R^\lambda} \left|\frac{u - (u)_\rho^\lambda}{R}\right|^2 dz + \left(\frac{R}{R-r}\right)^q \lambda^2\right)^{1-\theta_0} \\ &\leq c\left(\frac{R}{R-r}\right)^{\frac{2q}{n+2}} \lambda^2 + c\lambda^{p_0} \left(\frac{R}{R-r}\right)^{\frac{4}{n+2}} \left(\int_{Q_R^\lambda} \left|\frac{u - (u)_\rho^\lambda}{R}\right|^2 dz\right)^{\frac{2}{n+2}} \\ &\leq \varepsilon \int_{Q_R^\lambda} \left|\frac{u - (u)_\rho^\lambda}{R}\right|^2 dz + c\left(\frac{R}{R-r}\right)^{\frac{2q}{n+2}} \lambda^2 + c_\varepsilon \left(\frac{R}{R-r}\right)^{\frac{4}{n}} \lambda^2, \end{aligned}$$

for any  $\varepsilon \in (0, 1)$ . Then, since  $\rho \leq r < R \leq 4\rho$ , we have

$$\underbrace{\int_{Q_r^\lambda} \left|\frac{u - (u)_\rho^\lambda}{\rho}\right|^2 dz}_{=Z(r)} \leq \frac{1}{2} \underbrace{\int_{Q_R^\lambda} \left|\frac{u - (u)_\rho^\lambda}{\rho}\right|^2 dz}_{=Z(R)} + \underbrace{c\left[\left(\frac{4\rho}{R-r}\right)^{\frac{2q}{n+2}} + \left(\frac{4\rho}{R-r}\right)^{\frac{4}{n}}\right]}_{=X(1/(R-r))} \lambda^2.$$

Therefore, by applying a standard iteration Lemma 2.8, we obtain  $Z(r) \leq cX(1/(R - r))$ , i.e.

$$\int_{Q_r^\lambda} \left| \frac{u - (u)_\rho^\lambda}{\rho} \right|^2 dz \leq c \left[ \left( \frac{\rho}{R-r} \right)^{\frac{2q}{n+2}} + \left( \frac{\rho}{R-r} \right)^{\frac{4}{n}} \right] \lambda^2$$

for any  $\rho \leq r < R \leq 4\rho$ . We choose here  $(r, R) = (3\rho, 4\rho)$  and use it in (3.11) with  $(r, R) = (2\rho, 3\rho)$  to conclude  $T(2\rho, 3\rho) \leq c\lambda^2$ .  $\square$

Now, we derive a reverse Hölder inequality.

**Lemma 3.12.** *Let  $u$  be a weak solution to (1.1) with (1.2) and  $Q_{4\rho}^\lambda \Subset \Omega_I$  with  $\lambda, \rho > 0$ . Suppose that*

$$\varphi(\lambda) \leq \int_{Q_\rho^\lambda} \varphi(|Du|) dz \quad \text{and} \quad \int_{Q_{4\rho}^\lambda} \varphi(|Du|) dz \leq \varphi(\lambda). \tag{3.13}$$

Then there exist  $\theta = \theta(n, p, q) \in (0, 1)$  and  $c = c(n, N, p, q, L, v, \Lambda) > 0$  such that

$$\int_{Q_\rho^\lambda} \varphi(|Du|) dz \leq c \left( \int_{Q_{4\rho}^\lambda} \varphi(|Du|)^\theta dz \right)^{\frac{1}{\theta}}.$$

**Proof.** We denote  $p_0 := \frac{2n}{n+2}$ , and  $A_0$  and  $\Theta_0$  as in (2.10) with  $U := Q_{2\rho}^\lambda$ . By the Caccioppoli inequality (Lemma 3.1) with  $a := (u)_\rho^\lambda$ , we find that

$$\int_{Q_\rho^\lambda} \varphi(|Du|) dz \leq c\varphi_2(\lambda) \int_{Q_{2\rho}^\lambda} \left| \frac{u - (u)_\rho^\lambda}{\rho} \right|^2 dz + c \int_{Q_{2\rho}^\lambda} \varphi \left( \left| \frac{u - (u)_\rho^\lambda}{\rho} \right| \right) dz. \tag{3.14}$$

We then estimate the last two integrals.

By Lemma 3.9 for  $\psi := \varphi$ , considering also Lemma 2.9 with  $U = Q_{2\rho}^\lambda$  and Young’s inequality, we have that for any  $\delta \in (0, 1)$

$$\int_{Q_{2\rho}^\lambda} \varphi \left( \left| \frac{u - (u)_\rho^\lambda}{\rho} \right| \right) dz \leq c\varphi(A_0) + c\varphi(\lambda)^{(1-\theta_0)} \Theta_0^{\theta_0} \leq c_\delta \Theta_0 + c\delta\varphi(\lambda). \tag{3.15}$$

Using the same steps in the case  $\psi(t) := t^2$  and  $\theta_0 := \frac{p_0}{2}$ , we conclude that for any  $\delta \in (0, 1)$

$$\begin{aligned} \left( \int_{Q_{2\rho}^\lambda} \left| \frac{u - (u)_\rho^\lambda}{\rho} \right|^2 dz \right)^{\frac{1}{2}} &\leq cA_0 + c \left( \lambda^{2-p_0} \int_{Q_{2\rho}^\lambda} |Du|^{p_0} dz \right)^{\frac{1}{2}} \\ &\leq c_\delta \left( \int_{Q_{2\rho}^\lambda} |Du|^{p_0} dz \right)^{\frac{1}{p_0}} + cA_0 + \delta\lambda. \end{aligned}$$

In particular, we also have

$$\left( \int_{Q_{2\rho}^\lambda} \left| \frac{u - (u)_\rho^\lambda}{\rho} \right|^2 dz \right)^{\frac{1}{2}} \leq c\lambda.$$

Multiplying the previous two inequalities, and using Young’s inequality with (2.4) for the second step and Lemma 2.9, we obtain that for any  $\delta \in (0, 1)$

$$\begin{aligned} \varphi_2(\lambda) \int_{Q_{2\rho}^\lambda} \left| \frac{u - (u)_\rho^\lambda}{\rho} \right|^2 dz &\leq c\varphi_1(\lambda) \left[ c_\delta \left( \int_{Q_{2\rho}^\lambda} |Du|^{p_0} dz \right)^{\frac{1}{p_0}} + A_0 + \delta\lambda \right] \\ &\leq c_\delta \varphi \left( \left( \int_{Q_{2\rho}^\lambda} |Du|^{p_0} dz \right)^{\frac{1}{p_0}} \right) + c_\delta \varphi(A_0) + c\delta\varphi(\lambda) \\ &\leq c_\delta \Theta_0 + c\delta\varphi(\lambda), \end{aligned} \tag{3.16}$$

where the last step follows from Jensen’s inequality (Lemma 2.2) when  $\theta_0 \geq \frac{p_0}{p}$  so that  $\varphi(t^{1/p_0})^{\theta_0}$  satisfies (aInc)<sub>1</sub>.

Finally, inserting (3.15) and (3.16) into (3.14), we find that

$$\int_{Q_\rho^\lambda} \varphi(|Du|) dz \leq c_\delta \Theta_0 + c\delta\varphi(\lambda).$$

Choosing  $\delta$  so small that  $c\delta = \frac{1}{2}$  and absorbing the term in the left-hand side by (3.13) we have the reverse Hölder inequality.  $\square$

#### 4. Proof of higher integrability

Now we prove the main result, Theorem 1.5.

Step 1. Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be a weak  $\Phi$ -function and satisfy (aInc)<sub>p</sub> and (aDec)<sub>q</sub> with (1.3). In view of Remark 2.6, we can assume without loss of generality that  $\varphi$  is differentiable, strictly increasing and satisfies (aInc)<sub>p</sub> with  $L = 1$ . We also recall

$$\mathcal{D}(t) := \min\{t^2, \varphi(t)^{\frac{n}{2}+1} t^{-n}\} = \min\{1, \varphi_2(t)^{\frac{n+2}{2}}\} t^2 \tag{4.1}$$

from (1.6). Then  $\mathcal{D}$  is increasing and from (aInc)<sub>p</sub> of  $\varphi$  we have

$$C^{\min\{2, \frac{p(n+2)-2n}{2}\}} \mathcal{D}(t) = \min \left\{ C^2, C^{\frac{p(n+2)-2n}{2}} \right\} \mathcal{D}(t) \leq \mathcal{D}(Ct) \tag{4.2}$$

for all  $t > 0$  and  $C \geq 1$ .

Step 2. Fix  $Q_{2\rho} \in \Omega_I$ . We define

$$\lambda_0 := \mathcal{D}^{-1} \left( \int_{Q_{2\rho}} \varphi(|Du|) dz \right) \tag{4.3}$$

and, for  $0 < s \leq 2$  and  $\lambda > 0$ ,

$$E(s, \lambda) := \{z \in Q_{s\rho} : |Du(z)| > \lambda\}.$$

We next fix any  $1 \leq s_1 < s_2 \leq 2$  and any  $\lambda$  satisfying

$$\lambda \geq \lambda_1 := \left( \frac{40}{s_2 - s_1} \right)^{(n+2) \max\{\frac{1}{2}, \frac{2}{\rho(n+2)-2n}\}} \lambda_0. \tag{4.4}$$

With this  $\lambda$  we also define

$$r_\lambda := \min\{1, \varphi_2(\lambda)^{\frac{1}{2}}\} (s_2 - s_1) \rho. \tag{4.5}$$

We notice that  $Q_r^\lambda(w) \subset Q_{s_2\rho}$  for  $w \in E(s_1, \lambda)$  and  $r \leq r_\lambda$ . Then we prove a Vitali type covering of the super-level set  $E(s_1, \lambda)$  satisfying a balancing condition on each set.

**Lemma 4.6.** *For each  $1 \leq s_1 < s_2 \leq 2$  and  $\lambda \geq \lambda_1$ , there exist  $w_i \in E(s_1, \lambda)$  and  $r_i \in (0, \frac{r_\lambda}{20})$ ,  $i = 1, 2, 3, \dots$ , such that  $Q_{4r_i}^\lambda(w_i)$  are mutually disjoint,*

$$E(s_1, \lambda) \setminus N \subset \bigcup_{i=1}^{\infty} Q_{20r_i}^\lambda(w_i) \tag{4.7}$$

for some a Lebesgue measure zero set  $N$ ,

$$\int_{Q_{r_i}^\lambda(w_i)} \varphi(|Du|) dz = \varphi(\lambda) \quad \text{and} \quad \int_{Q_r^\lambda(w)} \varphi(|Du|) dz \leq \varphi(\lambda) \quad \text{for all } r \in (r_i, r_\lambda). \tag{4.8}$$

**Proof.** For  $w \in E(s_1, \lambda)$  and  $r \in [\frac{r_\lambda}{20}, r_\lambda)$ , using (4.3) we have

$$\int_{Q_r^\lambda(w)} \varphi(|Du|) dz \leq \frac{|Q_{2\rho}|}{|Q_r^\lambda|} \int_{Q_{2\rho}} \varphi(|Du|) dz \leq \frac{|Q_{2\rho}|}{|Q_{r_\lambda/20}|} \varphi_2(\lambda) \mathcal{D}(\lambda_0).$$

By (4.5), (4.2), (4.4) and (4.1),

$$\begin{aligned} \frac{|Q_{2\rho}|}{|Q_{r_\lambda/20}|} \varphi_2(\lambda) \mathcal{D}(\lambda_0) &\leq \left( \frac{40}{s_2 - s_1} \right)^{n+2} \max\{1, \varphi_2(\lambda)^{-\frac{n+2}{2}}\} \varphi_2(\lambda) \mathcal{D}(\lambda_0) \\ &\leq \mathcal{D}(\lambda) \max\{1, \varphi_2(\lambda)^{-\frac{n+2}{2}}\} \varphi_2(\lambda) \\ &= \min\{1, \varphi_2(\lambda)^{\frac{n+2}{2}}\} \lambda^2 \max\{1, \varphi_2(\lambda)^{-\frac{n+2}{2}}\} \varphi_2(\lambda) = \varphi(\lambda). \end{aligned}$$

Therefore we obtain that

$$\int_{Q_r^\lambda(w)} \varphi(|Du|) dz \leq \varphi(\lambda) \quad \text{for all } r \in [\frac{r_\lambda}{20}, r_\lambda).$$

On the other hand, by the parabolic Lebesgue differentiation theorem we see that for almost every  $w \in E(s, \lambda)$

$$\lim_{r \rightarrow 0^+} \int_{Q_r^\lambda(w)} \varphi(|Du|) dz = \varphi(|Du(w)|) > \varphi(\lambda).$$

Therefore, since the map  $r \mapsto \int_{Q_r^\lambda(w)} \varphi(|Du|) dz$  is continuous, one can find  $r_w \in (0, \frac{r_\lambda}{20})$  such that

$$\int_{Q_{r_w}^\lambda(w)} \varphi(|Du|) dz = \lambda, \quad \text{and} \quad \int_{Q_r^\lambda(w)} \varphi(|Du|) dz \leq \lambda \quad \text{for all } r \in (r_w, r_\lambda].$$

Consequently, applying Vitali’s covering lemma for  $\{Q_{r_w}^\lambda(w)\}$ , we have the conclusion.  $\square$

Step 3. By (4.8), we can apply Lemma 3.12, so that we have that for sufficiently small  $\delta \in (0, 1)$ ,

$$\begin{aligned} \varphi(\lambda) &= \int_{Q_{r_i}^\lambda(w_i)} \varphi(|Du|) dz \leq c \left( \int_{Q_{4r_i}^\lambda(w_i)} \varphi(|Du|)^\theta dz \right)^{\frac{1}{\theta}} \\ &\leq c\varphi(\delta\lambda) + c \left( \frac{1}{|Q_{4r_i}^\lambda|} \int_{Q_{4r_i}^\lambda(w_i) \cap E(s_2, \delta\lambda)} \varphi(|Du|)^\theta dz \right)^{\frac{1}{\theta}} \\ &\leq c\delta^p \varphi(\lambda) + \frac{c}{|Q_{4r_i}^\lambda|} \int_{Q_{4r_i}^\lambda(w_i) \cap E(s_2, \delta\lambda)} \varphi(|Du|)^\theta dz \left( \int_{Q_{4r_i}^\lambda} \varphi(|Du|) dz \right)^{1-\theta} \\ &\leq \frac{1}{2} \varphi(\lambda) + c \frac{\varphi(\lambda)^{1-\theta}}{|Q_{4r_i}^\lambda|} \int_{Q_{4r_i}^\lambda(w_i) \cap E(s_2, \delta\lambda)} \varphi(|Du|)^\theta dz. \end{aligned}$$

Then we absorb  $\varphi(\lambda)$  into the left-hand side. Using (4.8) again we have

$$\int_{Q_{20r_i}^\lambda(w_i)} \varphi(|Du|) dz \leq \varphi(\lambda) \leq c \frac{\varphi(\lambda)^{1-\theta}}{|Q_{4r_i}^\lambda|} \int_{Q_{4r_i}^\lambda(w_i) \cap E(s_2, \delta\lambda)} \varphi(|Du|)^\theta dz$$

and so

$$\int_{Q_{20r_i}^\lambda(w_i)} \varphi(|Du|) dz \leq c\varphi(\lambda)^{1-\theta} \int_{Q_{4r_i}^\lambda(w_i) \cap E(s_2, \delta\lambda)} \varphi(|Du|)^\theta dz.$$

Therefore, since  $\{Q_{20r_i}^\lambda(w_i)\}$  is a covering of  $E(s_1, \lambda)$  according to (4.7) and  $Q_{4r_i}^\lambda(w_i)$  are mutually disjoint,

$$\int_{E(s_1, \lambda)} \varphi(|Du|) dz \leq \sum_{i=1}^\infty \int_{Q_{20r_i}^\lambda(w_i)} \varphi(|Du|) dz \leq c\varphi(\lambda)^{1-\theta} \int_{E(s_2, \delta\lambda)} \varphi(|Du|)^\theta dz.$$

In addition,

$$\begin{aligned} \int_{E(s_1, \delta\lambda) \setminus E(s_1, \lambda)} \varphi(|Du|) dz &\leq \int_{E(s_1, \delta\lambda) \setminus E(s_1, \lambda)} \varphi(\lambda)^{1-\theta} \varphi(|Du|)^\theta dz \\ &\leq \int_{E(s_2, \delta\lambda)} \varphi(\lambda)^{1-\theta} \varphi(|Du|)^\theta dz. \end{aligned}$$

Combining these and replacing  $\delta\lambda$  by  $\lambda$ , we have

$$\int_{E(s_1, \lambda)} \varphi(|Du|) dz \leq c \int_{E(s_2, \lambda)} \varphi(\lambda)^{1-\theta} \varphi(|Du|)^\theta dz \quad \text{for all } \lambda > \delta\lambda_1. \tag{4.9}$$

Step 4. Let us set

$$\begin{aligned} |Du|_k &:= \min\{|Du|, k\} \quad \text{for } k \geq 0, \\ E_k(s, \lambda) &:= \{z \in Q_{s\rho} : |Du|_k(z) > \lambda\}. \end{aligned}$$

From now on, we assume that  $k > \lambda_1$ . Then we have from (4.9) that for  $\varepsilon > 0$ , which will be determined later,

$$\begin{aligned} I &:= \int_{\lambda_1}^\infty \varphi(\lambda)^{\varepsilon-1} \varphi'(\lambda) \int_{E_k(s_1, \lambda)} \varphi(|Du|_k)^{1-\theta} \varphi(|Du|)^\theta dz d\lambda \\ &\leq \int_{\lambda_1}^k \varphi(\lambda)^{\varepsilon-1} \varphi'(\lambda) \int_{E(s_1, \lambda)} \varphi(|Du|) dz d\lambda \\ &\leq c \int_{\lambda_1}^\infty \int_{E_k(s_2, \lambda)} \varphi(\lambda)^{\varepsilon-\theta} \varphi'(\lambda) \varphi(|Du|)^\theta dz d\lambda =: II, \end{aligned}$$

where we have used the facts that  $E_k(s, \lambda) = \emptyset$  if  $\lambda > k$  and  $E(s, \lambda) = E_k(s, \lambda)$  if  $\lambda \leq k$ . We then apply Fubini’s theorem to  $I$  and  $II$ , so that

$$\begin{aligned}
 I &= \int_{E_k(s_1, \lambda_1)} \varphi(|Du|_k)^{1-\theta} \varphi(|Du|)^\theta \int_{\lambda_1}^{|Du|_k} \varphi(\lambda)^{\varepsilon-1} \varphi'(\lambda) d\lambda dz \\
 &= \frac{1}{\varepsilon} \int_{E_k(s_1, \lambda_1)} \left[ \varphi(|Du|_k)^{1-\theta+\varepsilon} \varphi(|Du|)^\theta - \varphi(\lambda_1)^\varepsilon \varphi(|Du|_k)^{1-\theta} \varphi(|Du|)^\theta \right] dz
 \end{aligned}$$

and

$$\begin{aligned}
 II &= c \int_{E_k(s_2, \lambda_1)} \varphi(|Du|)^\theta \int_{\lambda_1}^{|Du|_k} \varphi(\lambda)^{\varepsilon-\theta} \varphi'(\lambda) d\lambda dz \\
 &= \frac{c}{1-\theta+\varepsilon} \int_{E_k(s_2, \lambda_1)} \left( \varphi(|Du|_k)^{1-\theta+\varepsilon} - \varphi(\lambda_1)^{1-\theta+\varepsilon} \right) \varphi(|Du|)^\theta dz \\
 &\leq \frac{c}{1-\theta} \int_{E_k(s_2, \lambda_1)} \varphi(|Du|_k)^{1-\theta+\varepsilon} \varphi(|Du|)^\theta dz.
 \end{aligned}$$

Therefore we have

$$\begin{aligned}
 \int_{E_k(s_1, \lambda_1)} \varphi(|Du|_k)^{1-\theta+\varepsilon} \varphi(|Du|)^\theta dz &\leq \varphi(\lambda_1)^\varepsilon \int_{E_k(s_1, \lambda_1)} \varphi(|Du|_k)^{1-\theta} \varphi(|Du|)^\theta dz \\
 &\quad + c\varepsilon \int_{Q_{s_2\rho}} \varphi(|Du|_k)^{1-\theta+\varepsilon} \varphi(|Du|)^\theta dz.
 \end{aligned}$$

At this stage, we choose  $\varepsilon = \varepsilon(n, N, p, q, L, \nu, \Lambda) > 0$  so small that  $c\varepsilon \leq \frac{1}{2}$ . On the other hand,

$$\int_{Q_{s_1\rho} \setminus E_k(s_1, \lambda_1)} \varphi(|Du|_k)^{1-\theta+\varepsilon} \varphi(|Du|)^\theta dz \leq \varphi(\lambda_1)^\varepsilon \int_{Q_{s_1\rho}} \varphi(|Du|_k)^{1-\theta} \varphi(|Du|)^\theta dz.$$

Combining the last two estimates, we have

$$\begin{aligned}
 \int_{Q_{s_1\rho}} \varphi(|Du|_k)^{1-\theta+\varepsilon} \varphi(|Du|)^\theta dz &\leq \frac{1}{2} \int_{Q_{s_2\rho}} \varphi(|Du|_k)^{1-\theta+\varepsilon} \varphi(|Du|)^\theta dz \\
 &\quad + \frac{c\varphi(\lambda_0)^\varepsilon}{(s_2 - s_1)^{\alpha_0}} \int_{Q_{2\rho}} \varphi(|Du|_k)^{1-\theta} \varphi(|Du|)^\theta dz,
 \end{aligned}$$

where  $\alpha_0 := \varepsilon q(n+2) \max\{\frac{1}{2}, \frac{2}{p(n+2)-2n}\}$ ; here we used (4.4) which yields  $\varphi(\lambda_1) \leq \frac{c}{(s_2 - s_1)^{\alpha_0}} \varphi(\lambda_0)$ . Applying the standard iteration Lemma 2.8 to this inequality, we find that

$$\begin{aligned} \int_{Q_\rho} \varphi(|Du|_k)^{1-\theta+\varepsilon} \varphi(|Du|)^\theta dz &\leq c\varphi(\lambda_0)^\varepsilon \int_{Q_{2\rho}} \varphi(|Du|_k)^{1-\theta} \varphi(|Du|)^\theta dz \\ &\leq c\varphi(\lambda_0)^\varepsilon \int_{Q_{2\rho}} \varphi(|Du|) dz. \end{aligned}$$

Finally, letting  $k \rightarrow \infty$  and recalling (4.3), we have

$$\begin{aligned} \int_{Q_\rho} \varphi(|Du|)^{1+\varepsilon} dz &\leq c\varphi(\lambda_0)^\varepsilon \int_{Q_{2\rho}} \varphi(|Du|) dz \\ &\leq c \left[ \varphi \left( \mathcal{D}^{-1} \left( \int_{Q_{2\rho}} \varphi(|Du|) dz \right) \right) \right]^\varepsilon \int_{Q_{2\rho}} \varphi(|Du|) dz. \end{aligned}$$

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