

Persistence of Hyperbolic Invariant Tori for Hamiltonian Systems¹

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A perturbation problem for hyperbolic invariant tori is considered, and a KAM type theorem about the existence of invariant tori is proved. In particular, the unperturbed system considered may admit the different dimensions of action and angle variables, and hence this generalizes the works of Graff, Zehnder, Hermann, and Parasyuk. © 2000 Academic Press

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1. INTRODUCTION AND STATEMENT OF THE RESULT

The classical KAM theorem concludes the persistence of invariant tori in nearly integrable Hamiltonian systems [1, 3, 5, 6, 8, 9, 11–15]. In recent years, there has been increasing interest in the persistence problem for Hamiltonian systems with distinct numbers of action-angle variables [4, 7, 10]. Such systems are usually called nonstandard Hamiltonian systems. Motivated by the works of Hermann [7], Graff [6], and Parasyuk [10], we consider the persistence of hyperbolic invariant tori for nonstandard systems in the present paper.

Let $G \subset \mathbb{R}^l$ be a closed bounded and connected domain and $T^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ the usual n -dimensional torus. Let $l + n$ be even. Denote

$$G_\rho = \{y : \operatorname{Re} y \in G, |\operatorname{Im} y| \leq \rho\},$$

$$\Sigma_\rho = \{x : \operatorname{Re} x \in T^n, |\operatorname{Im} x| \leq \rho\},$$

$$B_\rho(0) = \{z = (z_+, z_-) \in \mathbb{C}^{2m} : |z| \leq \rho\},$$

where $|\cdot|$ denotes the maximum norm of a vector in components.

In order to construct the latter KAM iteration, let us consider a complex symplectic manifold $(G_\rho \times \Sigma_\rho \times B_\rho(0), \omega^2)$. Here, ω^2 is a given 2-form such that $\omega^2(\cdot, E\omega^1(\cdot)) = \omega^1(\cdot)$, where ω^1 is a 1-form on $G_\rho \times \Sigma_\rho \times B_\rho(0)$,

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$E = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$ is a symplectic matrix, I is an analytic homeomorphism from the 1-form space to vector fields on $G_\rho \times \Sigma_\rho$, and J is a usual $2m$ -order standard symplectic matrix. For all smooth functions f_1, f_2 defined on $G_\rho \times \Sigma_\rho \times B_\rho(0)$, the 2-form ω^2 can be determined in the following way: $\{f_1, f_2\} = df_1(E df_2) = \omega^2(E df_2, E df_1)$, where $\{\cdot, \cdot\}$ denotes the usual Poisson bracket.

Let ω^2 be invariant relative to the quasi-periodic motion on T^n of $G_\rho \times \Sigma_\rho \times B_\rho(0)$, and let the functions y_i, y_j satisfy that $\{y_i, y_j\} = 0, i, j = 1, \dots, l$. The 2-form ω^2 and the homeomorphic matrix I are independent of the angle variable x .

Consider the Hamiltonian system

$$\begin{pmatrix} \dot{y} \\ \dot{x} \\ \dot{z}_+ \\ \dot{z}_- \end{pmatrix} = \begin{pmatrix} I(y) & 0 \\ 0 & J \end{pmatrix} \text{grad}^T H(y, x, z_+, z_-), \quad (1.1)$$

with

$$\begin{aligned} \text{grad}^T H(y, x, z_+, z_-) &= \left(\frac{\partial H}{\partial y_1}, \dots, \frac{\partial H}{\partial y_l}, \frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n}, \right. \\ &\quad \left. \frac{\partial H}{\partial z_{+1}}, \dots, \frac{\partial H}{\partial z_{+m}}, \frac{\partial H}{\partial z_{-1}}, \dots, \frac{\partial H}{\partial z_{-m}} \right)^T. \\ H(y, x, z_+, z_-) &= h(y) + \langle z_+, \Omega(y, x) z_- \rangle + P(y, x, z_+, z_-) \\ &= N(y, x, z_+, z_-) + P(y, x, z_+, z_-), \end{aligned} \quad (1.2)$$

where $\Omega(y, x)$ is an $(m \times m)$ -order real analytic matrix function on $G_\rho \times \Sigma_\rho$, $h(y)$ is a real analytic function on G_ρ , and $P(y, x, z_+, z_-)$ is a small enough analytic function on $G_\rho \times \Sigma_\rho \times B_\rho(0)$. We call N a normal form and P a perturbation of N .

For the unperturbed system, i.e., $P \equiv 0$, by Lemma 6 in Section 6,

$$\begin{aligned} \begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} &= I(y) \text{grad}_{(y, x)}^T h(y) + I(y) \text{grad}_{(y, x)}^T \langle z_+, \Omega(y, x) z_- \rangle, \\ \dot{z}_+ &= \Omega^T(y, x) z_+, \\ \dot{z}_- &= -\Omega(y, x) z_- \end{aligned} \quad (1.3)$$

admit invariant tori $\tau_{y_0} = \{(y, x, 0, 0) : y = y_0, x \in T^n\}$, $y_0 \in G$, where $\text{grad}_{(y, x)}^T H$ denotes the gradient vector of H for the variable (y, x) .

Let

$$(0, \dots, 0, \omega(y))^T = I(y) \operatorname{grad}_{(y, x)}^T h(y).$$

Each n -dimensional invariant torus τ_{y_0} of (1.3) with hyperbolic structure holds a quasi-periodic flow $x = x_0 + \omega(y_0)t$.

From now on, we assume that the following hold:

(H₁) On $G_\rho \times \Sigma_\rho$,

$$\operatorname{Re} \langle \gamma, \Omega \gamma \rangle \geq 2\mu |\gamma|^2, \quad \forall \gamma \in C^m,$$

where μ is a positive constant;

(H₂) $h(y), \Omega(y, x), P(y, x, z_+, z_-)$ are real analytic functions on $G_\rho \times \Sigma_\rho \times B_\rho(0)$;

(H₃) on G , $\omega(y)$ satisfies

$$\operatorname{rank} \left(\frac{\partial \omega}{\partial y} \right) = r \leq \min \{n, l\},$$

$$\operatorname{rank} \left\{ \omega, \frac{\partial^\alpha \omega}{\partial y^\alpha} : \forall \alpha, |\alpha| \leq n - r + 1 \right\} = n,$$

where $\partial^\alpha \omega / \partial y^\alpha = (\partial^{|\alpha|} \omega_1 / \partial y^\alpha, \dots, \partial^{|\alpha|} \omega_n / \partial y^\alpha)^T$, $\alpha = (\alpha_1, \dots, \alpha_l)$, $\alpha_i, i = 1, \dots, l$, are nonnegative integers, and $|\alpha| = \alpha_1 + \dots + \alpha_l$.

THEOREM A. Assume that (H₁)–(H₃) hold. Then there exist a nonempty Cantor set $G_* \subset G$ and a constant $M_0 = M(h, \Omega, \rho, G) > 0$ such that for any $M > 0$, whenever $\|P\| \leq M < M_0$ and $y_0 \in G_*$, the invariant torus τ_{y_0} of the unperturbed system $N(y, x, z_+, z_-)$ is preserved under the perturbation P except for a small deformation and a frequency drift, and each frequency $\omega^\infty(y)$ satisfies

$$|\omega^\infty(y) - \omega(y)| \leq M_0^{1/2}, \quad y \in G_*,$$

where $\omega^\infty(y_0)$ denotes the frequency of the drifted invariant torus corresponding to the invariant torus τ_{y_0} ; moreover, there exists the following measure estimate:

$$\operatorname{mes}_l(G \setminus G_*) = O(M_0^{1/72(n+1)(n-r+1)}).$$

Remark 1. If $n = l$ and $I(y) = J_{2n \times 2n}$, then Theorem A reduces to the result of [3]. For $r = n$ in (H₃), Graff [6] and Zehnder [15] gave some

important results. There, because of stronger nondegeneracy, each torus persists and the frequency remains unchanged.

Remark 2. For the system (1.2), if $n = l, m = 0$, Bruno [2] gave the following nondegenerate condition to guarantee existence of the invariant tori for the system (1.2):

$$\operatorname{rank}\left(\omega, \frac{\partial \omega}{\partial y}\right)=n.$$

In 1987, Rüssmann [12] proposed a geometric condition: the frequency ω does not lie in any hyperplane through the origin. This is called the Rüssmann conjecture. Recently, this conjecture has been proven by Xu *et al.* [14] and Sevryuk [13].

Remark 3. For $m = 0$ and $n \neq l$, Theorem A corresponds to the main results of [4] and [10].

The paper is arranged as follows: in Section 2. we give an outline of the proof of Theorem A; in Section 3 we describe a cycle of KAM steps. In Section 4 we give the measure estimate of the persisting tori. In Section 5 we illustrate Theorem A, and in Section 6 we list some preliminary lemmas.

2. OUTLINE OF THE PROOF OF THEOREM A

Since $G_\rho \times \Sigma_\rho \times B_\rho(0)$ is a compact set, there exists a positive constant $C_0 > 0$ such that on $G_\rho \times \Sigma_\rho \times B_\rho(0)$,

$$\max \left\{ \|h\|, \|\Omega\|, \left\|\frac{\partial h}{\partial y}\right\|, \|I\|, \|I \operatorname{grad}^T h\|, \left\|\frac{\partial \Omega}{\partial x}\right\|, \left\|\frac{\partial \Omega}{\partial y}\right\| \right\} < C_0, \tag{2.0}$$

where $\|\cdot\|$ denotes the usual supremum norm for a function on the given set in what follows.

Choose convergent sequences:

$$\begin{aligned} s_i &= M_i^{7/18}, \quad \delta_{i+1} = \delta_i^{8/7}, \quad r_{i+1} = r_i - 6 \delta_i, \quad M_{i+1} = M_i^{8/7}, \quad i = 0, 1, \dots, \\ r_0 &= \rho, \quad s_0 = \delta_0^{28(n+1)}, \quad s_0 = M_0^{7/18}. \end{aligned}$$

Rewrite (1.1) in the form

$$H(y, x, z) = h^1(y) + \langle z_+, \Omega^1(y, x) z_- \rangle + \bar{P}(y, x, z),$$

where

$$h^1(y) = h(y) + [P]_x(y, 0),$$

$$[P]_x(y, 0) = \int_{T^n} P(y, x, 0) dx,$$

$$\begin{aligned} \langle z_+, \Omega^1(y, x) z_- \rangle &= \langle z_+, (\Omega(y, x) + P_{z_+ z_-}(y, x, 0)) z_- \rangle \\ &\quad + \langle \text{grad}_{(y, x)}^T \langle z_+, \Omega(y, x) z_- \rangle, I(y) \text{grad}_{(y, x)}^T A^1 \rangle, \\ \bar{P}(y, x, z) &= P(y, x, z) - [P]_x(y, 0) - \langle z_+, P_{z_+ z_-}(y, x, 0) z_- \rangle, \\ (\underbrace{0, \dots, 0}_I, \omega^1(y))^T &= I(y) \text{grad}_{(y, x)}^T h^1(y), \end{aligned}$$

and A^1 will be determined below.

We assume $y_0 \in G$. Let

$$\begin{aligned} D_0 &= \{y : |y - y_0| \leq 4s_0, y \in G\} \times \{x : \text{Re } x \in T^n, |\text{Im } x| \leq r_0\} \\ &\quad \times \{(z_+, z_-) : |z_{\pm}| \leq 6s_0\}, \\ D_1 &= \{y : |y - y_0| \leq 4s_1, y \in G\} \times \{x : \text{Re } x \in T^n, |\text{Im } x| \leq r_1\} \\ &\quad \times \{(z_+, z_-) : |z_{\pm}| \leq 6s_1\}, \\ O_0 &= \{y : |\langle k, \omega(y) \rangle| \geq \delta_0 |k|^{-\tau}, 0 \neq k \in \mathbb{Z}^n, y \in G\}, \end{aligned}$$

where $\tau \geq n(n-r+1)$.

Choose $y_0 \in O_1$. We want to construct a canonical transformation:

$$\begin{aligned} \Phi^1 : (y^+, x^+, z_+^+, z_-^+) \in D_1 &\rightarrow (y, x, z_+, z_-) \in D, \\ \begin{pmatrix} y \\ x \end{pmatrix} &= \begin{pmatrix} y^+ \\ x^+ \end{pmatrix} + \int_0^1 I(y^+(t)) \text{grad}_{(y, x)}^T S(y^+(t), x^+(t), z_+^+(t), z_-^+(t)) dt, \\ \begin{pmatrix} z_+ \\ z_- \end{pmatrix} &= \begin{pmatrix} z_+^+ \\ z_-^+ \end{pmatrix} + \int_0^1 J \text{grad}_{(z_+, z_-)}^T S(y^+(t), x^+(t), z_+^+(t), z_-^+(t)) dt, \end{aligned} \tag{2.1}$$

where the generating function S will be determined below. Then Φ^1 takes H into H^1 ,

$$\begin{aligned} H^1(y^+, x^+, z^+) &= H \circ \Phi^1(y^+, x^+, z^+) \\ &= h^1(y^+) + \langle z_+^+, \Omega^1(y^+, x^+) z_-^+ \rangle \\ &\quad + P^1(y^+, x^+, z^+), \end{aligned} \tag{2.2}$$

and if $\|P\| \leq M_0$, we shall prove that the inequalities

$$\begin{aligned} \|h^1 - h\|_{D_0} &\leq M_0, & \|\Omega^1 - \Omega\|_{D_0} &\leq M_0^{1/6}, \\ \|P^1\|_{D_1} &\leq M_1, & \left\| \frac{\partial \Phi^1}{\partial(y^+, x^+, z^+)} - \text{Id} \right\|_{D_1} &\leq M_0^{1/6}, \end{aligned}$$

where Id denotes the unit matrix, hold.

An iteration process is defined as follows: Let

$$\begin{aligned} D_i &= \{y : |y - y_0| \leq 4s_i, y \in G\} \times \{x : \text{Re } x \in T^n, |\text{Im } x| \leq r_i\} \\ &\quad \times \{(z_+, z_-) : |z_{\pm}| < 6s_i\}, \quad i = 0, 1, 2, \dots \end{aligned}$$

We assume H is changed into H^i on D_i ,

$$H^i(y, x, z) = h^i(y) + \langle z_+, \Omega^i(y, x) z_- \rangle + P^i(y, x, z), \quad (2.2_i)$$

and satisfies

$$\begin{aligned} \text{Re} \langle \gamma, \Omega^i(y, x) \gamma \rangle &\geq \mu |\gamma|^2, \quad \gamma \in C^m, \\ \|P^i\| &\leq M_i. \end{aligned} \quad (2.3_i)$$

Denote

$$h^{i+1}(y) = h^i(y) + [P^i]_x(y, 0), \quad (2.4_{i+1})$$

$$\begin{aligned} \langle z_+, \Omega^{i+1}(y, x) z_- \rangle &= \langle z_+, (\Omega^i(y, x) + P^i_{z_+ z_-}(y, x, 0)) z_- \rangle \\ &\quad + \langle \text{grad}_{(y, x)}^T \langle z_+, \Omega^i(y, x) z_- \rangle, \\ &\quad I(y) \text{grad}_{(y, x)}^T A^{i+1} \rangle, \end{aligned} \quad (2.5_{i+1})$$

$$\begin{aligned} \bar{P}^i(y, x, z) &= P^i(y, x, z) - [P^i]_x(y, 0) \\ &\quad - \langle z_+, P^i_{z_+ z_-}(y, x, 0) z_- \rangle, \end{aligned} \quad (2.6_i)$$

$$(0, \underbrace{\dots, 0}_I, \omega^{i+1}(y))^T = I(y) \text{grad}_{(y, x)}^T h^{i+1}(y), \quad (2.7_{i+1})$$

where A^{i+1} will be determined below.

Define

$$O_i = \{y : |\langle k, \omega^i(y) \rangle| \geq \delta_i |k|^{-\tau}, 0 \neq k \in \mathbb{Z}^n, y \in G\}.$$

If $y \in O_i$, then with a suitable generating function $S^{i+1}(y, x, z)$ we can construct a canonical transformation $\Phi^{i+1}: (y^+, x^+, z^+) \in D_{i+1} \rightarrow (y, x, z) \in D$ of the form

$$\begin{aligned} \begin{pmatrix} y \\ x \end{pmatrix} &= \begin{pmatrix} y^+ \\ x^+ \end{pmatrix} + \int_0^1 I(y^+(t)) \operatorname{grad}_{(y, x)}^T S^{i+1}(y^+(t), x^+(t), z_+^+(t), z_-^+(t)) dt, \\ \begin{pmatrix} z_+ \\ z_- \end{pmatrix} &= \begin{pmatrix} z_+^+ \\ z_-^+ \end{pmatrix} + \int_0^1 J \operatorname{grad}_{(z_+, z_-)}^T S^{i+1}(y^+(t), x^+(t), z_+^+(t), z_-^+(t)) dt, \end{aligned} \quad (2.1_{i+1})$$

such that

$$\begin{aligned} H^{i+1}(y^+, x^+, z^+) &= H^i \circ \Phi^{i+1}(y^+, x^+, z^+) \\ &= h^{i+1}(y^+) + \langle z_+^+, \Omega^{i+1}(y^+, x^+) z_-^+ \rangle \\ &\quad + P^{i+1}(y^+, x^+, z^+), \end{aligned}$$

and on D^{i+1} ,

$$\|P^{i+1}\| \leq M_{i+1}, \quad (2.3_{i+1})$$

$$\|\Phi^{i+1} - \operatorname{Id}\| \leq M_i^{1/3}, \quad \left\| \frac{\partial \Phi^{i+1}}{\partial (y^+, x^+, z^+)} - \operatorname{Id} \right\| \leq M_i^{1/6}, \quad (2.8_{i+1})$$

$$\|h^{i+1} - h^i\| \leq M_i, \quad \|\Omega^{i+1} - \Omega^i\| \leq M_i^{1/6}. \quad (2.9_{i+1})$$

Choose

$$0 < \delta_0 < \min\{2^{-7}, \frac{1}{24}\rho_0\}. \quad (\text{A})$$

Then $r_i \geq r - \sum_{j=0}^{\infty} \delta_j \geq \frac{1}{2}\rho_0$, $i = 1, 2, \dots$. Let

$$\begin{aligned} D^\infty &= \{y = y_0\} \times \{|\operatorname{Im} x| \leq \tfrac{1}{2}\rho_0\} \times \{z_\pm = 0\}, \\ U_i &= \Phi^1 \circ \Phi^2 \circ \dots \circ \Phi^i, \quad U'_i = \Phi^{1'} \bullet \Phi^{2'} \bullet \dots \bullet \Phi^{i'}. \end{aligned}$$

Then $U_i: D_i \rightarrow D_0$. Hence if $\Phi^i, \Phi^{i'}$ satisfy (2.8_i), then for

$$0 < \delta_0 < \min\left\{\left(\frac{1}{4}\right)^{1/12(n+1)}, \left(\frac{1}{2}\right)^{7/24(n+1)}\right\}, \quad (\text{B})$$

we can prove that

$$U_\infty = \lim_{i \rightarrow \infty} U_i, \quad U'_\infty = \lim_{i \rightarrow \infty} U'_i$$

hold uniformly on D_∞ and $U_\infty: D_\infty \rightarrow D_0$. In fact, on D_∞

$$\begin{aligned}
\|U_i - \text{Id}\| &\leq \sum_{j=1}^i \|\Phi^j \circ \dots \circ \Phi^i - \Phi^{j+1} \circ \dots \circ \Phi^i\|_{D_i} \leq \sum_{j=1}^i \|\Phi^j - \text{Id}\|_{D_j} \\
&\leq \sum_{j=1}^i M_j^{1/3} \leq 2M_0^{1/3} < 2; \\
\|U'_i\| &\leq \sum_{j=0}^i (\|\Phi^{j'} - \text{Id}\| + \|\text{Id}\|) \leq \sum_{j=0}^\infty (1 + M_j^{1/6}) \leq \sum_{k=0}^\infty \left(\sum_{j=0}^\infty M_j^{1/6} \right)^k \\
&\leq \sum_{k=0}^\infty (2M_0^{1/6})^k \leq \sum_{k=0}^\infty \left(\frac{1}{2}\right)^k < 2.
\end{aligned}$$

Put $\text{Im } x = 0$. By Lemma 5, $U_\infty: T^n \rightarrow D_0$ is a continuous embedding and on T^n ,

$$x' = \omega^\infty(y_0),$$

where $\omega^\infty(y_0) = \omega(y_0) + \sum_{i=1}^\infty \omega^i(y_0)$. Applying (A), we see that $\omega^\infty(y_0)$ exists, and

$$|\omega^\infty(y_0) - \omega(y_0)| \leq 2M_0^{1/2}.$$

In addition, by (2.4) we obtain

$$\max \left\{ \left\| \frac{\partial h^{i+1}}{\partial y} \right\|, \left\| \frac{\partial}{\partial y} (I(y) \text{grad}^T h^{i+1}(y)) \right\| \right\} \leq 2C_0, \quad (2.10_{i+1})$$

provided δ is small.

3. INDUCTIVE ITERATIONS

To prove Theorem A, we only consider one cycle of the iteration scheme. To this end, assuming that (2.2_k) and (2.3_k) hold for k , we need to prove that they also hold for $k+1$. For simplicity, we omit “ k ” and rewrite “ $k+1$ ” as “ $+$ ”.

Rewrite $P(y, x, z)$ as

$$\begin{aligned}
P(y, x, z) &= P(y, x, 0) + \langle P_{z_+}(y, x, 0), z_+ \rangle + \langle P_{z_-}(y, x, 0), z_- \rangle \\
&\quad + \frac{1}{2} \langle z_+, P_{z_+ z_+}(y, x, 0) z_+ \rangle + \frac{1}{2} \langle z_-, P_{z_- z_-}(y, x, 0) z_- \rangle \\
&\quad + \langle z_+, P_{z_+ z_-}(y, x, 0) z_- \rangle + P^*(y, x, z). \tag{3.1}
\end{aligned}$$

On D , there exists a constant $C_1 > 0$, C_1 depending only on ρ, C_0, n, l, m such that

$$\|P^*\| \leq C_1 M. \quad (3.2)$$

Let the function $S(y, x, z)$ denote

$$\begin{aligned} S(y, x, z) = & A(y, x) + \langle z_+, B(y, x) \rangle + \langle z_-, C(y, x) \rangle \\ & + \frac{1}{2} \langle z_+, D(y, x) z_+ \rangle \\ & + \frac{1}{2} \langle z_-, F(y, x) z_- \rangle, \end{aligned}$$

where A, B, C, D, F are determined below.

We construct a canonical transformation Φ^+ on D^+ ,

$$\dot{W} = E \operatorname{grad}^T S(W), \quad W = (y, x, z_+, z_-). \quad (3.3)$$

$$\begin{aligned} \Phi^+ : \begin{pmatrix} y \\ x \end{pmatrix} &= \begin{pmatrix} y^+ \\ x^+ \end{pmatrix} + \int_0^1 (I \operatorname{grad}_{(y, x)}^T S) \circ \phi^t(y^+, x^+, z_+^+, z_-^+) dt, \\ \begin{pmatrix} z_+ \\ z_- \end{pmatrix} &= \begin{pmatrix} z_+^+ \\ z_-^+ \end{pmatrix} + \int_0^1 (J \operatorname{grad}_{(z_+, z_-)}^T S) \circ \phi^t(y^+, x^+, z_+^+, z_-^+) dt, \end{aligned} \quad (3.4)$$

where ϕ^t denotes the flow of (3.3). By

$$\frac{d}{dt} G \circ \phi^t = \{G, S\} \circ \phi^t,$$

and Taylor's formula, we have

$$\begin{aligned} H \circ \Phi^+ &= H \circ \phi^1 = (N + P - P^*) \circ \phi^1 + P^* \circ \phi^1 \\ &= N + P - P^* + \{N, S\} + \int_0^1 (1-t) \{ \{N, S\}, S \} \circ \phi^t dt \\ &\quad + \int_0^1 \{P - P^*, S\} \circ \phi^t dt + P^* \circ \phi^1. \end{aligned} \quad (3.5)$$

If $y_0 \in G_*$, then choose S such that

$$\begin{aligned} & \langle \operatorname{grad}_{(y, x)}^T h(y)|_{y=y_0}, I(y_0) \operatorname{grad}_{(y, x)}^T S \rangle \\ & + \langle \operatorname{grad}_{(y, x)}^T \langle z_+, \Omega z_- \rangle, I(y_0) \operatorname{grad}_{(y, x)}^T S \rangle \\ & + P - P^* - \langle z_+, P_{z_+ z_-}(y, x, 0) z_- \rangle - [P]_x(y, 0) = 0. \end{aligned} \quad (3.6)$$

Resolve (3.6) by

$$\partial A + P(y, x, 0) - [P]_x(y, 0) = 0, \quad (3.7)$$

$$\partial B + B\Omega + P_{z_+}(y, x, 0) = 0, \quad (3.8)$$

$$\partial C - C\Omega^T + P_{z_-}(y, x, 0) = 0, \quad (3.9)$$

$$\partial D + D\Omega + \Omega^T D + P_{z_+ z_+}(y, x, 0) = 0, \quad (3.10)$$

$$\partial F - F\Omega^T - \Omega F + P_{z_- z_-}(y, x, 0) = 0, \quad (3.11)$$

where $\partial = \sum_{i=1}^n \omega_i (\partial/\partial x_i)$, $I(y_0) \operatorname{grad}_{(y,x)}^T h(y)|_{y=y_0} = (\underbrace{0, \dots, 0}_I, \omega(y_0))^T$.

Put

$$\begin{aligned} \langle z_-, \Omega^+ z_+ \rangle &= \langle z_-, (\Omega + P_{z_+ z_-}(y, x, 0)) z_+ \rangle \\ &\quad + \langle \operatorname{grad}_{(y,x)}^T \langle z_-, \Omega z_+ \rangle, I(y) \operatorname{grad}_{(y,x)}^T A \rangle, \\ h^+ &= h + [P]_x(y, 0). \end{aligned} \quad (3.12)$$

Applying the definition of Φ^+ and (3.6), we obtain

$$\begin{aligned} H^+ &= H \circ \Phi^+ = H \circ \phi^1 = h^+(y) + \langle z_+, \Omega^+(y, x) z_- \rangle + P^* \circ \phi^1 \\ &\quad + \int_0^1 \{P - P^*, S\} \circ \phi^t dt + \int_0^1 (1-t) \{ \{N, S\}, S \} \circ \phi^t dt \\ &\quad + \langle \operatorname{grad}_{(y,x)}^T \langle z_+, \Omega z_- \rangle, I(y) \operatorname{grad}_{(y,x)}^T (S - A) \rangle \\ &\quad + \langle \operatorname{grad}_{(y,x)}^T h(y), I(y) \operatorname{grad}_{(y,x)}^T S \rangle \\ &\quad - \langle \operatorname{grad}_{(y,x)}^T h(y)|_{y=y_0}, I(y_0) \operatorname{grad}_{(y,x)}^T S \rangle \\ &= h^+(y) + \langle z_+, \Omega^+(y, x) z_- \rangle + P_1 + P_2 + P_3 + P_4 + P_5 \\ &= N^+ + P^+. \end{aligned} \quad (3.13)$$

If $\|P\| \leq M$ on D , then it also holds on the neighborhood $\{|y - y_0| \leq 4s\} \times \{|\operatorname{Im} x| \leq r\}$. By Cauchy's integral formula, we have

$$\max\{|P_{z_+}(y, x, 0)|, |P_{z_-}(y, x, 0)|\} \leq \frac{M}{s}, \quad (3.14)$$

$$\max\{|P_{z_+ z_+}(y, x, 0)|, |P_{z_- z_-}(y, x, 0)|\} \leq \frac{M}{s^2}. \quad (3.15)$$

By Lemmas 2–4 and (3.7)–(3.11), on $\{|y^+ - y_0| \leq 2s\} \times \{|\operatorname{Im} x| \leq r - 2\delta\}$,

$$\|A\| \leq C_3 M \delta^{-(2n+1)}, \quad (3.16)$$

$$\|B\|, \|C\| \leq C_4 M s^{-1}, \quad (3.17)$$

$$\|D\|, \|E\| \leq C_4 M s^{-2}, \quad (3.18)$$

where C_i ($i = 3, 4, \dots$) are constants independent of iteration procedures.

If $\|P\| \leq M$ on D , then by applying Cauchy's integral formula we have

$$\|[P]_x(\cdot, 0)\| \leq M, \quad \|[P]_{z_+ z_-}(\cdot, 0)\| \leq \frac{M}{s^2} \leq M^{7/36}.$$

So on D , by (3.12) and (3.16), we have

$$\begin{aligned} \|h^+ - h\| &\leq M, \\ \|\Omega^+ - \Omega\| &\leq M^{7/36} + 2C_0 \cdot \frac{1}{s\delta} \cdot C_0 \cdot \frac{\|A\|}{s\delta} \leq M^{7/36} + 2C_0^2 C_3 \cdot M^{13/72} \\ &\leq M^{1/6}, \end{aligned} \quad (3.19)$$

provided δ_0 satisfies the inequality

$$0 < \delta_0 < \min \left\{ \left(\frac{1}{2} \right)^{1/2(n+1)}, \left(\frac{1}{4C_0^2 C_3} \right)^{1/6(n+1)} \right\}. \quad (C)$$

By (3.12), (3.19), Cauchy's integral formula, and (2.0) in Section 2, for $(y, x) \in \{|y^+ - y_0| \leq 4s\} \times \{|\operatorname{Im} x| \leq r\}$,

$$\begin{aligned} |\Omega^+(y, x)| &\leq \|\Omega^0\|_{D_0} + \sum_{j=1}^k \|M_j^{1/6}\| \leq C_0 + \sum_{j=1}^k \delta_j^{12(n+1)} \\ &\leq C_0 + 2\delta_0 < 2C_0, \end{aligned} \quad (3.20)$$

$$\begin{aligned} \operatorname{Re} \langle \gamma, \Omega^+(y, x) \gamma \rangle &\geq \operatorname{Re} \langle \gamma, \Omega^0(y, x) \gamma \rangle - \sum_{j=1}^k \|P_{z_+ z_-}^j(\cdot, \cdot, 0)\|_{D_j} |\gamma|^2 \\ &\quad - \sum_{j=1}^k C_0 \frac{\|A^j\|}{s_j \delta_j} \cdot \frac{2C_0}{s_j \delta_j} |\gamma|^2 \\ &\geq 2\mu |\gamma|^2 - 2\delta_0 |\gamma|^2 - 4C_0^2 C_3 \delta_0 |\gamma|^2 \\ &\geq \mu |\gamma|^2, \quad \gamma \in C^m. \end{aligned} \quad (3.21)$$

In (3.20) and (3.21), we have used the inequality

$$0 < \delta_0 < \min \left\{ \left(\frac{1}{2} \right)^{7/12(n+1)}, C_0, \frac{\mu}{2 + 4C_0^2 C_3} \right\}. \quad (\text{D})$$

Using (3.16)–(3.18) on $\{|y - y_0| \leq 2s\} \times \{|\operatorname{Im} x| \leq r - 2\delta\} \times \{|z_{\pm}| \leq 6s\}$ implies

$$\|S\| \leq C_5 M \delta^{-(2n+1)}. \quad (3.22)$$

Also by (3.16)–(3.18) and on $\{|y - y_0| \leq s\} \times \{|\operatorname{Im} x| \leq r - 3\delta\} \times \{|z_{\pm}| \leq 6s\}$, by applying Cauchy's estimate we obtain

$$\|S_x\| \leq C_6 M \delta^{-(2n+1)}, \quad (3.23)$$

$$\|S_y\|, \|S_{z_+}\|, \|S_{z_-}\| \leq C_7 M s^{-1}, \quad (3.24)$$

$$\|S_{z_+ z_-}\| = 0, \quad \|S_{z_+ z_+}\|, \|S_{z_- z_-}\| \leq C_8 M s^{-2}. \quad (3.25)$$

Hence,

$$\|I \operatorname{grad}_{(y,x)}^T S\| \leq C_9 M s^{-1} \delta^{-(2n+2)}, \quad (3.26)$$

$$\|\operatorname{grad}_{(z_+, z_-)}^T S\| \leq C_7 M s^{-1}. \quad (3.27)$$

Thus, on $\{|y - y_0| \leq s\} \times \{|\operatorname{Im} x| \leq r - 3\delta\} \times \{|z_{\pm}| \leq 6s\}$,

$$|x - x^+| \leq C_9 M s^{-1} \delta^{-(2n+2)} = C_9 \delta^{42(n+1)}, \quad (3.28)$$

$$|y - y^+| \leq C_9 M s^{-1} \delta^{-(2n+2)} = C_9 s^{3/2}, \quad (3.29)$$

$$|z_+ - z_+^+|, |z_- - z_-^+| \leq C_7 M s^{-1} = C_7 s^{11/7}, \quad (3.30)$$

where $M = \delta^{72(n+1)}$, $s = \delta^{28(n+1)}$. Hence on D_+ ,

$$|x| \leq |x^+| + C_9 \delta^{42(n+1)} \leq r_+ + C_9 \delta^{42(n+1)} < r, \quad (3.31)$$

$$|y| \leq |y^+| + C_9 s^{3/2} < 4s_+ + C_9 s^{3/2} < 4s, \quad (3.32)$$

$$|z| \leq |z^+| + c_7 s^{11/7} < 6s_+ + C_7 s^{11/7} < 6s^{9/8} < 6s, \quad (3.33)$$

where we used the inequality

$$0 < \delta_0 < \min \left\{ \left(\frac{6}{C_9 + 1} \right)^{1/42(n+1)}, \left(\frac{4}{C_9 + 4} \right)^{1/4(n+1)}, \left(\frac{6}{C_7 + 6} \right)^{2/(n+1)} \right\}. \quad (\text{E})$$

By (3.31)–(3.33), we have $\Phi^+: D_+ \rightarrow D$.

We continue to estimate the perturbation P^+ on D_+ . Applying Talyor's expansion of P for z and (3.30) yields

$$\|P_1\| = \|P^* \circ \phi^1\| \leq C_{10} \left(\frac{6s^{9/8}}{s} \right)^3 M \leq \frac{1}{4} M^{8/7} = \frac{1}{4} M_+, \quad (3.34)$$

where we choose δ_0 such that

$$0 < \delta_0 < \left(\frac{1}{864C_{10} + 1} \right)^{3(n+1)/14}. \quad (F)$$

Next, using (2.0), (3.23), (3.25), and Cauchy's estimate, we derive

$$\begin{aligned} \|P_2\| &\leq \sup_{0 \leq t \leq 1} \|\{P - P^*, S\} \circ \phi^t\| \leq \sup_{D_+} |\langle \text{grad}^T(P - P^*), E \text{grad}^T S \rangle| \\ &\leq C_{11} \frac{1}{s^2 \delta} \|P - P^*\| \|S\| \leq C_{11}(1 + C_1) C_5 M^2 \frac{1}{s^2 \delta^{2(n+1)}} \\ &\leq C_{12} M^{8/7} \delta^{(26/7)(n+1)} < \frac{1}{4} M_+, \end{aligned} \quad (3.35)$$

where we used the inequality

$$0 < \delta_0 < \left(\frac{1}{4(C_{12} + 1)} \right)^{7/26(n+1)}. \quad (G)$$

According to Lemma 6, Cauchy's formula, and (2.10), we have

$$\begin{aligned} \|P_5\| &\leq \|\langle \text{grad}_{(y,x)}^T h(y)|_{y=y_0}, I(y_0) \text{grad}_{(y,x)}^T S \rangle \\ &\quad - \langle \text{grad}_{(y,x)}^T h(y), I(y) \text{grad}_{(y,x)}^T S \rangle\| \\ &\leq \langle \omega(y_0) - \omega(y), \text{grad}_{(y,x)}^T S \rangle\| \\ &\leq C_{20} \left\| \frac{\partial}{\partial y} I \bullet \text{grad}_{(y,x)}^T h \right\| |y - y_0| \bullet \frac{\|S\|}{\delta} \\ &\leq C_{21} M \delta^{-2(n+1)} \bullet s \leq \frac{1}{8} M_+, \end{aligned} \quad (3.36)$$

provided

$$0 < \delta_0 < \left(\frac{1}{8C_{21} + 1} \right)^{1/16(n+1)}. \quad (H)$$

Note that

$$\begin{aligned}
 \|\{N, S\}\| &\leq \|\langle \text{grad}_{(y,x)}^T h(y)|_{y=y_0}, I(y_0) \text{grad}_{(y,x)}^T S \rangle\| \\
 &\quad + \|\langle \text{grad}_z^T \langle z_+, \Omega z_- \rangle, J \text{grad}_z^T S \rangle\| \\
 &\quad + \|\langle \text{grad}_{(y,x)}^T \langle z_+, \Omega z_- \rangle, I(y) \text{grad}_{(y,x)}^T S \rangle\| \\
 &\quad + \|P_5\|.
 \end{aligned} \tag{3.37}$$

By (3.2), (3.23), (3.24), (3.6), and (2.0) in Section 2, on D_+ ,

$$\begin{aligned}
 &\|\langle \text{grad}_{(y,x)}^T h(y)|_{y=y_0}, I(y_0) \text{grad}_{(y,x)}^T S \rangle\| \\
 &\quad + \|\langle \text{grad}_{(z_+, z_-)}^T \langle z_+, \Omega z_- \rangle, J \text{grad}_z^T S \rangle\| \\
 &\quad + \|\langle \text{grad}_{(z_+, z_-)}^T \langle z_+, \Omega z_- \rangle, I(y) \text{grad}_{(y,x)}^T S \rangle\| \\
 &\leq \|P - P^*\| \\
 &\quad + \sup_{D_+} |\langle \text{grad}_{(z_+, z_-)}^T \langle z_+, \Omega z_- \rangle, I(y) \text{grad}_{(y,x)}^T (S - A) \rangle| + 2 \|P\| \\
 &\leq (3 + C_1) M + C_{13} s^2 \frac{1}{s\delta} \|S - A\| \leq (1 + C_1) M + C_{13} \frac{sM}{\delta^{2n+2}} \\
 &\leq (3 + C_1 + C_{13}) M.
 \end{aligned} \tag{3.38}$$

Using (3.36)–(3.38) we obtain

$$\{N, S\} \leq (4 + C_1 + C_{13}) M.$$

So, if

$$0 < \delta_0 < \left(\frac{1}{4C_5 C_{14} (3 + C_1 + C_{13})(n+1) + 1} \right)^{7/418(n+1)}, \tag{I}$$

then

$$\begin{aligned}
 \|P_3\| &\leq \|\{N, S\}, S\| \leq C_{14} \frac{\|S\| \|\{N, S\}\|}{s^2 \delta} \\
 &\leq C_{14} (4 + C_1 + C_{13}) C_5 M^2 \delta^{-(2n+2)} \leq \frac{1}{4} M^{8/7} \leq \frac{1}{4} M_+. \tag{3.39}
 \end{aligned}$$

Similarly, on D_+ ,

$$\begin{aligned} \|P_4\| &\leq \sup_{D_+} |\operatorname{grad}_{(y,x)}^T \langle z_+, \Omega z_- \rangle| \|I\| \|\operatorname{grad}_{(y,x)}^T (S-A)\| \\ &\leq C_{15} s^2 \frac{1}{s\delta} \|S-A\| \leq C_{16} \frac{sM}{\delta^{(2n+2)}} \\ &< \frac{1}{8} M^{8/7} \leq \frac{1}{8} M_+, \end{aligned} \quad (3.40)$$

where we used the inequality

$$0 < \delta_0 < \left(\frac{1}{8(C_{16} + 1)} \right)^{7/110(n+1)}. \quad (J)$$

By (3.34)–(3.36), (3.39), and (3.40), on D_+

$$\|P^+\| \leq \|P_1\| + \|P_2\| + \|P_3\| + \|P_4\| + \|P_5\| \leq M_+,$$

so we prove (2.3₊).

By (3.4), (3.22), and (2.0) from Section 2, applying the Cauchy integral formula we have that on $\{|y - y_0| \leq s\} \times \{|\operatorname{Im} x| \leq r - 3\delta\} \times \{|z_{\pm}| \leq 5s\}$,

$$\left\| \frac{\partial}{\partial(y, x, z_+, z_-)} (E \operatorname{grad}^T S) \right\| \leq C_{17} M s^{-2} \delta^{-(2n+2)} \leq C_{17} M^{7/36} \leq M^{1/6},$$

where we used the inequality

$$0 < \delta_0 < \left(\frac{1}{C_{17} + 1} \right)^{1/2(n+1)}. \quad (K)$$

Then on D_+ ,

$$\left\| \frac{\partial \Phi^+(y, x, z_+, z_-)}{\partial(y, x, z_+, z_-)} - \operatorname{Id} \right\| \leq M^{1/6}. \quad (3.41)$$

By (3.28)–(3.30), if

$$0 < \delta_0 < \left(\frac{1}{C_9 + C_7 + 1} \right)^{1/6(n+1)}, \quad (L)$$

then

$$\|\Phi^+ - \operatorname{id}\| \leq M^{1/3}. \quad (3.42)$$

Equations (3.41) and (3.42) imply that (2.8_i) holds. According to (2.4_i)–(2.7_i), for

$$0 < \delta_0 < \min \left\{ \left(\frac{1}{2} \right)^{7/44(n+1)}, \left(\frac{\varepsilon_0}{2} \right)^{144(n+1)} \right\}, \quad (\text{M})$$

$$\begin{aligned} \|\omega^i - \omega\| &\leq \sum_{j=1}^{\infty} \left\| \frac{\partial}{\partial y} [P^j]_x(y, 0) \right\|_{D_j} \\ &\leq \sum_{j=1}^{\infty} M_j^{11/18} \leq 2M_0^{11/18} < \varepsilon_0, \end{aligned} \quad (3.43)$$

where $\varepsilon_0 > 0$ is a constant in Lemma 1.

Thus we complete one inductive iteration process.

4. MEASURE ESTIMATE

Now we discuss the measure estimate in Theorem A. Put

$$G_* = \bigcap_{i=0}^{\infty} O_i. \quad (4.1)$$

1. $l = n$. According to (4.1) and Lemma 1,

$$\begin{aligned} \text{mes}_l(G \setminus G_*) &\leq \sum_{j=0}^{\infty} \text{mes}(G \setminus O_j) \\ &\leq \sum_{j=0}^{\infty} C\delta_j^{1/(l-r+1)} \leq 2C\delta_0^{1/(l-r+1)}, \end{aligned} \quad (4.2)$$

where we used the inequality

$$0 < \delta_0 < \left(\frac{1}{2} \right)^{7(l-r+1)}. \quad (\text{N})$$

2. $l < n$. Denote $\hat{y} = (y, y_{l+1}, \dots, y_n)$, $\hat{\omega}(\hat{y}) = \omega(y)$, $\hat{\omega}^i(\hat{y}) = \omega^i(y)$, $G' = G \times \underbrace{[1, 2] \times \dots \times [1, 2]}_{n-l}$. According to the degenerate condition in Theorem A:

$$\text{rank} \left(\frac{\partial \hat{\omega}}{\partial \hat{y}} \right) = r, \quad \hat{y} \in G', \quad (4.3)$$

$$\text{rank} \left\{ \bar{\omega}, \frac{\partial^\alpha \hat{\omega}}{\partial \hat{y}^\alpha} : \forall \alpha \in \mathbb{Z}_+^n, 0 < |\alpha| \leq n - r + 1 \right\} = n, \quad \hat{y} \in G'. \quad (4.4)$$

Define

$$O'_i = O_i \times \underbrace{[1, 2] \times \cdots \times [1, 2]}_{n-l},$$

$$G'_* = G_* \times \underbrace{[1, 2] \times \cdots \times [1, 2]}_{n-l}.$$

By (4.1), (4.3), (4.4), Lemma 1 and (L),

$$\text{mes}_n(G' \setminus G'_*) \leq C \sum_{i=0}^{\infty} \text{mes}_n(G' \setminus O'_i) \leq 2C\delta_0^{1/(n-r+1)}.$$

Applying Fubini's Theorem,

$$\text{mes}_n(G \setminus G_*) \leq C_{18}\delta_0^{1/(n-r+1)}.$$

3. $l > n$. For every $y \in \bar{G}$, according to condition (H_3) in Theorem A, there exist positive integers $i_1, \dots, i_r, \dots, i_l$, $1 \leq i_j \leq l$, $j \neq l$, $i_j \neq i_l$ and multiplex index $\alpha^1, \dots, \alpha^{n-r}$, $|\alpha^i| \geq 2$, $i = 1, \dots, n-r$, such that

$$\text{rank} \left\{ \frac{\partial \omega}{\partial y_{i_1}}, \dots, \frac{\partial \omega}{\partial y_{i_r}} \right\} = r, \quad (4.5)$$

$$\text{rank} \left\{ \frac{\partial \omega}{\partial y_{i_1}}, \dots, \frac{\partial \omega}{\partial y_{i_r}}, \frac{\partial^{\alpha^1} \omega}{\partial y^{\alpha^1}}, \dots, \frac{\partial^{\alpha^{n-r}} \omega}{\partial y^{\alpha^{n-r}}} \right\} = n. \quad (4.6)$$

Let $\tilde{\omega}(y) = (\omega(y), y_{i_{n+1}}, \dots, y_{i_l})$. Then

$$\text{rank} \left\{ \frac{\partial \tilde{\omega}}{\partial y_{i_1}}, \dots, \frac{\partial \tilde{\omega}}{\partial y_{i_r}} \right\} = r, \quad (4.7)$$

$$\begin{aligned} & \text{rank} \left\{ \frac{\partial \tilde{\omega}}{\partial y_{i_1}}, \dots, \frac{\partial \tilde{\omega}}{\partial y_{i_r}}, \frac{\partial^{\alpha^1} \tilde{\omega}}{\partial y^{\alpha^1}}, \dots, \frac{\partial^{\alpha^{n-r}} \tilde{\omega}}{\partial y^{\alpha^{n-r}}}, \frac{\partial \tilde{\omega}}{\partial y_{i_{n+1}}}, \dots, \frac{\partial \tilde{\omega}}{\partial y_{i_l}} \right\} \\ &= \text{rank} \begin{pmatrix} \frac{\partial \omega}{\partial y_{i_1}}, \dots, \frac{\partial \omega}{\partial y_{i_r}}, \frac{\partial^{\alpha^1} \omega}{\partial y^{\alpha^1}}, \dots, \frac{\partial^{\alpha^{n-r}} \omega}{\partial y^{\alpha^{n-r}}}, & * \\ 0, & I_{l-n} \end{pmatrix} = l, \end{aligned} \quad (4.8)$$

where I_{l-n} denotes the $l-n$ order unit matrix.

Define

$$\begin{aligned}\tilde{\omega}^i(y) &= (\omega^i(y), y_{i_{n+1}}, \dots, y_{i_l}), \\ O'_i &= \{y \in G : |\langle k, \omega^i(y) \rangle + \langle \bar{k}, \bar{y} \rangle| \geq \delta_i (|k| + |\bar{k}|)^{-\tau}, 0 \neq (k, \bar{k}) \in Z^l\},\end{aligned}$$

where $\bar{y} = (y_{i_{n+1}}, \dots, y_{i_l})$, $\bar{k} = (k_{n+1}, \dots, k_l)$, $\tau \geq n(n-r+1) - 1$.

By (4.1), (4.7), (4.8), Lemma 1, and (N),

$$\text{mes}_n(G \setminus O'_i) \leq C_0 \delta_i^{1/(n-r+1)}.$$

Obviously

$$O'_i \subset O_i.$$

Hence

$$\begin{aligned}\text{mes}_l(G \setminus G_*) &\leq \sum_{i=0}^\infty \text{mes}_l(G \setminus O_i) \leq \sum_{i=0}^\infty \text{mes}_l(G \setminus O'_i) \\ &\leq C_0 \sum_{i=0}^\infty \delta_i^{1/(n-r+1)} \leq 2C_0 \delta_0^{1/(n-r+1)}.\end{aligned}$$

By Conditions 1–3 and the definition of δ_0 , we have

$$\text{mes}_l(G \setminus G_*) = O(M_0^{1/72(n+1)(n-r+1)}).$$

Hence for M_0 small enough, choose $y_0 \in G_*$ so that the inductive iteration can continue. Thus the proof of Theorem A is complete.

5. SOME EXAMPLES

In this section we give some examples to illustrate Theorem A.

EXAMPLE 1. We consider the following unperturbed system

$$\begin{aligned}\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} &= I(y) \operatorname{grad}_{(y,x)}^T h(y) + I(y) \operatorname{grad}_{(y,x)}^T \langle z_+, \Omega(y, x) z_- \rangle, \\ \dot{z}_+ &= \Omega^T(y, x) z_+, \\ \dot{z}_- &= -\Omega(y, x) z_-, \end{aligned} \tag{5.1}$$

where $y \in R^1$, $x \in R^3$, $z = (z_+, z_-) \in R^4$, and

$$I(y) = \begin{pmatrix} 0 & y & y & -1 \\ y^2 + y & y & -1 & 0 \\ y & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega(y, x) = \begin{pmatrix} x_1^2 + 1 & -x_2 - x_3 \\ x_2 + x_3 & y^2 + 2 \end{pmatrix},$$

$$h(y) = y.$$

Then $I(y)$ is symplectic, in fact,

$$\begin{aligned} I^T(y) J I(y) &= \begin{pmatrix} 0 & y^2 + y & y & 1 \\ y & y & 1 & 0 \\ y & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 0 & y & y & -1 \\ y^2 + y & y & -1 & 0 \\ y & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & y & -y^2 - y & 0 \\ 0 & 1 & -y & -y \\ 0 & 0 & 1 & -y \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & y & y & -1 \\ y^2 + y & y & -1 & 0 \\ y & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = J \end{aligned}$$

and

$$(0, \omega(y))^T = I(y) \operatorname{grad}_{(y, x)}^T h(y) = \begin{pmatrix} 0 \\ y^2 + y \\ y \\ 1 \end{pmatrix};$$

hence

$$\begin{aligned}\operatorname{rank}\left(\frac{\partial\omega(y)}{\partial y}\right)&=\operatorname{rank}\begin{pmatrix}2y+1\\1\\0\end{pmatrix}=1=l,\\ \operatorname{rank}\left\{\omega(y),\frac{\partial\omega(y)}{\partial y},\frac{\partial^2\omega(y)}{\partial y^2}\right\}&=\operatorname{rank}\begin{Bmatrix}y^2+y&2y+1&2\\y&1&0\\1&0&0\end{Bmatrix}=3=n\\ \langle\mu,\Omega(y)\mu\rangle&\geqslant|\mu|^2.\end{aligned}$$

Consequently, $\omega(y)$, $\Omega(y,x)$ satisfy the conditions (H_1) – (H_3) in Theorem A, and the higher dimensional invariant torus of the system (5.1),

$$\tau_{y_0}=\{(y,x,0,0)\mid y=y_0,\,x\in T^3\}$$

persists under a small perturbation P .

EXAMPLE 2. We discuss the above system (5.1), where $y=(y_1,y_2)\in R^2$, $x=(x_1,x_2,x_3,x_4)\in R^4$, $z=(z_1,z_2)\in R^2$, and

$$\begin{aligned}h(y)&=y_1+y_2,\qquad \Omega(y,x)=\begin{pmatrix}x_1^2+1&-x_2-x_3\\x_2+x_3&y_1^2+2\end{pmatrix},\\ I(y)&=\begin{pmatrix}0&0&y_1^2&y_1^2&y_1&-1\\0&0&0&0&-1&0\\y_1^3+y_1^2&0&y_1&-1&0&0\\y_1^2&0&1&0&0&0\\y_1&1&0&0&0&0\\1&0&0&0&0&0\end{pmatrix}.\end{aligned}$$

Obviously $I(y)$ is symplectic. Thus

$$(0,\omega(y))^T=I(y)\operatorname{grad}_{(y,x)}^Th(y)=\begin{pmatrix}0\\0\\y_1^3+y_1^2\\y_1^2\\y_1+1\\1\end{pmatrix},$$

then

$$\begin{aligned} \operatorname{rank} \left(\frac{\partial \omega(y)}{\partial y} \right) &= \operatorname{rank} \begin{pmatrix} 3y_1^2 + 2y & 0 \\ 2y_1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 < 2 = l, \\ \operatorname{rank} \left\{ \omega(y), \frac{\partial \omega(y)}{\partial y}, \frac{\partial^2 \omega(y)}{\partial y^2}, \frac{\partial^3 \omega(y)}{\partial y^3} \right\} \\ &= \operatorname{rank} \begin{pmatrix} y_1^3 + y_1^2 & 3y_1^2 + 2y_1 & 6y_1 + 2 & 6 \\ y_1^2 & 2y_1 & 2 & 0 \\ y_1 + 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 4 = n \\ &\langle \mu, \Omega(y) \mu \rangle \geq |\mu|^2. \end{aligned}$$

Hence, $\omega(y)$, $\Omega(y)$ satisfy the conditions (H_1) – (H_3) in Theorem A, and the higher dimension invariant torus $\tau_{y_0} = \{(y, x, 0, 0) \mid y = y_0, x \in T^4\}$ of the system (5.1) persists under the small perturbation P .

EXAMPLE 3. We consider the system (5.1) with $x \in R^1$,

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in R^3,$$

$h(y) = \frac{1}{2} y_3^2$, and

$$I(y) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this case $m = 0$, $l = 3$, $n = 1$. By Theorem A, most of the one-dimensional tori persist under small perturbations.

6. PRELIMINARY LEMMAS

In this section, we list some lemmas which have been used in the previous sections.

LEMMA 1. [14]. Assume $\omega(y)$ and $\bar{\omega}(y)$ are the l -dimensional C^{l-r+1} functions on \bar{G} , where $G \subset \mathbb{R}^l$ is a connected open bounded set. Suppose on \bar{G} ,

$$\text{rank} \left(\frac{\partial \omega}{\partial y} \right) = r,$$

$$\text{rank} \left\{ \omega, \frac{\partial^\alpha \omega}{\partial y^\alpha} : \forall \alpha, |\alpha| \leq l - r + 1 \right\} = l,$$

then there exists $\varepsilon_0 > 0$ such that if $\|\bar{\omega}\|_{C^{l-r+1}(\bar{G})} \leq \varepsilon_0$ and if δ is enough small, the Cantor set

$$G_\delta = \{y : |\langle k, \omega(y) + \bar{\omega}(y) \rangle| \geq \delta |k|^{-\tau}, 0 \neq k \in \mathbb{Z}^n\}$$

possesses a positive measure, where $\tau > l(l-r+1) - 1$ is a given constant. Moreover,

$$\text{mes}_l(G \setminus G_\delta) \leq C\delta^{1/(l-r+1)},$$

where $C > 0$ is a constant independent of $\alpha, \bar{\omega}$, and δ .

LEMMA 2 [6]. Consider the equation

$$V_x \omega + V(x) \Phi(x) + A(x) V(x) = F(x),$$

where $\omega = (\omega_1, \dots, \omega_n)$, $x = (x_1, \dots, x_n)$, and $\Phi(x)$ and $A(x)$ are real analytic matrix functions on $\sum_r : \{|\text{Im } x| \leq r\}$. Assume $\text{Re} \langle \gamma, \Phi(x) \gamma \rangle \geq \mu |\gamma|^2$, $\text{Re} \langle \gamma, A(x) \gamma \rangle \geq \mu |\gamma|^2$ hold for all $\gamma \in C^l$. Then for every real analytic matrix $F(x)$, there exists a unique real analytic matrix $V(x)$ such that

$$|V(x)| \leq 2l^{1/2} \mu^{-1} |F(x)|.$$

LEMMA 3 [6]. Consider the equation

$$V_x(x, z) \omega + A(x, z) V(x, z) = f(x, z),$$

where $\omega = (\omega_1, \dots, \omega_n)$, $x = (x_1, \dots, x_n)$, $z = (z_1, \dots, z_l)$, and $A(x, z)$ is a real analytic matrix function on $\sum_{r,R} = \{|\text{Im } x| \leq r\} \times \{|z| \leq R\}$. Assume on $\sum_{r,R}$, that $\text{Re} \langle \gamma, A(x, z) \gamma \rangle \geq \mu |\gamma|^2$ holds for all $\gamma \in C^l$. Then for each real analytic matrix function $f(x, z)$ defined on $\sum_{r,R}$ the equation admits a unique solution $V(x, z)$ such that

$$|V(x, z)| \leq 2l^{1/2} \mu^{-1} |f(x, z)|.$$

LEMMA 4 [6]. Consider the equation

$$\partial V(x) + f(x) = 0,$$

where $\partial = \sum_{k=1}^n \omega_k (\partial / \partial x_k)$. Assume

- (1) f is a real analytic and 1-periodic function on Σ_r ;
- (2) $[f] = \int_{T^n} f(x) dx = 0$;
- (3) ω satisfies $|\langle k, \omega \rangle| \geq K |k|^{-\tau}$ for all $0 \neq k \in \mathbb{Z}^n$, where $K > 0$, $\tau > n$ are constants.

Then on $\Sigma_{r-2\delta}$, $0 < 2\delta < r < 1$, the equation admits a unique real analytic solution $U(x)$ such that

$$|U(x)| \leq C \delta^{-(2n+1)} \|f\|, \quad [U] = 0,$$

where $C = 4^n K^{-1} ((n+1) e^{-1})^{n+1}$.

LEMMA 5 [6]. Let $V_0(x)$ be a smooth vector field on D_0 . Define the flow:

$$\phi_0^t(x) : \frac{d}{dt} \phi_0^t(x) = V_0(\phi_0^t(x)), \quad \phi_0^0(x) = x.$$

Assume there exists an invertible transformation $T_i : D_i \rightarrow D_{i-1}$ with $\|\prod_{i=1}^{\infty} T'_i\| < \infty$, where T'_i denotes the Jacobian of T_i . The transformation

$$U_i = T_1 \circ \dots \circ T_i : D_i \rightarrow D_0$$

naturally reduces flows,

$$\phi_i^t = U_i^{-1} \circ \phi_0^t \circ U_i,$$

with corresponding vector fields V_i on D_i ,

$$V_i = \frac{d}{dt} (\phi_i^t(x))|_{t=0}.$$

Assume

- (1) V_i converges to V_{∞} as $i \rightarrow \infty$ and $\|V_i - V_{\infty}\| \leq C d_{i+1}$ on D_{∞} , where C is independent of i and $d_i = \text{dist}(D_i, \partial D_{i-1})$;
- (2) the segment $x = x_0 + vt$, $0 \leq t \leq 1$, belongs to D_{∞} ; and on this segment $V_{\infty} = V$;
- (3) $\|\partial V_i / \partial x\|_{D_i} \leq B$, where B is independent of i ;
- (4) $U_{\infty} = \lim_{i \rightarrow \infty} U_i$ exists and is continuous.

Then for $0 \leq t \leq 1/(B+C)$,

$$\phi_0^t(U_\infty(x_0)) = U_\infty(x_0 + vt) \subset D_0.$$

LEMMA 6 [10]. Consider the Hamiltonian system

$$\dot{z} = I(y) \operatorname{grad}^T H(z), \quad z = (y, x).$$

If $H(z) = H(y)$, then the equation admits the form

$$\dot{y} = 0, \quad \dot{x} = \omega(y),$$

where $\omega(y) = (\omega_1(y), \dots, \omega_n(y))^T$.

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