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Dichotomy spectra and Morse decompositions of linear nonautonomous differential equations[☆]

Martin Rasmussen

Department of Mathematics, Imperial College, London SW72AZ, United Kingdom

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ABSTRACT

Recently, the existence of Morse decompositions for nonautonomous dynamical systems was shown for three different time domains: the past, the future and—in the linear case—the entire time. In this article, notions of exponential dichotomy are discussed with respect to the three time domains. It is shown that an exponential dichotomy gives rise to an attractor–repeller pair in the projective space, which is a building block of a Morse decomposition. Moreover, based on the notions of exponential dichotomy, dichotomy spectra are introduced, and it is proved that the corresponding spectral manifolds lead to Morse decompositions in the projective space.

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1. Introduction

In the qualitative theory of dynamical systems, the study of linear systems is very important, since a comprehensive analysis of nonlinear systems via perturbation techniques requires linear theory. This is due to the fact that in many cases, stability properties of solutions can be derived from the linearization along the solution, the so-called variational equation.

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E-mail address: m.rasmussen@imperial.ac.uk.

In the linear analysis of nonautonomous systems, the concept of an exponential dichotomy is essential, which extends the idea of hyperbolicity to explicitly time-dependent systems. There have been extensive studies showing the significance of exponential dichotomies both in theory and applications. Based on the notion of an exponential dichotomy, R.J. Sacker und G.R. Sell founded a spectral theory for linear skew product flows with compact base flow in the 1970s, the so-called *Sacker–Sell spectral theory* (see [25]). Recently, the Sacker–Sell spectrum was adapted in [21] to arbitrary systems of linear differential equations.

In all the studies above, however, the focus is concentrated on the entire time axis. We want to consider also both the past and the future in this article. This is of importance in the situation of nonautonomous systems which cannot be embedded into the setting of skew product flows with a compact base space, since then, the past and the future are not related in form of recurrence properties. Moreover, the systems under consideration may only be defined on the half axis. Based on the notions of an exponential dichotomy on the half line (see [6]), a past and future dichotomy spectrum is introduced in this article for linear nonautonomous ordinary differential equations, and it is proved that the spectra consist of unions of closed intervals, whose number is bounded by the dimension of the system.

Another possibility to analyze linear nonautonomous systems is to consider the induced system on the projective space. This approach is also fairly classical; for instance, J. Selgrade found conditions for the existence of a finest Morse decomposition of the skew product flow on the projective space in 1975 (see [20]). Morse decompositions have been introduced by C.C. Conley in his famous article *Isolated Invariant Sets and the Morse Index* [5] in order to describe the global asymptotic behavior of dynamical systems on compact metric spaces. Recently, the existence of Morse decompositions for nonautonomous systems was shown for the above mentioned three time domains (see [17,18]). The construction is based on special notions of local attractivity and repulsivity which have been introduced in [16].

In this article, relationships between the concepts of exponential dichotomy, dichotomy spectra and Morse decompositions are pointed out also. In first instance, it is shown that the existence of an exponential dichotomy yields an attractor–repeller pair in the projective space which is a building block of a Morse decomposition. Then it is shown that the spectral manifolds form a Morse decomposition in the projective space.

This paper is organized as follows. The following section is devoted to preliminary definitions, and in Section 3, the relevant notions of nonautonomous attractivity and repulsivity are introduced. In Section 4, nonautonomous Morse decompositions are treated, and the different notions of exponential dichotomy are discussed in Section 5. Finally, the last section of this paper deals with properties of the dichotomy spectra.

Notation. Given a metric space (X, d) , we write $U_\varepsilon(x_0) = \{x \in X : d(x, x_0) < \varepsilon\}$ for the ε -neighborhood of a point $x_0 \in X$. For arbitrary nonempty sets $A, B \subset X$ and $x \in X$, let $d(x, A) := \inf\{d(x, y) : y \in A\}$ be the distance of x to A and $d(A|B) := \sup\{d(x, B) : x \in A\}$ be the Hausdorff semi-distance of A and B . Moreover, we set $\mathbb{R}_\kappa^+ := [\kappa, \infty)$ and $\mathbb{R}_\kappa^- := (-\infty, \kappa]$ for $\kappa \in \mathbb{R}$.

We denote by $\mathbb{R}^{N \times N}$ the set of all real $N \times N$ matrices, and we use the symbol $\mathbb{1}$ for the unit matrix. The Euclidean space \mathbb{R}^N is equipped with the Euclidean norm $\|\cdot\|$, which is induced by the scalar product $\langle \cdot, \cdot \rangle$, defined by $\langle x, y \rangle := \sum_{i=1}^N x_i y_i$. To introduce the real projective space \mathbb{P}^{N-1} of the \mathbb{R}^N , we say, two nonzero elements $x, y \in \mathbb{R}^N$ are equivalent if there exists a $c \in \mathbb{R}$ such that $x = cy$. The equivalence class of $x \in \mathbb{R}^N$ is denoted by $\mathbb{P}x$, and we call the set of all equivalent classes the projective space \mathbb{P}^{N-1} . Equipped with the metric $d_{\mathbb{P}} : \mathbb{P}^{N-1} \times \mathbb{P}^{N-1} \rightarrow [0, \sqrt{2}]$, given by

$$d_{\mathbb{P}}(\mathbb{P}v, \mathbb{P}w) = \min \left\{ \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\|, \left\| \frac{v}{\|v\|} + \frac{w}{\|w\|} \right\| \right\} \quad \text{for all } v, w \in \mathbb{R}^N,$$

the projective space is a compact metric space. For any $v \in \mathbb{P}^{N-1}$, we define $\mathbb{P}^{-1}v := \{x \in \mathbb{R}^N : \mathbb{P}x = v\} \cup \{0\}$. The $(N-1)$ -sphere of the \mathbb{R}^N is defined by $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : \|x\| = 1\}$. We make use of the following fundamental lemma, which follows from [2, Lemma B.1.17, p. 538].

Lemma 1.1. For all $\varepsilon > 0$, there exists a $\delta \in (0, 1)$ such that for all nonzero $v, w \in \mathbb{R}^N$ with $\langle v, w \rangle^2 / (\|v\|^2 \|w\|^2) \geq 1 - \delta$, we have $d_{\mathbb{P}}(\mathbb{P}v, \mathbb{P}w) \leq \varepsilon$.

2. Cocycles and nonautonomous sets

Throughout this paper, \mathbb{I} denotes a real interval of the form $(-\infty, 0]$, $[0, \infty)$ or \mathbb{R} , respectively. Given a metric space (X, d) , a cocycle is a mapping $\varphi: \mathbb{I} \times \mathbb{I} \times X \rightarrow X$ with

$$\varphi(\tau, \tau, \xi) = \xi \quad \text{and} \quad \varphi(t, \tau, \xi) = \varphi(t, s, \varphi(s, \tau, \xi))$$

for all $\tau, t, s \in \mathbb{I}$ and $\xi \in X$. For simplicity in notation, we write $\varphi(t, \tau)\xi$ instead of $\varphi(t, \tau, \xi)$. The set X is called *phase space*, and $\mathbb{I} \times X$ is called *extended phase space*. The general solution of a nonautonomous differential equation $\dot{x} = f(t, x)$ is a cocycle if the *right-hand side* $f: \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfies conditions guaranteeing global existence and uniqueness of solutions.

A subset M of the extended phase space $\mathbb{I} \times X$ is called *nonautonomous set*; we use the term *t-fiber* of M for the set $M(t) := \{x \in X: (t, x) \in M\}$, $t \in \mathbb{I}$. We call M *closed* or *compact* if all t -fibers are closed or compact, respectively. Finally, a nonautonomous set M is called *invariant* (w.r.t. the cocycle φ) if $\varphi(t, \tau, M(\tau)) = M(\tau + t)$ for all $t, \tau \in \mathbb{I}$.

In case $X = \mathbb{R}^N$, a cocycle φ is called *linear* if for given $\alpha, \beta \in \mathbb{R}$, we have

$$\varphi(t, \tau, \alpha x + \beta y) = \alpha \varphi(t, \tau, x) + \beta \varphi(t, \tau, y) \quad \text{for all } t, \tau \in \mathbb{I} \text{ and } x, y \in \mathbb{R}^N.$$

For instance, a linear cocycle is generated by a linear nonautonomous differential equation $\dot{x} = B(t)x$, where $B: \mathbb{I} \rightarrow \mathbb{R}^{N \times N}$ is continuous. Given a linear cocycle φ , there exists a corresponding matrix-valued function $\Phi: \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{R}^{N \times N}$ with $\Phi(t, \tau)x = \varphi(t, \tau, x)$ for all $t, \tau \in \mathbb{I}$ and $x \in \mathbb{R}^N$. We will also use the term *linear cocycle* for this function. Φ canonically induces a cocycle $\mathbb{P}\Phi$ on \mathbb{P}^{N-1} by the definition

$$\mathbb{P}\Phi(t, \tau)\mathbb{P}x := \mathbb{P}(\Phi(t, \tau)x) \quad \text{for all } t, \tau \in \mathbb{I} \text{ and } x \in \mathbb{R}^N$$

(see [2, Lemma 5.2.1, p. 149]).

Let $\gamma \in \mathbb{R}$ and \mathbb{I} be an interval of the form $(-\infty, 0]$, $[0, \infty)$ or \mathbb{R} , respectively. A function $g: \mathbb{I} \rightarrow \mathbb{R}^N$ is called γ^+ -*quasibounded* if \mathbb{I} is unbounded above and $\sup_{t \in \mathbb{I} \cap [0, \infty)} \|g(t)\|e^{-\gamma t} < \infty$. Accordingly, we say that a function $g: \mathbb{I} \rightarrow \mathbb{R}^N$ is γ^- -*quasibounded* if \mathbb{I} is unbounded below and we have $\sup_{t \in \mathbb{I} \cap (-\infty, 0]} \|g(t)\|e^{-\gamma t} < \infty$.

3. Nonautonomous attractivity and repulsivity

In this section, several notions of local attractivity and repulsivity are explained (see also [16]). The concepts are introduced for the past (past attractivity and repulsivity), the future (future attractivity and repulsivity) and the entire time (all-time attractivity and repulsivity).

Throughout this section, let (X, d) be a metric space and $\varphi: \mathbb{I} \times \mathbb{I} \times X \rightarrow X$ be a cocycle.

Note that the following notions of attractor are local forms of attractors which have been discussed since the 1990s. For instance, a past attractor is a local form of a pullback attractor (see, e.g., [4]), i.e., it attracts a neighborhood of itself in the sense of pullback attraction. Moreover, a future attractor is a local form of a forward attractor, and an all-time attractor is a local form of a uniform attractor as discussed, e.g., in [7].

Definition 3.1 (*Nonautonomous attractivity and repulsivity*). Let A and R be invariant and compact nonautonomous sets.

(i) In case \mathbb{I} is unbounded below, A is called *past attractor* if there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} d(\varphi(\tau, \tau - t)U_\eta(A(\tau - t)) | A(\tau)) = 0 \quad \text{for all } \tau \in \mathbb{I}.$$

(ii) In case \mathbb{I} is unbounded below, R is called *past repeller* if there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} d(\varphi(\tau - t, \tau)U_\eta(R(\tau)) | R(\tau - t)) = 0 \quad \text{for all } \tau \leq 0.$$

(iii) In case \mathbb{I} is unbounded above, A is called *future attractor* if there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} d(\varphi(\tau + t, \tau)U_\eta(A(\tau)) | A(\tau + t)) = 0 \quad \text{for all } \tau \geq 0.$$

(iv) In case \mathbb{I} is unbounded above, R is called *future repeller* if there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} d(\varphi(\tau, \tau + t)U_\eta(R(\tau + t)) | R(\tau)) = 0 \quad \text{for all } \tau \geq 0.$$

(v) In case $\mathbb{I} = \mathbb{R}$, A is called *all-time attractor* if there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} \sup_{\tau \in \mathbb{R}} d(\varphi(\tau + t, \tau)U_\eta(A(\tau)) | A(\tau + t)) = 0.$$

(vi) In case $\mathbb{I} = \mathbb{R}$, R is called *all-time repeller* if there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} \sup_{\tau \in \mathbb{R}} d(\varphi(\tau, \tau + t)U_\eta(R(\tau + t)) | R(\tau)) = 0.$$

Remark 3.2.

- (i) Every all-time attractor (repeller, respectively) is both a past attractor (repeller, respectively) and a future attractor (repeller, respectively).
- (ii) The notions of future attractivity and repulsivity can be derived from the concept of past attractivity and repulsivity via time reversal. A past attractor (repeller, respectively) corresponds to a future repeller (attractor, respectively) of the system under time reversal.
- (iii) The Hausdorff semi-distance d in Definition 3.1 can be replaced by the Hausdorff distance d_H , which for nonempty sets $A, B \subset X$ is defined by $d_H(A, B) := \max\{d(A|B), d(B|A)\}$.
- (iv) Every invariant and compact nonautonomous set of the differential equation $\dot{x} = x$ is a past repeller. Therefore, past repellers are not uniquely determined in general, in contrast to past attractors (see [16, Proposition 2.37]).

Example 3.3. We consider the linear nonautonomous differential equation

$$\dot{x} = a(t)x$$

with a continuous function $a : \mathbb{R} \rightarrow \mathbb{R}$. It is easy to see that every invariant and compact all-time nonautonomous set $M \subset \mathbb{R} \times \mathbb{R}$ is a

- past attractor if and only if $\lim_{t \rightarrow -\infty} \int_t^0 a(s) ds = -\infty$,
- past repeller if and only if $\lim_{t \rightarrow -\infty} \int_t^0 a(s) ds = \infty$,
- future attractor if and only if $\lim_{t \rightarrow \infty} \int_0^t a(s) ds = -\infty$,
- future repeller if and only if $\lim_{t \rightarrow \infty} \int_0^t a(s) ds = \infty$,
- all-time attractor if and only if $\lim_{t \rightarrow \infty} \sup_{\tau \in \mathbb{R}} \int_\tau^{\tau+t} a(s) ds = -\infty$,
- all-time repeller if and only if $\lim_{t \rightarrow \infty} \sup_{\tau \in \mathbb{R}} \int_\tau^{\tau+t} a(s) ds = \infty$.

4. Nonautonomous Morse decompositions

This section is devoted to a summary of the basic results from [17,18] concerning the existence of nonautonomous Morse decompositions for a linear cocycle $\Phi : \mathbb{I} \times \mathbb{I} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$. In addition to Φ , we also consider the induced system on the projective space $\mathbb{P}\Phi$, which was introduced in Section 2.

The first step towards a Morse decomposition is the construction of attractor–repeller pairs.

Theorem 4.1 (Existence of attractor–repeller pairs). *The following statements are fulfilled:*

- (i) Let \mathbb{I} be unbounded below and R be a past repeller of $\mathbb{P}\Phi$, i.e., there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} d(\mathbb{P}\Phi(\tau - t, \tau)U_\eta(R(\tau)) | R(\tau - t)) = 0 \quad \text{for all } \tau \leq 0.$$

Then the nonautonomous set R^* , defined by

$$R^*(\tau) := \bigcap_{t^* \geq 0} \overline{\bigcup_{t \geq t^*} \mathbb{P}\Phi(\tau, \tau - t)(\mathbb{P}^{N-1} \setminus U_\eta(R(\tau - t)))} \quad \text{for all } \tau \in \mathbb{I}, \quad (4.1)$$

is a past attractor, which is maximal outside R in the following sense: Any past attractor $A \supsetneq R^*$ has nonempty intersection with R . We call (R^*, R) a past attractor–repeller pair.

- (ii) Let \mathbb{I} be unbounded above and A be a future attractor of $\mathbb{P}\Phi$, i.e., there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} d(\mathbb{P}\Phi(\tau + t, \tau)U_\eta(A(\tau)) | A(\tau + t)) = 0 \quad \text{for all } \tau \geq 0.$$

Then the nonautonomous set A^* , defined by

$$A^*(\tau) := \bigcap_{t^* \geq 0} \overline{\bigcup_{t \geq t^*} \mathbb{P}\Phi(\tau, \tau + t)(\mathbb{P}^{N-1} \setminus U_\eta(A(\tau + t)))} \quad \text{for all } \tau \in \mathbb{I}, \quad (4.2)$$

is a future repeller, which is maximal outside A in the following sense: Any future repeller $R \supsetneq A^*$ has nonempty intersection with A . We call (A, A^*) a future attractor–repeller pair.

- (iii) Let $\mathbb{I} = \mathbb{R}$ and A be an all-time attractor of $\mathbb{P}\Phi$, i.e., there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} \sup_{\tau \in \mathbb{R}} d(\mathbb{P}\Phi(\tau + t, \tau)U_\eta(A(\tau)) | A(\tau + t)) = 0.$$

Then the nonautonomous set A^* , defined by (4.2) is an all-time repeller, which is maximal outside A in the following sense: Any all-time repeller $R \supsetneq A^*$ has nonempty intersection with A . We call (A, A^*) an all-time attractor–repeller pair.

- (iv) Let $\mathbb{I} = \mathbb{R}$ and R be an all-time repeller of $\mathbb{P}\Phi$, i.e., there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow \infty} \sup_{\tau \in \mathbb{R}} d(\mathbb{P}\Phi(\tau - t, \tau)U_\eta(R(\tau)) | R(\tau - t)) = 0.$$

Then the nonautonomous set R^* , defined by (4.1) is an all-time attractor, which is maximal outside R in the following sense: Any all-time attractor $A \supsetneq R^*$ has nonempty intersection with R . We call (R^*, R) an all-time attractor–repeller pair.

Proof. See [17, Theorem 4.3] and [18, Theorem 3.2]. \square

Remark 4.2.

- (i) In general, there is no formalism to obtain a past repeller from a past attractor and to get a future attractor from a future repeller (see [17, Example 4.4]).
- (ii) For an all-time attractor A , the relation $(A^*)^* = A$ is fulfilled, and an all-time repeller R fulfills $(R^*)^* = R$ (see [18, Theorem 3.2]).
- (iii) In [15], so-called *generalized attractor–repeller pairs* are introduced: Two invariant (w.r.t. the co-cycle φ) subsets \bar{A} and \bar{R} are called a generalized attractor–repeller pair if the following three conditions are fulfilled:
 - (a) $\bar{A}(t) \oplus \bar{R}(t) = \mathbb{R}^N$ for all $t \in \mathbb{R}$,
 - (b) given $\tau \in \mathbb{R}$, $0 \neq \xi \in \bar{A}(\tau)$ and $0 \neq \eta \in \bar{R}(\tau)$, we have

$$\frac{\|\varphi(t, \tau, \eta)\|}{\|\varphi(t, \tau, \xi)\|} \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{and} \quad \frac{\|\varphi(t, \tau, \xi)\|}{\|\varphi(t, \tau, \eta)\|} \rightarrow 0 \quad \text{as } t \rightarrow -\infty,$$

- (c) the angle between $\bar{A}(t)$ and $\bar{R}(t)$ is bounded below by a positive number.

All-time attractor–repeller pairs are also generalized attractor–repeller pairs, but in general, the reversal is not true.

The notion of an attractor–repeller pair is generalized by the following definition.

Definition 4.3 (*Nonautonomous Morse decompositions*). A set $\{M_1, M_2, \dots, M_n\}$ of nonautonomous sets, the so-called *Morse sets*, is called *past (future, all-time, respectively) Morse decomposition* of $\mathbb{P}\Phi$ if the representation

$$M_i = A_i \cap R_{i-1} \quad \text{for all } i \in \{1, \dots, n\}$$

is fulfilled with past (future, all-time, respectively) attractor–repeller pairs (A_i, R_i) , $i \in \{0, \dots, n\}$, fulfilling

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_n = \mathbb{I} \times \mathbb{P}^{N-1}$$

and

$$\mathbb{I} \times \mathbb{P}^{N-1} = R_0 \supsetneq R_1 \supsetneq \dots \supsetneq R_n = \emptyset.$$

The following theorem shows that Morse decompositions are crucial for the dynamical behavior of the nonautonomous dynamical system.

Theorem 4.4 (*Dynamical properties of nonautonomous Morse decompositions*). The following statements are fulfilled:

- (i) *Convergence in forward time.* Let $\{M_1, \dots, M_n\}$ be a future (all-time, respectively) Morse decomposition of $\mathbb{P}\Phi$. Then for all $(\tau, x) \in \mathbb{I} \times \mathbb{P}^{N-1}$, there exists an $i \in \{1, \dots, n\}$ with

$$\lim_{t \rightarrow \infty} d_{\mathbb{P}}(\mathbb{P}\Phi(\tau + t, \tau)x, M_i(\tau + t)) = 0.$$

- (ii) *Convergence in backward time.* Let $\{M_1, \dots, M_n\}$ be a past (all-time, respectively) Morse decomposition of $\mathbb{P}\Phi$. Then for all $(\tau, x) \in \mathbb{I} \times \mathbb{P}^{N-1}$, there exists an $i \in \{1, \dots, n\}$ with

$$\lim_{t \rightarrow \infty} d_{\mathbb{P}}(\mathbb{P}\Phi(\tau - t, \tau)x, M_i(\tau - t)) = 0.$$

Proof. See [17, Theorem 8.5] and [18, Theorem 4.4]. \square

Further convergence results for Morse decompositions can be found in [17, Theorem 5.6].

We conclude this section by stating a result concerning finest Morse decompositions, which is an analogon to the theorem of Selgrade (see [20]).

Theorem 4.5 (*Finest Morse decompositions*). *There exists a finest past (future, all-time, respectively) Morse decomposition $\{M_1, \dots, M_n\}$ of $\mathbb{P}\Phi$, i.e., the number of Morse sets of another past (future, all-time, respectively) Morse decomposition is bounded by n . Moreover, we have $n \leq N$, and the following decomposition is fulfilled:*

$$\mathbb{P}^{-1}M_1(t) \oplus \dots \oplus \mathbb{P}^{-1}M_n(t) = \mathbb{R}^N \quad \text{for all } t \in \mathbb{I}.$$

Proof. See [17, Theorem 8.7] and [18, Theorem 5.1]. \square

5. Notions of exponential dichotomy

In this section, several notions of exponential dichotomy are introduced with respect to the three different time domains. The concept of exponential dichotomy has been established in [13,14] in the late 1920s. In the sequel, many authors developed the theory; for fundamental work on this topic, we refer to [6,8–12,19,22–24].

Throughout this section, let \mathbb{I} be an interval of the form $(-\infty, 0]$, $[0, \infty)$ or \mathbb{R} , respectively, and consider a nonautonomous linear differential equation

$$\dot{x} = B(t)x \tag{5.1}$$

with a continuous function $B : \mathbb{I} \rightarrow \mathbb{R}^{N \times N}$. The linear cocycle of this equation is denoted by φ , and the corresponding functions Φ and $\mathbb{P}\Phi$ are defined as in Section 2.

We begin this section with some preliminary definitions. An invariant nonautonomous set $M \subset \mathbb{I} \times \mathbb{R}^N$ is called *linear integral manifold* of (5.1) if for each $t \in \mathbb{I}$, the sets $M(t)$ are linear subspaces of \mathbb{R}^N . Given linear integral manifolds M_1, M_2 of (5.1), the sets

$$M_1 \cap M_2 := \{(t, \xi) \in \mathbb{I} \times \mathbb{R}^N : \xi \in M_1(t) \cap M_2(t)\} \quad \text{and}$$

$$M_1 + M_2 := \{(t, \xi) \in \mathbb{I} \times \mathbb{R}^N : \xi \in M_1(t) + M_2(t)\}$$

are also linear integral manifolds of (5.1). A finite sum $M_1 + \dots + M_n$ of linear integral manifolds is called *Whitney sum* $M_1 \oplus \dots \oplus M_n$ if $M_i \cap M_j = \mathbb{I} \times \{0\}$ is satisfied for $i \neq j$. An *invariant projector* of (5.1) is a function $P : \mathbb{I} \rightarrow \mathbb{R}^{N \times N}$ with

$$P(t) = P(t)^2 \quad \text{and} \quad P(t)\Phi(t, \tau) = \Phi(t, \tau)P(\tau) \quad \text{for all } \tau, t \in \mathbb{I}.$$

The *range*

$$\mathcal{R}(P) := \{(t, \xi) \in \mathbb{I} \times \mathbb{R}^N : \xi \in \mathcal{R}(P(t))\}$$

and the *null space*

$$\mathcal{N}(P) := \{(t, \xi) \in \mathbb{I} \times \mathbb{R}^N : \xi \in \mathcal{N}(P(t))\}$$

of an invariant projector P are linear integral manifolds of (5.1) such that $\mathcal{R}(P) \oplus \mathcal{N}(P) = \mathbb{I} \times \mathbb{R}^N$. Since the fibres of $\mathcal{R}(P)$ and $\mathcal{N}(P)$ have the same dimension, we define the rank of P by

$$\text{rk } P := \dim \mathcal{R}(P) := \dim \mathcal{R}(P(t)) \quad \text{for all } t \in \mathbb{I},$$

and we set

$$\dim \mathcal{N}(P) := \dim \mathcal{N}(P(t)) \quad \text{for all } t \in \mathbb{R}.$$

Next, several notions of dichotomy are introduced for the linear system (5.1).

Definition 5.1 (*Nonhyperbolic exponential dichotomies*). Let $\gamma \in \mathbb{R}$ and $P_\gamma : \mathbb{I} \rightarrow \mathbb{R}^{N \times N}$ be an invariant projector of (5.1).

- (i) In case \mathbb{I} is unbounded below, we say that (5.1) admits a *nonhyperbolic past exponential dichotomy* with growth rate $\gamma \in \mathbb{R}$, constants $\alpha > 0$, $K \geq 1$ and projector P_γ if

$$\begin{aligned} \|\Phi(t, \tau)P_\gamma(\tau)\| &\leq Ke^{(\gamma-\alpha)(t-\tau)} \quad \text{for all } \tau \leq t \leq 0, \\ \|\Phi(t, \tau)(\mathbb{1} - P_\gamma(\tau))\| &\leq Ke^{(\gamma+\alpha)(t-\tau)} \quad \text{for all } t \leq \tau \leq 0. \end{aligned}$$

- (ii) In case \mathbb{I} is unbounded above, we say that (5.1) admits a *nonhyperbolic future exponential dichotomy* with growth rate $\gamma \in \mathbb{R}$, constants $\alpha > 0$, $K \geq 1$ and projector P_γ if

$$\begin{aligned} \|\Phi(t, \tau)P_\gamma(\tau)\| &\leq Ke^{(\gamma-\alpha)(t-\tau)} \quad \text{for all } 0 \leq \tau \leq t, \\ \|\Phi(t, \tau)(\mathbb{1} - P_\gamma(\tau))\| &\leq Ke^{(\gamma+\alpha)(t-\tau)} \quad \text{for all } 0 \leq t \leq \tau. \end{aligned}$$

- (iii) In case $\mathbb{I} = \mathbb{R}$, we say that (5.1) admits a *nonhyperbolic all-time exponential dichotomy* with growth rate $\gamma \in \mathbb{R}$, constants $\alpha > 0$, $K \geq 1$ and projector P_γ if

$$\begin{aligned} \|\Phi(t, \tau)P_\gamma(\tau)\| &\leq Ke^{(\gamma-\alpha)(t-\tau)} \quad \text{for all } \tau \leq t, \\ \|\Phi(t, \tau)(\mathbb{1} - P_\gamma(\tau))\| &\leq Ke^{(\gamma+\alpha)(t-\tau)} \quad \text{for all } t \leq \tau. \end{aligned}$$

We call a nonhyperbolic past (future, all-time, respectively) exponential dichotomy with growth rate $\gamma = 0$ also a *past (future, all-time, respectively) exponential dichotomy*.

Remark 5.2.

- (i) In the literature, an all-time exponential dichotomy is simply called *exponential dichotomy*. Furthermore, a past or future exponential dichotomy is called exponential dichotomy on half line \mathbb{R}_0^- or \mathbb{R}_0^+ , respectively (see, e.g., [6]).
- (ii) If (5.1) is almost periodic, then the notions for the past, future and entire time are identical (see [6, Proposition 3, p. 70]).

In the following proposition, the relationship between the above introduced notions of dichotomies is examined.

Proposition 5.3. *In case $\mathbb{I} = \mathbb{R}$, the following statements are fulfilled:*

- (i) *If (5.1) admits a nonhyperbolic all-time exponential dichotomy with growth rate γ , then it also admits both a nonhyperbolic past and future exponential dichotomy with growth rate γ .*
- (ii) *If (5.1) admits both a nonhyperbolic past and future exponential dichotomy with growth rate γ with the same invariant projector P , then it also admits an all-time exponential dichotomy with growth rate γ .*

Proof. (i) is obvious; for (ii), see [6, p. 19]. \square

Lemma 5.4 (Criteria for nonhyperbolic dichotomies). Suppose that (5.1) admits a nonhyperbolic past (future, all-time, respectively) exponential dichotomy with growth rate γ and projector P_γ . Then the following statements are fulfilled:

- (i) If $P_\gamma \equiv \mathbb{1}$, then (5.1) admits a nonhyperbolic past (future, all-time, respectively) exponential dichotomy with growth rate ζ and projector $P_\zeta \equiv \mathbb{1}$ for all $\zeta > \gamma$.
- (ii) If $P_\gamma \equiv 0$, then (5.1) admits a nonhyperbolic past (future, all-time, respectively) exponential dichotomy with growth rate ζ and projector $P_\zeta \equiv 0$ for all $\zeta < \gamma$.

Proof. The assertions follow directly from the monotonicity of the exponential function. \square

In case \mathbb{I} is unbounded above, we define

$$\mathcal{S}_\gamma := \{(\tau, \xi) \in \mathbb{I} \times \mathbb{R}^N : \Phi(\cdot, \tau)\xi \text{ is } \gamma^+\text{-quasibounded}\} \quad \text{for all } \gamma \in \mathbb{R},$$

and if \mathbb{I} is unbounded below, we set

$$\mathcal{U}_\gamma := \{(\tau, \xi) \in \mathbb{I} \times \mathbb{R}^N : \Phi(\cdot, \tau)\xi \text{ is } \gamma^-\text{-quasibounded}\} \quad \text{for all } \gamma \in \mathbb{R}.$$

It is obvious that \mathcal{S}_γ and \mathcal{U}_γ are linear integral manifolds of (5.1), and given $\gamma \leq \zeta$, the relations $\mathcal{S}_\gamma \subset \mathcal{S}_\zeta$ and $\mathcal{U}_\gamma \supset \mathcal{U}_\zeta$ are fulfilled.

Now we discuss the important relationship between the projectors of nonhyperbolic exponential dichotomies with growth rate γ and the sets \mathcal{S}_γ and \mathcal{U}_γ .

Proposition 5.5 (Dynamical properties). If (5.1) admits a nonhyperbolic past exponential dichotomy with growth rate γ , constants α , K and projector P_γ , then we have $\mathcal{N}(P_\gamma) = \mathcal{U}_\gamma$ and

$$\|\Phi(t, \tau)\xi\| \leq K \|\xi\| e^{\gamma(t-\tau)} \quad \text{for all } \tau \leq t \leq 0 \text{ and } \xi \in \mathcal{R}(P_\gamma(\tau)). \quad (5.2)$$

If (5.1) admits a nonhyperbolic future exponential dichotomy with growth rate γ , constants α , K and projector P_γ , then we have $\mathcal{R}(P_\gamma) = \mathcal{S}_\gamma$ and

$$\|\Phi(t, \tau)\xi\| \leq K \|\xi\| e^{\gamma(t-\tau)} \quad \text{for all } 0 \leq t \leq \tau \text{ and } \xi \in \mathcal{N}(P_\gamma(\tau)).$$

If (5.1) admits a nonhyperbolic all-time exponential dichotomy with growth rate γ and projector P_γ , then $\mathcal{N}(P_\gamma) = \mathcal{U}_\gamma$ and $\mathcal{R}(P_\gamma) = \mathcal{S}_\gamma$ are fulfilled.

Proof. Suppose that (5.1) admits a nonhyperbolic past exponential dichotomy with growth rate γ , constants α , K and projector P_γ . Hence, we have

$$\begin{aligned} \|\Phi(t, \tau)P_\gamma(\tau)\| &\leq Ke^{(\gamma-\alpha)(t-\tau)} \quad \text{for all } \tau \leq t \leq 0, \\ \|\Phi(t, \tau)(\mathbb{1} - P_\gamma(\tau))\| &\leq Ke^{(\gamma+\alpha)(t-\tau)} \quad \text{for all } t \leq \tau \leq 0. \end{aligned}$$

The first inequality implies (5.2). Now we prove the relation $\mathcal{N}(P_\gamma) = \mathcal{U}_\gamma$.

(\supseteq) We choose $(\tau, \xi) \in \mathcal{U}_\gamma$ arbitrarily. This implies $\|\Phi(t, \tau)\xi\| \leq Ce^{\gamma(t-\tau)}$ for all $t \leq \tau$ with some real constant $C > 0$. We write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathcal{R}(P_\gamma(\tau))$ and $\xi_2 \in \mathcal{N}(P_\gamma(\tau))$. Hence, for all $t \leq \tau$, we get

$$\begin{aligned} \|\xi_1\| &= \|\Phi(\tau, t)\Phi(t, \tau)P_\gamma(\tau)\xi\| = \|\Phi(\tau, t)P_\gamma(t)\Phi(t, \tau)\xi\| \\ &\leq Ke^{(\gamma-\alpha)(\tau-t)} \|\Phi(t, \tau)\xi\| \leq CK e^{(\gamma-\alpha)(\tau-t)} e^{\gamma(t-\tau)} = CK e^{-\alpha(\tau-t)}. \end{aligned}$$

The right-hand side of this inequality converges to zero in the limit $t \rightarrow -\infty$. Therefore, $\xi_1 = 0$, and this means $(\tau, \xi) \in \mathcal{N}(P_\gamma)$.

(\subseteq) We choose $(\tau, \xi) \in \mathcal{N}(P_\gamma)$. Thus, for all $t \leq \tau \leq 0$, the relation $\|\Phi(t, \tau)\xi\| \leq Ke^{(\gamma+\alpha)(t-\tau)}\|\xi\|$ is fulfilled. This means that $\Phi(\cdot, \tau)\xi$ is γ^- -quasibounded.

The assertions concerning the future exponential dichotomy are treated analogously. In case (5.1) admits an all-time exponential dichotomy, Proposition 5.3(i) yields that (5.1) also admits a past exponential dichotomy and a future exponential dichotomy. Hence, we obtain both $\mathcal{N}(P_\gamma) = \mathcal{U}_\gamma$ and $\mathcal{R}(P_\gamma) = \mathcal{S}_\gamma$. \square

Remark 5.6. According to this proposition, an invariant projector is uniquely determined only in case of a nonhyperbolic all-time exponential dichotomy. In addition, the null space of a projector of a past exponential dichotomy and the range of a projector of a future exponential dichotomy are uniquely determined. This implies that the rank of exponential dichotomies with the same growth rate is independent of the choice of the projector. For further information about the kind of nonuniqueness of ranges of projectors of past exponential dichotomies and null spaces of projectors of future exponential dichotomies, we refer to Lemma 6.1 in the next section.

The following proposition shows that the notions of dichotomy are consistent to the concepts of attractivity and repulsivity.

Proposition 5.7 (Nonhyperbolic dichotomies and the notions of attractivity and repulsivity). Suppose that (5.1) admits a nonhyperbolic past (future, all-time, respectively) exponential dichotomy with growth rate γ and invariant projector P_γ . Then the following statements are fulfilled:

- If $\gamma \leq 0$ and $\text{rk } P_\gamma \geq 1$, then no solution of (5.1) is past (future, all-time, respectively) repulsive.
- If $\gamma \geq 0$ and $\text{rk } P_\gamma \leq N - 1$, then no solution of (5.1) is past (future, all-time, respectively) attractive.
- If $\gamma \leq 0$ and $\text{rk } P_\gamma = N$, then every solution of (5.1) is past (future, all-time, respectively) attractive.
- If $\gamma \geq 0$ and $\text{rk } P_\gamma = 0$, then every solution of (5.1) is past (future, all-time, respectively) repulsive.

Proof. These assertions are direct consequences of Proposition 5.5. \square

For the rest of this section, the studies are concentrated on the induced system $\mathbb{P}\Phi$ on the real projective space \mathbb{P}^{N-1} .

Lemma 5.8. The following statements are fulfilled:

- (i) We suppose that (5.1) admits a nonhyperbolic past exponential dichotomy with projector P . Then there exists a $\beta > 0$ such that

$$U_\beta(\mathbb{P}\mathcal{R}(P(t))) \cap U_\beta(\mathbb{P}\mathcal{N}(P(t))) = \emptyset \quad \text{for all } t \leq 0.$$

Moreover, for all $\tau \leq 0$ and compact sets $C \subset \mathbb{S}^{N-1} \setminus \mathcal{N}(P(\tau))$, we have

$$\lim_{t \rightarrow -\infty} \frac{\sup_{v \in \mathbb{S}^{N-1} \cap \mathcal{N}(P(\tau))} \|\Phi(t, \tau)v\|}{\inf_{w \in C} \|\Phi(t, \tau)w\|} = 0.$$

- (ii) We suppose that (5.1) admits a nonhyperbolic future exponential dichotomy with projector P . Then there exists a $\beta > 0$ such that

$$U_\beta(\mathbb{P}\mathcal{R}(P(t))) \cap U_\beta(\mathbb{P}\mathcal{N}(P(t))) = \emptyset \quad \text{for all } t \geq 0.$$

Moreover, for all $\tau \geq 0$ and compact sets $C \subset \mathbb{S}^{N-1} \setminus \mathcal{R}(P(\tau))$, we have

$$\lim_{t \rightarrow \infty} \frac{\sup_{v \in \mathbb{S}^{N-1} \cap \mathcal{R}(P(\tau))} \|\Phi(t, \tau)v\|}{\inf_{w \in C} \|\Phi(t, \tau)w\|} = 0.$$

(iii) We suppose that (5.1) admits a nonhyperbolic all-time exponential dichotomy with projector P . Then there exists a $\beta > 0$ such that

$$U_\beta(\mathbb{P}\mathcal{R}(P(t))) \cap U_\beta(\mathbb{P}\mathcal{N}(P(t))) = \emptyset \quad \text{for all } t \in \mathbb{R},$$

and we have

$$\lim_{t \rightarrow -\infty} \sup_{\tau \in \mathbb{R}} \frac{\sup_{v \in \mathbb{S}^{N-1} \cap \mathcal{N}(P(\tau))} \|\Phi(t, \tau)v\|}{\inf_{w \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}U_\beta(\mathbb{P}\mathcal{R}(P(\tau)))} \|\Phi(t, \tau)w\|} = 0$$

and

$$\lim_{t \rightarrow \infty} \sup_{\tau \in \mathbb{R}} \frac{\sup_{v \in \mathbb{S}^{N-1} \cap \mathcal{R}(P(\tau))} \|\Phi(t, \tau)v\|}{\inf_{w \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}U_\beta(\mathbb{P}\mathcal{N}(P(\tau)))} \|\Phi(t, \tau)w\|} = 0.$$

Proof. (i) Suppose that (5.1) admits a nonhyperbolic past exponential dichotomy with growth rate γ , constants $\alpha > 0$, $K \geq 1$ and projector P . We define $\beta := \frac{1}{3K}$ and fix an arbitrary $\tau \in \mathbb{R}$. The remaining proof of (i) is divided into four steps.

Step 1. $U_\beta(\mathbb{P}\mathcal{R}(P(t))) \cap U_\beta(\mathbb{P}\mathcal{N}(P(t))) = \emptyset$ for all $t \leq 0$.

Assume that there exists a $t \leq 0$ such that $U_\beta(\mathbb{P}\mathcal{R}(P(t))) \cap U_\beta(\mathbb{P}\mathcal{N}(P(t))) \neq \emptyset$. Hence, there exist $x \in \mathbb{P}\mathcal{R}(P(t))$ and $y \in \mathbb{P}\mathcal{N}(P(t))$ with $d_{\mathbb{P}}(x, y) \leq 2\beta$. Due to the definition of $d_{\mathbb{P}}$, there exist $\tilde{x} \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}\{x\}$ and $\tilde{y} \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}\{y\}$ such that $\|\tilde{x} - \tilde{y}\| \leq 2\beta$. This yields

$$\frac{\|P(t)(\tilde{x} - \tilde{y})\|}{\|\tilde{x} - \tilde{y}\|} = \frac{\|\tilde{x}\|}{\|\tilde{x} - \tilde{y}\|} \geq \frac{1}{2\beta} = \frac{3K}{2},$$

and this is a contradiction, since Definition 3.1(i) implies $\|P(t)\| \leq K$.

Step 2. We have

$$\|\Phi(t, \tau)x\| \geq \frac{1}{K} e^{(\gamma - \alpha)(t - \tau)} \|x\| \quad \text{for all } 0 \geq \tau \geq t \text{ and } x \in \mathcal{R}(P(\tau)).$$

The assertion follows from

$$\|x\| = \|\Phi(\tau, t)\Phi(t, \tau)P(\tau)x\| \stackrel{\text{Definition 3.1(i)}}{\leq} K e^{(\gamma - \alpha)(\tau - t)} \|\Phi(t, \tau)x\|.$$

Step 3. Let $M \subset \mathbb{S}^{N-1} \setminus \mathcal{N}(P(t))$ be a compact set. For $w \in M$, we write $w = w_r + w_n$ with $w_r \in \mathcal{R}(P(t))$ and $w_n \in \mathcal{N}(P(t))$. Then

$$W_r(M) := \{w_r : w \in M\} = P(t)M$$

is bounded away from zero, and

$$W_n(M) := \{w_n : w \in M\} = (\mathbb{1} - P(t))M$$

is bounded.

Assume that the set $W_r(M) = P(t)M$ is not bounded away from zero. Then it contains 0, since it is compact, and thus, there exists a $w \in M$ with $w \in \mathcal{N}(P(t))$. This is a contradiction. Moreover, the set $W_n(M) = (\mathbb{1} - P(t))M$ is bounded, since it is compact.

Step 4. For all $\tau \leq 0$ and compact sets $C \subset \mathbb{S}^{N-1} \setminus \mathcal{N}(P(\tau))$, we have

$$\lim_{t \rightarrow -\infty} \frac{\sup_{v \in \mathbb{S}^{N-1} \cap \mathcal{N}(P(\tau))} \|\Phi(t, \tau)v\|}{\inf_{w \in C} \|\Phi(t, \tau)w\|} = 0.$$

For all $t \leq \tau$, we have

$$\begin{aligned} \frac{\sup_{v \in \mathbb{S}^{N-1} \cap \mathcal{N}(P(\tau))} \|\Phi(t, \tau)v\|}{\inf_{w \in C} \|\Phi(t, \tau)w\|} &\stackrel{\text{Definition 3.1(i)}}{\leq} \frac{\sup_{v \in \mathbb{S}^{N-1} \cap \mathcal{N}(P(\tau))} Ke^{(\gamma+\alpha)(t-\tau)} \|v\|}{\inf_{w \in C} \|\Phi(t, \tau)w_r + \Phi(t, \tau)w_n\|} \\ &\leq \sup_{w \in C} \frac{\frac{Ke^{(\gamma+\alpha)(t-\tau)}}{\|\Phi(t, \tau)w_r\|}}{1 - \frac{\|\Phi(t, \tau)w_n\|}{\|\Phi(t, \tau)w_r\|}}. \end{aligned}$$

Please note that for the last inequality, we require $w_r \neq 0$ for all $w \in C$. This is fulfilled, since $W_r(C)$ is bounded away from zero (cf. Step 3). Furthermore, using

$$\frac{Ke^{(\gamma+\alpha)(t-\tau)}}{\|\Phi(t, \tau)w_r\|} \stackrel{\text{Step 2}}{\leq} \frac{Ke^{(\gamma+\alpha)(t-\tau)}}{\frac{1}{K}e^{(\gamma-\alpha)(t-\tau)}\|w_r\|} = \frac{K^2e^{2\alpha(t-\tau)}}{\|w_r\|},$$

we obtain

$$\lim_{t \rightarrow -\infty} \sup_{w \in C} \frac{Ke^{(\gamma+\alpha)(t-\tau)}}{\|\Phi(t, \tau)w_r\|} = 0,$$

since $W_r(C)$ is bounded away from zero. Moreover, due to

$$\frac{\|\Phi(t, \tau)w_n\|}{\|\Phi(t, \tau)w_r\|} \stackrel{\text{Definition 3.1(i), Step 2}}{\leq} \frac{Ke^{(\gamma+\alpha)(t-\tau)}\|w_n\|}{\frac{1}{K}e^{(\gamma-\alpha)(t-\tau)}\|w_r\|} = \frac{K^2e^{2\alpha(t-\tau)}\|w_n\|}{\|w_r\|},$$

we get

$$\lim_{t \rightarrow -\infty} \sup_{w \in C} \frac{\|\Phi(t, \tau)w_n\|}{\|\Phi(t, \tau)w_r\|} = 0$$

(please note that Step 3 says that $W_n(C)$ is bounded and $W_r(C)$ is bounded away from zero). This implies the assertion.

(ii) and (iii) can be proved similarly to (i). \square

The following theorem says that ranges and null spaces of invariant projectors give rise to nonautonomous repellers and attractors.

Theorem 5.9 (Ranges and null spaces of invariant projectors as nonautonomous repellers and attractors). *We suppose that (5.1) admits a nonhyperbolic past (future, all-time, respectively) exponential dichotomy with projector P and consider the induced system $\mathbb{P}\Phi$ on the real projective space \mathbb{P}^{N-1} . Then the following statements are fulfilled:*

- (i) $\mathbb{P}\mathcal{R}(P)$ is a past (future, all-time, respectively) repeller,
- (ii) $\mathbb{P}\mathcal{N}(P)$ is a past (future, all-time, respectively) attractor,

(iii) in case of a nonhyperbolic past exponential dichotomy, the relation $\mathbb{P}\mathcal{N}(P) = \mathbb{P}\mathcal{R}(P)^*$ holds, and in case of a nonhyperbolic future exponential dichotomy, we have $\mathbb{P}\mathcal{R}(P) = \mathbb{P}\mathcal{N}(P)^*$. Hence, in case of a nonhyperbolic all-time exponential dichotomy, both $\mathbb{P}\mathcal{N}(P) = \mathbb{P}\mathcal{R}(P)^*$ and $\mathbb{P}\mathcal{R}(P) = \mathbb{P}\mathcal{N}(P)^*$ are fulfilled.

Proof. We concentrate on the case of a nonhyperbolic past exponential dichotomy, since the other cases can be treated analogously. The proof is divided into five steps.

Step 1. For all $\tau \in \mathbb{R}$ and compact sets $C \subset \mathbb{P}^{N-1}$ with $C \cap \mathbb{P}\mathcal{N}(P(\tau)) = \emptyset$, we have

$$\lim_{t \rightarrow -\infty} \inf_{0 \neq v \in \mathbb{P}^{N-1}C} \frac{\|\Phi(t, \tau)v_r\|}{\|\Phi(t, \tau)v\|} = \lim_{t \rightarrow -\infty} \sup_{0 \neq v \in \mathbb{P}^{N-1}C} \frac{\|\Phi(t, \tau)v_r\|}{\|\Phi(t, \tau)v\|} = 1,$$

where $v = v_a + v_r$ with $v_a \in \mathcal{N}(P(\tau))$ and $v_r \in \mathcal{R}(P(\tau))$.

The first assertion follows from

$$\begin{aligned} \lim_{t \rightarrow -\infty} \inf_{0 \neq v \in \mathbb{P}^{N-1}C} \frac{\|\Phi(t, \tau)v_r\|}{\|\Phi(t, \tau)v\|} &\geq \left(\lim_{t \rightarrow -\infty} \sup_{0 \neq v \in \mathbb{P}^{N-1}C} \frac{\|\Phi(t, \tau)v_a\|}{\|\Phi(t, \tau)v_r\|} + 1 \right)^{-1} \\ &= \left(\lim_{t \rightarrow -\infty} \sup_{v \in \mathbb{P}^{N-1}C, v_a \neq 0} \frac{\|v_a\| \|\Phi(t, \tau) \frac{v_a}{\|v_a\|}\|}{\|v_r\| \|\Phi(t, \tau) \frac{v_r}{\|v_r\|}\|} + 1 \right)^{-1} \stackrel{\text{Lemma 5.8(i)}}{=} 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow -\infty} \inf_{0 \neq v \in \mathbb{P}^{N-1}C} \frac{\|\Phi(t, \tau)v_r\|}{\|\Phi(t, \tau)v\|} &\leq \left(\lim_{t \rightarrow -\infty} \sup_{0 \neq v \in \mathbb{P}^{N-1}C} \left| 1 - \frac{\|\Phi(t, \tau)v_a\|}{\|\Phi(t, \tau)v_r\|} \right| \right)^{-1} \\ &= \left(\lim_{t \rightarrow -\infty} \sup_{v \in \mathbb{P}^{N-1}C, v_a \neq 0} \left| 1 - \frac{\|v_a\| \|\Phi(t, \tau) \frac{v_a}{\|v_a\|}\|}{\|v_r\| \|\Phi(t, \tau) \frac{v_r}{\|v_r\|}\|} \right| \right)^{-1} \stackrel{\text{Lemma 5.8(i)}}{=} 1. \end{aligned}$$

In both relations, Lemma 5.8(i) is applicable, because the set $\{v_a: v \in \mathbb{P}^{N-1}C \cap \mathbb{S}^{N-1}\}$ is compact and the set $\{v_r: v \in \mathbb{P}^{N-1}C \cap \mathbb{S}^{N-1}\}$ is bounded away from zero. This is due to the fact that $\{v_a: v \in \mathbb{P}^{N-1}C \cap \mathbb{S}^{N-1}\} = P(\tau)(\mathbb{P}^{N-1}C \cap \mathbb{S}^{N-1})$ and $\{v_r: v \in \mathbb{P}^{N-1}C \cap \mathbb{S}^{N-1}\} = (\mathbb{1} - P(\tau))(\mathbb{P}^{N-1}C \cap \mathbb{S}^{N-1})$. The assertion

$$\lim_{t \rightarrow -\infty} \sup_{0 \neq v \in \mathbb{P}^{N-1}C} \frac{\|\Phi(t, \tau)v_r\|}{\|\Phi(t, \tau)v\|} = 1$$

follows analogously.

Step 2. For all $\tau \in \mathbb{R}$ and compact sets $C \subset \mathbb{P}^{N-1}$ with $C \cap \mathbb{P}\mathcal{N}(P(\tau)) = \emptyset$, we have

$$\lim_{t \rightarrow -\infty} d_{\mathbb{P}}(\mathbb{P}\Phi(t, \tau)C | \mathbb{P}\mathcal{R}(P(t))) = 0.$$

With v_a and v_r defined as in Step 1, for all $t \leq \tau$ and $v \in \mathbb{S}^{N-1} \cap \mathbb{P}^{N-1}C$, we consider the expression

$$\begin{aligned} &\frac{\langle \Phi(t, \tau)v, \Phi(t, \tau)v_r \rangle^2}{\|\Phi(t, \tau)v\|^2 \|\Phi(t, \tau)v_r\|^2} \\ &= \frac{(\langle \Phi(t, \tau)v_a, \Phi(t, \tau)v_r \rangle + \langle \Phi(t, \tau)v_r, \Phi(t, \tau)v_r \rangle)^2}{\|\Phi(t, \tau)v\|^2 \|\Phi(t, \tau)v_r\|^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{\langle \Phi(t, \tau)v_a, \Phi(t, \tau)v_r \rangle^2 + \|\Phi(t, \tau)v_r\|^4 + 2\langle \Phi(t, \tau)v_a, \Phi(t, \tau)v_r \rangle \|\Phi(t, \tau)v_r\|^2}{\|\Phi(t, \tau)v\|^2 \|\Phi(t, \tau)v_r\|^2} \\
&= \frac{\langle \Phi(t, \tau)v_a, \Phi(t, \tau)v_r \rangle^2}{\|\Phi(t, \tau)v\|^2 \|\Phi(t, \tau)v_r\|^2} + \frac{\|\Phi(t, \tau)v_r\|^2}{\|\Phi(t, \tau)v\|^2} + \frac{2\langle \Phi(t, \tau)v_a, \Phi(t, \tau)v_r \rangle}{\|\Phi(t, \tau)v\|^2}.
\end{aligned}$$

Using the Cauchy–Schwarz inequality, we obtain the following relations:

$$\begin{aligned}
0 &\leq \lim_{t \rightarrow -\infty} \sup_{v \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}C} \frac{\langle \Phi(t, \tau)v_a, \Phi(t, \tau)v_r \rangle^2}{\|\Phi(t, \tau)v\|^2 \|\Phi(t, \tau)v_r\|^2} \\
&\leq \lim_{t \rightarrow -\infty} \sup_{v \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}C} \frac{\|\Phi(t, \tau)v_a\|^2}{\|\Phi(t, \tau)v\|^2} \stackrel{\text{Lemma 5.8(i)}}{=} 0
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \lim_{t \rightarrow -\infty} \sup_{v \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}C} 2 \frac{|\langle \Phi(t, \tau)v_a, \Phi(t, \tau)v_r \rangle|}{\|\Phi(t, \tau)v\|^2} \\
&\leq \lim_{t \rightarrow -\infty} \sup_{v \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}C} 2 \frac{\|\Phi(t, \tau)v_a\| \|\Phi(t, \tau)v_r\|}{\|\Phi(t, \tau)v\| \|\Phi(t, \tau)v\|} \\
&\stackrel{\text{Step 1}}{=} \lim_{t \rightarrow -\infty} \sup_{v \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}C} 2 \frac{\|\Phi(t, \tau)v_a\|}{\|\Phi(t, \tau)v\|} \stackrel{\text{Lemma 5.8(i)}}{=} 0.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\lim_{t \rightarrow -\infty} \inf_{v \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}C} \frac{\langle \Phi(t, \tau)v, \Phi(t, \tau)v_r \rangle^2}{\|\Phi(t, \tau)v\|^2 \|\Phi(t, \tau)v_r\|^2} \\
&= \lim_{t \rightarrow -\infty} \inf_{v \in \mathbb{S}^{N-1} \cap \mathbb{P}^{-1}C} \left(\frac{\langle \Phi(t, \tau)v_a, \Phi(t, \tau)v_r \rangle^2}{\|\Phi(t, \tau)v\|^2 \|\Phi(t, \tau)v_r\|^2} + \frac{\|\Phi(t, \tau)v_r\|^2}{\|\Phi(t, \tau)v\|^2} + \frac{2\langle \Phi(t, \tau)v_a, \Phi(t, \tau)v_r \rangle}{\|\Phi(t, \tau)v\|^2} \right) \\
&\stackrel{\text{Step 1}}{=} 1.
\end{aligned}$$

Using Lemma 1.1, this implies the assertion.

Step 3. $\mathbb{P}\mathcal{R}(P)$ is a past repeller.

This is a direct consequence of Step 2 and the fact that there exists a $\beta > 0$ such that $U_\beta(\mathbb{P}\mathcal{R}(P(t))) \cap U_\beta(\mathbb{P}\mathcal{N}(P(t))) = \emptyset$ for all $t \leq 0$ (see Lemma 5.8(i)).

Step 4. The relation $\mathbb{P}\mathcal{N}(P) = \mathbb{P}\mathcal{R}(P)^*$ is fulfilled.

Since $\mathbb{P}\mathcal{R}(P)$ is a past repeller, there exists an $\eta > 0$ such that

$$\lim_{t \rightarrow -\infty} d(\mathbb{P}\Phi(t, \tau)U_\eta(\mathbb{P}\mathcal{R}(P(\tau))) | \mathbb{P}\mathcal{R}(P(t))) = 0 \quad \text{for all } \tau \leq 0.$$

We choose $\varepsilon > 0$ and $\tau \in \mathbb{R}$ arbitrarily and consider the compact set $C := \mathbb{P}^{N-1} \setminus U_\varepsilon(\mathbb{P}\mathcal{N}(P(\tau)))$. Due to Step 2, we have

$$\lim_{t \rightarrow -\infty} d(\mathbb{P}\Phi(t, \tau)C | \mathcal{R}(P(t))) = 0.$$

This implies that there exists a $t_0 < \tau$ such that $\mathbb{P}\Phi(t, \tau)C \subset U_\eta(\mathbb{P}\mathcal{R}(P(t)))$ for all $t \leq t_0$. Thus,

$$d_{\mathbb{P}}(\mathbb{P}\Phi(\tau, t)(\mathbb{P}^{N-1} \setminus U_\eta(\mathcal{R}(P(t)))) | \mathbb{P}\mathcal{N}(P(\tau))) \leq \varepsilon \quad \text{for all } t \leq t_0,$$

and therefore,

$$\mathbb{PN}(P(\tau)) = \bigcap_{t^* \leq 0} \overline{\bigcup_{t \leq t^*} \mathbb{P}\Phi(\tau, t)(\mathbb{P}^{N-1} \setminus U_\eta(\mathbb{PR}(P(t))))}.$$

The assertion follows from Theorem 4.1(i). \square

6. Dichotomy spectra

In the previous section, notions of dichotomy have been introduced by localizing attractive and repulsive directions. To classify the strength of attractivity and repulsivity of linear systems, the concept of the dichotomy spectrum is essential. For linear skew product flows with compact base sets, the so-called Sacker–Sell spectrum (see [25]) has become widely accepted. In [21], this spectrum has been adapted for arbitrary classes of linear differential and difference equations, respectively. In addition to this dichotomy spectrum, two other kinds of spectra are introduced in this section, which represent the behavior of the linear system in the past and future.

Throughout this section, let \mathbb{I} be an interval of the form $(-\infty, 0]$, $[0, \infty)$ or \mathbb{R} , respectively, and consider a nonautonomous linear differential equation

$$\dot{x} = B(t)x \tag{6.1}$$

with a continuous function $B : \mathbb{I} \rightarrow \mathbb{R}^{N \times N}$. The linear cocycle of this equation is denoted by φ , and the corresponding functions Φ and $\mathbb{P}\Phi$ are defined as in Section 2.

As indicated in Remark 5.6, an invariant projector is uniquely determined only in case of a nonhyperbolic all-time exponential dichotomy. The degree of nonuniqueness of projectors of past and future exponential dichotomies is described in the following lemma, which is adapted from [1, Lemma 2.4].

Lemma 6.1. *The following statements are fulfilled:*

- (i) *Suppose that (6.1) admits a nonhyperbolic past exponential dichotomy with growth rate γ and projector P , and let \bar{P} be another invariant projector with*

$$\sup_{t \leq 0} \|\bar{P}(t)\| < \infty \quad \text{and} \quad \mathcal{N}(P) = \mathcal{N}(\bar{P}).$$

Then (6.1) also admits a nonhyperbolic past exponential dichotomy with growth rate γ and projector \bar{P} .

- (ii) *Suppose that (6.1) admits a nonhyperbolic future exponential dichotomy with growth rate γ and projector P , and let \bar{P} be another invariant projector with*

$$\sup_{t \geq 0} \|\bar{P}(t)\| < \infty \quad \text{and} \quad \mathcal{R}(P) = \mathcal{R}(\bar{P}).$$

Then (6.1) also admits a nonhyperbolic future exponential dichotomy with growth rate γ and projector \bar{P} .

Proof. (i) Suppose that (6.1) admits a nonhyperbolic past exponential dichotomy with growth rate γ , constants $\alpha > 0$, $K \geq 1$ and projector P , and let \bar{P} be given as above. First, we observe that $\sup_{t \leq 0} \|P(t)\| \leq K$, and we define $M := \sup_{t \leq 0} \|\bar{P}(t)\|$. The relation $\mathcal{N}(P) = \mathcal{N}(\bar{P})$ implies the two equations

$$(\mathbb{1} - \bar{P}) = (\mathbb{1} - P)(\mathbb{1} - \bar{P}) \quad \text{and} \quad \bar{P} = (\mathbb{1} - P + \bar{P})P.$$

The first equation yields for all $\tau \leq 0$ and $t \leq \tau$

$$\begin{aligned}\|\Phi(t, \tau)(\mathbb{1} - \bar{P}(\tau))\| &= \|\Phi(t, \tau)(\mathbb{1} - P(\tau))(\mathbb{1} - \bar{P}(\tau))\| \\ &\leq \|\Phi(t, \tau)(\mathbb{1} - P(\tau))\| \|\mathbb{1} - \bar{P}(\tau)\| \leq K(1 + M)e^{(\gamma + \alpha)(t - \tau)}.\end{aligned}$$

Using the invariance of P and \bar{P} , the second equation implies

$$\begin{aligned}\|\Phi(t, \tau)\bar{P}(\tau)\| &= \|\Phi(t, \tau)(\mathbb{1} - P(\tau) + \bar{P}(\tau))P(\tau)\| \\ &\leq \|(\mathbb{1} - P(t) + \bar{P}(t))\| \|\Phi(t, \tau)P(\tau)\| \\ &\leq K(1 + K + M)e^{(\gamma - \alpha)(t - \tau)}\end{aligned}$$

for all $t \leq 0$ and $\tau \leq t$.

The assertion (ii) can be proved similarly. \square

For the definition of the dichotomy spectra, it is crucial for which growth rates, the linear system (6.1) admits a nonhyperbolic exponential dichotomy. We will not exclude growth rates $\gamma = \pm\infty$ from our considerations, i.e., we say that (6.1) admits a nonhyperbolic dichotomy with growth rate ∞ if there exists a $\gamma \in \mathbb{R}$ such that (6.1) admits a nonhyperbolic dichotomy with growth rate γ and projector $P_\gamma \equiv \mathbb{1}$. Accordingly, we say that (6.1) admits a nonhyperbolic dichotomy with growth rate $-\infty$ if there exists a $\gamma \in \mathbb{R}$ such that (6.1) admits a nonhyperbolic dichotomy with growth rate γ and projector $P_\gamma \equiv 0$.

Definition 6.2 (*Dichotomy spectra*). Consider the linear system (6.1), $\dot{x} = B(t)x$.

(i) The *past dichotomy spectrum* of (6.1) is defined by

$$\Sigma^{\leftarrow} := \{\gamma \in \overline{\mathbb{R}}: (6.1) \text{ does not admit a nonhyperbolic past exponential dichotomy with growth rate } \gamma\}.$$

(ii) The *future dichotomy spectrum* of (6.1) is defined by

$$\Sigma^{\rightarrow} := \{\gamma \in \overline{\mathbb{R}}: (6.1) \text{ does not admit a nonhyperbolic future exponential dichotomy with growth rate } \gamma\}.$$

(iii) The *all-time dichotomy spectrum* of (6.1) is defined by

$$\Sigma^{\leftrightarrow} := \{\gamma \in \overline{\mathbb{R}}: (6.1) \text{ does not admit a nonhyperbolic all-time exponential dichotomy with growth rate } \gamma\}.$$

The corresponding *resolvent sets* are defined as follows:

$$\rho^{\leftarrow} := \overline{\mathbb{R}} \setminus \Sigma^{\leftarrow}, \quad \rho^{\rightarrow} := \overline{\mathbb{R}} \setminus \Sigma^{\rightarrow} \quad \text{and} \quad \rho^{\leftrightarrow} := \overline{\mathbb{R}} \setminus \Sigma^{\leftrightarrow}.$$

Remark 6.3.

- (i) The all-time dichotomy spectrum without $\{-\infty, \infty\}$, i.e., $\Sigma^{\leftrightarrow} \cap \mathbb{R}$, coincides with the dichotomy spectrum for differential equations introduced in [21].
- (ii) From Proposition 5.3, we obtain directly $\Sigma^{\leftarrow} \subset \Sigma^{\leftrightarrow}$ and $\Sigma^{\rightarrow} \subset \Sigma^{\leftrightarrow}$.

- (iii) If (6.1) is almost periodic, then the spectra for the past, future and entire time are identical, and they coincide with the Sacker–Sell spectrum from [25] (cf. also Remark 5.2(ii)).

The aim of the following lemma is to analyze the topological structure of the resolvent sets.

Lemma 6.4. *We suppose that $\rho := \rho^{\leftarrow}, \rho^{\rightarrow}, \rho^{\leftrightarrow}$, respectively. Then $\rho \cap \mathbb{R}$ is open, more precisely, for all $\gamma \in \rho \cap \mathbb{R}$, there exists an $\varepsilon > 0$ such that $U_\varepsilon(\gamma) \subset \rho$. Furthermore, the relation $\text{rk } P_\zeta = \text{rk } P_\gamma$ is fulfilled for all $\zeta \in U_\varepsilon(\gamma)$ and every invariant projector P_γ and P_ζ of the nonhyperbolic dichotomies of (6.1) with growth rates γ and ζ , respectively.*

Proof. We choose $\gamma \in \rho$ arbitrarily, and let $\mathbb{I} = \mathbb{R}_0^-$ ($\mathbb{I} = \mathbb{R}_0^+$, $\mathbb{I} = \mathbb{R}$, respectively) be the interval corresponding to the time domain. Since (6.1) admits a nonhyperbolic past (future, all-time, respectively) exponential dichotomy with growth rate γ , there exist an invariant projector P_γ and constants $\alpha > 0$, $K \geq 1$ such that

$$\begin{aligned} \|\Phi(t, \tau)P_\gamma(\tau)\| &\leq Ke^{(\gamma-\alpha)(t-\tau)} \quad \text{for all } t \geq \tau \text{ and } t \in \mathbb{I}, \\ \|\Phi(t, \tau)(\mathbb{1} - P_\gamma(\tau))\| &\leq Ke^{(\gamma+\alpha)(t-\tau)} \quad \text{for all } t \leq \tau \text{ and } \tau \in \mathbb{I}. \end{aligned}$$

We set $\varepsilon := \frac{\alpha}{2}$ and choose $\zeta \in U_\varepsilon(\gamma)$. Thus,

$$\begin{aligned} \|\Phi(t, \tau)P_\gamma(\tau)\| &\leq Ke^{(\zeta-\frac{\alpha}{2})(t-\tau)} \quad \text{for all } t \geq \tau \text{ and } t \in \mathbb{I}, \\ \|\Phi(t, \tau)(\mathbb{1} - P_\gamma(\tau))\| &\leq Ke^{(\zeta+\frac{\alpha}{2})(t-\tau)} \quad \text{for all } t \leq \tau \text{ and } \tau \in \mathbb{I}. \end{aligned}$$

This yields $\zeta \in \rho$. Since the ranks of the projectors of past (future, all-time, respectively) exponential dichotomies with the same growth rate are equal (see Remark 5.6), we have $\text{rk } P_\zeta = \text{rk } P_\gamma$ for any projector P_ζ of the nonhyperbolic exponential dichotomy with growth rate ζ . \square

Lemma 6.5. *Assume that $\rho := \rho^{\leftarrow}, \rho^{\rightarrow}, \rho^{\leftrightarrow}$, respectively, let $\gamma_1, \gamma_2 \in \rho \cap \mathbb{R}$ with $\gamma_1 < \gamma_2$, and choose invariant projectors P_{γ_1} and P_{γ_2} for the corresponding nonhyperbolic exponential dichotomies with growth rates γ_1 and γ_2 . Then the relation $\text{rk } P_{\gamma_1} \leq \text{rk } P_{\gamma_2}$ holds. Moreover, $[\gamma_1, \gamma_2] \subset \rho$ is fulfilled if and only if $\text{rk } P_{\gamma_1} = \text{rk } P_{\gamma_2}$.*

Proof. The relation $\text{rk } P_{\gamma_1} \leq \text{rk } P_{\gamma_2}$ is a direct consequence of Proposition 5.5, since $\mathcal{S}_{\gamma_1} \subset \mathcal{S}_{\gamma_2}$ and $\mathcal{U}_{\gamma_1} \supset \mathcal{U}_{\gamma_2}$. Assume now that $[\gamma_1, \gamma_2] \subset \rho$. Arguing negatively, we suppose that $\text{rk } P_{\gamma_1} \neq \text{rk } P_{\gamma_2}$. We choose invariant projectors P_γ for the nonhyperbolic dichotomies of (6.1) with growth rate γ for all $\gamma \in (\gamma_1, \gamma_2)$ and define

$$\zeta_0 := \sup\{\zeta \in [\gamma_1, \gamma_2]: \text{rk } P_\zeta \neq \text{rk } P_{\gamma_2}\}.$$

Due to Lemma 6.4, there exists an $\varepsilon > 0$ such that $\text{rk } P_{\zeta_0} = \text{rk } P_\zeta$ for all $\zeta \in U_\varepsilon(\zeta_0)$. This is a contradiction to the definition of ζ_0 . Conversely, let $\text{rk } P_{\gamma_1} = \text{rk } P_{\gamma_2}$. We first treat the case $\rho = \rho^{\leftarrow}$. Because of $\text{rk } P_{\gamma_1} = \text{rk } P_{\gamma_2}$, Proposition 5.5 yields that $\mathcal{N}(P_{\gamma_1}) = \mathcal{N}(P_{\gamma_2})$. Due to Lemma 6.1, P_{γ_2} is an invariant projector of the nonhyperbolic past exponential dichotomy with growth rate γ_1 . Thus, we have

$$\|\Phi(t, \tau)P_{\gamma_2}(\tau)\| \leq K_1 e^{(\gamma_1-\alpha_1)(t-\tau)} \quad \text{for all } \tau \leq t \leq 0$$

for some $K_1 \geq 1$ and $\alpha_1 > 0$. P_{γ_2} is also projector of the nonhyperbolic past exponential dichotomy with growth rate γ_2 . Hence,

$$\|\Phi(t, \tau)(\mathbb{1} - P_{\gamma_2}(\tau))\| \leq K_2 e^{(\gamma_2+\alpha_2)(t-\tau)} \quad \text{for all } t \leq \tau \leq 0$$

is fulfilled for some $K_2 \geq 1$ and $\alpha_2 > 0$. For all $\gamma \in [\gamma_1, \gamma_2]$, these two inequalities imply by setting $K := \max\{K_1, K_2\}$ and $\alpha := \min\{\alpha_1, \alpha_2\}$ that

$$\begin{aligned}\|\Phi(t, \tau)P_{\gamma_2}(\tau)\| &\leq Ke^{(\gamma-\alpha)(t-\tau)} \quad \text{for all } \tau \leq t \leq 0, \\ \|\Phi(-t, \tau)(\mathbb{1} - P_{\gamma_2}(\tau))\| &\leq Ke^{(\gamma+\alpha)(t-\tau)} \quad \text{for all } t \leq \tau \leq 0.\end{aligned}$$

This means $\gamma \in \rho$, and thus, $[\gamma_1, \gamma_2] \subset \rho$. The cases $\rho = \rho^{\rightarrow}, \rho^{\leftarrow}$ are treated analogously. \square

For arbitrarily chosen $a \in \mathbb{R}$, we define

$$\begin{aligned}[-\infty, a] &:= (-\infty, a] \cup \{-\infty\}, \quad [a, \infty] := [a, \infty) \cup \{\infty\}, \\ [-\infty, -\infty] &:= \{-\infty\}, \quad [\infty, \infty] := \{\infty\}, \quad \text{and} \quad [-\infty, \infty] = \overline{\mathbb{R}}.\end{aligned}$$

We now state the main result of this section.

Theorem 6.6 (Spectral Theorem). *Given $\Sigma := \Sigma^{\leftarrow}, \Sigma^{\rightarrow}$ or Σ^{\leftrightarrow} , respectively. Then there exists an $n \in \{1, \dots, N\}$ such that*

$$\Sigma = [a_1, b_1] \cup \dots \cup [a_n, b_n]$$

with $-\infty \leq a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_n \leq b_n \leq \infty$.

Proof. Due to Lemma 6.4, the resolvent set $\rho \cap \mathbb{R}$ is open. Thus, $\Sigma \cap \mathbb{R}$ is the disjoint union of closed intervals. The relation $(-\infty, b_1] \subset \Sigma$ implies $[-\infty, b_1] \subset \Sigma$, because the assumption of the existence of a $\gamma \in \mathbb{R}$ such that (6.1) admits a nonhyperbolic dichotomy with growth rate γ and projector $P_\gamma \equiv 0$ leads to $(-\infty, \gamma] \subset \rho$ using Lemma 5.4, and this is a contradiction. Analogously, it follows from $[a_n, \infty) \subset \Sigma$ that $[a_n, \infty] \subset \Sigma$. To show the relation $n \leq N$, we assume to the contrary that $n \geq N + 1$. Thus, there exist

$$\zeta_1 < \zeta_2 < \dots < \zeta_N \in \rho$$

such that the $N + 1$ intervals $(-\infty, \zeta_1), (\zeta_1, \zeta_2), \dots, (\zeta_N, \infty)$ have nonempty intersection with the spectrum Σ . It follows from Lemma 6.5 that

$$0 \leq \text{rk } P_{\zeta_1} < \text{rk } P_{\zeta_2} < \dots < \text{rk } P_{\zeta_N} \leq N$$

is fulfilled for invariant projectors P_{ζ_i} of the nonhyperbolic dichotomy with growth rate ζ_i , $i \in \{1, \dots, n\}$. This implies either $\text{rk } P_{\zeta_1} = 0$ or $\text{rk } P_{\zeta_N} = N$. Thus, either

$$[-\infty, \zeta_1] \cap \Sigma = \emptyset \quad \text{or} \quad [\zeta_N, \infty] \cap \Sigma = \emptyset$$

is fulfilled, and this is a contradiction. To show $n \geq 1$, we assume that $\Sigma = \emptyset$. This implies $\{-\infty, \infty\} \subset \rho$. Thus, there exist $\zeta_1, \zeta_2 \in \mathbb{R}$ such that (6.1) admits a nonhyperbolic dichotomy with growth rate ζ_1 and projector $P_{\zeta_1} \equiv 0$ and a nonhyperbolic dichotomy with growth rate ζ_2 and projector $P_{\zeta_2} \equiv \mathbb{1}$. Applying Lemma 6.5, we get $(\zeta_1, \zeta_2) \cap \Sigma \neq \emptyset$. This contradiction yields $n \geq 1$ and finishes the proof of this theorem. \square

In the following example, dichotomy spectra of scalar equations are studied.

Example 6.7. We consider scalar linear differential equations of the form

$$\dot{x} = a(t)x,$$

where $a: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. We have $\Phi(t, \tau) = \exp(\int_{\tau}^t a(s) ds)$ for all $t, \tau \in \mathbb{R}$. The Spectral Theorem says that the all-time, past and future dichotomy spectra consist of exactly one closed interval. The following examples show that there are several possibilities. For simplicity, we first define $\Phi_{\gamma}(t, \tau) := e^{-\gamma(t-\tau)} \Phi(t, \tau)$ for $\gamma \in \mathbb{R}$ and note that the linear cocycle Φ_{γ} admits a past (future, all-time, respectively) exponential dichotomy (with growth rate 0) if and only if Φ admits a past (future, all-time, respectively) exponential dichotomy with growth rate γ .

- (i) $\Sigma^{\leftarrow} = \Sigma^{\rightarrow} = \Sigma^{\leftrightarrow} = \{\infty\}$ for $a(t) := |t|$ for all $t \in \mathbb{R}$.

Proof. For $\gamma \in \mathbb{R}$, we have

$$\Phi_{\gamma}(t, \tau) = \exp\left(\int_{\tau}^{\tau+t} (|s| - \gamma) ds\right) \quad \text{for all } t, \tau \in \mathbb{R}.$$

Since for all $s \in \mathbb{R}$ with $|s| \geq \gamma + 1$, the relation $|s| - \gamma \geq 1$ is fulfilled, Φ_{γ} admits a nonhyperbolic exponential dichotomy on the intervals $\mathbb{R}_{-|\gamma|-1}^{-}$ and $\mathbb{R}_{|\gamma|+1}^{+}$ with growth rate 0, constants $\alpha = 1$, $K = 1$ and invariant projector 0. This implies that $\Sigma^{\leftarrow} = \Sigma^{\rightarrow} = \{\infty\}$. The remaining assertion $\Sigma^{\leftrightarrow} = \{\infty\}$ is a consequence of Proposition 5.3(ii).

- (ii) $\Sigma^{\leftarrow} = \{-\infty\}$, $\Sigma^{\rightarrow} = \{\infty\}$ and $\Sigma^{\leftrightarrow} = \mathbb{R}$ for $a(t) := t$ for all $t \in \mathbb{R}$.

Proof. The assertions concerning the past and future dichotomy spectrum are proved analogously to (i). Concerning the all-time dichotomy spectrum, we assume to the contrary that there exists a $\gamma \in \mathbb{R}$ such that Φ_{γ} admits an all-time exponential dichotomy. Please note that the relation

$$\Phi_{\gamma}(t, \tau) = \exp\left(\frac{1}{2}t^2 + \tau t + \gamma t\right) \quad \text{for all } t, \tau \in \mathbb{R}$$

holds. For the corresponding invariant projector P_{γ} , there are only the possibilities $P_{\gamma} \equiv 0$ or $P_{\gamma} \equiv \mathbb{1}$. In case $P_{\gamma} \equiv \mathbb{1}$, the dichotomy estimate

$$\Phi_{\gamma}(t, 0) = \exp\left(\frac{1}{2}t^2 + \gamma t\right) \leq Ke^{-\alpha t} \quad \text{for all } t \geq 0$$

yields a contradiction in the limit $t \rightarrow \infty$. Analogously, the case $P_{\gamma} \equiv 0$ is treated.

- (iii) $\Sigma^{\leftarrow} = [-\infty, \beta]$, $\Sigma^{\rightarrow} = \{\beta\}$ and $\Sigma^{\leftrightarrow} = [-\infty, \beta]$ for

$$a(t) := \begin{cases} \beta, & t \geq -1, \\ \beta - n + n(t + 2^{2n} + 1), & t \in [-2^{2n} - 1, -2^{2n}] \text{ f.s. } n \in \mathbb{N}_0, \\ \beta - n, & t \in [-2^{2n+1}, -2^{2n} - 1] \text{ f.s. } n \in \mathbb{N}_0, \\ \beta - n(t + 2^{2n+1} + 1), & t \in [-2^{2n+1} - 1, -2^{2n+1}] \text{ f.s. } n \in \mathbb{N}_0, \\ \beta, & t \in [-2^{2(n+1)}, -2^{2n+1} - 1] \text{ f.s. } n \in \mathbb{N}_0. \end{cases}$$

Proof. The statement concerning Σ^{\rightarrow} is clear. To compute Σ^{\leftarrow} , assume to the contrary that for some $\gamma \leq \beta$, Φ_{γ} admits a past exponential dichotomy with projector P_{γ} . In the one-dimensional context, there are only the possibilities $P_{\gamma} \equiv 0$ or $P_{\gamma} \equiv \mathbb{1}$. In case $P_{\gamma} \equiv \mathbb{1}$, we have the dichotomy estimate

$$\Phi_{\gamma}(t, \tau) = \exp\left(\int_{\tau}^{\tau+t} (a(s) - \gamma) ds\right) \leq Ke^{-\alpha t} \quad \text{for all } \tau \leq 0 \text{ and } 0 \leq t \leq -\tau$$

for some $K \geq 1$ and $\alpha > 0$. We choose $n \in \mathbb{N}_0$ such that $K \exp(-\alpha(2^{2n+1} - 1)) < 1$. Then

$$\Phi_\gamma(2^{2n+1} - 1, -2^{2(n+1)}) = \exp\left(\int_{-2^{2(n+1)}}^{-2^{2n+1}-1} \underbrace{(\beta - \gamma)}_{\geq 0} ds\right) \geq 1.$$

This is a contradiction. In case $P_\gamma \equiv 0$, we have the dichotomy estimate

$$\Phi_\gamma(-t, \tau) = \exp\left(\int_\tau^{\tau-t} (a(s) - \gamma) ds\right) \leq K e^{-\alpha t} \quad \text{for all } \tau \leq 0 \text{ and } t \geq 0$$

for some $K \geq 1$ and $\alpha > 0$. We choose $n \in \mathbb{N}_0$ such that $K \exp(-\alpha(2^{2n} - 1)) < 1$ and $\beta - n - \gamma \leq 0$. Then

$$\Phi_\gamma(-2^{2n+1}, -2^{2n} - 1) = \exp\left(\int_{-2^{2n}-1}^{-2^{2n+1}} \underbrace{(\beta - n - \gamma)}_{\leq 0} ds\right) \geq 1.$$

This is also a contradiction. It is easy to see that for $\gamma > \beta$, Φ_γ admits a past exponential dichotomy with projector $P_\gamma \equiv \mathbb{1}$. Hence, we have $\Sigma^\leftarrow = [-\infty, \beta]$. Due to Remark 6.3(iii), $\Sigma^\leftrightarrow \supset \Sigma^\leftarrow \cup \Sigma^\rightarrow = [-\infty, \beta]$ is fulfilled. It is also easily shown that for $\gamma > \beta$, Φ_γ admits an all-time exponential dichotomy with projector $P_\gamma \equiv \mathbb{1}$. Thus, we obtain $\Sigma^\leftrightarrow = [-\infty, \beta]$.

(iv) $\Sigma^\leftarrow = \{\beta\}$, $\Sigma^\rightarrow = [\beta, \infty]$ and $\Sigma^\leftrightarrow = [\beta, \infty]$ for

$$a(t) := \begin{cases} \beta, & t \leq 1, \\ \beta + n(t - 2^{2n}), & t \in [2^{2n}, 2^{2n} + 1] \text{ f.s. } n \in \mathbb{N}_0, \\ \beta + n, & t \in [2^{2n} + 1, 2^{2n+1}] \text{ f.s. } n \in \mathbb{N}_0, \\ \beta + n - n(t - 2^{2n+1}), & t \in [2^{2n+1}, 2^{2n+1} + 1] \text{ f.s. } n \in \mathbb{N}_0, \\ \beta, & t \in [2^{2n+1} + 1, 2^{2(n+1)}] \text{ f.s. } n \in \mathbb{N}_0. \end{cases}$$

Proof. See proof of (iii).

(v) $\Sigma^\leftarrow = \{\beta\}$, $\Sigma^\rightarrow = [\beta, \delta]$ and $\Sigma^\leftrightarrow = [\beta, \delta]$ for

$$a(t) := \begin{cases} \beta, & t \leq 1, \\ \beta + (t - 2^{2n})(\delta - \beta), & t \in [2^{2n}, 2^{2n} + 1] \text{ f.s. } n \in \mathbb{N}_0, \\ \delta, & t \in [2^{2n} + 1, 2^{2n+1}] \text{ f.s. } n \in \mathbb{N}_0, \\ \delta + (t - 2^{2n+1})(\beta - \delta), & t \in [2^{2n+1}, 2^{2n+1} + 1] \text{ f.s. } n \in \mathbb{N}_0, \\ \beta, & t \in [2^{2n+1} + 1, 2^{2(n+1)}] \text{ f.s. } n \in \mathbb{N}_0. \end{cases}$$

Proof. See proof of (iii).

The following theorem says that each interval of the past (future, all-time, respectively) spectrum corresponds to a linear integral manifold.

Theorem 6.8 (Spectral manifolds). *Let*

$$\Sigma := \Sigma^\leftarrow, \Sigma^\rightarrow, \Sigma^\leftrightarrow = [a_1, b_1] \cup \dots \cup [a_n, b_n],$$

respectively, define the invariant projectors $P_{\gamma_0} := 0$, $P_{\gamma_n} := \mathbb{1}$, and for $i \in \{1, \dots, n-1\}$, choose $\gamma_i \in (b_i, a_{i+1})$ and projectors P_{γ_i} of the nonhyperbolic dichotomy of (6.1) with growth rate γ_i . Then the sets

$$\mathcal{W}_i := \mathcal{R}(P_{\gamma_i}) \cap \mathcal{N}(P_{\gamma_{i-1}}) \quad \text{for all } i \in \{1, \dots, n\}$$

are linear integral manifolds, the so-called spectral manifolds, such that

$$\mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_n = \mathbb{R} \times \mathbb{R}^N$$

and $\mathcal{W}_i \neq \mathbb{R} \times \{0\}$ for $i \in \{1, \dots, n\}$.

Proof. The sets $\mathcal{W}_1, \dots, \mathcal{W}_n$ are obviously linear integral manifolds. We suppose that there exists an $i \in \{1, \dots, n\}$ with $\mathcal{W}_i = \mathbb{R} \times \{0\}$. In case $i = 1$ or $i = n$, Lemma 5.4 implies $[-\infty, \gamma_1] \cap \Sigma = \emptyset$ or $[\gamma_{n-1}, \infty] \cap \Sigma = \emptyset$, and this is a contradiction. In case $1 < i < n$, due to Lemma 6.5, we obtain

$$\begin{aligned} \dim \mathcal{W}_i &= \dim(\mathcal{R}(P_{\gamma_i}) \cap \mathcal{N}(P_{\gamma_{i-1}})) \\ &= \text{rk } P_{\gamma_i} + N - \text{rk } P_{\gamma_{i-1}} - \dim(\mathcal{R}(P_{\gamma_i}) + \mathcal{N}(P_{\gamma_{i-1}})) \geq 1, \end{aligned}$$

and this is also a contradiction. We now prove $\mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_n = \mathbb{R} \times \mathbb{R}^N$. W.l.o.g., we assume $\Sigma = \Sigma^<, \Sigma^< \cup \Sigma^>$. For $1 \leq i < j \leq n$, due to Proposition 5.5, the relations $\mathcal{W}_i \subset \mathcal{R}(P_{\gamma_i})$ and $\mathcal{W}_j \subset \mathcal{N}(P_{\gamma_{j-1}}) \subset \mathcal{N}(P_{\gamma_i})$ are fulfilled. This yields

$$\mathcal{W}_i \cap \mathcal{W}_j \subset \mathcal{R}(P_{\gamma_i}) \cap \mathcal{N}(P_{\gamma_i}) = \mathbb{R} \times \{0\},$$

and we obtain

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^N &= \mathcal{W}_1 + \mathcal{N}(P_{\gamma_1}) = \mathcal{W}_1 + \mathcal{N}(P_{\gamma_1}) \cap (\mathcal{R}(P_{\gamma_2}) + \mathcal{N}(P_{\gamma_2})) \\ &= \mathcal{W}_1 + \mathcal{N}(P_{\gamma_1}) \cap \mathcal{R}(P_{\gamma_2}) + \mathcal{N}(P_{\gamma_2}) = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{N}(P_{\gamma_2}). \end{aligned}$$

Here, we used the fact that linear subspaces $E, F, G \subset \mathbb{R}^N$ with $E \supset G$ fulfill $E \cap (F + G) = (E \cap F) + G$. It follows inductively that

$$\mathbb{R} \times \mathbb{R}^N = \mathcal{W}_1 + \dots + \mathcal{W}_n + \mathcal{N}(P_{\gamma_n}) = \mathcal{W}_1 + \dots + \mathcal{W}_n.$$

This finishes the proof of this theorem. \square

We conclude this paper with the conclusion that the spectral manifolds give rise to a Morse decomposition in the projective space.

Theorem 6.9 (Spectral manifolds and Morse decompositions). *Let*

$$\Sigma = \Sigma^<, \Sigma^>, \Sigma^< \cup \Sigma^> = [a_1, b_1] \cup \dots \cup [a_n, b_n],$$

respectively, define the invariant projectors $P_{\gamma_0} := 0$, $P_{\gamma_n} := \mathbb{1}$, and for $i \in \{1, \dots, n-1\}$, choose $\gamma_i \in (b_i, a_{i+1})$ and projectors P_{γ_i} of the nonhyperbolic dichotomy of (6.1) with growth rate γ_i . Then the sets

$$M_i := \mathbb{P}(\mathcal{R}(P_{\gamma_i}) \cap \mathcal{N}(P_{\gamma_{i-1}})) \quad \text{for all } i \in \{1, \dots, n\}$$

are the Morse sets of a past (future, all-time, respectively) Morse decomposition of $\mathbb{P}\Phi$.

Proof. This is a direct consequence of Theorem 5.9. \square

Remark 6.10. It is possible that the above Morse decomposition defined by the spectral intervals is coarser than the finest Morse decomposition of Theorem 4.5 (see also [3]).

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