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Godunov-type approximation for a general resonant balance law with large data ☆

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Abstract

We consider the Cauchy problem for the 2×2 nonstrictly hyperbolic system

$$\begin{cases} a_t = 0, \\ u_t + f(a,u)_x - g(a,u)a_x = 0, \end{cases} \quad (a,u)(t=0,\cdot) = (a_0, u_0).$$

For possibly large, discontinuous and resonant data, the generalized solution to the Riemann problem is introduced, interaction estimates are carried out using an original change of variables and the convergence of Godunov approximations is shown. Uniqueness is addressed relying on a suitable extension of Kružkov's techniques.

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1. Introduction

The main concern of the present paper is with the Cauchy problem in $\mathbb{R}$ for the following 2×2 nonstrictly hyperbolic system of balance laws:

$$\begin{cases} a_t = 0, \\ u_t + f(a,u)_x - g(a,u)a_x = 0, \end{cases} \tag{1.1}$$

completed with the initial data

$$a(0,\cdot) = a_0, \quad u(0,\cdot) = u_0, \tag{1.2}$$

under the forthcoming assumptions (P_1)–(P_5). Like the authors of [8,13–15,19,27], we are especially interested in nonlinear resonance, that is, when wave speeds coalesce. Since this is equivalent to the existence of points where f_u vanishes, let $\mathscr{T}$ denote their set: $\mathscr{T} = \{(a,u) : f_u(a,u) = 0\}$. We shall work in the sequel under some rather general assumptions which read:

(P_1) f, g are smooth functions; $\forall a \lim_{u \to \pm\infty} f(a,u) = +\infty$;
(P_2) $\mathscr{T}$ is a graph: there exists a $\mathbf{C}^1$ map $\tau : \mathbb{R} \to \mathbb{R}$ such that $f_u(a,\tau(a)) = 0$ for all $a \in \mathbb{R}$;
(P_3) $f_u(a,u) \cdot (u - \tau(a)) > 0$, for all $(a,u) \in \mathbb{R}^2 \backslash \mathscr{T}$;
(P_4) $f_a - g \neq 0$ for all $(a,u) \in \mathscr{T}$;
(P_5) for any $(a_0,u_0) \in \mathbb{R}^2 \backslash \mathscr{T}$, the solution to the Cauchy problem

$$\frac{du(a)}{da} = \frac{g - f_a}{f_u}, \quad u(a_0) = u_0, \tag{1.3}$$

does not blow up to infinity on bounded intervals.

Let us pause to state a few comments. System (1.1) can be viewed either as a 2×2 hyperbolic system, not in conservation form, or as a different way to write a general scalar balance laws with source term,

$$u_t + f(a(x),u)_x = a'(x)g(a(x),u), \tag{1.4}$$

as advocated for instance in [4] to derive a simple model of shallow-water flow. Another case of special interest for (1.1) lies in the modeling of one-dimensional flow in a nozzle as pointed out in [19]; in this context, f, g do not depend on a:

$$u_t + f(u)_x = k(x)g(u), \quad k(x) = a'(x) \geqslant 0. \tag{1.5}$$

We shall develop this case first in Section 2.1: as in [11], we can consider (1.5) within a limiting process when $a'(x)$ concentrates onto a Dirac comb. The classical compensated compactness results, [26], imply strong compactness of the entropy solutions u but the lack of BV estimates prevents us from defining rigorously the product $g(a,u)a_x$ within the framework of [17]. Hence taking advantage of the linear degeneracy of the nonconservative field, we shall interpret it following integral curves of the associated eigenvector, see e.g. [3,5].

These equations are endowed with a source term $g(a,u)a_x$. Of course, in the case it vanishes, $g \equiv 0$, one recovers a conservation law with the flux function being space dependent as in [2,7,15,18,21], this dependence being possibly discontinuous.

Concerning our assumptions, we stress that (P_3) is less restrictive than the usual genuine nonlinearity requirement; however it excludes the linear cases already investigated in e.g. [6,12,23] for which the theory is quite different, see also [21]. Observe that condition (P_3) is a consequence of (P_1) and the definition of $\mathscr{T}$. Conditions (P_2)–(P_5) will allow us to define and study the integral curves of the steady-state problem for (1.1) in Section 2.2. Within the special framework (1.5) considered in [19], our condition (P_5) is equivalent to the one required there to ensure existence for arbitrary large initial data (see Remark 2.7).

Our plan is as follows. We aim at first establishing the existence of global solutions to (1.1)–(1.2). This will be carried out by extending and simplifying former studies, [13,19,27]. In [13] (see also [27]), the Cauchy problem for (1.1) has been solved for small data, that is to say, in a neighborhood of some given point $U^* = (a^*, u^*) \in \mathscr{T}$ In Section 3, we shall consider global solutions of the Riemann problem (3.1) for (1.1). This raises the problem of interpreting the nonconservative term $a'g$; this will be tackled as previously indicated for the special case (1.5).

Interaction estimates are to be carefully derived in Section 4 by means of an original change of variables; it sheds light on some computations already present in [13,19]. Relying on these stepping stones, a Godunov scheme is applied to build approximate solutions in Section 5; when ignited with a suitable discretization of the initial data, this scheme has the property of preserving the stationary solutions: this is the so-called Well-Balanced property, see [4,9,10,14,22]. Compactness of the approximate sequence is then established, Theorems 5.1, 5.2; the initial data are assumed to be bounded, without any smallness assumption.

The delicate question of uniqueness is finally to be raised (and partly solved) by means of Kružkov's techniques under a refined condition that $a(x)$ is absolutely continuous ($a' \in \mathbf{L}^1(\mathbb{R})$). In this case, the classical weak formulation for (1.1) does make sense and the limit is found to be a generalized solution in the sense of Kružkov [16]: see Theorem 6.1.

In Section 7, related stability estimates are also obtained, Theorem 7.1; however, an additional boundedness assumption is needed: see (7.1) and the counterexample in Section 8.

Let us finally mention an alternative approach to the treatment of resonance inside balance laws relying on their kinetic formulations: consult [4,22,29].

2. Motivation and preliminaries

2.1. A localization process for the a variable

As announced in the introduction, we first restrict ourselves to the special case (1.5) (the source term could also read $g(x,u)$). We assume furthermore that there exists a constant $M \in \mathbb{R}^+$ such that $ug(u) \leqslant 0$ for $|u| \geqslant M$ and that the function $u \mapsto f(u)$ is strictly convex (however, (P_3) would even suffice). The condition on the source term guarantees that blow-up cannot occur for the weak solutions of (1.5). Following the ideas of [11], we aim at deriving a meaning for $g(u)a_x$ when a is discontinuous relying on [17], *i.e.* within a limiting process. Let $K \in C^2(\mathbb{R})$ stand for an anti-derivative of k and, given a parameter $h > 0$, the nondecreasing function $a^\varepsilon(x)$ belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ for $\varepsilon > 0$ and is defined as follows:

$$a^\varepsilon(x) = \begin{cases} K(jh), & x \in \left[\, jh, \left(j + \frac{1}{2} - \frac{\varepsilon}{2}\right) h \right[, \\ K\left(\left(j + \frac{1}{2}\right) h \left(1 - \frac{1}{\varepsilon}\right) + \frac{x}{\varepsilon}\right), & x \in \left[\left(j + \frac{1}{2} - \frac{\varepsilon}{2}\right) h, \left(j + \frac{1}{2} + \frac{\varepsilon}{2}\right) h \right[, \\ K((j+1)h), & x \in \left[\left(j + \frac{1}{2} + \frac{\varepsilon}{2}\right) h, (j+1)h \right[. \end{cases} \tag{2.1}$$

The Kružkov's theory (see Theorem 7.1 in this paper) ensures that for each $\varepsilon > 0$, there exists a unique entropy solution u^ε of (1.5) and (2.1). But since

$$a^\varepsilon \xrightarrow{\varepsilon \to 0} a^h \stackrel{\text{def}}{=} \sum_j K(jh) \mathbf{1}_{x \in [(j-\frac{1}{2})h, (j+\frac{1}{2})h[},$$

the term lying at the right-hand side of (1.5) becomes ambiguous in the limit $\varepsilon \to 0$. As formerly done, one could establish that for $u_0, a^\varepsilon \in \mathrm{BV}(\mathbb{R})$, the total variation in space of the Riemann invariants for (1.1)

$$a; \quad w(u,a) = \phi^{-1}(\phi(u) - a), \quad \phi' = f'/g,$$

decays as time increases. However, as a consequence of resonance ($f_u = 0$), this implies no estimate on u^ε. Thus we turn to a weaker compactness framework based on $\mathbf{L}^\infty$ estimates. As a special case of (6.1), Kružkov's entropy inequalities hold for any $\varepsilon > 0$, $k \in \mathbb{R}$. We select $k = \max(M, ||u_0||_{\mathbf{L}^\infty(\mathbb{R})})$ and by integration, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \max(u^\varepsilon(x,t) - k, 0)\, dx \leqslant \int_{\mathbb{R}} \max(\operatorname{sgn}(u^\varepsilon - k), 0) g(u^\varepsilon)(x,t) a^\varepsilon_x \, dx \leqslant 0,$$

together with

$$\frac{d}{dt} \int_{\mathbb{R}} \min(0, u^\varepsilon(x,t) - k)\, dx \geqslant \int_{\mathbb{R}} \min(0, \operatorname{sgn}(u^\varepsilon - k)) g(u^\varepsilon)(x,t) a^\varepsilon_x \, dx \geqslant 0.$$

This gives a maximum principle which is uniform in ε and reads:

$$||u^\varepsilon||_{\mathbf{L}^\infty} \leqslant \max(M, ||u_0||_{\mathbf{L}^\infty(\mathbb{R})}).$$

At this level, we observe that under our hypotheses, $g(u^\varepsilon)a_x^\varepsilon$ is a bounded measure; hence we get, following [26], for all entropy–entropy flux pairs η, q satisfying $q' = \eta' f_u$,

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{compact in } H^{-1}_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+).$$

We deduce by the strict convexity of the function $u \mapsto f(u)$ that the sequence of entropy solutions u^ε is relatively compact in all L^p for $p < +\infty$. Since a^ε is constant in time, it is in $\mathrm{BV}(\mathbb{R})$ and compact in $\mathbf{L}^1_{\text{loc}}(\mathbb{R})$. This is enough to pass to the limit in (6.1) and get the limit problem in the sense of measures:

$$\eta(u)_t + q(u)_x \leqslant \overline{\eta' g} a_x^h, \quad \eta'(u^\varepsilon)g(u^\varepsilon)a_x^\varepsilon \rightharpoonup \overline{\eta' g} a_x^h.$$

Since this holds for Kružkov's entropies $\eta(u) = |u - k|$, $k \in \mathbb{R}$, the weak formulation follows. Unfortunately, we cannot follow the results of [17] to deduce also the meaning of the right-hand side term.

2.2. The stationary solutions

For smooth solutions, (1.1) corresponds to the quasilinear system

$$U_t + A(U)U_x = 0, \quad U = (a, u), \quad A(U) = \begin{pmatrix} 0 & 0 \\ f_a - g & f_u \end{pmatrix}. \tag{2.2}$$

The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = f_u$; the corresponding eigenvectors are $\mathbf{r}_1 = (f_u, g - f_a)$, $\mathbf{r}_2 = (0, 1)$. Observe that one of the characteristic fields is linearly degenerate, while the other one, due to condition (P_3), is, roughly speaking, genuinely nonlinear "around resonance points" and its integral curves are parallel straight lines. The strict hyperbolicity is lost along the transonic curve $\mathscr{T}$, and there the corresponding eigenvectors become parallel to each other. In the case $a = \bar{a}$ constant, the system reduces to the scalar homogeneous conservation law with parameter $\bar{a}$:

$$u_t + f(\bar{a}, u)_x = 0.$$

Let us introduce some notation. Denote by Ω^+ and Ω^- the following regions:

$$\Omega^+ = \{(a, u) : f_u(a, u) > 0\}, \quad \Omega^- = \{(a, u) : f_u(a, u) < 0\}. \tag{2.3}$$

We shall consider the case of a positive source along resonant states: $g - f_a|_{\mathscr{T}} > 0$, the other case being symmetric under the transformation $a \mapsto -a$. Let us introduce some special solutions of system (1.1): the stationary ones, which correspond to the integral curves of the linearly degenerate field,

$$f(a, u)_x = a_x g(a, u).$$

Regular solutions satisfy $f_u u_x = (g - f_a)a_x$, which is locally equivalent either to

$$\frac{du}{da} = \frac{g - f_a}{f_u} \tag{2.4}$$

or to

$$\frac{da}{du} = \frac{f_u}{g - f_a}. \tag{2.5}$$

(Observe that, by hypotheses, the two quantities f_u and $g - f_a$ do not vanish at the same points.) Denote by $\phi(a; a_0, u_0)$ the solution to (2.4) with Cauchy data $u(a_0) = u_0$:

$$\frac{du}{da} = \frac{g - f_a}{f_u}, \quad u(a_0) = u_0, \quad (a_0, u_0) \notin \mathcal{T}, \tag{2.6}$$

and denote by (α, β) its maximal interval of existence, we will also use the notation $\phi^{\pm}(a; a_0, u_0)$ for $(a_0, u_0) \in \Omega^{\pm}$. It is clear that

$$-\infty \leqslant \alpha(a_0, u_0) < a_0 < \beta(a_0, u_0) \leqslant +\infty.$$

Proposition 2.1. *For any $(a_0, u_0) \notin \mathcal{T}$, the maximal interval of existence of $\phi(a) = \phi(a; a_0, u_0)$ is unbounded to the right*: $\beta(a_0, u_0) = +\infty$.

Proof. Assume that $(a_0, u_0) \in \Omega^+$, the other case being similar. Since we are in Ω^+, $\phi(a) > \tau(a)$ holds where the solution ϕ exists. If, by contradiction $\beta(a_0, u_0) = a^* < +\infty$, then standard o.d.e. theorems ensure that $(a, \phi(a))$ approaches the boundary of Ω^+ as a approaches a^* to the left. But (P_5) prevents that $\phi(a) \xrightarrow{a \to a^*} +\infty$, hence we necessarily have $\phi(a) \xrightarrow{a \to a^*} \tau(a^*)$. Since $(g - f_a)(a^*, \tau(a^*)) > 0$, (2.4) implies $\phi'(a) \xrightarrow{a \to a^*} +\infty$. Therefore take $a_0 < a' < a'' < a^*$ and, using Lagrange theorem, compute

$$\inf_{a \in (a', a^*)} \phi'(a) \leqslant \frac{\phi(a'') - \phi(a')}{a'' - a'} \leqslant \frac{\phi(a'') - \tau(a')}{a'' - a'} \xrightarrow{a'' \to a^*} \frac{\tau(a^*) - \tau(a')}{a^* - a'} \tag{2.7}$$

which, when $a' \to a^*$, gives the desired contradiction since τ is $\mathbf{C}^1$. □

On the contrary, the maximal solutions to (2.4) may not be defined as $a \to -\infty$:

Proposition 2.2. *For any $(a_0, u_0 = \tau(a_0)) \in \mathcal{T}$, there exist two unique solutions $\phi^+(a) = \phi^+(a; a_0, u_0)$, $\phi^-(a) = \phi^-(a; a_0, u_0)$ of* (2.4) *with the following properties*:

ϕ^+, ϕ^- are maximally defined on $(a_0, +\infty)$; $\phi^+(a) \in \Omega^+$, $\phi^-(a) \in \Omega^-$,

$$\lim_{a \to a_0+} \phi^{\pm}(a) = u_0, \quad \left|\frac{d\phi^{\pm}}{da}(a)\right| \to +\infty \quad \text{as } a \to a_0. \tag{2.8}$$

Proof. Indeed, denote by $a(u) = a(u; u_0, a_0)$ the solution to the Cauchy problem for (2.5) with data $a(u_0) = a_0$. Due to (P_4), $a(u)$ is locally defined in a neighborhood of u_0 with $a'(u_0) = 0$. Then the curves $a(u)$ and $\tau(a)$ are transverse at the point (a_0, u_0) and $a(u) \in \Omega^+$ with $a'(u) > 0$ for $u > u_0$. As a consequence, $a(u)$ is invertible on $[u_0, u_0 + \delta]$, for some $\delta > 0$. Denote by $\phi^+(a)$ the inverse function, which is defined on the right of a_0, $[a_0, a_0 + \gamma]$. It is clear that $\phi^+(a)$ is a solution to (2.4) for $a > a_0$ and satisfies (2.8); by Proposition 2.1 it can be prolonged on $(a_0, +\infty)$. A totally similar procedure leads to the definition of $\phi^-(a)$. □

Let us introduce the following sets:

$$\Omega_1^+ = \{(a, u) \in \Omega^+ : \alpha(a, u) > -\infty\}, \quad \Omega_1^- = \{(a, u) \in \Omega^- : \alpha(a, u) > -\infty\}, \tag{2.9}$$

$$D_2^+ = \{(a, u) \in \Omega^+ : \alpha(a, u) = -\infty\}, \quad D_2^- = \{(a, u) \in \Omega^- : \alpha(a, u) = -\infty\}. \tag{2.10}$$

It is clear that $\Omega_1^+ \cup D_2^+ = \Omega^+$ and $\Omega_1^- \cup D_2^- = \Omega^-$.

The region Ω_1^+ contains all those integral curves of (2.4), inside the supersonic region Ω^+, which are *not* globally defined on $\mathbb{R}$. On the other hand, D_2^+ contains the globally defined solutions, and it may be empty, depending on the system under consideration.

Lemma 2.3. *The following holds: either the set D_2^+ (the set D_2^-) is empty, or there exists a $\mathbf{C}^1$ curve, $\tau^+(a)$ ($\tau^-(a)$), which is a global solution to* (2.4) *and satisfies*

$$\Omega_1^+ = \{(a, u) : \tau(a) < u < \tau^+(a)\}, \quad D_2^+ = \{(a, u) : u \geqslant \tau^+(a)\},$$

$$(\Omega_1^- = \{(a, u) : \tau^-(a) < u < \tau(a)\}, \quad D_2^- = \{(a, u) : u \leqslant \tau^-(a)\}).$$

Proof. For any u close to $\tau(0)$, $u > \tau(0)$, one has that $\alpha(0, u)$ is finite: $\alpha(0, u) > -\infty$. Indeed, take any $a_0 < 0$ and consider the solution $\phi^+(a) = \phi^+(a; a_0, \tau(a_0))$ introduced by Proposition 2.2, which tends to $\tau(a_0)$ as $a \to a_0+$. Since $\phi^+(0) > \tau(0)$, by uniqueness, the trajectories passing through the points $(0, \bar{u})$, with $\tau(0) < \bar{u} < \phi^+(0)$, must lie below $\phi^+(a)$; then $\alpha(0, \bar{u})$ must be finite (greater then a_0).

Arguing in the same way, one can observe that $\alpha(0, u)$ is monotone decreasing as u increases, $u > \tau(0)$. Let us set

$$\tilde{u} \doteq \sup\{u = \phi^+(0; a_0, \tau(a_0)),\ a_0 < 0\} = \sup\{u; \alpha(0, u) > -\infty\}.$$

If $\tilde{u} = +\infty$, then $D_2^+ = \emptyset$. On the other hand, if $\tilde{u} < +\infty$, the trajectory of (2.4) passing through the point $(0, \tilde{u})$ is globally defined on $\mathbb{R}$ and satisfies the required properties.

A completely similar procedure works for Ω^-. $\square$

Lemma 2.4. *The map $(a, u) \mapsto \alpha(a, u)$ defined on the set $\Omega_1 \doteq \Omega_1^+ \cup \mathscr{T} \cup \Omega_1^-$ has the following properties*:

- *it is a $\mathbf{C}^1$ map*;
- *if D_2^+ is not empty, then as $(a, u) \to (a_0, \tau^+(a_0))$ with $(a, u) \in \Omega_1$, one has $\alpha(a, u) \to -\infty$; the analogous holds if $D_2^- \neq \emptyset$*;
- *$u \mapsto \alpha(a, u)$ is a strictly decreasing map for $(a, u) \in \Omega_1^+$ and a strictly increasing map for $(a, u) \in \Omega_1^-$, $\alpha(a, u) = a$ for $(a, u) \in \mathscr{T}$.*

Proof. The graphs of ϕ^+ and ϕ^- are the integral curves of the degenerate characteristic field which can be written in a normalized form as

$$h \doteq \frac{1}{\sqrt{f_u^2 + (g - f_a)^2}} \begin{pmatrix} f_u \\ g - f_a \end{pmatrix}. \tag{2.11}$$

Hypotheses (P$_3$) and (P$_4$) ensure that h is a smooth vector field with norm 1. Fix $U_0 = (a_0, u_0) \in \Omega_1$ and define $U(t; U_0) = (U_1(t; U_0), U_2(t; U_0))$ as the solution of the autonomous Cauchy problem

$$\begin{cases} \dot{U} = h(U), \\ U(0) = U_0. \end{cases} \tag{2.12}$$

The curve $t \mapsto U(t; U_0)$ describes the integral curve of the linearly degenerate field passing through the point U_0. Standard results of the o.d.e. theory ensure that the map $(t; U_0) \mapsto U(t; U_0)$ is $\mathbf{C}^1$. Since U_0 is in Ω_1, the integral curve crosses the graph of τ in one and only one point, hence we can define the implicit function $\bar{t}(a_0, u_0)$ as

$$G(\bar{t}(a_0, u_0); a_0, u_0) \doteq U_2(\bar{t}(a_0, u_0); a_0, u_0) - \tau[U_1(\bar{t}(a_0, u_0); a_0, u_0)] = 0. \tag{2.13}$$

But $(t; a_0, u_0) \mapsto G(t; a_0, u_0)$ is $\mathbf{C}^1$, moreover

$$\frac{\partial G}{\partial t}(\bar{t}(a_0, u_0); a_0, u_0) = (h_2 - \tau' h_1)(U(\bar{t}(a_0, u_0); a_0, u_0)) = 1, \tag{2.14}$$

since $h_1 = 0$ on $\mathscr{T}$. Therefore the implicit function theorem ensures that $\bar{t}(\cdot, \cdot)$ is $\mathbf{C}^1$ and consequently also the function

$$(a_0, u_0) \mapsto U_1(\bar{t}(a_0, u_0); a_0, u_0) = \alpha(a_0, u_0) \tag{2.15}$$

is $\mathbf{C}^1$. As regards the other assertions, they follow directly from definition (2.9) and standard considerations based on the uniqueness of the solutions to (2.6). $\square$

2.3. A new set of variables

It will prove convenient to recast system (1.1) into the following set of coordinates:

$$\theta = \theta(a), \quad z = \begin{cases} 2 + [\phi(0; a, u) - \tau^+(0)] & \text{if } (a, u) \in D_2^+, \\ \pm[1 - \theta(\alpha(a, u))] & \text{if } (a, u) \in \Omega_1^\pm, \\ -2 + [\phi(0; a, u) - \tau^-(0)] & \text{if } (a, u) \in D_2^-, \end{cases} \tag{2.16}$$

where θ is a fixed function that satisfies

$$\theta : \mathbb{R} \to (-1, 1), \mathbf{C}^1, \text{ increasing, surjective.} \tag{2.17}$$

For instance, convenient choices are given by:

$$\theta(a) = \tanh(a) \quad \text{or} \quad \theta(a) = \frac{2}{\pi}\arctan(a).$$

Observe that the quantities $\phi(0; a, u)$, $\alpha(a, u)$ are constant along integral curves of (2.4).

Lemma 2.5. *The variables* (z, θ) *are continuous Riemann coordinates for system* (1.1) *on the set* Ω^+ *and on* Ω^- *(defined at* (2.3)); *the map* $u \mapsto z(a, u)$ *is strictly increasing for all* $a \in \mathbb{R}$ *and is discontinuous on* $\mathcal{T}$.

Proof. First we observe that $\nabla_{(a,u)}\theta \cdot \mathbf{r}_2 = \theta_u = 0$. Concerning z, observe that it is constant along integral curves of the stationary equation. By Lemma 2.4 and standard arguments for o.d.e.'s, it turns out that z is continuous on $\Omega_1^\pm$, $D_2^\pm$. Assume that $D_2^+ \neq \emptyset$; the following holds:

$$\text{as } (a, u) \to (a_0, \tau^+(a_0)), \quad (a, u) \in \Omega_1^+ \;\Rightarrow\; z \to 2.$$

On the other hand, as $(a, u) \to (a_0, \tau^+(a_0))$ from "above":

$$\phi(0; a, u) \to \phi(0; a_0, \tau^+(a_0)) = \phi(0; 0, \tau^+(0)) = \tau^+(0).$$

This gives the continuity of z across the graph of τ^+ and then on Ω^+. With similar arguments, one can deduce the continuity of z on Ω^-. Then, it is easy to check that z cannot be continuously extended across the transonic curve $\mathcal{T}$. Indeed,

$$\lim_{(a,u) \to (a_0, u_0), (a,u) \in \Omega^\pm} z(a, u) = \pm(1 - \theta(a_0)) \quad \text{for any } (a_0, u_0) \in \mathcal{T}. \tag{2.18}$$

Finally, the uniqueness of the solution to (2.6) and Lemma 2.4 imply that $u \mapsto z(a,u)$ is strictly increasing for $a \in \mathbb{R}$ fixed. Hence the map $(z,\theta) : \Omega^+ \cup \Omega^- \to \mathbb{R}^2$ is injective and consequently (z,θ) is a coordinate system on $\Omega^+ \cup \Omega^-$. $\square$

In the following, a key role will be played by the function

$$w(a,u) \doteq \begin{cases} z(a,u) + \theta(a) - 1 = z + \theta - 1 & \text{if } (a,u) \in \Omega^+, \\ 0 & \text{if } (a,u) \in \mathscr{T}, \\ z(a,u) - \theta(a) + 1 = z - \theta + 1 & \text{if } (a,u) \in \Omega^-. \end{cases} \tag{2.19}$$

More explicitly, using (2.16) we get

$$w(a,u) = \begin{cases} 1 + \theta(a) + [\phi(0;a,u) - \tau^+(0)] & \text{if } (a,u) \in D_2^+, \\ \pm[\theta(a) - \theta(\alpha(a,u))] & \text{if } (a,u) \in \Omega_1^\pm \cup \mathscr{T}, \\ -1 - \theta(a) + [\phi(0;a,u) - \tau^-(0)] & \text{if } (a,u) \in D_2^-. \end{cases} \tag{2.20}$$

Let us now introduce the two maps $\boldsymbol{\Phi}, \boldsymbol{\Psi}$ defined as follows:

$$\begin{cases} \boldsymbol{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2, \\ \boldsymbol{\Phi}(a,u) = (w,\theta), \end{cases} \qquad \begin{cases} \boldsymbol{\Psi} : \mathbb{R}^2 \backslash \mathscr{T} \to \mathbb{R}^2, \\ \boldsymbol{\Psi}(a,u) = (z,\theta). \end{cases} \tag{2.21}$$

Lemma 2.6. *Let $w(a,u)$, $z(a,u)$ the maps defined at* (2.19) *and* (2.16) *respectively.*

(a) *w is continuous w.r.t. (a,u). The map $u \to w(a,u)$ is monotone increasing, for all a.*
(b) *The map $\boldsymbol{\Phi}(a,u)$ is bijective and bi-continuous from $\mathbb{R}^2$ to* $\operatorname{Im}\boldsymbol{\Phi}$.
(c) *For a suitable choice of θ, the maps $\boldsymbol{\Phi}$ and θ^{-1} are locally Lipschitz.*

Proof. Concerning (a), the first property follows from Lemma 2.5 and (2.18); the second holds because of the analogous property of z, and implies that $\boldsymbol{\Phi}$ is injective on $\mathbb{R}^2$. Concerning (b), we have to prove that the inverse of $\boldsymbol{\Phi}$ is continuous. Assume that $\boldsymbol{\Phi}(a_n,u_n) = (w_n,\theta_n) \to (\bar{w},\bar{\theta}) = \boldsymbol{\Phi}(\bar{a},\bar{u})$, then $a_n \to \bar{a}$ as $n \to \infty$, since θ is continuous and depends only on a.

Let us prove that also $u_n \to \bar{u}$; this necessarily follows from the continuity of $\boldsymbol{\Phi}$ if u_n is bounded. On the other hand, assume that, possibly passing to a sub-sequence, $u_n \to +\infty$. If $(a_n,u_n) \in \Omega_1^+$ (because D_2^+ is empty), then $\alpha(a_n,u_n) \to -\infty$, which implies the contradiction $(w_n,\theta_n) \to (1+\bar{\theta},\bar{\theta}) \notin \operatorname{Im}\boldsymbol{\Phi}$. On the other hand, if D_2^+ is not empty, then $\phi(0;a_n,u_n) \to +\infty$, which again is impossible because it implies $w_n \to +\infty$.

(c) Lemma 2.4 ensures that the map

$$A(u) \doteq \begin{cases} \alpha(0,u) & \text{for } u \in (\tau^-(0), \tau(0)], \\ -\alpha(0,u) & \text{for } u \in [\tau(0), \tau^+(0)) \end{cases} \tag{2.22}$$

is $\mathbf{C}^1$, strictly increasing and surjective on $\mathbb{R}$. In (2.22) we have posed $\tau^-(0) = -\infty$ if $D_2^- = \emptyset$ and $\tau^+(0) = +\infty$ if $D_2^+ = \emptyset$. Observe that $A'(\tau(0)) = 0$. Now define

$$\theta(a) = -1 + c\int_{-\infty}^{a} \frac{d\xi}{A'[A^{-1}(-\xi)] + A'[A^{-1}(\xi)] + \xi^2 + 1}, \tag{2.23}$$

where $c>0$ is a normalization constant which makes θ satisfy (2.17). Definition (2.20), Lemma 2.4 and standard results on o.d.e. ensure that w is $\mathbf{C}^1$ when restricted to the closed set $D_2^+ \cup D_2^-$ or to the open set $\Omega_1 = \Omega_1^+ \cup \mathscr{T} \cup \Omega_1^-$. Since w is also continuous on all $\mathbb{R}^2$, to have local Lipschitz continuity we have only to show that its derivatives are bounded near the boundary of Ω_1. Therefore suppose $D_2^+ \neq \emptyset$ and take $(a_0, \tau^+(a_0)) \in \partial\Omega_1$, if $\delta>0$ is sufficiently small, the map $(a,u) \mapsto \phi(0; a, u)$ is well defined and $\mathbf{C}^1$ on $B_\delta(a_0, \tau^+(a_0))$, the closed neighborhood of the point $(a_0, \tau^+(a_0))$ with radius δ. Hence the map $(a,u) \mapsto \phi(0; a, u)$ has bounded first derivatives on the set $\Omega_1 \cap B_\delta(a_0, \tau^+(a_0))$. The conclusion follows from the identity

$$\theta[\alpha(a,u)] = \theta[\alpha(0, \phi(0; a, u))] \tag{2.24}$$

since by construction, the map $u \mapsto \theta[\alpha(0,u)]$ has bounded first derivative. Finally from $\theta'>0$ it follows that θ^{-1} is locally Lipschitz continuous. $\square$

Now assume that the sets D_2^+, D_2^- are both non-empty (see Lemma 2.3). As a consequence, w and z are unbounded from above and below, then $\operatorname{Im}\boldsymbol{\Phi} = \mathbb{R} \times (-1,1)$. Let us define

$$\mathscr{R} = \{(z,\theta) \in \mathbb{R}^2 :\ \theta \in (-1,1),\ |z| \geqslant 1 - \theta\}, \tag{2.25}$$

where, in order to have a unique representation for $\mathscr{T}$, we identify the points

$$(-1+\theta, \theta) \sim (1-\theta, \theta), \quad \theta \in (-1,1), \tag{2.26}$$

moreover we define

$$\boldsymbol{\Psi}(a,u) = (-1+\theta(a), \theta(a)) \sim (1-\theta(a), \theta(a)) \quad \text{for any } (a,u) \in \mathscr{T}.$$

We have then

$$\mathscr{R} = \boldsymbol{\Psi}(\mathbb{R}^2 \backslash \mathscr{T}) \cup \tilde{\mathscr{T}} = \boldsymbol{\Psi}(\mathbb{R}^2), \quad \tilde{\mathscr{T}} \doteq \{(z,\theta) \in \mathscr{R}, |z| = 1-\theta\}.$$

If $\psi \in \tilde{\mathscr{T}}$, we will denote by ψ^+ the representative with z positive and by ψ^- the one with z negative. The representatives of points $\psi \in \mathscr{R} \backslash \tilde{\mathscr{T}}$ are unique. Moreover, we will use the notation

$$\mathscr{R}^+ = \boldsymbol{\Psi}(\Omega^+) \cup \tilde{\mathscr{T}}, \quad \mathscr{R}^- = \boldsymbol{\Psi}(\Omega^-) \cup \tilde{\mathscr{T}}.$$

Finally observe that, if either D_2^+ or D_2^- is empty, or both are empty, w and z become bounded from above, below or both, respectively. For instance,

$$D_2^+ = D_2^- = \emptyset \;\Rightarrow\; \mathscr{R} = \{(z,\theta) : \theta \in (-1,1), |z| \geqslant 1-\theta, |z| < 2\}.$$

Remark 2.7. In the special case of (1.5) already considered in [11,19], with $f(0) = f'(0) = 0$, assume that $g(u)$ never vanishes (note that we require only $g(0) \neq 0$, (P_4)). If Ψ denotes the anti-derivative of f'/g with $\Psi(0) = 0$, it is easy to see that

$$\alpha(a,u) = a - \Psi(u) = \text{const.} \tag{2.27}$$

along the integral curves of the stationary equation. In [19] it was required that

$$\lim_{|u| \to +\infty} |\Psi(u)| = +\infty \tag{2.28}$$

to ensure the existence of the solution for arbitrarily large initial data. In this context, (2.28) is equivalent to (P_5). Indeed, take an integral curve of the stationary equation, i.e. a solution $u(a)$ to (1.3). Whenever $u(a)$ is defined, we have

$$a - \Psi(u(a)) = \text{const.} \tag{2.29}$$

If (P_5) does not hold, then there exists an integral curve $u(a)$ and a value a_1 such that (for instance) $\lim_{a \to a_1^-} u(a) = +\infty$, therefore taking the limit as a tends to a_1 from the left in (2.29) we obtain that Ψ has to be bounded as $u \to +\infty$ and hence condition (2.28) cannot hold.

On the other hand, let (P_5) hold and take an integral curve $u(a)$ defined on $(a_0, +\infty)$. Now we can let a go to $+\infty$ in (2.29). It is easy to see that $\Psi(u(a))$ and consequently $u(a)$ must be unbounded and therefore (2.28) holds.

3. The Riemann problem

3.1. The general self-similar solution

Let us consider the Riemann problem for (1.1),

$$U(x,0) = \begin{cases} U_1 = (a_1, u_1), & x < 0, \\ U_2 = (a_2, u_2), & x > 0, \end{cases} \tag{3.1}$$

for any $U_1, U_2 \in \mathbb{R}^2$. It will be solved, as usual, connecting the two states with waves of increasing speed. We will focus on the (z,θ) coordinates since in these variables the characteristic curves of the 1st (*standing waves*) and 2nd family (*homogeneous waves*) are straight lines with the variable z and θ constant, respectively. We will denote by $\psi = (z,\theta)$ the points of the set $\mathscr{R}$.

In this section, our goal is to find at least one solution of the Riemann problem. However, in general the solution is not unique and one may have multiple solutions depending on the shape of the graphs of the functions $\bar{z}$ and $\tilde{z}$ defined below. In [13], some assumptions on f, g are discussed concerning uniqueness/non-uniqueness of the solution.

Anyway, in the case when a is absolutely continuous, we will prove, in Section 5, the uniqueness and continuous dependence of the exact solution of the Cauchy problem, even if the approximate solutions (constructed with the Riemann problem as building block) are not uniquely defined,[1] under some additional assumptions on the source term $a'g$.

We consider a number of cases, which cover all the possible situations, depending on the relative location of the left and right states $\psi_1 = (z_1, \theta_1)$ and $\psi_2 = (z_2, \theta_2)$. A set of consecutive homogeneous waves, which have increasing velocities and connect two states with the same θ, will be denoted by O^+, O^0 and O^- if all the velocities of the waves in the set are respectively positive, zero and negative (remember that away from the resonance the flux may be non convex and hence more then one wave may be needed to solve the homogeneous Riemann problem). While a standing wave, which always has velocity zero, will be denoted by s if subsonic ($z<0$) and by S if supersonic ($z>0$).

Observe that hypothesis (P_3) implies that any homogeneous wave which connects states in region $\mathscr{R}^+$ has strictly positive velocity whereas any homogeneous wave which connects states in region $\mathscr{R}^-$ has strictly negative velocity.

1. $\psi_1 \in \mathscr{R}^+$: The hypotheses on f imply that there exists a unique continuous function $\bar{z}(\theta)$ such that

$$\begin{cases} \bar{z}(\theta)<0, \quad \theta>\max\{1-z_1, -1\}, \\ (f\circ\boldsymbol{\Psi}^{-1})(\bar{z}(\theta), \theta) = (f\circ\boldsymbol{\Psi}^{-1})(z_1, \theta). \end{cases}$$

This function describes what in [13] is called the 0-*speed shock curve* corresponding to the standing wave $\theta \mapsto (z_1, \theta)$. All the homogeneous waves with increasing velocities connecting the point (z_1, θ) to the point (z, θ) with $z<z_1$ have positive, zero or negative speeds if respectively $z>\bar{z}(\theta)$, $z=\bar{z}(\theta)$ or $z<\bar{z}(\theta)$. Indeed all the velocities of the waves in the set must have the same sign of the shock which crosses the resonance (condition (P_3) ensures that the resonance can be crossed only by a shock). If the shock which crosses the resonance has zero velocity, then it turns out to be the only component of the set of waves. We now consider five regions, which depend on the state ψ_1:

$$\begin{aligned} R_1(\psi_1) &= \{(z,\theta)\in\mathscr{R}^+ : \theta\geqslant 1-z_1\}, \\ R_2(\psi_1) &= \{(z,\theta)\in\mathscr{R}^- : z\leqslant\bar{z}(\theta_1)\}, \\ R_3(\psi_1) &= \{(z,\theta)\in\mathscr{R}^- : z\geqslant\bar{z}(\theta), \theta>\max\{1-z_1,-1\}\}, \\ R_4(\psi_1) &= [\mathscr{R}^-]\backslash[R_2(\psi_1)\cup R_3(\psi_1)], \\ R_5(\psi_1) &= [\mathscr{R}^+]\backslash R_1(\psi_1). \end{aligned} \tag{3.2}$$

[1] This highlights one big difficulty in simulating numerically (1.1) in a resonant setting; consult also [28].

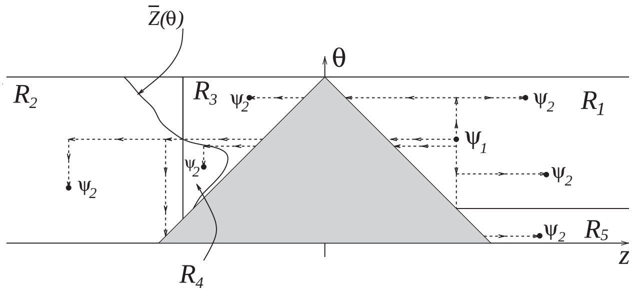

Fig. 1. The Riemann problem for a supersonic left state ψ_1.

The Riemann problem has qualitatively different solutions depending on which region ψ_2 lies. The five cases are discussed referring to Fig. 1.

- $\psi_2 \in R_1(\psi_1) \cup R_3(\psi_1)$:

$$(z_1, \theta_1) \overset{S}{\rightarrow} (z_1, \theta_2) \overset{O^+}{\rightarrow} (z_2, \theta_2).$$

- $\psi_2 \in R_2(\psi_1)$:

$$(z_1, \theta_1) \overset{O^-}{\rightarrow} (z_2, \theta_1) \overset{s}{\rightarrow} (z_2, \theta_2).$$

- $\psi_2 \in R_4(\psi_1)$:

In this case one has $z_2 > \bar{z}(\theta_1)$, and either $z_2 < \bar{z}(\theta_2)$ or $\theta_2 \leqslant 1 - z_1$.

Hence there exists θ^* between θ_1 and θ_2 such that $z_2 = \bar{z}(\theta^*)$. So the solution of the Riemann problem is given by:

$$(z_1, \theta_1) \overset{S}{\rightarrow} (z_1, \theta^*) \overset{O^0}{\rightarrow} (z_2, \theta^*) \overset{s}{\rightarrow} (z_2, \theta_2).$$

- $\psi_2 \in R_5(\psi_1) = \{(z, \theta) \in \mathscr{R}^+ : \theta < 1 - z_1\}$.

If $z_1 \geqslant 2$, this region is empty. If not, note that $(-1 + \theta_2, \theta_2) \notin R_3(\psi_1)$. Then the waves, in the solution of the Riemann problem, that connect ψ_1 to the state $(-1 + \theta_2, \theta_2)$ have nonpositive velocity. Hence we have only to add homogeneous waves with positive velocity connecting $(-1 + \theta_2, \theta_2) \sim (1 - \theta_2, \theta_2)$ to (z_2, θ_2).

2. $\psi_1 \in \mathscr{R}^-$: The hypotheses on f imply that there exists a continuous function $\tilde{z}(\theta)$ such that

$$\begin{cases} \tilde{z}(\theta) < 0, \ \ \theta \geqslant \theta_1, \\ (f \circ \boldsymbol{\Psi}^{-1})(\tilde{z}(\theta), \theta) = (f \circ \boldsymbol{\Psi}^{-1})(1 - \theta_1, \theta). \end{cases}$$

This function describes the 0-*speed shock curve* which corresponds to the standing wave $\theta \mapsto (1-\theta_1, \theta)$. The homogeneous waves with increasing velocities connecting the point $(1-\theta_1, \theta)$ to the point (z, θ) with $z < 1-\theta_1$ have positive, zero or negative speed if respectively $z > \tilde{z}(\theta)$, $z = \tilde{z}(\theta)$ or $z < \tilde{z}(\theta)$.

We now consider six regions depending on the state ψ_1:

$$
\begin{aligned}
&\bar{R}_1(\psi_1) = \{(z,\theta) \in \mathscr{R}^- : z \leqslant \theta_1 - 1\},\\
&\bar{R}_2(\psi_1) = \{(z,\theta) \in \mathscr{R}^+ : \theta \leqslant \theta_1\},\\
&\bar{R}_3(\psi_1) = \{(z,\theta) \in \mathscr{R}^+ : \theta \geqslant \theta_1,\ z \geqslant 1-\theta_1\},\\
&\bar{R}_4(\psi_1) = \{(z,\theta) \in \mathscr{R}^+ : \theta \geqslant \theta_1,\ z \leqslant 1-\theta_1\},\\
&\bar{R}_5(\psi_1) = \{(z,\theta) \in \mathscr{R}^- : \theta \geqslant \theta_1, z \geqslant \max(\tilde{z}(\theta), \theta_1 - 1)\},\\
&\bar{R}_6(\psi_1) = [\mathscr{R}^-] \backslash [\bar{R}_1(\psi_1) \cup \bar{R}_5(\psi_1)].
\end{aligned}
\tag{3.3}
$$

The Riemann problem has qualitatively different solutions depending on which region ψ_2 lies. The six cases are discussed referring to Fig. 2.

- $\psi_2 \in \bar{R}_1(\psi_1)$:

$$(z_1, \theta_1) \xrightarrow{O^-} (z_2, \theta_1) \xrightarrow{s} (z_2, \theta_2).$$

- $\psi_2 \in \bar{R}_2(\psi_1)$:

$$(z_1, \theta_1) \xrightarrow{O^-} (\theta_2 - 1, \theta_1) \xrightarrow{s} (\theta_2 - 1, \theta_2) \sim (1-\theta_2, \theta_2) \xrightarrow{O^+} (z_2, \theta_2).$$

- $\psi_2 \in \bar{R}_3(\psi_1) \cup \bar{R}_4(\psi_1) \cup \bar{R}_5(\psi_1)$:

$$(z_1, \theta_1) \xrightarrow{O^-} (\theta_1 - 1, \theta_1) \sim (1-\theta_1, \theta_1) \xrightarrow{S} (1-\theta_1, \theta_2) \xrightarrow{O^+} (z_2, \theta_2).$$

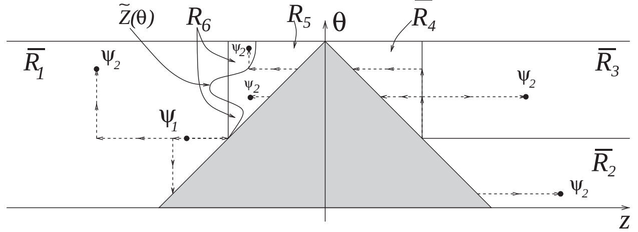

Fig. 2. The Riemann problem for a subsonic left state ψ_1.

- $\psi_2 \in \bar{R}_6(\psi_1)$:

In this case one has $z_2 > \theta_1 - 1 = \tilde{z}(\theta_1)$, and $z_2 < \tilde{z}(\theta_2)$, hence there exists θ^* between θ_1 and θ_2 such that $z_2 = \tilde{z}(\theta^*)$. So the solution of the Riemann problem is given by

$$(z_1, \theta_1) \xrightarrow{O^-} (\theta_1 - 1, \theta_1) \sim (1 - \theta_1, \theta_1) \xrightarrow{S} (1 - \theta_1, \theta^*) \xrightarrow{O^0} (z_2, \theta^*) \xrightarrow{s} (z_2, \theta_2).$$

Remark 3.1. Concerning invariant domains, it is easy to note the following property. For any pair of states ψ_1, ψ_2, consider the closed rectangle D, in the z–θ plan, that has the two states as vertices and each edge parallel to one of the axes. If all vertices of D belong to $\mathscr{R}$, then D is invariant w.r.t. the solution of the Riemann problem; if not, it can happen that D is not invariant (for instance if $\psi_1 \in \mathscr{R}^+$, $\psi_2 \in R_5(\psi_1)$).

In this second case, an invariant domain is given by the smallest rectangle $\tilde{D}$, with each edge parallel to one of the axes, which contains ψ_1 and ψ_2 and has all four vertices in $\mathscr{R}$. This larger rectangle $\tilde{D}$ does not increase the sup-norm: if $(z, \theta) \in \tilde{D}$, then

$$\min\{\theta_1, \theta_2\} \leqslant \theta \leqslant \max\{\theta_1, \theta_2\}, \quad |z| \leqslant \max\{|z_1|, |z_2|\}.$$

In the original variables a, u, the invariant domains are described by the regions bounded by the graphs of two standing waves, see Fig. 3. Such domains are not necessarily convex w.r.t. both variables, but their sections for a fixed are intervals. Observe that there are invariant domains which are not bounded in the original variables (a, u); this corresponds to allow the vertices belong to $\{\theta = \pm 1\}$. In particular, the domains

$$\mathscr{I}_{z_1, z_2} = \{(a, u);\ z_1 \leqslant z(a, u) \leqslant z_2\}, \quad z_1, z_2 \in \mathbb{R} \tag{3.4}$$

are invariant if and only if: either $|z_1|$, $|z_2| \geqslant 2$, or $|z_1| < 2$ and $z_2 = -z_1$.

3.2. Entropy dissipation across the standing wave

Going back to the original variables, let us focus on the jump relation on the fluxes at $x = 0$. For any U_1, U_2, let $U(x, t) = W(x/t)$ be a self-similar solution solving the

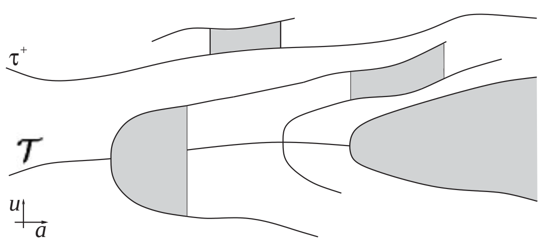

Fig. 3. Some invariant domains in the original variables (a, u).

Riemann problem. Denote by $U_- = W(0-)$ and $U_+ = W(0+)$ the left and right trace at $x = 0$, respectively. In the simplest case, a single standing wave connects U_- to U_+, and

$$f(U_+) - f(U_-) = \int_{a_-}^{a_+} g(a, \phi(a; U_-))\, da.$$

(ϕ was defined in Section 2.2). More generally, there exist two intermediate states $U_\pm^* = (a^*, u_\pm^*)$, with a^* possibly coinciding with a_- or a_+, such that

$$U_- \xrightarrow{S} (a^*, u_-^*) \xrightarrow{O^0} (a^*, u_+^*) \xrightarrow{s} U_+.$$

Since $f(U_+^*) = f(U_-^*)$, the jump relation becomes

$$\begin{aligned} f(U_+) - f(U_-) &= f(U_+) - f(U_+^*) + f(U_-^*) - f(U_-) \\ &= \int_{a_-}^{a_+} g(a, \tilde{\phi}(a; U_-, U_+))\, da, \end{aligned} \tag{3.5}$$

where $\tilde{\phi}$ is the function, possibly discontinuous at one point, defined by (assume $a_- < a_+$, the definition for $a_- > a_+$ is similar)

$$\tilde{\phi}(a; U_-, U_+) = \begin{cases} \phi(a_-; U_-) = U_-, & a \leqslant a_-, \\ \phi(a; U_-), & a_- \leqslant a \leqslant a^*, \\ \phi(a; U_+), & a^* < a \leqslant a_+, \\ \phi(a_+; U_+) = U_+, & a \geqslant a_+. \end{cases} \tag{3.6}$$

We stress that such a function $\tilde{\phi}$ matches the "asymptotic profiles" computed by the authors of [20] as the long-time behavior of a scalar conservation law of type (1.5) in a bounded domain. The jump relation (3.5) is closely related to the ones derived in [17] relying on the "families of locally Lipschitz paths". However this theory does not apply directly to our resonant problem because of the discontinuities appearing in $\tilde{\phi}$.

We end this section with a result about the entropy dissipation along the *zero waves*, as we call the discontinuities located at $\{x = 0\}$. This generalizes a previous work (see Remark 3.1 in [9]) where convenient hypotheses forbid resonance.

Proposition 3.2. *Let $\eta(u)$ be a smooth convex function and $q(a, u)$ the corresponding entropy flux*:

$$q(a, u) = \int_k^u \eta'(\bar{u}) f_u(a, \bar{u})\, d\bar{u},$$

where $k \in \mathbb{R}$ is arbitrary. Then, along $\{x = 0\}$, we have the inequality:

$$q(U_+) - q(U_-) \leqslant \int_{a_-}^{a^+} [q_a + \eta'(g - f_a)](a, \tilde{\phi}(a; U_-, U_+))\, da. \tag{3.7}$$

Proof. Following [9], we compute the jump of the entropy flux:

$$\begin{aligned} q(U_+) - q(U_-) &= \int_{a_-}^{a^+} \left[q_a + q_u \frac{g - f_a}{f_u} \right] (a, \tilde{\phi}(a; U_-, U_+))\, da + q(U_+^*) - q(U_-^*) \\ &= \int_{a_-}^{a^+} [q_a + \eta'(g - f_a)](a, \tilde{\phi}(a; U_-, U_+)) da + q(U_+^*) - q(U_-^*). \end{aligned}$$

The proposition is proved by the following inequality:

$$q(U_+^*) - q(U_-^*) = q(a^*, u_+^*) - q(a^*, u_-^*) \leqslant 0,$$

which can be shown observing that u_-^* and u_+^* are connected by an entropic shock with velocity zero and that $u \mapsto q(a^*, u)$ is an entropy flux for the scalar law $u_t + f(a^*, u)_x = 0$. $\square$

4. Interaction estimates

In this section we introduce a suitable definition of wave-strength, which is equivalent to the total variation measured in the singular variables w, θ. Moreover we will prove that the sum of all the wave-strengths is non increasing in time. These two properties guarantee that Helly's compactness theorem can be applied to approximate solutions.

If the two states ψ_1 and ψ_2 are connected by a single zero wave (z constant) or a set of consecutive homogeneous waves (θ constant), we define the size σ of the jump in the following way:

$$\sigma = \begin{cases} |z_1 - z_2| & \text{for homogeneous waves with } z_1 z_2 > 0, \\ |z_1 - z_2| - 2 + 2\theta_1 & \text{for homogeneous waves with } z_1 z_2 < 0, \\ |\theta_1 - \theta_2| & \text{for standing waves with } z_1(\theta_2 - \theta_1) > 0, \\ 3|\theta_1 - \theta_2| & \text{for standing waves with } z_1(\theta_2 - \theta_1) < 0. \end{cases} \tag{4.1}$$

Recalling the definition of w, (2.19), we have that $\sigma = |w(\psi_1) - w(\psi_2)|$ in the first three cases above, and $\sigma = 3|w(\psi_1) - w(\psi_2)|$ in the last one.

Observe that the size of homogeneous waves is the variation of the z variable across $\mathscr{R}$, while the size of the standing waves is given by the variation of the θ variable with a weight of 1 for supersonic waves with increasing θ and subsonic waves with decreasing θ and a weight of 3 in the other two cases. Temple (see for instance [13,27]) chooses respectively the weights 2 and 4. Here we follow Liu [19]

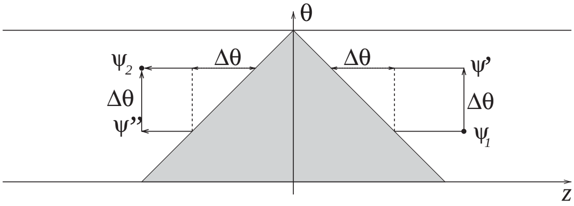

Fig. 4. Explanation of the weights choice.

and observe that in the choice of the weight the key point is that the second must be equal to the first increased by 2. This condition is easily understood in the (z,θ) coordinates. For instance, referring to Fig. 4, consider the four points in $\mathscr{R}$: $\psi_1 = (z_1,\theta_1)$, $\psi_2 = (z_2,\theta_2)$, $\psi' = (z_1,\theta_2)$ and $\psi'' = (z_2,\theta_1)$ with $z_1>0$, $z_2<0$ and $\theta_2>\theta_1$. The size of the jump connecting ψ' to ψ_2 is equal to the size of the jump connecting ψ_1 to ψ'' plus $2(\theta_2-\theta_1)=\Delta\theta$. Hence if we want the sum of the sizes of the waves connecting ψ_1 to ψ_2 not to depend on the path, the size of the standing wave connecting ψ'' to ψ_2 has to be equal to the size of the standing wave connecting ψ_1 to ψ' plus $2\Delta\theta$. Obviously, the same is not true if one wants to go from ψ_2 to ψ_1. But the entropy condition on the homogeneous conservation law implies that we cannot go with a single set of waves with positive (negative) velocities from a subsonic state to a supersonic one.

To get interaction estimates, the next step is now to show rigorously that three states ψ_1, ψ_m and ψ_2 being given, the sum of the sizes of the waves which solve the Riemann problem with left and right states ψ_1 and ψ_2 are less or equal to the sum of the sizes of the waves which solve the Riemann problem with left and right states ψ_1 and ψ_m plus the sum of the sizes of the waves which solve the Riemann problem with left and right states ψ_m and ψ_2. The rigorous proof involves the study of several cases and is carried out by Temple [27]. However the introduction of the "Riemann coordinates" (z,θ) simplifies the proof and reduces the number of cases to be considered, hence we give here a new proof in this coordinate system.

We base our proof on the set $\mathscr{G}$ of continuous and piecewise $\mathbf{C}^1$ oriented curves in $\mathscr{R}$ with at most a finite number of intersection with $\mathscr{T}$. We denote these curves by using the Greek letter γ. Given two states $\psi_1,\psi_2\in\mathscr{R}$ we denote by $\mathscr{G}(\psi_1,\psi_2)$ the set of curves in $\mathscr{G}$ which connect ψ_1 to ψ_2. Observe that the solutions of Riemann problems with left and right states ψ_1 and ψ_2 are elements of $\mathscr{G}(\psi_1,\psi_2)$.

In the (z,θ)-plane we define the two norms

$$|\psi|_* = |z| + |\theta|, \qquad |\psi|_\dagger = |z| + 2|\theta|.$$

Then we give the following two definitions of curve length:

$$\ell_*(\gamma) = \int_a^b |\psi_\gamma'(t)|_*\,dt, \qquad \ell_\dagger(\gamma) = \int_a^b |\psi_\gamma'(t)|_\dagger\,dt,$$

where $\psi(t)$ is any regular (continuous and piecewise $\mathbf{C}^1$) parameterization of γ. Finally we define the functional

$$\mathscr{F}(\gamma) = \ell_\dagger(\gamma) - \int_\gamma \operatorname{sgn} z \, d\theta.$$

Observe that if γ is a curve made of consecutive simple waves, then the sum of the sizes of all the waves is given by $\mathscr{F}(\gamma)$.

Given two states $\psi_1, \psi_2 \in \mathscr{R}$ we define the functional

$$\Lambda(\psi_1, \psi_2) = \inf\{\mathscr{F}(\gamma) : \gamma \in \mathscr{G}(\psi_1, \psi_2)\}. \tag{4.2}$$

If we are able to show that the sum of the sizes of the waves that solve the Riemann problem with left and right states ψ_1 and ψ_2 is given by $\Lambda(\psi_1, \psi_2)$ then we are done since from (4.2) it follows directly that

$$\Lambda(\psi_1, \psi_2) \leqslant \Lambda(\psi_1, \psi_m) + \Lambda(\psi_m, \psi_2), \quad \forall \psi_m \in \mathscr{R}. \tag{4.3}$$

For the proof of this assertion, some other auxiliary functionals are needed. We define a metric d_* and the two functionals Λ^+, Λ^- in the following way:

$$\begin{aligned} d_*(\psi_1, \psi_2) &= \inf\{\ell_*(\gamma) : \gamma \in \mathscr{G}(\psi_1, \psi_2)\}, \quad \forall \psi_1, \psi_2 \in \mathscr{R}, \\ \Lambda^+(\psi_1, \psi_2) &= \inf\{\mathscr{F}(\gamma) : \gamma \in \mathscr{G}(\psi_1, \psi_2) \text{ and } \gamma \subset \mathscr{R}^+\}, \quad \forall \psi_1, \psi_2 \in \mathscr{R}^+, \\ \Lambda^-(\psi_1, \psi_2) &= \inf\{\mathscr{F}(\gamma) : \gamma \in \mathscr{G}(\psi_1, \psi_2) \text{ and } \gamma \subset \mathscr{R}^-\}, \quad \forall \psi_1, \psi_2 \in \mathscr{R}^-. \end{aligned} \tag{4.4}$$

From the definitions it is easy to see that Λ^+, Λ^- and $3d_*$ are upper bounds for the functional Λ, while the metric d_* is a lower bound, in particular

$$d_*(\psi_1, \psi_2) \leqslant \Lambda(\psi_1, \psi_2) \leqslant 3d_*(\psi_1, \psi_2), \quad \forall \psi_1, \psi_2 \in \mathscr{R}. \tag{4.5}$$

Moreover, since $\operatorname{sgn} z \, d\theta$ is an exact differential form when restricted to $\mathscr{R}^+$ or to $\mathscr{R}^-$, we have the following expression for the functionals $\Lambda^\pm$:

$$\begin{aligned} \Lambda^+(\psi_1, \psi_2) &= |\psi_1 - \psi_2|_\dagger + \theta_1 - \theta_2, \\ \Lambda^-(\psi_1, \psi_2) &= |\psi_1 - \psi_2|_\dagger + \theta_2 - \theta_1. \end{aligned} \tag{4.6}$$

Now we state two theorems which give explicit values of the metric d_* and the functional Λ (their proofs are developed in Appendix A).

Theorem 4.1. *Given two states $\psi_1 = (z_1, \theta_1)$, $\psi_2 = (z_2, \theta_2)$ in $\mathscr{R}$, define $\theta_m = \min(\theta_1, \theta_2)$ and $\psi_m = (1 - \theta_m, \theta_m) \in \tilde{\mathscr{T}}$. Then*

$$d_*(\psi_1, \psi_2) = \begin{cases} |\psi_1 - \psi_2|_* & \text{if } z_1 z_2 > 0, \\ d_*(\psi_1, \psi_m) + d_*(\psi_m, \psi_2) & \text{if } z_1 z_2 < 0. \end{cases} \tag{4.7}$$

Moreover, having set $\Delta w \doteq w(\psi_1) - w(\psi_2)$, $\Delta\theta \doteq \theta_1 - \theta_2$,

$$\tfrac{1}{2}(|\Delta w| + |\Delta\theta|) \leqslant d_*(\psi_1, \psi_2) \leqslant 2(|\Delta w| + |\Delta\theta|). \tag{4.8}$$

Theorem 4.2. *Let the two states* $\psi_1 = (z_1, \theta_1)$, $\psi_2 = (z_2, \theta_2)$ *be given in* $\mathscr{R}$. *Then the functional* Λ *has the following properties*:

(a) *If* $\psi_1 \in \mathscr{R}^-$ *and* $\psi_2 \in \mathscr{R}^+$

$$\Lambda(\psi_1, \psi_2) = d_*(\psi_1, \psi_2). \tag{4.9}$$

(b) *If* $\psi_1, \psi_2 \in \mathscr{R}^+$

$$\Lambda(\psi_1, \psi_2) = \begin{cases} \Lambda^+(\psi_1, \psi_2) & \textit{for } 1 - z_1 - \theta_2 \leqslant 0, \\ \Lambda^+(\psi_1, \psi_2) - 2[1 - z_1 - \theta_2] & \textit{for } 1 - z_1 - \theta_2 > 0, \end{cases} \tag{4.10}$$

(c) *If* $\psi_1, \psi_2 \in \mathscr{R}^-$

$$\Lambda(\psi_1, \psi_2) = \begin{cases} \Lambda^-(\psi_1, \psi_2) & \textit{for } 1 + z_2 - \theta_1 \leqslant 0, \\ \Lambda^-(\psi_1, \psi_2) - 2[1 + z_2 - \theta_1] & \textit{for } 1 + z_2 - \theta_1 > 0. \end{cases} \tag{4.11}$$

(d) *If* $\psi_1 \in \mathscr{R}^+$ *and* $\psi_2 \in \mathscr{R}^-$, *define* $\theta_m = \min(\theta_1, \theta_2)$ *and* $\theta_M = \max(\theta_1, \theta_2)$. *Then*

$$\Lambda(\psi_1, \psi_2) = \Lambda(\psi_1, \psi) + \Lambda(\psi, \psi_2) \quad \textit{for any } \psi = (1 - \theta, \theta), \quad \theta \in [\theta_m, \theta_M]. \tag{4.12}$$

Now to conclude the interaction estimates, it is enough to check that the sum of the wave sizes in the solutions of the Riemann problem described in the previous section is equal to the value of the functional Λ given in the previous theorem. We shall omit this proof since it is straightforward.

Remark 4.3. Observe that if $\psi_0, \ldots, \psi_n$ are the intermediate states in the solution of a Riemann problem with left and right state respectively equal to ψ_0 and ψ_n, one has

$$\Lambda(\psi_1, \psi_n) = \sum_{i=1}^{n} \Lambda(\psi_{i-1}, \psi_i).$$

5. Convergence of approximate solutions

In this section we are about to define a family of approximate solutions to our system (1.1) with Cauchy data (1.2). Let θ be a function satisfying (2.17) and denote by $\theta_0 = \theta(a_0)$, $w_0 = w(a_0, u_0)$ the corresponding initial data in the θ–w variables. The assumptions on a_0, u_0 are the following:

$$a_0,\ u_0 \in \mathbf{L}^\infty(\mathbb{R}), \quad \text{Tot.Var.}(w_0, \theta_0) < +\infty. \tag{5.1}$$

We remark that, in general, the total variation of (w_0, θ_0) is not equivalent to the one of the original variables (a_0, u_0). If θ is chosen to satisfy (2.23), then (c) of Lemma 2.6 holds; as a consequence, assumption (5.1) is met if we require a more explicit condition on the initial data: Tot.Var.$(a_0, u_0) < +\infty$.

Let us fix a spatial mesh length $\Delta x > 0$ and define for all $j \in \mathbb{Z}$: $x_j \doteq j\Delta x$,

$$a^{\Delta x}(x,0) = a_0(x_j+), \quad u^{\Delta x}(x,0) = u_0(x_j+) \quad x_j < x < x_{j+1}. \tag{5.2}$$

Observe that, assuming (5.1), $a^{\Delta x}$, $u^{\Delta x}$ are well defined. Moreover, with this choice of the approximated initial data, the stationary solutions are preserved. If (5.1) holds, there exists a compact set $K_1 \subset \mathbb{R}^2$ that contains $(a_0, u_0)(x), (a_0^{\Delta x}, u_0^{\Delta x})(x)$ for all x, it is not restrictive to assume that K_1 is an invariant domain (see Remark 3.1 and Fig. 3). Then (z_0, θ_0) and $(z_0^{\Delta x}, \theta_0^{\Delta x})$ take the values in the invariant rectangle $K = \boldsymbol{\Psi}(K_1) \subset \mathscr{R}$. In the w, θ variables, K_1 corresponds to the compact set $\tilde{K} = \boldsymbol{\Phi}(K_1) \subset \operatorname{Im} \boldsymbol{\Phi}$. Now, we define

$$\lambda \doteq 2 \sup_{(a,u) \in K_1} \left| \frac{\partial f}{\partial u} \right|, \quad \Delta t = \lambda \Delta x$$

and discretize $\mathbb{R} \times [0, +\infty)$ by introducing the points of a grid, (x_j, t_n), $t_n = n\Delta t$. Denote by $R_{j,n} = [x_j, x_{j+1}) \times [t_n, t_{n+1})$ the unit cell.

Then, proceeding as in [13] with the only difference of the choice of the initial data (we use (5.2) instead of (27) in [13]), a family $(a^{\Delta x}, u^{\Delta x})(x,t) \in K_1$ is defined by the Godunov method. Note that K_1 is not necessarily convex as a subset of $\mathbb{R}^2$ but for any constant a it is an interval, which is invariant under the average w.r.t. u. Since the corresponding $(z^{\Delta x}, \theta^{\Delta x})(x,t) \in K$, we can always solve the Riemann problems at every time step, hence the approximate solutions are defined for all times.

At each time t_n, the approximate solution is discontinuous at the points x_j. We introduce the functional

$$F(t) = F(t_n) = \sum_{j \in \mathbb{Z}} \Lambda(\psi_{j-1}, \psi_j)(t_n), \quad t_n \leqslant t < t_{n+1}, \tag{5.3}$$

where $\psi_j = (z^{\Delta x}, \theta^{\Delta x})(t_n, x)$, $x \in (x_j, x_{j+1})$. By using (4.5), (4.8) and Remark 4.3 we find

$$\tfrac{1}{2}\text{Tot.Var.}\{(w^{\Delta x}, \theta^{\Delta x})(t)\} \leqslant F(t) \leqslant 6\,\text{Tot.Var.}\{(w^{\Delta x}, \theta^{\Delta x})(t)\}.$$

and consequently from (5.2):

$$F(0) \leqslant 6\text{Tot.Var.}\{(w^{\Delta x}, \theta^{\Delta x})(0)\} \leqslant 6\text{Tot.Var.}\{(w_0, \theta_0)\}. \tag{5.4}$$

We have the following theorem.

Theorem 5.1. *Assume* (P_1)–(P_5) *and* (5.1). *Then for all* $\Delta x > 0$, $(a^{\Delta x}, u^{\Delta x})(x,t)$ *is defined for all times and*

$$\text{Tot.Var.}\{(w^{\Delta x}, \theta^{\Delta x})(t)\} \leqslant 2F(t) \leqslant 2F(0).$$

Moreover, for a sub-sequence $\Delta x_j \to 0$, *we have that* $(w^{\Delta x_j}, \theta^{\Delta x_j}) \to (w, \theta_0)$ *and*

$$(a^{\Delta x_j}, u^{\Delta x_j}) \to (a_0, u) \quad \textit{in } \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R} \times [0, +\infty)),$$

where w *and* u *are bounded measurable functions satisfying*:

(a) $(a_0, u) = \boldsymbol{\Phi}^{-1}(w, \theta_0)$;
(b) *the maps* $t \mapsto w(\cdot, t)$, $t \mapsto u(\cdot, t)$ *are continuous in* $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$ *and* $w(\cdot, 0) = w_0$, $u(\cdot, 0) = u_0$;
(c) *the following estimate holds*:

$$\text{Tot.Var.}\{(w, \theta_0)(t)\} \leqslant 12\text{Tot.Var.}\{(w_0, \theta_0)\}. \tag{5.5}$$

Proof. From the previous analysis, $(w^{\Delta x}, \theta^{\Delta x}) \in \tilde{K}$, hence the $\mathbf{L}^\infty$ norm is uniformly bounded for all $\Delta x > 0$. The proof that $F(t)$ is nonincreasing follows from the analogous one in [18, Theorem 1, p. 833], because of the property that $u \mapsto w(a, u)$ is strictly increasing, for all a (see (a) of Lemma 2.6).

Moreover, the uniform bound on the total variation implies that $(w^{\Delta x}, \theta^{\Delta x})$ are (approximately) $\mathbf{L}^1$-Lipschitz continuous in time, uniformly in Δx.

Then, by Helly's theorem, a sub-sequence $(w^{\Delta x_j}, \theta^{\Delta x_j})$ converges in $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R} \times [0, +\infty))$ to a measurable function (w, θ) with values in $\tilde{K}$. Helly's theorem ensures also that the map $t \mapsto (w, \theta)(\cdot, t)$ is $\mathbf{L}^1$-Lipschitz continuous. Moreover by construction we have $\theta(x, t) = \theta_0(x)$ and $w(x, 0) = w_0(x)$ for any $(x, t) \in \mathbb{R} \times [0, +\infty)$. Now, by using (5.4) and the continuity of $\boldsymbol{\Phi}^{-1}$ (see Proposition B.1), all the other conclusions follow. $\square$

In the previous result the coefficient a was assumed to be bounded. However, it is also interesting to consider the case of possibly unbounded a; for instance, if $a(x) = x$, a very general framework is achieved:

$$u_t + f(x, u)_x = g(x, u),$$

but at the price of dealing with an unbounded variable. Let us make a further assumption:

(P_6) for any compact K in $\mathbb{R}$, $f_u(a,u)$ is uniformly bounded on $\mathbb{R}\times K$.

Condition (P_6) is trivially satisfied if f does not depend on a. It ensures that signals evolve with a finite speed of propagation in time (see [24, p. 44], of the French version).

We have the following theorem.

Theorem 5.2. *Assume* (P_1)–(P_6),

- $a_0\in\mathbf{L}^\infty_{\mathrm{loc}}(\mathbb{R})$, $u_0\in\mathbf{L}^\infty(\mathbb{R})$, Tot.Var.$(w_0,\theta_0)<+\infty$;
- (a_0,u_0) *takes its values in an invariant domain* $\mathscr{I}$ *having the form* $\mathscr{I}=\{(a,u);\ z_1\leqslant z(a,u)\leqslant z_2\}$ (*see Remark* 3.1);
- $\mathscr{I}$ *is uniformly bounded w.r.t. u.*

Then the same conclusion of Theorem 5.1 *holds.*

Proof. Define $\lambda\doteq 2\sup_{(a,u)\in\mathscr{I}}|\frac{\partial f}{\partial u}|$, which is finite because of (P_6) and the assumptions on $\mathscr{I}$. Then define $(a^{\Delta x},u^{\Delta x})$ as in the proof of Theorem 5.1; by construction, $(a^{\Delta x},u^{\Delta x})(x,t)\in\mathscr{I}$ for all $(x,t)\in\mathbb{R}\times[0,+\infty)$.

The corresponding sequence $(w^{\Delta x},\theta^{\Delta x})$ satisfies the assumptions of Helly's theorem; therefore, there exists a sub-sequence $(w^{\Delta x_j},\theta^{\Delta x_j})$ that converges to a function (w,θ) in $\mathbf{L}^1_{\mathrm{loc}}$ as $j\to\infty$, and (5.5) is satisfied.

For any compact subset H of $\mathbb{R}\times[0,+\infty)$, $(a^{\Delta x},u^{\Delta x})(H)$ is contained in a compact set $K\subset\mathscr{I}$ independent of Δx; then we can proceed as before and get a function (a,u) in the limit, with $u(x,t)$ bounded. $\square$

6. Entropy inequalities and consistency

In this section we show that, under some additional regularity assumptions on $a=a_0$, the limit function (a,u) of Theorem 5.1 satisfies Kružkov entropy inequalities and hence is a weak solution of (1.1) (or, equivalently, of (1.4)).

Assume that $a\in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$; then a is absolutely continuous on compact sets and the classical Kružkov entropy inequality [16] makes sense; within our hypotheses, it can be written as

$$\begin{aligned}\int_{\mathbb{R}}\int_0^{+\infty}&\{|u(x,t)-k|\varphi_t(x,t)\\&+\operatorname{sgn}[u(x,t)-k][f(a(x),u(x,t))-f(a(x),k)]\varphi_x(x,t)\\&+\operatorname{sgn}[u(x,t)-k][g(a(x),u(x,t))-f_a(a(x),k)]a'(x)\varphi(x,t)\}\,dx\,dt\geqslant 0,\end{aligned}\qquad(6.1)$$

where φ is a nonnegative $\mathbf{C}^1$ function with compact support in $\mathbb{R}\times(0,+\infty)$ and k is a real constant (for definiteness we assume $\mathrm{sgn}(0)=0$). Recall that the validity of (6.1) for $|k|$ sufficiently large implies that u is a distributional solution of (1.4).

We have the following theorem:

Theorem 6.1. *In the assumption of Theorem* 5.1, *if* $a \doteq a_0 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$, *then the limit functions* (a,u) *satisfy the Kružkov entropy inequalities* (6.1) *for any nonnegative* $\mathbf{C}^1$ *function* φ *with compact support in* $\mathbb{R}\times(0,+\infty)$ *and for any constant* $k\in\mathbb{R}$.

Proof. Fix a constant k and a nonnegative $\mathbf{C}^1$ function φ with compact support in $\mathbb{R}\times(0,+\infty)$. For notational purpose define

$$\bar{\eta}(v)=|v-k|, \quad \bar{q}(b,v)=\mathrm{sgn}(v-k)[f(b,v)-f(b,k)].$$

Let $E\in\mathbf{C}^2(\mathbb{R})$ be such that: $E''\geqslant 0$, $E(v)=|v|$ for $|v|\geqslant 1$ and $E'(0)=0$. Then the sequence $\eta^{\varepsilon}(v)=\varepsilon E(\frac{v-k}{\varepsilon})$ converges to $\bar{\eta}(v)$ in $\mathbf{C}^0(\mathbb{R})$, as $\varepsilon\to 0$.

If we fix

$$\eta=\eta^{\varepsilon}, \quad q=q^{\varepsilon}(b,v)=\int_k^v \eta'(\bar{v})f_u(b,\bar{v})\,d\bar{v}, \tag{6.2}$$

then, for any constant b, $(\eta, q(b,\cdot))$ is an entropy–entropy flux pair for the homogeneous scalar law $u_t+f(b,u)_x=0$.

Let (a^{ν},u^{ν}) be an approximating sequence that converges to (a,u) in $\mathbf{L}^1_{\mathrm{loc}}$, as $\nu\to+\infty$, let Δx_{ν} be the corresponding mesh size and consider the quantity

$$I_{\varepsilon,\nu}=\int_{\mathbb{R}\times\mathbb{R}^+}[\eta(u^{\nu})\varphi_t+q(a^{\nu},u^{\nu})\varphi_x+\eta'(u)[g(a,u)-f_a(a,k)]a'\varphi]\,dx\,dt.$$

Clearly, as $\nu\to+\infty$, the Lipschitz continuity of η and q implies:

$$I_{\varepsilon,\nu}\to I_{\varepsilon}=\int_{\mathbb{R}\times\mathbb{R}^+}[\eta(u)\varphi_t+q(a,u)\varphi_x+\eta'(u)[g(a,u)-f_a(a,k)]a'\varphi]\,dx\,dt. \tag{6.3}$$

Now we restrict our attention on the integration over each cell $R_{j,n}=[x_j,x_{j+1})\times[t_n,t_{n+1})$. Since a^{ν} is constant over $R_{j,n}$, the couple $(\eta, q(a^{\nu},\cdot))$ is an entropy–entropy flux pair for the scalar law $u_t+f(a^{\nu},u)_x=0$ in that cell, moreover, by construction, u^{ν} is an entropy solution of the scalar law in the same cell. Therefore, observing that

$a^\nu(x_j+) = a^\nu(x_{j+1}-) = a(x_j)$ and integrating by parts, we can compute

$$\begin{aligned}
&\int_{R_{j,n}} [\eta(u^\nu)\varphi_t + q(a^\nu, u^\nu)\varphi_x]\, dx\, dt \\
&\quad \geqslant \int_{t_n}^{t_{n+1}} [q(a(x_j), u^\nu(x_{j+1}-, t))\varphi(x_{j+1}, t) - q(a(x_j), u^\nu(x_j+, t))\varphi(x_j, t)]\, dt \\
&\qquad + \int_{x_j}^{x_{j+1}} [\eta(u^\nu(x, t_{n+1}-))\varphi(x, t_{n+1}) - \eta(u^\nu(x, t_n))\varphi(x, t_n)]\, dx. \qquad (6.4)
\end{aligned}$$

Now integrating over every cell $R_{j,n}$ and then rearranging the summations we get

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}^+} [\eta(u^\nu)\varphi_t + q(a^\nu, u^\nu)\varphi_x]\, dx\, dt = \sum_{j\in\mathbb{Z}} \sum_{n\geqslant 0} \int_{R_{j,n}} \ldots dx\, dt \\
&\geqslant - \sum_{n\geqslant 1} \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} [\eta(u^\nu(x, t_n)) - \eta(u^\nu(x, t_n-))]\varphi(x, t_n)\, dx \\
&\quad - \sum_{j\in\mathbb{Z}} \int_0^{+\infty} [q(a(x_{j+1}), u^\nu(x_{j+1}+, t)) - q(a(x_j), u^\nu(x_{j+1}-, t))]\varphi(x_{j+1}, t) dt \\
&\doteq J^1_{\varepsilon,\nu} + J^2_{\varepsilon,\nu}. \qquad (6.5)
\end{aligned}$$

The first term: $J^1_{\varepsilon,\nu}$ By Jensen's inequality and the definition of $u^\nu(x, t_n)$ in the Godunov scheme, one has for any $\bar{x} \in (x_j, x_{j+1})$:

$$\eta(u^\nu(\bar{x}, t_n)) = \eta\left(\frac{1}{\Delta x_\nu} \int_{x_j}^{x_{j+1}} u^\nu(x, t_n-)\, dx\right) \leqslant \frac{1}{\Delta x_\nu} \int_{x_j}^{x_{j+1}} \eta(u^\nu(x, t_n-))\, dx,$$

and hence for some $\bar{x} \in (x_j, x_{j+1})$:

$$\begin{aligned}
&\int_{x_j}^{x_{j+1}} [\eta(u^\nu(x, t_n)) - \eta(u^\nu(x, t_n-))]\varphi(x, t_n)\, dx \\
&\quad \leqslant ||\varphi||_{\mathbf{C}^1} ||\eta||_{\mathbf{C}^1} \Delta x_\nu \int_{x_j}^{x_{j+1}} |u^\nu(x, t_n) - u^\nu(x, t_n-)|\, dx \\
&\qquad + \varphi(x_j, t_n)\left[\Delta x_\nu \eta(u^\nu(\bar{x}, t_n)) - \int_{x_j}^{x_{j+1}} \eta(u^\nu(x, t_n-))\, dx\right] \\
&\quad \leqslant ||\varphi||_{\mathbf{C}^1} ||\eta||_{\mathbf{C}^1} \frac{1}{\lambda} \int_{R_{j,n-1}} |u^\nu(x, t_n) - u^\nu(x, t_n-)|\, dt\, dx.
\end{aligned}$$

Finally, if we sum this last inequality over $n \geqslant 1$ and $j \in \mathbb{Z}$ we get

$$J^1_{\varepsilon,\nu} \geqslant - ||\varphi||_{\mathbf{C}^1} ||\eta||_{\mathbf{C}^1} \frac{1}{\lambda} \int_A |\tilde{u}^\nu(x, t) - \bar{u}^\nu(x, t)|\, dt\, dx,$$

where A is a suitable bounded open set containing the support of φ and $\tilde{u}^\nu$, $\bar{u}^\nu$ are defined by

$$\begin{aligned} \tilde{u}^\nu(x,t) &= u^\nu(x,t_n) \\ \bar{u}^\nu(x,t) &= u^\nu(x,t_n-) \end{aligned} \quad \text{for } (x,t)\in R_{j,n-1}, \quad j\in\mathbb{Z},\ n\geqslant 1.$$

Proposition B.2 ensures that $\tilde{u}^\nu$ and $\bar{u}^\nu$ converge to u in $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$, since, for any $(x,t)\in R_{j,n-1}$, $\tilde{u}^\nu(x,t)$ and $\bar{u}^\nu(x,t)$ belong to the convex hull of $u^\nu([x_j,x_{j+1}],t_n-)$, hence we have for instance for all $(t,x)\in R_{j,n-1}$

$$|w(a^\nu(x),\tilde{u}^\nu(x,t)) - w^\nu(x,t_n-)|\leqslant \text{Tot.Var.}\{w^\nu(\cdot,t_n-),[x_j,x_{j+1}]\}.$$

Therefore we get

$$\liminf_{\nu\to+\infty} J^1_{\varepsilon,\nu}\geqslant 0. \tag{6.6}$$

The second term: $J^2_{\varepsilon,\nu}$ We first observe that the two states

$$U_j^{\nu-}(t)\doteq(a(x_j),u^\nu(x_{j+1}-,t)) \quad \text{and} \quad U_j^{\nu+}(t)\doteq(a(x_{j+1}),u^\nu(x_{j+1}+,t))$$

are connected by a zero wave. Recalling (3.6), the map $a\mapsto\phi(a;U_j^{\nu-},U_j^{\nu+})$ is a Borel function. Hence, applying Proposition 3.2 and the (absolutely continuous) change of variable $a=a(x)$ (see [25]), we obtain

$$\begin{aligned} q(a(x_{j+1}),u^\nu(x_{j+1}+,t)) - q(a(x_j),u^\nu(x_{j+1}-,t)) &= q(U_j^{\nu+}(t)) - q(U_j^{\nu-}(t)) \\ \leqslant \int_{x_j}^{x_{j+1}} [q_a+\eta'(g-f_a)](a(x),\tilde{\phi}(a(x);U_j^{\nu-}(t),U_j^{\nu+}(t)))\cdot a'(x)\,dx \\ = \int_{x_j}^{x_{j+1}} [q_a+\eta'(g-f_a)](a(x),\hat{u}^\nu(x,t))\cdot a'(x)\,dx, \end{aligned} \tag{6.7}$$

where we have defined

$$\hat{u}^\nu(x,t) = \tilde{\phi}(a(x);U_j^{\nu-}(t),U_j^{\nu+}(t)) \quad \text{for } (x,t)\in[x_j,x_{j+1})\times\mathbb{R}^+.$$

Now, we define also

$$\hat{\varphi}^\nu(x,t) = \varphi(x_{j+1},t) \quad \text{for}(x,t)\in[x_j,x_{j+1})\times\mathbb{R}^+$$

and, using (6.7), we get

$$J^2_{\varepsilon,\nu}\geqslant -\int_{\mathbb{R}\times\mathbb{R}^+} a'(x)[q_a+\eta'(g-f_a)](a(x),\hat{u}^\nu(x,t))\hat{\varphi}^\nu(x,t)\,dx\,dt.$$

We claim that $\hat{u}^\nu \to u$ in $\mathbf{L}^1_{\text{loc}}$. This, together with the fact that $\hat{\varphi}^\nu \to \varphi$ uniformly, allows us to obtain

$$\liminf_{\nu\to+\infty} J^2_{\varepsilon,\nu} \geqslant -\int_{\mathbb{R}\times\mathbb{R}^+} a'(x)[q_a + \eta'(g - f_a)](a(x), u(x,t))\varphi(x,t)\,dx\,dt. \tag{6.8}$$

It remains to prove the claim regarding $\hat{u}^\nu(x,t)$. If $x \in [x_j, x_{j+1}]$ and $a(x_j) \leqslant a(x_{j+1})$, define

$$\bar{a}^\nu(x) = \begin{cases} a(x) & \text{if } a(x_j) \leqslant a(x) \leqslant a(x_{j+1}), \\ a(x_j) & \text{if } a(x) \leqslant a(x_j), \\ a(x_{j+1}) & \text{if } a(x) \geqslant a(x_{j+1}). \end{cases}$$

The definition of $\bar{a}^\nu(x)$ if $a(x_j) \geqslant a(x_{j+1})$ is analogous. Clearly one has $||\bar{a}^\nu u||_\infty \leqslant ||a^\nu u||_\infty$ and $\bar{a}^\nu u \to a$ in $\mathbf{L}^1_{\text{loc}}(\mathbb{R})$ (note that $|\bar{a}^\nu(x) - a(x)| \leqslant |a^\nu(x) - a(x)|$). Moreover, recalling (3.6), we observe that

$$\tilde{\phi}(a(x); U_j^{\nu-}, U_j^{\nu+}) = \tilde{\phi}(\bar{a}^\nu(x); U_j^{\nu-}, U_j^{\nu+}), \quad x \in [x_j, x_{j+1}].$$

Fix now $(x,t) \in R_{j,n}$, from definition (3.6) we know that $\tilde{\phi}$, as a function of a, has at most one discontinuity point located between $a(x_j)$ and $a(x_{j+1})$. Suppose hence that $\tilde{\phi}$ is continuous on the interval with $a(x)$ and $a(x_{j+1})$ as extrema (if it is not continuous in this interval, we take the other interval which has as extrema $a(x)$ and $a(x_j)$). Hence we can compute

$$\begin{aligned} &|w(\bar{a}^\nu(x), \hat{u}^\nu(x,t)) - w^\nu(x_{j+1}+, t)| \\ &\quad = |w(\bar{a}^\nu(x), \tilde{\phi}(\bar{a}^\nu(x); U_j^{\nu-}(t), U_j^{\nu+}(t))) - w(U_j^{\nu+}(t))| \\ &\quad = |\theta(\bar{a}^\nu(x)) - \theta(\bar{a}^\nu(x_{j+1}))| \\ &\quad \leqslant |\theta(a(x_j)) - \theta(a(x_{j+1}))| = \text{Tot.Var.}\{\theta^\nu, [x_j, x_{j+1}]\} \end{aligned} \tag{6.9}$$

since along a simple standing wave one has $|\Delta z| = 0$ and hence $|\Delta w| = |\Delta\theta|$. The last inequality shows that we can apply Proposition B.2 and obtain that $\hat{u}^\nu \to u$ in $\mathbf{L}^1_{\text{loc}}$ as $\nu \to +\infty$.

Concluding the estimates: Now, putting (6.8), (6.6) and (6.5) into (6.3), we obtain

$$\begin{aligned} I_\varepsilon = \lim_{\nu\to+\infty} I_{\varepsilon,\nu} &\geqslant \liminf_{\nu\to+\infty} J^1_{\varepsilon,\nu} + \liminf_{\nu\to+\infty} J^2_{\varepsilon,\nu} \\ &\quad + \int_{\mathbb{R}\times\mathbb{R}^+} \eta'(u)[g(a,u) - f_a(a,k)]a'\varphi\,dx\,dt \\ &\geqslant \int_{\mathbb{R}\times\mathbb{R}^+} \{\eta'(u)[f_a(a,u) - f_a(a,k)] - q_a(a,u)\}a'\varphi\,dx\,dt. \end{aligned} \tag{6.10}$$

But we have the inequality:

$$|\eta'(u)[f_a(a,u) - f_a(a,k)] - q_a(a,u)| = \left| \int_k^u [\eta'(u) - \eta'(\xi)] f_{au}(a,\xi)\, d\xi \right|$$
$$\leqslant 2\varepsilon ||E||_{\mathbf{C}^1} ||f||_{\mathbf{C}^2} \tag{6.11}$$

because $\eta'(u) = \eta'(\xi)$ for $u, \xi \notin [k-\varepsilon, k+\varepsilon]$. Therefore the following inequality holds:

$$I_\varepsilon \geqslant -2\varepsilon ||E||_{\mathbf{C}^1} ||f||_{\mathbf{C}^2} \int_{\mathbb{R}\times\mathbb{R}^+} |a'| \varphi \, dx\, dt.$$

which implies

$$\liminf_{\varepsilon\to 0} I_\varepsilon \geqslant 0.$$

Finally, the integrand in (6.3) converges pointwise as $\varepsilon \to 0$, hence by the dominated convergence theorem we can pass to the limit and complete the proof of Theorem 6.1. □

7. Uniqueness for $a \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$

We are now in the position to study uniqueness and stability for (1.4), in the spirit of Kružkov ([16, Theorems 1 and 2]. To apply directly these results, the coefficient a should be assumed to be more regular, $a \in \mathbf{C}^1(\mathbb{R})$. However, by refining the proof, the estimate for uniqueness can be recovered for $a \in W^{1,1}_{\mathrm{loc}}$ in the case of $a' g_u(a,u)$ bounded from above. Notice that in this case, a' has no atoms; hence the product $a' g(a,u)(.,t)$ is a well-defined $\mathbf{L}^1(\mathbb{R})$ function for $t>0$ (cf. with [11,17,29]).

Theorem 7.1. *Assume that f, g are smooth. Let $M, R>0$ be two positive constants; let $a \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ satisfy*

$$\gamma \doteq \sup\{a'(x) g_u(a(x), w) : x \in [-R, R], |w| \leqslant M\} < +\infty \tag{7.1}$$

and define

$$L \doteq \sup\{f_u(a(x), w) : x \in [-R, R], |w| \leqslant M\}. \tag{7.2}$$

If $u(t,v)$, $v(t,x) \in \mathbf{L}^\infty(\mathbb{R} \times [0, +\infty))$ are such that

- $||u||_\infty, ||v||_\infty \leqslant M$;
- *they satisfy* (6.1) *for all $k \in \mathbb{R}$ and $\varphi \in \mathbf{C}^1_c(\mathbb{R} \times (0, +\infty))$, $\varphi \geqslant 0$;*
- *$t \mapsto u(\cdot, t)$, $t \mapsto v(\cdot, t)$ are continuous in $\mathbf{L}^1_{\mathrm{loc}}$;*

then for any $t \in [0, R/L]$ *the Kružkov estimate holds*:

$$\int_{-R+Lt}^{R-Lt} |u(x,t) - v(x,t)|\, dx \leqslant e^{\gamma t} \int_{-R}^{R} |u(x,0) - v(x,0)|\, dx. \tag{7.3}$$

Proof. We assume the reader familiar with the proof and notations of [16, Theorem 1]. Let $\phi \geqslant 0$ be $\mathbf{C}^1$ with compact support in $\mathbb{R} \times (0, +\infty)$; the proof aims at deriving the inequality

$$\begin{aligned} 0 \leqslant \int_{\mathbb{R}\times(0,+\infty)} & \{|u(x,t) - v(x,t)|\phi_t(x,t) + \operatorname{sgn}[u(x,t) - v(x,t)] \\ & \cdot [f(a(x), u(x,t)) - f(a(x), v(x,t))]\phi_x(x,t) - \operatorname{sgn}[u(x,t) - v(x,t)] \\ & \cdot [g(a(x), v(x,t)) - g(a(x), u(x,t))] \cdot a'(x)\phi(x,t)\}\, dx\, dt. \end{aligned} \tag{7.4}$$

From (7.4), we use (7.1) to get a Gronwall-type estimate and obtain (7.3) following exactly [16], since those computations are not affected by the lower regularity of a'. In the rest of the proof, assumption (7.1) is not needed.

1. Let $N, T > 0$ be two constants such that the support of ϕ is contained in $\Omega \doteq (-N, N) \times (0, T)$. Define the $\mathbf{C}^1$ function φ as

$$\varphi(x,t;y,\tau) = \phi\Big(\frac{x+y}{2}, \frac{t+\tau}{2}\Big)\lambda_h\Big(\frac{x-y}{2}, \frac{t-\tau}{2}\Big), \tag{7.5}$$

where $\lambda_h(a,b) = \delta_h(a) \cdot \delta_h(b)$,

$$\delta_h(a) = \frac{1}{h}\delta\Big(\frac{a}{h}\Big), \quad \delta : \mathbb{R} \to [0,1] \in \mathbf{C}^\infty, \int_{-\infty}^{\infty} \delta(x)\, dx = 1, \delta(x) = 0 \quad \forall x \in [-1,1].$$

Observe that if h is sufficiently small, φ has compact support contained in the open set $\mathscr{G} = \Omega \times \Omega$.

The analogous to inequality (3.4) in [16] is given by

$$\sum_{i=1}^{4} \int_{\mathscr{G}} P_i^h(x,t;y,\tau)\ dx\, dt\, dy\, d\tau \geqslant 0 \tag{7.6}$$

with

$$\begin{aligned} P_i^h(x,t;y,\tau) = {} & F_i(x,t;y,\tau;u(x,t),v(y,\tau)) \\ & \times \lambda_h\Big(\frac{x-y}{2}, \frac{t-\tau}{2}\Big), \quad i = 1,2,4 \end{aligned} \tag{7.7}$$

(the term P_3^h will be considered later on) and

$$F_1(x,t;y,\tau;u,v) = |u-v|\phi_t\Big(\frac{x+y}{2},\frac{t+\tau}{2}\Big),$$
$$F_2(x,t;y,\tau;u,v) = \mathrm{sgn}[u-v][f(a(x),u)-f(a(y),v)]\phi_x\Big(\frac{x+y}{2},\frac{t+\tau}{2}\Big),$$
$$F_4(x,t;y,\tau;u,v) = \mathrm{sgn}[u-v][g(a(x),u)a'(x)-g(a(y),v)a'(y)]\phi\Big(\frac{x+y}{2},\frac{t+\tau}{2}\Big).$$

Now, if $|u|,|v_1|,|v_2|\leqslant M$, then easy computations show that there exists a constant $C>0$ depending on f,g,ϕ,M and $||a||_{\mathbf{L}^\infty((-N,N))}$ such that for $i=1,2,4$ we have

$$\begin{aligned}&|F_i(x,t;y,\tau;u,v_1)-F_i(x,t;x,t;u,v_2)|\\ &\quad\leqslant C[(1+|a'(x)|+|a'(y)|)\cdot(|t-\tau|+|x-y|+|a(x)-a(y)|+|v_1-v_2|)\\ &\qquad+|a'(x)-a'(y)|]\doteq F(x,t;y,\tau;v_1,v_2).\end{aligned}$$

2. We claim that

$$\lim_{h\to 0}\int_{\mathscr{G}} F(x,t;y,\tau;v(y,\tau),v(x,t))\lambda_h\Big(\frac{x-y}{2},\frac{t-\tau}{2}\Big)\,dx\,dt\,dy\,d\tau=0. \tag{7.8}$$

If (7.8) holds, following [16] we obtain for $i=1,2,4$,

$$\lim_{h\to 0}\int_{\mathscr{G}} P_i^h(x,t;y,\tau)\,dx\,dt\,dy\,d\tau = 4\int_\Omega dx\,dtF_i(x,t;x,t;u(x,t),v(x,t)). \tag{7.9}$$

To prove (7.8), we make the change of variables:

$$\begin{cases}\xi=x,\\ s=t,\\ \eta=(x-y)/2,\\ \sigma=(t-\tau)/2,\end{cases}\qquad\begin{cases}x=\xi,\\ t=s,\\ y=\xi-2\eta,\\ \tau=s-2\sigma.\end{cases} \tag{7.10}$$

Because of the presence of λ_h in integral (7.8), in the new variables, integral is restricted to the set $(\xi,s;\eta,\sigma)\in\Omega\times[-h,h]\times[-h,h]$. Now we can analyze the various terms in (7.8):

- The continuity of the translations in $\mathbf{L}^1$ implies:

$$\begin{aligned}&\int_\Omega d\xi\,ds\int_{-h}^h d\eta\int_{-h}^h d\sigma|a'(\xi)-a'(\xi-2\eta)|\lambda_h(\eta,\sigma)\\ &\quad=\int_{-h}^h d\eta\delta_h(\eta)||a'(\cdot)-a'(\cdot-2\eta)||_{\mathbf{L}^1((-N,N))}\overset{h\to 0}{\longrightarrow}0.\end{aligned}$$

- The uniform continuity of a implies

$$\begin{aligned}
&\int_\Omega d\xi\, ds \int_{-h}^{h} d\eta \int_{-h}^{h} d\sigma (1+|a'(\xi)|+|a'(\xi-2\eta)|)\\
&\quad\cdot (|2\sigma|+|2\eta|+|a(\xi)-a(\xi-2\eta)|)\lambda_h(\eta,\sigma)\\
&\quad\leqslant 2T\int_{-N-h}^{N+h}(1+2|a'(\xi)|)\,d\xi\cdot\left(4h+\sup_{|\xi|\leqslant N,\ |\eta|\leqslant h}|a(\xi)-a(\xi-2\eta)|\right)\\
&\quad\xrightarrow{h\to 0} 0.
\end{aligned}$$

- Concerning the last term

$$\begin{aligned}
\int_\Omega d\xi\, ds \int_{-h}^{h} d\eta \int_{-h}^{h} d\sigma (1+|a'(\xi)|+|a'(\xi-2\eta)|)\\
\times |v(\xi,s)-v(\xi-2\eta,s-2\sigma)|\lambda_h(\eta,\sigma),
\end{aligned}$$

simple changes of variables show that it can be written as the sum of integrals of the form

$$\int_{-h}^{h} d\eta \int_{-h}^{h} d\sigma \delta_h(\eta)\delta_h(\sigma)\int_{\mathbb{R}^2}\psi(\xi,s)|w(\xi,s)-w(\xi-2\eta,s-2\sigma)|\,d\xi\,ds$$

with $\psi\in\mathbf{L}^1(\mathbb{R}^2)$ and $w\in\mathbf{L}^\infty(\mathbb{R}^2)$. The continuity of the translations in $\mathbf{L}^1_{\mathrm{loc}}$ ensures that there exists a sub-sequence $(\eta_i,\sigma_i)\to(0,0)$ such that $w(\xi-2\eta_i,s-2\sigma_i)\to w(\xi,s)$ a.e. $(\xi,s)\in\mathbb{R}^2$. Hence the dominated convergence theorem and the uniqueness of the real limit $\ell=0$ imply

$$\lim_{(\eta,\sigma)\to(0,0)}\int_{\mathbb{R}^2}\psi(\xi,s)|w(\xi,s)-w(\xi-2\eta,s-2\sigma)|\,d\xi\,ds\to\ell=0. \tag{7.11}$$

Therefore also the last term in (7.8) tends to zero as $h\to 0$.

3. Now we consider the term P_3^h and want to show that

$$\int_{\mathscr{G}} P_3^h(x,t;y,\tau)\,dx\,dt\,dy\,d\tau \xrightarrow{h\to 0} 0; \tag{7.12}$$

this fact, together with (7.6) and (7.9), gives finally (7.4). The term P_3^h can be written as (see (3.4) in [16])

$$\begin{aligned}
&P_3^h(x,t;y,\tau)\\
&\quad = \operatorname{sgn}[u(x,t)-v(y,\tau)]\cdot\frac{\partial}{\partial x}\{\varphi(x,t;y,\tau)[f(a(y),v(y,\tau))-f(a(x),v(y,\tau))]\}\\
&\qquad + \operatorname{sgn}[u(x,t)-v(y,\tau)]\cdot\frac{\partial}{\partial y}\{\varphi(x,t;y,\tau)[f(a(y),u(x,t))-f(a(x),u(x,t))]\}.
\end{aligned}$$

Consider now the two functions

$$\begin{aligned}
Q_1^h(x,t;y,\tau) \doteq{}& \operatorname{sgn}[u(x,t)-v(y,\tau)]\phi\left(\frac{x+y}{2},\frac{t+\tau}{2}\right)\\
&\times[f_a(a(x),u(x,t))-f_a(a(x),v(y,\tau))]\frac{\partial}{\partial y}\\
&\times\left[[a(y)-a(x)]\lambda_h\left(\frac{x-y}{2},\frac{t-\tau}{2}\right)\right],
\end{aligned}$$

$$\begin{aligned}
Q_2^h(x,t;y,\tau) \doteq{}& \operatorname{sgn}[u(x,t)-v(x,t)]\phi(x,t)\\
&\times[f_a(a(x),u(x,t))-f_a(a(x),v(x,t))]\frac{\partial}{\partial y}\\
&\times\left[[a(y)-a(x)]\lambda_h\left(\frac{x-y}{2},\frac{t-\tau}{2}\right)\right].
\end{aligned}$$

Since, for h sufficiently small, Q_2^h is a total derivative (with respect to the variable y) of a function with compact support in $\mathscr{G}$, then one has

$$\int_{\mathscr{G}} Q_2^h(x,t;y,\tau)\,dx\,dt\,dy\,d\tau = 0. \tag{7.13}$$

Now we can estimate

$$\begin{aligned}
|Q_1^h(x,t;y,\tau)-Q_2^h(x,t;y,\tau)| \leqslant{}& C_1[h+|v(y,\tau)-v(x,t)|]\\
&\times\left[|a'(y)|\lambda_h\left(\frac{x-y}{2},\frac{t-\tau}{2}\right)+|a(y)-a(x)|\right.\\
&\left.\times\,\delta_h\left(\frac{t-\tau}{2}\right)\frac{1}{h^2}\mathbf{1}_{[-h,h]}\left(\frac{x-y}{2}\right)\right],
\end{aligned}$$

where $C_1>0$ depends only on $f,g,\phi,M,||a||_{\mathbf{L}^\infty((-N,N))}$ and $||\delta||_{\mathbf{C}^1}$ but not on h. We have already seen that the integral of the term which has λ_h as a factor tends to zero as $h\to 0$. The other two terms can be analyzed with the change of variables (7.10):

- Using the Fubini theorem we compute:

$$\begin{aligned}
&\int_{\mathscr{G}} h|a(y)-a(x)|\delta_h\left(\frac{t-\tau}{2}\right)\frac{1}{h^2}\mathbf{1}_{[-h,h]}\left(\frac{x-y}{2}\right)dx\,dt\,dy\,d\tau\\
&\leqslant 4\int_{\Omega} d\xi\,ds\int_{-h}^{h}d\eta\int_{-h}^{h}d\sigma\delta_h(\sigma)\frac{1}{h}\left|\int_{\xi}^{\xi-2\eta}a'(z)\,dz\right|\\
&\leqslant \frac{4T}{h}\int_{-N}^{N}d\xi\int_{-h}^{h}d\eta\int_{\xi-2h}^{\xi+2h}|a'(z)|\,dz\\
&\leqslant 8T\int_{-N}^{N}d\xi\int_{-2h}^{2h}|a'(\xi+z)|\,dz\leqslant 32hT\int_{-N-h}^{N+h}|a'(\xi)|\,d\xi\overset{h\to 0}{\longrightarrow}0.
\end{aligned}$$

- Using the $\mathbf{L}^1$ continuity of translations, (7.11) and again the Fubini theorem we get

$$\begin{aligned}
&\int_{\mathscr{G}}|v(x,t)-v(y,\tau)|\cdot|a(y)-a(x)|\delta_h\left(\frac{t-\tau}{2}\right)\frac{1}{h^2}\mathbf{1}_{[-h,h]}\left(\frac{x-y}{2}\right)dx\,dt\,dy\,d\tau\\
&\leqslant\frac{4}{h^2}\int_{\Omega}d\xi\,ds\int_{-h}^{h}d\eta\int_{-h}^{h}d\sigma\delta_h(\sigma)|v(\xi,s)-v(\xi-2\eta,s-2\sigma)|\int_{-2h}^{2h}|a'(\xi+z)|\,dz\\
&\leqslant\frac{1}{h}\int_{-h}^{h}d\eta\int_{-h}^{h}d\sigma\delta_h(\sigma)\int_{\mathbb{R}^2}d\xi\,ds\mathbf{1}_{\Omega}(\xi,s)|a'(\xi)|\cdot|v(\xi,s)-v(\xi-2\eta,s-2\sigma)|\\
&\quad+\frac{8MT}{h}\int_{-N}^{N}d\xi\int_{-2h}^{2h}dz|a'(\xi+z)-a'(\xi)|\overset{h\to 0}{\longrightarrow}0.
\end{aligned}$$

Therefore we have proved:

$$\int_{\mathscr{G}}|Q_1^h(x,t;y,\tau)-Q_2^h(x,t;y,\tau)|\,dx\,dt\,dy\,d\tau\overset{h\to 0}{\longrightarrow}0, \tag{7.14}$$

which, recalling (7.13), implies that

$$\int_{\mathscr{G}}Q_1^h(x,t;y,\tau)\,dx\,dt\,dy\,d\tau\overset{h\to 0}{\longrightarrow}0. \tag{7.15}$$

With the help of the inequality

$$\begin{aligned}
&|f(a_2,u)-f(a_1,u)-[f(a_2,v)-f(a_1,v)]-[f_a(a_1,u)-f_a(a_1,v)]\cdot[a_2-a_1]|\\
&\leqslant||f||_{\mathbf{C}^3}|a_2-a_1|^2\cdot|u-v|
\end{aligned}$$

and carrying out the derivatives in the definitions of P_3^h, Q_1^h we compute

$$
\begin{aligned}
&|P_3^h(x,t;y,\tau) - Q_1^h(x,t;y,\tau)| \\
&\quad \leqslant C_2 \left\{ [(1+|a'(y)|)|a(y)-a(x)| + |a'(y)-a'(x)|] \cdot \lambda_h\left(\frac{x-y}{2}, \frac{t-\tau}{2}\right) \right. \\
&\qquad \left. + |a(x)-a(y)|^2 \cdot \delta_h\left(\frac{t-\tau}{2}\right) \frac{1}{h^2} \mathbf{1}_{[-h,h]}\left(\frac{x-y}{2}\right) \right\},
\end{aligned}
\tag{7.16}
$$

where, as usual the constant $C_2 > 0$ does not depend on h. We have already proved that the integral of the first term in the right-hand side of (7.16) tends to zero as $h \to 0$. Concerning the last term we compute

$$
\begin{aligned}
&\int_{\mathscr{G}} |a(x)-a(y)|^2 \cdot \delta_h\left(\frac{t-\tau}{2}\right) \frac{1}{h^2} \mathbf{1}_{[-h,h]}\left(\frac{x-y}{2}\right) dx\, dt\, dy\, d\tau \\
&\quad = \frac{4T}{h^2} \int_{-N}^{N} d\xi \int_{-h}^{h} d\eta |a(\xi) - a(\xi - 2\eta)|^2 \\
&\quad \leqslant \frac{4T}{h^2} \sup_{|\xi| \leqslant N,\, |\eta| \leqslant h} |a(\xi) - a(\xi - 2\eta)| \cdot \int_{-N}^{N} d\xi \int_{-h}^{h} d\eta \int_{-2h}^{2h} dz |a'(\xi + z)| \\
&\quad \leqslant 4T \int_{-N-h}^{N+h} d\xi |a'(\xi)| \cdot \sup_{|\xi| \leqslant N,\, |\eta| \leqslant h} |a(\xi) - a(\xi - 2\eta)| \xrightarrow{h \to 0} 0.
\end{aligned}
$$

Therefore, recalling (7.15), we get (7.12), completing the proof. □

8. Concluding remarks

We close this paper with some final remarks.

1. When the coefficient a is discontinuous, by means of Theorem 5.1 we deduce the existence of a limit function that we can regard as a solution, in some sense, of (1.1), (1.2); however, this case is not covered by our Kružkov-type results, hence we cannot obtain uniqueness.

2. Condition (7.1) in Theorem 7.1 is necessary to have the bare uniqueness of the entropic solutions, as can be seen by the following Cauchy problem:

$$
\begin{cases}
u_t + \left(\dfrac{u^2}{2}\right)_x = \dfrac{u}{\sqrt{4x}} \cdot \mathbf{1}_{(0,1)}(x), \\
u(0,x) = 0
\end{cases}
\tag{8.1}
$$

which has the following two entropic solutions for $(x,t)\in\mathbb{R}\times[0,3]$:

$$u_1(x,t)=0,\quad u_2(x,t)=\begin{cases}0 & \text{for } x\notin\left(0,\dfrac{t^2}{16}\right),\\ \sqrt{x} & \text{for } x\in\left(0,\dfrac{t^2}{16}\right).\end{cases}$$

This example is very simple but does not satisfy our condition (P_4). To obtain a counterexample satisfying all our conditions (P_1)–(P_5), it is enough to substitute u in the source with $1+u$ which does not vanishes on the resonance point $u=0$. Then for the initial data $u(0,x)=1-2\cdot\mathbf{1}_{(0,+\infty)}(x)$ there are again two entropic solutions. We do not describe in details this counterexample since it only requires some straightforward technicalities.

3. Another way to find a solution is to approximate a by smooth functions a^ε which also satisfy (5.1) with a uniform upper bound on the total variation of the transformed variables. The corresponding solutions u^ε are compact in the sense described in our paper, hence we can extract a converging sub-sequence to a "solution" u. If moreover $a_x g_u(a,w)$ is bounded from above and we can choose an approximating sequence a^ε with $a^\varepsilon_x g_u(a^\varepsilon,w)$ uniformly bounded from above, then (7.3) holds for the approximate solutions with the same ε. Therefore with a diagonalization argument (as described for instance in [1]) one can show the existence of a Lipschitz (with respect to the initial data) semigroup of "solutions" for a discontinuous a. But this implies neither uniqueness for the semigroup nor for the entropy solutions. To have uniqueness one should also characterize the semigroup's trajectories, as done in [1].

Appendix A. Interaction estimates: technical proofs

Proof of Theorem 4.1. Suppose first $z_1z_2>0$. If γ' is the segment joining ψ_1 and ψ_2 one has $\ell_*(\gamma')=|\psi_1-\psi_2|_*$ hence $d_*(\psi_1,\psi_2)\leqslant|\psi_1-\psi_2|_*$. For the converse inequality take $\gamma\in\mathscr{G}(\psi_1,\psi_2)$ and a parametrization $\psi_\gamma(t)=(z_\gamma(t),\theta_\gamma(t))$. We call $\bar\gamma\in\mathscr{G}(\psi_1,\psi_2)$ the curve which has as a parametrization the function $\psi_{\bar\gamma}(t)=(z_{\bar\gamma}(t),\theta_\gamma(t))$ where $z_{\bar\gamma}(t)$ is defined by

$$z_{\bar\gamma}(t)=\begin{cases}z_\gamma(t) & \text{if } z_\gamma(t)z_1>0,\\ -z_\gamma(t) & \text{if } z_\gamma(t)z_1<0.\end{cases}$$

Clearly one has $\ell_*(\gamma)=\ell_*(\bar\gamma)$ and since $\bar\gamma$ is entirely contained in $\mathscr{R}^+$ or $\mathscr{R}^-$, the length $\ell_*(\bar\gamma)$ is greater or equal then $|\psi_1-\psi_2|_*$, proving the first equality in (4.7).

Concerning the other equality, suppose that $\psi_1\in\mathscr{R}^+$ and $\psi_2\in\mathscr{R}^-$, the other case being similar. Take $\gamma\in\mathscr{G}(\psi_1,\psi_2)$, one must have $\gamma\cap\tilde{\mathscr{T}}\neq\emptyset$ (we use the Greek letter γ also to denote the support of the curve). Hence take $\bar\psi=(1-\bar\theta,\bar\theta)\in\gamma\cap\tilde{\mathscr{T}}$ and

$\gamma_1 \in \mathscr{G}(\psi_1, \bar{\psi})$ and $\gamma_2 \in \mathscr{G}(\bar{\psi}, \psi_2)$ such that $\gamma = \gamma_1 \cup \gamma_2$. Applying the first in (4.7) we get

$$\ell_*(\gamma) = \ell_*(\gamma_1) + \ell_*(\gamma_2) \geqslant d_*(\psi_1, \bar{\psi}) + d_*(\bar{\psi}, \psi_2) = |\psi_1 - \bar{\psi}^+|_* + |\bar{\psi}^- - \psi_2|_*.$$

Therefore we can write

$$\begin{aligned}\ell_*(\gamma) &\geqslant |\psi_1 - \bar{\psi}^+|_* + |\bar{\psi}^- - \psi_2|_* \\ &= |z_1 - (1 - \bar{\theta})| + |\theta_1 - \bar{\theta}| + | - 1 + \bar{\theta} - z_2| + |\bar{\theta} - \theta_2| \doteq h(\bar{\theta}).\end{aligned}$$

Observe that $h(\theta)$ is continuous, piecewise linear; since $\theta_1 \geqslant 1 - z_1$ and $\theta_2 \geqslant 1 + z_2$, the function $h(\theta)$ is minimized on an interval which has $\theta_m = \min(\theta_1, \theta_2)$ as an extremum. So we have

$$\ell_*(\gamma) \geqslant h(\theta_m) = d_*(\psi_1, \psi_m) + d_*(\psi_m, \psi_2)$$

which yields the desired equality. Finally, let us prove (4.8). If we take a $\mathbf{C}^1$ function $(z(t), \theta(t))$, by (2.19) we easily find

$$\tfrac{1}{2}(|\theta'(t)| + |w'(t)|) \leqslant |z'(t)| + |\theta'(t)| \leqslant 2(|w'(t)| + |\theta'(t)|).$$

Then, using the definition of d_* and simple inequalities, we get (4.8) $\square$

Proof of Theorem 4.2. (a) Take $\theta_m = \min(\theta_1, \theta_2)$ and set $\psi_m = (1 - \theta_m, \theta_m)$. Since $\theta_m \leqslant \theta_1$ and $\theta_m \leqslant \theta_2$, then the equalities $\Lambda^-(\psi_1, \psi_m) = d_*(\psi_1, \psi_m)$ and $\Lambda^+(\psi_m, \psi_2) = d_*(\psi_m, \psi_2)$ hold. Hence applying (4.7) we get

$$\begin{aligned}\Lambda(\psi_1, \psi_2) \leqslant \Lambda(\psi_1, \psi_m) + \Lambda(\psi_m, \psi_2) &\leqslant \Lambda^-(\psi_1, \psi_m) + \Lambda^+(\psi_m, \psi_2) \\ &= d_*(\psi_1, \psi_m) + d_*(\psi_m, \psi_2) \\ &= d_*(\psi_1, \psi_2) \qquad \text{(A.1)}\end{aligned}$$

which, together with (4.5), proves (4.9).

(b) If $\theta_2 \geqslant \theta_1$, (4.10) follows from the equality $d_*(\psi_1, \psi_2) = \Lambda^+(\psi_1, \psi_2)$.

Take now $\theta_2 < \theta_1$ and hence $1 - \theta_1 \leqslant \min(z_1, z_2)$. Then take a path $\gamma \in \mathscr{G}(\psi_1, \psi_2)$. If $\gamma \subset \mathscr{R}^+$, then $\mathscr{F}(\gamma) \geqslant \Lambda^+(\psi_1, \psi_2)$. If instead $\gamma \not\subset \mathscr{R}^+$ then there exists a state $\bar{\psi} = (1 - \bar{\theta}, \bar{\theta}) \in \tilde{\mathscr{T}} \cap \gamma$ and two curves $\gamma_1 \in \mathscr{G}(\psi_1, \bar{\psi})$ and $\gamma_2 \in \mathscr{G}(\bar{\psi}, \psi_2)$ satisfying $\gamma = \gamma_1 \cup \gamma_2$ and $\gamma_1 \subset \mathscr{R}^+$. Since $\bar{\psi} \in \tilde{\mathscr{T}} \subset \mathscr{R}^-$ we can apply (4.9) to obtain:

$$\mathscr{F}(\gamma) = \mathscr{F}(\gamma_1) + \mathscr{F}(\gamma_2) \geqslant \Lambda^+(\psi_1, \bar{\psi}) + \Lambda(\bar{\psi}, \psi_2) = \Lambda^+(\psi_1, \bar{\psi}) + d_*(\bar{\psi}, \psi_2) = h(\bar{\theta}),$$

where the function h is defined by

$$h(\theta) = |z_1 - (1 - \theta)| + 2|\theta_1 - \theta| + \theta_1 - \theta + |1 - \theta - z_2| + |\theta - \theta_2|.$$

The function h is minimized on an interval which has θ_1 as an extremum, hence we can write

$$\begin{aligned}\mathscr{F}(\gamma) \geqslant h(\theta_1) &= z_1 - (1-\theta_1) + z_2 - (1-\theta_1) + \theta_1 - \theta_2 \\ &= |z_1 - z_2| + 2\min_{i=1,2}\{z_i - (1-\theta_1)\} + \theta_1 - \theta_2. \qquad \text{(A.2)}\end{aligned}$$

We have two cases:

- $1 - z_1 - \theta_2 \leqslant 0$: then

  $$z_1 - (1-\theta_1) \geqslant \theta_1 - \theta_2, \quad z_2 - (1-\theta_1) \geqslant 1 - \theta_2 - (1-\theta_1) = \theta_1 - \theta_2$$

  which implies

  $$\mathscr{F}(\gamma) \geqslant |z_1 - z_2| + 3|\theta_1 - \theta_2| = \Lambda^+(\psi_1, \psi_2).$$

- $1 - z_1 - \theta_2 > 0$: then $z_2 \geqslant 1 - \theta_2 > z_1$ and

  $$\min_{i=1,2}\{z_i - (1-\theta_1)\} = z_1 - (1-\theta_1) = \theta_1 - \theta_2 + z_1 - 1 + \theta_2$$

  which implies

  $$\mathscr{F}(\gamma) \geqslant |z_1 - z_2| + 3|\theta_1 - \theta_2| - 2[1 - z_1 - \theta_2] = \Lambda^+(\psi_1, \psi_2) - 2[1 - z_1 - \theta_2].$$

Putting together all these inequalities we have for any $\gamma \in \mathscr{G}(\psi_1, \psi_2)$:

$$\mathscr{F}(\gamma) \geqslant \begin{cases} \Lambda^+(\psi_1, \psi_2) & \text{for } 1 - z_1 - \theta_2 \leqslant 0, \\ \Lambda^+(\psi_1, \psi_2) - 2[1 - z_1 - \theta_2] & \text{for } 1 - z_1 - \theta_2 > 0. \end{cases} \qquad \text{(A.3)}$$

Finally (4.10) is obtained observing that there exists a path $\bar{\gamma} \in \mathscr{G}(\psi_1, \psi_2)$ (i.e. the solution of the Riemann problem described in Section 3 for the cases $\psi_1 \in \mathscr{R}^+$, $\psi_2 \in R_1(\psi_1)$ and $\psi_2 \in R_5(\psi_1)$) for which inequality (A.3) becomes actually an equality.

(c) The proof of this point can be carried out in exactly the same way than point (b).

(d) Take $\gamma \in \mathscr{G}(\psi_1, \psi_2)$, the hypothesis implies that there exists a state $\bar{\psi} = (1 - \bar{\theta}, \bar{\theta}) \in \tilde{\mathscr{T}} \cap \gamma$ and two curves $\gamma_1 \in \mathscr{G}(\psi_1, \bar{\psi})$ and $\gamma_2 \in \mathscr{G}(\bar{\psi}, \psi_2)$ satisfying $\gamma = \gamma_1 \cup \gamma_2$. Hence we obtain

$$\mathscr{F}(\gamma) = \mathscr{F}(\gamma_1) + \mathscr{F}(\gamma_2) \geqslant \Lambda(\psi_1, \bar{\psi}) + \Lambda(\bar{\psi}, \psi_2) = h(\bar{\theta}).$$

Since $\psi_1, \bar{\psi} \in \mathscr{R}^+$ and $\bar{\psi}, \psi_2 \in \mathscr{R}^-$ we can apply (4.10) and (4.11) to explicit the function h:

$$\begin{aligned} h(\theta) = {} & |z_1 - (1-\theta)| + 2|\theta_1 - \theta| + \theta_1 - \theta - 2(1 - z_1 - \theta)\Theta[1 - z_1 - \theta] \\ & + |\theta - 1 - z_2| + 2|\theta - \theta_2| + \theta_2 - \theta - 2(1 + z_2 - \theta)\Theta[1 + z_2 - \theta], \qquad \text{(A.4)} \end{aligned}$$

where we have denoted by Θ the Heaviside function. Using the identity $2x\Theta(x) = x + |x|$ for all $x \in \mathbb{R}$, we find that $h(\theta) = \text{const.} + 2|\theta - \theta_1| + 2|\theta - \theta_2|$, which is minimized for $\theta \in [\theta_m, \theta_M]$. Therefore for any $\gamma \in \mathscr{G}(\psi_1, \bar{\psi})$, $\psi = (1-\theta, \theta)$ with $\theta \in [\theta_m, \theta_M]$, we have

$$\mathscr{F}(\gamma) \geqslant h(\theta) = \Lambda(\psi_1, \psi) + \Lambda(\psi, \psi_2)$$

which proves (4.12). □

Appendix B. Technicalities for the Godunov scheme

Proposition B.1. *Let $(X, \mathscr{S}, \mu)$ be a measure space, let $F : K \subset \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function with K compact and let a sequence $f_v : X \to \mathbb{R}^n$ of integrable functions satisfy*:

- *$f_v \to f$ in $\mathbf{L}^1(X, \mathbb{R}^n)$*;
- *$f_v(x) \in K$ a.e. on X*;

then, if $\alpha \in \mathbf{L}^1(X, \mathbb{R})$, one has that $f(x) \in K$ a.e. on X and $\alpha F(f_v) \to \alpha F(f)$ in $\mathbf{L}^1(X, \mathbb{R}^m)$.

Proof. Take a sub-sequence $\alpha F(f_{v_i})$, we can extract a sub-sequence $f_{v_{i_k}}$ converging almost everywhere to f, hence $f(x) \in K$ a.e. $x \in X$. Moreover the continuity of F implies that $\alpha F(f_{v_{i_k}})$ converges almost everywhere to $\alpha F(f)$. Since $F(K)$ is compact, $|F(f_{v_{i_k}})|$ is uniformly bounded a.e. by a constant R, hence $|\alpha F(f_{v_{i_k}})| \leqslant R|\alpha|$ a.e. and the dominated convergence theorem ensures that $\alpha F(f_{v_{i_k}}) \to \alpha F(f)$ in $\mathbf{L}^1(X, \mathbb{R}^m)$. The arbitrariness of the choice of the initial sub-sequence concludes the proof. □

Proposition B.2. *In the assumptions of Theorem* 5.1, *consider a sequence (a^v, u^v) that converges to (a_0, u) in $\mathbf{L}^1_{\text{loc}}$ as $v \to +\infty$, and let $(w^v, \theta^v) = \boldsymbol{\Phi}(a^v, u^v)$.*

Let $(\bar{a}^v(x), \bar{u}^v(x,t))$ be a sequence of measurable functions, such that

(a) *$\bar{a}^v \to a_0$ in $\mathbf{L}^1_{\text{loc}}$ as $v \to +\infty$,*
(b) Tot.Var.$\{\theta(\bar{a}^v)\} \leqslant C \cdot$ Tot.Var.$\{\theta_0\}$, *for all v,*

(c) *there exists a positive integer* N *such that, for any* $\nu \in \mathbb{N}$, $j \in \mathbb{Z}$, $n \in \mathbb{N}$ *and for all* $(x,t) \in R_{j,n}$, *one has*

$$\inf_{(y,s) \in [x_{j-N}, x_{j+N}] \times [t_n, t_{n+1}]} |w(\bar{a}^\nu(x), \bar{u}^\nu(x,t)) - w^\nu(y,s)| \leqslant C \,\mathrm{Tot.Var.}\{(w^\nu, \theta^\nu)(\cdot, \bar{t}_n), [x_{j-N}, x_{j+N}]\},$$

where $\bar{t}_n = \frac{t_n + t_{n+1}}{2}$,

for a suitable constant $C > 0$. *Then*

$$\bar{u}^\nu \to u \quad in \quad \mathbf{L}^1_{\mathrm{loc}} \quad as \quad \nu \to +\infty.$$

Proof. In the proof of Theorem 5.1 we showed that $w^\nu(x,t) \to w(x,t) \doteq w(a_0(x), u(x,t))$ in $\mathbf{L}^1_{\mathrm{loc}}$. Now we prove the same property for $\bar{w}^\nu(x,t) \doteq w(\bar{a}^\nu(x), \bar{u}^\nu(x,t))$.

Let $K \subset \mathbb{R} \times [0,T]$ be a compact set and compute

$$\int_K |\bar{w}^\nu(x,t) - w^\nu(x,t)|\, dx\, dt = \sum_{j \in \mathbb{Z}} \sum_{n \geqslant 0} \int_{K \cap R_{j,n}} |\bar{w}^\nu(x,t) - w^\nu(x,t)|\, dx\, dt. \tag{B.1}$$

Fix $\delta > 0$, for all $(x,t) \in R_{j,n}$, by hypothesis, there exists (y^*, s^*) (possibly depending on (x,t)) such that $|y^* - x_j| \leqslant N \Delta x_\nu$, $s^* \in [t_n, t_{n+1})$ and satisfying:

$$|\bar{w}^\nu(x,t) - w^\nu(y^*, s^*)| \leqslant C \cdot \mathrm{Tot.Var.}\{(w^\nu, \theta^\nu)(\cdot, \bar{t}_n), [x_{j-N}, x_{j+N}]\} + \frac{\delta}{2^{|j|+n}}.$$

By construction, there exists a point $(x^*, s^*) \in R_{j,n}$ such that $w^\nu(x,t) = w^\nu(x^*, s^*)$, hence we can compute

$$\begin{aligned} |\bar{w}^\nu(x,t) - w^\nu(x,t)| &\leqslant |\bar{w}^\nu(x,t) - w^\nu(y^*, s^*)| + |w^\nu(y^*, s^*) - w^\nu(x^*, s^*)| \\ &\leqslant (C+1)\mathrm{Tot.Var.}\{(w^\nu, \theta^\nu)(\cdot, \bar{t}_n), [x_{j-N}, x_{j+N}]\} + \frac{\delta}{2^{|j|+n}}. \end{aligned}$$

Putting this inequality in (B.1), because of the arbitrariness of δ, we obtain

$$\begin{aligned} \int_K |\bar{w}^\nu(x,t) - w^\nu(x,t)|\, dx\, dt &\leqslant (C+1) \sum_{n \geqslant 0, \bar{t}_n \leqslant T} \Delta x_\nu \Delta t_\nu \, 2N \mathrm{Tot\,Var.}\{(w^\nu, \theta^\nu)(\cdot, \bar{t}_n)\} \\ &\leqslant (C+1) 4NF(0) \cdot T \Delta x_\nu \to 0. \end{aligned}$$

Therefore $\bar{w}^\nu \to w$ in $\mathbf{L}^1_{\mathrm{loc}}$ as $\nu \to +\infty$. Finally, we apply twice Proposition B.1, first to show that $\theta(\bar{a}^\nu) \to \theta_0$, and again to prove the convergence of $(\bar{a}^\nu, \bar{u}^\nu) = \boldsymbol{\Phi}^{-1}(\bar{w}^\nu, \theta(\bar{a}^\nu))$ to (a_0, u). $\square$

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