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J. Differential Equations 229 (2006) 317–352

Journal of Differential Equations

www.elsevier.com/locate/jde

Nonlinear geometric optics for contact discontinuities and shock waves in one-dimensional Euler system for gas dynamics ☆

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Received 15 October 2005; revised 2 March 2006

Abstract

The aim of this paper is to study the rigorous theory of nonlinear geometric optics for a contact discontinuity and a shock wave to the Euler system for one-dimensional gas dynamics. For the problem of a contact discontinuity and a shock wave perturbed by a small amplitude, high frequency oscillatory wave train, under suitable stability assumptions, we obtain that the perturbed problem has still a shock wave and a contact discontinuity, and we give their asymptotic expansions.
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MSC: 35B05; 35L65; 35L67

Keywords: Nonlinear geometric optics; Conservation laws; Shock wave; Contact discontinuity

1. Introduction

The topic of nonlinear geometric optics is to study high frequency oscillatory waves in nonlinear problems by using the method of multiple scales, and to rigorously justify the asymptotic properties of oscillations. Thanks to its width and importance in applied mathematics, there is a rich literature devoted to the study of this topic (see [3,4,12] and references quoted therein).

☆ Research partially supported by the NNSF (10131050) of China and the University of Donghua.
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0022-0396/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2006.03.002

This paper deals with the nonlinear geometric optics for gas dynamics in one space variable with background states being a shock wave and a contact discontinuity. The problem with background states being shock waves was studied by Corli [1] and Wang [10] for the case of a single shock in one space variable. Williams [12] established a rigorous theory on the nonlinear geometric optics for a single multidimensional shock. The case of two shock waves for a 2×2 conservation law in one space variable was studied by the second author in [11], and the work of [11] is generalized to the general $N \times N$ conservation law in one space variable by Peng and Wang [7] recently, which has many prototypes in physics and mechanics such as gas dynamical systems. Under the assumption of simpler form similar to [8], Corli [2] has established a rigorous result for the case of a contact discontinuity in one space variable. As already mentioned, until now, most of the analysis has either been concerned with the nonlinear geometric optics for hyperbolic conservation laws with background states being shock waves, or one with background being contact discontinuity. In this paper, we first study the case of a shock wave and a contact discontinuity as background states. For simplicity, this paper is devoted to the study of one-dimensional gas dynamics. It is not difficult to see that our investigation can be generalized to the general $M \times M$ system by combining the ideas of [2,7] with this article.

Comparing with the other works, the main difficulty that we encounter is that here we are not only concerned with noncharacteristic problem for shock wave, but also concerned with characteristic problem due to the presence of a contact discontinuity. In order to overcome this difficulty, for the noncharacteristic problem, we use the method given by Wang in [11], while for the characteristic problem, we use the one given by Corli in [2]. What we need to do is how to technically combine the two methods. It is worth to strengthen, to solve nonlinear problem, we need stability conditions on shock wave and contact discontinuity, which are always valid for weak shock and weak contact discontinuity.

Let us now describe the content of this paper. In Section 2, we present the problem of shock wave and contact discontinuity as well as that of leading profiles by using the method of multiple scales and state our main result. Then, in Section 3, we shall study the problem of the oscillatory shock wave and contact discontinuity. We study the problem of leading profiles in Section 4. Finally, Section 5 is devoted to the proof of the asymptotic expansions of the oscillation problem, which gives the nonlinear geometric optics.

2. Formulations of the problem and the main results

2.1. Formulations of the problem

We consider the following one-dimensional non-isentropic gas dynamical system

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p) = 0, \\ \partial_t\big(\rho e + \frac{\rho v^2}{2}\big) + \partial_x\big(\rho e v + \frac{\rho v^3}{2} + p v\big) = 0, \end{cases} \tag{2.1.1}$$

where ρ, v, e and p represent the mass density, the flow velocity, the internal energy per unit mass and the pressure, respectively. These functions are linked by some constitutive laws, such as $p = p(\rho, S)$ and $e = e(\rho, S)$ with S being the entropy. Assume that ρ, p, p_ρ and p_S are strictly positive. If we let $u = (\rho, v, S)$, then (2.1.1) can be written as

$$\partial_t g(u) + \partial_x f(u) = 0, \tag{2.1.2}$$

where $g(u) = (\rho, \rho v, \frac{\rho v^2}{2} + \rho e)$ and $f(u) = (\rho v, p + \rho v^2, \frac{\rho v^3}{2} + \rho e v + p v)$. As is well known, it is strictly hyperbolic with eigenvalues $\lambda_1 = v - c$, $\lambda_2 = v$, and $\lambda_3 = v + c$ with $c = \sqrt{p_\rho}$ being the sound speed. Let $r_1 = (\rho, -c, 0)^{\mathrm{T}}$, $r_2 = (p_S, 0, -c^2)^{\mathrm{T}}$ and $r_3 = (\rho, c, 0)^{\mathrm{T}}$ be the right eigenvectors with respect to $\{\lambda_j\}_{j=1}^3$. Obviously, only the second mode λ_2 is linearly degenerate. A set of Riemann invariants with linearly independent gradients is $\{v, p\}$ (see [9, Chapter 15]).

Assume that the Riemann problem for (2.1.1) with piecewise constant initial data

$$u(0,x) = \begin{cases} u_- = (\rho_-, v_-, S_-), & x < 0, \\ u_+ = (\rho_+, v_+, S_+), & x > 0, \end{cases} \tag{2.1.3}$$

admits a weak solution

$$\underline{\mathrm{u}} = \begin{cases} u_-, & -\infty < x < q_0 t, \\ u_\star, & q_0 t < x < \sigma t, \\ u_+, & \sigma t < x < \infty, \end{cases} \tag{2.1.4}$$

where $u_\star = (\rho_\star, v_\star, S_\star)$, q_0 and σ are constants, $(u_-, u_\star, q_0)$ and $(u_\star, u_+, \sigma)$ are a contact discontinuity and a shock wave, respectively. Thus, one has that

$$v_\star = v_-, \qquad p(\rho_\star, S_\star) = p(\rho_-, S_-), \qquad q_0 = v_- = v_\star, \tag{2.1.5}$$

$$v_+ + c_+ < \sigma < v_\star + c_\star, \qquad \sigma > v_\star, \tag{2.1.6}$$

with $c_\triangle = \sqrt{p_\rho(u_\triangle)}$ denoting the sound speed at the state $u_\triangle$ for $\triangle \in \{-, \star, +\}$, and $\underline{\mathrm{u}}$ satisfies the Rankine–Hugoniot condition

$$\sigma\big(g(u_+) - g(u_\star)\big) = f(u_+) - f(u_\star). \tag{2.1.7}$$

Denote by $r_k^\sharp = r_k(u_\sharp)$ and $l_k^\sharp = l_k(u_\sharp)$ $(k = 1, 2, 3)$ with $l_k(u_\sharp)$ being the left eigenvectors associated with $\lambda_k(u_\sharp)$ for (2.1.1) satisfying the normalization $r_k^\sharp \cdot l_j^\sharp = \delta_{kj}$. The purpose of this article is to study the stability and asymptotic behavior of (2.1.4) under the perturbation of highly oscillatory waves. For this, first we suppose that the following stability conditions on the contact discontinuity and the shock (2.1.4).

(H1) The matrix

$$M = \big(r_1^\star, r_2^\star, \big(\nabla g(u_\star)\big)^{-1}\big(g(u_+) - g(u_\star)\big)\big) \tag{2.1.8}$$

is nonsingular.

(H2) For the standard unit vector $\vec{e}_2 = (0, 1, 0)$, we have

$$0 < \vec{e}_2 M^{-1} r_3^\star < 1. \tag{2.1.9}$$

Remark 2.1. When $\underline{\mathrm{u}}$ are weak enough, hypotheses (H1) and (H2) hold always. (H1) is the one-dimensional stability condition given by Majda in [5], while (H2) is similar to the stability condition given by Métiver in [6].

By perturbing the initial data $u_\pm$ with some small amplitude oscillating functions $\epsilon u^\epsilon_{\pm,0}$, we are lead to the following Cauchy problem

$$\begin{cases} \partial_t g(U^\epsilon) + \partial_x f(U^\epsilon) = 0, & t > 0,\ x \in \mathbb{R}, \\ U^\epsilon(0, x) = \begin{cases} u_+ + \epsilon u^\epsilon_{+,0}(x), & x > 0, \\ u_- + \epsilon u^\epsilon_{-,0}(x), & x < 0, \end{cases} \end{cases} \tag{2.1.10}$$

where $\epsilon > 0$ is small enough, and $u^\epsilon_{\pm,0} \in C^1$ satisfy some compatibility conditions, which will be given precisely.

Before giving assumptions on problem (2.1.10), we first introduce some notations as in [10]. Given a small closed neighborhood $\omega \subset \{t = 0\}$ of the origin, suppose Ω is the closure of a determinacy domain of ω for the Cauchy problem of (2.1.2) when $|u - \underline{u}| < \delta$ for a fixed small constant $\delta > 0$. Let $u \in C^k(\Omega)$, we define

$$\|u\|^\epsilon_{k,\Omega} = \sum_{|\alpha| \leqslant k} \epsilon^\alpha \left\| \partial^\alpha_{t,x} u \right\|_{L^\infty(\Omega)}.$$

A family $u^\epsilon \in C^k(\Omega)$ are bounded in $C^k_\epsilon(\Omega)$ if the norms $\|u^\epsilon\|^\epsilon_{k,\Omega}$ are bounded, and $\phi^\epsilon(t)$ are bounded in $\tilde{C}^k_\epsilon([0, T])$ if $\phi^\epsilon \in C^k([0, T])$ and $\|d_t \phi^\epsilon\|^\epsilon_{k-1,[0,T]}$ are bounded for $k \geqslant 1$.

Let $C^0_p(\mathbb{R}^q)$ be the space of continuous almost periodic functions in $\theta \in \mathbb{R}^q$. Denote by $\mathcal{C}^0(\Omega : \mathbb{R}^q) = C^0(\Omega : C^0_p(\mathbb{R}^q))$ the space of continuous functions from Ω into $C^0_p(\mathbb{R}^q)$. For $k \in \mathbb{N}$, define the space $\mathcal{C}^k(\Omega : \mathbb{R}^q)$ of those functions $U \in \mathcal{C}^0(\Omega : \mathbb{R}^q)$ whose derivatives $\partial^\alpha_{(t,x;\theta)} U$ belong to $\mathcal{C}^0(\Omega : \mathbb{R}^q)$ for any $|\alpha| \leqslant k$.

For problem (2.1.10), we assume that there are $U_{\pm,0} \in \mathcal{C}^1(\omega^\pm : \mathbb{R})$ such that

$$\left\| u^\epsilon_{\pm,0}(x) - U_{\pm,0}\left(x, \frac{x}{\epsilon}\right) \right\|^\epsilon_{1,\omega^\pm} = o(1), \quad \text{when } \epsilon \to 0, \tag{2.1.11}$$

where $\omega^+ = \omega \cap \{x > 0\}$ and $\omega^- = \omega \cap \{x < 0\}$, which immediately implies the boundedness of $u^\epsilon_{\pm,0}$ in $C^1_\epsilon(\omega^\pm)$.

We are going to study whether the structures of contact discontinuity and shock wave are conserved under such perturbations, i.e., whether there exists a local solution which contains a contact discontinuity and a shock wave

$$U^\epsilon(t, x) = \begin{cases} u_- + \epsilon u^\epsilon_-(t, x), & x < q_0 t + \epsilon q^\epsilon(t), \\ u_\star + \epsilon u^\epsilon_\star(t, x), & q_0 t + \epsilon q^\epsilon(t) < x < \sigma t + \epsilon \varphi^\epsilon(t), \\ u_+ + \epsilon u^\epsilon_+(t, x), & x > \sigma t + \epsilon \varphi^\epsilon(t), \end{cases} \tag{2.1.12}$$

to problem (2.1.10). Under the suitable conditions (H1) and (H2) of the background state $\underline{u}$, the answer to this problem is in the affirmative.

Under the assumption of (2.1.11), $u^\epsilon_+(t, x)$ can be easily determined by (2.1.10). Obviously, $\Omega^+ = \{x > \sigma t + \epsilon \varphi^\epsilon(t)\}$ is the determinacy domains ω^+. Using the same way as in [11, p. 1626], we obtain the solution $U^\epsilon_+(t, x) = u_+ + \epsilon u^\epsilon_+(t, x)$ to (2.1.10) in Ω^+ and the asymptotic expansion

$$u_+^\epsilon(t,x) = U_+\left(t,x;\frac{t}{\epsilon},\frac{x}{\epsilon}\right) + o(1), \quad \text{in } C_\epsilon^1(\Omega_T^+), \tag{2.1.13}$$

where $U_+ \in \mathcal{C}^1(\Omega_T^+ : \mathbb{R}^2)$ satisfies an integro-differential system (see [11, (2.18)]), with $\Omega_T^+ = \Omega^+ \cap \{t < T\}$.

The next aim is to study the local existence of U^ϵ in the form of (2.1.12) to problem (2.1.10), and to study the asymptotic properties of $(u_-^\epsilon(t,x), u_\star^\epsilon(t,x), q^\epsilon(t), \varphi^\epsilon(t))$ with respect to ϵ.

Denote by $\Omega^- = \{x < q_0 t + \epsilon q^\epsilon(t)\}$ and $\Omega^\star = \{q_0 t + \epsilon q^\epsilon(t) < x < \sigma t + \epsilon\varphi^\epsilon(t)\}$. From the Rankine–Hugoniot condition, we know that $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon)$ satisfy

$$\begin{cases} \partial_t u_\sharp^\epsilon + A(u_\sharp + \epsilon u_\sharp^\epsilon)\partial_x u_\sharp^\epsilon = 0, \quad (t,x) \in \Omega^\sharp,\ \sharp = -, \star, +, \\ v_-^\epsilon(t,x) = v_\star^\epsilon(t,x), \quad \text{on } x = q_0(t) + \epsilon q^\epsilon(t), \\ p(\rho_\star + \epsilon\rho_\star^\epsilon, S_\star + \epsilon S_\star^\epsilon) = p(\rho_- + \epsilon\rho_-^\epsilon, S_- + \epsilon S_-^\epsilon), \quad \text{on } x = q_0 t + \epsilon q^\epsilon(t), \\ (\sigma + \epsilon d_t\varphi^\epsilon(t))(g(u_+ + \epsilon u_+^\epsilon) - g(u_\star + \epsilon u_\star^\epsilon)) = f(u_+ + \epsilon u_+^\epsilon) - f(u_\star + \epsilon u_\star^\epsilon), \\ \quad \text{on } x = \sigma t + \epsilon\varphi^\epsilon(t), \\ \varphi^\epsilon(0) = 0, \qquad u_-^\epsilon(0,x) = u_{-,0}^\epsilon(x), \end{cases} \tag{2.1.14}$$

where $A(u_\sharp + \epsilon u_\sharp^\epsilon) = (\nabla g(u_\sharp + \epsilon u_\sharp^\epsilon))^{-1}\nabla f(u_\sharp + \epsilon u_\sharp^\epsilon)$.

In order to transform problem (2.1.14) into one with fixed boundaries, we introduce the transformations

$$\mathrm{T}^-: \quad \begin{cases} y = t, \\ z = x - q_0 t - \epsilon q^\epsilon(t), \end{cases}$$

for $u_-^\epsilon(t,x)$ in Ω^-, and

$$\mathrm{T}^\star: \quad \begin{cases} y = t, \\ z = t\frac{x - q_0 t - \epsilon q^\epsilon(t)}{\sigma t + \epsilon\varphi^\epsilon(t) - q_0 t - \epsilon q^\epsilon(t)}, \end{cases}$$

for $u_\star^\epsilon(t,x)$ in $\Omega^\star$.

By computation for (2.1.14), it is easy to see that $\tilde{u}_\sharp^\epsilon(y,z) = u_\sharp^\epsilon(t,x)$ ($\sharp = -, \star$) satisfy the following boundary value problem on $\Omega^\sharp$

$$\begin{cases} L_\sharp^\epsilon(u_\sharp^\epsilon, \varphi^\epsilon)u_\sharp^\epsilon = 0, \quad (y,z) \in \Omega^\sharp, & \\ \gamma_1 v_-^\epsilon(t) = \gamma_1 v_\star^\epsilon(t), & \text{on } z = 0, \\ p(\rho_\star + \epsilon\gamma_1\rho_\star^\epsilon(t), S_\star + \epsilon\gamma_1 S_\star^\epsilon(t)) = p(\rho_- + \epsilon\gamma_1\rho_-^\epsilon(t), S_- + \epsilon\gamma_1 S_-^\epsilon(t)), & \text{on } z = 0, \\ \mathcal{F}^\epsilon(t, \gamma_2 u_\star^\epsilon, d_t\varphi^\epsilon, \varphi^\epsilon) = 0, & \text{on } z = t, \\ \varphi^\epsilon(0) = 0, \qquad u_-^\epsilon(0,z) = u_{-,0}^\epsilon(z), & \end{cases} \tag{2.1.15}$$

where tildes are dropped, $\gamma_1 v_\sharp^\epsilon(t) = v_\sharp^\epsilon(t,0)$, $\gamma_2 v_\sharp^\epsilon(t) = v_\sharp^\epsilon(t,t)$,

$$\begin{aligned} \mathcal{F}^\epsilon\big(t, \gamma_2 u_\star^\epsilon, d_t\varphi^\epsilon, \varphi^\epsilon\big) = \frac{1}{\epsilon}\Big(&\big(\sigma + \epsilon d_t\varphi^\epsilon(t)\big)\Big(g\big(u_+ + \epsilon u_+^\epsilon\big(t, \sigma t + \epsilon\varphi^\epsilon(t)\big)\big) - g\big(u_\star + \epsilon\gamma_2 u_\star^\epsilon(t)\big)\Big) \\ &- f\big(u_+ + \epsilon u_+^\epsilon\big(t, \sigma t + \epsilon\varphi^\epsilon(t)\big)\big) + f\big(u_\star + \epsilon\gamma_2 u_\star^\epsilon(t)\big)\Big), \end{aligned} \tag{2.1.16}$$

and $L^\epsilon_\sharp(u^\epsilon_\sharp,\varphi^\epsilon)=\partial_y+N_\sharp\partial_z$, where

$$N_-=A\big(u_-+\epsilon u^\epsilon_-\big)-\lambda_2\big(u_-+\epsilon u^\epsilon_-\big)I,\qquad N_\star=\left(\frac{\partial z}{\partial t}\right)_\star I+\left(\frac{\partial z}{\partial x}\right)_\star A\big(u_\star+\epsilon u^\epsilon_\star\big),\tag{2.1.17}$$

with

$$\left(\frac{\partial z}{\partial t}\right)_\star=-\frac{q^\epsilon(t)+\epsilon z d_t\theta^\epsilon(y)}{\sigma-q_0+\epsilon\theta^\epsilon(y)},\qquad \left(\frac{\partial z}{\partial x}\right)_\star=\frac{1}{\sigma-q_0+\epsilon\theta^\epsilon(y)},\tag{2.1.18}$$

and $\theta^\epsilon(t)=t^{-1}(\varphi^\epsilon(t)-q^\epsilon(t))$ with $q^\epsilon(t)$ satisfying

$$\begin{cases} d_tq^\epsilon(t)=\frac{1}{\epsilon}(\lambda_2(u_\star+\epsilon\gamma_1u^\epsilon_\star)-\lambda_2(u_\star)),\\ q^\epsilon(0)=0.\end{cases}\tag{2.1.19}$$

At first, we give a result for the coefficients of $L^\epsilon_\sharp$ as in [11, Lemma 2.1].

Lemma 2.1.

(1) *The matrices N_- and $N_\star$ are smooth with respect to arguments ϵu^ϵ_- $(\epsilon u^\epsilon_\star,\epsilon d_t\varphi^\epsilon,\epsilon\theta^\epsilon,$ $\epsilon z d_t\theta^\epsilon)$ around the origin, and at the origin, respectively,*

$$N_-(0)=A_--q_0I,\qquad N_\star(0)=(\sigma-q_0)^{-1}(A_\star-q_0I).\tag{2.1.20}$$

(2) *For any given $(q^\epsilon,\varphi^\epsilon)\in\tilde{C}^2_\epsilon([0,T])$ with $q^\epsilon(0)=\varphi^\epsilon(0)=0$, it follows that $\{\theta^\epsilon(y),$ $zd_t\theta^\epsilon(y)\}_{\epsilon>0}$ are bounded in $C^1_\epsilon(\Omega^\sharp)$.*

Setting $R_\sharp=(r^\sharp_1,r^\sharp_2,r^\sharp_3)$, $L_\sharp=(l^\sharp_1,l^\sharp_2,l^\sharp_3)^{\mathrm T}$ and $\Lambda_\sharp=\mathrm{diag}[\lambda^\sharp_1,\lambda^\sharp_2,\lambda^\sharp_3]$, we have $L_\sharp R_\sharp=I$ and $L_\sharp A(u_\sharp)R_\sharp=\Lambda_\sharp$. Suppose that the solutions $(u^\epsilon_-,u^\epsilon_\star,\varphi^\epsilon)$ of problem (2.1.15) have the forms

$$\begin{cases} u^\epsilon_\sharp(y,z)=R_\sharp U_\sharp\big(y,z;\frac{y}{\epsilon},\frac{z}{\epsilon}\big)+\epsilon R_\sharp V_\sharp\big(y,z;\frac{y}{\epsilon},\frac{z}{\epsilon}\big)+o(\epsilon),\\ \varphi^\epsilon(t)=\varphi\big(t,\frac{t}{\epsilon}\big)+\epsilon\phi\big(t,\frac{t}{\epsilon}\big)+o(\epsilon),\end{cases}\tag{2.1.21}$$

where $U_\sharp(y,z;\xi,\eta)$, $V_\sharp(y,z;\xi,\eta)$ and $(\varphi(t,\tau),\phi(t,\tau))$ are almost periodic in $(\xi,\eta)\in\mathbb{R}^2$ and $\tau\in\mathbb{R}$, respectively. For convenience, let $q^\epsilon(t)$ admit the formal expansion

$$q^\epsilon(t)=q\left(t,\frac{t}{\epsilon}\right)+\epsilon Q\left(t,\frac{t}{\epsilon}\right)+o(\epsilon),\tag{2.1.22}$$

with (q,Q) being almost periodic in $\tau\in\mathbb{R}$. Obviously, the profiles of $q^\epsilon(t)$ can be obtained by those of $u^\epsilon_\star$ from problem (2.1.19). As in [11, pp. 1630–1635], we can formally deduce the problem of $(U_-,U_\star,\varphi)$ from (2.1.15).

Set $\xi=y/\epsilon$, $\eta=z/\epsilon$ and $\tau=t/\epsilon$. Plugging the formal expressions (2.1.21) into the boundary conditions in (2.1.15), grouping each power of ϵ and using the Rankine–Hugoniot condition, it follows

$$\partial_\tau\varphi=0,\tag{2.1.23}$$

which means the leading term φ of shock front φ^ϵ do not oscillate. Moreover, from (2.1.19), we can obtain

$$\partial_\tau q = 0. \tag{2.1.24}$$

In order to formulate the problem of $U_\sharp$, let us define $\mathbb{E}_\sharp = \mathrm{diag}[\mathbb{E}_\sharp^1, \mathbb{E}_\sharp^2, \mathbb{E}_\sharp^3]$ by

$$\begin{cases} \mathbb{E}_-^k u(y,z;\xi,\eta) = \lim_{\rho\to\infty} \frac{1}{2\rho}\int_{-\rho}^{\rho} u(y,z;\xi+s,\eta-(q_0-\lambda_k^-)s)\,ds, \\ \mathbb{E}_\star^k u(y,z;\xi,\eta) = \lim_{\rho\to\infty} \frac{1}{2\rho}\int_{-\rho}^{\rho} u(y,z;\xi+s,\eta-\frac{1}{\sigma-q_0}(q_0-\lambda_k^\star)s)\,ds, \end{cases} \tag{2.1.25}$$

the mean value operators for any $u \in \mathcal{C}^0(\Omega:\mathbb{R}^2)$.

Denote by

$$\begin{cases} P_-(\partial_\xi,\partial_\eta) = \partial_\xi + (\Lambda_- - q_0 I)\partial_\eta, \\ P_\star(\partial_\xi,\partial_\eta) = \partial_\xi + \frac{1}{\sigma-q_0}(\Lambda_\star - q_0 I)\partial_\eta. \end{cases} \tag{2.1.26}$$

Then, as usual, we have

$$\begin{array}{ll} (1) & \mathbb{E}_\sharp U_\sharp = U_\sharp \text{ is equivalent to } P_\sharp(\partial_\xi,\partial_\eta)U_\sharp = 0, \quad \text{and} \\ (2) & \text{for any } V_\sharp \in \mathcal{C}^1(\Omega^\sharp:\mathbb{R}^2), \quad \mathbb{E}_\sharp P_\sharp(\partial_\xi,\partial_\eta)V_\sharp = 0. \end{array} \tag{2.1.27}$$

By formal analysis, it follows that the leading terms of $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon)$ satisfy the problem:

$$\begin{cases} \mathbb{E}_\sharp U_\sharp = U_\sharp, \\ \mathbb{E}_-(P_-(\partial_y,\partial_z)U_- + L_- B_-(R_- U_-, R_-\partial_\eta U_-) - \chi_1 \partial_\eta U_-) = 0, \\ \mathbb{E}_\star(P_\star(\partial_y,\partial_z)U_\star + \frac{1}{\sigma-q_0} L_\star B_\star(R_\star U_\star, R_\star \partial_\eta U_\star) + (dI + h\Lambda_\star)\partial_\eta U_\star) = 0, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \quad \text{on } z=\eta=0, \\ \chi_2(g(u_+) - g(u_\star)) + \sigma(\nabla g(u_+)\mathbf{U}_+ - \nabla g(u_\star)R_\star \gamma_2 u_\star) \\ \quad = \nabla f(u_+)\mathbf{U}_+ - \nabla f(u_\star) R_\star \gamma_2 U_\star, \quad \text{on } z=t,\ \eta=\tau, \\ U_-|_{y=\xi=0} = U_{-,0}(z;\eta), \end{cases} \tag{2.1.28}$$

where $\chi_1(t,\tau) = d_t q(t) + \partial_\tau Q(t,\tau)$ and $\chi_2(t,\tau) = d_t\varphi(t) + \partial_\tau \phi(t,\tau)$ are leading terms of the contact discontinuity speed $\partial_t q^\epsilon(t)$ and the shock speed $\partial_t \varphi^\epsilon(t)$, respectively, $\mathbf{U}_+(t,\tau) = U_+(t,\sigma t;\tau,\sigma\tau+\varphi(t))$, $(d,h) \in C^1$ is given by the following formula:

$$\begin{cases} d = \frac{(q_0 y + \sigma z - q_0 z)(\varphi - q)}{(\sigma-q_0)^2 y^2} - \frac{(y-z)\chi_1 + z\chi_2}{(\sigma-q_0)y}, \\ h = \frac{q-\varphi}{(\sigma-q_0)^2 y} \end{cases} \tag{2.1.29}$$

and

$$\mathcal{S} = \begin{pmatrix} c_\star & c_- \\ \rho_\star c_\star & -\rho_- c_- \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} c_\star & c_- \\ -\rho_\star c_\star & \rho_- c_- \end{pmatrix}. \tag{2.1.30}$$

Similar to the computation in [11, p. 1634], we know that (2.1.28) is equivalent to the following problem:

$$\begin{cases} \mathbb{E}_\sharp U_\sharp = U_\sharp, \\ P_-(\partial_y, \partial_z)U_- + \mathbb{E}_-(L_-B_-(R_-U_-, R_-\partial_\eta U_-)) - d_t q \partial_\eta U_- = 0, \\ P_\star(\partial_y, \partial_z)U_\star + \frac{1}{\sigma - q_0}\mathbb{E}_\star(L_\star B_\star(R_\star U_\star, R_\star \partial_\eta U_\star)) + (\tilde{d} I + h\Lambda_\star)\partial_\eta U_\star = 0, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2(g(u_+) - g(u_\star)) + \sigma(\nabla g(u_+)\mathbf{U}_+ - \nabla g(u_\star) R_\star \gamma_2 U_\star) \\ \quad = \nabla f(u_+)\mathbf{U}_+ - \nabla f(u_\star) R_\star \gamma_2 U_\star, \quad \text{on } z = t,\ \eta = \tau, \\ U_-|_{y=\xi=0} = U_{-,0}(z; \eta), \end{cases} \tag{2.1.31}$$

where

$$\tilde{d} = \frac{(q_0 y + \sigma z - q_0 z)(\varphi - q)}{(\sigma - q_0)^2 y^2} - \frac{(y - z)d_t q + z d_t \varphi}{(\sigma - q_0) y} \tag{2.1.32}$$

is independent of (ξ, η).

To solve (2.1.31), we should first determine (q, φ). By acting the mean value operator

$$\bar{u}(y, z) = \mathbb{E}_0 u(y, z; \xi, \eta) = \lim_{\rho \to \infty} \frac{1}{(2\rho)^2} \int_{-\rho}^{\rho} \int_{-\rho}^{\rho} u(y, z; \xi, \eta)\, d\xi\, d\eta,$$

on (2.1.31), it follows that $(\bar{U}_-, \bar{U}_\star)$ and (q, φ) satisfy the following linear problem

$$\begin{cases} P_\sharp(\partial_y, \partial_z)\bar{U}_\sharp = 0, \\ \mathcal{S}(\gamma_1 \bar{U}_{\star,3}, \gamma_1 \bar{U}_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 \bar{U}_{\star,1}, \gamma_1 \bar{U}_{-,3})^{\mathrm{T}}, \quad \text{on } z = 0, \\ d_t\varphi(g(u_+) - g(u_\star)) + (\sigma \nabla g(u_+) - \nabla f(u_+))\bar{\mathbf{U}}_+ \\ \quad - (\sigma \nabla g(u_\star) - \nabla f(u_\star)) R_\star \gamma_2 \bar{U}_\star = 0, \quad \text{on } z = t, \\ \varphi(0) = 0, \qquad U_-|_{y=0} = U_{-,0}(z) \end{cases} \tag{2.1.33}$$

and

$$\begin{cases} d_t q(t) = \int_0^t (-c_- \gamma_1 \bar{U}_{-,1} + c_- \gamma_1 \bar{U}_{-,3})\, ds, \\ q(0) = 0, \end{cases} \tag{2.1.34}$$

where $\gamma_1 \bar{U}_{\sharp,j}(t) = \bar{U}_{\sharp,j}(t, 0)$ $(j = 1, 3)$, $\gamma_2 \bar{U}_\star(t) = \bar{U}_\star(t, t)$ and $\bar{\mathbf{U}}_+ = U_+(t, \sigma t)$.

Finally, for functions h and $\tilde{d}$ given by (2.1.29) and (2.1.32), respectively, similar to [11, Lemma 2.2]. We have:

Lemma 2.2. *If $q, \varphi \in C^2([0, T])$ and $q(0) = \varphi(0) = 0$, then $\tilde{d}, h \in W^{1,\infty}(\Omega_T^\star)$.*

2.2. Compatibility conditions and main results

As mentioned above, compatibility conditions of (2.1.15), (2.1.31) and (2.1.33) are needed. Note that (2.1.33) is deduced from (2.1.31) with $\chi_2(t,\tau)=d_t\varphi(t)+\partial_\tau\phi(t,\tau)$, the compatibility conditions of (2.1.33) immediately follow from those of (2.1.31).

Two compatibility conditions for problem (2.1.15) are:

(1) There exists $u^\epsilon_\star(0,0)$ such that

$$\begin{cases} u^\epsilon_{-,2}(0,0)=u^\epsilon_{\star,2}(0,0),\\ p(\rho_\star+\epsilon u^\epsilon_{\star,1}(0,0),S_\star+\epsilon u^\epsilon_{\star,3}(0,0))=p(\rho_-+\epsilon u^\epsilon_{-,1}(0,0),S_-+\epsilon u^\epsilon_{-,3}(0,0)),\\ (\sigma+\epsilon d_t\varphi^\epsilon(0))(g(u_++\epsilon u^\epsilon_+(0,0))-g(u_\star+\epsilon u^\epsilon_\star(0,0)))-f(u_++\epsilon u^\epsilon_+(0,0))\\ \quad +f(u_\star+\epsilon u^\epsilon_\star(0,0))=0. \end{cases} \tag{2.2.1}$$

(2) There exist $\partial_z u^\epsilon_\star(0,0)$ such that

$$\begin{cases} (N_-(0)u^{\epsilon\prime}_{-,0})_2=(N_\star(0)\partial_z u^\epsilon_\star(0,0))_2,\\ \nabla p(\rho_-+\epsilon u^\epsilon_{-,1}(0,0),S_-+\epsilon u^\epsilon_{-,3}(0,0))((N_-(0)u^{\epsilon\prime}_{-,0})_1,(N_-(0)u^{\epsilon\prime}_{-,0})_3)^{\mathrm T}\\ \quad =\nabla p(\rho_\star+\epsilon u^\epsilon_{\star,1}(0,0),S_\star+\epsilon u^\epsilon_{\star,3}(0,0))((N_\star(0)\partial_z u^\epsilon_\star(0,0))_1,(N_\star(0)\partial_z u^\epsilon_\star(0,0))_3)^{\mathrm T},\\ d_t^2\varphi^\epsilon(0)(g(u_++\epsilon u^\epsilon_{+,0}(0))-g(u_\star+\epsilon u^\epsilon_\star(0,0)))-f(u_++\epsilon u^\epsilon_{+,0}(0))\\ \quad +f(u_\star+\epsilon u^\epsilon_\star(0,0))+(\sigma+\epsilon d_t\varphi^\epsilon(0))\big\{(\nabla g(u_++\epsilon u^\epsilon_{+,0}(0))\\ \quad -\nabla f(u_++\epsilon u^\epsilon_{+,0}(0)))u^{\epsilon\prime}_{+,0}(0)-(\nabla g(u_\star+\epsilon u^\epsilon_\star(0,0))\\ \quad -\nabla f(u_\star+\epsilon u^\epsilon_\star(0,0)))(I-N^\epsilon(0))\partial_z u^\epsilon_\star(0,0)\big\}=0, \end{cases} \tag{2.2.2}$$

where $(.)_k$ denote the kth component of $(.)$, $u^\epsilon_\star(0,0)$ and $d_t\varphi^\epsilon(0)$ are determined by (2.2.1).

From (2.1.27), we know that $U_-(y,z;\xi,\eta)=\tilde U(y,z,\theta)$ with $\theta=(q_0I-\Lambda_-)\xi+\eta$ and $U_\star(y,z;\xi,\eta)=\tilde U_\star(y,z,\theta)$ with $\theta=(q_0I-\Lambda_\star)\xi+(\sigma-q_0)\eta$. Thus two compatibility conditions for problem (2.1.31) are:

(1) There exists $\tilde U_\star(0,0;(q_0I-\Lambda_\star)\xi)$ and $\tilde U_\star(0,0;(q_0I-\Lambda_\star)\xi+(\sigma-q_0)\eta)$ such that

$$\begin{cases} \mathcal S(\bar U_{\star,3}(0,0;(q_0-\lambda^\star_3)\xi),\bar U_{-,1}(0,0;(q_0-\lambda^-_1)\xi))^{\mathrm T}\\ \quad =\mathcal J(\bar U_{\star,1}(0,0;(q_0-\lambda^\star_1)\xi),\bar U_{-,3}(0,0;(q_0-\lambda^-_3)\xi))^{\mathrm T},\\ \chi_2(0,\eta)(g(u_+)-g(u_\star))+(\sigma\nabla g(u_+)-\nabla f(u_+))U_{+,0}(\eta,\sigma\eta)\\ \quad -(\sigma\nabla g(u_\star)-\nabla f(u_\star))R_\star\bar U_\star(0,0;(q_0I-\Lambda_\star)\xi+(\sigma-q_0)\eta)=0. \end{cases} \tag{2.2.3}$$

(2) There exists $\partial_z\bar U_\star(0,0,(q_0I-\Lambda_\star)\xi)$ and $\partial_z\bar U_\star(0,0,(q_0I-\Lambda_\star)\xi+(\sigma-q_0)\eta)$ such that

$$\begin{cases} \mathcal S(\square_{\star,3},\square_{-,1})^{\mathrm T}=\mathcal J(\square_{\star,1},\square_{-,3})^{\mathrm T},\\ \partial_t\chi_2(0,\eta)(g(u_+)-g(u_\star))+(\sigma\nabla g(u_+)-\nabla f(u_+))U'_{+,0}(\eta,\sigma\eta)\\ \quad -(\sigma\nabla g(u_\star)-\nabla f(u_\star))R_\star(\partial_y+\partial_z)\bar U_\star(0,0;(q_0I-\Lambda_\star)\xi+(\sigma-q_0)\eta)=0, \end{cases} \tag{2.2.4}$$

where

$$\square_- = (\Lambda_- - q_0 I)\partial_z \bar U_- + E_-\big(L_- B_-(R_- \bar U_-, R_- \partial_\theta \bar U_-)\big) - d_t q|_{y=z=0}\partial_\theta \bar U_-,$$

$$\square_\star = \frac{\Lambda_\star - q_0 I}{\sigma - q_0}\partial_z \bar U_\star + \frac{1}{\sigma - q_0}E_\star\big(L_\star B_\star\big(R_\star \bar U_\star, R_\star(\sigma - q_0)\partial_\theta \bar U_\star\big)\big) + (\tilde d I + h\Lambda_\star)|_{y=z=0}\partial_\theta \bar U_\star,$$

with $\bar U_\sharp = \bar U_\sharp(0,0;(q_0 I - \Lambda_\sharp)\xi)$, $\partial_z \bar U_\sharp = \partial_z \bar U_\sharp(0,0;(q_0 I - \Lambda_\sharp)\xi)$, $\partial_\theta \bar U_\sharp = \partial_\theta \bar U_\sharp(0,0;(q_0 I - \Lambda_\sharp)\xi)$, $\square_{\sharp,j}$ $(j=1,3)$ denote the jth component of $\square_\sharp$, and

$$\begin{aligned}
&\partial_y \bar U_\star\big(0,0;(q_0 I - \Lambda_\star)\xi + (\sigma - q_0)\eta\big)\\
&\quad = -\frac{\Lambda_\star - q_0 I}{\sigma - q_0}\partial_z \bar U_\star - \frac{1}{\sigma - q_0}\mathbb{E}_\star\big(L_\star B_\star\big(R_\star \bar U_\star, R_\star(\sigma - q_0)\partial_\theta \bar U_\star\big)\big)\\
&\qquad - (\tilde d I + h\Lambda_\star)|_{y=z=0}(\sigma - q_0)\partial_\theta \bar U_{-\star},
\end{aligned}$$

with $\bar U_\star = \bar U_\sharp(0,0;(q_0 I - \Lambda_\star)\xi + (\sigma - q_0)\eta)$, $\partial_z \bar U_\star = \partial_z \bar U_\star(0,0;(q_0 I - \Lambda_\star)\xi + (\sigma - q_0)\eta)$, $\partial_\theta \bar U_\star = \partial_\theta \bar U_\star(0,0;(q_0 I - \Lambda_\star)\xi + (\sigma - q_0)\eta)$.

The following proposition explains that there indeed exist a class of functions such that the compatibility conditions for problem (2.1.15) up to order one are satisfied. Before giving the proposition, for convenience, let us introduce some notations:

$$r_k^{\sharp,\epsilon} := r_k\big(u_\sharp + \epsilon a_{\sharp,0}^\epsilon\big), \qquad \lambda_k^{\sharp,\epsilon} := \lambda_k\big(u_\sharp + \epsilon a_{\sharp,0}^\epsilon\big), \qquad p_\rho^\sharp := p_\rho\big(\rho_\sharp + \epsilon a_{\sharp,0,1}^\epsilon, S_\sharp + \epsilon a_{\sharp,0,3}^\epsilon\big),$$

$$A = \begin{pmatrix} \rho_- p_\rho^-(\lambda_1^- - q_0) & -\frac{(\lambda_3^\star - q_0)}{\sigma - q_0}\rho_\star p_\rho^\star \\ -(\lambda_1^- - q_0)c_- & -\frac{(\lambda_3^\star - q_0)c_\star}{\sigma - q_0} \end{pmatrix}, \qquad B = \begin{pmatrix} -\rho_- p_\rho^-(\lambda_3^- - q_0) & \frac{(\lambda_1^\star - q_0)}{\sigma - q_0}\rho_\star p_\rho^\star \\ -(\lambda_3^- - q_0)c_- & -\frac{(\lambda_1^\star - q_0)c_\star}{\sigma - q_0} \end{pmatrix},$$

$$g_+^\epsilon(0) := \big(\sigma + \epsilon\sigma^\epsilon\big)\big(\nabla g\big(u_\star + \epsilon a_{\star,0}^\epsilon\big)\big)^{-1}\big(\nabla g\big(u_+ + \epsilon u_{+,0}^\epsilon(0)\big) - \nabla f\big(u_+ + \epsilon u_{+,0}^\epsilon(0)\big)\big)u_{+,0}^{\epsilon\,\prime}(0),$$

$$\begin{aligned}
M^\epsilon = \Big(&-\big(\sigma + \epsilon\sigma^\epsilon\big)\big(1 - \lambda_1^{\star,\epsilon}\big)r_1^{\star,\epsilon}, -\big(\sigma + \epsilon\sigma^\epsilon\big)\big(1 - \lambda_2^{\star,\epsilon}\big)r_2^{\star,\epsilon}, \big(\nabla g\big(u_\star + \epsilon a_{\star,0}^\epsilon\big)\big)^{-1}\\
&\times\big(g\big(u_+ + \epsilon u_{+,0}^\epsilon(0)\big) - g\big(u_\star + \epsilon a_{\star,0}^\epsilon\big) - f\big(u_+ + \epsilon u_{+,0}^\epsilon(0) + f\big(u_\star + \epsilon a_{\star,0}^\epsilon\big)\big)\big)\big)\Big).
\end{aligned}$$

Proposition 2.1. *Suppose that $u_\sharp^\epsilon(0,0) = a_{\sharp,0}^\epsilon$ and $d_t\varphi^\epsilon(0) = \sigma^\epsilon$ satisfying the zeroth order compatibility condition* (2.2.1) *with $\{a_{\sharp,0}^\epsilon, \sigma^\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ being bounded. Then the first order compatibility* (2.2.2) *is equivalent to the following fact*:

$$\tilde u_{-,0,1}^{\epsilon\,\prime}(0) = \vec e_1 A^{-1} B\big(\tilde u_{-,0,3}^{\epsilon\,\prime}(0), \partial_z \tilde u_{\star,1}^\epsilon(0,0)\big)^{\mathrm T}, \tag{2.2.5}$$

i.e., there is a special relation between $\tilde u_{-,0,1}^{\epsilon\,\prime}(0)$ and $g_+^\epsilon(0)$, where

$$\partial_z \tilde u_{\star,1}^\epsilon(0,0) = \vec e_1 C^{-1}\big(\big(-\alpha_1^\epsilon\beta_2^\epsilon, -\alpha_2^\epsilon\beta_2^\epsilon\big)^{\mathrm T}\tilde u_{-,0,3}^{\epsilon\,\prime}(0) + \big(\vec e_1\big(M^\epsilon\big)^{-1}, \vec e_2\big(M^\epsilon\big)^{-1}\big)^{\mathrm T} g_+^\epsilon(0)\big). \tag{2.2.6}$$

Here

$$C = \begin{pmatrix} 1 - \alpha_1^\epsilon\beta_1^\epsilon & 0 \\ -\alpha_2^\epsilon\beta_1^\epsilon & 1 \end{pmatrix}$$

with

$$\alpha_1^\epsilon := \vec{e}_1\big(M^\epsilon\big)^{-1}\big(\sigma+\epsilon\sigma^\epsilon\big)\big(1-\lambda_3^{\star,\epsilon}\big)\frac{\lambda_3^{\star,\epsilon}-q_0}{\sigma-q_0}r_3^{\star,\epsilon}, \qquad \beta_1^\epsilon := \vec{e}_2A^{-1}\frac{\lambda_1^\star-q_0}{\sigma-q_0}\begin{pmatrix}\rho_\star p_\rho^\star\\ -c_\star\end{pmatrix},$$

$$\alpha_2^\epsilon := \vec{e}_2\big(M^\epsilon\big)^{-1}\big(\sigma+\epsilon\sigma^\epsilon\big)\big(1-\lambda_3^{\star,\epsilon}\big)\frac{\lambda_3^{\star,\epsilon}-q_0}{\sigma-q_0}r_3^{\star,\epsilon}, \qquad \beta_2^\epsilon := \vec{e}_2A^{-1}\big(\lambda_3^- - q_0\big)\begin{pmatrix}\rho_- p_\rho^-\\ -c_-\end{pmatrix}.$$

Proof. Let us diagonalize condition (2.2.2). Setting $R_\sharp^\epsilon = (r_1(u_\sharp+\epsilon a_{\sharp,0}^\epsilon), r_2(u_\sharp+\epsilon a_{\sharp,0}^\epsilon), r_3(u_\sharp+\epsilon a_{\sharp,0}^\epsilon))$, obviously we have $(R_\sharp^\epsilon)^{-1} = (l_1(u_\sharp+\epsilon a_{\sharp,0}^\epsilon), l_2(u_\sharp+\epsilon a_{\sharp,0}^\epsilon), l_3(u_\sharp+\epsilon a_{\sharp,0}^\epsilon))^{\mathrm{T}}$. In (2.2.2), by taking the transformation

$$\tilde{u}_\sharp^\epsilon(y,z) = \big(R_\sharp^\epsilon\big)^{-1}u_\sharp^\epsilon(y,z),$$

it follows that (2.2.2) is equal to the following equation

$$\begin{cases} A(\tilde{u}_{-,0,1}^{\epsilon\prime}(0), \partial_z\tilde{u}_{\star,3}^\epsilon(0,0))^{\mathrm{T}} = B(\tilde{u}_{-,0,3}^{\epsilon\prime}(0), \partial_z\tilde{u}_{\star,1}^\epsilon(0,0))^{\mathrm{T}},\\ M^\epsilon(\partial_z\tilde{u}_{\star,1}^\epsilon(0,0), \partial_z\tilde{u}_{\star,2}^\epsilon(0,0), d_t^2\varphi^\epsilon(0))^{\mathrm{T}}\\ \quad = (\sigma+\epsilon\sigma^\epsilon)(1-\lambda_3^{\star,\epsilon})\frac{\lambda_3^{\star,\epsilon}-q_0}{\sigma-q_0}r_3^{\star,\epsilon}\partial_z\tilde{u}_{\star,3}^\epsilon(0,0) + g_+^\epsilon(0). \end{cases} \tag{2.2.7}$$

The matrices A and M^ϵ are nonsingular, thus, from (2.2.7), it follows

$$\begin{aligned} &C\big(\partial_z\tilde{u}_{\star,1}^\epsilon(0,0), \partial_z\tilde{u}_{\star,2}^\epsilon(0,0)\big)^{\mathrm{T}}\\ &\quad = \big(-\alpha_1^\epsilon\beta_2^\epsilon, -\alpha_2^\epsilon\beta_2^\epsilon\big)^{\mathrm{T}}\tilde{u}_{-,0,3}^{\epsilon\prime}(0) + \big(\bar{e}_1\big(M^\epsilon\big), \bar{e}_2\big(M^\epsilon\big)^{-1}\big)^{\mathrm{T}}g_+^\epsilon(0). \end{aligned} \tag{2.2.8}$$

Thanks to (2.1.6) and (H2), by computation, we know that the matrix C is nonsingular. Thus, from (2.2.8), we obtain (2.2.6). Moreover, from (2.2.7), we prove (2.2.5). □

In the same way, we can give another proposition, which implies that there indeed exist a class of functions such that the compatibility conditions for problem (2.1.31) up to order one are also satisfied. For simplicity, we omit it here.

In the following, we state the main assumption of this paper.

(MA). Given the initial data $u_{\pm,0}^\epsilon(x) \in C^1(\omega^\pm)$ satisfying the compatibility conditions (2.2.1) and (2.2.2) for problem (2.1.15) for any $\epsilon \in (0,\epsilon_0]$, then there are $U_\pm(x,\eta) \in \mathcal{C}^1(\omega^\pm : \mathbb{R})$ satisfying the compatibility conditions (2.2.3) and (2.2.4), such that the asymptotic property (2.1.11) holds.

The main results of this paper are stated as follows.

Theorem 2.1. *Under the above assumption* (MA)*, we have*:

(1) *there are constants T, $\epsilon_0 > 0$ such that problem* (2.1.15) *has unique solutions* $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon)$ *bounded in* $C_\epsilon^1(\Omega_T^-) \times C_\epsilon^1(\Omega_T^\star) \times \tilde{C}_\epsilon^2([0,T])$ *for any* $\epsilon \in (0,\epsilon_0]$;

(2) *there are unique solutions* $(U_-, U_\star, \chi_2) \in \mathcal{C}^1(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^1(\Omega_T^\star : \mathbb{R}^2) \times \mathcal{C}^1([0,T] : \mathbb{R})$ *and* $\varphi(t) \in C^2([0,T])$ *to problems* (2.1.31) *and* (2.1.33);

(3) *when* $\epsilon \to 0$, *the asymptotic expansions*

$$\left\| u_\sharp^\epsilon(y,z) - R_\sharp U_\sharp\left(y,z; \frac{y}{\epsilon}, \frac{z}{\epsilon}\right)\right\|_{1,\Omega_T^\star}^\epsilon = o(1) \tag{2.2.9}$$

and

$$\left\| d_t\varphi^\epsilon(t) - \chi_2\left(t, \frac{t}{\epsilon}\right)\right\|_{1,[0,T]}^\epsilon = o(1), \qquad \left\|\varphi^\epsilon(t) - \varphi(t)\right\|_{L^\infty([0,T])} = o(1) \tag{2.2.10}$$

hold.

Remark 2.2.

(1) From this theorem and (2.1.34), it follows that there exists $\chi_1 \in \mathcal{C}^1([0,T] : \mathbb{R})$ such that

$$\left\| d_t q^\epsilon(t) - \chi_1\left(t, \frac{t}{\epsilon}\right)\right\|_{1,[0,T]}^\epsilon = o(1), \qquad \left\| q^\epsilon(t) - q(t)\right\|_{L^\infty[0,T]} = o(1). \tag{2.2.11}$$

(2) From this theorem, we can easily obtain the asymptotic expansions of the perturbed contact discontinuity $x = q_0 t + \epsilon q^\epsilon(t)$ and the perturbed shock front $x = \sigma t + \epsilon\varphi^\epsilon(t)$.

(3) From this theorem and the results of $u_\pm^\epsilon$, we can also obtain the existence of the perturbed contact discontinuity and shock solution U^ϵ in the form of (2.1.12) to problem (2.1.10) as well as their asymptotic properties.

3. Existence of oscillatory waves

This section concerns the local existence of solutions $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon)$ to problem (2.1.15) by using an iterative scheme. It is necessary to construct the first approximate solution $(u_-^{\epsilon,0}, u_\star^{\epsilon,0}, \varphi^{\epsilon,0}) \in C_\epsilon^1(\Omega_T^-) \times C_\epsilon^1(\Omega_T^\star) \times \tilde{C}_\epsilon^2([0,T])$ to this problem by the following proposition, whose proof can be given similar to [7, Proposition 2.1] by using compatibility conditions.

Proposition 3.1. *Under the assumption* (MA) *in Section* 2.2, *there are approximate solutions* $(u_-^{\epsilon,0}, u_\star^{\epsilon,0}, \varphi^{\epsilon,0})$ *to problem* (2.1.15), *such that they are bounded in* $C_\epsilon^1(\Omega_T^-) \times C_\epsilon^1(\Omega_T^\star) \times \tilde{C}_\epsilon^2([0,T])$ *and satisfy*

$$\begin{cases} L_\sharp^\epsilon(u_\sharp^{\epsilon,0}, \varphi^{\epsilon,0}) u_\sharp^{\epsilon,0} = 0, \quad (y,z) \in \Omega^\sharp, \\ \gamma_1 u_{-,2}^{\epsilon,0}\big|_{y=t=0} = \gamma_1 u_{\star,2}^{\epsilon,0}\big|_{y=t=0}, \\ d_t^k \mathcal{G}^\epsilon(t, \gamma_1 u_-^{\epsilon,0}, \gamma_1 u_\star^{\epsilon,0})\big|_{y=t=0}, \qquad k = 0,1, \\ d_t^k \mathcal{F}^\epsilon(t, \gamma_2 u_\star^{\epsilon,0}, d_t\varphi^{\epsilon,0}, \varphi^{\epsilon,0})\big|_{y=z=t=0}, \quad k = 0,1, \\ u_-^{\epsilon,0}(0,z) = u_{-,0}^\epsilon(z), \end{cases} \tag{3.1}$$

where

$$\mathcal{G}^\epsilon\big(t,\gamma_1 u_-^\epsilon,\gamma_1 u_\star^\epsilon\big)=p\big(\rho_\star+\epsilon\gamma_1\rho_\star^\epsilon,S_\star+\epsilon\gamma_1 S_\star^\epsilon\big)-p\big(\rho_-+\epsilon\gamma_1\rho_-^\epsilon,S_-+\epsilon\gamma_1 S_-^\epsilon\big).\tag{3.2}$$

For any fixed $u_\sharp^\epsilon\in C_\epsilon^1(\Omega_T^\sharp)$ and $\varphi^\epsilon\in\tilde{C}_\epsilon^2([0,T])$, define the Fréchet derivative of $\mathcal{G}^\epsilon$ and $\mathcal{F}^\epsilon$ with respect to $(\gamma_1 u_-^\epsilon,\gamma_1 u_\star^\epsilon)$ and $(\gamma_2 u_\star^\epsilon,d_t\varphi^\epsilon)$

$$\begin{cases}G^\epsilon_{(\gamma_1 u_-^\epsilon,\gamma_1 u_\star^\epsilon)}(w_-,w_\star^1)\\ \quad=p_\rho(\rho_\star+\epsilon\gamma_1\rho_\star^\epsilon,S_\star+\epsilon\gamma_1 S_\star^\epsilon)w_{\star,1}^1+p_S(\rho_\star+\epsilon\gamma_1\rho_\star^\epsilon,S_\star+\epsilon\gamma_1 S_\star^\epsilon)w_{\star,3}^1\\ \qquad-p_\rho(\rho_-+\epsilon\gamma_1\rho_-^\epsilon,S_-+\epsilon\gamma_1 S_-^\epsilon)w_{-,1}-p_S(\rho_-+\epsilon\gamma_1\rho_-^\epsilon,S_-+\epsilon\gamma_1 S_-^\epsilon)w_{-,3},\\ F^\epsilon_{(\gamma_2 u_\star^\epsilon,d_t\varphi^\epsilon)}(w_\star^2,\tau)=(g(u_++\epsilon\mathbf{u}_+^\epsilon)-g(u_\star+\epsilon\gamma_2 u_\star^\epsilon))\tau\\ \qquad\qquad-((\sigma+\epsilon d_t\varphi^\epsilon)\nabla g(u_\star+\epsilon\gamma_2 u_\star^\epsilon)-\nabla f(u_\star+\epsilon\gamma_2 u_\star^\epsilon))w_\star^2,\end{cases}\tag{3.3}$$

where $\mathbf{u}_+^\epsilon(t)=u_+^\epsilon(t,\sigma t+\epsilon\varphi^\epsilon(t))$.

For problem (2.1.15), we take the following iteration scheme:

$$\begin{cases}L_\sharp^\epsilon(u_\sharp^{\epsilon,\nu},\varphi^{\epsilon,\nu})u_\sharp^{\epsilon,\nu+1}=0,\quad (y,z)\in\Omega^\sharp,\\ \gamma_1 u_{-,2}^{\epsilon,\nu+1}=\gamma_1 u_{\star,2}^{\epsilon,\nu+1},\quad \text{on } z=0,\\ G^\epsilon_{(\gamma_1 u_-^{\epsilon,\nu},\gamma_1 u_\star^{\epsilon,\nu})}(\gamma_1 u_-^{\epsilon,\nu+1},\gamma_1 u_\star^{\epsilon,\nu+1})\\ \quad=-\mathcal{G}^\epsilon(t,\gamma_1 u_-^{\epsilon,\nu},\gamma_1 u_\star^{\epsilon,\nu})+G^\epsilon_{(\gamma_1 u_-^{\epsilon,\nu},\gamma_1 u_\star^{\epsilon,\nu})}(\gamma_1 u_-^{\epsilon,\nu},\gamma_1 u_\star^{\epsilon,\nu}),\quad \text{on } z=0,\\ F^\epsilon_{(\gamma_2 u_\star^{\epsilon,\nu},d_t\varphi^{\epsilon,\nu})}(\gamma_2 u_\star^{\epsilon,\nu+1},d_t\varphi^{\epsilon,\nu+1})\\ \quad=-\mathcal{F}^\epsilon(t,\gamma_2 u_\star^{\epsilon,\nu},d_t\varphi^{\epsilon,\nu})+F^\epsilon_{(\gamma_2 u_\star^{\epsilon,\nu},d_t\varphi^{\epsilon,\nu})}(\gamma_2 u_\star^{\epsilon,\nu},d_t\varphi^{\epsilon,\nu}),\quad \text{on } z=t,\\ \varphi^{\epsilon,\nu+1}(0)=0,\qquad u_-^{\epsilon,\nu+1}(0,z)=u_{-,0}^\epsilon(z),\end{cases}\tag{3.4}$$

where the first approximate solutions $(u_-^{\epsilon,0},u_\star^{\epsilon,0},\varphi^{\epsilon,0})$ are given by Proposition 3.1.

To study the iterative problem (3.4), we first consider the linear problem:

$$\begin{cases}L_\sharp^\epsilon(u_\sharp^\epsilon,\varphi^\epsilon)w_\sharp^\epsilon=f_\sharp^\epsilon,\quad (y,z)\in\Omega^\sharp,\\ \gamma_1 w_{-,2}^\epsilon(t)=\gamma_1 w_{\star,2}^\epsilon(t),\qquad \text{on } z=0,\\ G^\epsilon_{(\gamma_1 u_-^\epsilon,\gamma_1 u_\star^\epsilon)}(\gamma_1 w_-^\epsilon,\gamma_1 w_\star^\epsilon)=g_1^\epsilon(t),\qquad \text{on } z=0,\\ F^\epsilon_{(\gamma_2 u_\star^\epsilon,d_t\varphi^\epsilon)}(\gamma_2 w_\star^\epsilon,d_t\phi^\epsilon)=g_2^\epsilon(t),\qquad \text{on } z=t,\\ \phi^\epsilon(0)=0,\qquad w_-^\epsilon(0,z)=w_{-,0}^\epsilon(z).\end{cases}\tag{3.5}$$

Let us diagonalize problem (3.5). Denote by $R_\sharp^\epsilon=(r_1(u_\sharp+\epsilon u_\sharp^\epsilon),r_2(u_\sharp+\epsilon u_\sharp^\epsilon),r_3(u_\sharp+\epsilon u_\sharp^\epsilon))$, and its inverse $(R_\sharp^\epsilon)^{-1}=(l_1(u_\sharp+\epsilon u_\sharp^\epsilon),l_2(u_\sharp+\epsilon u_\sharp^\epsilon),l_3(u_\sharp+\epsilon u_\sharp^\epsilon))^{\mathrm{T}}$. By taking the transformation

$$\tilde{w}_\sharp^\epsilon=\big(R_\sharp^\epsilon\big)^{-1}w_\sharp^\epsilon\tag{3.6}$$

in problem (3.5), and using the fact $(R_\sharp^\epsilon)^{-1}(\partial R_\sharp^\epsilon)=-(\partial(R_\sharp^\epsilon)^{-1})R_\sharp^\epsilon$, it follows that $\tilde{w}_\sharp^\epsilon$ satisfy

$$\begin{cases} \tilde{L}^\epsilon_\sharp(u^\epsilon_\sharp, \varphi^\epsilon)\tilde{w}^\epsilon_\sharp = (R^\epsilon_\sharp)^{-1} f^\epsilon_\sharp + (\tilde{L}^\epsilon_\sharp(u^\epsilon_\sharp, \varphi^\epsilon)(R^\epsilon_\sharp)^{-1})R^\epsilon_\sharp \tilde{w}^\epsilon_\sharp, \\ (\gamma_1 \tilde{w}^\epsilon_{\star,3}, \gamma_1 \tilde{w}^\epsilon_{-,1})^{\mathrm{T}} = \Pi^\epsilon_1(t)(\gamma_1 \tilde{w}^\epsilon_{\star,1}, \gamma_1 \tilde{w}^\epsilon_{-,3})^{\mathrm{T}} + \tilde{g}^\epsilon_1(t), \quad \text{on } z = 0, \\ M^\epsilon(\gamma_2 \tilde{w}^\epsilon_{\star,1}, \gamma_2 \tilde{w}^\epsilon_{\star,2}, d_t\phi^\epsilon)^{\mathrm{T}} \\ \quad = \tilde{g}^\epsilon_2(t) + (\sigma + \epsilon d_t\varphi^\epsilon - \lambda_3(u_\star + \epsilon\gamma_2 u^\epsilon_\star))r_3(u_\star + \epsilon\gamma_2 u^\epsilon_\star)\gamma_2 \tilde{w}^\epsilon_{\star,3}(t), \quad \text{on } z = t, \\ \phi^\epsilon(0) = 0, \qquad \tilde{w}^\epsilon_-(0, z) = (R^\epsilon_-)^{-1} w^\epsilon_{-,0}(z), \end{cases} \tag{3.7}$$

where $\Pi^\epsilon_1(t) = (\mathcal{S}^\epsilon(t))^{-1}\mathcal{J}^\epsilon(t)$ is 2×2 matrix with

$$\mathcal{S}^\epsilon(t) = \begin{pmatrix} c^2_{\star,\epsilon}(\rho_\star + \epsilon\gamma_1 u^\epsilon_{\star,1}) & -c^2_{-,\epsilon}(\rho_- + \epsilon\gamma_1 u^\epsilon_{-,1}) \\ c_{\star,\epsilon} & c_{-,\epsilon} \end{pmatrix},$$
$$\mathcal{J}^\epsilon(t) = \begin{pmatrix} -c^2_{\star,\epsilon}(\rho_\star + \epsilon\gamma_1 u^\epsilon_{\star,1}) & -c^2_{-,\epsilon}(\rho_- + \epsilon\gamma_1 u^\epsilon_{-,1}) \\ c_{\star,\epsilon} & c_{-,\epsilon}, \end{pmatrix}$$

(here $c^2_{\sharp,\epsilon} := p_\rho(\rho_\sharp + \epsilon\gamma_1 u^\epsilon_{\sharp,2}, S_\sharp + \epsilon\gamma_1 u^\epsilon_{\sharp,3})$),

$$\begin{aligned} M^\epsilon = \Big(&\big(\lambda_1\big(u_\star + \epsilon\gamma_2 u^\epsilon_\star\big) - \sigma - \epsilon d_t\varphi^\epsilon\big)r_1\big(u_\star + \epsilon\gamma_2 u^\epsilon_\star\big), \\ &\big(\lambda_2\big(u_\star + \epsilon\gamma_2 u^\epsilon_\star\big) - \sigma - \epsilon d_t\varphi^\epsilon\big)r_2\big(u_\star + \epsilon\gamma_2 u^\epsilon_\star\big), \\ &\big(\nabla g\big(u_\star + \epsilon\gamma_2 u^\epsilon_\star\big)\big)^{-1}\big(g\big(u_+ + \epsilon\mathbf{u}^\epsilon_+\big) - g\big(u_\star + \epsilon\gamma_2 u^\epsilon_\star\big)\big)\Big), \end{aligned} \tag{3.8}$$
$$\tilde{g}^\epsilon_1(t) = \big(\big(\mathcal{S}^\epsilon(t)\big)^{-1} g^\epsilon_1(t)\ 0\big)^{\mathrm{T}}, \qquad \tilde{g}^\epsilon_2 = \big(\nabla g\big(u_\star + \epsilon\gamma_2 u^\epsilon_\star\big)\big)^{-1} g^\epsilon_2(t),$$

and

$$\tilde{L}^\epsilon_\sharp\big(u^\epsilon_\sharp, \varphi^\epsilon\big) = \partial_y + N^\epsilon_- \partial_z,$$

with

$$N^\epsilon_- = \Lambda\big(u_- + \epsilon u^\epsilon_-\big) - \lambda_2\big(u_- + \epsilon u^\epsilon_-\big)I, \qquad N^\epsilon_\star = \left(\frac{\partial z}{\partial t}\right)_\star I + \left(\frac{\partial z}{\partial x}\right)_\star \Lambda\big(u_\star + \epsilon u^\epsilon_\star\big), \tag{3.9}$$

and $((\frac{\partial z}{\partial t})_\star, (\frac{\partial z}{\partial x})_\star)$ being given in (2.1.18).

From (2.1.18), when $\epsilon \in (0, \epsilon_0]$, we have $N^\epsilon_\star = \mathrm{diag}[a^\epsilon_{\star,1}, a^\epsilon_{\star,2}, a^\epsilon_{\star,3}]$ with

$$a^\epsilon_{\star,1} < 0 = a^\epsilon_{\star,2} < 1 < a^\epsilon_{\star,3}. \tag{3.10}$$

In order to study problem (3.7), we first consider the following diagonal problem:

$$\begin{cases} \partial_y w^\epsilon_\sharp + N^\epsilon_\sharp \partial_z w^\epsilon_\sharp = f^\epsilon_\sharp, \\ (\gamma_1 w^\epsilon_{\star,3}, \gamma_1 w^\epsilon_{-,1})^{\mathrm{T}} = \Pi^\epsilon_1(t)(\gamma_1 w^\epsilon_{\star,1}, \gamma_1 w^\epsilon_{-,3})^{\mathrm{T}} + g^\epsilon_1(t), \quad \text{on } z = 0, \\ M^\epsilon(\gamma_2 w^\epsilon_{\star,1}, \gamma_2 w^\epsilon_{\star,2}, d_t\phi^\epsilon)^{\mathrm{T}} \\ \quad = g^\epsilon_2(t) + (\sigma + \epsilon d_t\varphi^\epsilon - \lambda_3(u_\star + \epsilon\gamma_2 u^\epsilon_\star))r_3(u_\star + \epsilon\gamma_2 u^\epsilon_\star)\gamma_2 w^\epsilon_{\star,3}(t), \quad \text{on } z = t, \\ \phi^\epsilon(0) = 0, \qquad w^\epsilon_-(0, z) = w^\epsilon_{-,0}(z), \end{cases} \tag{3.11}$$

where the notations are the same as in (3.7).

For problem (3.11), similar to [2, p. 182], it is easy to solve $w^\epsilon_{-,2}$ and $w^\epsilon_{-,3}$ in Ω_T^-. Denote by $(s, \gamma^\epsilon_{-,1}(s; y, z))$ the characteristic curve of the operator $\partial_y + (\lambda_1(u_- + \epsilon u^\epsilon_-) - \lambda_2(u_- + \epsilon u^\epsilon_-))\partial_z$ passing through (y, z) at $s = y$. If we define the set

$$\mathcal{B}^\epsilon_{-,1} = \{(y, z) \in \Omega_T^-:\ z - \gamma^\epsilon_{-,1}(y; 0, 0) \geqslant 0\},$$
$$\mathcal{C}^\epsilon_{-,1} = \{(y, z) \in \Omega_T^-:\ z - \gamma^\epsilon_{-,1}(y; 0, 0) \leqslant 0\},$$

obviously, $\Omega_T^- = \mathcal{B}^\epsilon_{-,1} \cup \mathcal{C}^\epsilon_{-,1}$. In $\mathcal{C}^\epsilon_{-,1}$, $w^\epsilon_{-,1}(y, z)$ can be given similar to [2, p. 183], while in $\mathcal{B}^\epsilon_{-,1}$, $w^\epsilon_{-,1}(y, z)$ is a problem with its boundary condition being coupled together with those of $w^\epsilon_{-,3}$ and $w^\epsilon_{\star,1}$ on $\{z = 0\}$. The following idea is similar to [11], we want to obtain a functional equation about $\gamma_2 w^\epsilon_{\star,3} = w^\epsilon_{\star,3}|_{y=z}$, which implies the existence of $w^\epsilon_{\star,1}$ and $w^\epsilon_{\star,2}$ in $\Omega_T^\star$, and $w^\epsilon_{-,1}(y, z)$ in $\mathcal{C}^\epsilon_{-,1}$.

Let us derive the equation of $\gamma_2 w^\epsilon_{\star,3}$. Set $\Pi^\epsilon_1(t) = (m_{ij})_{2\times 2}$, $g^\epsilon_1(t) = (g^\epsilon_{11}(t), g^\epsilon_{12}(t))^{\mathrm{T}}$. Then the boundary conditions about $\gamma_1 w^\epsilon_{\star,3}$ and $\gamma_2 w^\epsilon_{\star,1}$ in (3.11) can be written as

$$\begin{cases} \gamma_1 w^\epsilon_{\star,3}(t) = m_{11}\gamma_1 w^\epsilon_{\star,1} + m_{12}\gamma_1 w^\epsilon_{-,3} + g^\epsilon_{11}(t) := m_{11}\gamma_1 w^\epsilon_{\star,1} + h^\epsilon_1(t), \\ \gamma_2 w^\epsilon_{\star,1}(t) = \vec{e}_1 (M^\epsilon)^{-1}(g^\epsilon_2(t) + (\sigma + \epsilon d_t\varphi^\epsilon - \lambda_3(u_\star + \epsilon\gamma_2 u^\epsilon_\star)) \\ \qquad \times r_3(u_\star + \epsilon\gamma_2 u^\epsilon_\star)\gamma_2 w^\epsilon_{\star,3}(t)). \end{cases} \tag{3.12}$$

Set $b^\epsilon_{\star,1} = -a^\epsilon_{\star,1}$, $b^\epsilon_{\star,3} = 1/a^\epsilon_{\star,3}$ and $f^\epsilon_\star = (f^\epsilon_{\star,1}, f^\epsilon_{\star,2}, a^\epsilon_{\star,3} f^\epsilon_{\star,3})^{\mathrm{T}}$, where $a^\epsilon_{\star,1}$ and $a^\epsilon_{\star,3}$ are given in (3.10). Then the equations about $w^\epsilon_{\star,1}$, $w^\epsilon_{\star,2}$ in (3.11) can be written as

$$\begin{cases} \partial_y w^\epsilon_{\star,1} - b^\epsilon_{\star,1}\partial_z w^\epsilon_{\star,1} = f^\epsilon_{\star,1}, \\ b^\epsilon_{\star,3}\partial_y w^\epsilon_{\star,3} + \partial_z w^\epsilon_{\star,3} = f^\epsilon_{\star,3}. \end{cases} \tag{3.13}$$

For any $(y, z) \in \Omega_T^\star$, let $s \to (y^\epsilon(s; y, z), s)$ ($(s, z^\epsilon(s; y, z))$, respectively) be the characteristic curve of $b^\epsilon_{\sharp,3}\partial_y + \partial_z$ ($\partial_y - b^\epsilon_{\sharp,1}\partial_z$, respectively) through (y, z), where $y^\epsilon(s; y, z)$ and $z^\epsilon(s; y, z)$ are solutions to the following problems

$$\begin{cases} d_s y^\epsilon(s; y, z) = b^\epsilon_{\star,3}(y^\epsilon(s; y, z), s), \\ y^\epsilon(z; y, z) = y \end{cases} \tag{3.14}$$

and

$$\begin{cases} d_s z^\epsilon(s; y, z) = -b^\epsilon_{\star,1}(s, z^\epsilon(s; y, z)), \\ z^\epsilon(y; y, z) = z. \end{cases} \tag{3.15}$$

Set $Y^\epsilon(y, z) = y^\epsilon(0; y, z)$ and $Z^\epsilon(y, z) = z^\epsilon(t; y, z)$. Then we have

$$y \leqslant Y^\epsilon(y, z) \leqslant y + z, \qquad z \leqslant Z^\epsilon(y, z) \leqslant y + z, \tag{3.16}$$

and the solutions of $w^\epsilon_{\star,1}$, $w^\epsilon_{\star,3}$ in (3.13) can be expressed as

$$\begin{cases} w^\epsilon_{\star,1}(y,z) = \gamma_2 w^\epsilon_{\star,1}(Z^\epsilon(y,z)) + F^\epsilon_{\star,1}(y,z), \\ w^\epsilon_{\star,3}(y,z) = \gamma_1 w^\epsilon_{\star,3}(Y^\epsilon(y,z)) + F^\epsilon_{\star,3}(y,z), \end{cases} \tag{3.17}$$

with

$$\begin{cases} F^\epsilon_{\star,1}(y,z) = \int_t^y f^\epsilon_{\star,1}(s, z^\epsilon(s;y,z))\,ds, \\ F^\epsilon_{\star,3}(y,z) = \int_0^z f^\epsilon_{\star,3}(y^\epsilon(s;y,z), s)\,ds. \end{cases} \tag{3.18}$$

From (3.17), we obtain

$$\begin{cases} \gamma_1 w^\epsilon_{\star,1}(t) = \gamma_2 w^\epsilon_{\star,1}(Z^\epsilon(t,0)) + F^\epsilon_{\star,1}(t,0), \\ \gamma_2 w^\epsilon_{\star,3}(t) = \gamma_1 w^\epsilon_{\star,3}(Y^\epsilon(t,t)) + F^\epsilon_{\star,3}(t,t). \end{cases} \tag{3.19}$$

Combining (3.19) with (3.12), it follows

$$\begin{aligned} \gamma_2 w^\epsilon_{\star,3} = {} & m_{11}\vec{e}_1\left(M^\epsilon\right)^{-1}\left(\sigma + \epsilon d_t\varphi^\epsilon - \lambda_3\left(u_\star + \epsilon\gamma_2 u^\epsilon_\star\right)\right) r_3\left(u_\star + \epsilon\gamma_2 u^\epsilon_\star\right)\gamma_2 w^\epsilon_{\star,3}\left(X^\epsilon(t),0\right) \\ & + h^\epsilon(t), \end{aligned} \tag{3.20}$$

where $X^\epsilon(t) = Z^\epsilon(Y^\epsilon(t,t),0)$,

$$h^\epsilon(t) = m_{11}h^\epsilon_3\left(Z^\epsilon\left(Y^\epsilon(t,t),0\right)\right) + m_{11}F^\epsilon_{\star,1}\left(Y^\epsilon(t,t),0\right) + h^\epsilon_1\left(Y^\epsilon(t,t)\right) + F^\epsilon_{\star,3}(t,t). \tag{3.21}$$

At first, for the functional equation (3.20), similar to [11, Lemma 3.3], it is easy to obtain the following result:

Lemma 3.1. *Given any $u^\epsilon_\star \in C^1_\epsilon(\Omega^\star_{T_0})$, $\varphi^\epsilon \in \tilde{C}^2_\epsilon([0,T_0])$ and $h^\epsilon \in C^0_\epsilon([0,T_0])$, there is a unique solution $\gamma_2 w^\epsilon_{\star,3} \in C^0_\epsilon([0,T_0])$ to Eq.* (3.20)*, and the estimate*

$$\left\|\gamma_2 w^\epsilon_{\star,3}\right\|_T \leqslant C_0\left\|h^\epsilon\right\|_T, \tag{3.22}$$

is valid for any $T \in (0,T_0]$.

Then, for problem (3.11), we sum-up the above derivation in the following proposition:

Proposition 3.2. *Let $u^\epsilon_\sharp$, $f^\epsilon_\sharp$, $g^\epsilon = (g^\epsilon_1, g^\epsilon_2)$ and $u^\epsilon_{\pm,0}$ be some families of functions bounded in $C^1_\epsilon(\Omega^\sharp_T)$, $C^0_\epsilon(\Omega^\sharp_T)$, $(C^0_\epsilon([0,T]))^2$ and $C^0_\epsilon(\omega^\pm)$, respectively, for some $T \in (0,T_0]$, assume moreover that $\epsilon\|u^\epsilon_\sharp\| \leqslant \eta$ for $\epsilon \in (0,\epsilon_0]$ and the zeroth order compatibility condition hold. Then problem* (3.11) *has a unique solution $w^\epsilon_\sharp$ bounded in $C^0_\epsilon(\Omega^\sharp_T)$, and such that*

$$\left\|w^\epsilon_\sharp(t)\right\|_T \leqslant C\left(\left\|u^\epsilon_{-,0}\right\| + \left\|\mathbf{u}^\epsilon_+\right\|_T + \left\|g^\epsilon\right\|_T + \int_0^T \left\|f^\epsilon_\sharp\right\|_s\,ds\right), \tag{3.23}$$

for some positive constant C, which implies

$$\|w^\epsilon(t)\|_T + \|d_t\phi^\epsilon\|_T \leqslant C\left(\|u^\epsilon_{-,0}\| + \|\mathbf{u}^\epsilon_+\|_T + \|g^\epsilon\|_T + \int_0^T \|f^\epsilon\|_s\,ds\right), \tag{3.24}$$

where $w^\epsilon = (w^\epsilon_-, w^\epsilon_\star)$, $f^\epsilon = (f^\epsilon_-, f^\epsilon_\star)$.

Now, let us turn to the study of problem (3.5).

Proposition 3.3. *Let us take $u^\epsilon_\sharp$, $f^\epsilon_\sharp$, g^ϵ and $u^\epsilon_{\pm,0}$ as in Proposition 3.2, and $\varphi^\epsilon \in \tilde{C}^0_\epsilon([0,T])$, in particular let $\epsilon\|u^\epsilon_\sharp\| \leqslant \eta$ and the zeroth order compatibility condition hold. Then problem (3.5) has a unique solution $w^\epsilon_\sharp$ bounded in $C^0_\epsilon(\Omega^\sharp_T)$, $\phi^\epsilon \in \tilde{C}^0_\epsilon([0,T])$ and there exists some constant C such that*

$$\|w^\epsilon(t)\|_T + \|d_t\phi^\epsilon\|_T \leqslant Ce^{CMT}\left(\|u^\epsilon_{-,0}\| + \|\mathbf{u}^\epsilon_+\|_T + \|g^\epsilon\|_T + \int_0^T \|f^\epsilon\|_s\,ds\right), \tag{3.25}$$

where $M > 1 + \epsilon\|\nabla u^\epsilon\|_T + \|d_t\varphi^\epsilon\|^\epsilon_{1,[0,T]}$ with $u^\epsilon = (u^\epsilon_-, u^\epsilon_\star)$.

Proof. From the above discussions, we know that it is sufficient to consider the diagonal problem (3.7), which is solved by the iterative scheme:

$$\begin{cases} \tilde{L}^\epsilon_\sharp(u^\epsilon_\sharp, \varphi^\epsilon)\tilde{w}^{\epsilon,\nu+1}_\sharp = (R^\epsilon_\sharp)^{-1} f^\epsilon_\sharp - (\tilde{L}^\epsilon_\sharp(u^{\epsilon,\nu}_\sharp, \varphi^\epsilon)(R^\epsilon_\sharp)^{-1})R^\epsilon_\sharp \tilde{w}^{\epsilon,\nu}_\sharp, & \\ (\gamma_1\tilde{w}^{\epsilon,\nu+1}_{\star,3}, \gamma_1\tilde{w}^{\epsilon,\nu+1}_{-,1})^{\mathrm{T}} = \Pi^\epsilon_1(t)(\gamma_1\tilde{w}^{\epsilon,\nu+1}_{\star,1}, \gamma_1\tilde{w}^{\epsilon,\nu+1}_{-,3})^{\mathrm{T}} + g^\epsilon_1(t), & \text{on } z=0, \\ M^\epsilon(\gamma_2\tilde{w}^{\epsilon,\nu+1}_{\star,1}, \gamma_2\tilde{w}^{\epsilon,\nu+1}_{\star,2}, d_t\phi^\epsilon)^{\mathrm{T}} & \\ \quad = \tilde{g}^\epsilon_2(t) + (\sigma + \epsilon d_t\varphi^\epsilon - \lambda_3(u_\star + \epsilon\gamma_2 u^\epsilon_\star))r_3(u_\star + \epsilon\gamma_2 u^\epsilon_\star)\gamma_2\tilde{w}^{\epsilon,\nu+1}_{\star,3}(t), & \text{on } z=t, \\ \phi^{\epsilon,\nu+1}(0) = 0, \qquad \tilde{w}^{\epsilon,\nu+1}_-(0,z) = (R^\epsilon_-)^{-1} w^\epsilon_{-,0}(z), & \end{cases} \tag{3.26}$$

where the first approximate solution $(\tilde{w}^{\epsilon,0}_-, \tilde{w}^{\epsilon,0}_\star, \phi^{\epsilon,0}) \in C^1_\epsilon(\Omega^-_T) \times C^1_\epsilon(\Omega^\star_T) \times \tilde{C}^2_\epsilon([0,T])$ can be constructed.

In view of Proposition 3.2, for each ν we can find a solution $\tilde{w}^{\epsilon,\nu+1}_\star$ to problem (3.26), and estimate (3.24) gives

$$\|\tilde{w}^{\epsilon,\nu+1}\|_T + \|d_t\phi^{\epsilon,\nu+1}\|_T \leqslant C_0\left(\|u^\epsilon_{-,0}\| + \|\mathbf{u}^\epsilon_+\|_T + \|g^\epsilon\|_T + \int_0^T \left(\|f^\epsilon\|_s + k_s\|\tilde{w}^{\epsilon,\nu}\|_s\right)ds\right),$$

which implies

$$\|\tilde{w}^{\epsilon,\nu}\|_T + \|d_t\phi^{\epsilon,\nu}\|_T \leqslant Ce^{CMT}\left(\|u^\epsilon_{-,0}\| + \|\mathbf{u}^\epsilon_+\|_T + \|g^\epsilon\|_T + \int_0^T \|f^\epsilon\|_s\,ds\right), \tag{3.27}$$

where $M > 1+\epsilon\|\nabla u^\epsilon\|_T + \|d_t\varphi^\epsilon\|^\epsilon_{1,[0,T]}$. From this estimate, it is easy to prove that the sequence $\{\tilde{w}_\sharp^{\epsilon,\nu+1}\}$ is a Cauchy sequence in $C^1_\epsilon(\Omega^\sharp_T)$, and its limits is solution to (3.7). Therefore we have found a solution $(w^\epsilon_-, w^\epsilon_\star, \phi^\epsilon)$ to (3.7) and estimate (3.25) follows from (3.27) with possibly another constant C. □

We pass now to smooth C^1 solution, let us consider again problem (3.5).

Proposition 3.4. *Let $u^\epsilon_\sharp$, $f^\epsilon_\sharp$, $g^\epsilon = (g^\epsilon_1, g^\epsilon_2)$, $u^\epsilon_{\pm,0}$ and $\varphi^\epsilon(t)$ be some families of functions and assume that they are bounded in $C^1_\epsilon(\Omega^\sharp_T)$, $C^1_\epsilon(\Omega^\sharp_T)$, $(C^1_\epsilon([0,T]))^2$, $C^1_\epsilon(\omega^\pm)$ and $\tilde{C}^2_\epsilon([0,T])$, respectively, for some $T \in (0, T_0]$. Moreover, assume that $\epsilon\|u^\epsilon_\sharp\| \leqslant \eta$ for $\epsilon \in (0,\epsilon_0]$ and the compatibility condition of problem* (3.5) *up to order one are satisfied. Then problem* (3.5) *has a unique solution $(w^\epsilon_-, w^\epsilon_\star, \phi^\epsilon)$ bounded in $C^1_\epsilon(\Omega^-_T) \times C^1_\epsilon(\Omega^\star_T) \times \tilde{C}^2_\epsilon([0,T])$; $w^\epsilon = (w^\epsilon_-, w^\epsilon_\star)$ and ϕ^ϵ satisfy* (3.25) *and*

$$\begin{aligned}\|\epsilon\nabla w^\epsilon\|_T + \|\epsilon d_t^2\phi^\epsilon\|_T \leqslant Ce^{CMT}\Bigg(&\epsilon M\big(\|u^\epsilon_{-,0}\| + \|\mathbf{u}^\epsilon_+\|_T + \|g^\epsilon\|_T\big) + \epsilon\|f^\epsilon(0)\| \\ &+ \epsilon\big(\|(u^\epsilon_{-,0})'\| + \|d_t\mathbf{u}^\epsilon_+\|_T + \|d_t g^\epsilon\|_T\big) \\ &+ \int_0^T \big(\epsilon M\|f^\epsilon\|_s + \epsilon\|\nabla f^\epsilon\|_s\big)\,ds\Bigg),\end{aligned} \tag{3.28}$$

for some constants T and $M > 1+\epsilon\|\nabla u^\epsilon\|_T + \|d_t\varphi^\epsilon\|^\epsilon_{1,[0,T]}$.

Proof. From Proposition 3.3, we know that problem (3.5) has a continuous solution $(w^\epsilon_-, w^\epsilon_\star, \phi^\epsilon)$, which is obtained after a change of variables from the limit of the sequence $\{\tilde{w}^{\epsilon,\nu}_\sharp\}$ defined through (3.26). At the present assumptions, we see that the data $u^\epsilon_\sharp$, $\tilde{f}^\epsilon_\sharp = (R^\epsilon_\sharp)^{-1}f^\epsilon_\sharp$, g^ϵ, $\tilde{u}^\epsilon_{-,0}$ entering in (3.26) are continuously differentiate function, while $\tilde{m}^\epsilon_\sharp = (\tilde{L}^\epsilon_\sharp(u^\epsilon_\sharp, \varphi^\epsilon)(R^\epsilon)^{-1})R^\epsilon_\sharp$ is barely continuous. However, thanks to the particular form of $\tilde{m}^\epsilon_\sharp$, we can deduce that $\tilde{w}^{\epsilon,\nu}_\sharp$ is continuously differentiable for each ν by applying [7, Lemma 3.5]. Next, similar to [3, Lemma 6.2.7], we can prove that, for fixed ϵ, the sequence $\{\nabla w^{\epsilon,\nu}_-, \nabla w^{\epsilon,\nu}_\star, d^2_t\phi^{\epsilon,\nu}, \nu \in N\}$ is bounded in L^∞ and then the sequence $\{\tilde{w}^{\epsilon,\nu}_-, \tilde{w}^{\epsilon,\nu}_\star, \phi^{\epsilon,\nu}\}$ is equicontinuous. Then Ascoli's theorem is applied and existence of a C^1 solution $\{w^\epsilon_-, w^\epsilon_\star, \phi^\epsilon\}$ to (3.5) is proved.

It remains to prove the C^1 estimate (3.28) and here we cannot proceed the case of shock waves in [1,10,11], i.e., derive with respect to y system (3.5), estimate $\partial_y w^\epsilon_\sharp$, then recover an estimate for $\partial_z w^\epsilon_\sharp$, by inversing the matrix $N_\sharp$, since the boundary is characteristic. In order to overcome this difficulty, we will decouple the problem, solving at first the noncharacteristic components (which need boundary conditions) and then the characteristic component (which need not).

Let us split function u into $u = (u^I, u_2)$ with $u^I = (u_1, u_3)$ Since λ_2 is linearly degenerate, then $\lambda_2 = \lambda_2(u^I)$, and the system $\partial_y u + A(u)\partial_z u = 0$ can be written as

$$\begin{cases}\partial_y u^I_\sharp + C(u)\partial_z u^I = 0,\\ \partial_y u_2 + c(u)\partial_z u^I + \lambda_2(u^I)\partial_z u_2 = 0,\end{cases} \tag{3.29}$$

where $C(u)$ is an 2×2 matrix, $c(u)$ a 2-row vector. The vector c does not vanish, in general.

Thus problem (3.5) can be written as

$$\begin{cases} \partial_y w_\sharp^{\epsilon,I} + C(u_\sharp + \epsilon u_\sharp^\epsilon)\partial_z w_\sharp^{\epsilon,I} = f_\sharp^{\epsilon,I}, \\ \partial_y w_{\sharp,2}^\epsilon + c(u_\sharp + \epsilon u_\sharp^\epsilon)\partial_z w_\sharp^{\epsilon,I} + \lambda_2(u_\sharp + \epsilon u_\sharp^\epsilon)\partial_z w_{\sharp,2}^\epsilon = f_{\sharp,2}^\epsilon, \\ w_\star^{\epsilon,I} - w_-^{\epsilon,I} = g_1^\epsilon, \quad \text{on } z=0, \\ F^\epsilon_{(\gamma_2 u_\star^\epsilon, d_t\varphi^\epsilon)}(\gamma_2 w_\star^\epsilon, d_t\phi^\epsilon) = g_2^\epsilon(t), \quad \text{on } z=t, \\ \phi^\epsilon(0)=0, \qquad w_-^\epsilon(0,z) = w_{-,0}^\epsilon(z), \end{cases} \tag{3.30}$$

where C and c are defined by

$$N_\sharp = \begin{pmatrix} C(u_\sharp + \epsilon u_\sharp^\epsilon) & 0 \\ c(u_\sharp + \epsilon u_\sharp^\epsilon) & \lambda_2(u_\star + \epsilon u_\star^\epsilon) \end{pmatrix},$$

with C and c being given by (3.29), 0 standing for a null 2-column vector.

Define $z_\sharp^\epsilon = \epsilon\partial_z w_\sharp^\epsilon$, then $z_\sharp^\epsilon$ and $\epsilon d_t\phi^\epsilon$ are a solution to

$$\begin{cases} \partial_y z_\sharp^\epsilon + C(u_\sharp + \epsilon u_\sharp^\epsilon)\partial_z z_\sharp^{\epsilon,I} + \partial_z(C(u_\sharp + \epsilon u_\sharp^\epsilon)) z_\sharp^{\epsilon,I} = \epsilon\partial_z f_\sharp^{\epsilon,I}, \\ \partial_y z_{\sharp,2}^\epsilon + c(u_\sharp + \epsilon u_\sharp^\epsilon)\partial_z z_\sharp^{\epsilon,I} + \lambda_2(u_\sharp + \epsilon u_\sharp^\epsilon)\partial_z\partial_z z_{\sharp,2}^\epsilon + \partial_z(c(u_\sharp + \epsilon u_\sharp^\epsilon)) z_\sharp^{\epsilon,I} \\ \quad + \partial_z(\lambda_2(u_\sharp + \epsilon u_\sharp^\epsilon)) z_{\sharp,2}^\epsilon = \epsilon\partial_z f_{\sharp,2}^\epsilon, \\ z_\star^{\epsilon,I} - z_-^{\epsilon,I} = \epsilon\partial_z w_\star^{\epsilon,I} - \epsilon\partial_z w_-^{\epsilon,I}, \quad \text{on } z=0, \\ \nabla g(u_\star + \epsilon\gamma_2 u_\star^\epsilon)\cdot\epsilon\gamma_2\partial_z u_\star^\epsilon \cdot \epsilon d_t\phi^\epsilon + ((\sigma + \epsilon d_t\varphi^\epsilon)\nabla g(u_\star + \epsilon\gamma_2 u_\star^\epsilon) \\ \qquad - \nabla f(u_\star + \epsilon\gamma_2 u_\star^\epsilon))\cdot\epsilon\gamma_2 z_\star^\epsilon \\ \quad = -\partial_z[(\sigma + \epsilon d_t\varphi^\epsilon)\nabla g(u_\star + \epsilon\gamma_2 u_\star^\epsilon) - \nabla f(u_\star + \epsilon\gamma_2 u_\star^\epsilon)]\cdot\epsilon\gamma_2 w_\star^\epsilon, \quad \text{on } z=t, \\ \phi^\epsilon(0)=0, \qquad z_-^\epsilon(0,z) = \epsilon\partial_z w_{-,0}^\epsilon(z), \end{cases} \tag{3.31}$$

which is of the form

$$m_\sharp^\epsilon(y,z) = \begin{pmatrix} \partial_z(C(u_\sharp + \epsilon u_\sharp^\epsilon)) & 0 \\ \partial_z(c(u_\sharp + \epsilon u_\sharp^\epsilon)) & \partial_z(\lambda_2(u_\sharp + \epsilon u_\sharp^\epsilon)) \end{pmatrix},$$

and since $u_\sharp^\epsilon \in C_\epsilon^1(\Omega_T^\sharp)$, then $m_\sharp^\epsilon$ is bounded in $C_\epsilon^0(\Omega_T^\sharp)$, i.e., $\|m_\sharp^\epsilon\| \leqslant M$ for some $M \geqslant C(1 + \epsilon\|\nabla w_\sharp^\epsilon\|_T + \epsilon\|d_t^2\phi^\epsilon\|_T)$. Using Proposition 3.3, we obtain

$$\epsilon\|\partial_z w^\epsilon\|_T + \epsilon\|d_t^2\phi^\epsilon\|_T \leqslant Ce^{CMT}\Bigg(\epsilon\|(u_{-,0}^\epsilon)'\| + \epsilon\|d_t\mathbf{u}_+^\epsilon\|_T + \epsilon\|\partial_z w^{\epsilon,I}|_{z=0}\|_T + \epsilon\|\gamma_2 w_\star^\epsilon\| + \int_0^T \epsilon\|\partial_z f^\epsilon(s)\|_s\,ds\Bigg). \tag{3.32}$$

Here, it is easy to inspect that the second compatibility condition for (3.5) is just the first one for (3.30). Since the problem for $w_\sharp^{\epsilon,I}$

$$\begin{cases} \partial_y w_\sharp^{\epsilon,I} + C(u_\sharp + \epsilon u_\sharp^\epsilon)\partial_z w_\sharp^{\epsilon,I} = f_\sharp^{\epsilon,I}, \\ w_\star^{\epsilon,I} - w_-^{\epsilon,I} = g_1^\epsilon, \quad \text{on } z=0, \quad \text{and} \quad w_-^{\epsilon,I}(0,z) = w_{-,0}^{\epsilon,I}(z), \end{cases} \tag{3.33}$$

noncharacteristic in consequence of strict hyperbolicity, similar to [1], it is easy to find C^1 estimates for $w^{\epsilon,I}$:

$$\epsilon \|\partial_z w^{\epsilon,I}\|_T \leqslant Ce^{CMT}\Bigg(\epsilon M\big(\|u_{-,0}^{\epsilon,I}\| + \|g_1^\epsilon\|_T\big) + \epsilon\|f^{\epsilon,I}(0)\| + \epsilon\big(\|(u_{-,0}^{\epsilon,I})'\| + \|d_t g_1^\epsilon\|_T\big) + \int_0^T \big(\epsilon M\|f^{\epsilon,I}\|_s + \epsilon\|\nabla f^{\epsilon,I}\|_s\big)\,ds\Bigg). \tag{3.34}$$

Substituting (3.34) and (3.25) into (3.32), we obtain an analogous estimate holds for $\epsilon\|\partial_z w^\epsilon\|$ with possibly another constant C. At last, we can obtain the estimate of $\epsilon\|\partial_y w^\epsilon\|$ from the interior equation in (3.29). Proposition 3.4 is therefore proved. □

The following result is devoted to the study of the iterative scheme (3.4), from which we immediately obtain the conclusion of Theorem 2.1(1).

Theorem 3.1. *Let $u_{\pm,0}^\epsilon$ be a bounded families in $C_\epsilon^1(\omega^\pm)$ satisfying the compatibility conditions* (2.2.1) *and* (2.2.2). *Then there exist some $T \in (0, T_0]$ and $\epsilon_0 > 0$ such that problem* (2.1.15) *has a unique solution $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon)$ for $\epsilon \in (0, \epsilon_0)$, which are bounded in $C_\epsilon^1(\Omega_T^-) \times C_\epsilon^1(\Omega_T^\star) \times \tilde{C}_\epsilon^2([0,T])$.*

Proof. We consider the iterative scheme (3.4) starting with $(u_-^{\epsilon,0}, u_\star^{\epsilon,0}, \varphi^{\epsilon,0})$, where $(u_-^{\epsilon,0}, u_\star^{\epsilon,0}, \varphi^{\epsilon,0})$ being given by construction and $q^{\epsilon,\nu}(t) = \epsilon^{-1}\int_0^t (\lambda_2(u_\sharp + \epsilon u_\sharp^{\epsilon,\nu}(s,0)) - \lambda_2(u_\sharp))\,ds$. One first prove that $\epsilon(\|u^{\epsilon,\nu}\|_T + \|d_t\varphi^{\epsilon,\nu}\|_T) \leqslant \eta$ and the sequences $\{u^{\epsilon,\nu}\}$, $\{\varphi^{\epsilon,\nu}\}$ are bounded in $C_\epsilon^1(\Omega_T^-) \times C_\epsilon^1(\Omega_T^\star)$ and $\tilde{C}_\epsilon^2([0,T])$, respectively, by induction on ν if T and ϵ_0 are sufficiently small. In fact, by the boundedness of $u_{\pm,0}^\epsilon$ in $C_\epsilon^1(\omega^\pm)$ and Proposition 3.4, it is easy to prove these facts. Then it can be showed that for ϵ fixed, $\nabla u^{\epsilon,\nu}$ and $d_t^2\varphi^{\epsilon,\nu}$ are not only uniformly bounded, but also equicontinuous, which implies the convergence $u^{\epsilon,\nu} \to u^\epsilon$ and $\varphi^{\epsilon,\nu} \to \varphi^\epsilon$, $q^{\epsilon,\nu} \to q^\epsilon$ also hold in $C_\epsilon^1(\Omega_T^-) \times C_\epsilon^1(\Omega_T^\star)$ and $(\tilde{C}_\epsilon^2([0,T]))^2$ by Ascoli's theorem once again. Thus, we conclude that $(u^\epsilon, q^\epsilon, \varphi^\epsilon) \in C_\epsilon^1(\Omega_T^-) \times C_\epsilon^1(\Omega_T^\star) \times (\tilde{C}_\epsilon^2([0,T]))^2$ are the solution to problem (2.1.15). For the detailed proof, we refer the reader to [7, Lemma 3.1, Theorem 3.1], or, more generally to [3]. Theorem 3.1 is therefore proved. □

4. Existence of profiles

In this section, we consider problem (2.1.31). At first, it is easy to construct the first approximate solutions $(U_-^0, U_\star^0, \chi_2^0) \in \mathcal{C}^1(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^1(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^1([0,T] : \mathbb{R})$ satisfying the compatibility conditions of (2.1.31). Moreover, the following asymptotics

$$\begin{cases} u_\sharp^{\epsilon,0}(y,z) = R_\sharp U_\sharp^0\big(y,z; \frac{y}{\epsilon}, \frac{z}{\epsilon}\big) + o(1), & \text{in } L^\infty(\Omega_T), \\ d_t\varphi^{\epsilon,0}(t) = \chi_2^0\big(t, \frac{t}{\epsilon}\big) + o(1), & \text{in } L^\infty[0,T], \end{cases} \tag{4.1}$$

holds when $\epsilon \to 0$, where $u_\sharp^{\epsilon,0}(y,z) \in C^1_\epsilon(\Omega_T^\sharp)$ and $\varphi^{\epsilon,0}(t) \in \tilde{C}_\epsilon([0,T])$ are the approximate solutions constructed in Section 3. For sake of simplicity, the construction of the first approximate solutions and proof of the asymptotic (4.1) are omitted here. In order to reduce overlaps, we shall refer the reader to [10, Section 4.1]. Without loss of generality, in the remainder of this paper, we suppose that $A_\sharp = (\nabla g(u_\sharp))^{-1}\nabla f(u_\sharp)$ is the diagonal matrix

$$A_\sharp = \Lambda_\sharp = \operatorname{diag}\left[\lambda_1^\sharp, \lambda_2^\sharp, \lambda_3^\sharp\right], \tag{4.2}$$

which can be easily obtained by using a transformation similar to (3.6). In this diagonal case, $r_k^\sharp = \vec{e}_k$ $(k = 1,2,3)$ are the standard unit vectors.

Obviously, the nonlinear problem (2.1.31) can be written as

$$\begin{cases} \mathbb{E}_\sharp U_\sharp = U_\sharp, \\ P_-(\partial_y,\partial_z)U_- - d_t q\partial_\eta U_- + E_- B_-(U_-, \partial_\eta U_-) = 0, \\ P_\star(\partial_y,\partial_z)U_\star + (\tilde{d}_\star I + h_\star \Lambda_\star)\partial_\eta U_\star + \frac{1}{\sigma - q_0} E_\star B_\star(U_\star, \partial_\eta U_\star) = 0, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2(g(u_+) - g(u_\star)) + (\sigma\nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ - (\sigma\nabla g(u_\star) - \nabla f(u_\star))\gamma_2 U_\star = 0, \\ \quad \text{on } z = t,\ \eta = \tau, \\ U_-|_{y=0} = U_{-,0}(z;\theta), \end{cases} \tag{4.3}$$

where $\mathbf{U}_+(t,\tau) = U_+(t,\sigma t;\tau,\sigma t + \varphi^\epsilon(t))$ and $\tilde{d}$ being given by (2.1.32) with $(q,\varphi) \in (C^2([0,T]))^2$ being determined by problems (2.1.33) and (2.1.34).

For the linear problem (2.1.33) and problem (2.1.34), it is easy to get the existence of $\bar{U}_\sharp \in \mathcal{C}^1(\Omega_T^\sharp)$ and $(q,\varphi) \in (C^2([0,T]))^2$.

We solve the nonlinear problem (4.2) by the iterative scheme:

$$\begin{cases} \mathbb{E}_\sharp U_\sharp^{\nu+1} = U_\sharp^{\nu+1}, \\ P_-(\partial_y,\partial_z)U_-^{\nu+1} - d_t q\partial_\eta U_-^{\nu+1} + E_- B_-(U_-^\nu, \partial_\eta U_-^{\nu+1}) = 0, \\ P_\star(\partial_y,\partial_z)U_\star^{\nu+1} + (\tilde{d}_\star I + h_\star \Lambda_\star)\partial_\eta U_\star^{\nu+1} + \frac{1}{\sigma - q_0} E_\star B_\star(U_\star^\nu, \partial_\eta U_\star^{\nu+1}) = 0, \\ \mathcal{S}(\gamma_1 U_{\star,3}^{\nu+1}, \gamma_1 U_{-,1}^{\nu+1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}^{\nu+1}, \gamma_1 U_{-,3}^{\nu+1})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2^{\nu+1}(g(u_+) - g(u_\star)) + (\sigma\nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ - (\sigma\nabla g(u_\star) - \nabla f(u_\star))\gamma_2 U_\star^{\nu+1} = 0, \\ \quad \text{on } z = t,\ \eta = \tau, \\ U_-^{\nu+1}|_{y=0} = U_{-,0}(z;\theta), \end{cases} \tag{4.4}$$

where $(U_-^0, U_\star^0, \chi_2^0)$ being given by construction. It is easy to verify that the compatibility conditions here up to order one are valid for each $\nu \geqslant 0$.

For any $u(y,z;\xi,\eta) \in \mathcal{C}^0(\Omega_T : \mathbb{R}^2)$, define the mean value operator $E_k^\sharp$, $E_3^\sharp$ as follows:

$$\begin{cases} E_k^\sharp u(y,z;\xi,\eta) = \lim_{\rho\to\infty} \frac{1}{2\rho}\int_{-\rho}^{+\rho} u(y,z;\xi+s,\eta - b_k^\sharp s)\,ds, \quad k = 1,2, \\ E_3^\sharp u(y,z;\xi,\eta) = \lim_{\rho\to\infty} \frac{1}{2\rho}\int_{-\rho}^{+\rho} u(y,z;\xi - b_3^\sharp s,\eta + s)\,ds. \end{cases} \tag{4.5}$$

As Joly et al. in [3, Proposition 6.3.1], we can establish the following lemma.

Lemma 4.1. *For any* $(u_\sharp, v_\sharp) \in (\mathcal{C}^1(\Omega_T : \mathbb{R}^2))^2$ *satisfying* $E_\sharp u_\sharp = u_\sharp$, *if we denote by* $B_\sharp(u,v)$ *the bilinear form*

$$B_\sharp(u,v) = \left(B_\sharp^1(u,v), B_\sharp^2(u,v), B_\sharp^3(u,v)\right)^{\mathrm{T}},$$

with $B_\sharp^k(u,v) = \sum_{i,l=1}^3 b_{\sharp,k}^{il} u_i v_l$, *then*

$$\begin{cases} E_\sharp^p B_\sharp^p(\partial_\eta u_\sharp, v_\sharp) = \gamma_\sharp^p(v_\sharp)\partial_\eta u_\sharp^p + \Xi_\sharp^p(u_\sharp, (\partial_\xi - b_p^\sharp \partial_\eta)v_\sharp), \quad p = 1,2, \\ E_\sharp^3 B_\sharp^3(\partial_\eta u_\sharp, v_\sharp) = \gamma_\sharp^3(v_\sharp)\partial_\eta u_\sharp^3 + \Xi_\sharp^3(u_\sharp, (\partial_\eta - b_3^\sharp \partial_\xi)v_\sharp), \end{cases} \tag{4.6}$$

where

$$\gamma_\sharp^k(v_\sharp) = E_\sharp^k\left(\sum_{l=1}^3 b_{\sharp,k}^{kl} v_{\sharp,l}\right) \quad \text{and}$$

$$\Xi_\sharp^k(u_\sharp, v_\sharp) = E_\sharp^k\left(\sum_{i\neq k, l\neq k} \gamma_{\sharp,k}^{il} u_{\sharp,i} v_{\sharp,l}\right), \quad k = 1,2,3, \tag{4.7}$$

with

$$\gamma_{\sharp,p}^{il} = b_{\sharp,p}^{il} \frac{1}{b_p^\sharp - b_i^\sharp}, \quad i \neq p,\ p = 1,2 \quad \text{and} \quad \gamma_{\sharp,3}^{il} = b_{\sharp,3}^{il} \frac{1}{b_1^\sharp b_3^\sharp - 1}, \quad i \neq 3. \tag{4.8}$$

Moreover, if $E_\sharp v_\sharp = v_\sharp$, *then*

$$\Xi_\sharp^k(u_\sharp, v_\sharp) = E_\sharp^k\left(\sum_{i\neq k} \gamma_{\sharp,k}^{il} u_{\sharp,i} v_{\sharp,i}\right), \quad i \neq k,\ k = 1,2,3. \tag{4.9}$$

Applying Lemma 4.1 for problem (4.4), it follows that (4.4) can be written as

$$\begin{cases} \mathbb{E}_\sharp U_\sharp^{\nu+1} = U_\sharp^{\nu+1}, \\ P_-(\partial_y, \partial_z)U_-^{\nu+1} + (-d_t q + \gamma_-(U_-^\nu))\partial_\eta U_-^{\nu+1} + \Xi_-(U_-^{\nu+1}, \partial_\eta U_-^\nu) = 0, \\ P_\star(\partial_y, \partial_z)U_\star^{\nu+1} + \left(\tilde{d}_\star I + h_\star \Lambda_\star + \frac{1}{\sigma - q_0}\gamma_\star(U_\star^\nu)\right)\partial_\eta U_\star^{\nu+1} + \frac{1}{\sigma - q_0}\Xi_\star(U_\star^{\nu+1}, \partial_\eta U_\star^\nu) = 0, \\ \mathcal{S}(\gamma_1 U_{\star,3}^{\nu+1}, \gamma_1 U_{-,1}^{\nu+1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}^{\nu+1}, \gamma_1 U_{-,3}^{\nu+1})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2^{\nu+1}(g(u_+) - g(u_\star)) + (\sigma \nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ \\ \quad - (\sigma \nabla g(u_\star) - \nabla f(u_\star))\gamma_2 U_\star^{\nu+1} = 0, \quad \text{on } z = t,\ \eta = \tau, \\ U_-^{\nu+1}|_{y=0} = U_{-,0}(z;\theta), \end{cases} \tag{4.10}$$

where $(U_-^0, U_\star^0, \chi_2^0)$ can be constructed, $\gamma_\sharp(U_\sharp^\nu) = \mathrm{diag}[\gamma_\sharp^1(U_\sharp^\nu), \gamma_\sharp^2(U_\sharp^\nu), \gamma_\sharp^3(U_\sharp^\nu)]$ and

$$\Xi_\sharp\left(U_\sharp^{\nu+1}, \partial_\theta U_\sharp^\nu\right) = \begin{pmatrix} \Xi_\sharp^1(U_\sharp^{\nu+1}, (\partial_\xi - b_1^\sharp \partial_\eta)U_\sharp^\nu) \\ \Xi_\sharp^2(U_\sharp^{\nu+1}, (\partial_\xi - b_2^\sharp \partial_\eta)U_\sharp^\nu) \\ \Xi_\sharp^3(U_\sharp^{\nu+1}, (\partial_\eta - b_3^\sharp \partial_\xi)U_\sharp^\nu) \end{pmatrix}.$$

To study the iteration problem (4.10), we consider the linear problem:

$$\begin{cases} \mathbb{E}_\sharp U_\sharp = U_\sharp, \\ P_-(\partial_y, \partial_z)U_- + (-d_t q + \gamma_-(V_-))\partial_\eta U_- + \Xi_-(U_-, \partial_\eta V_-) = E_- F_-, \\ P_\star(\partial_y, \partial_z)U_\star + \big(\tilde{d}_\star I + h_\star \Lambda_\star + \frac{1}{\sigma - q_0}\gamma_\star(V_\star)\big)\partial_\eta U_\star + \frac{1}{\sigma - q_0}\Xi_\star(U_\star, \partial_\eta V_\star) = E_\star F_\star, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2(g(u_+) - g(u_\star)) + (\sigma \nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ \\ \quad - (\sigma \nabla g(u_\star) - \nabla f(u_\star))\gamma_2 U_\star = 0, \qquad \text{on } z = t,\ \eta = \tau, \\ U_-|_{y=0} = U_{-,0}(z; \theta), \end{cases} \tag{4.11}$$

where $V = (V_-, V_\star) \in \mathcal{C}^1(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^1(\Omega_T^\star : \mathbb{R}^2)$, $f = (f_-, f_\star) \in \mathcal{C}^1(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^1(\Omega_T^\star : \mathbb{R}^2)$ and $U_\pm \in \mathcal{C}^1(\omega^+; \mathbb{R})$ satisfying the compatibility conditions of (4.11) up to order one.

In order to study the linear problem (4.11), let us first consider the diagonal systems

$$\begin{cases} \mathbb{E}_\sharp U_\sharp = U_\sharp, \\ P_-(\partial_y, \partial_z)U_- + (-d_t q + \gamma_-(V_-))\partial_\eta U_- = E_- F_-, \\ P_\star(\partial_y, \partial_z)U_\star + \big(\tilde{d}_\star I + h_\star \Lambda_\star + \frac{1}{\sigma - q_0}\gamma_\star(V_\star)\big)\partial_\eta U_\star = E_\star F_\star, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2((\nabla g(u_\star))^{-1}(g(u_+) - g(u_\star)) + (\nabla g(u_\star))^{-1}(\sigma \nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ \\ \quad - (\sigma I - \Lambda_\star))R_\star \gamma_2 U_\star = 0, \qquad \text{on } z = t,\ \eta = \tau, \\ U_-|_{y=0} = U_{-,0}(z; \theta), \end{cases} \tag{4.12}$$

where the notations and the assumption are the same as in (4.11). Using condition (H1) with $r_k^\sharp = \vec{e}_k$ for problem (4.12), it follows that (4.12) can be written as

$$\begin{cases} \mathbb{E}_\sharp U_\sharp = U_\sharp, \\ P_-(\partial_y, \partial_z)U_- + (-d_t q + \gamma_-(V_-))\partial_\eta U_- = E_- F_-, \\ P_\star(\partial_y, \partial_z)U_\star + \big(\tilde{d}_\star I + h_\star \Lambda_\star + \frac{1}{\sigma - q_0}\gamma_\star(V_\star)\big)\partial_\eta U_\star = E_\star F_\star, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \qquad \text{on } z = \eta = 0, \\ \bar{M}(\gamma_2 U_{\star,1}, \gamma_2 U_{\star,2}, \chi_2)^{\mathrm{T}} = (\sigma - \lambda_3^\star) r_3^\star \gamma_2 U_{\star,3} + g_2(t, \tau), \quad \text{on } z = t,\ \eta = \tau, \\ U_-|_{y=0} = U_{-,0}(z; \theta), \end{cases} \tag{4.13}$$

where $\bar{M} = ((\lambda_1^\star - \sigma) r_1^\star, (\lambda_2^\star - \sigma) r_2^\star, (\nabla g(u_\star))^{-1}(g(u_+) - g(u_\star)))$, $g_2(t, \tau) = (\nabla g(u_\star))^{-1} \times (\nabla f(u_+) - \sigma \nabla g(u_+))\mathbf{U}_+$.

Similar to Section 3, our aim is to obtain a functional equation of $U_{\star,1}$ and $U_{\star,3}$ on $z = 0$.

From $E_\sharp U_\sharp = U_\sharp$, it follows that $U_{\sharp,l}(y, z; \xi, \eta)$ $(l = 1, 2)$ (respectively $U_{\sharp,3}$) are functions of $(y, z; b_l^\sharp \xi + \eta)$ (respectively of $(y, z; \xi + b_3^\sharp \eta)$) with $U_{\sharp,k}(y, z; \theta)$ $(k = 1, 2, 3)$ being almost periodic in $\theta \in \mathbb{R}$.

Set

$$\begin{cases} k_l^- = -d_t q + \gamma_-^l(V_-), & l = 1, 2, \qquad k_3^- = \frac{1}{q_0 - \lambda_3^-}(-d_t q + \gamma_-^3(V_-)), \\ k_l^\star = \tilde{d} + h\lambda_l^\star + \frac{1}{\sigma - q_0}\gamma_\star^l(V_\star), & l = 1, 2, \qquad k_3^\star = -\frac{\sigma - q_0}{\lambda_3^\star - q_0}(\tilde{d} + h\lambda_3^\star + \frac{1}{\sigma - q_0}\gamma_\star^3(V_\star)), \end{cases}$$

and

$$\begin{cases} E_- f_- = (F_{-,1}, F_{-,2}, (\lambda_3^- - q_0)F_{-,3})^{\mathrm{T}}, \\ E_\star f_\star = \big(F_{\star,1}, F_{\star,2}, \frac{\sigma - q_0}{\lambda_3^\star - q_0} F_{\star,3}\big)^{\mathrm{T}}. \end{cases}$$

By computation, it follows that (4.12) can be written as

$$\begin{cases} \partial_y U_{\sharp,l} - b_l^\sharp \partial_z U_{\sharp,l} + k_l^\sharp \partial_\theta U_{\sharp,l} = F_{\sharp,l}, \\ b_3^\sharp \partial_y U_{\sharp,3} + \partial_z U_{\sharp,3} + k_3^\sharp \partial_\theta U_{\sharp,3} = F_{\sharp,3}, \\ \gamma_1 U_{\star,3} = n_{11}\gamma_1 U_{\star,1} + n_{12}\gamma_1 U_{-,3}, & \text{on } z = \eta = 0, \\ \gamma_1 U_{-,1} = n_{21}\gamma_1 U_{\star,1} + n_{22}\gamma_1 U_{-,3}, & \text{on } z = \eta = 0, \\ \bar{M}(\gamma_2 U_{\star,1}, \gamma_2 U_{\star,2}, \chi_2)^{\mathrm{T}} = (\sigma - \lambda_3^\star) r_3^\star \gamma_2 U_{\star,3} + g_2(t,\tau), & \text{on } z = t,\ \eta = \tau, \\ U_-|_{y=0} = U_{-,0}(z;\theta), \end{cases} \tag{4.14}$$

where the numbers (n_{ij}) are the entries of the matrix $\Pi_1 = \mathcal{S}^{-1}\mathcal{J}$.

Obviously, the boundary condition in (4.14) can be decoupled into

$$\chi_2(t,\tau) = \vec{e}_2 \bar{M}^{-1}\big(\sigma - \lambda_2^\star\big) r_2^\star \gamma_2 U_{\star,3} + \vec{e}_2 \bar{M}^{-1} g_2(t,\tau) \tag{4.15}$$

and

$$\begin{cases} \gamma_1 U_{\star,3}(t,\tau) = n_{11}\gamma_1 U_{\star,1} + n_{12}\gamma_1 U_{-,3}, \\ \gamma_1 U_{-,1}(t,\tau) = n_{21}\gamma_1 U_{\star,1} + n_{22}\gamma_1 U_{-,3}, \\ \gamma_2 U_{\star,1}(t,\tau) = \vec{e}_1 \bar{M}^{-1}(\sigma - \lambda_3^\star) r_3^\star + \vec{e}_1 \bar{M}^{-1} g_2(t,\tau), \\ \gamma_2 U_{\star,2}(t,\tau) = \vec{e}_2 \bar{M}^{-1}(\sigma - \lambda_3^\star) r_3^\star + \vec{e}_2 \bar{M}^{-1} g_2(t,\tau). \end{cases} \tag{4.16}$$

Similar to Section 3, $U_{-,2}$, $U_{-,3}$ can be given. While for $U_{-,1}$, $U_{\star,k}$ $(k = 1,2,3)$, if we obtain $\gamma_1 U_{\star,1}$ or $\gamma_2 U_{\star,3}$, they can be obtained. In the following, we will obtain a functional equation of $\gamma_2 U_{\star,3}$.

For any $(y,z) \in \Omega_T^\star$, set $y_3^\star(s) = y + b_3^\star(z - s)$ and $z_1^\star(s) = (t - z) + b_1^\star(y - s)$ be the characteristic curves of the vectors $b_3^\star \partial_y + \partial_z$ and $\partial_y - b_1^\star \partial_z$, respectively, through (y,z). Let $s \to (y_3^\star(s), s; \mu_3^\star(s; y, z, \theta))$ ($(s, z_1^\star(s); \mu_1^\star(s; y, z, \theta))$, respectively) be the characteristic curve of $b_3^\star \partial_y + \partial_z + k_3^\star \partial_\theta$ ($\partial_y - b_1^\star \partial_z + k_1^\star \partial_\theta$, respectively) through $(y, z; \theta) \in \Omega_T^\star \times \mathbb{R}$, where $\mu_3^\star$ and $\mu_1^\star$ are solutions to the following problems

$$\begin{cases} d_s \mu_3^\star(s; y, z, \theta) = k_3^\star(y_3^\star(s), s; \mu_3^\star(s; y, z, \theta)), \\ \mu_3^\star(z; y, z, \theta) = \theta \end{cases} \tag{4.17}$$

and

$$\begin{cases} d_s \mu_1^\star(s; y, z, \theta) = k_1^\star(s, z_1^\star(s); \mu_1^\star(s; y, z, \theta)), \\ \mu_1^\star(y; y, z, \theta) = \theta. \end{cases} \tag{4.18}$$

Obviously, the solutions $(U_{\sharp,1}, U_{\sharp,3})$ can be expressed as

$$\begin{cases} U_{\star,1}(y,z;b_1^\star\xi+\theta) = U_{\star,1}(t,z_1^\star(t);\mu_1^\star(t)) + F_{\star,1}(y,z;\theta), \\ U_{\star,3}(y,z;\xi+b_3^\star\theta) = U_{\star,3}(y_3^\star(0),0;\mu_3^\star(0)) + F_{\star,3}(y,z;\theta), \end{cases} \tag{4.19}$$

where $y_3^\sharp(0) = y + b_3^\sharp z$, $z_1^\sharp(t) = t - z + b_1^\sharp(y-t)$, and

$$\begin{cases} F_{\sharp,1}(y,z;\theta) = \int_t^y f_{\sharp,1}(s,z_1^\sharp(s);\mu_1^\sharp(s))\,ds, \\ F_{\sharp,3}(y,z;\theta) = \int_0^z f_{\sharp,3}(y_3^\sharp(s),s;\mu_3^\sharp(s))\,ds, \end{cases} \tag{4.20}$$

with $\mu_1^\sharp(s) = \mu_1^\sharp(s;y,z,\theta)$ and $\mu_3^\sharp(s) = \mu_3^\sharp(s;y,z,\theta)$.

From (4.19), we have

$$\begin{cases} \gamma_1 U_{\star,1}(t,b_1^\star\tau) = \gamma_2 U_{\star,1}(t,\mu_1^\star(t;t,0,b_1^\star\tau)) + F_{\star,1}(t,0;b_1^\star\tau), \\ \gamma_2 U_{\star,3}(t,(1+b_3^\star)\tau) = \gamma_1 U_{\star,3}((1+b_3^\star)t,\mu_3^\star(0;t,t,(1+b_3^\star))) + F_{\star,3}(t,t;(1+b_3^\star)). \end{cases} \tag{4.21}$$

Combining (4.21) with the first and third lines in (4.16), it gives rise to

$$\gamma_2 U_{\star,3}\big(t,\big(1+b_3^\star\big)\tau\big) = n_{11}\vec{e}_1\bar{M}^{-1}\big(\sigma-\lambda_3^\star\big)r_3^\star\gamma_2 U_{\star,3}\big(\big(1+b_3^\star\big)t,\theta(t,\tau)\big) + H(t,\tau), \tag{4.22}$$

where

$$\theta(t,\tau) = \big(1+b_1^\star\big)^{-1}\big(1+b_3^\star\big)\mu_1^\star\big(\big(1+b_3^\star\big)t;\big(1+b_3^\star\big)t,0,b_1^\star\mu_3^\star\big(0;t,t,\big(1+b_3^\star\big)\tau\big)\big) \tag{4.23}$$

and

$$\begin{aligned} H(t,\tau) = {} & n_{11}\vec{e}_1\bar{M}^{-1}g_2\big(\big(1+b_3^\star\big)t,\big(1+b_3^\star\big)^{-1}\theta(t,\tau)\big) \\ & + n_{11}F_{\star,l}\big(\big(1+b_3^\star\big)t,0;b_l^\star\mu_3^\star\big(0;t,t,\big(1+b_3^\star\big)\tau\big)\big) \\ & + n_{12}\gamma_1 U_{-,3}\big(\big(1+b_3^\star\big)t;\mu_3^\star\big(0;t,t,\big(1+b_3^\star\big)\tau\big)\big) + F_{\star,3}\big(t,t;\big(1+b_3^\star\big)\tau\big). \end{aligned} \tag{4.24}$$

Denote by $M = \|V_\star\|_{1,T}$ for a given $V_\star \in \mathcal{C}^1(\Omega_{T_0^\star}:\mathbb{R}^2)$, for the functional equation (4.22), similar to [11, Lemma 4.2], it follows:

Lemma 4.2.

(1) *Given any $V_\star \in \mathcal{C}^1(\Omega_T^\star:\mathbb{R}^2)$ and $H \in \mathcal{C}^0([0,T_0]:\mathbb{R})$, such that $U_{\pm,0} \in \mathcal{C}^0([0,T_0]:\mathbb{R})$, there is a unique solution $\gamma_2 U_{\star,3} \in \mathcal{C}^0([0,T_0]:\mathbb{R})$ to Eq.* (4.21)*, and the estimate*

$$\|\gamma_2 U_{\star,3}\|_T \leqslant C_0\|H\|_T \tag{4.25}$$

holds for any $T \in (0,T_0]$.

(2) *If* $H \in \mathcal{C}^1([0, T_0]:\mathbb{R})$, $U_\pm \in \mathcal{C}^0([0, T_0]:\mathbb{R})$, *then there exists* $T_1 \in (0, T_0]$ *depending only upon* M_{T_0}, *such that the solution* $\gamma_2 U_{\star,3}$ *of* (4.21) *belong to* $\mathcal{C}^1([0, T_0]:\mathbb{R})$, *and the estimate*

$$\|\gamma_2 U_{\star,3}\|_{1,T} \leqslant C_0 \|H\|_{1,T} \tag{4.26}$$

holds for any $T \in (0, T_1]$.

Then for problem (4.12), similar to Section 3, we sum up the following proposition.

Proposition 4.1. *For any given* $V = (V_-, V_\star) \in \mathcal{C}^1(\Omega^-_{T_0}:\mathbb{R}^2) \times \mathcal{C}^1(\Omega^\star_{T_0}:\mathbb{R}^2)$, $F = (F_-, F_\star) \in \mathcal{C}^0(\Omega^-_{T_0}:\mathbb{R}^2) \times \mathcal{C}^1(\Omega^\star_{T_0}:\mathbb{R}^2)$, $U_{\pm,0} \in \mathcal{C}^1(\omega^\pm:\mathbb{R})$ *and* $G(t,\tau) \in \mathcal{C}^0([0, T_0]:\mathbb{R})$, *assume that the zeroth compatibility condition holds of* (4.12). *Then there are unique solutions* $U = (U_-, U_\star) \in \mathcal{C}^0(\Omega^-_T:\mathbb{R}^2) \times \mathcal{C}^0(\Omega^\star_{T_0}:\mathbb{R}^2)$ *and* $\chi_2 \in \mathcal{C}^0([0, T]:\mathbb{R})$ *to problem* (4.12), *moreover, the estimate*

$$\|\chi_2\|_T + \|U\|_T \leqslant C\left(\|U_{-,0}\| + \|\mathbf{U}_+\|_T + \|G\|_T + \int_0^T \|F\|_s\, ds\right) \tag{4.27}$$

holds for any $T \in (0, T_0]$.

Let us turn to consider the linear problem (4.11). Similar to Proposition 3.3, we have:

Proposition 4.2. *Let us take* V, F, $U_\pm$ *and* G *as in Proposition* 4.1, *assume that the zeroth compatibility condition of* (4.11) *holds. Then problem* (4.11) *has a unique solution* $U = (U_-, U_\star) \in \mathcal{C}^0(\Omega^-_T:\mathbb{R}^2) \times \mathcal{C}^0(\Omega^\star_{T_0}:\mathbb{R}^2)$ *and* $\chi_2 \in \mathcal{C}^0([0, T]:\mathbb{R})$. *Moreover, for any* $T \in (0, T_0]$, *we have*

$$\|\chi_2\|_T + \|U\|_T \leqslant C e^{CMT}\left(\|U_{-,0}\| + \|\mathbf{U}_+\|_T + \|G\|_T + \int_0^T \|F\|_s\, ds\right), \tag{4.28}$$

where $M = 1 + \|V\|_T$.

Proof. As usual, the existence is proved by means of the iterative

$$\begin{cases} \mathbb{E}_\sharp U_\sharp^{\nu+1} = U_\sharp^{\nu+1}, \\ P_-(\partial_y, \partial_z) U_-^{\nu+1} + (-d_t q + \gamma_-(V_-))\partial_\eta U_-^{\nu+1} + \varXi_-(U_-^\nu, \partial_\eta V_-) = E_- F_-, \\ P_\star(\partial_y, \partial_z) U_\star^{\nu+1} + \big(\tilde{d}_\star I + h_\star \Lambda_\star + \frac{1}{\sigma - q_0}\gamma_\star(V_\star)\big)\partial_\eta U_\star^{\nu+1} + \frac{1}{\sigma - q_0}\varXi_\star(U_\star^\nu, \partial_\eta V_\star) = E_\star F_\star, \\ \mathcal{S}(\gamma_1 U_{\star,3}^{\nu+1}, \gamma_1 U_{-,1}^{\nu+1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}^{\nu+1}, \gamma_1 U_{-,3}^{\nu+1})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2^{\nu+1}(g(u_+) - g(u_\star)) + (\sigma \nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ \\ \quad - (\sigma \nabla g(u_\star) - \nabla f(u_\star))\gamma_2 U_\star^{\nu+1} = 0, \qquad \text{on } z = t,\ \eta = \tau, \\ U_-^{\nu+1}|_{y=0} = U_{-,0}(z), \end{cases} \tag{4.29}$$

where $U^0 = (U_-^0, U_\star^0) \in \mathcal{C}^1(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^1(\Omega_{T_0}^\star : \mathbb{R}^2)$ and $\chi_2^0 \in \mathcal{C}^1([0, T_0] : \mathbb{R})$ are the first approximate solutions constructed in a way similar to [7]. Note that

$$\left\| \Xi_\sharp\left(U_\sharp^\nu, \partial_\theta V_\sharp\right)\right\|_s \leqslant C_0 \|V_\sharp\|_{1,s} \left\|U_\sharp^\nu\right\|_s,$$

then one proceeds as in the second part of Proposition 3.3, exploiting Proposition 4.1. □

Moreover, parallelly to Proposition 3.4 in Section 3, we have:

Proposition 4.3. *Assume that V, F, $U_\pm$ and G are $\mathcal{C}^1$-smooth, if two compatibility conditions of* (4.11) *hold, then the solution U and χ_2 of problem* (4.11) *are $\mathcal{C}^1$-smooth and satisfies the following estimate*

$$\|\chi_2\|_{1,T} + \|U\|_{1,T} \leqslant Ce^{CMT}\left(\|U_{-,0}\|_1 + \|\mathbf{U}_+\|_{1,T} + \left\|F(0)\right\| + \|G\|_{1,T} + \int_0^T \|F\|_{1,s}\, ds\right), \tag{4.30}$$

where $M = 1 + \|V\|_{1,T}$.

Proof. Proof of this proposition follows from the same item in Proposition 3.4 and the estimate $\|\Xi_\sharp(U_\sharp^\nu, \partial_\theta V_\sharp)\|_{1,T} \leqslant C_0 \|V_\sharp\|_{1,T} \|U_\sharp^\nu\|_{1,T}$. □

At last, let us turn to the study of the iterative scheme (4.10), from which we immediately obtain the conclusion of Theorem 2.1(2).

Theorem 4.1. *If $U_{\pm,0} \in \mathcal{C}(\omega^\pm : \mathbb{R})$ and two compatibility conditions hold for* (2.1.31)*, then there exists $T \in (0, T_0]$ and a unique solution $U \in \mathcal{C}^1(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^1(\Omega_T^\star : \mathbb{R}^2)$ and $\chi_2 \in \mathcal{C}^1([0, T] : \mathbb{R})$ of this problem.*

Proof. The proof parallels that of Theorem 3.1. The iterative scheme we take into account here is (4.10) (notice that the first approximate solution is given by construction). At first, using Proposition 4.3, we solve problem (4.10) for each ν, has L^∞ bounds. Then these bounds prove that $\{U^\nu, \chi_2^\nu\}$ is a Cauchy sequence in L^∞ and hence the existence of a continuous almost periodic solution U and χ_2 to (2.1.31). $\mathcal{C}^1$-smoothness is proved once more through Ascoli's theorem. □

5. Asymptotic properties

In this section, we study the asymptotic properties of the oscillatory shock wave and contact discontinuity solutions $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon)$ to problem (2.1.15), which gives the proof of Theorem 2.1(3). Let $T > 0$ be the smaller one between those obtained in Theorems 3.1 and 4.1.

At first, we give a result, whose proof can be found in [11, Proposition 5.1].

Proposition 5.1. *Suppose that $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon) \in C_\epsilon^1(\Omega_T^-) \times C_\epsilon^1(\Omega_T^\star) \times \tilde{C}_\epsilon^0([0, T])$, $(U_-, U_\star, \chi_2) \in \mathcal{C}^1(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^1(\Omega_T^\star : \mathbb{R}^2) \times \mathcal{C}^1([0, T] : \mathbb{R})$ and $\varphi \in C^2([0, T])$ are solutions to Goursat problems* (2.1.15), (2.1.31) *and* (2.1.33)*, respectively, and $u_\sharp^\epsilon$ satisfy the asymptotics*

$$\left\| u_\sharp^\epsilon(y,z) - R_\sharp U_\sharp\left(y,z;\frac{y}{\epsilon},\frac{z}{\epsilon}\right)\right\|_{1,\Omega_T}^\epsilon = o(1), \quad \textit{when } \epsilon \to 0. \tag{5.1}$$

Then, when $\epsilon \to 0$, *we have*

$$\begin{cases} \left\| d_t\varphi^\epsilon(t) - \chi_2\left(t,\frac{t}{\epsilon}\right)\right\|_{1,[0,T]}^\epsilon = o(1), \\ \|\varphi^\epsilon(t) - \varphi(t)\|_{L^\infty[0,T]} = o(1). \end{cases} \tag{5.2}$$

By Proposition 5.1, in order to finish the proof of Theorem 2.1, we only need to establish the asymptotics (5.1).

We start with the following linear diagonal problem

$$\begin{cases} \tilde{L}_\sharp^\epsilon(v_\sharp^\epsilon,\Phi^\epsilon)u_\sharp^\epsilon = f_\sharp^\epsilon, \\ (\gamma_1 u_{\star,3}^\epsilon, \gamma_1 u_{-,1}^\epsilon)^{\mathrm{T}} = \Pi_1^\epsilon(t)(\gamma_1 u_{\star,1}^\epsilon, \gamma_1 u_{-,3}^\epsilon)^{\mathrm{T}} + g_1^\epsilon(t), & \text{on } z=0, \\ d_t\varphi^\epsilon(g(u_+) - g(u_\star)) + (\sigma I - A_+)\mathbf{u}_+^\epsilon - (\sigma I - \Lambda_\star)\gamma_2 u_\star^\epsilon = g_2^\epsilon(t), & \text{on } z=t, \\ u_-^\epsilon(0,z) = u_{-,0}^\epsilon(z), \end{cases} \tag{5.3}$$

where $\mathbf{u}_+^\epsilon = u_+^\epsilon(t,\sigma t + \epsilon\Phi^\epsilon(t))$, $\tilde{L}_\sharp^\epsilon(v_\sharp^\epsilon,\Phi^\epsilon) = \partial_y + N_\star^\epsilon\partial_z$ and

$$N_-^\epsilon = \Lambda\left(u_- + \epsilon v_-^\epsilon\right) - \lambda_2\left(u_\sharp + \epsilon v_\sharp^\epsilon\right)I, \qquad N_\star^\epsilon = \left(\frac{\partial z}{\partial t}\right)_\star I + \left(\frac{\partial z}{\partial x}\right)_\star \Lambda\left(u_\star + \epsilon v_\star^\epsilon\right), \tag{5.4}$$

with $((\frac{\partial z}{\partial t})_\star, (\frac{\partial z}{\partial x})_\star)$ being given in (2.1.18) by replacing Φ^ϵ for φ^ϵ.

For problem (5.3), suppose that $v^\epsilon = (v_-^\epsilon, v_\star^\epsilon)$, $f^\epsilon = (f_-^\epsilon, f_\star^\epsilon)$, Φ^ϵ, $g^\epsilon = (g_1^\epsilon, g_2^\epsilon)$ and $u_{\pm,0}^\epsilon$ are bounded as in Proposition 3.2, and u_+^ϵ satisfies the asymptotic property (2.1.13). Assume that $V_\sharp \in \mathcal{C}^1(\Omega_T^\sharp : \mathbb{R}^2)$, $F_\sharp \in \mathcal{C}^0(\Omega_T^\sharp : \mathbb{R}^2)$ and $K \in \mathcal{C}^0([0,T] : \mathbb{R})$ such that

$$E_\sharp V_\sharp = V_\sharp \tag{5.5}$$

and

$$\begin{cases} v_\sharp^\epsilon(y,z) - V_\sharp\left(y,z;\frac{y}{\epsilon},\frac{z}{\epsilon}\right) = o(1), & \text{in } L^\infty(\Omega_T^\sharp), \\ f_\sharp^\epsilon(y,z) - F_\sharp\left(y,z;\vec{\psi}\left(\frac{y}{\epsilon},\frac{z}{\epsilon}\right)\right) = o(1), & \text{in } L^\infty(\Omega_T^\sharp), \\ d_t\Phi^\epsilon(t) - K\left(t,\frac{t}{\epsilon}\right) = o(1), & \text{in } L^\infty([0,T]), \\ (g_1^\epsilon(t), g_2^\epsilon(t)) = o(1), & \text{in } L^\infty([0,T]), \end{cases} \tag{5.6}$$

when $\epsilon \to 0$, where $\vec{\psi}(\xi,\eta) = (\vec{\psi}_-(\xi,\eta), \vec{\psi}_\star(\xi,\eta), \psi(\xi,\eta))$ with $\vec{\psi}_\sharp(\xi,\eta) = (b_1^\sharp\xi + \eta, b_2^\sharp\xi + \eta, \xi + b_3^\sharp\eta)$ and $\psi(\xi,\eta) = \xi + \eta$.

The profile problem related to (5.3) will be

$$\begin{cases} \mathbb{E}_\sharp U_\sharp = U_\sharp, \\ P_-(\partial_y, \partial_z)U_- + (-d_t q + \gamma_-(V_-))\partial_\eta U_- = E_- F_-, \\ P_\star(\partial_y, \partial_z)U_\star + \big(\tilde{d} I + h_\star \Lambda_\star + \frac{1}{\sigma - q_0}\gamma_\star(V_\star)\big)\partial_\eta U_\star = E_\star F_\star, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2((\nabla g(u_\star))^{-1}(g(u_+) - g(u_\star))) + (\nabla g(u_\star))^{-1}(\sigma \nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ \\ \quad - (\sigma I - \Lambda_\star)R_\star \gamma_2 U_\star = 0, \quad \text{on } z = t,\ \eta = \tau, \\ U_-|y = 0 = U_{-,0}(z), \end{cases} \tag{5.7}$$

where $\mathbf{U}_+(t,\tau) = U_+(t, \sigma t; \tau, \sigma\tau + \Phi(t))$ with

$$\Phi(t) = \int_0^t (E_0 K)(s)\, ds, \tag{5.8}$$

$$\tilde{d}_\star = \frac{(q_0 y + \sigma z - q_0 z)(\Phi - q)}{(\sigma - q_0)^2 y^2} - \frac{(y - z)d_t q + z d_t \Phi}{(\sigma - q_0) y}, \tag{5.9}$$

and $\gamma_\sharp(V_\sharp) = \mathrm{diag}[\gamma_\sharp^1(V_\sharp), \gamma_\sharp^2(V_\sharp), \gamma_\sharp^3(V_\sharp)]$ with

$$\gamma_\sharp^k(V_\sharp) = E_\sharp^k\left(\sum_{p=1}^{3} \frac{\partial \lambda_k}{\partial v_p}(u_\sharp) V_{\sharp,p}\right). \tag{5.10}$$

If we assume that the zeroth order compatibility condition holds for problem (5.3) and one for problem (5.7), then Propositions 3.2, 4.1 apply and provide solutions $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon) \in C_\epsilon^1(\Omega_T^-) \times C_\epsilon^1(\Omega_T^\star) \times \tilde{C}_\epsilon^1([0,T])$ and $(U_-, U_\star, \chi_2) \in \mathcal{C}^0(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^0(\Omega_T^\star : \mathbb{R}^2) \times \mathcal{C}^1([0,T] : \mathbb{R})$.

Proposition 5.2. *Under the above assumption and notations, we have the following asymptotic developments*

$$\begin{cases} u_\sharp^\epsilon(y,z) - U_\sharp\big(y, z; \frac{y}{\epsilon}, \frac{z}{\epsilon}\big) = o(1), & \text{in } L^\infty(\Omega_T^\sharp), \\ d_t \varphi^\epsilon(t) - \chi\big(t, \frac{t}{\epsilon}\big) = o(1), & \text{in } L^\infty([0,T]), \end{cases} \tag{5.11}$$

when $\epsilon \to 0$.

Proof. The asymptotic development for the components $u_{-,2}^\epsilon$, $u_{-,3}^\epsilon$ are deduced from [1] by using the nonstationary phase lemma in [3] and [2, Lemma 2.4.1]. Then $u_{-,i}^\epsilon(y,z) - U_{-,i}(y,z; \frac{y}{\epsilon}, \frac{z}{\epsilon}) = o(1)$ $(i = 2,3)$ in $L^\infty(\Omega_T^-)$. Similarly, asymptotic development of the component $u_{-,1}^\epsilon$ holds also in the region $\mathcal{C}_{-,1}^\epsilon$. For the region $\mathcal{B}_{-,1}^\epsilon$, the boundary conditions are needed. For the components $u_{\star,k}^\epsilon$ $(k = 1,2,3)$, the boundary conditions are needed also in the region $\Omega_T^\star$. In fact, if we know the asymptotic development of $\gamma_2 u_{\star,3}^\epsilon$ in $\Omega_T^\star$, then we will know the one of $u_{-,1}^\epsilon$ in $\mathcal{B}_{-,1}^\epsilon$, $u_{\star,1}^\epsilon$, $u_{\star,2}^\epsilon$ in $\Omega_T^\star$ and $d_t\varphi^\epsilon(t)$. The proof of Proposition 5.2 is thus completed. □

In the following, we are devoted to the study of asymptotic develop of $\gamma_2 u^{\epsilon}_{\star,3}$ in $\Omega^{\star}_T$. Similar to the derivation [11, pp. 1669–1670], if we set

$$d^{\epsilon}=\frac{(q_0y+\sigma z-q_0z)(\Phi^{\epsilon}-q^{\epsilon})}{(\sigma-q_0)^2y^2}-\frac{(y-z)d_tq^{\epsilon}+zd_t\Phi^{\epsilon}}{(\sigma-q_0)y},\qquad h^{\epsilon}=\frac{q^{\epsilon}-\Phi^{\epsilon}}{(\sigma-q_0)y},\tag{5.12}$$

$$\begin{cases} b^{\epsilon}_{\star,1}=-\frac{\lambda_1(u_\star+\epsilon v^{\epsilon}_\star)-q_0+\epsilon(d^{\epsilon}+h^{\epsilon}\lambda_1(u_\star+\epsilon v^{\epsilon}_\star))}{\sigma-q_0},\\ b^{\epsilon}_{\star,3}=\frac{\sigma-q_0}{\lambda_3(u_\star+\epsilon v^{\epsilon}_\star)-q_0+\epsilon(d^{\epsilon}+h^{\epsilon}\lambda_3(u_\star+\epsilon v^{\epsilon}_\star))},\end{cases}\tag{5.13}$$

and

$$f^{\epsilon}_\star=\left(f^{\epsilon}_{\star,1},f^{\epsilon}_{\star,2},\frac{\sigma-q_0}{\lambda^{\star}_3-q_0}f^{\epsilon}_{\star,N}\right)^{\mathrm T},\tag{5.14}$$

then the equations of $u^{\epsilon}_{\star,1}$, $u^{\epsilon}_{\star,3}$ in (5.3) are written as

$$\begin{cases} \partial_y u^{\epsilon}_{\star,1}-b^{\epsilon}_{\star,1}\partial_z u^{\epsilon}_{\star,1}=f^{\epsilon}_{\star,1}+o(1),\\ b^{\epsilon}_{\star,3}\partial_y u^{\epsilon}_{\star,3}+\partial_z u^{\epsilon}_{\star,3}=f^{\epsilon}_{\star,3}+o(1),\\ (\gamma_1u^{\epsilon}_{\star,3},\gamma_1u^{\epsilon}_{-,1})^{\mathrm T}=\Pi^{\epsilon}_1(t)(\gamma_1u^{\epsilon}_{\star,1},\gamma_1u^{\epsilon}_{-,3})^{\mathrm T}+g^{\epsilon}_1(t), & \text{on } z=0,\\ d_t\varphi^{\epsilon}(g(u_+)-g(u_\star))+(\sigma I-A_+)\mathbf{u}^{\epsilon}_+-(\sigma I-\Lambda_\star)\gamma_2u^{\epsilon}_\star=g^{\epsilon}_2(t), & \text{on } z=t.\end{cases}\tag{5.15}$$

On the other hand, the equation of $U_{\star,1}$, $U_{\star,3}$ in (5.7) can be written as

$$\begin{cases} (\partial_y-b^{\star}_1\partial_z)U_{\star,1}+\big(\tilde d+h\lambda^{\star}_1+\frac{1}{\sigma-q_0}\gamma^1_\star(V_\star)\big)\partial_\theta U_{\star,1}=E^1_\star F_{\star,1},\\ (b^{\star}_3\partial_y+\partial_z)U_{\star,3}+\frac{\sigma-q_0}{\lambda^{\star}_3-q_0}\big(\tilde d+h\lambda^{\star}_3+\frac{1}{\sigma-q_0}\gamma^3_\star(V_\star)\big)\partial_\theta U_{\star,3}=\frac{\sigma-q_0}{\lambda^{\star}_3-q_0}E^3_\star F_{\star,3},\\ \mathcal S(\gamma_1U_{\star,3},\gamma_1U_{-,1})^{\mathrm T}=\mathcal J(\gamma_1U_{\star,1},\gamma_1U_{-,3})^{\mathrm T},\quad \text{on } z=\eta=0,\\ \chi_2((\nabla g(u_\star))^{-1}(g(u_+)-g(u_\star)))+(\nabla g(u_\star))^{-1}(\sigma\nabla g(u_+)-\nabla f(u_+))\mathbf U_+\\ \quad-(\sigma I-\Lambda_\star)R_\star\gamma_2U_\star=0,\quad \text{on } z=t,\ \eta=\tau,\end{cases}\tag{5.16}$$

where we denote by $\theta=b^{\star}_1\xi+\eta$ in the first equation, and $\theta=\xi+b^{\star}_3\eta$ in the second equation.

For any $(y,z)\in\Omega^{\star}_T$, set $y^{\star}_3(s)$ and $z^{\star}_1(s)$ be the characteristic curves as in Section 3.4. Let $s\to(y^{\star}_3(s)+\epsilon y^{\epsilon}_{\star,3}(s;y,z),s)$ ($s\to(s,z^{\star}_1(s)+\epsilon z^{\epsilon}_{\star,l,}(s;y,z))$, respectively) be the characteristic curve of $b^{\epsilon}_{\star,3}\partial_y+\partial_z$ (respectively of $\partial_y-b^{\epsilon}_{\star,l}\partial_z$) through $(y,z)\in\Omega^{\star}_T$, where $(y^{\epsilon}_{\star,3},z^{\epsilon}_{\star,l})$ satisfies

$$\begin{cases} d_sy^{\epsilon}_{\star,3}(s;y,z)=\epsilon^{-1}(b^{\star}_3-b^{\epsilon}_{\star,3}(y^{\star}_3(s)+\epsilon y^{\epsilon}_{\star,3}(s;y,z),s)),\\ y^{\epsilon}_{\star,3}(z;y,z)(z;y,z)=0\end{cases}\tag{5.17}$$

and

$$\begin{cases} d_sz^{\epsilon}_{\star,1}(s;y,z)=\epsilon^{-1}(b^{\star}_1-b^{\epsilon}_{\star,1}(s,z^{\star}_1(s)+\epsilon z^{\epsilon}_{\star,1}(s;y,z))),\\ z^{\epsilon}_{\star,1}(y;y,z)=0.\end{cases}\tag{5.18}$$

Let $s\to(y^{\star}_3(s),s;\theta+Y^{\star}_3(s;y,z,\theta))$ ($(s,z^{\star}_1(s);\theta+Z^{\star}_1(s;y,z,\theta))$, respectively) be the characteristic curve of $b^{\epsilon}_{\star,3}\partial_y+\partial_z+\frac{\sigma-q_0}{\lambda^{\star}_3-q_0}(\tilde d+h\lambda^{\star}_3+\frac{1}{\sigma-q_0}\gamma^3_\star(V_\star))\partial_\theta$ (respectively of $\partial_y-b^{\epsilon}_{\star,1}\partial_z+$

$(\tilde{d}+h\lambda_1^\star+\frac{1}{\sigma-q_0}\gamma_\star^1(V_\star))\partial_\theta)$ through $(y,z;\theta)\in\Omega_T^\star$, where $(Y_3^\star, Z_1^\star)$ satisfies the following problem:

$$\begin{cases} d_s Y_3^\star(s;y,z,\theta)=\frac{\sigma-q_0}{\lambda_3^\star-q_0}\big(\tilde{d}+h\lambda_3^\star+\frac{1}{\sigma-q_0}\gamma_\star^3(V_\star)\big)(y_3^\star(s),s;\theta+Y_3^\star(s)), \\ Y_3^\star(z;y,z,\theta)=0 \end{cases} \tag{5.19}$$

and

$$\begin{cases} d_s Z_1^\star(s;y,z,\theta)=\big(\tilde{d}+h\lambda_1^\star+\frac{1}{\sigma-q_0}\gamma_\star^1(V_\star)\big)(s,z_1^\star(s);\theta+Z_1^\star(s)), \\ Z_1^\star(y;y,z,\theta)=0. \end{cases} \tag{5.20}$$

Denote by $\Omega_{T,y}^\star=\{(s;y,z)\mid z\leqslant s\leqslant t,\ (y,z)\in\Omega_T^\star\}$, and $\Omega_{T,z}^\star=\{(s;y,z)\mid t\leqslant s\leqslant y,\ (y,z)\in\Omega_T^\star\}$. Similar to [11, Lemma 5.1], we have:

Lemma 5.1. *There are unique solutions $y_{\star,3}^\epsilon\in C_\epsilon^1(\Omega_{T,y}^\star)$, $z_{\star,1}^\epsilon\in C_\epsilon^1(\Omega_{T,z}^\star)$, $Y_3^\star\in\mathcal{C}^1(\Omega_{T,y}^\star:\mathbb{R})$ and $Z_1^\star\in\mathcal{C}^1(\Omega_{T,z}^\star:\mathbb{R})$ to problems* (5.17), (5.18), (5.19) *and* (5.20)*, respectively. Moreover, the following asymptotic development holds in L^∞ when $\epsilon\to 0$,*

$$\begin{cases} y_{\star,3}^\epsilon(s;y,z)-Y_3^\star\big(s;y.z,\frac{y+b_3^\star z}{\epsilon}\big)=o(1), \\ z_{\star,1}^\epsilon(s;y,z)-Z_1^\star\big(s;y.z,\frac{b_1^\star y+z}{\epsilon}\big)=o(1). \end{cases} \tag{5.21}$$

As derivation in Sections 2 and 3, problems (5.15) and (5.16) can be transformed into a system of functional equations respectively. Here, for simplicity, we need not to give the whole system of functional equations. However, the functional equations of $\gamma_2 u_{\star,3}(t)$ and $\gamma_2 U_{\star,3}(t,\tau)$ are needed.

As in [11, Lemma 5.2], applying the above Lemma 5.1, we give the following result.

Lemma 5.2. *Let $\gamma_2 u_{\star,3}^\epsilon(t)\in C_\epsilon^0([0,T])$ and $\gamma_2 U_{\star,3}(t,\tau)\in\mathcal{C}^0([0,T]:\mathbb{R})$ be the unique solutions to the functional equations*

$$\gamma_2 u_{\star,3}^\epsilon(t)=\eta^\epsilon\gamma_2 u_{\star,3}^\epsilon\big(X^\epsilon(t)\big)+h^\epsilon(t) \tag{5.22}$$

and

$$\gamma_2 U_{\star,3}(t,\tau)=\eta\gamma_2 U_{\star,3}(t,\tau)\big(\big(1+b_3^\star\big)t,\theta(t,\tau)\big)+H(t,\tau), \tag{5.23}$$

respectively, where $\eta^\epsilon=m_{11}\vec{e}_1(M^\epsilon)^{-1}(\sigma+\epsilon d_t\varphi^\epsilon-\lambda_3(u_\star+\epsilon\gamma_2 v_\star^\epsilon)r_3(u_\star+\epsilon\gamma_2 v_\star^\epsilon))$, $\eta=n_{11}\vec{e}_1(\bar{M})^{-1}(\sigma-\lambda_3^\star)$ and every notation is the same as in Sections 3 *and* 4*, then when $\epsilon\to 0$, we have*

$$\gamma_2 u_{\star,3}^\epsilon(t)-\gamma_2 U_{\star,3}\left(t,\frac{t}{\epsilon}\right)=o(1),\quad in\ L^\infty\big([0,T]\big). \tag{5.24}$$

Before passing to the asymptotics for solutions of general linear problems like (3.4), in order to shorten the proof of the next proposition, we first give a lemma by considering the following semilinear problem with linear diagonal principal part

$$\begin{cases} \tilde{L}_\sharp^\epsilon(v_\sharp^\epsilon, \Phi^\epsilon)u_\sharp^\epsilon + m_\sharp(\epsilon v_\sharp^\epsilon, w_\sharp^\epsilon)u_\sharp^\epsilon + Q_\sharp(\epsilon v_\sharp^\epsilon, u_\sharp^\epsilon) = f_\sharp^\epsilon, \\ (\gamma_1 u_{\star,3}^\epsilon, \gamma_1 u_{-,1}^\epsilon)^{\mathrm{T}} = \Pi_1^\epsilon(t)(\gamma_1 u_{\star,1}^\epsilon, \gamma_1 u_{-,3}^\epsilon)^{\mathrm{T}} + g_1^\epsilon(t), & \text{on } z=0, \\ d_t\varphi^\epsilon(g(u_+) - g(u_\star)) + (\sigma I - A_+)\mathbf{u}_+^\epsilon - (\sigma I - \Lambda_\star)\gamma_2 u_\star^\epsilon = g_2^\epsilon(t), & \text{on } z=t, \\ u_-^\epsilon(0,z) = u_{-,0}^\epsilon(z), \end{cases} \tag{5.25}$$

where the notations and assumptions are the same as in (5.3), and

$$m_\sharp(v,w) = \sum_l m_{\sharp,l}(v)w_{\sharp,l}, \qquad Q_{\sharp,k}(v,u) = \sum_{i,p} Q_{\sharp,k}^{ip}(v)u_{\sharp,i}u_{\sharp,p}, \tag{5.26}$$

where $m_\sharp(v,w)$ are linear in w and $Q_\sharp(v,u)$ are quadratic form in u, with $m_{\sharp,l}$ and $(Q_{\sharp,k}^{ip})_{rn}$ being (3×3)-matrices. We suppose that $w_\sharp^\epsilon$ is bounded in $C_\epsilon^0(\Omega_T^\sharp)$ and the asymptotic expansion

$$w_\sharp^\epsilon(y,z) - W_\sharp\left(y, z; \vec{\psi}\left(\frac{y}{\epsilon}, \frac{z}{\epsilon}\right)\right) = o(1), \quad \text{in } L^\infty\big(\Omega_T^\sharp\big), \tag{5.27}$$

hold with $W_\sharp(y,z;\theta) \in \mathcal{C}^0$. Then for problem (5.25), we have:

Lemma 5.3. *For the solution $u_\sharp^\epsilon$ and $\varphi^\epsilon(t)$ to problem* (5.25)*, there is $T_1 \in [0,T]$ such that in $L^\infty(\Omega_{T_1})$, the asymptotic properties*

$$\begin{cases} u_\sharp^\epsilon(y,z) - U_\sharp\big(y, z; \frac{y}{\epsilon}, \frac{z}{\epsilon}\big) = o(1), \\ d_t\varphi^\epsilon(t) - \chi_2\big(t, \frac{t}{\epsilon}\big) = o(1), \quad k=1,2, \end{cases} \tag{5.28}$$

hold, where $T_1 = T$ when $Q(v,u) = 0$ in problem (5.25)*, $(U_-, U_\star, \chi_2) \in \mathcal{C}^0(\Omega_{T_1}^- : \mathbb{R}^2) \times \mathcal{C}^0(\Omega_{T_1}^\star : \mathbb{R}^2) \times \mathcal{C}^0([0,T_1] : \mathbb{R})$ are unique solutions to the problem*

$$\begin{cases} E_\sharp U_\sharp = U_\sharp \\ \partial_y U_{-,k} + (\lambda_k^- - q_1)\partial_z U_{-,k} + (-d_t q + \gamma_-^k(V_-))\partial_\eta U_{-,k} \\ \quad + E_-^k\big(\sum_{l,p} \bar{m}_{-,l}^{kp} W_{-,l} U_{-,p} + \sum_{i,p} \bar{Q}_{-,k}^{ip} U_{-,i} U_{-,p}\big) = E_-^k F_{-,k}, \quad k=1,2,3, \\ \partial_y U_{\star,k} - \frac{q_0 - \lambda_k^\star}{\sigma - q_0}\partial_z U_{\star,k} + \big(\tilde{d} + h\lambda_k^\star + \frac{1}{\sigma - q_0}\gamma_\star^k(V_\star)\big)\partial_\eta U_{\star,k} \\ \quad + E_\star^k\big(\sum_{l,p} \bar{m}_{\star,l}^{kp} W_{\star,l} U_{\star,p} + \sum_{i,p} \bar{Q}_{\star,k}^{ip} U_{\star,i} U_{\star,p}\big) = E_\star^k F_{\star,k}, \qquad k=1,2, \\ \frac{\sigma - q_0}{\lambda_3^\star - q_0}\partial_y U_{\star,3} + \partial_z U_{\star,3} + \big(\tilde{d} + h\lambda_3^\star + \frac{1}{\sigma - q_0}\gamma_\star^3(V_\star)\big)\partial_\eta U_{\star,3} \\ \quad + E_\star^3\big(\sum_{l,p} \bar{m}_{\star,l}^{3p} W_{\star,l} U_{\star,p} + \sum_{i,p} \bar{Q}_{\star,3}^{ip} U_{\star,i} U_{\star,p}\big) = E_\star^3 F_{\star,3}, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2((\nabla g(u_\star))^{-1}(g(u_+) - g(u_\star))) + (\nabla g(u_\star))^{-1}(\sigma \nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ \\ \quad - (\sigma I - \Lambda_\star)R_\star \gamma_2 U_\star = 0, \quad \text{on } z = t,\ \eta = \tau, \\ U_-|y=0 = U_{-,0}(z;\theta), \end{cases} \tag{5.29}$$

with the notation being the same as in (5.7). $\bar{m}_{\sharp,l} = m_{\sharp,l}(0)$ *and* $\bar{Q}_{\sharp,k}^{ip} = Q_{\sharp,k}^{ip}(0)$ $(k=1,2,3)$.

Proof. Under the zeroth order compatibility condition for problems (5.25) and (5.29), existence of a solution $(u_-^\epsilon, u_-^\epsilon, \varphi^\epsilon)$ bounded in $C_\epsilon^0(\Omega_T^-) \times C_\epsilon^0(\Omega_T^\star) \times C_\epsilon^1([0,T])$ to problem (5.25) is a byproduct of the proof of Proposition 3.3, while existence of a solution $(U_-, U_\star, \chi_2) \in \mathcal{C}^0(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^0(\Omega_T^\star : \mathbb{R}^2) \times \mathcal{C}^1([0,T] : \mathbb{R})$ to problem (5.29) follows from Proposition 4.2. The solution $(u_-^\epsilon, u_-^\epsilon, \varphi^\epsilon)$ are constructed by way of an iterative scheme similar to (3.26), and the limits are uniform in ϵ. The profile $(U_-, U_\star, \chi_2)$ are obtained by an iterative scheme similar to (4.29). Then the asymptotic properties can be easily obtained by using Proposition 5.1 to parallel terms of both sequence and pass to the limit (see [3, Proposition 6.5.1]). □

With this lemma, we can consider the following linear problem with nondiagonal principal part

$$\begin{cases} L_\sharp^\epsilon(v_\sharp^\epsilon, \Phi^\epsilon)u_\sharp^\epsilon + m_\sharp(\epsilon v_\sharp^\epsilon, w_\sharp^\epsilon)u_\sharp^\epsilon = f_\sharp^\epsilon, \\ (\gamma_1 u_{\star,3}^\epsilon, \gamma_1 u_{-,1}^\epsilon)^{\mathrm{T}} = \Pi_1^\epsilon(t)(\gamma_1 u_{\star,1}^\epsilon, \gamma_1 u_{-,3}^\epsilon)^{\mathrm{T}} + g_1^\epsilon(t), & \text{on } z=0, \\ d_t\varphi^\epsilon(g(u_+) - g(u_\star)) + (\sigma I - A_+)\mathbf{u}_+^\epsilon - (\sigma I - \Lambda_\star)\gamma_2 u_\star^\epsilon = g_2^\epsilon(t), & \text{on } z=t, \\ u_-^\epsilon(0,z) = u_{-,0}^\epsilon(z), \end{cases} \tag{5.30}$$

where $L_\sharp^\epsilon(v_\sharp^\epsilon, \Phi^\epsilon) = \partial_y + N_\sharp^\epsilon \partial_z$ with

$$N_-^\epsilon = A\big(u_- + \epsilon v_-^\epsilon\big) - \lambda_2\big(u_- + \epsilon v_-^\epsilon\big)I, \qquad N_\star^\epsilon = \left(\frac{\partial z}{\partial t}\right)_\star I + \left(\frac{\partial z}{\partial x}\right)_\star A\big(u_\star + \epsilon v_\star^\epsilon\big), \tag{5.31}$$

and $((\frac{\partial z}{\partial t})_\star, (\frac{\partial z}{\partial x})_\star)$ being given in (2.1.18) by replacing Φ^ϵ for φ^ϵ, and all hypotheses are the same as in Proposition 5.1, Lemma 5.3. Moreover, we have

$$v_\sharp^\epsilon(y,z) - V_\sharp\left(y, z; \frac{y}{\epsilon}, \frac{z}{\epsilon}\right) = o(1), \quad \text{in } L^\infty\big(\Omega_T^\sharp\big), \tag{5.32}$$

with $V_\sharp \in \mathcal{C}^1(\Omega_T^\sharp : \mathbb{R}^2)$ being the same as in (5.5).

Proposition 5.3. *For the solutions $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon) \in C_\epsilon^0(\Omega_T^-) \times C_\epsilon^0(\Omega_T^\star) \times \tilde{C}_\epsilon^1([0,T])$ of* (5.30)*, we have the following asymptotic properties*:

$$\begin{cases} u_\sharp^\epsilon(y,z) - R_\sharp U_\sharp\big(y, z; \frac{y}{\epsilon}, \frac{z}{\epsilon}\big) = o(1), & \text{in } L^\infty(\Omega_T^\sharp), \\ d_t\varphi^\epsilon(t) - \chi_2\big(t, \frac{t}{\epsilon}\big) = o(1), & \text{in } L^\infty([0,T]), \end{cases} \tag{5.33}$$

where $(U_-, U_\star, \chi_2) \in \mathcal{C}^0(\Omega_T^- : \mathbb{R}^2) \times \mathcal{C}^0(\Omega_T^\star : \mathbb{R}^2) \times \mathcal{C}^0([0,T] : \mathbb{R})$ are unique solutions to the problem

$$\begin{cases} E_\sharp U_\sharp = U_\sharp, \\ P_-(\partial_y, \partial_z)U_- - d_t q \partial_\eta U_- + E_-\big(B_-(\partial_\eta U_-, V_-) + \sum_l \bar{m}_{-,l} U_- W_{-,l}\big) = E_- F_-, \\ P_\star(\partial_y, \partial_z)U_\star + (\tilde{d} I + h\Lambda_\star)\partial_\eta U_\star + \frac{1}{\sigma - q_0} E_\star\big(B_\star(\partial_\eta U_\star, V_\star) + \sum_l \bar{m}_{\star,l} U_\star W_{\star,l}\big) = E_\star F_\star, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2((\nabla g(u_\star))^{-1}(g(u_+) - g(u_\star))) + (\nabla g(u_\star))^{-1}(\sigma \nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ \\ \quad - (\sigma I - \Lambda_\star) R_\star \gamma_2 U_\star = 0, \quad \text{on } z = t,\ \eta = \tau, \\ U_-|y = 0 = U_{-,0}(z; \theta). \end{cases} \tag{5.34}$$

Clearly the zeroth order compatibility conditions are satisfied for problems (5.30) *and* (5.34).

Proof. The idea of this proposition is that one can diagonalize problem (5.30), then for diagonalized problem, one can easily obtained results by using Lemma 5.3. For the detailed process, one can be found Proposition 5.4 in [11] and Proposition 2.2.4 in [2]. □

So far we have been concerned with L^∞ asymptotics, we give now a result about C^1_ϵ asymptotics. Let us consider the asymptotics of derivation of solutions to problem (3.4), i.e., (5.30) with $m_\sharp = 0$, where $v^\epsilon_\sharp$ is the same as in (5.32), $f^\epsilon_\sharp$, Φ^ϵ and g^ϵ_k $(k = 1, 2)$ are bounded in $C^1_\epsilon(\Omega^\sharp_T)$, $\tilde{C}^2_\epsilon([0, T])$ and $C^1_\epsilon([0, T])$, respectively. Under the assumptions of Proposition 3.4, let $u^\epsilon_\sharp$, φ^ϵ be its solution. We already know from the previous proposition that $u^\epsilon_\sharp$, φ^ϵ has $U_\sharp$, χ_2 as profiles, with errors in $L^\infty(\Omega^\sharp_T)$; to reach $C^1_\epsilon(\Omega^\sharp_T)$, we strengthen the assumption on the asymptotic expansions (5.6), replacing $L^\infty(\Omega^\sharp_T)$ with $C^1_\epsilon(\Omega^\sharp_T)$.

Proposition 5.4. *Let* $(u^\epsilon_-, u^\epsilon_\star, \varphi^\epsilon) \in C^1_\epsilon(\Omega^-_T) \times C^1_\epsilon(\Omega^\star_T) \times \tilde{C}^2_\epsilon([0, T])$ *be the solution to* (3.5) *and* $(U_-, U_\star, \chi_2) \in \mathcal{C}^1(\Omega^-_T : \mathbb{R}^2) \times \mathcal{C}^1(\Omega^\star_T : \mathbb{R}^2) \times \mathcal{C}^2([0, T] : \mathbb{R})$ *be the solutions to the following problem*

$$\begin{cases} E_\sharp U_\sharp = U_\sharp, \\ P_-(\partial_y, \partial_z)U_- + E_-(-d_t q \partial_\eta)U_- + B_-(\partial_\eta U_-, V_-) = E_- F_-, \\ P_\star(\partial_y, \partial_z)U_\star + E_\star(\tilde{d} I + h\Lambda_\star)\partial_\eta U_\star + \frac{1}{\sigma - q_0} B_\star(\partial_\eta U_\star, V_\star) = E_\star F_\star, \\ \mathcal{S}(\gamma_1 U_{\star,3}, \gamma_1 U_{-,1})^{\mathrm{T}} = \mathcal{J}(\gamma_1 U_{\star,1}, \gamma_1 U_{-,3})^{\mathrm{T}}, \quad \text{on } z = \eta = 0, \\ \chi_2((\nabla g(u_\star))^{-1}(g(u_+) - g(u_\star))) + (\nabla g(u_\star))^{-1}(\sigma \nabla g(u_+) - \nabla f(u_+))\mathbf{U}_+ \\ \quad - (\sigma I - \Lambda_\star) R_\star \gamma_2 U_\star = 0, \quad \text{on } z = t,\ \eta = \tau, \\ U_-|y = 0 = U_{-,0}(z; \theta). \end{cases} \tag{5.35}$$

Under the above mentioned assumptions and two compatibility conditions for both problems. Then we have

$$u^\epsilon_\sharp(y, z) - R_\sharp U_\sharp\left(y, z; \frac{y}{\epsilon}, \frac{z}{\epsilon}\right) = o(1), \tag{5.36}$$

in $C^1_\epsilon(\Omega^\sharp_T)$ *when* $\epsilon \to 0$.

Proof. From the proof of Proposition 3.4, we know that $z_\sharp^\epsilon = \epsilon\partial_z u_\sharp^\epsilon$ satisfies (3.29). This proposition falls under the frame of Proposition 5.3, except for the boundary term $\epsilon\partial_z u_\sharp^{\epsilon,I}$, whose asymptotic properties are not yet know. Thus what the first step we do is to establish the C^1 asymptotic for $u_\sharp^{\epsilon,I}$ solving the noncharacteristic problem

$$\begin{cases} \partial_y u_\sharp^{\epsilon,I} + C(u_\sharp + \epsilon v_\sharp^\epsilon)\partial_z u_\sharp^{\epsilon,I} = f_\sharp^{\epsilon,I}, & \\ u_\star^{\epsilon,I} - u_-^{\epsilon,I} = g_1^\epsilon(t), & \text{on } z=0, \\ F^\epsilon_{(\gamma_2 v_\star^\epsilon, d_t\varphi^\epsilon)}(\gamma_2 u_\star^\epsilon, d_t\phi^\epsilon) = g_2^\epsilon(t), & \text{on } z=t, \\ u_-^{\epsilon,I}(0,z) = u_{-,0}^{\epsilon,I}(z). & \end{cases} \tag{5.37}$$

For the noncharacteristic problem (5.37), we can apply the method similar to [11, Proposition 5.5] and deduce that $u_\sharp^{\epsilon,I}(y,z) - R_\sharp U_\sharp^I(y,z;\frac{y}{\epsilon},\frac{z}{\epsilon}) = o(1)$ in $C_\epsilon^1(\Omega_T^\sharp)$, where $U^I = \sum U_{\sharp,k} r_{\sharp,k}^I$ are profiles of $u_\sharp^{\epsilon,I}$. This gives the asymptotics we needed. Now we can come back to problem (3.29). The remainder proof is similar to [2, Proposition 2.4.5], we omit it here. Proposition 5.4 is thus proved. □

Finally, we turn to the item (3) in Theorem 2.1. Let $(u_-^\epsilon, u_\star^\epsilon, \varphi^\epsilon)$ be the solutions to (2.1.15) and $(U_-, U_\star, \chi_2)$ that of (2.1.31). We may suppose that both are defined for $T \in (0, T_0]$ for some $T_0 > 0$.

Theorem 5.1. *Under the assumptions of Theorem* 2.1, *we have the following asymptotic expansions*:

$$\begin{cases} u_\sharp^\epsilon(y,z) - R_\sharp U_\sharp\big(y,z;\frac{y}{\epsilon},\frac{z}{\epsilon}\big) = o(1), \\ d_t\varphi^\epsilon(t) - \chi_2\big(t,\frac{t}{\epsilon}\big) = o(1), \end{cases} \tag{5.38}$$

in $C_\epsilon^1(\Omega_T^\sharp)$ *when* $\epsilon \to 0$.

Proof. We only provide a brief sketch of the proof, since it is similar to that of Proposition 5.4. Problem (2.1.15) was solved by means of the iterative scheme (3.2), while problem (2.1.31) used the scheme (4.4). Applying Proposition 5.4, it is easy to prove that

$$\begin{cases} u_\sharp^{\epsilon,\nu}(y,z) - R_\sharp U_\sharp^\nu\big(y,z;\frac{y}{\epsilon},\frac{z}{\epsilon}\big) = o(1), \\ d_t\varphi^{\epsilon,\nu}(t) - \chi_2^\nu\big(t,\frac{t}{\epsilon}\big) = o(1), \end{cases} \tag{5.39}$$

in $C_\epsilon^1(\Omega_T^\sharp)$ when $\epsilon \to 0$ for each ν, where the case of $\nu = 0$ is valid by (4.1). By combining (5.39) with the uniform convergence of $(u_-^{\epsilon,\nu}, u_\star^{\epsilon,\nu}, \varphi^{\epsilon,\nu})$ in $C^0(\Omega_T^-) \times C^0(\Omega_T^\star) \times \tilde{C}^1([0,T])$, it immediately follows the part of L^∞-norms in (5.38). The asymptotic property of derivation of $(u_-^{\epsilon,\nu}, u_\star^{\epsilon,\nu}, \varphi^{\epsilon,\nu})$ will be obtained directly from the study of the nonlinear problem (2.1.15). In fact, we differentiate (2.1.15) with respect to z and find that $z_\sharp^\epsilon = \epsilon\partial_z u_\sharp^\epsilon$ satisfies a semilinear problem, the existence for such a problem is obtained by following the lines of Proposition 3.3. Also the associated profile is semilinear, and has a solution $Z_\sharp$. Then an asymptotic result show that $z_\sharp^\epsilon(y,z) - Z_\sharp(y,z;\frac{y}{\epsilon},\frac{z}{\epsilon}) = o(1)$ in $L^\infty(\Omega_T^\sharp)$ (compare with Proposition 5.3). At last, one can check that $\partial_\theta U_\sharp$ are solutions of the problem for $Z_\sharp$, and then $\partial_\theta U_\sharp = Z_\sharp$. This completes the proof of Theorem 5.1. $\square$

Corollary 5.1. *Under the hypotheses of Theorem* 2.1*, there exists* $\chi_1 \in \mathcal{C}^1([0,T]:\mathbb{R})$ *such that the asymptotic expansions*

$$\left\| d_t q^\epsilon(t) - \chi_1\left(t, \frac{t}{\epsilon}\right)\right\|_{1,[0,T]}^\epsilon = o(1), \tag{5.40}$$

when $\epsilon \to 0$.

Acknowledgment

The authors are grateful to the referee for his/her valuable comments and suggestions which have led to an improvement of the presentation.

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