



# A second-order Cauchy problem in a scale of Banach spaces and application to Kirchhoff equations

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## Abstract

In a scale of Banach spaces we study the Cauchy problem for the equation  $u'' = A(Bu(t), u)$ , where  $A$  is a bilinear operator and  $B$  is a completely continuous operator. Obtained results are applied to prove existence of solutions in the Gevrey class for Kirchhoff equations.

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## 1. Introduction

This paper is devoted to the study of existence results and application of a class of second order Cauchy problems in a scale of Banach spaces.

Existence and uniqueness results for Cauchy problems of first order in a scale of Banach spaces have been studied by Ovsjannikov, Treves, Nirenberg, Nishida, Deimling and others and found various applications to differential equations, to physics and mechanics (see [2,5–10] and references therein). Barkova and Zabreiko [1] have obtained similar results for second order Cauchy problems which satisfy the Lipschitz condition.

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In Section 2 of the present paper, we shall be concerned with existence results for a class of second order equations satisfying a compactness condition.

Our abstract results will be applied to prove the existence of solutions in the Gevrey class for generalized Kirchhoff equations considered in [3,4]. The authors of the papers [3,4] have used the complicated method based on formal norms of Leray–Waelbroeck. Our method of reducing to the equations in a scale of Banach spaces seems to be simpler and allows us to lighten the assumptions on the data of the problem and also to give estimates for lifespan of the solution which are more exact than obtained in [4].

## 2. A second-order Cauchy problem in a scale of Banach spaces

Throughout this section let us given a scale of Banach spaces  $(E_\lambda, |\cdot|_\lambda)$ ,  $\lambda \in [a, b] \subset (0, +\infty)$  such that  $\lambda < \lambda'$  implies  $E_{\lambda'} \subset E_\lambda$  and  $|u|_\lambda \leq |u|_{\lambda'}$  for all  $u \in E_{\lambda'}$ . The main difficulty in studying Cauchy problems in a scale of Banach spaces consists in that operators under consideration map any  $E_\lambda$  not into itself, but into a whole family of larger spaces  $E_\beta$ ,  $\beta < \lambda$ . To overcome this difficulty we apply standard assumptions and arguments of Ovsjannikov, Nirenberg, Nishida and Barkova, Zabreiko.

First we will study the existence and estimate for solutions to the following linear Cauchy problem

$$\begin{aligned} u'' &= A(t)u + f(t), \\ u(0) &= u_0, \quad u'(0) = u_1. \end{aligned} \quad (1)$$

**Theorem 1.** *Let the following assumptions be satisfied:*

(1) *For any pair  $(\lambda, \beta)$ ,  $a \leq \lambda < \beta \leq b$ , the operator  $A : I = [0, T] \rightarrow L(E_\beta, E_\lambda)$  is continuous and there exists a number  $M > 0$ , independent of  $t, \lambda, \beta$  such that*

$$|A(t)u|_\lambda \leq \frac{M}{(\beta - \lambda)^2} \cdot |u|_\beta \quad \text{for all } u \in E_\beta.$$

(2)  $u_0, u_1 \in E_b$ ,  $f \in C(I, E_b)$ .

*Then for any  $\lambda \in (a, b)$  there exists a number  $T_\lambda = \min\{T, (b - \lambda)/\sqrt{Me}\}$  such that problem (1) has a unique solution  $u : [0, T_\lambda] \rightarrow E_\lambda$  satisfying*

$$|u(t) - \bar{u}(t)|_\lambda \leq \frac{K(t)(b - \lambda)}{2(b - \lambda - t\sqrt{Me})}, \quad (2)$$

$$|u'(t) - u_1|_\lambda \leq Tg(t) + \frac{4M}{\sqrt{Me}} \left( \frac{c}{b - \lambda - t\sqrt{Me}} + \frac{2K(t)(b - \lambda)}{(b - \lambda - t\sqrt{Me})^2} \right) \quad (3)$$

for  $t \in [0, T_\lambda)$ . Here,

$$\begin{aligned}\bar{u}(t) &= u_0 + tu_1, \quad c = \sup\{|\bar{u}(t)|_b : t \in [0, T]\}, \\ g(t) &= \sup\{|f(s)|_b : s \in [0, t]\}, \quad K(t) = c + \frac{(b-\lambda)^2}{2Me} g(t).\end{aligned}\quad (4)$$

**Proof.** Fix  $\lambda \in (a, b)$ . We replace problem (1) by the following equivalent integral equation:

$$u(t) = \bar{u}(t) + \int_0^t ds \int_0^s (A(r)u(r) + f(r)) dr := Fu(t). \quad (5)$$

Consider the successive approximations  $u_0(t) = \bar{u}(t)$ ,  $u_n(t) = Fu_{n-1}(t)$ . Since  $\bar{u}, f \in C(I, E_b)$  we have  $u_n \in C(I, E_\beta)$  for all  $n$  and all  $\beta \in [\lambda, b)$ . We shall prove by induction

$$|u_n(t) - u_{n-1}(t)|_\beta \leq K(t) \left( \frac{Met^2}{(b-\beta)^2} \right)^n. \quad (6)$$

We have

$$\begin{aligned}|u_1(t) - \bar{u}(t)|_\beta &\leq \int_0^t ds \int_0^s \left( \frac{M}{(b-\beta)^2} |\bar{u}(r)|_b + |f(r)|_b \right) dr \\ &\leq \left( \frac{Mc}{(b-\beta)^2} + g(t) \right) \frac{t^2}{2} \leq K(t) \frac{Met^2}{(b-\beta)^2}.\end{aligned}$$

If the inequality holds for  $n$  then

$$\begin{aligned}|u_{n+1}(t) - u_n(t)|_\beta &\leq \frac{M}{\varepsilon^2} \int_0^t ds \int_0^s |u_n(r) - u_{n-1}(r)|_{\beta+\varepsilon} dr \\ &\leq \frac{M}{\varepsilon^2} \int_0^t ds \int_0^s K(r) \left( \frac{Mer^2}{(b-\beta-\varepsilon)^2} \right)^n dr \\ &\leq \frac{K(t)(Mer^2)^{n+1}}{\varepsilon^2(b-\beta-\varepsilon)^{2n}(2n+1)(2n+2)e}.\end{aligned}\quad (7)$$

Choosing  $\varepsilon = (b-\beta)/(2n+1)$  we get

$$\varepsilon^2(b-\beta-\varepsilon)^2 = (b-\beta)^{2n+2} \left( \frac{2n}{2n+1} \right)^{2n} \frac{1}{(2n+1)^2} > \frac{(b-\beta)^{2n+2}}{(2n+1)(2n+2)e}. \quad (8)$$

Combining (7) with (8) we obtain (6) with  $n$  replaced by  $n+1$ .

Consider a number  $t \in [0, T_\lambda)$  and choose  $\beta > \lambda$  such that  $Met^2 < (b - \beta)^2$ . Inequality (6) shows that the sequence  $\{u_n\}$  converges in  $C([0, t], E_\beta)$  to a function  $u$ . Taking limit in  $E_\lambda$ -norm as  $n \rightarrow \infty$  in the equality  $u_n(t) = Fu_{n-1}(t)$  we see that the obtained function  $u : [0, T_\lambda) \rightarrow E_\lambda$  satisfies (5), hence it is a solution of problem (1). Next we verify estimates (2), (3). For simplicity of notations we set  $d = \sqrt{Me}$ ; we have from (6)

$$|u_n(t) - \bar{u}(t)|_\lambda \leq K(t) \sum_{i=1}^n \left( \frac{dt}{b - \lambda} \right)^{2n}$$

and by letting  $n \rightarrow \infty$

$$|u(t) - \bar{u}(t)|_\lambda \leq \frac{K(t)t^2d^2}{(b - \lambda)^2 - d^2t^2} \leq \frac{K(t)(b - \lambda)}{2(b - \lambda - dt)}$$

for  $0 \leq t < (b - \lambda)/d$ . From (5) we obtain

$$\begin{aligned} |u'(t) - u_1|_\lambda &= \left| \int_0^t (A(s)u(s) + f(s))ds \right|_\lambda \\ &\leq Tg(t) + \int_0^t \frac{M}{(\lambda(s) - \lambda)^2} |u(s)|_{\lambda(s)} ds, \end{aligned} \quad (9)$$

where  $\lambda(s) = (b + \lambda - ds)/2$ . By applying (2) we get

$$\begin{aligned} |u(s)|_{\lambda(s)} &\leq c + \frac{K(s)(b - \lambda(s))}{2(b - \lambda(s) - ds)} = c + \frac{K(s)(b - \lambda + ds)}{2(b - \lambda - ds)} \\ &\leq c + \frac{K(s)(b - \lambda)}{b - \lambda - ds} \quad \text{for } 0 \leq s < (b - \lambda)/d. \end{aligned}$$

Consequently, from (9) we deduce

$$\begin{aligned} |u'(t) - u_1|_\lambda &\leq Tg(t) + \int_0^t \frac{4M}{(b - \lambda - ds)^2} \left[ c + \frac{K(s)(b - \lambda)}{b - \lambda - ds} \right] ds \\ &\leq Tg(t) + 4M \left[ c \int_0^t \frac{ds}{(b - \lambda - ds)^2} + K(t)(b - \lambda) \int_0^t \frac{ds}{(b - \lambda - ds)^3} \right] \\ &\leq Tg(t) + \frac{4M}{d} \left( \frac{c}{b - \lambda - dt} + \frac{2K(t)(b - \lambda)}{(b - \lambda - dt)^2} \right) \end{aligned}$$

for  $t \in [0, T_\lambda)$ . Thus, (3) is established.

Finally we prove uniqueness. Let  $v : [0, T'] \rightarrow E_\lambda$  be a solution of problem (1). Fix  $\lambda' < \lambda$ , we may repeat arguments in the proof of existence with  $\lambda, b, u_n$  replaced by  $\lambda', \lambda$  and  $u_n - v$  respectively, to obtain for the function  $u - v$  estimate (2) with  $\bar{u}(t) = f(t) \equiv 0$ . Consequently,  $u(t) = v(t)$  for  $0 \leq t < \min\{T', (\lambda - \lambda')/d\}$  and hence  $u(t) = v(t)$  for  $0 \leq t < T'$  by standard reasons. The proof is complete.  $\square$

Theorem 1 will be needed in proving of the following main result of the paper.

**Theorem 2.** *Let the following assumptions are satisfied:*

(1) *For any pair  $(\lambda, \beta)$ ,  $a \leq \lambda < \beta < b$  the operator  $A : E_\lambda \times E_\beta \rightarrow E_\lambda$  is bilinear and there exists a number  $M > 0$  independent of  $\lambda, \beta$  such that*

$$|A(u, v)|_\lambda \leq \frac{M}{(\beta - \lambda)^2} |u|_\lambda |v|_\beta \quad \text{for all } u \in E_\lambda, \quad \text{all } v \in E_\beta.$$

(2) *The operator  $B$  is completely continuous from  $C^1([0, T], E_a)$  into  $C([0, T], E_b)$  equipped with the usual norms. Moreover*

$$\sup\{|Bu(t)|_b : t \in [0, T], u \in C^1([0, T], E_a)\} = L < \infty.$$

(3)  $u_0, u_1 \in E_b$ .

*Then for any  $\lambda \in (a, b)$  there exists a number  $T_\lambda = \min\{T, \frac{b-\lambda}{4\sqrt{MLE}}\}$  such that the Cauchy problem*

$$\begin{aligned} u'' &= A(Bu(t), u), \\ u(0) &= u_0, u'(0) = u_1 \end{aligned} \tag{10}$$

*has a solution  $u : [0, T_\lambda] \rightarrow E_\lambda$ .*

**Proof.** Set  $I = [0, T]$  we first observe that from hypothesis (2) and continuity of the imbedding  $E_\lambda \hookrightarrow E_a$ , the operator  $B$  also is completely continuous from  $C^1(I, E_\lambda)$  into  $C(I, E_b)$  for any  $\lambda \in [a, b]$ . Fix  $\lambda \in (a, b)$ , for every  $u \in C^1(I, E_\lambda)$  we consider the following linear Cauchy problem

$$\begin{aligned} v'' &= A(Bu(t), v), \\ v(0) &= u_0, v'(0) = u_1. \end{aligned} \tag{11}$$

For  $\lambda \leq \gamma < \beta \leq b$  and  $v \in E_\beta$  we have

$$\begin{aligned} |A(Bu(t), v) - A(Bu(s), v)|_\gamma &\leq \frac{M}{(\beta - \gamma)^2} |Bu(t) - Bu(s)|_b |v|_\beta, \\ |A(Bu(t), v)|_\gamma &\leq \frac{M|Bu(t)|_\gamma |v|_\beta}{(\beta - \gamma)^2} \leq \frac{ML}{(\beta - \gamma)^2} |v|_\beta. \end{aligned}$$

Therefore, the operator  $t \mapsto A(Bu(t), \cdot)$  from  $I$  into  $L(E_\beta, E_\gamma)$  satisfies assumption (1) in Theorem 1. Consequently, for each  $\beta \in [\lambda, b)$  there exists  $T'_\beta = \min\{T, (b - \beta)/$

$\sqrt{MLE}\}$  so that problem (11) has a unique solution  $v := Fu : [0, T'_\beta) \rightarrow E_\beta$ , satisfying

$$|Fu(t) - \bar{u}(t)|_\beta \leq \frac{c(b - \beta)}{2(b - \beta - dt)}, \quad (12)$$

$$|(Fu)'(t) - u_1|_\beta \leq \frac{4ML}{d} \left( \frac{c}{b - \beta - dt} + \frac{2c(b - \beta)}{(b - \beta - dt)^2} \right) \quad (13)$$

for  $t \in [0, T'_\beta)$ , where  $\bar{u}(t) = u_0 + tu_1$ ,  $c = \sup\{|\bar{u}(t)|_b : t \in I\}$ ,  $d = \sqrt{MLE}$ . In order to study continuity and compactness of the operator  $F$  we shall estimate  $w = Fu_1 - Fu_2$ . Clearly,  $w$  satisfies

$$\begin{aligned} w'' &= A(Bu_1(t), w) + A(Bu_1(t) - Bu_2(t), Fu_2(t)), \\ w(0) &= w'(0) = 0. \end{aligned} \quad (14)$$

We will consider the Cauchy problem (14) in the scale  $(E_\beta, |\cdot|_\beta)$ ,  $\beta \in [\lambda, \lambda + \varepsilon]$  with  $\varepsilon > 0$  choosing later. By applying to problem (14) the estimates of type (2),(3) with notations (4) in Theorem 1 we get

$$\begin{aligned} |w(t)|_\lambda &\leq \frac{\varepsilon^3}{4d^2(\varepsilon - dt)} \sup_{s \in [0, t]} |f(s)|_{\lambda + \varepsilon}, \\ |w'(t)|_\lambda &\leq \left( T'_{\lambda + \varepsilon} + \frac{4ML\varepsilon^3}{d^3(\varepsilon - dt)^2} \right) \sup_{s \in [0, t]} |f(s)|_{\lambda + \varepsilon} \end{aligned} \quad (15)$$

for  $0 \leq t < \min\{T, \varepsilon/d\}$ , where  $f(t) = A(Bu_1(t) - Bu_2(t), Fu_2(t))$ . We have by assumption (1) of the theorem

$$|f(t)|_{\lambda + \varepsilon} \leq \frac{M}{\delta^2} |Bu_1(t) - Bu_2(t)|_{\lambda + \varepsilon} |Fu_2(t)|_{\lambda + \varepsilon + \delta} \quad (16)$$

and by (12)

$$|Fu_2(t)|_{\lambda + \varepsilon + \delta} \leq c + \frac{c(b - \lambda - \varepsilon - \delta)}{2(b - \lambda - \varepsilon - \delta - dt)}$$

for  $0 \leq t < \min\{T, (b - \lambda - \varepsilon - \delta)/d\}$ . By choosing  $\varepsilon = (b - \lambda)/3$ ,  $\delta = (b - \lambda)/6$  we obtain

$$\begin{aligned} |Fu_2(t)|_{\lambda + \varepsilon + \delta} &\leq c + c(b - \lambda)/2(b - \lambda - 2dt) \quad \text{for } t < \min\{T, (b - \lambda)/2d\} \\ &\leq 2c \quad \text{for } t < \min\{T, (b - \lambda)/4d\}. \end{aligned} \quad (17)$$

Finally, for  $0 \leq t < T_\lambda = \min\{T, (b - \lambda)/4d\}$  we have from (15)–(17)

$$\begin{aligned} |Fu_1(t) - Fu_2(t)|_\lambda &\leq \frac{8Mc}{d^2} \sup_{s \in [0, t]} |Bu_1(s) - Bu_2(s)|_b, \\ |(Fu_1 - Fu_2)'(t)|_\lambda &\leq \frac{48Mc(d^2 + 32ML)}{d^3(b - \lambda)} \sup_{s \in [0, t]} |Bu_1(s) - Bu_2(s)|_b \end{aligned} \quad (18)$$

and from (12), (13)

$$|Fu(t) - \bar{u}(t)|_\lambda \leq \frac{2}{3}c, \quad |(Fu)'(t) - u_1|_\lambda \leq \frac{176MLc}{9d(b - \lambda)}. \quad (19)$$

Now we end the proof by proving that the operator  $F$  has a fixed point. We set  $X = C^1([0, T_\lambda], E_\lambda)$  equipped with the norm  $\|u\| = \sup\{|u(t)|_\lambda + |u'(t)|_\lambda : t \in [0, T_\lambda]\}$ . We have from (19)  $F(X) \subset \bar{B}(\bar{u}, r)$  for some  $r > 0$ , and from (18)

$$\|Fu_1 - Fu_2\| \leq K \sup_{t \in [0, T_\lambda]} |Bu_1(t) - Bu_2(t)|_b$$

for some constant  $K > 0$ . Since  $B$  is completely continuous, so is  $F$ . Therefore,  $F$  has a fixed point in  $X$  by the Schauder theorem. The theorem is proved.  $\square$

### 3. Application to Kirchhoff equations

#### 3.1. The scale of spaces of functions in the Gevrey class

Let  $\Omega \subset \mathbb{R}^n$  be an open subset, we denote by  $\mathcal{A}(\Omega)$  the class of all real functions  $u \in C^\infty(\Omega)$  satisfying

$$\exists K > 0 \exists c > 0 : \quad \|D^\alpha u\| \leq K \frac{\alpha!}{c^{|\alpha|}} \quad \text{for all } \alpha \in \mathbb{N}^n, \quad (20)$$

where we set  $\|v\| = \sup\{|v(x)| : x \in \Omega\}$  and  $\alpha! = \alpha_1! \dots \alpha_n!$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$  for  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ .

For any  $\lambda > 0$  we denote by  $E_\lambda$  the space of all functions  $u \in C^\infty(\Omega)$  such that

$$|u|_\lambda := \sum_{\alpha \in \mathbb{N}^n} \|D^\alpha u\| \frac{\lambda^{|\alpha|}}{\alpha!} < \infty.$$

It is known that the family  $(E_\lambda, |\cdot|_\lambda)$ ,  $\lambda > 0$  forms a scale of Banach spaces. Moreover, if a function  $u$  satisfies condition (20), then for  $\lambda < c$  we have

$$|u|_\lambda = \sum_{\alpha} \|D^\alpha u\| \frac{c^{|\alpha|}}{\alpha!} \left(\frac{\lambda}{c}\right)^{|\alpha|} \leq K \sum_{i=0}^{\infty} (i+1)(\lambda/c)^i < \infty,$$

hence  $u \in E_\lambda$ . Thus, we have  $\mathcal{A}(\Omega) = \cup\{E_\lambda : \lambda > 0\}$ .

**Lemma 1.** *The scale  $(E_\lambda, |\cdot|_\lambda)$ ,  $\lambda \in [a, b]$  has the following properties:*

- (1) *If  $u, v \in E_\lambda$  then  $uv \in E_\lambda$  and one has  $|uv|_\lambda \leq |u|_\lambda |v|_\lambda$*
- (2) *There exists a constant  $M > 0$ , depending only on  $a, b$  such that for  $a \leq \lambda < \beta \leq b$  one has*

$$|\Delta u|_\lambda \leq \frac{M}{(\beta - \lambda)^2} |u|_\beta, \quad u \in E_\beta,$$

where  $\Delta$  is the Laplacian.

**Proof.** (1) We have

$$D^\alpha(uv) = \sum_{\delta \leq \alpha} C_\alpha^\delta D^\delta u D^{\alpha-\delta} v,$$

where we define  $\delta = (\delta_1, \dots, \delta_n) \leq \alpha = (\alpha_1, \dots, \alpha_n)$  if  $\delta_i \leq \alpha_i$  for all  $i = \overline{1, n}$  and then  $\alpha - \delta = (\alpha_1 - \delta_1, \dots, \alpha_n - \delta_n)$ . Hence

$$\sum_{\alpha} \|D^\alpha uv\| \frac{\lambda^{|\alpha|}}{\alpha!} \leq \sum_{\alpha} \sum_{\delta \leq \alpha} \|D^\delta u\| \frac{\lambda^{|\delta|}}{\delta!} \|D^{\alpha-\delta} v\| \frac{\lambda^{|\alpha-\delta|}}{(\alpha-\delta)!}. \quad (21)$$

By the rule for multiplication of two series, the right-hand side of (21) is equal to  $|u|_\lambda |v|_\lambda$  and hence  $|uv|_\lambda \leq |u|_\lambda |v|_\lambda$ .

(2) For a multi-index  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  we set  $\alpha + 2 = (\alpha_1 + 2, \alpha_2, \dots, \alpha_n)$ ; then we have

$$\left\| D^\alpha \left( \frac{\partial^2 u}{\partial x_1^2} \right) \right\| \frac{\lambda^{|\alpha|}}{\alpha!} = \|D^{\alpha+2} u\| \frac{\beta^{|\alpha+2|}}{(\alpha+2)!} \left(\frac{\lambda}{\beta}\right)^{|\alpha+2|} \frac{(\alpha_1+1)(\alpha_1+2)}{\lambda^2}. \quad (22)$$

Since

$$\sup \left\{ t^2 \left( \frac{\lambda}{\beta} \right)^t : t \geq 0 \right\} = \frac{4}{e^2 (\ln(\lambda/\beta))^2} \leq \frac{4\beta^2}{e^2 (\beta - \lambda)^2}$$



we obtain

$$\left(\frac{\lambda}{\beta}\right)^{|\alpha|+2} \frac{(\alpha_1+1)(\alpha_1+2)}{\lambda^2} \leq \left(\frac{\lambda}{\beta}\right)^{|\alpha|+2} \frac{(|\alpha|+2)^2}{\lambda^2} \leq \frac{4\beta^2}{e^2\lambda^2(\beta-\lambda)^2}.$$

Consequently, from (22) we deduce

$$|\Delta u|_\lambda \leq 4n \left(\frac{b}{ea}\right)^2 \frac{|u|_\beta}{(\beta-\lambda)^2}.$$

The lemma is proved.  $\square$

### 3.2. Cauchy problem for generalized Kirchhoff equations

Following the paper [4] we consider the Cauchy problem

$$\begin{aligned} D_t^2 u(t, x) &= f(t, x, \int_P |\nabla_x u|^2 dx) \Delta_x u(t, x), \quad (t, x) \in \Omega_T = [0, T] \times \Omega, \\ u(0, x) &= u_0(x), \quad D_t u(0, x) = u_1(x), \quad \forall x \in \Omega, \end{aligned} \quad (23)$$

where  $P, \Omega$  are open subsets in  $\mathbb{R}^n$  and  $P \subset \Omega$  is bounded. Under the function  $f : \Omega_T \times \mathbb{R}^+ \rightarrow \mathbb{R}$  we assume the following hypotheses:

- (H<sub>1</sub>)  $f(t, \cdot, u) \in C^\infty(\Omega)$  for all  $(t, u) \in [0, T] \times \mathbb{R}^+$  and for all  $\alpha \in \mathbb{N}^n$  the operator-function  $u \mapsto D_x^\alpha f(\cdot, \cdot, u)$  belongs to  $C(\mathbb{R}^+, C(\Omega_T))$   
 (H<sub>2</sub>) There are  $c > 0, K > 0$  such that

$$|D_x^\alpha f(t, x, u)| \leq K \frac{\alpha!}{c^{|\alpha|}}$$

for all  $(t, x, u) \in \Omega_T \times \mathbb{R}^+$  and all  $\alpha \in \mathbb{N}^n$ .

In the paper [4] the following hypotheses on the function  $f$  are proposed:

- (H<sub>1</sub>')  $D_x^\alpha f \in C(\Omega_T \times \mathbb{R}^+, \mathbb{R})$  for all  $\alpha \in \mathbb{N}^n$ .  
 (H<sub>2</sub>') There are  $c > 0, K > 0$  such that

$$|D_x^\alpha f(t, x, u)|, \left| D_x^\alpha \frac{\partial}{\partial u} f(t, x, u) \right| \leq K \frac{\alpha!}{c^{|\alpha|}}$$

for all  $(t, x, u) \in \Omega_T \times \mathbb{R}^+$  and all  $\alpha \in \mathbb{N}^n$ .

Clearly, hypothesis (H<sub>2</sub>') is more restrictive than (H<sub>2</sub>) and from the Mean value theorem we see that (H<sub>1</sub>') together (H<sub>2</sub>') imply (H<sub>1</sub>).

**Lemma 2.** *Let hypotheses  $(H_1)$ ,  $(H_2)$  be satisfied. Then the operator*

$$Bu(t) = f(t, x, \int_P |\nabla_x u|^2 dx)$$

*is completely continuous from  $C^1([0, T], E_a)$  into  $C([0, T], E_b)$  with  $0 < a < b < c$ ; moreover,  $\sup\{|Bu(t)|_b : u \in C^1([0, T], E_a), t \in [0, T]\} < \infty$ . Here, in  $C^1([0, T], E_a)$  and  $C([0, T], E_b)$  we consider the usual norms:*

$$\|u\|_a = \sup\{|u(t)|_a + |u'(t)|_a : t \in [0, T]\}, \quad \|u\|_b = \sup\{|u(t)|_b : t \in [0, T]\}.$$

**Proof.** Set  $I = [0, T]$ , we first prove that the operator

$$F : C^1(I, E_a) \rightarrow C(I, \mathbb{R}), \quad Fu(t) = \int_P |\nabla_x u(t, x)|^2 dx$$

is completely continuous. Let  $V \subset C^1(I, E_a)$  be a bounded subset and  $\|u\|_a \leq r$  for all  $u \in V$ . Since  $|\frac{\partial}{\partial x_i} u(t, x)| \leq \frac{1}{a} |u(t, \cdot)|_a$ , we get

$$\left| \left( \frac{\partial u(t, x)}{\partial x_i} \right)^2 - \left( \frac{\partial v(t, x)}{\partial x_i} \right)^2 \right| \leq \frac{2r}{a^2} |u(t, \cdot) - v(t, \cdot)|_a \quad \text{for } u, v \in V.$$

Therefore,

$$|Fu(t) - Fv(t)| \leq \frac{2nrmsP}{a^2} \|u - v\|_a \quad \text{for all } t \in I, \text{ all } u, v \in V,$$

which proves continuity of the operator  $F$ . Analogously, by the Mean value theorem we have

$$\left| \left( \frac{\partial u(t, x)}{\partial x_i} \right)^2 - \left( \frac{\partial u(s, x)}{\partial x_i} \right)^2 \right| \leq \frac{2r^2}{a^2} |t - s| \quad \text{for } u \in V, t, s \in I$$

and hence

$$|Fu(t) - Fu(s)| \leq \frac{2nr^2msP}{a^2} |t - s| \quad \text{for all } t, s \in I, \text{ all } v \in V.$$

Thus, the set  $F(V)$  is relatively compact in  $C(I, \mathbb{R})$  by the Ascoli–Arzèla theorem.

Finally, we prove continuity and boundedness of the operator

$$G : C(I, \mathbb{R}) \rightarrow C(I, E_b), \quad Gu(t) = f(t, x, u(t)).$$

It follows from hypothesis (H<sub>2</sub>) that

$$|Gu(t)|_b = \sum_{\alpha} \|D^{\alpha} f(t, x, u(t))\| \frac{b^{|\alpha|}}{\alpha!} \leq K \sum_{\alpha} \left(\frac{b}{c}\right)^{|\alpha|}$$

for all  $t \in I$  and all  $u \in C(I, \mathbb{R})$ , and hence,  $\sup\{\|Gu\|_b : u \in C(I, \mathbb{R})\} < \infty$ . Let the sequence  $\{u_n\}$  be convergent in  $C(I, \mathbb{R})$  to a function  $u$  and  $|u_m(t)| \leq r$ ,  $|u(t)| \leq r$  for all  $t \in I$ , all  $m \in \mathbb{N}$ . Given  $\varepsilon > 0$ , first we choose  $n_0$  so large that

$$\sum_{|\alpha| \geq n_0+1} \|D^{\alpha} f(t, x, u_m(t)) - D^{\alpha} f(t, x, u(t))\| \frac{b^{|\alpha|}}{\alpha!} \leq 2K \sum_{|\alpha| \geq n_0+1} \left(\frac{b}{c}\right)^{|\alpha|} < \varepsilon/2.$$

By hypothesis (H<sub>1</sub>), the operator-functions  $u \mapsto D^{\alpha} f(\cdot, \cdot, u)$ ,  $|\alpha| \leq n_0$  from  $\mathbb{R}^+$  to  $C(\Omega_T)$  are uniformly continuous on  $[0, r]$ . Consequently, for  $m$  sufficiently large we have

$$\sum_{|\alpha| \leq n_0} \|D^{\alpha} f(t, x, u_m(t)) - D^{\alpha} f(t, x, u(t))\| \frac{b^{|\alpha|}}{\alpha!} < \varepsilon/2 \quad \text{for all } t \in I.$$

Therefore,  $\lim_{m \rightarrow \infty} \sup_{t \in I} |Gu_m(t) - Gu(t)|_b = 0$ . The lemma is proved.  $\square$

**Definition.** We write  $u \in C^2(I, \mathcal{A}(\Omega))$  if  $u \in C^2(I, E_{\lambda})$  for some  $\lambda > 0$ .

**Theorem 3.** Assume hypotheses (H<sub>1</sub>), (H<sub>2</sub>) are satisfied and  $u_0, u_1 \in \mathcal{A}(\Omega)$ . Then there exists  $T' \leq T$  such that the Cauchy problem for generalized Kirchhoff equation (23) has a solution  $u \in C^2([0, T'], \mathcal{A}(\Omega))$ .

**Proof.** Consider the scale  $(E_{\lambda, |\cdot|_{\lambda}})$ ,  $\lambda \in [a, b]$ , where  $E_{\lambda}$  has been defined in Section 3.1 and  $b < c$  is chosen so that  $u_0, u_1 \in E_b$ . The Cauchy problem (23) has the form of (10) with operator  $B$  defined in Lemma 2 and  $A(u, v) = u \cdot \Delta v$ . By Lemmas 1, 2 all assumptions of Theorem 2 are fulfilled and hence, problem (23) has a solution.

**Remark.** From the estimate  $T_{\lambda} = \min\{T, (b - \lambda)/4\sqrt{MLE}\}$  for lifespan of the solution in Theorem 2, we have the following conclusions:

- (1) If the function  $f$  is sufficiently small (i.e. if the number  $K$  in hypothesis (H<sub>2</sub>) is small) then  $T_{\lambda} = T$  because the constant  $L$  is small. Thus, the Kirchhoff equation (23) has a global solution.

- (2) If  $f(t, x, u) = \varepsilon g(t, x, u)$  and  $g$  satisfies  $(H_2)$  for  $(t, x, u) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^+$  then  $L = \mathcal{O}(\varepsilon)$ . Hence, for the lifespan  $T'$  in Theorem 3 we obtain estimate  $T' \geq m.(1/\varepsilon)^{1/2}$  which is more exact than the estimate  $T' \geq m.(1/\varepsilon)^{1/6}$  in [4].

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