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A Characteristic Equation for Non-autonomous Partial Functional Differential Equations

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We characterize the exponential dichotomy of non-autonomous partial functional differential equations by means of a spectral condition extending known characteristic equations for the autonomous or time-periodic case. From this we deduce robustness results. We further study the almost periodicity of solutions to the inhomogeneous equation. Our approach is based on the spectral theory of evolution semigroups. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

For the autonomous partial functional differential equation

$$\begin{aligned} \dot{u}(t) &= Au(t) + Lu_t, \quad && t \geqslant 0, \\ u(t) &= \phi(t), && -r \leqslant t \leqslant 0, \end{aligned} \tag{1.1}$$

there is a well-developed semigroup approach; in particular, a powerful spectral theory is available. Here we assume that A generates a C_0-semigroup $V(\cdot)$ on a Banach space X. Further, $r \geqslant 0$, $\phi \in E := C([-r,0],X)$, $L \in \mathscr{L}(E,X)$, and we let $u_t(\xi) := u(t+\xi)$ for $\xi \in [-r,0]$, $t \geqslant 0$, and $u : [-r,\infty) \to$

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X. Then the operator

$$A_L := \frac{d}{d\xi},\ D(A_L) := \{\phi \in C^1([-r,0],X) : \phi(0) \in D(A),$$
$$\phi'(0) = A\phi(0) + L\phi\}, \tag{1.2}$$

generates a C_0-semigroup $U_L(\cdot)$ on E and the function

$$u(t) := \begin{cases} [U_L(t)\phi](0), & t \geqslant 0, \\ \phi(t), & -r \leqslant t \leqslant 0, \end{cases}$$

solves (1.1) for $\phi \in D(A_L)$. We point out that the semigroup $U_L(\cdot)$ describes the time evolution of the history function of the solution; i.e., $u_t = U_L(t)\phi$. The spectrum of A_L is determined by the relation

$$\lambda \in \sigma(A_L) \quad \Leftrightarrow \quad \lambda \in \sigma(A + L_\lambda), \tag{1.3}$$

where $L_\lambda \in \mathscr{L}(X)$ is given by $L_\lambda x := L(e^{\lambda \cdot} x)$ for $\lambda \in \mathbb{C}$. Moreover, if $t \mapsto V(t)$ is continuous in operator norm for $t > 0$ (e.g., if $V(\cdot)$ is analytic or compact), then

$$\sigma(U_L(t)) \setminus \{0\} = \exp(t\sigma(A_L)). \tag{1.4}$$

For these results we refer the reader to [7, Section VI.6] and also to [28, Chap. 3] and the references therein. Combining (1.3) and (1.4) with standard spectral theory, one sees that the solution semigroup $U_L(\cdot)$ is exponentially stable if and only if

$$\sup\{\operatorname{Re} \lambda\colon\ \lambda \in \sigma(A + L_\lambda)\} < 0$$

and that $U_L(\cdot)$ has exponential dichotomy if and only if

$$\{\lambda \in \mathbb{C}\colon\ \lambda \in \sigma(A + L_\lambda)\} \cap i\mathbb{R} = \emptyset$$

(provided that $V(\cdot)$ is norm continuous for $t > 0$). In this way one obtains exponential stability and dichotomy of the history function u_t. Thus one can consider (1.3) as a *generalized characteristic equation* which extends the classical results for the case $X = \mathbb{C}^n$ as presented in e.g. [13, Chap. 7].

In this paper we want to prove analogous characterizations for the non-autonomous problem

$$\begin{aligned} \dot{u}(t) &= A(t)u(t) + L(t)u_t, && t \geqslant s, \\ u(t) &= \phi(t - s), && s - r \leqslant t \leqslant s, \end{aligned} \tag{1.5}$$

where the linear operators $A(t)$, $t \in \mathbb{R}$, generate an evolution family $V(t,s)$, $t \geqslant s$, on X and $L(\cdot)$ belongs to $C_b(\mathbb{R}, \mathscr{L}_s(E,X))$, the space of uniformly bounded and strongly continuous operator-valued functions. In the next section we review the existence theory of (1.5); typically this problem is solved by an evolution family $U_L(t,s)$ on E which is generated by operators $A_L(t)$ given as in (1.2).

However, even in the case of ordinary differential equations (i.e., $X = \mathbb{C}^n$, $r = 0$, and $L(t) = 0$, thus $A_L(t) = A(t)$) it is known that the location of the spectra of $A_L(t)$ does not influence the asymptotic behavior of solutions. This can be seen by, e.g., Example VI.9.9 in [7] where time-periodic equations are considered. But we note that for a certain class of periodic problems one can prove a characteristic equation involving the spectrum of the monodromy operator $U_L(p,0)$; see [13, Section 8.3; 11] and Corollary 3.7.

In the present work we derive in Theorem 3.5 a generalized characteristic equation for (1.5) which is formulated on function spaces like $C_0(\mathbb{R},X)$ and determines the exponential dichotomy of (1.5). As a consequence we obtain the above-mentioned characteristic equations for autonomous and periodic problems. Theorem 3.5 further allows us to characterize those delay perturbations such that $U_L(\cdot,\cdot)$ inherits the exponential dichotomy of $V(\cdot,\cdot)$; see Theorem 4.1 and Corollary 4.4 which extend results in [6, 8, 18]. In this context we also refer the reader to the recent papers [3, 12, 14]. Moreover, our characteristic equation is closely related to the qualitative behavior of the solutions to the inhomogeneous problem

$$\dot{u}(t) = A(t)u(t) + L(t)u_t + g(t), \qquad t \in \mathbb{R}. \tag{1.6}$$

For instance, let $L(t)$ be periodic and let $A(t)$ generate a periodic evolution family with exponential dichotomy. Then the solution u is almost periodic if the inhomogeneity g is almost periodic; see Theorem 4.6. Here we generalize results from, e.g., [1, 11, 28] to the non-autonomous setting. Finally, the influence of positivity is explored in Theorem 4.10 for a certain class of periodic problems. In the last section we discuss a retarded parabolic partial differential equation with time-periodic coefficients.

Our approach is based on the so-called evolution semigroup associated with (1.5) which is introduced in the next section. Concerning unexplained concepts and notation we refer the reader to [7].

2. PREREQUISITES

Let Z be a Banach space. A family $\{W(t,s);\ -\infty < s \leqslant t < \infty\} \subseteq \mathscr{L}(Z)$ is called an *evolution family* if $W(t,s) = W(t,r)W(r,s)$, $W(s,s) = I$, and (t,s)

$\mapsto W(t,s)$ is strongly continuous for $-\infty < s \leqslant r \leqslant t < \infty$. Its *exponential growth bound* is given by

$$\omega(W) := \inf\{w \in \mathbb{R}\colon \|W(t,s)\| \leqslant M_w e^{w(t-s)} \text{ for } t \geqslant s\}.$$

The evolution family is said to be *exponentially bounded* if $\omega(W) < \infty$ and *exponentially stable* if $\omega(W) < 0$. We denote by $Q = I - P$ the complementary projection of a projection P. An evolution family $W(\cdot,\cdot)$ has an *exponential dichotomy* if there are a projection-valued function $P(\cdot) \in C_b(\mathbb{R}, \mathscr{L}_s(X))$ and constants $N, \delta > 0$ such that

(a) $P(t)W(t,s) = W(t,s)P(s)$,

(b) the restriction $W_Q(t,s)\colon Q(s)X \to Q(t)X$ of $W(t,s)$ has the inverse $W_Q(s,t)$, and

(c) $\|W(t,s)P(s)\| \leqslant Ne^{-\delta(t-s)}$ and $\|W_Q(s,t)Q(t)\| \leqslant Ne^{-\delta(t-s)}$,

for $t \geqslant s$. We note that the projections $P(t)$, $t \in \mathbb{R}$, are uniquely determined by (a)–(c); see [26, Corollary 3.3].

We also deal with *p-periodic evolution families* $W(\cdot,\cdot)$ which means that there exists a constant $p > 0$ such that $W(t+p, s+p) = W(t,s)$ for $t \geqslant s$. In this case it is known that

$$\sigma(W(p,0)) \setminus \{0\} = \sigma(W(t+p,t)) \setminus \{0\},$$

for $t \in \mathbb{R}$, that $\omega(W) = \frac{1}{p} \ln r(W(p,0))$, and that $W(\cdot,\cdot)$ has an exponential dichotomy if and only if $\mathbb{T} \cap \sigma(W(p,0)) = \emptyset$, where $\mathbb{T} = \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$; see [15, Section 7.2].

Given an exponentially bounded evolution family $W(\cdot,\cdot)$ on Z, we define the associated *evolution semigroup* $T_W(\cdot)$ on $C_b(\mathbb{R}, Z)$ by setting

$$(T_W(t)f)(s) := W(s, s-t)f(s-t), \qquad t \geqslant 0,\ s \in \mathbb{R},\ f \in C_b(\mathbb{R}, Z).$$

Note that $\omega(T_W) = \omega(W)$. This semigroup is not strongly continuous on $C_b(\mathbb{R}, X)$. We are thus looking for closed subspaces of $C_b(\mathbb{R}, Z)$ which are invariant under $T_W(\cdot)$ and the group of translations and on which $T_W(\cdot)$ is strongly continuous. It is easy to see that the space $C_0(\mathbb{R}, Z)$ satisfies these requirements for each exponentially bounded evolution family. This situation is thoroughly studied in the monograph [5]; see also the survey given in [7, Section VI.9].

If $W(\cdot,\cdot)$ is p-periodic, we can also choose the subspaces $P_p(\mathbb{R}, Z)$ of p-periodic functions and $AP(\mathbb{R}, Z)$ of almost periodic functions; i.e.,

$$AP(\mathbb{R}, Z) = \overline{\mathrm{lin}}\{e^{i\eta\cdot}z\colon \eta \in \mathbb{R}, z \in Z\},$$

where the closure is taken in $C_b(\mathbb{R}, Z)$. In the case $P_p(\mathbb{R}, Z)$ one can verify the required properties in a straightforward way; for $AP(\mathbb{R}, Z)$ we refer to e.g.

[19, Lemma 2]. We further recall that each $f \in AP(\mathbb{R}, Z)$ has a unique representation

$$f(t) = \sum_{k=0}^{\infty} e^{i\eta_k t} z_k \quad \text{with } z_k \in Z,\ \eta_k \in \mathbb{R}, \text{ and } \operatorname{sp}(f) = \overline{\{\eta_k \colon k \in \mathbb{N}\}}, \tag{2.1}$$

where the series converges uniformly in $t \in \mathbb{R}$; see [17, Sec. 2.3] and [2, Prop. 2.3]. Here the *spectrum* of a function $f \in L^\infty(\mathbb{R}, Z)$ is defined by

$$\operatorname{sp}(f) := \{\eta \in \mathbb{R} \colon \forall\, \varepsilon > 0 \exists \varphi \in L^1(\mathbb{R}) \text{ with } \operatorname{supp}(\hat{\varphi}) \subseteq [\eta - \varepsilon, \eta + \varepsilon] \\ \text{and } \varphi \star f \neq 0\},$$

where $\hat{\varphi}$ denotes the Fourier transform of φ and $\varphi \star f$ denotes the convolution of φ and f. Note that $\operatorname{sp}(f) \subseteq \frac{2\pi}{p}\mathbb{Z}$ if f is p-periodic; cf. [22, (0.48)]. We can now introduce the subspaces

$$AP_{\Lambda_p}(\mathbb{R}, Z) := \{f \in AP(\mathbb{R}, Z) \colon \ \operatorname{sp}(f) \subseteq \Lambda_p\}$$

for a non-empty closed subset Λ_p of $\mathbb{R}$ with $\Lambda_p + \frac{2\pi}{p}\mathbb{Z} = \Lambda_p$ and

$$AP_{\Lambda_\infty}(\mathbb{R}, Z) := \{f \in AP(\mathbb{R}, Z) \colon \ \operatorname{sp}(f) \subseteq \Lambda_\infty\}$$

for a non-empty closed subset Λ_∞ of $\mathbb{R}$. In particular, $P_p(\mathbb{R}, Z) = AP_{\Lambda_p}(\mathbb{R}, Z)$ if $\Lambda_p = \frac{2\pi}{p}\mathbb{Z}$ and $AP(\mathbb{R}, Z) = AP_{\Lambda_p}(\mathbb{R}, Z)$ if $\Lambda_p = \mathbb{R}$. Due to [22, Theorem 0.8], the space $AP_{\Lambda_p}(\mathbb{R}, Z)$ is closed in $AP(\mathbb{R}, Z)$. It is invariant under translations and $T_W(t)$ because of [22, Propositions 0.4 & 0.5] and [4, Lemma 3.6] (or Lemma 3.3 below) provided that $W(\cdot,\cdot)$ is p-periodic. Similarly, one verifies these assertions for the space $AP_{\Lambda_\infty}(\mathbb{R}, Z)$ if $W(t,s) = W(t-s)$ for a C_0-semigroup $W(\cdot)$.

Convention. Throughout this paper $F(\mathbb{R}, Z)$ stands for $C_0(\mathbb{R}, Z)$ if we deal with an exponentially bounded evolution family $W(\cdot,\cdot)$ on Z, for $AP_{\Lambda_p}(\mathbb{R}, Z)$ if $W(\cdot,\cdot)$ is p-periodic, and for $AP_{\Lambda_\infty}(\mathbb{R}, Z)$ if $W(\cdot,\cdot)$ is a semigroup. In these cases the restriction of the evolution semigroup to $F(\mathbb{R}, Z)$ is also denoted by $T_W(\cdot)$ and its generator by G_W.

The spectrum of the generator G_W can be used to characterize certain asymptotic properties of the evolution family $W(\cdot,\cdot)$ as stated in the following proposition. Part (a) is due to R. Rau, Y. Latushkin, and S. Montgomery-Smith; see [5, Theorems 3.13 & 3.17] or [7, Theorems VI.9.15 & VI.9.18]. For part (b) we refer the reader to [16, Props. 3.1 & 3.2] or [19, Prop. 1], for part (c) to [19, Lemmas 2 and 4], and for part (d) to [4, Corollary 3.9]. Part (e) can be proved following the arguments in the proof of [4, Theorem 3.8].

PROPOSITION 2.1. *For an exponentially bounded evolution family $W(\cdot,\cdot)$ on the Banach space Z the following assertions hold.*

(*a*) *$W(\cdot,\cdot)$ has an exponential dichotomy if and only if G_W is invertible on $C_0(\mathbb{R},Z)$. Moreover, $\omega(W) = s(G_W)$.*

(*b*) *Let $W(\cdot,\cdot)$ be p-periodic. Then $1 \in \rho(W(p,0))$ if and only if G_W is invertible on $P_p(\mathbb{R},Z)$. Moreover, $\omega(W) = s(G_W)$.*

(*c*) *Let $W(\cdot,\cdot)$ be p-periodic. Then $W(\cdot,\cdot)$ has an exponential dichotomy if and only if G_W is invertible on $AP(\mathbb{R},X)$.*

(*d*) *Let $W(\cdot,\cdot)$ be p-periodic. If $\sigma(W(p,0)) \cap \overline{\{e^{i\eta p}\colon \eta \in \Lambda_p\}} = \emptyset$, then G_W is invertible on $AP_{\Lambda_p}(\mathbb{R},Z)$.*

(*e*) *Let $W(t,s) = W(t-s)$ for a C_0-semigroup $W(\cdot)$. If $\sigma(W(t_0)) \cap \overline{\{e^{i\eta t_0}\colon \eta \in \Lambda_\infty\}} = \emptyset$ for some $t_0 > 0$, then G_W is invertible on $AP_{\Lambda_\infty}(\mathbb{R},Z)$.*

Using basic semigroup theory one sees that, given $u, f \in F(\mathbb{R},Z)$ and $\lambda \in \mathbb{C}$, one has $u \in D(G_W)$ and $(\lambda - G_W)u = f$ if and only if

$$e^{-\lambda t}T_W(t)u - u = -\int_0^t e^{-\lambda\sigma}T_W(\sigma)f\,d\sigma, \qquad t \geqslant 0. \tag{2.2}$$

Further, the resolvent of G_W is given by

$$[R(\lambda,G_W)f](t) = \left[\int_0^\infty e^{-\lambda s}T_W(s)f\,ds\right](t) = \int_{-\infty}^t e^{-\lambda(t-\tau)}W(t,\tau)f(\tau)d\tau, \tag{2.3}$$

for $\operatorname{Re}\lambda > \omega(W)$, $t \in \mathbb{R}$, and $f \in F(\mathbb{R},Z)$.

Throughout this paper we use the following assumptions, where $E = C([-r,0],X)$ for a fixed $r \geqslant 0$.

(A1) $V(\cdot,\cdot)$ is an exponentially bounded evolution family on a Banach space X and $L(\cdot) \in C_b(\mathbb{R},\mathscr{L}_s(E,X))$.

(A2) $V(\cdot,\cdot)$ is a p-periodic evolution family on X and $L(\cdot) \in C_b(\mathbb{R},\mathscr{L}_s(E,X))$ is p-periodic.

(A3) $V(t,s) = V(t-s)$, $t \geqslant s$, for a C_0-semigroup $V(\cdot)$ on X and $L(\cdot) \equiv L \in \mathscr{L}(E,X)$.

Then we can define operators $U(t,s)$ on E by setting

$$(U(t,s)\phi)(\xi) := \begin{cases} V(t+\xi,s)\phi(0), & s-t \leqslant \xi, \\ \phi(t+\xi-s), & \xi \leqslant s-t, \end{cases} \tag{2.4}$$

for $t \geqslant s$, $\xi \in [-r,0]$, and $\phi \in E$. Note that $U(\cdot,\cdot)$ is an evolution family on E with $\omega(U) = \omega(V)$ and that $U(\cdot,\cdot)$ is p-periodic in the case (A2) and is given by a semigroup in the case (A3). We are looking for a function $u \in C([s-r,\infty),X)$ such that

$$\begin{aligned} u(t) &= V(t,s)\phi(0) + \textstyle\int_s^t V(t,\tau)L(\tau)u_\tau d\tau, \quad t \geqslant s, \\ u(t) &= \phi(t-s), \quad s-r \leqslant t \leqslant s, \end{aligned} \tag{2.5}$$

for $\phi \in E$ and $s \in \mathbb{R}$, where $u_t = u(t+\cdot) \in E$. Assume for a moment that there are operators $A(t)$, $t \in \mathbb{R}$, such that $V(t,s)D(A(s)) \subseteq D(A(t))$ and $V(\cdot,s)x \in C^1([s,\infty),X)$ with derivative $A(t)V(t,s)x$ for $x \in D(A(s))$ and $t \geqslant s$. (In this case we say that $A(\cdot)$ *generates* $V(\cdot,\cdot)$. Assumptions implying this property can be found in most monographs on evolution equations; see [7, Sect. VI.9.a] for references.) Then a solution u of (1.5) belonging to $C^1([s,\infty),X)$ also satisfies (2.5) as can be seen by a standard argument. We therefore call u the *mild solution* of (1.5) (or (2.5)). Conversely, imposing appropriate conditions on $A(\cdot)$, $L(\cdot)$, and ϕ, one can deduce that a solution of (2.5) is differentiable and fulfills (1.5); cf. [23; 28, Chap. 2] or Section 5.

The problem (2.5) can easily be solved if (A1) holds. In fact, by a straightforward fixed-point argument one obtains a unique exponentially bounded evolution family $U_L(\cdot,\cdot)$ on E such that

$$(U_L(t,s)\phi)(\xi) = (U(t,s)\phi)(\xi) + \int_s^{(t+\xi)\vee s} V(t+\xi,\tau)L(\tau)U_L(\tau,s)\phi d\tau, \tag{2.6}$$

for $\xi \in [-r,0]$, $\phi \in E$, and $t \geqslant s$, where $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$ for $a,b \in \mathbb{R}$. By uniqueness, $U_L(\cdot,\cdot)$ is p-periodic in the case (A2) and is given by a semigroup $U_L(\cdot)$ in the case (A3). Moreover, $U_L(\cdot)$ is generated by the operator A_L defined in (1.2); cf. Lemma VI.6.2 and VI.6.5 in [7]. Observe that Gronwall's inequality yields

$$\|U_L(t,s)\| \leqslant M \exp[(w + M\|L(\cdot)\|_\infty)(t-s)] \tag{2.7}$$

for $t \geqslant s$, where $w > \omega(V)$, $M := M_0 e^{|w|r}$, and $\|V(t,s)\| \leqslant M_0 e^{w(t-s)}$. Given $\phi \in E$ and $s \in \mathbb{R}$, we now set

$$u(t) := \begin{cases} [V(t,s)\phi](0), & t \geqslant s, \\ \phi(t-s), & s-r \leqslant t \leqslant s. \end{cases}$$

Then (2.6) shows that $u_t = U_L(t,s)\phi$ and that u solves (2.5). These facts are proved in [10, Theorem 3.2] or [27, Proposition 3.2]. Related results are contained in [6, 8, 21, 23, 24]. We note that in [6, 21, 24] the evolution family $U_L(\cdot,\cdot)$ is constructed by using general well-posedness results for the operators $A_L(t)$ given as in (1.2) and that in [6, 8, 24] nonlinear problems are treated. The proof given in [23] is based on the perturbation theory for Hille–Yosida operators whose basic result is stated below.

A linear operator A on Z is called a *Hille–Yosida operator* if there are constants $w \in \mathbb{R}$ and $K \geqslant 0$ such that

$$(w, \infty) \subseteq \rho(A) \qquad \text{and} \qquad \|(\lambda - w)^n R(\lambda, A)^n\| \leqslant K \text{ for } \lambda > w, \ n \in \mathbb{N}.$$

It is known that the part A_0 of A in $Z_0 := \overline{D(A)}$ generates a C_0-semigroup $S_0(\cdot)$ on Z_0 and that $\rho(A_0) = \rho(A)$; see e.g. [20, Theorem 3.1.10] and [7, Prop. IV.2.17]. The proof of the following perturbation result can be found in [20, Sect. 4.1] (see also the references therein for related approaches).

Proposition 2.2. *Let A be a Hille–Yosida operator on a Banach space Z and $B \in \mathscr{L}(Z_0, Z)$. Then $A + B$ with $D(A + B) = D(A)$ is a Hille–Yosida operator on Z. The C_0-semigroup $S_B(\cdot)$ generated by the part of $A + B$ in Z_0 is the unique solution of*

$$S_B(t)z = S_0(t)z + \lim_{\lambda \to \infty} \int_0^t S_0(t - \tau)\lambda R(\lambda, A)BS_B(\tau)z\,d\tau, \quad t \geqslant 0, \ z \in Z_0. \tag{2.8}$$

3. THE CHARACTERISTIC EQUATION

Assume that (A1), (A2), or (A3) holds. We want to study the asymptotic behavior of the evolution family $U_L(\cdot, \cdot)$ obtained in the previous section. Proposition 2.1 indicates that it will be useful to employ the evolution semigroups

$$(T_L(t)f)(s) = U_L(s, s - t)f(s - t), \qquad f \in F(\mathbb{R}, E),$$

and

$$(T_V(t)\varphi)(s) = V(s, s - t)\varphi(s - t), \qquad \varphi \in F(\mathbb{R}, X),$$

and their generators G_L on $F(\mathbb{R}, E)$ and G_V on $F(\mathbb{R}, X)$, respectively. In Proposition 3.4 below we express G_L in terms of G_V and $L(\cdot)$. This relation leads to a generalized characteristic equation determining $\sigma(G_L)$; see Theorem 3.5.

At first we deduce the desired representation of G_L in a heuristic way. Assume for a moment that the evolution family $V(\cdot, \cdot)$ is generated by operators $A(\cdot)$ on X. Then the evolution family $U_L(\cdot, \cdot)$ should be generated by $A_L(t)$ given as in (1.2). We recall that the evolution semigroup on $C_0(\mathbb{R}, Z)$ corresponding to a well-posed Cauchy problem $\dot{u}(t) = B(t)u(t), \ u(s) = x$, on Z is generated by the closure of $-\frac{d}{dt} + B(\cdot)$ defined on the intersection of the maximal domains of the derivative $-\frac{d}{dt}$ and the multiplication operator $B(\cdot)$ on $C_0(\mathbb{R}, Z)$; see [5, Theorem 3.12] or [27, Prop. 1.14]. In our case this means that

$$G_L f = -\frac{\partial}{\partial t} f + \frac{\partial}{\partial \xi} f \qquad \text{and} \qquad \left(\frac{\partial}{\partial \xi} f\right)(t,0) = A(t)f(t,0) + L(t)f(t) \quad (3.1)$$

for functions f contained in a core of G_L. In view of the first identity we define

$$\partial f := \left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial \xi}\right) f,$$

$$D(\partial) := \{f : \mathbb{R} \times [-r, 0] \to X\colon\ f \in F(\mathbb{R}, E) \text{ is differentiable in direction } \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \partial f \in F(\mathbb{R}, E)\}.$$

It is reasonable to expect that $D(G_L) \subset D(\partial)$. Unfortunately, the boundary condition in (3.1) does not make sense for all $f \in D(\partial)$. To circumvent this problem, we subtract $\frac{\partial}{\partial t} f(t,0)$ on both sides of the boundary condition and use the expression for G_V indicated above. Then the second identity in (3.1) becomes

$$\delta_0 \partial f = G_V \delta_0 f + L(\cdot) f, \qquad \text{where } \delta_0 f := f(\cdot, 0).$$

This boundary condition makes sense for

$$f \in D := \{f \in D(\partial)\colon\ \delta_0 f \in D(G_V)\}$$

even if we only assume that (A1) holds. In fact, it turns out to be the correct one, but instead of the above heuristic arguments we have to develop a completely different method partly inspired by [23].

As a preliminary step we compute the generator G_0 of the evolution semigroup $T_0(\cdot)$ on $F(\mathbb{R}, E)$ associated with the evolution family $U(\cdot,\cdot)$ given in (2.4). Observe that $\omega(T_0) = \omega(U) = \omega(V)$.

LEMMA 3.1. *The generator G_0 on $F(\mathbb{R}, E)$ is given by $G_0 f = \partial f$ with $D(G_0) = \{f \in D\colon\ \delta_0 \partial f = G_V \delta_0 f\} =: D_0$.*

Proof. Let $\lambda > \omega(V)$ and $f = R(\lambda, G_0) g$ for $g \in F(\mathbb{R}, E)$. Then

$$\begin{aligned} f(t,\xi) &= \int_{-\infty}^{t} e^{-\lambda(t-\tau)} (U(t,\tau) g(\tau,\cdot))(\xi) d\tau \\ &= \int_{-\infty}^{t+\xi} e^{-\lambda(t-\tau)}\, V(t+\xi,\tau) g(\tau,0) d\tau + \int_{t+\xi}^{t} e^{-\lambda(t-\tau)} g(\tau, t+\xi-\tau) d\tau \end{aligned} \quad (3.2)$$

for $t \in \mathbb{R}$ and $\xi \in [-r, 0]$ due to (2.3) and (2.4). This equality implies $f \in D(\partial)$ and $\partial f = \lambda f - g = G_0 f$. Moreover, (2.3) and (3.2) yield $\delta_0 f = R(\lambda, G_V)\delta_0 g$ so that $f \in D$ and

$$\delta_0 \partial f = \lambda \delta_0 f - \delta_0 g = \lambda R(\lambda, G_V)\delta_0 g - \delta_0 g = G_V R(\lambda, G_V)\delta_0 g = G_V \delta_0 f.$$

Thus, $D(G_0) \subseteq D_0$ and $G_0 = \partial$ on $D(G_0)$. It remains to show that $\lambda - \partial$ is injective on D_0. If $\lambda f = \partial f$ for $f \in D_0$, then $\lambda \delta_0 f = \delta_0 \partial f = G_V \delta_0 f$, and hence $\delta_0 f = 0$. Now, considering the function $h(s) := f(t - s, \xi + s)$ for $s \in [0, -\xi]$, $t \in \mathbb{R}$, and $\xi \in [-r, 0)$, one easily deduces $f = 0$. ∎

On $\mathscr{F} := F(\mathbb{R}, X \times E) = F(\mathbb{R}, X) \times F(\mathbb{R}, E)$, we define the operator

$$\mathscr{G} := \begin{pmatrix} 0 & G_V \delta_0 - \delta_0 \partial \\ 0 & \partial \end{pmatrix} \qquad \text{with } D(\mathscr{G}) := \{0\} \times D. \tag{3.3}$$

Lemma 3.1 shows that $\mathscr{F}_0 = \overline{D(\mathscr{G})} = \{0\} \times F(\mathbb{R}, E) \cong F(\mathbb{R}, E)$ and that the part $\mathscr{G}_0$ of $\mathscr{G}$ in $F(\mathbb{R}, E)$ coincides with G_0. We need some preparations in order to determine the resolvent of $\mathscr{G}$. Note that $R(\lambda, G_V)$ leaves $F(\mathbb{R}, X)$ invariant. Let $\varphi \in F(\mathbb{R}, X)$ and $\lambda > \omega(V)$. In the cases (A2) and (A3) the identity (2.1) shows that

$$[R(\lambda, G_V)\varphi](t) = \sum_{k=0}^{\infty} e^{i\eta_k t} x_k \qquad \text{for appropriate } \eta_k \in \mathbb{R},\ x_k \in X.$$

This series converges in X uniformly for $t \in \mathbb{R}$. Set $e_\mu(\xi) := e^{\mu \xi}$ for $\mu \in \mathbb{C}$ and $\xi \in [-r, 0]$. Then

$$e_\lambda [R(\lambda, G_V)\varphi]_t = \sum_{k=0}^{\infty} e^{i\eta_k t} e_{\lambda + i\eta_k} x_k,$$

and the series converges in E uniformly for $t \in \mathbb{R}$. Thus we can define in all the cases (A1), (A2), and (A3) the bounded operator

$$E_\lambda : F(\mathbb{R}, X) \to F(\mathbb{R}, E), \qquad [E_\lambda \varphi](t, \xi) := e^{\lambda \xi}(R(\lambda, G_V)\varphi)(t + \xi), \tag{3.4}$$

for $t \in \mathbb{R}$ and $\xi \in [-r, 0]$. Observe that

$$E_\lambda \varphi \in D,\ \partial E_\lambda \varphi = \lambda E_\lambda \varphi \text{ and } \delta_0 E_\lambda \varphi = R(\lambda, G_V)\varphi. \tag{3.5}$$

LEMMA 3.2. *The operator* $\mathcal{G}$ *defined in* (3.3) *is a Hille–Yosida operator on* $\mathcal{F} = F(\mathbb{R}, X \times E)$ *with resolvent*

$$R(\lambda, \mathcal{G}) = \begin{pmatrix} 0 & 0 \\ E_\lambda & R(\lambda, G_0) \end{pmatrix} =: \mathcal{R}_\lambda \quad \textit{for } \lambda > \omega(V).$$

Proof. Let $\lambda > \omega(V)$. Lemma 3.1 and (3.5) imply $\mathcal{R}_\lambda \mathcal{F} \subseteq D(\mathcal{G})$ and $(\lambda - \mathcal{G})\ \mathcal{R}_\lambda = I$. If $(\lambda - \mathcal{G})\binom{0}{f} = 0$, then $f \in D(G_0)$ and $(\lambda - \partial)f = (\lambda - G_0)f = 0$. Hence, $f = 0$. This yields $\lambda \in \rho(\mathcal{G})$ and $R(\lambda, \mathcal{G}) = \mathcal{R}_\lambda$. Since

$$R(\lambda, \mathcal{G})^n = \begin{pmatrix} 0 & 0 \\ R(\lambda, G_0)^{n-1} E_\lambda & R(\lambda, G_0)^n \end{pmatrix} \quad \text{for } n \in \mathbb{N}.$$

$\mathcal{G}$ is a Hille–Yosida operator. ∎

We now perturb $\mathcal{G}$ by the bounded multiplication operator

$$\mathcal{L} : F(\mathbb{R}, E) \to \mathcal{F}, \quad \mathcal{L}f := \begin{pmatrix} L(\cdot)f \\ 0 \end{pmatrix}.$$

In the cases (A2) and (A3), the representation (2.1) of almost-periodic functions and the following lemma actually imply that $L(\cdot)$ maps $F(\mathbb{R}, E)$ into $F(\mathbb{R}, X)$. The lemma is a slight extension of [4, Lemma 3.6] and can be proved in the same way.

LEMMA 3.3. *Let* $q > 0$ *and let* Λ_q *be a closed subset of* $\mathbb{R}$ *such that* $\Lambda_q = \Lambda_q + \frac{2\pi}{q}\mathbb{Z}$. *For two Banach spaces* Z_1 *and* Z_2, *let* $h \in AP_{\Lambda_q}(\mathbb{R}, Z_1)$ *and let* $S : \mathbb{R} \to \mathcal{L}(Z_1, Z_2)$ *be strongly continuous and q-periodic. Then the spectrum of* $S(\cdot)h(\cdot)$ *is contained in* Λ_q.

By Proposition 2.2 the sum $\mathcal{G}_L := \mathcal{G} + \mathcal{L}$ is again a Hille–Yosida operator and its part $\tilde{G}_L$ in $\mathcal{F}_0 = F(\mathbb{R}, E)$ generates a C_0-semigroup $\tilde{T}_L(\cdot)$. In the next result we see that $\tilde{G}_L = G_L$.

PROPOSITION 3.4. *Assume that* (A1), (A2), *or* (A3) *holds. Then the generator* G_L *of the evolution semigroup* $T_L(\cdot)$ *on* $F(\mathbb{R}, E)$ *is given by*

$$G_L f = \partial f \quad \textit{for } f \in D(G_L) = \{f \in D\colon G_V \delta_0 f - \delta_0 \partial f + L(\cdot)f = 0\}.$$

Proof. We have to show that $G_L = \tilde{G}_L$ or that the corresponding semigroups $T_L(\cdot)$ and $\tilde{T}_L(\cdot)$ coincide. Combining (2.8) with Lemma 3.2, we see that $\tilde{T}_L(\cdot)$ is uniquely given by

$$\tilde{T}_L(t)f = T_0(t)f + \lim_{\lambda\to\infty} \int_0^t T_0(t-\tau)\lambda E_\lambda L(\cdot)\tilde{T}_L(\tau)f d\tau, \tag{3.6}$$

for $f \in F(\mathbb{R}, E)$ and $t \geqslant 0$. On the other hand, (2.6) and the definition of $T_L(t)$ yield

$$(U_L(t,s)\phi)(\xi) = (U(t,s)\phi)(\xi) + \int_s^{(t+\xi)\vee s} V(t+\xi,\tau)[L(\cdot)T_L(\tau-s)f](\tau)d\tau,$$

for $\phi = f(s) \in E$, $f \in F(\mathbb{R}, E)$, $t \geqslant s$, and $\xi \in [-r, 0]$. Since $\lambda R(\lambda, G_V)$ converges strongly to I in $F(\mathbb{R}, X)$ as $\lambda \to \infty$, we can write

$$\begin{aligned}(U_L(t,s)\phi)(\xi) = {} & (U(t,s)\phi)(\xi) \\ & + \lim_{\lambda\to\infty} \int_s^{(t+\xi)\vee s} V(t+\xi,\tau)[\lambda R(\lambda, G_V)L(\cdot)T_L(\tau-s)f](\tau)d\tau,\end{aligned}$$

where the limit is uniform for $\xi \in [-r, 0]$ and for t, s with $0 \leqslant t - s \leqslant t_0$. Note that

$$\begin{aligned}&\left\|\int_{(t+\xi)\vee s}^t \lambda e^{\lambda(t+\xi-\tau)}[R(\lambda, G_V)L(\cdot)T_L(\tau-s)f](t+\xi)d\tau\right\| \\ &\leqslant \frac{c}{\lambda - w} e^{\tilde{w}(t-s)}\|f\|_\infty \int_{t+\xi}^t \lambda e^{\lambda(t+\xi-\tau)}d\tau \leqslant \frac{c}{\lambda-w} e^{\tilde{w}(t-s)}\|f\|_\infty,\end{aligned}$$

for $\lambda > w > w(V)$ with $\lambda \geqslant 0$ and for suitable constants $c, \tilde{w} \geqslant 0$. As a result,

$$\begin{aligned}(U_L(t,s)\phi)(\xi) = {} & (U(t,s)\phi)(\xi) \\ & + \lim_{\lambda\to\infty} \int_s^{(t+\xi)\vee s} V(t+\xi,\tau)[\lambda R(\lambda, G_V)L(\cdot)T_L(\tau-s)f](\tau)d\tau \\ & + \lim_{\lambda\to\infty} \int_{(t+\xi)\vee s}^t e^{\lambda(t+\xi-\tau)}[\lambda R(\lambda, G_V)L(\cdot)T_L(\tau-s)f](t+\xi)d\tau\end{aligned}$$

for $t \geqslant s$ and $\xi \in [-r, 0]$. Using (2.4) and (3.4), this leads to

$$U_L(t,s)\phi = U(t,s)\phi + \lim_{\lambda\to\infty} \int_s^t U(t,\tau)[\lambda E_\lambda L(\cdot)T_L(\tau-s)f](\tau)d\tau,$$

or equivalently

$$U_L(s, s-t)f(s-t) = U(s, s-t)f(s-t)$$
$$+ \lim_{\lambda\to\infty} \int_{s-t}^{s} U(s,\tau)[\lambda E_\lambda L(\cdot)T_L(t+\tau-s)f](\tau)d\tau$$

for $s \in \mathbb{R}$ and $t \geqslant 0$. Here both limits exist in E and the second one is uniform for $s \in \mathbb{R}$. Using the definition of the evolution semigroups, we obtain

$$T_L(t)f = T_0(t)f + \lim_{\lambda\to\infty} \int_0^t T_0(t-\tau)\lambda E_\lambda L(\cdot)T_L(\tau)fd\tau \tag{3.7}$$

so that $T_L(t) = \tilde{T}_L(t)$ by (3.6). ∎

We can now compute the resolvent of G_L and thereby establish the desired characteristic equation. We set

$$H_\lambda : F(\mathbb{R}, E) \to F(\mathbb{R}, E), \qquad [H_\lambda f](t,\xi) := \int_\xi^0 e^{\lambda(\xi-\tau)} f(\xi - \tau + t, \tau)d\tau,$$
$$\hat{L}_\lambda : F(\mathbb{R}, X) \to F(\mathbb{R}, X), \qquad [\hat{L}_\lambda \varphi](t) := L(t)(e_\lambda \varphi_t), \tag{3.8}$$

where $\lambda \in \mathbb{C}$ and $[e_\lambda\varphi_t](\xi) = e^{\lambda\xi}\varphi(t+\xi)$ for $\xi \in [-r, 0]$ and $t \geqslant 0$. (Use (2.1) and Lemma 3.3 to check that H_λ and $\hat{L}_\lambda$ are well-defined in the cases (A2) and (A3).)

THEOREM 3.5. *Assume that* (A1), (A2), *or* (A3) *holds. Let G_V be the generator of the evolution semigroup on $F(\mathbb{R}, X)$ associated with $V(\cdot,\cdot)$ and let G_L be the generator of the evolution semigroup on $F(\mathbb{R}, E)$ induced by the evolution family $U_L(\cdot,\cdot)$ on E given by* (2.6). *Then*

$$\lambda \in \sigma(G_L) \iff \lambda \in \sigma(G_V + \hat{L}_\lambda), \tag{3.9}$$

for $\lambda \in \mathbb{C}$. Moreover, for $\lambda \in \rho(G_L)$ the resolvent of G_L is given by

$$[R(\lambda, G_L)g](t) = e_\lambda[R(\lambda, G_V + \hat{L}_\lambda)(\delta_0 g + L(\cdot)H_\lambda g)]_t + [H_\lambda g](t), \tag{3.10}$$

for $g \in F(\mathbb{R}, E)$.

Proof. In view of Proposition 3.4, $\lambda - G_L$ is invertible if and only if for every $g \in F(\mathbb{R}, E)$ there exists a unique

$$f \in D \qquad \text{with } \delta_0\partial f = G_V\delta_0 f + L(\cdot)f \text{ and } \lambda f - \partial f = g. \tag{3.11}$$

Considering $h(s) = f(t-s, \xi+s)$ for $s \in [0, -\xi]$, $t \in \mathbb{R}$, and $\xi \in [-r, 0)$, we see that f satisfies (3.11) if and only if there exists a unique $\varphi \in D(G_V)$ such that

$$f(t,\xi) = e^{\lambda\xi}\varphi(t+\xi) + \int_\xi^0 e^{\lambda(\xi-\tau)} g(\xi - \tau + t, \tau) d\tau$$

and

$$\lambda\varphi - G_V\varphi - \hat{L}_\lambda\varphi = g(\cdot, 0) + L(\cdot)H_\lambda g =: S_\lambda g. \tag{3.12}$$

In other words, $\lambda \in \rho(G_L)$ is equivalent to the condition

$$\forall g \in F(\mathbb{R}, E) \ \exists! \varphi \in D(G_V) \text{ such that } (\lambda - G_V - \hat{L}_\lambda)\varphi = S_\lambda g.$$

Thus (3.9) follows from the surjectivity of $S_\lambda : F(\mathbb{R}, E) \to F(\mathbb{R}, X)$. To verify that S_λ is onto, take functions of the form $[e_\mu\psi](t,\xi) = e^{\mu\xi}\psi(t)$ for $t \in \mathbb{R}$ and $\xi \in [-r, 0]$, where $\psi \in F(\mathbb{R}, X)$ and $\mu > \lambda$. Since $S_\lambda(e_\mu\psi) = \psi + L(\cdot)H_\lambda(e_\mu\psi)$ and

$$\|L(\cdot)H_\lambda(e_\mu\psi)\|_\infty \leqslant \frac{1 \vee e^{-\lambda r}}{\mu - \lambda}\|L(\cdot)\|_\infty \|\psi\|_\infty,$$

the operator $\psi \mapsto S_\lambda(e_\mu\psi)$ is invertible on $F(\mathbb{R}, X)$ for sufficiently large μ. Finally, (3.10) is an immediate consequence of (3.12). ∎

Theorem 3.5 and Proposition 2.1(a) imply the following characterization of exponential stability and dichotomy, where we take $F = C_0$.

COROLLARY 3.6. *Assume that* (A1) *holds. Then* $U_L(\cdot,\cdot)$ *is exponentially stable if and only if* $\sup\{\operatorname{Re}\lambda \colon \lambda \in \sigma(G_V + \hat{L}_\lambda)\} < 0$, *and* $U_L(\cdot,\cdot)$ *has exponential dichotomy if and only if* $0 \in \rho(G_V + \hat{L}_0)$.

We point out that the above result allows us to study the exponential dichotomy of the retarded non-autonomous problem (2.5) on X by means of a spectral condition on $C_0(\mathbb{R}, X)$. As mentioned in the Introduction, a spectral condition on the space X itself is possible only in certain situations. We now show that (3.9) implies the known characteristic equations on X. We first study the following special case of (A2).

(A2′) $V(\cdot,\cdot)$ is a p-periodic evolution family on X and $L(t)\phi = B(t)\phi(-p)$, $\phi \in E$, for $B(\cdot + p) = B(\cdot) \in C(\mathbb{R}, \mathscr{L}_s(X))$. Let $r = p$.

Since in this case the evolution family $U_L(\cdot,\cdot)$ on E solving (2.5) is p-periodic, its exponential behavior is determined by the spectrum of the monodromy operator $U_L(p,0)$. To exploit this fact, we consider the p-periodic evolution family $V_L^\lambda(\cdot,\cdot)$ on X solving

$$V_L^\lambda(t,s)x = V(t,s)x + \int_s^t V(t,\sigma)e^{-\lambda p}B(\sigma)V_L^\lambda(\sigma,s)x d\sigma, \tag{3.13}$$

for $\lambda \in \mathbb{C}$, $x \in X$, and $t \geqslant s$. (Observe that (3.13) is of the form (2.6) with $r = 0$.) The next corollary (partially) extends [13, Theorem 8.3.1], where finitely many delay terms $B_k(t)u(t-kp)$, $k = 0, 1, \ldots, m$, are considered for $X = \mathbb{C}^n$. Moreover, in [11, Theorem 5.9] the case $A(t) \equiv A$ is treated for a general Banach space X.

COROLLARY 3.7. *Assume that the condition* (A2′) *holds, and let* $\lambda \in \mathbb{C}$. *Then*

$$e^{\lambda p} \in \sigma(U_L(p,0)) \quad \textit{if and only if} \quad e^{\lambda p} \in \sigma(V_L^\lambda(p,0)). \tag{3.14}$$

In particular, $\omega(U_L) = \sup\{\operatorname{Re}\lambda\colon e^{\lambda p} \in \sigma(V_L^\lambda(p,0))\}$ *and* $U_L(\cdot,\cdot)$ *has an exponential dichotomy if and only if* $\{\lambda \in \mathbb{C}\colon e^{\lambda p} \in \sigma(V_L^\lambda(p,0))\} \cap i\mathbb{R} = \emptyset$.

Proof. Let $\Lambda_p = \frac{2\pi}{p}\mathbb{Z}$ and consider G_L on $F(\mathbb{R},E) = P_p(\mathbb{R},E)$. Using a rescaling argument, we derive from Proposition 2.1(b) that $e^{\lambda p} \in \sigma(U_L(p,0))$ if and only if $\lambda \in \sigma(G_L)$. Moreover, $\lambda \in \sigma(G_L)$ if and only if $\lambda \in \sigma(G_V + \hat{L}_\lambda)$ by Theorem 3.5. Observe that $G_V + \hat{L}_\lambda$ generates a strongly continuous semigroup $T_L^\lambda(\cdot)$ on $P_p(\mathbb{R},X)$ which is uniquely given by

$$T_L^\lambda(t)\psi = T_V(t)\psi + \int_s^t T_V(t-\sigma)\hat{L}_\lambda T_L^\lambda(\sigma)\psi d\sigma \tag{3.15}$$

for $t \geqslant s$ and $\psi \in P_p(\mathbb{R},X)$. But from (3.13) we deduce that the evolution semigroup on $P_p(\mathbb{R},X)$ corresponding to $V_L^\lambda(\cdot,\cdot)$ also satisfies (3.15). Hence $T_L^\lambda(\cdot)$ coincides with this evolution semigroup. Proposition 2.1(b) now implies that $\lambda \in \sigma(G_V + \hat{L}_\lambda)$ if and only if $e^{\lambda p} \in \sigma(V_L^\lambda(p,0))$, so that (3.14) is established. The final assertions follow from [15, Sect. 7.2]. ∎

If we restrict ourself to the autonomous situation in (A3), we recover [7, Prop. VI.6.7]. Related results can be found in [28, p. 82] if $V(t)$ is compact for each $t > 0$ and in [13, Lemma 7.2.1] if $X = \mathbb{C}^n$. Recall that $L_\lambda \in \mathscr{L}(X)$ is given by $L_\lambda x = Le_\lambda x$.

COROLLARY 3.8. *Assume that the condition* (A3) *holds. Let* A *generate* $V(\cdot)$ *and define the operator* A_L *on* E *as in* (1.2). *Then* $\lambda \in \sigma(A_L)$ *if and only if* $\lambda \in \sigma(A + L_\lambda)$ *for* $\lambda \in \mathbb{C}$.

Proof. Let $\Lambda_\infty = \{0\}$. Then $AP_{\Lambda_\infty}(\mathbb{R}, X)$ is simply the space of constant X-valued functions and hence can be identified with X. Therefore $\hat{L}_\lambda = L_\lambda$, $G_V = A$, and $G_L = A_L$. Theorem 3.5 now implies the assertion. ∎

Using the standard spectral theory of semigroups, one can deduce asymptotic properties of $U_L(\cdot)$ from the above corollary under additional hypotheses on A or X; see [7, Sect. VI.6].

4. APPLICATIONS

Theorem 3.5 allows to study robustness of exponential dichotomy under delay perturbations. Recall from Proposition 2.1(a) that G_V is invertible on $C_0(\mathbb{R}, X)$ if $V(\cdot,\cdot)$ has exponential dichotomy on X.

THEOREM 4.1. *Assume that the condition* (A1) *holds and let* $U_L(\cdot,\cdot)$ *be given by* (2.6). *If* $V(\cdot,\cdot)$ *has an exponential dichotomy on* X, *then the following assertions are equivalent*:

(a) $U_L(\cdot,\cdot)$ *has an exponential dichotomy on* E,
(b) $1 \in \rho(\hat{L}_0 R(0, G_V))$ *on* $C_0(\mathbb{R}, X)$,
(c) $1 \in \rho(R(0, G_V)\hat{L}_0)$ *on* $C_0(\mathbb{R}, X)$.

Proof. Assertion (a) is equivalent to $0 \in \rho(G_V + \hat{L}_0)$ by Theorem 3.5. Since $G_V + \hat{L}_0 = (I - \hat{L}_0 R(0, G_V))G_V$, the operator $G_V + \hat{L}_0$ is invertible if and only if (b) holds. The second equivalence follows from the general fact that $1 - TS$ has the inverse $1 + T(1 - ST)^{-1}S$ if $1 \in \rho(ST)$ for two bounded operators T and S. ∎

Of course, condition (b) or (c) is satisfied if

$$r(\hat{L}_0 R(0, G_V)) < 1 \quad \text{or} \quad r(R(\lambda, G_V)\hat{L}_0) < 1, \tag{4.1}$$

respectively, or if $\|\hat{L}_0\|$ is small; see Corollary 4.4. In these cases we can also show that the dimension of the unstable subspace of $V(\cdot,\cdot)$ is inherited by $U_L(\cdot,\cdot)$. We first need the following result whose straightforward proof is omitted; see [10, Lemma 3.11].

LEMMA 4.2. *Assume that* (A1) *holds and that* $V(\cdot,\cdot)$ *has an exponential dichotomy on* X *with projections* $Q_V(t)$ *on the unstable subspaces. Then* $U(\cdot,\cdot)$ *defined in* (2.4) *has an exponential dichotomy on* E *with projections* $[Q_U(t)\phi](\xi) = V_Q(t+\xi, t)Q_V(t)\phi(0)$ *for* $\phi \in E$, $t \in \mathbb{R}$, *and* $\xi \in [-r, 0]$. *In particular*, $\dim Q_U(t)E = \dim Q_V(t)X$. *Moreover*, $U_Q(s,t)Q_U(t)\phi = V(s+\cdot, t)Q_V(t)\phi(0)$.

PROPOSITION 4.3. *Assume that* (A1) *holds and that* $V(\cdot,\cdot)$ *has an exponential dichotomy on* X *with projections* $Q_V(t)$ *on the unstable subspaces. If one of the conditions in* (4.1) *holds, then* $U_L(\cdot,\cdot)$ *has an exponential dichotomy on* E *with projections* $Q_L(t)$ *on the unstable subspaces and* dim $Q_L(t)E = \dim Q_V(s)X$ *for* $t, s \in \mathbb{R}$.

Proof. Due to Theorem 4.1, Property (b) in the definition of exponential dichotomy, and Lemma 4.2, it remains to show that $\dim Q_L(0)E = \dim Q_U(0)E$. Let $\varepsilon \in [0,1]$. Clearly, (4.1) still holds if we replace $L(t)$ with $\varepsilon L(t)$, $t \in \mathbb{R}$. So due to Theorem 4.1 the evolution family $U_\varepsilon(\cdot,\cdot)$ given by (2.6) for $\varepsilon L(t)$ has exponential dichotomy with projections $Q_\varepsilon(t)$ on the unstable subspaces, where $Q_0(t) = Q_U(t)$ and $Q_1(t) = Q_L(t)$. In view of [9, Lemma II.4.3], it suffices to prove the continuity of $[0,1] \ni \varepsilon \mapsto Q_\varepsilon(0) \in \mathscr{L}(E)$.

Let $T_\varepsilon(\cdot)$ be the evolution semigroup on $C_0(\mathbb{R}, E)$ corresponding to $U_\varepsilon(\cdot,\cdot)$. Note that $T_\varepsilon(t)$ is uniformly bounded for $t, \varepsilon \in [0,1]$ by (2.7). The identity (3.7) yields

$$T_\varepsilon(t)f - T_\eta(t)f = (\varepsilon - \eta)\lim_{\lambda\to\infty} \int_0^t T_0(t-\tau)\lambda E_\lambda L(\cdot)T_\varepsilon(\tau)f\,d\tau$$

$$+\ \eta \lim_{\lambda\to\infty} \int_0^t T_0(t-\tau)\lambda E_\lambda L(\cdot)(T_\varepsilon(\tau) - T_\eta(\tau))f\,d\tau$$

for $f \in C_0(\mathbb{R}, E)$ and $t, \varepsilon, \eta \in [0,1]$. Since $\|\lambda E_\lambda\| \leqslant c$ for a constant $c > 0$ independent of $\lambda \geqslant \omega(V) + 1$, Gronwall's Lemma shows that $\varepsilon \mapsto T_\varepsilon(1)$ is continuous in the operator norm. The assertion now follows from the formula

$$Q_\varepsilon(\cdot) = I - \frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda, T_\varepsilon(1))\,d\lambda,$$

see e.g. [5, Theorem 6.41] and (the proof of) [7, Theorem VI.9.18]. ∎

Proposition 4.3 is applicable to the following situation, cf. [6, Theorem 4; 8, Theorem 2; 18, Lemma 2.1]. It is not difficult to establish refined versions of the next corollary using the same arguments.

COROLLARY 4.4. *Assume that* (A1) *holds. Let* $V(\cdot,\cdot)$ *have an exponential dichotomy on* X *with projections* $P_V(t)$ *such that*

$$\|V(t,s)P_V(s)\| \leqslant Me^{-\int_s^t \alpha(\tau)d\tau} \quad \textit{and} \quad \|V_Q(s,t)Q_V(t)\| \leqslant Me^{-\int_s^t \alpha(\tau)d\tau}$$

for $t \geqslant s$ *and some* $M \geqslant 1$ *and* $\alpha \in C(\mathbb{R})$ *with* $\alpha(t) \geqslant \delta > 0$. *If* $\|L(t)\| \leqslant \beta(t),\ t \in \mathbb{R}$, *and*

$$a := \sup_{t \in \mathbb{R}} \frac{2M\beta(t)}{\alpha(t)} < 1,$$

then $U_L(\cdot,\cdot)$ *has exponential dichotomy with projections* $Q_L(t)$ *on the unstable subspaces satisfying* $\dim Q_L(t)E = \dim Q_V(s)X$ *for* $t, s \in \mathbb{R}$.

Proof. By Proposition 4.3 it suffices to show that $\|R(0, G_V)\hat{L}_0\| < 1$ on $C_0(\mathbb{R}, X)$. The inverse of G_V is given by

$$[R(0, G_V)\varphi](t) = \int_{-\infty}^{t} V(t,\sigma)P_V(\sigma)\varphi(\sigma)d\sigma - \int_{t}^{\infty} V_Q(t,\sigma)Q_V(\sigma)\varphi(\sigma)d\sigma,$$

for $t \in \mathbb{R}$ and $\varphi \in C_0(\mathbb{R}, X)$; see e.g. [5, Remark 4.26] or [7, Theorem VI.9.18]. Hence,

$$\begin{aligned} &\|[R(0, G_V)\hat{L}_0\varphi](t)\| \\ &\leqslant M \int_{-\infty}^{t} e^{-\int_\sigma^t \alpha(\tau)d\tau}\beta(\sigma)\|\varphi\|_\infty d\sigma + M\int_t^\infty e^{-\int_t^\sigma \alpha(\tau)d\tau}\beta(\sigma)\|\varphi\|_\infty d\sigma \\ &= M \int_{-\infty}^{t} \alpha(\sigma)e^{-\int_\sigma^t \alpha(\tau)d\tau}\frac{\beta(\sigma)}{\alpha(\sigma)}\|\varphi\|_\infty d\sigma + M\int_t^\infty \alpha(\sigma)e^{-\int_t^\sigma \alpha(\tau)d\tau}\frac{\beta(\sigma)}{\alpha(\sigma)}\|\varphi\|_\infty d\sigma \\ &\leqslant a\|\varphi\|_\infty. \quad \blacksquare \end{aligned}$$

We now turn our attention to the inhomogeneous problem (1.6) on $\mathbb{R}$. Again we study an integrated version of it: Given $g \in C_b(\mathbb{R}, X)$, we are looking for a continuous function $u : \mathbb{R} \to X$ such that

$$u(t) = V(t,s)u(s) + \int_s^t V(t,\tau)(L(\tau)u_\tau + g(\tau))d\tau, \quad t \geqslant s. \tag{4.2}$$

Here we do not fix an initial function $\phi \in E$ or an initial time $s \in \mathbb{R}$, and we refer the reader to [11] and also to [13, Sect. 6.2; 28, Sect. 4.2] for further information and references. We want to show that there is a unique $u \in F(\mathbb{R}, X)$ solving (4.2) if $g \in F(\mathbb{R}, X)$; i.e., the solution inherits properties like almost periodicity. It turns out that (3.9) allows us to characterize this assertion by the exponential dichotomy of $U_L(\cdot,\cdot)$. For undelayed problems such results have a long history going back to a paper by O. Perron from 1930; see Sections 4.3 and 7.3 and the corresponding notes in [5]. Concerning almost periodic solutions to inhomogeneous problems we refer the reader to [1, 2, 4, 11, 13, 19, 22, 25, 28], and the references therein. In particular, in [1, Theorem 8] and [28, Theorem 4.3.3] (for the autonomous case) and [11, Sect. 5] (if $A(t) \equiv A$) it was shown that the exponential dichotomy of $U_L(\cdot,\cdot)$ implies the above-mentioned inheritance of almost periodicity in the case of slightly differing retarded problems.

THEOREM 4.5. *Assume that* (A1) *holds. Then* $U_L(\cdot,\cdot)$ *has an exponential dichotomy on* E *if and only if for every* $g \in C_0(\mathbb{R}, X)$ *there exists a unique function* $u \in C_0(\mathbb{R}, X)$ *solving* (4.2).

Proof. From Corollary 3.6 we know that $U_L(\cdot,\cdot)$ has an exponential dichotomy if and only if $G_V + \hat{L}_0$ is invertible on $C_0(\mathbb{R}, X)$. This is the case if and only if for every $g \in C_0(\mathbb{R}, X)$ there exists a unique $u \in D(G_V)$ satisfying $(G_V + \hat{L}_0)u = -g$. By (2.2) this is equivalent to

$$u = T_V(\tau)u + \int_0^\tau T_V(\sigma)(\hat{L}_0 u + g)d\sigma, \qquad \tau \geqslant 0.$$

We now obtain the assertion by evaluating this formula at $t \in \mathbb{R}$ for $\tau = t - s \geqslant 0$. ∎

In the same way the following two theorems can be derived from Proposition 2.1 and Theorem 3.5 by varying the function spaces $F(\mathbb{R}, X)$ and $F(\mathbb{R}, E)$.

THEOREM 4.6. *Assume that* (A2) *holds. Then*:

(*a*) $U_L(\cdot,\cdot)$ *has an exponential dichotomy if and only if for every* $g \in AP(\mathbb{R}, X)$ *there exists a unique* $u \in AP(\mathbb{R}, X)$ *satisfying* (4.2).

(*b*) $1 \in \rho(U_L(p, 0))$ *if and only if for every* $g \in P_p(\mathbb{R}, X)$ *there exists a unique* $u \in P_p(\mathbb{R}, X)$ *satisfying* (4.2).

(*c*) *If* $\sigma(U_L(p,0)) \cap \overline{\{e^{i\eta p} \colon \eta \in \Lambda_p\}} = \emptyset$, *then for every* $g \in AP_{\Lambda_p}(\mathbb{R}, X)$ *there exists a unique* $u \in AP_{\Lambda_p}(\mathbb{R}, X)$ *satisfying* (4.2), *where* $\emptyset \neq \Lambda_p = \Lambda_p + \frac{2\pi}{p}\mathbb{Z} \subseteq \mathbb{R}$ *is closed.*

THEOREM 4.7. *Assume that* (A3) *holds and let* $\emptyset \neq \Lambda_\infty \subseteq \mathbb{R}$ *be closed. If* $\sigma(U_L(t_0)) \cap \overline{\{e^{i\eta t_0} \colon \eta \in \Lambda_\infty\}} = \emptyset$ *for some* $t_0 > 0$, *then for every* $g \in AP_{\Lambda_\infty}(\mathbb{R}, X)$ *there exists a unique* $u \in AP_{\Lambda_\infty}(\mathbb{R}, X)$ *satisfying* (4.2).

Observe that the spectral conditions in Theorems 4.6(c) and 4.7 allow for a non-empty intersection of the spectrum of the monodromy operator and the unit circle provided that a "non-resonance" condition holds.

Finally, we study the influence of positivity on stability properties of $U_L(\cdot,\cdot)$ in the case (A2′). To that purpose we assume that X is a Banach lattice. Thus $E = C([-r, 0], X)$ with the canonical order is a Banach lattice as well. The same holds for $F(\mathbb{R}, Z)$, where $F = C_0, P_p$ and $Z = X, E$. We first establish that $U_L(t,s)$ is positive.

PROPOSITION 4.8. *Assume that* (A1) *holds and that* $V(t,s) \geqslant 0$ *and* $L(t) \geqslant 0$ *for* $\infty > t \geqslant s > -\infty$. *Then the operators* $U_L(t,s)$ *given by* (2.6) *are positive.*

Proof. Clearly, the evolution semigroup $T_V(\cdot)$ on $C_0(\mathbb{R}, X)$ corresponding to $V(\cdot,\cdot)$ is positive. Thus $R(\lambda, G_V) \geqslant 0$ for $\lambda > \omega(V)$ by, e.g., [7, Theorem VI.1.8]. Since $\hat{L}_\lambda \geqslant 0$ and

$$R(\lambda, G_V + \hat{L}_\lambda) = R(\lambda, G_V) \sum_{n=0}^{\infty} [\hat{L}_\lambda R(\lambda, G_V)]^n$$

for large $\lambda > \omega(V)$, the operator $R(\lambda, G_V + \hat{L}_\lambda)$ is positive. Now (3.10) shows that $R(\lambda, G_L) \geqslant 0$ for these λ and so the evolution semigroup $T_L(\cdot)$ is positive by, e.g., [7, Theorem VI.1.8]. Hence, $U_L(t,s) \geqslant 0$. ∎

The following result can be proved exactly as were Lemma VI.6.12–Corollary VI.6.16 in [7].

LEMMA 4.9. *Assume that* (A1) *or* (A2) *holds. Let* G_V *and* G_L *be the generators of the evolution semigroups on* $F(\mathbb{R}, X)$ *and* $F(\mathbb{R}, E)$ *induced by* $V(\cdot,\cdot)$ *and* $U_L(\cdot,\cdot)$, *respectively, where* $F = C_0$ *in Case* (A1) *and* $F = P_p$ *in Case* (A2). *Suppose that* $V(t,s) \geqslant 0$ *and* $L(t) \geqslant 0$ *for* $t \geqslant s$. *Then* $s(G_L) < 0$ *if and only if* $s(G_V + \hat{L}_0) < 0$.

We can now show in the case (A2′) that $\omega(U_L) = \omega(V_L)$ for the evolution family $V_L(\cdot,\cdot)$ on X given by

$$V_L(t,s)x = V(t,s)x + \int_s^t V(t,\sigma)B(\sigma)V_L(\sigma,s)x\,d\sigma. \tag{4.3}$$

A corresponding result for autonomous problems is due to W. Kerscher and R. Nagel; see [7, Example VI.6.18].

THEOREM 4.10. *Assume that* (A2′) *holds and let* $V(t,s)$ *and* $B(t)$ *be positive for* $\infty > t \geqslant s > -\infty$. *Then the evolution family* $U_L(\cdot,\cdot)$ *given by* (2.6) *is exponentially stable on* E *if and only if the evolution family* $V_L(\cdot,\cdot)$ *given by* (4.3) *is exponentially stable on* X.

Proof. We have already seen in the proof of Corollary 3.7 that $G_V + \hat{L}_0$ generates the evolution semigroup on $P_p(\mathbb{R}, X)$ corresponding to $V_L(t,s)$.

Proposition 2.1(b) then yields $s(G_V + \hat{L}_0) = \omega(V_L)$ and $s(G_L) = \omega(U_L)$ so that the assertion follows from Lemma 4.9. ∎

In order to illustrate the consequences of Theorem 4.10, consider the non-autonomous retarded Cauchy problem

$$\begin{aligned} &\frac{d}{dt}u(t) = A(t)u(t) + B(t)u(t-p), \quad t \geqslant s, \\ &u_s = \phi \in C([-p,0],X), \end{aligned} \tag{4.4}$$

where the operators $A(t)$, $t \in \mathbb{R}$, generate a positive evolution family $V(\cdot,\cdot)$ on X, $p > 0$, and $0 \leqslant B(\cdot + p) = B(\cdot) \in C(\mathbb{R}, \mathscr{L}_s(X))$. Then Theorem 4.10 says that the solution of (4.4) is uniformly exponentially stable if and only if the solution of the undelayed Cauchy problem

$$\frac{d}{dt}v(t) = A(t)v(t) + B(t)v(t), \quad t \geqslant s, \ v(s) = x,$$

on X is uniformly exponentially stable. In other words, the delay does not influence the stability if positivity is present.

5. A PERIODIC PARTIAL DIFFERENTIAL EQUATION WITH DELAY

We investigate the retarded parabolic differential equation

$$\frac{\partial}{\partial t}u(t,x) = k(t)\frac{\partial^2}{\partial x^2}u(t,x) - a(t)u(t,x) - b(t)u(t-1,x) + g(t,x), \quad t,x \in \mathbb{R}, \tag{5.1}$$

on $X = L^q(\mathbb{R})$, $1 < q < \infty$, for 1-periodic coefficients $a, b, k \in C(\mathbb{R})$ with $k \geqslant \delta > 0$ and an inhomogeneity $g \in C_b(\mathbb{R}, X)$. Clearly, the operators

$$A(t)\varphi := k(t)\varphi'' - a(t)\varphi \quad \text{with } D(A(t)) := W^{2,q}(\mathbb{R}), \ t \in \mathbb{R},$$

generate the evolution family

$$V(t,s) = e^{-\int_s^t a(\tau)d\tau} S\left(\int_s^t k(\tau)d\tau\right), \qquad t \geqslant s,$$

on X, where $S(\cdot)$ is the analytic semigroup generated by the second derivative $A\phi = \phi''$ on X. We further define $E := C([-1,0],X)$ and

$$L(t): E \to X, \quad L(t)\phi := b(t)\phi(-1), \ t \in \mathbb{R}.$$

Note that $V(\cdot,\cdot)$ and $L(\cdot)$ satisfy (A2′). Mild solutions of (5.1) are defined as in (4.2). By the standard regularity theory of analytic semigroups one sees that a mild solution is continuously differentiable and solves (5.1) if $g(t) \in D((1-A)^\alpha)$ and $t \mapsto (1-A)^\alpha g(t) \in X$ is continuous for $t \in \mathbb{R}$ and some $\alpha \in (0,1)$.

We compute the spectrum of the monodromy operator $U_L(1,0)$, where we set $\bar{a} := \int_0^1 a(\tau)d\tau$, $\bar{b} := \int_0^1 b(\tau)d\tau$, and $\bar{k} := \int_0^1 k(\tau)d\tau$.

Proposition 5.1. *$e^\lambda \in \sigma(U_L(1,0))$ if and only if there exist $l \in \mathbb{Z}$ and $\rho \in \mathbb{R}_+$ such that*

$$\lambda + 2\pi i l = -e^{-\lambda}\bar{b} - \bar{a} - \rho\bar{k}. \tag{5.2}$$

Proof. The evolution family $(V_L^\lambda(t,s))_{t \geqslant s}$ defined in (3.13) is given by

$$V_L^\lambda(t,s) = e^{-\int_s^t e^{-\lambda}b(\tau)d\tau}V(t,s), \quad \text{for } t \geqslant s.$$

Corollary 3.7 shows that $e^\lambda \in \sigma(U_L(1,0))$ if and only if $e^\lambda \in \sigma(e^{-e^{-\lambda}\bar{b}}V(1,0))$. Since $S(\cdot)$ satisfies the spectral mapping theorem, we deduce from $\sigma(A) = \mathbb{R}_-$ that $\sigma(V(1,0)) = \{e^{-\bar{a}-\rho\bar{k}} \colon \rho \in \mathbb{R}_+\}$. This yields the assertion. ∎

Assume that $(\bar{a}+\rho\bar{k}, \bar{b})$ belongs to the interior of the shaded region in Fig. 1 for some $\rho \geqslant 0$, where the upper boundary is given by

$$\alpha = -\beta\cos z, \ \ \beta \sin z = z, \quad 0 < z < \pi.$$

Then Theorem A.5 in [13] shows that all $\lambda \in \mathbb{C}$ satisfying (5.2) have negative real part; cf. [13, p. 135]. This yields $\operatorname{Re}\lambda < 0$ for each λ satisfying (5.2) for some $\rho \in \mathbb{R}_+$ provided that $(\bar{a}, \bar{b})$ is contained in the interior of the shaded region in Fig. 1. Hence $U_L(\cdot,\cdot)$ is exponentially stable and, in particular, has an exponential dichotomy. As an example one may take $\bar{a} = 0$ and $\bar{b} = 1$. Note that in this situation $V(\cdot,\cdot)$ does not have an exponential dichotomy.

In view of the above discussion, our next theorem is an immediate consequence of Theorem 4.5 and Theorem 4.6(a).

Theorem 5.2. *Assume that $(\bar{a}, \bar{b})$ belongs to the interior of the shaded region shown in Fig.* 1. *Then the following holds.*

(a) *If $g \in C_0(\mathbb{R}, L^q(\mathbb{R}))$, then there exists exactly one mild solution $u \in C_0(\mathbb{R}, L^q(\mathbb{R}))$ of* (5.1).

(b) If *$g \in AP(\mathbb{R}, L^q(\mathbb{R}))$, then there exists exactly one mild solution $u \in AP(\mathbb{R}, L^q(\mathbb{R}))$ of* (5.1).

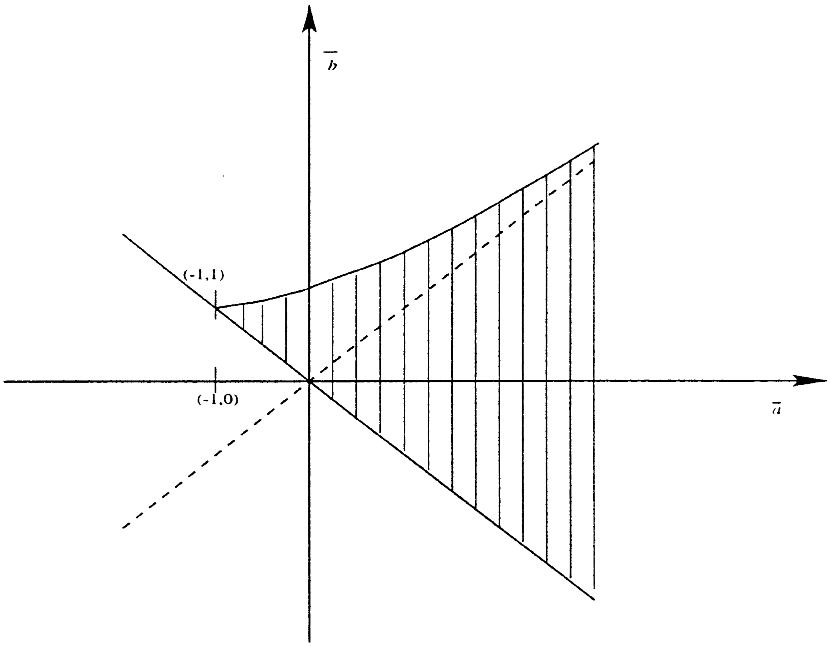

FIGURE 1

Similarly, Proposition 5.1 and Theorem 4.6(b) lead to a condition for the existence of a unique periodic solution.

THEOREM 5.3. *If* $g \in P_1(\mathbb{R}, L^q(\mathbb{R}))$ *and* $\bar{b} + \bar{a} > 0$, *then there exists exactly one mild solution* $u \in P_1(\mathbb{R}, L^q(\mathbb{R}))$ *of* (5.1).

REFERENCES

1. B. Amir and L. Maniar, Asymptotic behaviour of hyperbolic Cauchy problems for Hille–Yosida operators with an application to retarded differential equations, *Quaest. Math.* **23** (2000), 343–357.
2. W. Arendt and C.J.K. Batty, Almost periodic solutions of first and second order Cauchy problems, *J. Differential Equations* **137** (1997), 363–383.
3. A. Batkái, Hyperbolicity for linear partial differential equations with delay, preprint.
4. C.J.K. Batty, W. Hutter, and F. Räbiger, Almost periodicity of mild solutions of inhomogeneous Cauchy problems, *J. Differential Equations* **156** (1999), 309–327.
5. C. Chicone and Y. Latushkin, Evolution Semigroups in Dynamical Systems and Differential Equations, American Mathematical Society, Providence, RI, 1999.
6. J. Dyson and R. Villella-Bressan, Functional differential equations and non-linear evolution operators, *Proc. Roy. Soc. Edinburgh A* **75** (1976), 223–234.

7. K. Engel and R. Nagel, ‘‘One Parameter Semigroups for Linear Evolution Equations’’, Springer-Verlag, New York/Berlin, 2000.
8. W.E. Fitzgibbon, Stability for abstract nonlinear Volterra equations involving finite delay, *J. Math. Anal. Appl.* **60** (1977), 429–434.
9. I. Gohberg, S. Goldberg, and M.A. Kaashoek, ‘‘Classes of Linear Operators’’, Vol. I, Birkhäuser, Basel, 1990.
10. G. Gühring, ‘‘Asymptotic Properties of Nonautonomous Evolution Equations and Nonautonomous Retarded Equations’’, Ph.D. thesis, Universität Tübingen, 1999.
11. G. Gühring and F. Räbiger, Asymptotic properties of mild solutions of nonautonomous evolution equations with applications to retarded differential equations, *Abstr. Appl. Anal.* **4** (1999), 169–194.
12. I. Győri, F. Hartung, and J. Turi, Preservation of stability in delay equations under delay perturbations, *J. Math. Anal. Appl.* **220** (1998), 290–312.
13. J.K. Hale and S.M. Verduyn Lunel, ‘‘Introduction to Functional Differential Equations’’, Springer-Verlag, New York/Berlin, 1993.
14. J.K. Hale and S.M. Verduyn Lunel, Effects of small delays on stability and control, *in* Operator Theory and Analysis (Amsterdam, 1997),’’ *Oper. Theory Adv. Appl.* **122**, Birkhäuser, Basel, 2001, pp. 275–301.
15. D. Henry, ‘‘Geometric Theory of Semilinear Parabolic Equations’’, Springer-Verlag, New York/Berlin, 1981.
16. W. Hutter, Hyperbolicity of almost periodic evolution families, *Tübinger Berichte zur Funktional-analysis* **6** (1997), 92–109.
17. B.M. Levitan and V.V. Zhikov, ‘‘Almost Periodic Functions and Differential Equations’’, Cambridge University Press, Cambridge, UK, 1982.
18. X.-B. Lin, Exponential dichotomies and homoclinic orbits in functional differential equations, *J. Differential Equations* **63** (1986), 227–254.
19. T. Naito and Nguyen Van Minh, Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations, *J. Differential Equations* **152** (1999), 358–376.
20. J.V. Neerven, ‘‘The Adjoint of a Semigroup of Linear Operators’’, Springer-Verlag, New York/Berlin, 1992.
21. Nguyen Thanh Lan, ‘‘Nonautonomous Operator Matrices’’, Ph.D. thesis, Universität Tübingen, 1998.
22. J. Prüss, ‘‘Evolutionary Integral Equations and Applications’’, Birkhäuser, Basel, 1993.
23. A. Rhandi, Extrapolation methods to solve non-autonomous retarded partial differential equations, *Studia Math.* **126** (1997), 219–233.
24. W.M. Ruess, Existence of solutions to partial functional differential equations with delay, *in* ‘‘Theory and Applications of Nonlinear Operators of Accretive and Monotone Type’’ (A.G. Kartsatos, Ed.), Dekker, New York, 1996, pp. 259–288.
25. W.M. Ruess and W.H. Summers, Almost periodicity and stability for solutions to functional differential equations with infinite delay, *Differential Integral Equations* **9** (1996), 1225–1252.
26. R. Schnaubelt, Sufficient conditions for exponential stability and dichotomy of evolution equations, *Forum Math.* **11** (1999), 543–566.
27. R. Schnaubelt, ‘‘Exponential Dichotomy of Non-autonomous Evolution Equations’’, Habilitation thesis, Universität Tübingen, 1999.
28. J. Wu, ‘‘Theory and Applications of Partial Functional Differential Equations’’, Springer-Verlag, Berlin/New York, 1996.