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# Global existence of solutions to the compressible Navier–Stokes equation around parallel flows

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The initial boundary value problem for the compressible Navier–Stokes equation is considered in an infinite layer of $\mathbf{R}^n$. It is proved that if $n \geqslant 3$, then strong solutions to the compressible Navier–Stokes equation around parallel flows exist globally in time for sufficiently small initial perturbations, provided that the Reynolds and Mach numbers are sufficiently small. The proof is given by a variant of the Matsumura–Nishida energy method based on a decomposition of solutions associated with a spectral property of the linearized operator.



## 1. Introduction

This paper deals with the initial boundary value problem for the compressible Navier–Stokes equation

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - \left(\mu + \mu'\right)\nabla \operatorname{div} v + \nabla P(\rho) = \rho \mathbf{g} \tag{1.2}$$

in an $n$-dimensional infinite layer $\Omega_\ell = \mathbf{R}^{n-1} \times (0, \ell)$:

$$\Omega_\ell = \left\{x = \left(x', x_n\right);\ x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1},\ 0 < x_n < \ell\right\} \quad (n \geqslant 3).$$

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Here $\rho = \rho(x,t)$ and $v = {}^\top(v^1(x,t), \dots, v^n(x,t))$ denote the unknown density and velocity at time $t \geqslant 0$ and position $x \in \Omega_\ell$, respectively; $P = P(\rho)$ is the pressure that is assumed to be a smooth function of $\rho$ satisfying

$$P'(\rho_*) > 0$$

for a given constant $\rho_* > 0$; $\mu$ and $\mu'$ are the viscosity coefficients that are assumed to be constants and satisfy $\mu > 0$, $\frac{2}{n}\mu + \mu' \geqslant 0$; and $\boldsymbol{g} = \boldsymbol{g}(x_n)$ is an external force which has the form

$$\boldsymbol{g} = {}^\top\big(g^1(x_n), 0, \dots, 0, g^n(x_n)\big),$$

where $g^j$ $(j = 1, n)$ are given smooth function of $x_n$. Here and in what follows the superscript ${}^\top\cdot$ stands for the transposition.

The system (1.1)–(1.2) is considered under the boundary condition

$$v|_{x_n=0} = 0, \qquad v|_{x_n=\ell} = V^1\boldsymbol{e}_1, \tag{1.3}$$

and the initial condition

$${}^\top(\rho, v)|_{t=0} = {}^\top(\rho_0, v_0). \tag{1.4}$$

Here $V^1$ is a given constant and $\boldsymbol{e}_1$ is the unit vector $\boldsymbol{e}_1 = {}^\top(1, 0, \dots, 0) \in \mathbf{R}^n$.

Under suitable smallness conditions on $g^n$, problem (1.1)–(1.3) has a stationary solution ${}^\top(\bar\rho_s, \bar v_s)$ of parallel flow with properties:

$$\bar\rho_s = \bar\rho_s(x_n), \qquad \bar v_s = {}^\top\big(\bar v_s^1(x_n), 0, \dots, 0\big),$$
$$\|\rho_s - \rho_*\|_{C^0[0,\ell]} \leqslant C\frac{\ell\|g^n\|_{C^0[0,\ell]}}{P'(\rho_*)},$$
$$\|v_s^1\|_{C^0[0,\ell]} \leqslant C\bigg(\frac{\rho_*\ell^2\|g^1\|_{C^0[0,\ell]}}{\mu} + |V^1|\bigg).$$

Here and in what follows $\|\cdot\|_{C^k[a,b]}$ denotes the usual $C^k$ norm of functions on the interval $[a,b]$. Typical examples are the plane Couette flow:

$$\rho_s = \rho_*, \qquad v_s^1 = \frac{V^1}{\ell}x_n$$

when $\boldsymbol{g} = 0$ and $V^1 \neq 0$; and the Poiseuille flow:

$$\rho_s = \rho_*, \qquad v_s^1 = \frac{\rho_* g^1}{2\mu}x_n(\ell - x_n)$$

when $\boldsymbol{g} = g^1\boldsymbol{e}_1$ with a constant $g^1 \neq 0$ and $V^1 = 0$. If $g^n$ is sufficiently small, then ${}^\top(\bar\rho_s, \bar v_s)$ can be obtained as a perturbation of the superposition of the plane Couette flow and Poiseuille flow.

The purpose of this paper is to show the global existence of solutions to (1.1)–(1.4) when the initial value ${}^\top(\rho_0, v_0)$ is sufficiently close to a parallel flow ${}^\top(\rho_s, v_s)$.

To state our result more precisely, we set $V = \frac{\rho_*\ell^2\|g^1\|_{C^0[0,\ell]}}{\mu} + |V^1| > 0$ and introduce the parameters:

$$\nu = \frac{\mu}{\rho_*\ell V}, \qquad \nu' = \frac{\mu'}{\rho_*\ell V}, \qquad \gamma = \frac{\sqrt{P'(\rho_*)}}{V}.$$

We note that $Re = 1/\nu$ and $Ma = 1/\gamma$ are the Reynolds and Mach numbers. We will prove that if $\nu \gg 1$, $\gamma \gg 1$ and $\|\boldsymbol{g}\|_{C^m[0,\ell]} \ll 1$ for some $m \in \mathbf{N}$ satisfying $m \geqslant [n/2]+1$, then (1.1)–(1.4) has a unique global solution ${}^\top(\rho, v)$ such that ${}^\top(\rho - \overline{\rho}_s, v - \overline{v}_s) \in C([0,\infty); H^m)$, provided that ${}^\top(\rho_0 - \overline{\rho}_s, v_0 - \overline{v}_s) \in H^m$ is small enough. Here $H^m$ denotes the $L^2$ Sobolev space on $\Omega_\ell$ of order $m$.

In the case of the plane Couette flow, the results mentioned above was proved in [3]. Therefore, the result of this paper is an extension of that of [3] to the case of general parallel flows for $n \geqslant 3$. The main difference to [3] arises in the following point. A nondimensional form of the equations for the perturbation ${}^\top(\phi, w) = {}^\top(\frac{\gamma^2}{\rho_*}(\rho - \overline{\rho}_s), \frac{1}{V}(v - \overline{v}_s))$ is written in the form

$$\partial_t \phi + v_s \cdot \nabla \phi + \gamma^2 \operatorname{div}(\rho_s w) = f^0, \tag{1.5}$$

$$\partial_t w - \frac{\nu}{\rho_s}\Delta w - \frac{\tilde{\nu}}{\rho_s}\nabla \operatorname{div} w + \nabla\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}\phi\right) + \left(\frac{\nu}{\gamma^2 \rho_s^2}\Delta v_s\right)\phi + v_s \cdot \nabla w + w \cdot \nabla v_s = \boldsymbol{f}, \tag{1.6}$$

$$w|_{x_n=0,1} = 0, \tag{1.7}$$

$${}^\top(\phi, w)|_{t=0} = {}^\top(\phi_0, w_0). \tag{1.8}$$

Here $\tilde{\nu} = \nu + \nu'$; $f^0$ and $\boldsymbol{f}$ denote the nonlinearities; the domain $\Omega_\ell$ is transformed into $\Omega_1$. Eq. (1.6) has a lower order term $(\frac{\nu}{\gamma^2 \rho_s^2}\Delta v_s)\phi$ which is absent in the case of the plane Couette flow [3]. We note that the Poincaré inequality holds for $w$ but not for $\phi$ in general. So, due to the appearance of this term, a direct application of the Matsumura–Nishida energy method [9] does not work even though the coefficient $(\frac{\nu}{\gamma^2 \rho_s^2}\Delta v_s)$ is assumed to be sufficiently small. To overcome this, we employ a decomposition of solutions essentially introduced in the linearized analysis in [7]. We write (1.5)–(1.8) in the form

$$\begin{cases} \partial_t u + Lu = \boldsymbol{F}, & w|_{x_n=0,1} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.9}$$

where $u = {}^\top(\phi, w)$; $L$ is the linearized operator; $\boldsymbol{F}$ denotes the nonlinearity; and $u_0 = {}^\top(\phi_0, w_0)$. The Fourier transform of (1.9) in $x' \in \mathbf{R}^{n-1}$ can be written as

$$\begin{cases} \partial_t \hat{u} + \hat{L}_{\xi'}\hat{u} = \hat{\boldsymbol{F}}, & \hat{w}|_{x_n=0,1} = 0, \\ \hat{u}|_{t=0} = \hat{u}_0, \end{cases} \tag{1.10}$$

where $\xi' = (\xi_1, \ldots, \xi_{n-1}) \in \mathbf{R}^{n-1}$ is the dual variable. When $\xi' = 0$, the operator $\hat{L}_0$ (as an operator on $H^1(0,1) \times L^2(0,1)$) has a one-dimensional kernel spanned by a function $u^{(0)} = {}^\top(\phi^{(0)}(x_n), w^{(0)}(x_n))$ with $\int_0^1 \phi^{(0)}(x_n)\,dx_n = 1$. We then define a projection $P_1$ by

$$P_1 u = \mathscr{F}^{-1}\left(\hat{\chi}_1(\xi')\int_0^1 \hat{\phi}(\xi', x_n)\,dx_n\right)u^{(0)}(x_n)$$

for $u = {}^\top(\phi, w)$. Here $\mathscr{F}^{-1}$ denotes the inverse Fourier transform and $\hat{\chi}_1$ is a cut off function: $\hat{\chi}_1(\xi) = 1$ for $|\xi| \leqslant 1$ and $\hat{\chi}_1(\xi) = 0$ for $|\xi| > 1$. We decompose the solution $u(t)$ of (1.9) into

$$u = P_1 u + P_\infty u \equiv \sigma_1 u^{(0)} + u_\infty, \tag{1.11}$$

where $P_\infty = I - P_1$ and $P_1 u = \sigma_1 u^{(0)}$ with

$$\sigma_1 = \sigma_1(x', t) = \mathscr{F}^{-1}\left(\hat{\chi}_1(\xi') \int_0^1 \hat{\phi}(\xi', x_n, t)\, dx_n\right).$$

We can show, by a variant of the Matsumura–Nishida energy method, that

$$\|u(t)\|_{H^m}^2 + \int_0^t \|\partial_{x'}\sigma_1\|_{L^2}^2 + \|\partial_x \phi_\infty\|_{H^{m-1}}^2 + \|\partial_x w_\infty\|_{H^m}^2\, d\tau \leqslant C\|u_0\|_{H^m}^2 \tag{1.12}$$

for sufficiently small $\|u_0\|_{H^m}$, provided that $\nu \gg 1$, $\gamma \gg 1$ and $\|\mathbf{g}\|_{C^m[0,\ell]} \ll 1$. An advantage of the decomposition (1.11) is that the Poincaré inequality $\|u_\infty\|_{L^2} \leqslant C\|\partial_x u_\infty\|_{L^2}$ holds for $u_\infty$-part and that differentiations of any order in $x$ variable are bounded operators on the subspace Range$(P_1)$, i.e., $\|\partial_x^k(\sigma_1 u^{(0)})\|_{L^2} \leqslant C_k\|\sigma_1 u^{(0)}\|_{L^2}$ for any $k = 1, 2, \ldots$. Using these properties we can establish the a priori estimate by a variant of the Matsumura–Nishida energy method.

Once $H^m$-energy bound (1.12) is obtained, then the following decay and asymptotic behavior can be shown as in the case of the plane Couette flow [3] by using the linearized analysis in [7]:

$$\|u(t)\|_{L^2} = O\left(t^{-\frac{n-1}{4}}\right),$$
$$\left\|u(t) - \left(\sigma u^{(0)}\right)(t)\right\|_{L^2} = O\left(t^{-\frac{n-1}{4}-\frac{1}{2}} L(t)\right)$$

as $t \to \infty$. Here $\sigma = \sigma(x', t)$ is a function given by

$$\sigma(x', t) = \mathscr{F}^{-1}\left(e^{-(ia_0\xi_1 + \kappa_0|\xi_1|^2 + \kappa_1|\xi''|^2)t} \int_0^1 \hat{\phi}_0(\xi', x_n)\, dx_n\right)$$

with some constants $a_0 \in \mathbf{R}$, $\kappa_0 > 0$ and $\kappa_1 > 0$, where $\xi'' = (\xi_2, \ldots, \xi_{n-1}) \in \mathbf{R}^{n-2}$; and $L(t) = \log(1+t)$ when $n = 3$ and $L(t) = 1$ when $n \geqslant 4$. In this paper we concentrate on the proof of the global existence of solutions and do not consider the decay and asymptotic behavior of perturbations.

We remark that in contrast to the case of the plane Couette flow [3], we here restrict ourselves to the case $n \geqslant 3$. The case $n = 2$ is different from the case $n \geqslant 3$; and we will study the case $n = 2$ elsewhere.

This paper is organized as follows. In Section 2, we rewrite problem (1.1)–(1.4) into a nondimensional form and show the existence of parallel stationary solutions. In Section 3, we state the main result of this paper. In Section 4, we introduce the decomposition (1.11) of the solution and examine some properties of $P_1$. Section 5 is devoted to deriving the a priori estimate.

## 2. Nondimensionalization and parallel flows

In this section we rewrite the problem into a nondimensional form and show the existence of stationary parallel flows.

### *2.1. Notations*

We first introduce some notation which will be used throughout the paper. For a domain $D$ and $1 \leqslant p \leqslant \infty$ we denote by $L^p(D)$ the usual Lebesgue space on $D$ and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Let $m$ be a nonnegative integer. The symbol $H^m(D)$ denotes the $m$-th order $L^2$ Sobolev space on $D$

with norm $\|\cdot\|_{H^m(D)}$. $C_0^m(D)$ stands for the set of all $C^m$ functions which have compact support in $D$. We denote by $H_0^1(D)$ the completion of $C_0^1(D)$ in $H^1(D)$.

We simply denote by $L^p(D)$ (resp., $H^m(D)$) the set of all vector fields $w = {}^\top(w^1, \dots, w^n)$ on $D$ with $w^j \in L^p(D)$ (resp., $H^m(D)$), $j = 1, \dots, n$, and its norm is also denoted by $\|\cdot\|_{L^p(D)}$ (resp., $\|\cdot\|_{H^m(D)}$). For $u = {}^\top(\phi, w)$ with $\phi \in H^k(D)$ and $w = {}^\top(w^1, \dots, w^n) \in H^m(D)$, we define $\|u\|_{H^k(D)\times H^m(D)}$ by $\|u\|_{H^k(D)\times H^m(D)} = \|\phi\|_{H^k(D)} + \|w\|_{H^m(D)}$. When $k = m$, we simply write $\|u\|_{H^k(D)\times H^k(D)} = \|u\|_{H^k(D)}$.

Later we will transform the problem into a nondimensional form; and then $\Omega_\ell$ will be transformed into $\Omega \equiv \Omega_1 = \mathbf{R}^{n-1} \times (0, 1)$.

In case $D = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^m(\Omega)$) as $L^p$ (resp., $H^m$). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$.

In case $D$ is the interval $(0, 1)$ we denote the norm of $L^p(0, 1)$ by $|\cdot|_p$. The norm of $H^m(0, 1)$ is denoted by $|\cdot|_{H^m}$.

The inner product of $L^2(\Omega)$ is denoted by

$$(f, g) = \int_\Omega f(x)g(x)\,dx, \quad f, g \in L^2(\Omega).$$

We also denote the inner product of $L^2(0, 1)$ by

$$(f, g) = \int_0^1 f(x_n)g(x_n)\,dx_n, \quad f, g \in L^2(0, 1),$$

if no confusion occurs. We further introduce a weighted inner product $\langle\langle\cdot,\cdot\rangle\rangle$ defined by

$$\langle\langle u_1, u_2\rangle\rangle = \int_\Omega \phi_1\phi_2 \frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\,dx + \int_\Omega w_1 \cdot w_2\rho_s\,dx$$

for $u_j = {}^\top(\phi_j, w_j) \in L^2(\Omega)^{n+1}$ $(j = 1, 2)$; and, also, $\langle\cdot,\cdot\rangle$ defined by

$$\langle u_1, u_2\rangle = \int_0^1 \phi_1\phi_2 \frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\,dx_n + \int_0^1 w_1 \cdot w_2\rho_s\,dx_n$$

for $u_j = {}^\top(\phi_j, w_j) \in L^2(0, 1)^{n+1}$ $(j = 1, 2)$. Here $\rho_s = \rho_s(x_n)$ denotes the density of the parallel flow whose existence will be proved in Proposition 2.1 below. We note that $\langle\langle\cdot,\cdot\rangle\rangle$ and $\langle\cdot,\cdot\rangle$ define inner products of $L^2(\Omega)^{n+1}$ and $L^2(0, 1)^{n+1}$, respectively, since $0 < \rho_1 \leqslant \rho_s \leqslant \rho_2$ and $\tilde{P}'(\rho_s) > 0$ for $\rho_1 \leqslant \rho_s \leqslant \rho_2$ from Proposition 2.1. Furthermore, we denote the mean value of $f \in L^1(0, 1)$ by $\langle\cdot\rangle$:

$$\langle f\rangle = \int_0^1 f(x_n)\,dx_n.$$

For $u = {}^\top(\phi, w) \in L^1(0, 1)$ with $w = {}^\top(w^1, \dots, w^n)$ we define $\langle u\rangle$ by

$$\langle u\rangle = \langle\phi\rangle + \langle w^1\rangle + \dots + \langle w^n\rangle.$$

We denote the $k \times k$ identity matrix by $I_k$. We also define $(n+1) \times (n+1)$ diagonal matrices $Q_0$, $Q_n$ and $\tilde{Q}$ by

$$Q_0 = \operatorname{diag}(1, 0, \ldots, 0), \qquad Q_n = \operatorname{diag}(0, \ldots, 0, 1)$$

and

$$\tilde{Q} = \operatorname{diag}(0, 1, \ldots, 1).$$

Note that

$$\langle Q_0 u \rangle = \langle \phi \rangle \quad \text{for } u = {}^\top(\phi, w).$$

We often write $x \in \Omega$ as

$$x = {}^\top\big(x', x_n\big), \quad x' = {}^\top(x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}.$$

Partial derivatives of a function $u$ in $x$, $x'$, $x_n$ and $t$ are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of $u$ in $x$ as $\partial_x^k u = (\partial_x^\alpha u;\ |\alpha| = k)$.

For a function $f = f(x')$ ($x' \in \mathbf{R}^{n-1}$), we denote its Fourier transform by $\hat{f}$ or $\mathscr{F} f$:

$$\hat{f}\big(\xi'\big) = (\mathscr{F} f)\big(\xi'\big) = \int\limits_{\mathbf{R}^{n-1}} f\big(x'\big) e^{-i\xi' \cdot x'}\, dx'.$$

The inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$\big(\mathscr{F}^{-1} f\big)\big(x'\big) = (2\pi)^{-(n-1)} \int\limits_{\mathbf{R}^{n-1}} f\big(\xi'\big) e^{i\xi' \cdot x'}\, d\xi'.$$

*2.2. Nondimensional form of equations*

We introduce the following dimensionless variables:

$$x = \ell \tilde{x}, \qquad t = \frac{\ell}{V} \tilde{t}, \qquad v = V \tilde{v}, \qquad \rho = \rho_* \tilde{\rho}, \qquad P = \rho_* V^2 \tilde{P}$$

with

$$V = \frac{\rho_* \ell^2 |g^1|_\infty}{\mu} + \big|V^1\big| > 0.$$

Then the problem (1.1)–(1.4) is transformed into the following dimensionless problem on the layer $\Omega \equiv \Omega_1 = \mathbf{R}^{n-1} \times (0, 1)$:

$$\partial_{\tilde{t}} \tilde{\rho} + \operatorname{div}(\tilde{\rho} \tilde{v}) = 0, \tag{2.1}$$

$$\tilde{\rho}\big(\partial_{\tilde{t}} \tilde{v} + \tilde{v} \cdot \nabla \tilde{v}\big) - \nu \Delta \tilde{v} - \big(\nu + \nu'\big) \nabla \operatorname{div} \tilde{v} + \tilde{P}'(\tilde{\rho}) \nabla \tilde{\rho} = \tilde{\rho} \tilde{\boldsymbol{g}}, \tag{2.2}$$

$$\tilde{v}|_{\tilde{x}_n = 0} = 0, \qquad \tilde{v}|_{\tilde{x}_n = 1} = \frac{V^1}{V} \boldsymbol{e}^1, \tag{2.3}$$

$${}^\top(\tilde{\rho}, \tilde{v})|_{\tilde{t} = 0} = {}^\top(\tilde{\rho}_0, \tilde{v}_0). \tag{2.4}$$

Here div, $\nabla$ and $\Delta$ are the divergence, gradient and Laplacian with respect to $\tilde{x}$; $\tilde{\boldsymbol{g}}(\tilde{x}_n) = \frac{\ell}{V^2}\boldsymbol{g}(\ell\tilde{x}_n)$; and $\nu$, $\nu'$ and $\gamma$ are the nondimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \qquad \nu' = \frac{\mu'}{\rho_* \ell V}, \qquad \gamma = \sqrt{\tilde{P}'(1)} = \frac{\sqrt{P'(\rho_*)}}{V}.$$

**Remark.** The Reynolds number $Re$ and Mach number $Ma$ are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively.

### 2.3. Existence of stationary parallel flow

One can see that if $|\tilde{g}^n|_\infty$ is small enough, then a stationary solution ${}^\top(\rho_s, v_s) = {}^\top(\rho_s(\tilde{x}_n), v_s^1(\tilde{x}_n)\boldsymbol{e}_1)$ exists. More precisely, substituting $(\tilde{\rho}, \tilde{v}) = (\rho_s(\tilde{x}_n), v_s^1(\tilde{x}_n)\boldsymbol{e}_1)$ into (2.1)–(2.3), we have

$$-\nu\partial_{\tilde{x}_n}^2 v_s^1 = \rho_s \tilde{g}^1, \tag{2.5}$$

$$\partial_{\tilde{x}_n}\big(\tilde{P}(\rho_s)\big) = \rho_s \tilde{g}^n, \tag{2.6}$$

$$v_s^1\big|_{\tilde{x}_n=0} = 0, \qquad v_s^1\big|_{\tilde{x}_n=1} = \frac{V^1}{V}. \tag{2.7}$$

We will look for solutions of (2.5)–(2.7) with

$$\int_0^1 \rho_s(\tilde{x}_n)\, d\tilde{x}_n = 1. \tag{2.8}$$

**Proposition 2.1.** *Assume that $\tilde{P}'(\rho) > 0$ for $\rho_1 \leqslant \rho \leqslant \rho_2$ with some $0 < \rho_1 < 1 < \rho_2$. Let $\Phi(\rho) = \int_1^\rho \frac{\tilde{P}'(\eta)}{\eta}\, d\eta$ for $\rho_1 \leqslant \rho \leqslant \rho_2$ and let $\Psi(r) = \Phi^{-1}(r)$ for $r_1 \leqslant r \leqslant r_2$. Here $\Phi^{-1}$ denotes the inverse function of $\Phi$ and $r_j = \Phi(\rho_j)$ $(j = 1, 2)$.*

*If*

$$\big|\tilde{g}^n\big|_\infty \leqslant C \min\left\{|r_1|, r_2, \frac{1}{4\gamma^2\|\Psi''\|_{C^0[r_1,r_2]}}\right\},$$

*then there exists a smooth stationary solution ${}^\top(\rho_s, v_s) = {}^\top(\rho_s(\tilde{x}_n), v_s^1(\tilde{x}_n)\boldsymbol{e}_1)$ of (2.5)–(2.8) satisfying*

$$\rho_1 \leqslant \rho_s(\tilde{x}_n) \leqslant \rho_2, \qquad |\rho_s - 1|_\infty \leqslant C\frac{|\tilde{g}^n|_\infty}{\gamma^2},$$

$$\big|\partial_{x_n}^k v_s^1\big|_\infty \leqslant C(1+\rho_2), \quad k = 0, 1, 2.$$

*Furthermore, if $\|\tilde{g}^n\|_{C^{k-1}[0,1]} \leqslant \eta$, then*

$$\big|\partial_{\tilde{x}_n}^k \rho_s\big|_\infty \leqslant C_k \big\|\tilde{g}^n\big\|_{C^{k-1}[0,1]} \quad \textit{for } k = 1, 2, \ldots,$$

*and*

$$\big|\partial_{\tilde{x}_n}^k v_s\big|_\infty \leqslant \frac{C_k}{\nu}\|\tilde{\boldsymbol{g}}\|_{C^{k-2}[0,1]} \quad \textit{for } k = 3, 4, \ldots.$$

*Here $C_k$ are positive constants depending on $k$, $\eta$, $\|\Psi\|_{C^k[r_1,r_2]}$ and $\rho_2$.*

*In particular,*

$$|\partial_{x_n}\rho_s|_\infty \leqslant \frac{C}{\gamma^2}\left|\tilde{g}^n\right|_\infty,$$

$$\left|\tilde{P}'(\rho_s)-\gamma^2\right|_\infty \leqslant C\left\|\tilde{P}''\right\|_{C^0[\rho_1,\rho_2]}\frac{|\tilde{g}^n|_\infty}{\gamma^2}.$$

**Outline of proof.** We proceed as in [8, Proof of Lemma 2.1]. In the proof we omit "tilde" of $\tilde{x}_n$. We also denote $|g|_\infty$ by $\|g\|$. We first observe that for $\Phi$ and $\Psi$ there hold

$$\Phi(1)=0, \qquad \Psi(0)=\Phi^{-1}(0)=1, \quad r_1<0<r_2,$$

$$\Phi'(1)=\tilde{P}'(1)=\gamma^2, \qquad \Psi'(0)=\frac{1}{\Phi'(1)}=\frac{1}{\gamma^2},$$

$$\left|\Psi'(r)-\frac{1}{\gamma^2}\right| \leqslant \left\|\Psi''\right\|_{C^0[r_1,r_2]}r \quad (r_1\leqslant r\leqslant r_2),$$

$$\left|\Psi'(r)\right| \leqslant \left\|\Psi''\right\|_{C^0[r_1,r_2]}r+\frac{1}{\gamma^2} \quad (r_1\leqslant r\leqslant r_2).$$

We set $g(x_n)=\int_0^{x_n}\tilde{g}^n(y_n)\,dy_n$. By (2.6), we have

$$\frac{\tilde{P}'(\rho_s)}{\rho_s}\rho_s'=\tilde{g}^n.$$

It then follows that $\rho_s$ is given by

$$\int_1^{\rho(x_n)}\frac{\tilde{P}'(\eta)}{\eta}\,d\eta=\alpha+g(x_n) \tag{2.9}$$

with some constant $\alpha$ which is determined by $g$ through (2.8). In terms of $\Psi$, (2.9) is written as

$$\rho_s(x_n)=\Psi\big(\alpha+g(x_n)\big). \tag{2.10}$$

By (2.8) and (2.10), problem (2.6), (2.8) is reduced to

$$G(\alpha,g)=0, \tag{2.11}$$

where $G:\mathbf{R}\times C^0[r_1,r_2]\to\mathbf{R}$ is defined by

$$G(\alpha,g)=\int_0^1\big(\Psi\big(\alpha+g(x_n)\big)-1\big)\,dx_n.$$

Observe that

$$\partial_\alpha G(\alpha, g) = \int_0^1 \Psi'\big(\alpha + g(x_n)\big)\, dx_n,$$

$$\partial_g G(\alpha, g)h = \int_0^1 \Psi'\big(\alpha + g(x_n)\big)h(x_n)\, dx_n,$$

$$G(0,0) = 0, \qquad \partial_\alpha G(0,0) = \frac{1}{\gamma^2}, \qquad \partial_g G(0,0)h = \frac{1}{\gamma^2}\int_0^1 h(x_n)\, dx_n.$$

We can show a unique existence of a solution $\alpha = \alpha(g)$ of (2.11) for a suitably given $g$, together with estimate on $\alpha = \alpha(g)$ in terms of $g$, which leads to the desired estimates for $\rho_s$.

Let us solve (2.11) by contraction mapping principle. We define $\Gamma(\alpha, g)$ by

$$\Gamma(\alpha, g) = \alpha - \big(\partial_\alpha G(0,0)\big)^{-1} G(\alpha, g) = \alpha - \gamma^2 G(\alpha, g).$$

Note that (2.11) is equivalent to

$$\alpha = \Gamma(\alpha, g);$$

and that $\Gamma(0,0) = 0$.

One can show that $\Gamma : X \times Y \to X$ is a uniform contraction with $X = \{\alpha \in \mathbf{R};\ |\alpha| \leqslant \delta\}$ and $Y = \{g \in C^0[0,1];\ \|g\| \leqslant \delta\}$, where $\delta = \min\{|r_1|, r_2, 1/(4\gamma^2\|\Psi''\|_{C^0[r_1,r_2]})\}$; and hence, for each $g \in Y$, $\Gamma$ has a unique fixed point $\alpha = \alpha(g) \in X$. Furthermore, it can be seen that $\alpha(0) = 0$ and $|\alpha(g)| \leqslant \|g\|$.

The estimates for $\rho_s(x_n)$ are obtained as follows. We see from (2.10)

$$\rho_s(x_n) - 1 = \Psi\big(\alpha + g(x_n)\big) - \Psi(0) = \int_0^1 \Psi'\big(\theta\big(\alpha + g(x_n)\big)\big)\big(\alpha + g(x_n)\big)\, d\theta.$$

This implies

$$\begin{aligned}\big|\rho_s(x_n) - 1\big| &\leqslant \left(\big\|\Psi''\big\|_{C^0[r_1,r_2]}\big(\big|\alpha(g)\big| + \|g\|\big) + \frac{1}{\gamma^2}\right)\big(\big|\alpha(g)\big| + \|g\|\big) \\ &\leqslant \frac{3}{2\gamma^2}\|g\| \leqslant C\frac{|\tilde{g}^n|_\infty}{\gamma^2}.\end{aligned}$$

Moreover,

$$\big|\partial_{x_n}^k \rho_s(x_n)\big| = \big|\partial_{x_n}^k\big(\Psi\big(\alpha + g(x_n)\big)\big)\big| \leqslant C\big\|\tilde{g}^n\big\|_{C^{k-1}[0,1]}\big(1 + \big\|\tilde{g}^n\big\|_{C^{k-1}[0,1]}\big)^{k-1},$$

where $C$ depends on $\|\Psi\|_{C^k[r_1,r_2]}$.

Once $\rho_s(x_n)$ is obtained, then $v_s^1(x_n)$ is given by

$$v_s^1(x_n) = v_{s,C}^1(x_n) + v_{s,P}^1(x_n), \tag{2.12}$$

where

$$v_{s,C}^1 = \frac{V^1}{V} x_n, \qquad v_{s,P}^1(x_n) = \frac{1}{\nu} \int_0^1 G(x_n, y_n) \rho_s(y_n) \tilde{g}^1(y_n)\, dy_n \tag{2.13}$$

with

$$G(x_n, y_n) = \begin{cases} (1 - x_n) y_n & (0 < y_n < x_n), \\ x_n (1 - y_n) & (x_n < y_n < 1). \end{cases} \tag{2.14}$$

Since $|V^1|/V \leqslant 1$ and $|\tilde{g}^1|_\infty/\nu \leqslant 1$, we see from (2.13)–(2.15) that $|\partial_{x_n}^k v_s^1|_\infty \leqslant C(1 + \rho_2)$ for $k = 0, 1, 2$. The estimates for $k \geqslant 3$ can be obtained by differentiating (2.5). This completes the proof. □

## 3. Main result

In this section we rewrite the equations into the ones for the perturbation and state the main result of this paper.

We first rewrite the system (2.1)–(2.4) into the one for the perturbation. We set $\tilde{\rho} = \rho_s + \gamma^{-2}\phi$ and $\tilde{v} = v_s + w$ in (2.1)–(2.4). Then omitting "tildes" in $\tilde{t}$, $\tilde{x}$ we arrive at the initial boundary value problem for the perturbation $u = {}^\top(\phi, w)$:

$$\partial_t \phi + v_s \cdot \nabla \phi + \gamma^2 \operatorname{div}(\rho_s w) = f^0, \tag{3.1}$$

$$\partial_t w - \frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w + \nabla \left( \frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi \right) + \left( \frac{\nu}{\gamma^2 \rho_s^2} \Delta v_s \right) \phi + v_s \cdot \nabla w + w \cdot \nabla v_s = \boldsymbol{f}, \tag{3.2}$$

$$w|_{x_n = 0,1} = 0, \tag{3.3}$$

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{3.4}$$

Here $\tilde{\nu} = \nu + \nu'$; and $f^0$ and $\boldsymbol{f} = {}^\top(\boldsymbol{f}', f^n)$, $\boldsymbol{f}' = {}^\top(f^1, \dots, f^{n-1})$, denote the nonlinearities:

$$f^0 = -\operatorname{div}(\phi w),$$

$$\begin{aligned} \boldsymbol{f} = {} & -w \cdot \nabla w + \frac{\nu \phi}{(\phi + \gamma^2 \rho_s) \rho_s} \left( -\Delta w + \left( \frac{1}{\gamma^2 \rho_s} \Delta v_s \right) \phi \right) - \frac{\tilde{\nu} \phi}{(\phi + \gamma^2 \rho_s) \rho_s} \nabla \operatorname{div} w \\ & + \frac{\phi}{\gamma^2 \rho_s} \nabla \left( \frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi \right) - \frac{1}{2\gamma^4 \rho_s} \nabla \big( \tilde{P}''(\rho_s) \phi^2 \big) + \tilde{P}^3(\rho_s, \phi, \partial_{xx} \phi), \end{aligned}$$

where

$$\begin{aligned} \tilde{P}_3 = {} & \frac{\phi^3}{\gamma^4 (\phi + \gamma^2 \rho_s) \rho_s^3} \nabla \tilde{P}(\rho_s) - \frac{1}{2\gamma^6 \rho_s} \nabla \big( \phi^3 P_3(\rho_s, \phi) \big) \\ & + \frac{\phi}{2\gamma^6 \rho_s^2} \nabla \left( \tilde{P}''(\rho_s) \phi^2 + \frac{1}{\gamma^2} \phi^3 P_3(\rho, \phi) \right) \\ & - \frac{\phi^2}{\gamma^2 (\phi + \gamma^2 \rho_s) \rho_s^2} \nabla \left( \frac{\tilde{P}'(\rho_s)}{\gamma^2} \phi + \frac{1}{2\gamma^4} \tilde{P}''(\rho_s) \phi^2 + \frac{1}{2\gamma^6} \phi^3 P_3(\rho_s, \phi) \right) \end{aligned}$$

with

$$P_3(\rho_s, \phi) = \int_0^1 (1-\theta)^2 P'''\big(\theta\gamma^{-2}\phi + \rho_s\big)\, d\theta.$$

Before stating the main result we mention the compatibility condition for $u_0 = {}^\top(\phi_0, w_0)$. We will look for a solution $u = (\phi, w)$ of (3.1)–(3.4) in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j})$ satisfying $\int_0^t \|\partial_x w\|_{H^m}^2\, d\tau < \infty$ for all $t \geqslant 0$ with $m \geqslant [n/2]+1$. Therefore, we need to require the compatibility condition for the initial value $u_0 = {}^\top(\phi_0, w_0)$, which is formulated as follows.

Let $u = {}^\top(\phi, w)$ be a smooth solution of (3.1)–(3.4). Then $\partial_t^j u = {}^\top(\partial_t^j \phi, \partial_t^j w)$ $(j \geqslant 1)$ is inductively determined by

$$\partial_t^j \phi = -v_s \cdot \nabla \partial_t^{j-1}\phi - \gamma^2 \operatorname{div}\big(\rho_s \partial_t^{j-1} w\big) + \partial_t^{j-1} f^0$$

and

$$\begin{aligned} \partial_t^j w = {}& \frac{\nu}{\rho_s}\Delta\partial_t^{j-1} w + \frac{\tilde{\nu}}{\rho_s}\nabla \operatorname{div} \partial_t^{j-1} w - \nabla\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\partial_t^{j-1}\phi\right) \\ & - \left(\frac{\nu}{\gamma^2\rho_s^2}\Delta v_s\right)\partial_t^{j-1}\phi - v_s\cdot\nabla\partial_t^{j-1} w - \partial_t^{j-1} w\cdot\nabla v_s + \partial_t^{j-1}\boldsymbol{f}. \end{aligned}$$

From these relations we see that $\partial_t^j u|_{t=0} = {}^\top(\partial_t^j\phi, \partial_t^j w)|_{t=0}$ is inductively given by $u_0 = {}^\top(\phi_0, w_0)$ in the following way:

$$\partial_t^j u\big|_{t=0} = {}^\top\big(\partial_t^j\phi, \partial_t^j w\big)\big|_{t=0} = {}^\top(\phi_j, w_j) = u_j,$$

where

$$\phi_j = -v_s\cdot\nabla\phi_{j-1} - \gamma^2\operatorname{div}(\rho_s w_{j-1}) + f_{j-1}^0(u_0, \ldots, u_{j-1}, \partial_x u_0, \ldots, \partial_x u_{j-1}),$$

$$\begin{aligned} w_j = {}& \frac{\nu}{\rho_s}\Delta w_{j-1} + \frac{\tilde{\nu}}{\rho_s}\nabla\operatorname{div} w_{j-1} - \nabla\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi_{j-1}\right) \\ & - \left(\frac{\nu}{\gamma^2\rho_s^2}\Delta v_s\right)\phi_{j-1} - v_s\cdot\nabla w_{j-1} - w_{j-1}\cdot\nabla v_s \\ & + \boldsymbol{f}_{j-1}\big(u_0, \ldots, u_{j-1}, \ldots, \partial_x u_{j-1}, \ldots, \partial_x^2 w_{j-1}\big). \end{aligned}$$

Here $f_l^0(u_0, \ldots, u_l, \ldots)$ is a certain polynomial in $u_0, \ldots, u_l, \ldots; \ldots\ldots$, and so on.

By the boundary condition $w|_{x_n=0,1} = 0$ in (3.3), we necessarily have $\partial_t^j w|_{x_n=0,1} = 0$, and hence,

$$w_j|_{x_n=0,1} = 0.$$

Assume that $u = {}^\top(\phi, w)$ is a solution of (3.1)–(3.4) in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0, T_0]; H^{m-2j})$ for some $T_0 > 0$. Then, from the above observation, we need the regularity $u_j = {}^\top(\phi_j, w_j) \in H^{m-2j}$ for $j = 0, \ldots, [m/2]$, which, indeed, follows from the fact that $u_0 = {}^\top(\phi_0, w_0) \in H^m$ with $m \geqslant [n/2] + 1$. Furthermore, it is necessary to require that $u_0 = {}^\top(\phi_0, w_0)$ satisfies the *$\hat{m}$-th order compatibility condition*:

$$w_j \in H_0^1 \quad \text{for } j = 0, 1, \ldots, \ \hat{m} = \left[\frac{m-1}{2}\right].$$

We are ready to state our main result of this paper.

**Theorem 3.1.** *Assume that $n \geqslant 3$. Let $m$ be an integer satisfying $m \geqslant [n/2] + 1$. Then there are positive numbers $\nu_0$, $\gamma_0$ and $\tilde{\omega}$ such that if $\nu \geqslant \nu_0$, $\gamma^2/(\nu + \tilde{\nu}) \geqslant \gamma_0^2$ and $\|\tilde{\boldsymbol{g}}\|_{C^m[0,1]} \leqslant \tilde{\omega}$, then the following assertion holds. There is a positive number $\varepsilon_0$ such that if $u_0 = {}^\top(\phi_0, w_0) \in H^m$ satisfies $\|u_0\|_{H^m} \leqslant \varepsilon_0$ and the $\hat{m}$-th compatibility condition, then there exists a unique global solution $u(t) = {}^\top(\phi(t), w(t))$ of* (3.1)–(3.4) *in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j})$ which satisfies*

$$\big\|u(t)\big\|_{H^m}^2 + \int_0^t \|\partial_{x'}\sigma_1\|_2^2 + \|\partial_x \phi_\infty\|_{H^{m-1}}^2 + \|\partial_x w_\infty\|_{H^m}^2 \, d\tau \leqslant C\|u_0\|_{H^m}^2 \tag{3.5}$$

*uniformly for $t \geqslant 0$. Here $\sigma_1 = \sigma_1(x', t) = \mathscr{F}^{-1}(\hat{\chi}_1\langle\hat{\phi}(t)\rangle)$; $\hat{\chi}_1(\xi') = 1$ for $|\xi'| \leqslant 1$ and $\hat{\chi}_1(\xi') = 0$ for $|\xi'| > 1$; $u^{(0)} = u^{(0)}(x_n)$ is the function given in Lemma* 4.1 *below*; *and $u_\infty = {}^\top(\phi_\infty, w_\infty) = u - \sigma_1 u^{(0)}$.*

**Remark.** Once (3.5) is obtained for $m \geqslant [n/2] + 2$, then one can establish the decay estimates

$$\big\|u(t)\big\|_2 = O\big(t^{-\frac{n-1}{4}}\big),$$
$$\big\|u(t) - \big(\sigma u^{(0)}\big)(t)\big\|_2 = O\big(t^{-\frac{n-1}{4}-\frac{1}{2}} L(t)\big)$$

as $t \to \infty$, provided that $u_0 = (\phi_0, w_0) \in H^m \cap L^1$ with $\|u_0\|_{H^m \cap L^1} \ll 1$. Here $\sigma = \sigma(x', t) = \mathscr{F}^{-1}(e^{-(ia_0\xi_1 + \kappa_0|\xi_1|^2 + \kappa_1|\xi''|^2)t}\langle\hat{\phi}_0\rangle)$ with some constants $a_0 \in \mathbf{R}$, $\kappa_0 > 0$ and $\kappa_1 > 0$; and $L(t) = 1$ when $n \geqslant 4$ and $L(t) = \log(1 + t)$ when $n = 3$. In fact, this can be proved in a similar manner to the case of the plane Couette flow [3] by using (3.5) and the decay estimates for the linearized problem given in [7]. We note that in [7] it is considered the special case of a Poiseuille type flow, but one can easily see that the argument in [7] is valid for general parallel flows given in Proposition 2.1 if $\|\tilde{\boldsymbol{g}}\|_{C^1[0,1]}$ is sufficiently small.

As in [9,5], Theorem 3.1 is proved by showing the local existence of solutions and the a priori estimate such as (3.5). The local existence is proved by applying the local solvability result in [4]; and we will derive the a priori estimate in Proposition 5.1 below. So, the remaining part of this paper will be devoted to establish the necessary a priori estimate. To do so, in Section 4, we introduce a decomposition of solutions; and then, in Section 5, we establish the a priori estimate by a variant of the Matsumura–Nishida energy method.

## 4. Decomposition of solutions

We write (3.1)–(3.4) as

$$\partial_t u + Lu = \boldsymbol{F}, \qquad w|_{x_n=0,1} = 0, \qquad u|_{t=0} = u_0. \tag{4.1}$$

Here $L$ is the operator of the form

$$L = A + B + C_0,$$

where

$$A=\begin{pmatrix}0 & 0\\ 0 & -\frac{\nu}{\rho_s}\Delta I_n-\frac{\tilde{\nu}}{\rho_s}\nabla\,\mathrm{div}\end{pmatrix},\qquad B=\begin{pmatrix}v_s^1\partial_{x_1} & \gamma^2\,\mathrm{div}(\rho_s\,\cdot)\\ \nabla(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\,\cdot) & v_s^1\partial_{x_1}I_n\end{pmatrix},$$

$$C_0=\begin{pmatrix}0 & 0\\ \frac{\nu\Delta v_s^1}{\gamma^2\rho_s^2}\boldsymbol{e}_1 & (\partial_{x_n}v_s^1)\boldsymbol{e}_1{}^{\top}\boldsymbol{e}_n\end{pmatrix},$$

and $\boldsymbol{F}={}^{\top}(f^0,\boldsymbol{f})$ with $\boldsymbol{f}={}^{\top}(\boldsymbol{f}',f^n)$ is the nonlinearity. Note that

$$\langle\!\langle Au,u\rangle\!\rangle=\nu\|\nabla w\|_2^2+\tilde{\nu}\|\mathrm{div}\,w\|_2^2,\qquad \langle\!\langle Bu_1,u_2\rangle\!\rangle=-\langle\!\langle u_1,Bu_2\rangle\!\rangle \tag{4.2}$$

for $u,u_1,u_2\in H^1\times(H^2\cap H_0^1)$.

In the analysis of this paper we will decompose the solution by a projection operator associated with the linearized operator. To do so, we consider the Fourier transform of (3.1)–(3.4) in $x'\in\mathbf{R}^{n-1}$:

$$\partial_t\hat{\phi}+i\xi_1v_s^1\hat{\phi}+i\gamma^2\xi'\cdot(\rho_s\hat{w}')+\gamma^2\partial_{x_n}(\rho_s\hat{w}^n)=\hat{f}^0, \tag{4.3}$$

$$\begin{aligned}&\partial_t\hat{w}'+\frac{\nu}{\rho_s}\big(|\xi'|^2-\partial_{x_n}^2\big)\hat{w}'-i\frac{\tilde{\nu}}{\rho_s}\xi'\big(i\xi'\cdot\hat{w}'+\partial_{x_n}\hat{w}^n\big)+i\xi'\bigg(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\hat{\phi}\bigg)\\&\quad+\frac{\nu\Delta v_s^1}{\gamma^2\rho_s^2}\hat{\phi}\boldsymbol{e}_1'+i\xi_1v_s^1\hat{w}'+\big(\partial_{x_n}v_s^1\big)\hat{w}^n\boldsymbol{e}_1'=\hat{\boldsymbol{f}}',\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\partial_t\hat{w}^n+\frac{\nu}{\rho_s}\big(|\xi'|^2-\partial_{x_n}^2\big)\hat{w}^n-\frac{\tilde{\nu}}{\rho_s}\partial_{x_n}\big(i\xi'\cdot\hat{w}'+\partial_{x_n}\hat{w}^n\big)\\&\quad+\partial_{x_n}\bigg(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\hat{\phi}\bigg)+i\xi_1v_s^1\hat{w}^n=\hat{f}^n,\end{aligned} \tag{4.5}$$

$$\hat{w}|_{x_n=0,1}=0, \tag{4.6}$$

$$\hat{u}|_{t=0}=\hat{u}_0={}^{\top}(\hat{\phi}_0,\hat{w}_0). \tag{4.7}$$

Here $\hat{\phi}=\hat{\phi}(\xi',x_n,t)$ and $\hat{w}=\hat{w}(\xi',x_n,t)$ are the Fourier transform of $\phi=\phi(x',x_n,t)$ and $w=w(x',x_n,t)$ in $x'\in\mathbf{R}^{n-1}$ with $\xi'\in\mathbf{R}^{n-1}$ being the dual variable. We thus arrive at the following problem

$$\partial_t\hat{u}+\hat{L}_{\xi'}\hat{u}=\hat{\boldsymbol{F}},\qquad \hat{u}|_{t=0}=\hat{u}_0 \tag{4.8}$$

with a parameter $\xi'\in\mathbf{R}^{n-1}$. Here $\hat{L}_{\xi'}$ is the operator on $H^1(0,1)\times L^2(0,1)$ of the form

$$\hat{L}_{\xi'}=\hat{A}_{\xi'}+\hat{B}_{\xi'}+\hat{C}_0,$$

where

$$\hat{A}_{\xi'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s}(|\xi'|^2 - \partial_{x_n}^2) I_{n-1} + \frac{\tilde{\nu}}{\rho_s}\xi'^\top\xi' & -i\frac{\tilde{\nu}}{\rho_s}\xi'\partial_{x_n} \\ 0 & -i\frac{\tilde{\nu}}{\rho_s}{}^\top\xi'\partial_{x_n} & \frac{\nu}{\rho_s}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_s}\partial_{x_n}^2 \end{pmatrix},$$

$$\hat{B}_{\xi'} = \begin{pmatrix} i\xi_1 v_s^1 & i\gamma^2\rho_s{}^\top\xi' & \gamma^2\partial_{x_n}(\rho_s\,\cdot) \\ i\xi'\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s} & i\xi_1 v_s^1 I_{n-1} & 0 \\ \partial_{x_n}(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\,\cdot) & 0 & i\xi_1 v_s^1 \end{pmatrix},$$

$$\hat{C}_0 = C_0$$

with domain

$$D(\hat{L}_{\xi'}) = H^1(0,1) \times \big(H^2(0,1) \cap H_0^1(0,1)\big).$$

We will make use of a spectral property of the linearized operator $\hat{L}_0 = \hat{A}_0 + \hat{B}_0 + \hat{C}_0$ concerned with problem (4.3)–(4.7) with $\xi' = 0$.

We also introduce a formal adjoint operator $\hat{L}_0^*$ of $\hat{L}_0$ with respect to the inner product $\langle\cdot,\cdot\rangle$:

$$\hat{L}_0^* = \hat{A}_0 - \hat{B}_0 + \hat{C}_0^*,$$

with domain $D(\hat{L}_0^*) = D(\hat{L}_0)$, where

$$\hat{C}_0^* = \begin{pmatrix} 0 & \frac{\gamma^2\nu\Delta v_s^1}{\tilde{P}'(\rho_s)}{}^\top\boldsymbol{e}_1' & 0 \\ 0 & 0 & 0 \\ 0 & (\partial_{x_n} v_s^1)^\top\boldsymbol{e}_1' & 0 \end{pmatrix}.$$

**Lemma 4.1.** *(See [7].) Let ${}^\top(\rho_s, v_s)$ be a stationary solution obtained in Proposition 2.1. Then the following assertions hold.*

(i) *$\lambda = 0$ is a simple eigenvalue of $\hat{L}_0$ and $\hat{L}_0^*$.*
(ii) *The eigenspaces for $\lambda = 0$ of $\hat{L}_0$ and $\hat{L}_0^*$ are spanned by $u^{(0)}$ and $u^{(0)*}$ respectively, where*

$$u^{(0)} = {}^\top\big(\phi^{(0)}, w^{(0),1}\boldsymbol{e}_1', 0\big)$$

*and*

$$u^{(0)*} = {}^\top\big(\phi^{(0)*}, 0, 0\big)$$

*with*

$$\phi^{(0)}(x_n) = \alpha_0\frac{\gamma^2\rho_s(x_n)}{\tilde{P}'(\rho_s(x_n))}, \qquad \alpha_0 = \left(\int_0^1 \frac{\gamma^2\rho_s}{\tilde{P}'(\rho_s)}\,dx_n\right)^{-1},$$

$$w^{(0),1}(x_n) = \frac{1}{\nu\gamma^2}\int_0^1 G(x_n, y_n)\tilde{g}^1(y_n)\phi^{(0)}(y_n)\,dy_n,$$

$$\phi^{(0)*}(x_n) = \frac{\gamma^2}{\alpha_0}\phi^{(0)}(x_n).$$

*Here*

$$G(x_n, y_n) = \begin{cases} (1-x_n)y_n & (0 < y_n < x_n), \\ x_n(1-y_n) & (x_n < y_n < 1). \end{cases}$$

(iii) *The eigenprojection $\hat{\Pi}^{(0)}$ for $\lambda = 0$ of $\hat{L}_0$ is given by*

$$\hat{\Pi}^{(0)} u = \langle u, u^{(0)*}\rangle u^{(0)} = \langle \phi\rangle u^{(0)} \quad \text{for } u = {}^\top(\phi, w).$$

(iv) *Let $u^{(0)}$ be written as $u^{(0)} = u_0^{(0)} + u_1^{(0)}$, where*

$$u_0^{(0)} = {}^\top\big(\phi^{(0)}, 0, 0\big), \qquad u_1^{(0)} = {}^\top\big(0, w^{(0),1}\boldsymbol{e}_1', 0\big).$$

*Then $u^{(0)*} = \frac{\gamma^2}{\alpha_0} u_0^{(0)}$ and*

$$\langle u, u^{(0)}\rangle = \frac{\alpha_0}{\gamma^2}\langle\phi\rangle + \big(w^1, w^{(0),1}\rho_s\big)$$

*for $u = {}^\top(\phi, w', w^n)$.*

**Remark 4.2.** We note that if $|\tilde{g}^n|_\infty$ is sufficiently small, then

$$\phi^{(0)} = O(1) > 0, \qquad \alpha_0 = O(1) > 0, \qquad w^{(0),1} = O\big(1/\gamma^2\big).$$

**Proof of Lemma 4.1.** Lemma 4.1 can be proved in the same way as the proof of [7, Lemma 4.3]. So we here only derive the expression of $u^{(0)}$ given (ii).

If $\hat{L}^{(0)} u = 0$, then

$$\begin{cases} \gamma^2 \partial_{x_n}\big(\rho_s w^n\big) = 0, \\ -\dfrac{\nu}{\rho_s}\partial_{x_n}^2 w' + \dfrac{\nu\partial_{x_n}^2 v_s^1}{\gamma^2\rho_s^2}\phi\boldsymbol{e}_1' + \big(\partial_{x_n} v_s^1\big) w^n \boldsymbol{e}_1' = 0, \\ -\dfrac{\nu+\tilde{\nu}}{\rho_s}\partial_{x_n}^2 w^n + \partial_{x_n}\left(\dfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi\right) = 0, \\ w|_{x_n=0,1} = 0. \end{cases}$$

The first equation with the boundary condition $w^n|_{x_n=0,1} = 0$ implies that $w^n = 0$. Then, by the third equation, we see that $\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi = c$ for some constant $c$. We thus take $c = \alpha_0$. Since $-\nu\partial_{x_n}^2 v_s^1 = \rho_s\tilde{g}^1$, the second equation, together with the boundary condition $w'|_{x_n=0,1} = 0$, yields $w' = w^{(0),1}\boldsymbol{e}_1'$, where

$$w^{(0),1}(x_n) = \frac{1}{\nu\gamma^2}\int_0^1 G(x_n, y_n)\tilde{g}^1(y_n)\phi^{(0)}(y_n)\,dy_n.$$

We thus obtain the expression of $u^{(0)}$ in (ii). □

We now introduce a projection operator $P_1$. We define the projection $P_1$ by

$$P_1 u = \mathscr{F}^{-1}\big(\hat{\chi}_1\hat{\Pi}^{(0)}\hat{u}\big) = \mathscr{F}^{-1}\big(\hat{\chi}_1\langle\hat{Q}_0 u\rangle\big)u^{(0)},$$

i.e.,

$$P_1 u = \mathscr{F}^{-1}\big(\hat{\chi}_1\langle\hat{\phi}\rangle\big)u^{(0)} \quad \text{for } u = {}^\top(\phi, w),$$

and $P_\infty$ by

$$P_\infty = I - P_1.$$

Here $\hat{\chi}_1(\xi') = 1$ for $|\xi'| \leqslant 1$ and $\hat{\chi}_1(\xi') = 0$ for $|\xi'| > 1$.

We state several properties of $P_1$ and $P_\infty$ which are easily seen but useful in the subsequent analysis. One can easily see that differentiations in $x$ of any order are bounded on the range of $P_1$. As for the $P_\infty$-part, one can see that the Poincaré inequality holds. In fact, let $P_\infty u = {}^\top(\phi_\infty, w_\infty)$. Since $\langle\hat{\phi}_\infty(\xi', \cdot)\rangle = 0$ for $|\xi'| \leqslant 1$, we have $|\hat{\phi}_\infty(\xi', \cdot)|_2 \leqslant C|\partial_{x_n}\hat{\phi}_\infty(\xi', \cdot)|_2$ for $|\xi'| \leqslant 1$. We also have $\int_{|\xi'|>1} |\hat{\phi}_\infty(\xi', \cdot)|_2^2\, d\xi' \leqslant \int_{\mathbf{R}^{n-1}} |\xi'|^2|\hat{\phi}_\infty(\xi', \cdot)|_2^2\, d\xi'$. Therefore, by the Plancherel theorem,

$$\|\phi_\infty\|_2 \leqslant C\|\hat{\phi}_\infty\|_2 \leqslant C\big\{\|\partial_{x_n}\hat{\phi}_\infty\|_2 + \big\|\widehat{(\partial_{x'}\phi_\infty)}\big\|_2\big\} \leqslant C\|\partial_x\phi_\infty\|_2.$$

As for $w_\infty$, we note that if $w|_{x_n=0,1} = 0$, then $w_\infty|_{x_n=0,1} = 0$, since $w^{(0),1}|_{x_n=0,1} = 0$. Therefore, we have $\|w_\infty\|_2 \leqslant \|\partial_{x_n} w_\infty\|_2$ if $w|_{x_n=0,1} = 0$. We write these properties in the following lemma.

**Lemma 4.3.** *There hold the following assertions.*

(i) $P_j^2 = P_j$ $(j = 1, \infty)$.
(ii) $\|\partial_{x'}^k P_1 u\|_2 \leqslant \|P_1 u\|_2$ *for all* $k = 0, 1, 2, \ldots.$
(iii) $\|\partial_{x_n}^l P_1 u\|_2 \leqslant C_l\|P_1 u\|_2$ *for all* $l = 0, 1, 2, \ldots.$
(iv) *Let* $u = {}^\top(\phi, w)$ *and let* $P_\infty u = {}^\top(\phi_\infty, w_\infty)$. *Then*

$$\|\phi_\infty\|_2 \leqslant C\|\partial_x\phi_\infty\|_2.$$

*Furthermore, if* $w|_{x_n=0,1} = 0$, *then*

$$\|w_\infty\|_2 \leqslant C\|\partial_x w_\infty\|_2.$$

*Therefore, if* $w|_{x_n=0,1} = 0$, *then*

$$\|P_\infty u\|_2 \leqslant C\|\partial_x P_\infty u\|_2.$$

We now decompose the solution $u(t)$ of (3.1)–(3.4) into the $P_1$ and $P_\infty$ parts. Let $T > 0$ and let $u(t) = {}^\top(\phi(t), w(t))$ be a solution of (3.1)–(3.4), i.e., a solution of (4.1), in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0, T]; H^{m-2j})$ with $\partial_t^j w \in L^2(0, T; H^{m+1-2j})$, $0 \leqslant j \leqslant [(m+1)/2]$. We decompose $u(t)$ as

$$u(t) = \big(\sigma_1 u^{(0)}\big)(t) + u_\infty(t),$$

where

$$\big(\sigma_1 u^{(0)}\big)(t) = P_1 u(t), \qquad \sigma_1 = \sigma_1\big(x', t\big) = \mathscr{F}^{-1}\big(\hat{\chi}_1\big\langle\hat{\phi}(t)\big\rangle\big),$$
$$u_\infty(t) = P_\infty u(t).$$

Note that $\sigma_1$ is a function of $x'$ and $t$ and $u^{(0)}$ is a function of $x_n$. Furthermore, as for $u_\infty$-part, we have the Poincaré inequality $\|u_\infty\|_2 \leqslant \|\partial_x u_\infty\|_2$ by Lemma 4.3 since $w|_{x_n 0,1} = 0$.

We now deduce the equations for $\sigma_1(t)$ and $u_\infty(t)$. To do so, we introduce some notations. We define $\langle\cdot\rangle_1$ by

$$\langle f\rangle_1 = \mathscr{F}^{-1}\big(\hat{\chi}_1\langle\hat{f}\rangle\big).$$

We also denote

$$\tilde{\mathcal{M}} = L - \hat{L}_0 = \tilde{A} + \tilde{B}$$

with

$$\tilde{A} = A - \hat{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu}{\rho_s}\Delta' I_{n-1} - \frac{\tilde{\nu}}{\rho_s}\nabla'{}^\top\nabla' & -\frac{\tilde{\nu}}{\rho_s}\nabla'\partial_{x_n} \\ 0 & -\frac{\tilde{\nu}}{\rho_s}\partial_{x_n}{}^\top\nabla' & -\frac{\nu}{\rho_s}\Delta' \end{pmatrix},$$

$$\tilde{B} = B - \hat{B}_0 = \begin{pmatrix} v_s^1\partial_{x_1} & \gamma^2\rho_s{}^\top\nabla' & 0 \\ \nabla'\big(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\,\cdot\big) & v_s^1\partial_{x_1} I_{n-1} & 0 \\ 0 & 0 & v_s^1\partial_{x_1} \end{pmatrix},$$

where $\nabla' = {}^\top(\partial_{x_1},\dots,\partial_{x_{n-1}})$, $\Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$. We note the relations

$$P_1 L = P_1\tilde{\mathcal{M}}, \qquad L\big(\sigma_1 u^{(0)}\big) = \tilde{\mathcal{M}}\big(\sigma_1 u^{(0)}\big), \qquad Q_0\tilde{\mathcal{M}} = Q_0\tilde{B},$$

and

$$\langle\!\langle \tilde{B}u_1, u_2\rangle\!\rangle = -\langle\!\langle u_1, \tilde{B}u_2\rangle\!\rangle. \tag{4.9}$$

Applying $P_1$ and $P_\infty$ to (4.1), we have

$$\partial_t\sigma_1 + \big\langle Q_0\tilde{B}\big(\sigma_1 u^{(0)} + u_\infty\big)\big\rangle_1 = \langle Q_0\boldsymbol{F}\rangle_1, \tag{4.10}$$

$$\partial_t u_\infty + Lu_\infty + \tilde{\mathcal{M}}\big(\sigma_1 u^{(0)}\big) - \big\langle Q_0\tilde{B}\big(\sigma_1 u^{(0)} + u_\infty\big)\big\rangle_1 u^{(0)} = \boldsymbol{F}_\infty, \tag{4.11}$$

$$w_\infty|_{x_n=0,1} = 0, \tag{4.12}$$

$$\sigma_1|_{t=0} = \sigma_{1,0}, \qquad u_\infty|_{t=0} = u_{\infty,0}. \tag{4.13}$$

Here $u_\infty = {}^\top(\phi_\infty, w_\infty) = P_\infty u$, $\sigma_{1,0} = \langle\phi_0\rangle_1$, and $u_{\infty,0} = P_\infty u_0$. In what follows we will denote $\boldsymbol{F}_\infty = {}^\top(f_\infty^0, \boldsymbol{f}_\infty)$, $\boldsymbol{f}_\infty = {}^\top(\boldsymbol{f}'_\infty, f_\infty^n)$.

We note that if $u$ is a solution of (4.1) in $\bigcap_{j=0}^{[\frac{m}{2}]} C([0,T]; H^{m-2j})$, then $w_\infty|_{x_n=0,1} = 0$ since $w^{(0),1}|_{x_n=0,1} = 0$; and furthermore, $u_\infty \in \bigcap_{j=0}^{[\frac{m}{2}]} C([0,T]; H^{m-2j})$ by Lemma 4.3.

We close this section with several lemmas, which will be used in the next section to derive the a priori estimate.

**Lemma 4.4.**

(i) $\langle\partial_{x'} f\rangle_1 = \partial_{x'}\langle f\rangle_1$ *and* $\|\partial_{x'}\langle f\rangle_1\|_2 \leqslant \|\langle f\rangle_1\|_2$.

(ii) *Let* $\sigma = \sigma(x')$ *with* $\mathrm{supp}(\hat{\sigma}) \subset \{|\xi'| \leqslant 1\}$. *Then*

$$\big(\langle Q_0 \tilde{B} u\rangle_1, \sigma\big) = -\frac{\gamma^2}{\alpha_0}\langle\!\langle u, \tilde{B}\big(\sigma u_0^{(0)}\big)\rangle\!\rangle.$$

(iii) $\langle\!\langle \langle f\rangle_1 u_0^{(0)}, u_\infty\rangle\!\rangle = 0$ *for* $u_\infty \in \mathrm{Range}(P_\infty)$.

**Proof.** It is easy to see (i). Let us prove (ii). Since $\langle u\rangle = \langle u, u^{(0)*}\rangle$ and $Q_0 u^{(0)*} = u^{(0)*} = \frac{\gamma^2}{\alpha_0} u_0^{(0)}$, we have, by the Plancherel theorem and (4.9),

$$\begin{aligned}
\big(\langle Q_0 \tilde{B} u\rangle_1, \sigma\big) &= (2\pi)^{-(n-1)}\langle\!\langle \mathscr{F} Q_0 \tilde{B} u, \hat{\sigma} u^{(0)*}\rangle\!\rangle \\
&= \langle\!\langle Q_0 \tilde{B} u, \sigma u^{(0)*}\rangle\!\rangle \\
&= -\langle\!\langle u, \tilde{B}\big(\sigma u^{(0)*}\big)\rangle\!\rangle \\
&= -\frac{\gamma^2}{\alpha_0}\langle\!\langle u, \tilde{B}\big(\sigma u_0^{(0)}\big)\rangle\!\rangle.
\end{aligned}$$

This proves (ii). As for (iii), since $u_0^{(0)} = {}^\top(\phi^{(0)}, 0)$ and $\hat{\chi}_1\langle\hat{\phi}_\infty\rangle = 0$, we have

$$\begin{aligned}
\langle\!\langle \langle f\rangle_1 u_0^{(0)}, u_\infty\rangle\!\rangle &= (2\pi)^{-(n-1)}\frac{\alpha_0}{\gamma^2}\big(\hat{\chi}_1\langle\hat{f}\rangle, \hat{\phi}_\infty\big) \\
&= (2\pi)^{-(n-1)}\frac{\alpha_0}{\gamma^2}\big(\hat{\chi}_1\langle\hat{f}\rangle, \langle\hat{\phi}_\infty\rangle\big)_{L^2(\mathbf{R}^{n-1})} \\
&= 0.
\end{aligned}$$

This completes the proof. □

**Lemma 4.5.** *There hold the following assertions.*

(i) $\|\langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty)\rangle_1\|_2^2 \leqslant C(\|\partial_{x'}\sigma_1\|_2^2 + \|\partial_{x'}\phi_\infty\|_2^2 + \gamma^4\|\partial_{x'} w_\infty\|_2^2)$.
(ii) *If* $w_\infty^n|_{x_n=0,1} = 0$, *then* $\langle Q_0 \tilde{B} u_\infty\rangle_1 = \langle Q_0 B u_\infty\rangle_1 = \langle v_s^1 \partial_{x_1}\phi_\infty + \gamma^2 \,\mathrm{div}(\rho_s w_\infty)\rangle_1$.
(iii) *If* $w_\infty^n|_{x_n=0,1} = 0$, $\|\tilde{\mathbf{g}}\|_{C^m[0,1]} \leqslant \tilde{\omega}$ *and* $2j + k \leqslant m$, *then*

$$\begin{aligned}
&\big\|\partial_{x'}^k \partial_t^j \big\langle Q_0 \tilde{B}\big(\sigma_1 u^{(0)} + u_\infty\big)\big\rangle_1\big\|_2^2 \\
&\quad\leqslant C\big\{\big\|\partial_{x'}^p \partial_t^j \sigma_1\big\|_2^2 + \big\|\partial_{x'}^q \partial_t^j \phi_\infty\big\|_2^2 + \gamma^4\big\|\mathrm{div}(\partial_{x'}^r \partial_t^j w_\infty)\big\|_2^2 + \gamma^4\tilde{\omega}^2\big\|\partial_{x'}^s \partial_t^j w_\infty\big\|_2^2\big\}
\end{aligned}$$

*for* $0 \leqslant p, q \leqslant k + 1$, $0 \leqslant r, s \leqslant k$.

**Proof.** A direct computation gives (i), By integration by parts, we see $\langle \mathrm{div}(\rho_s w_\infty)\rangle_1 = \langle \nabla' \cdot (\rho_s w'_\infty)\rangle_1$. This gives (ii). By using (ii) and Lemma 4.4, one can obtain (iii) by a direct computation. This completes the proof. □

## 5. A priori estimate

In this section we establish the a priori estimate such as (3.5). We introduce the following notations:

$$[\![ f(t) ]\!]_m = \left( \sum_{j=0}^{[\frac{m}{2}]} \| \partial_t^j f(t) \|_{H^{m-2j}}^2 \right)^{\frac{1}{2}},$$

$$|\!|\!| Df(t) |\!|\!|_m = \begin{cases} \|\partial_x f(t)\|_2 & \text{for } m = 0, \\ ([\![\partial_x f(t)]\!]_m^2 + [\![\partial_t f(t)]\!]_{m-1}^2)^{\frac{1}{2}} & \text{for } m \geqslant 1. \end{cases}$$

We will prove the following estimate.

**Proposition 5.1.** *Let $m$ be an integer satisfying $m \geqslant [n/2] + 1$. There are positive numbers $\nu_0$, $\gamma_0$, $\tilde{\omega}$ and $\varepsilon_1$ such that the following assertion holds.*

*Let $T > 0$ be any given number and let $u(t)$ be a solution of (4.1) in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,T]; H^{m-2j})$ with $\int_0^T |\!|\!| Dw_\infty |\!|\!|_m^2 \, d\tau < \infty$. If $[\![u(t)]\!]_m \leqslant \varepsilon_1$ for $t \in [0,T]$, then there holds the inequality*

$$[\![ u(t) ]\!]_m^2 + \int_0^t \left( |\!|\!| D\sigma_1 |\!|\!|_{m-1}^2 + |\!|\!| D\phi_\infty |\!|\!|_{m-1}^2 + |\!|\!| Dw_\infty |\!|\!|_m^2 \right) d\tau \leqslant C \|u_0\|_{H^m}^2$$

*for $t \in [0,T]$ with $C > 0$ independent of $T$, provided that $\nu \geqslant \nu_0$, $\gamma^2/(\nu + \tilde{\nu}) \geqslant \gamma_0^2$ and $\|\tilde{\mathbf{g}}\|_{C^m[0,1]} \leqslant \tilde{\omega}$.*

As in [9,5], one can prove Theorem 3.1 by combining the a priori estimate given in Proposition 5.1 and the local existence result in [4].

To prove Proposition 5.1, we first derive fundamental estimates in the energy method (Section 5.1); we then combine them to obtain the $H^m$-energy inequality (Section 5.2); and we finally estimate the nonlinearities to complete the a priori estimate given in Proposition 5.1 (Section 5.3).

Throughout this section we assume that $u(t) = (\sigma_1 u^{(0)})(t) + u_\infty(t)$ is a solution of (4.1) in $\bigcap_{j=0}^{[\frac{m}{2}]} C([0,T]; H^{m-2j})$ with $\int_0^T |\!|\!| Dw_\infty |\!|\!|^2 \, d\tau < \infty$ for an arbitrarily fixed $T > 0$.

We also assume that

$$\|\tilde{\mathbf{g}}\|_{C^m[0,1]} \leqslant \tilde{\omega}$$

for a constant $\tilde{\omega} > 0$. This implies that

$$|\rho_s - 1|_\infty + \left| \frac{\tilde{P}'(\rho_s)}{\gamma^2} - 1 \right|_\infty + \|\partial_{x_n} \rho_s\|_{C^m[0,1]} + \left\| \partial_{x_n} \left( \frac{\tilde{P}'(\rho_s)}{\gamma^2} \right) \right\|_{C^m[0,1]} \leqslant C\tilde{\omega}$$

by Proposition 2.1.

### *5.1. Fundamental estimates*

We introduce some quantities. Let $E^{(0)}[u]$ and $D^{(0)}[w]$ be defined by

$$E^{(0)}[u] = \frac{\alpha_0}{\gamma^2} \|\sigma_1\|_2^2 + \frac{1}{\gamma^2} \left\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}} \phi_\infty \right\|_2^2 + \|\sqrt{\rho_s} w_\infty\|_2^2$$

for $u = \sigma_1 u^{(0)} + u_\infty$ with $u_\infty = {}^\top(\phi_\infty, w_\infty)$; and

$$D^{(0)}[w] = \nu\|\nabla w\|_2^2 + \tilde{\nu}\|\operatorname{div} w\|_2^2.$$

Note that

$$\langle\langle Au, u\rangle\rangle = D^{(0)}[w]$$

for $u = {}^\top(\phi, w)$ with $w|_{x_n=0,1} = 0$.

In what follows we will denote the tangential derivatives $\partial_t^j \partial_{x'}^k$ by $T_{j,k}$:

$$T_{j,k}u = \partial_t^j \partial_{x'}^k u;$$

and, for operators $A$ and $B$, we will denote the commutator $AB - BA$ by $[A, B]$.

We begin with the $L^2$ energy estimates for tangential derivatives.

**Proposition 5.2.** *There is a constant $\nu_0 > 0$ such that if $\nu \geqslant \nu_0$ and $\tilde{\omega} \leqslant 1$, then the following estimate holds for $0 \leqslant 2j + k \leqslant m$:*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E^{(0)}[T_{j,k}u] + \frac{3}{4}D^{(0)}[T_{j,k}w_\infty] \\
&\quad\leqslant R_{j,k}^{(1)} + C\left\{\left(\frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4}\right)\|\partial_{x'}T_{j,k}\sigma_1\|_2^2 + \left(\frac{1}{\gamma^2} + \frac{\nu}{\gamma^4}\right)\|\tilde{T}_{j,k}\phi_\infty\|_2^2\right\},
\end{aligned} \tag{5.1}$$

*where*

$$R_{j,k}^{(1)} = \frac{\alpha_0}{\gamma^2}\big(\langle Q_0 T_{j,k}\boldsymbol{F}\rangle_1, T_{j,k}\sigma_1\big) + \langle\langle T_{j,k}\boldsymbol{F}_\infty, T_{j,k}u_\infty\rangle\rangle,$$

*and*

$$\tilde{T}_{j,k}\phi_\infty = \begin{cases} \partial_x\phi_\infty & (j = k = 0), \\ T_{j,k}\phi_\infty & (2j + k \geqslant 1). \end{cases}$$

**Proof.** We consider the case $j = k = 0$. Recall that $u^{(0)} = u_0^{(0)} + u_1^{(0)}$ with $u_0^{(0)} = {}^\top(\phi^{(0)}, 0)$ and $u_1^{(0)} = {}^\top(0, w^{(0),1}\boldsymbol{e}_1)$. (See Lemma 4.1.)

We take the inner product of (4.10) with $\sigma_1$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\sigma_1\|_2^2 + \big(\big\langle Q_0\tilde{B}\big(\sigma_1 u^{(0)} + u_\infty\big)\big\rangle_1, \sigma_1\big) = \big(\langle Q_0\boldsymbol{F}\rangle_1, \sigma_1\big).$$

By integration by parts, we have

$$\begin{aligned}
\big(\big\langle Q_0\tilde{B}\big(\sigma_1 u^{(0)}\big)\big\rangle_1, \sigma_1\big) &= \big(\big\langle v_s^1\partial_{x_1}\sigma_1\phi^{(0)}\big\rangle_1 + \big\langle\gamma^2\rho_s\partial_{x_1}\sigma_1 w^{(0),1}\big\rangle_1, \sigma_1\big) \\
&= \big(\big\langle v_s^1\phi^{(0)}\big\rangle + \big\langle\gamma^2\rho_s w^{(0),1}\big\rangle\big)(\partial_{x_1}\sigma_1, \sigma_1) \\
&= 0.
\end{aligned}$$

We also see from Lemma 4.4(ii) that

$$\big(\langle Q_0\tilde{B}u_\infty\rangle_1,\sigma_1\big)=-\frac{\gamma^2}{\alpha_0}\big\langle\!\big\langle u_\infty,\tilde{B}\big(\sigma_1u_0^{(0)}\big)\big\rangle\!\big\rangle.$$

We thus obtain

$$\frac{1}{2}\frac{d}{dt}\|\sigma_1\|_2^2-\frac{\gamma^2}{\alpha_0}\big\langle\!\big\langle u_\infty,\tilde{B}\big(\sigma_1u_0^{(0)}\big)\big\rangle\!\big\rangle=\big(\langle Q_0\boldsymbol{F}\rangle_1,\sigma_1\big). \tag{5.2}$$

We next take the inner product of (4.11) with $u_\infty$ to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\gamma^2}\left\|\sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}\phi_\infty\right\|_2^2+\|\sqrt{\rho_s}w_\infty\|_2^2\right)+\langle\!\langle Lu_\infty,u_\infty\rangle\!\rangle+\big\langle\!\big\langle\tilde{\mathcal{M}}\big(\sigma_1u^{(0)}\big),u_\infty\big\rangle\!\big\rangle\\&\quad-\big\langle\!\big\langle\big\langle Q_0\tilde{B}\big(\sigma_1u^{(0)}+u_\infty\big)\big\rangle_1u^{(0)},u_\infty\big\rangle\!\big\rangle=\langle\!\langle\boldsymbol{F}_\infty,u_\infty\rangle\!\rangle.\end{aligned} \tag{5.3}$$

By (4.2), we see that

$$\langle\!\langle Lu_\infty,u_\infty\rangle\!\rangle=D^{(0)}[w_\infty]+\langle\!\langle C_0u_\infty,u_\infty\rangle\!\rangle.$$

By Lemma 4.4(iii), we have

$$\big\langle\!\big\langle\big\langle Q_0\tilde{B}\big(\sigma_1u^{(0)}+u_\infty\big)\big\rangle_1u^{(0)},u_\infty\big\rangle\!\big\rangle=\big\langle\!\big\langle\big\langle Q_0\tilde{B}\big(\sigma_1u^{(0)}+u_\infty\big)\big\rangle_1u_1^{(0)},u_\infty\big\rangle\!\big\rangle.$$

It then follows from (5.3) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\gamma^2}\left\|\sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}\phi_\infty\right\|_2^2+\|\sqrt{\rho_s}w_\infty\|_2^2\right)+D^{(0)}[w_\infty]+\langle\!\langle C_0u_\infty,u_\infty\rangle\!\rangle\\&\quad+\big\langle\!\big\langle\tilde{\mathcal{M}}\big(\sigma_1u^{(0)}\big),u_\infty\big\rangle\!\big\rangle-\big\langle\!\big\langle\big\langle Q_0\tilde{B}\big(\sigma_1u^{(0)}+u_\infty\big)\big\rangle_1u_1^{(0)},u_\infty\big\rangle\!\big\rangle=\langle\!\langle\boldsymbol{F}_\infty,u_\infty\rangle\!\rangle.\end{aligned} \tag{5.4}$$

We add $\frac{\alpha_0}{\gamma^2}\times$ (5.2) to (5.3). Then, since

$$\big\langle\!\big\langle\tilde{\mathcal{M}}\big(\sigma_1u^{(0)}\big),u_\infty\big\rangle\!\big\rangle=\big\langle\!\big\langle\tilde{A}\big(\sigma_1u^{(0)}\big),u_\infty\big\rangle\!\big\rangle+\big\langle\!\big\langle\tilde{B}\big(\sigma_1u_0^{(0)}\big),u_\infty\big\rangle\!\big\rangle+\big\langle\!\big\langle\tilde{B}\big(\sigma_1u_1^{(0)}\big),u_\infty\big\rangle\!\big\rangle,$$

the term $\langle\!\langle\tilde{B}(\sigma_1u_0^{(0)}),u_\infty\rangle\!\rangle$ is canceled, and hence, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E^{(0)}[u]+D^{(0)}[w_\infty]+\langle\!\langle C_0u_\infty,u_\infty\rangle\!\rangle+\big\langle\!\big\langle\tilde{A}\big(\sigma_1u^{(0)}\big),u_\infty\big\rangle\!\big\rangle+\big\langle\!\big\langle\tilde{B}\big(\sigma_1u_1^{(0)}\big),u_\infty\big\rangle\!\big\rangle\\&\quad-\big\langle\!\big\langle\big\langle Q_0\tilde{B}\big(\sigma_1u^{(0)}+u_\infty\big)\big\rangle_1u_1^{(0)},u_\infty\big\rangle\!\big\rangle\\&=\frac{\alpha_0}{\gamma^2}\big(\langle Q_0\boldsymbol{F}\rangle_1,\sigma_1\big)+\langle\!\langle\boldsymbol{F}_\infty,u_\infty\rangle\!\rangle.\end{aligned} \tag{5.5}$$

A direct calculation gives, if $\nu\geqslant\nu_0$, then

$$\big|\langle\!\langle C_0u_\infty,u_\infty\rangle\!\rangle\big|\leqslant\frac{1}{16}D^{(0)}[w_\infty]+C\frac{\nu}{\gamma^4}\|\phi_\infty\|_2^2. \tag{5.6}$$

Since $w^{(0),1} = O(\gamma^{-2})$, by integration by parts, we have

$$\begin{aligned}\left|\langle\!\langle \tilde{A}(\sigma_1 u^{(0)}), u_\infty\rangle\!\rangle\right| &\leqslant \frac{C}{\gamma^2}\left(\nu\left\|\nabla'\sigma_1\right\|_2\|\nabla w_\infty\|_2 + \tilde{\nu}\|\partial_{x_1}\sigma_1\|_2\|\operatorname{div} w_\infty\|_2\right) \\ &\leqslant \frac{1}{16}D^{(0)}[w_\infty] + C\frac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_{x'}\sigma_1\|_2^2, \qquad (5.7)\end{aligned}$$

and, also,

$$\begin{aligned}\left|\langle\!\langle \tilde{B}(\sigma_1 u_1^{(0)}), u_\infty\rangle\!\rangle\right| &\leqslant \frac{C}{\gamma^2}\left\|\nabla'\sigma_1\right\|_2\left(\|\phi_\infty\|_2 + \|w_\infty\|_2\right) \\ &\leqslant \frac{1}{16}D^{(0)}[w_\infty] + C\left\{\left(\frac{1}{\gamma^2} + \frac{1}{\nu\gamma^4}\right)\|\partial_{x'}\sigma_1\|_2^2 + \frac{1}{\gamma^2}\|\phi_\infty\|_2^2\right\}. \qquad (5.8)\end{aligned}$$

By Lemma 4.5(iii), we find that

$$\begin{aligned}&\langle\!\langle\langle Q_0\tilde{B}(\sigma_1 u^{(0)} + u_\infty)\rangle_1 u_1^{(0)}, u_\infty\rangle\!\rangle \\ &\quad\leqslant \frac{C}{\gamma^2}\left\|\langle Q_0\tilde{B}(\sigma_1 u^{(0)} + u_\infty)\rangle_1\right\|_2\|w_\infty\|_2 \\ &\quad\leqslant \frac{C}{\gamma^2}\left\{\|\partial_{x'}\sigma_1\|_2 + \|\phi_\infty\|_2 + \gamma^2\|\operatorname{div} w_\infty\|_2 + \gamma^2\tilde{\omega}\|w_\infty\|_2\right\}\|w_\infty\|_2.\end{aligned}$$

So, if $\nu \geqslant \nu_0$ for some $\nu_0 > 0$, then

$$\langle\!\langle\langle Q_0\tilde{B}(\sigma_1 u^{(0)} + u_\infty)\rangle_1 u_1^{(0)}, u_\infty\rangle\!\rangle \leqslant \frac{1}{16}D^{(0)}[w_\infty] + \frac{C}{\nu\gamma^4}\left(\|\partial_{x'}\sigma_1\|_2^2 + \|\phi_\infty\|_2^2\right). \qquad (5.9)$$

Since $\|\phi_\infty\|_2 \leqslant \|\partial_x\phi_\infty\|_2$ by Lemma 4.3(iv), we deduce from (5.5)–(5.9) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E^{(0)}[u] + \frac{3}{4}D^{(0)}[w_\infty] \\ &\quad\leqslant R_{0,0}^{(1)} + C\left\{\left(\frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4}\right)\|\partial_{x'}\sigma_1\|_2^2 + \left(\frac{1}{\gamma^2} + \frac{\nu}{\gamma^4}\right)\|\partial_x\phi_\infty\|_2^2\right\}. \qquad (5.10)\end{aligned}$$

This proves (5.1) for $j = k = 0$. The case $2j + k \geqslant 1$ can be proved similarly by applying $T_{j,k}$ to (4.10) and (4.11) and using Lemma 4.4. This completes the proof. $\square$

We next derive the $H^1$-parabolic estimates for $w_\infty$. We define $J[u]$ by

$$J[u] = -2\langle\!\langle \sigma_1 u^{(0)} + u_\infty, B\tilde{Q}u_\infty\rangle\!\rangle \quad \text{for } u = \sigma_1 u^{(0)} + u_\infty.$$

It is easy to see that if $\gamma^2 \geqslant 1$ and $\tilde{\omega} \leqslant 1$, then

$$\begin{aligned}\left|J[u]\right| &\leqslant C\left\{\frac{1}{\gamma^2}\left(\|\sigma_1\|_2 + \|\phi_\infty\|_2\right)\left\|\gamma^2\operatorname{div}(\rho_s w_\infty)\right\|_2 + \left(\frac{1}{\gamma^2}\|\sigma_1\|_2 + \|w_\infty\|_2\right)\|\partial_x w_\infty\|_2\right\} \\ &\leqslant \frac{b_0\gamma^2}{\nu}E^{(0)}[u] + \frac{1}{2}D^{(0)}[w_\infty]\end{aligned}$$

for some constant $b_0 > 0$.

Let $b_1$ be a positive constant (to be determined later) and define $E^{(1)}[u]$ by

$$E^{(1)}[u] = \frac{2b_1\gamma^2}{\nu} E^{(0)}[u] + D^{(0)}[w_\infty] + J[u].$$

Note that if $b_1 \geqslant b_0$, then $E^{(1)}[u]$ is equivalent to $E^{(0)}[u] + D^{(0)}[w_\infty]$.

**Proposition 5.3.** *There exists $b_1 \geqslant b_0$ such that if $\nu \geqslant \nu_0$, $\gamma^2 \geqslant 1$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geqslant 1$ and $\tilde{\omega} \leqslant 1$, then the following estimate holds for $0 \leqslant 2k + j \leqslant m - 1$:*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E^{(1)}[T_{j,k}u] + \frac{b_1\gamma^2}{\nu}D^{(0)}[T_{j,k}w_\infty] + \frac{1}{2}\|\sqrt{\rho_s}\partial_t T_{j,k}w_\infty\|_2^2 \\
&\quad \leqslant R_{j,k}^{(2)} + C\left\{\left(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\gamma^2}\right)\|\partial_{x'}T_{j,k}\sigma_1\|_2^2 \right. \\
&\qquad \left. + \left(\frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4}\right)\|\tilde{T}_{j,k}\phi_\infty\|_2^2 + \frac{1}{\gamma^2}\|\partial_{x'}T_{j,k}\phi_\infty\|_2^2\right\},
\end{aligned} \tag{5.11}$$

*where*

$$\begin{aligned}
R_{j,k}^{(2)} &= \frac{2b_1\alpha_0}{\nu}\big(\langle Q_0 T_{j,k}\boldsymbol{F}\rangle_1, T_{j,k}\sigma_1\big) + \frac{2b_1\gamma^2}{\nu}\langle\langle T_{j,k}\boldsymbol{F}_\infty, T_{j,k}u_\infty\rangle\rangle \\
&\quad + C\left\{\frac{1}{\gamma^2}\|T_{j,k}Q_0\boldsymbol{F}\|_2^2 + \|T_{j,k}\tilde{Q}\,\boldsymbol{F}\|_2^2\right\}.
\end{aligned}$$

**Proof.** We consider the case $j = k = 0$. We take the inner product of (4.11) with $\partial_t\tilde{Q}\,u_\infty$ to obtain

$$\begin{aligned}
&\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 + \langle\langle Lu_\infty, \partial_t\tilde{Q}\,u_\infty\rangle\rangle + \big\langle\big\langle\tilde{\mathcal{M}}\big(\sigma_1 u^{(0)}\big), \partial_t\tilde{Q}\,u_\infty\big\rangle\big\rangle \\
&\quad - \big\langle\big\langle\big\langle Q_0\tilde{B}\big(\sigma_1 u^{(0)} + u_\infty\big)\big\rangle_1 u^{(0)}, \partial_t\tilde{Q}\,u_\infty\big\rangle\big\rangle \\
&\quad = \langle\langle\boldsymbol{F}_\infty, \partial_t\tilde{Q}\,u_\infty\rangle\rangle.
\end{aligned} \tag{5.12}$$

Let us first consider $\langle\langle Lu_\infty, \partial_t\tilde{Q}\,u_\infty\rangle\rangle$ on the left of (5.12) which is written as

$$\langle\langle Lu_\infty, \partial_t\tilde{Q}\,u_\infty\rangle\rangle = \langle\langle Au_\infty, \partial_t\tilde{Q}\,u_\infty\rangle\rangle + \langle\langle Bu_\infty, \partial_t\tilde{Q}\,u_\infty\rangle\rangle + \langle\langle C_0u_\infty, \partial_t\tilde{Q}\,u_\infty\rangle\rangle. \tag{5.13}$$

The first term on the right of (5.13) is written as

$$\langle\langle Au_\infty, \partial_t\tilde{Q}\,u_\infty\rangle\rangle = \frac{1}{2}\frac{d}{dt}D^{(0)}[w_\infty]. \tag{5.14}$$

As for the second term,

$$\begin{aligned}
\langle\langle Bu_\infty, \partial_t\tilde{Q}\,u_\infty\rangle\rangle &= -\frac{d}{dt}\langle\langle u_\infty, B\tilde{Q}\,u_\infty\rangle\rangle + \langle\langle\partial_t u_\infty, B\tilde{Q}\,u_\infty\rangle\rangle \\
&= -\frac{d}{dt}\langle\langle u_\infty, B\tilde{Q}\,u_\infty\rangle\rangle + \langle\langle\partial_t Q_0u_\infty, B\tilde{Q}\,u_\infty\rangle\rangle + \langle\langle\partial_t\tilde{Q}\,u_\infty, B\tilde{Q}\,u_\infty\rangle\rangle.
\end{aligned}$$

By (4.11), we have

$$\partial_t\phi_\infty = -\bigl\{v_s^1\partial_{x_1}\phi_\infty + \gamma^2\operatorname{div}(\rho_s w_\infty) + \bigl(v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1}\bigr)\partial_{x_1}\sigma_1 \\ - \bigl\langle Q_0\tilde{B}\bigl(\sigma_1 u^{(0)} + u_\infty\bigr)\bigr\rangle_1\phi^{(0)}\bigr\} + f_\infty^0.$$

Therefore, using the Poincaré inequality, if $\tilde{\omega}\leqslant 1$, then we have

$$\begin{aligned}\bigl|\langle\!\langle\partial_t Q_0 u_\infty, B\tilde{Q}u_\infty\rangle\!\rangle\bigr| &\leqslant C\bigl\{\|\partial_{x_1}\phi_\infty\|_2 + \gamma^2\|\partial_x w_\infty\|_2 + \|\partial_{x'}\sigma_1\|_2\bigr\}\bigl\|\operatorname{div}(\rho_s w_\infty)\bigr\|_2 \\ &\quad + C\|Q_0\boldsymbol{F}_\infty\|_2\bigl\|\operatorname{div}(\rho_s w_\infty)\bigr\|_2 \\ &\leqslant C\left\{\frac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \frac{1}{\gamma^2}\bigl(\|\partial_{x'}\phi_\infty\|_2^2 + \|\partial_{x'}\sigma_1\|_2^2 + \|Q_0\boldsymbol{F}_\infty\|_2^2\bigr)\right\}.\end{aligned}$$

We also have

$$\begin{aligned}\bigl|\langle\!\langle\partial_t\tilde{Q}u_\infty, B\tilde{Q}u_\infty\rangle\!\rangle\bigr| &\leqslant \|\sqrt{\rho_s}\partial_t w_\infty\|_2\bigl\|\sqrt{\rho_s}v_s^1\partial_{x_1}w_\infty\bigr\|_2 \\ &\leqslant \frac{1}{12}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 + \frac{C}{\nu}D^{(0)}[w_\infty].\end{aligned}$$

We thus obtain, if $\gamma^2\geqslant 1$, then

$$\begin{aligned}\langle\!\langle Bu_\infty, \partial_t\tilde{Q}u_\infty\rangle\!\rangle &\geqslant -\frac{d}{dt}\langle\!\langle u_\infty, B\tilde{Q}u_\infty\rangle\!\rangle - \frac{1}{12}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 \\ &\quad - C\left\{\frac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \frac{1}{\gamma^2}\bigl(\|\partial_{x'}\phi_\infty\|_2^2 + \|\partial_{x'}\sigma_1\|_2^2 + \|Q_0\boldsymbol{F}_\infty\|_2^2\bigr)\right\}.\end{aligned}\tag{5.15}$$

As for the third term on the right of (5.13), we have

$$\begin{aligned}\langle\!\langle C_0 u_\infty, \partial_t\tilde{Q}u_\infty\rangle\!\rangle &\leqslant C\left\{\frac{\nu}{\gamma^2}\|\phi_\infty\|_2 + \|w_\infty\|_2\right\}\|\sqrt{\rho_s}\partial_t w_\infty\|_2 \\ &\leqslant \frac{1}{12}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 + C\left\{\frac{\nu^2}{\gamma^4}\|\phi_\infty\|_2^2 + \|w_\infty\|_2^2\right\} \\ &\leqslant \frac{1}{12}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 + +C\left\{\frac{\nu^2}{\gamma^4}\|\phi_\infty\|_2^2 + \frac{1}{\nu}D^{(0)}[w_\infty]\right\}.\end{aligned}\tag{5.16}$$

It follows from (5.13)–(5.16) that if $\gamma^2\geqslant 1$, then

$$\begin{aligned}\langle\!\langle Lu_\infty, \partial_t\tilde{Q}u_\infty\rangle\!\rangle &\geqslant \frac{1}{2}\frac{d}{dt}\bigl(D^{(0)}[w_\infty] - 2\langle\!\langle u_\infty, B\tilde{Q}u_\infty\rangle\!\rangle\bigr) - \frac{1}{6}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 \\ &\quad - C\left\{\frac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \frac{\nu^2}{\gamma^4}\|\phi_\infty\|_2^2\right. \\ &\quad \left. + \frac{1}{\gamma^2}\bigl(\|\partial_{x'}\phi_\infty\|_2^2 + \|\partial_{x'}\sigma_1\|_2^2 + \|Q_0\boldsymbol{F}_\infty\|_2^2\bigr)\right\}.\end{aligned}\tag{5.17}$$

We next consider $\langle\!\langle\tilde{\mathcal{M}}(\sigma_1 u^{(0)}), \partial_t\tilde{Q}u_\infty\rangle\!\rangle$ in (5.12):

$$\bigl\langle\!\bigl\langle\tilde{\mathcal{M}}\bigl(\sigma_1 u^{(0)}\bigr), \partial_t\tilde{Q}u_\infty\bigr\rangle\!\bigr\rangle = \bigl\langle\!\bigl\langle\tilde{A}\bigl(\sigma_1 u^{(0)}\bigr), \partial_t\tilde{Q}u_\infty\bigr\rangle\!\bigr\rangle + \bigl\langle\!\bigl\langle\tilde{B}\bigl(\sigma_1 u^{(0)}\bigr), \partial_t\tilde{Q}u_\infty\bigr\rangle\!\bigr\rangle.$$

Each term on the right is estimated as follows. Since $\|\partial_{x'}^k \sigma_1\|_2 \leqslant \|\partial_{x'}\sigma_1\|_2$ for $k \geqslant 1$ by Lemma 4.4(i) and $w^{(0),1} = O(\gamma^{-2})$, we have

$$\begin{aligned}\left|\left\langle\!\left\langle \tilde{A}\left(\sigma_1 u^{(0)}\right), \partial_t \tilde{Q} u_\infty\right\rangle\!\right\rangle\right| &\leqslant C\frac{\nu+\tilde{\nu}}{\gamma^2}\left(\|\partial_{x'}\sigma_1\|_2 + \left\|\partial_{x'}^2\sigma_1\right\|_2\right)\|\sqrt{\rho_s}\partial_t w_\infty\|_2 \\ &\leqslant \frac{1}{12}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 + C\frac{(\nu+\tilde{\nu})^2}{\gamma^4}\|\partial_{x'}\sigma_1\|_2^2.\end{aligned}$$

By Lemma 4.1(ii), we have $\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)} = \alpha_0$, and so $\partial_{x_n}(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)}) = 0$. This gives $\hat{B}_0 u^{(0)} = 0$, which, in turn, gives $\hat{B}_0(\sigma_1 u^{(0)}) = \sigma_1 \hat{B}_0 u^{(0)} = 0$. It follows that $\tilde{B}(\sigma_1 u^{(0)}) = \tilde{B}(\sigma_1 u^{(0)}) + \hat{B}_0(\sigma_1 u^{(0)}) = B(\sigma_1 u^{(0)})$. Therefore, we see, by (4.2),

$$\begin{aligned}\left\langle\!\left\langle \tilde{B}\left(\sigma_1 u^{(0)}\right), \partial_t \tilde{Q} u_\infty\right\rangle\!\right\rangle &= \left\langle\!\left\langle B\left(\sigma_1 u^{(0)}\right), \partial_t \tilde{Q} u_\infty\right\rangle\!\right\rangle \\ &= -\frac{d}{dt}\left\langle\!\left\langle \sigma_1 u^{(0)}, B\tilde{Q} u_\infty\right\rangle\!\right\rangle + \left\langle\!\left\langle \partial_t\sigma_1 u^{(0)}, B\tilde{Q} u_\infty\right\rangle\!\right\rangle.\end{aligned}$$

As for the second term on the right, we see from (4.10)

$$\partial_t\sigma_1 = -\left\langle Q_0\tilde{B}\left(\sigma_1 u^{(0)} + u_\infty\right)\right\rangle_1 + \langle Q_0 \boldsymbol{F}\rangle_1.$$

This, together with Lemma 4.5(ii), implies

$$\begin{aligned}\left|\left\langle\!\left\langle \partial_t\sigma_1 u^{(0)}, B\tilde{Q} u_\infty\right\rangle\!\right\rangle\right| &\leqslant C\left\{\|\partial_{x'}\sigma_1\|_2 + \|\partial_{x'}\phi_\infty\|_2 + \gamma^2\left\|\mathrm{div}(\rho_s w_\infty)\right\|_2 + \left\|\langle Q_0\boldsymbol{F}\rangle_1\right\|_2\right\} \\ &\quad\times\left\{\left\|\mathrm{div}(\rho_s w_\infty)\right\|_2 + \frac{1}{\gamma^2}\|\partial_{x_1} w_\infty\|_2\right\} \\ &\leqslant C\left\{\frac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \frac{1}{\gamma^2}\left(\|\partial_{x'}\sigma_1\|_2^2 + \|\partial_{x'}\phi_\infty\|_2^2 + \left\|\langle Q_0\boldsymbol{F}\rangle_1\right\|_2^2\right)\right\}.\end{aligned}$$

We thus obtain

$$\begin{aligned}\left\langle\!\left\langle \tilde{\mathcal{M}}\left(\sigma_1 u^{(0)}\right), \partial_t \tilde{Q} u_\infty\right\rangle\!\right\rangle &\geqslant -\frac{d}{dt}\left\langle\!\left\langle \sigma_1 u^{(0)}, B\tilde{Q} u_\infty\right\rangle\!\right\rangle - \frac{1}{12}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 \\ &\quad - C\left\{\frac{\gamma^2}{\nu}D^{(0)}[w_\infty] + \frac{1}{\gamma^2}\left(\|\partial_{x'}\sigma_1\|_2^2 + \|\partial_{x'}\phi_\infty\|_2^2\right.\right. \\ &\quad \left.\left. + \left\|\langle Q_0\boldsymbol{F}\rangle_1\right\|_2^2\right) + \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\|\partial_{x'}\sigma_1\|_2^2\right\}.\end{aligned} \tag{5.18}$$

We also have

$$\begin{aligned}&\left|\left\langle\!\left\langle \left\langle Q_0\tilde{B}\left(\sigma_1 u^{(0)} + u_\infty\right)\right\rangle_1 u^{(0)}, \partial_t\tilde{Q} u_\infty\right\rangle\!\right\rangle\right| \\ &\quad\leqslant \frac{C}{\gamma^2}\left\{\|\partial_{x'}\sigma_1\|_2 + \|\partial_{x'}\phi_\infty\|_2 + \gamma^2\left\|\mathrm{div}(\rho_s w_\infty)\right\|_2\right\}\|\sqrt{\rho_s}\partial_t w_\infty\|_2 \\ &\quad\leqslant \frac{1}{12}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 + C\left\{\frac{1}{\nu}D^{(0)}[w_\infty] + \frac{1}{\gamma^4}\left(\|\partial_{x'}\sigma_1\|_2^2 + \|\partial_{x'}\phi_\infty\|_2^2\right)\right\}\end{aligned} \tag{5.19}$$

and

$$\big|\langle\!\langle \boldsymbol{F}_\infty, \partial_t \tilde{Q} u_\infty \rangle\!\rangle\big| \leqslant \frac{1}{12}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 + C\|\tilde{Q}\,\boldsymbol{F}_\infty\|_2^2. \tag{5.20}$$

It then follows from (5.12), (5.17)–(5.20) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(D^{(0)}[w_\infty] + J[u]\big) + \frac{1}{2}\|\sqrt{\rho_s}\partial_t w_\infty\|_2^2 \\
&\quad \leqslant C R_{0,0}^{(2)} + C\frac{\gamma^2}{\nu}D^{(0)}[w_\infty] \\
&\qquad + C\left\{\left(\frac{1}{\gamma^2} + \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\right)\|\partial_{x'}\sigma_1\|_2^2 + \frac{\nu^2}{\gamma^4}\|\phi_\infty\|_2^2 + \frac{1}{\gamma^2}\|\partial_{x'}\phi_\infty\|_2^2\right\}.
\end{aligned} \tag{5.21}$$

We take $b_1 > 0$ in such a way that $b_1 \geqslant \max\{b_0, 2C\}$. Adding $\frac{2b_1\gamma^2}{\nu} \times$ (5.1) to (5.21), we obtain (5.11) for $j = k = 0$. The case $2j + k \geqslant 1$ can be obtained by replacing $u_\infty$ with $T_{j,k}u_\infty$. This completes the proof. □

We next derive the dissipative estimates for $x_n$-derivatives of $\phi_\infty$.

**Proposition 5.4.** *If $\tilde{\omega} \leqslant \min\{1, \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\}$, then the following estimate holds for $0 \leqslant 2j + k + l \leqslant m - 1$:*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2}\left\|\sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right\|_2^2 + \frac{1}{2(\nu+\tilde{\nu})}\left\|\frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right\|_2^2 \\
&\quad \leqslant R_{j,k,l}^{(3)} + C\frac{\nu+\tilde{\nu}}{\gamma^4}\|K_{j,k,l}\|_2^2,
\end{aligned} \tag{5.22}$$

*where*

$$R_{j,k,l}^{(3)} = \left|\frac{1}{2\gamma^2}\left(\operatorname{div}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}w\right), \big|T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\big|^2\right)\right| + \frac{\nu+\tilde{\nu}}{\gamma^4}\|H_{j,k,l}\|_2^2$$

*with*

$$\|H_{j,k,l}\|_2^2 \leqslant C\left\{\big\|\big[T_{j,k}\partial_{x_n}^{l+1}, w\big]\cdot\nabla\phi_\infty\big\|_2^2 + \big\|T_{j,k}\partial_{x_n}^{l+1}\tilde{f}_\infty^0\big\|_2^2 + \left\|\frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}T_{j,k}\partial_{x_n}^l f_\infty^n\right\|_2^2\right\},$$

*and*

$$\tilde{f}_\infty^0 = -\phi\operatorname{div} w - w\cdot\nabla\big(\sigma_1\phi^{(0)}\big) - \langle Q_0\boldsymbol{F}\rangle_1\phi^{(0)};$$

*and $K_{j,k,l}$ is estimated as*

$$\begin{aligned}
\frac{\nu+\tilde{\nu}}{\gamma^4}\|K_{j,k,l}\|_2^2 \leqslant C\Bigg\{&\frac{\nu^2}{\nu+\tilde{\nu}}\big\|T_{j,k+1}\partial_{x_n}^l\partial_x w_\infty\big\|_2^2 + \frac{1}{\nu+\tilde{\nu}}\big\|\sqrt{\rho_s}\partial_t T_{j,k}\partial_{x_n}^l w_\infty\big\|_2^2 \\
&+ (\nu+\tilde{\nu})\tilde{\omega}^2\left(\sum_{p=0}^{l-1}\big\|T_{j,k+1}\partial_{x_n}^p\partial_x w_\infty\big\|_2^2 + \sum_{p=0}^{l}\big\|T_{j,k}\partial_{x_n}^p\partial_x w_\infty\big\|_2^2\right)
\end{aligned}$$

$$+\frac{1}{\nu+\tilde{\nu}}\sum_{p=0}^{l}\left\|T_{j,k+1}\partial_{x_n}^p w_\infty\right\|_2^2$$
$$+\frac{\nu+\tilde{\nu}}{\gamma^4}\left(\sum_{p=0}^{l}\left\|T_{j,k}\partial_{x_n}^p\partial_x\phi_\infty\right\|_2^2+\|\partial_{x'}T_{j,k}\sigma_1\|_2^2\right)\Bigg\}.$$

**Proof.** The first equation of (4.11) is written as

$$\partial_t\phi_\infty+(v_s+w)\cdot\nabla\phi_\infty+\gamma^2\rho_s\nabla'\cdot w'_\infty+\gamma^2\partial_{x_n}\left(\rho_s w_\infty^n\right)$$
$$+\left(v_s^1\phi^{(0)}+\gamma^2\rho_s w^{(0),1}\right)\partial_{x_1}\sigma_1-\left\langle Q_0\tilde{B}\left(\sigma_1 u^{(0)}+u_\infty\right)\right\rangle_1\phi^{(0)}=\tilde{f}_\infty^0.$$

Applying $T_{j,k}\partial_{x_n}^{l+1}$ to this equation, we have

$$\begin{aligned}
&\partial_t\left(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right)+(v_s+w)\cdot\nabla\left(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right)+\gamma^2\rho_s T_{j,k}\partial_{x_n}^{l+2}w_\infty^n\\
&\quad=-\left[T_{j,k}\partial_{x_n}^{l+1},v_s+w\right]\cdot\nabla\phi_\infty-\gamma^2\partial_{x_n}^{l+1}\left(\rho_s\nabla'\cdot T_{j,k}w'_\infty\right)\\
&\qquad-\gamma^2\left[\partial_{x_n}^{l+2},\rho_s\right]T_{j,k}w_\infty^n-\partial_{x_n}^{l+1}\left(v_s^1\phi^{(0)}+\gamma^2\rho_s w^{(0),1}\right)\partial_{x_1}T_{j,k}\sigma_1\\
&\qquad+\left\langle Q_0\tilde{B}\left(T_{j,k}\sigma_1 u^{(0)}+T_{j,k}u_\infty\right)\right\rangle_1\partial_{x_n}^{l+1}\phi^{(0)}+T_{j,k}\partial_{x_n}^{l+1}\tilde{f}_\infty^0.
\end{aligned}\tag{5.23}$$

The $(n+1)$-th equation of (4.11) is written as

$$\begin{aligned}
&-\frac{\nu+\tilde{\nu}}{\rho_s}\partial_{x_n}^2 w_\infty^n+\partial_{x_n}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi_\infty\right)\\
&\quad=-\left\{\partial_t w_\infty^n-\frac{\nu}{\rho_s}\Delta' w_\infty^n-\frac{\tilde{\nu}}{\rho_s}\partial_{x_n}\nabla'\cdot w'_\infty+v_s^1\partial_{x_1}w_\infty^n-\frac{\tilde{\nu}}{\rho_s}\left(\partial_{x_n}w^{(0),1}\right)\partial_{x_1}\sigma_1\right\}+f_\infty^n.
\end{aligned}$$

Applying $T_{j,k}\partial_{x_n}^l$ to this equation, we have

$$\begin{aligned}
&-\frac{\nu+\tilde{\nu}}{\rho_s}T_{j,k}\partial_{x_n}^{l+2}w_\infty^n+\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\\
&\quad=(\nu+\tilde{\nu})\left[\partial_{x_n}^l,\frac{1}{\rho_s}\right]\partial_{x_n}^2T_{j,k}w_\infty^n-\left[\partial_{x_n}^{l+1},\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\right]T_{j,k}\phi_\infty\\
&\qquad-\left\{\partial_t T_{j,k}\partial_{x_n}^l w_\infty^n-T_{j,k}\partial_{x_n}^l\left(\frac{\nu}{\rho_s}\Delta' w_\infty^n+\frac{\tilde{\nu}}{\rho_s}\partial_{x_n}\nabla'\cdot w'_\infty\right)\right.\\
&\qquad\left.+T_{j,k}\partial_{x_n}^l\left(v_s^1\partial_{x_1}w_\infty^n\right)-\partial_{x_n}^l\left(\frac{\tilde{\nu}}{\rho_s}\partial_{x_n}w^{(0),1}\right)\partial_{x_1}T_{j,k}\sigma_1\right\}+T_{j,k}\partial_{x_n}^l f_\infty^n.
\end{aligned}\tag{5.24}$$

Adding $\frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}\times$ (5.24) to (5.23), we have

$$\begin{aligned}
&\partial_t\left(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right)+(v_s+w)\cdot\nabla\left(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right)+\frac{\rho_s\tilde{P}'(\rho_s)}{\nu+\tilde{\nu}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\\
&\quad=H_{j,k,l}+K_{j,k,l},
\end{aligned}\tag{5.25}$$

where

$$H_{j,k,l} = -\left[T_{j,k}\partial_{x_n}^{l+1}, w\right]\cdot\nabla\phi_\infty + T_{j,k}\partial_{x_n}^{l+1}\tilde{f}_\infty^0 + \frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}T_{j,k}\partial_{x_n}^l f_\infty^n,$$

$$\begin{aligned}K_{j,k,l} = {}&-\left[\partial_{x_n}^{l+1}, v_s^1\right]\cdot\nabla T_{j,k}\phi_\infty - \gamma^2\left[\partial_{x_n}^{l+1},\rho_s\right]\nabla'\cdot T_{j,k}w'_\infty \\ &-\gamma^2\left[\partial_{x_n}^{l+2},\rho_s\right]T_{j,k}w_\infty^n - \partial_{x_n}^{l+1}\left(v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1}\right)\partial_{x_1}T_{j,k}\sigma_1 \\ &+\left\langle Q_0\tilde{B}\left(T_{j,k}\sigma_1 u^{(0)} + T_{j,k}u_\infty\right)\right\rangle_1\partial_{x_n}^{l+1}\phi^{(0)} \\ &+\gamma^2\rho_s^2\left[\partial_{x_n}^l, \frac{1}{\rho_s}\right]\partial_{x_n}^2 T_{j,k}w_\infty^n - \frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}\left[\partial_{x_n}^{l+1}, \frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\right]T_{j,k}\phi_\infty \\ &-\frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}\left\{\partial_t T_{j,k}\partial_{x_n}^l w_\infty^n - \frac{\nu}{\rho_s}\Delta' T_{j,k}\partial_{x_n}^l w_\infty^n + \frac{\nu}{\rho_s}\partial_{x_n}^{l+1}\nabla'\cdot T_{j,k}w'_\infty\right. \\ &-\nu\left[\partial_{x_n}^l, \frac{1}{\rho_s}\right]\Delta' T_{j,k}w_\infty^n - \tilde{\nu}\left[\partial_{x_n}^l, \frac{1}{\rho_s}\right]\partial_{x_n}\nabla'\cdot T_{j,k}w'_\infty \\ &\left.+T_{j,k}\partial_{x_n}^l\left(v_s^1\partial_{x_1}w_\infty^n\right) - \partial_{x_n}^l\left(\frac{\tilde{\nu}}{\rho_s}\partial_{x_n}w^{(0),1}\right)\partial_{x_1}T_{j,k}\sigma_1\right\}.\end{aligned}$$

Multiplying (5.25) by $\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty$ and integrating over $\Omega$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2}\left\|\sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right\|_2^2 + \frac{1}{\nu+\tilde{\nu}}\left\|\frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right\|_2^2 \\ &\quad= -\left((v_s+w)\cdot\nabla\left(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right), T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\right) \\ &\qquad+\left(H_{j,k,l}, T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\right) + \left(K_{j,k,l}, T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\right).\end{aligned}\tag{5.26}$$

Since $v_s = v_s^1\boldsymbol{e}_1$, an integration by parts gives

$$\begin{aligned}&\left(v_s\cdot\nabla\left(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right), T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\right) \\ &\quad= \frac{1}{2}\left(v_s^1, \partial_{x_1}\left(\left|T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right|^2\right)\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\right) = 0.\end{aligned}\tag{5.27}$$

Also, we have

$$\begin{aligned}&\left(w\cdot\nabla\left(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right), T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\right) \\ &\quad= -\frac{1}{2\gamma^2}\left(\operatorname{div}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}w\right), \left|T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right|^2\right).\end{aligned}\tag{5.28}$$

Since $\tilde{P}'(\rho_s) = O(\gamma^2)$ and $\rho_s \geqslant \rho_1$ $(>0)$, the last two terms on the right of (5.26) are estimates as

$$\left|\left(H_{j,k,l}, T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\right)+\left(K_{j,k,l}, T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\right)\right|$$
$$\leqslant \frac{C}{\gamma^2}\left\|\frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right\|_2\left(\|H_{j,k,l}\|_2+\|K_{j,k,l}\|_2\right)$$
$$\leqslant \frac{1}{2(\nu+\tilde{\nu})}\left\|\frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right\|_2^2+C\frac{\nu+\tilde{\nu}}{\gamma^4}\left(\|H_{j,k,l}\|_2^2+\|K_{j,k,l}\|_2^2\right). \tag{5.29}$$

Noting that $|\partial_{x_n}^{p+1}\phi^{(0)}|_\infty \leqslant C\tilde{\omega}$, one can see that $K_{j,k,l}$ has the desired estimate if $\tilde{\omega}^2 \leqslant \min\{1, \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\}$, and hence, estimate (5.22) follows from (5.26)–(5.29). This completes the proof. □

In order to obtain the dissipative estimates for higher order $x_n$-derivatives of $w_\infty$ and the tangential derivatives of $\phi_\infty$, we consider the material derivative of $\phi_\infty$. We denote the material derivative of $\phi_\infty$ by $\dot{\phi}_\infty$:

$$\dot{\phi}_\infty = \partial_t\phi_\infty + (v_s + w)\cdot\nabla\phi_\infty.$$

We have the following estimates.

**Proposition 5.5.**

(i) *If* $\tilde{\omega}^2 \leqslant \min\{1, \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\}$ *then the following estimate holds for* $0 \leqslant 2j+k+l \leqslant m-1$:

$$\frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2}\left\|\sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right\|_2^2+\frac{1}{4(\nu+\tilde{\nu})}\left\|\frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right\|_2^2+c_0\frac{\nu+\tilde{\nu}}{\gamma^4}\left\|T_{j,k}\partial_{x_n}^{l+1}\dot{\phi}_\infty\right\|_2^2$$
$$\leqslant R_{j,k,l}^{(3)}+C\frac{\nu+\tilde{\nu}}{\gamma^4}\|K_{j,k,l}\|_2^2, \tag{5.30}$$

*where* $c_0$ *is a positive constant; and* $R_{j,k,l}^{(3)}$ *and* $K_{j,k,l}$ *satisfy the same estimates as in Proposition* 5.3.

(ii) *Let* $0 \leqslant q \leqslant k$. *Then*

$$\frac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\dot{\phi}_\infty\|_2^2 \leqslant C\left\{R_{j,k}^{(4)}+D^{(0)}[T_{j,k}w_\infty]+(\nu+\tilde{\nu})\tilde{\omega}^2\|T_{j,k}w_\infty\|_2^2\right.$$
$$\left.+\frac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_{x_1}T_{j,k}\sigma_1\|_2^2+\frac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_{x_1}T_{j,q}\phi_\infty\|_2^2\right\}, \tag{5.31}$$

*where* $R_{j,k}^{(4)} = \frac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\tilde{f}_\infty^0\|_2^2$.

**Proof.** By (5.25), we have

$$T_{j,k}\partial_{x_n}^{l+1}\dot{\phi}_\infty = -\frac{\rho_s\tilde{P}'(\rho_s)}{\nu+\tilde{\nu}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty+\tilde{H}_{j,k,l}+\tilde{K}_{j,k,l},$$

where

$$\tilde{H}_{j,k,l} = \left[T_{j,k}\partial_{x_n}^{l+1}, w\right]\cdot\nabla\phi_\infty+H_{j,k,l},$$
$$\tilde{K}_{j,k,l} = \left[\partial_{x_n}^{l+1}, v_s\right]\cdot\nabla T_{j,k}\phi_\infty+K_{j,k,l}.$$

It follows that

$$\frac{\nu+\tilde{\nu}}{\gamma^4}\big\|T_{j,k}\partial_{x_n}^{l+1}\dot{\phi}_\infty\big\|_2^2 \leqslant C\left\{\frac{1}{\nu+\tilde{\nu}}\left\|\frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\right\|_2^2+\frac{\nu+\tilde{\nu}}{\gamma^4}\big(\|\tilde{H}_{j,k,l}\|_2^2+\|\tilde{K}_{j,k,l}\|_2^2\big)\right\}. \tag{5.32}$$

Since $\tilde{H}_{j,k,l}$ and $\tilde{K}_{j,k,l}$ have the same estimates as those for $H_{j,k,l}$ and $K_{j,k,l}$, respectively, we obtain (5.30) by adding (5.22) to $c_0\times(5.32)$ with $c_0>0$ satisfying $c_0C\leqslant\frac{1}{4}$. This proves (i).

As for (ii), we see from the first equation of (4.11) that

$$\begin{aligned}T_{j,k}\dot{\phi}_\infty &= -\rho_s\gamma^2\operatorname{div}(T_{j,k}w_\infty)-\gamma^2(\partial_{x_n}\rho_s)T_{j,k}w_\infty^n\\ &\quad-\big(v_s^1\phi^{(0)}+\gamma^2\rho_s w^{(0),1}\big)\partial_{x_1}T_{j,k}\sigma_1\\ &\quad+\big\langle Q_0\tilde{B}\big(T_{j,k}\sigma_1u^{(0)}+T_{j,k}u_\infty\big)\big\rangle_1\phi^{(0)}+T_{j,k}\tilde{f}_\infty^0,\end{aligned}$$

from which one can obtain (5.31). This completes the proof. □

We next derive the dissipative estimates for $\sigma_1$.

**Proposition 5.6.** *There are positive constants $\nu_0$ and $\gamma_0$ such that if $\nu\geqslant\nu_0$ and $\gamma^2/(\nu+\tilde{\nu})\geqslant\gamma_0^2$, then the following estimate holds for $0\leqslant 2j+k\leqslant m-1$:*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\frac{\nu}{\gamma^2(\nu+\tilde{\nu})}\|T_{j,k}\sigma_1\|_2^2+\frac{\alpha_1}{2(\nu+\tilde{\nu})}\|\partial_{x'}T_{j,k}\sigma_1\|_2^2\\ &\quad\leqslant R_{j,k}^{(5)}+C\left\{\frac{1}{\nu+\tilde{\nu}}\|\partial_tT_{j,k}w_\infty\|_2^2+D^{(0)}[T_{j,k}w_\infty]\right.\\ &\qquad\left.+\left(\frac{1}{\nu+\tilde{\nu}}+\frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})}\right)\|\partial_{x_n}T_{j,p}\phi_\infty\|_2^2\right\},\end{aligned} \tag{5.33}$$

*where $\alpha_1>0$ is a constant; $p$ is any integer satisfying $0\leqslant 2j+p+1\leqslant m$ and $0\leqslant p\leqslant k$; and*

$$R_{j,k}^{(5)}=\frac{\nu}{\gamma^2(\nu+\tilde{\nu})}(Q_0T_{j,k}\boldsymbol{F},T_{j,k}\sigma_1)-\frac{1}{\nu+\tilde{\nu}}\big(\rho_s(-\Delta)^{-1}\big(\rho_s\nabla'\cdot T_{j,k}\boldsymbol{f}'_\infty\big),T_{j,k}\sigma_1\big).$$

*Here $(-\Delta)^{-1}$ is the inverse of $-\Delta$ on $L^2(\Omega)$ with domain $D(-\Delta)=H^2(\Omega)\cap H_0^1(\Omega)$.*

**Proof.** It is convenient to consider the Fourier transform of (4.10) in $x'$-variable:

$$\partial_t(\widehat{T_{j,k}\sigma_1})+\big\langle Q_0\tilde{B}_{\xi'}\big(\widehat{T_{j,k}\sigma_1}u^{(0)}+\widehat{T_{j,k}u_\infty}\big)\big\rangle=\langle Q_0\widehat{T_{j,k}\boldsymbol{F}}\rangle,$$

where $|\xi'|\leqslant 1$ and

$$\tilde{B}_{\xi'}=\hat{B}_{\xi'}-\hat{B}_0=\begin{pmatrix} i\xi_1v_s^1 & i\gamma^2\rho_s{}^\top\xi' & 0\\ i\xi'\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s} & i\xi_1v_s^1I_{n-1} & 0\\ 0 & 0 & i\xi_1v_s^1\end{pmatrix}.$$

We write this equation as

$$\partial_t(\widehat{T_{j,k}\sigma_1}) + \gamma^2\big\langle \rho_s i\xi' \cdot \widehat{T_{j,k}w'_\infty}\big\rangle + \big\langle Q_0\tilde{B}_{\xi'}\big(\widehat{T_{j,k}\sigma_1}u^{(0)}\big)\big\rangle$$
$$= \langle Q_0\widehat{T_{j,k}\boldsymbol{F}}\rangle - \big\langle i\xi_1 v_s^1 \widehat{T_{j,k}\phi_\infty}\big\rangle \tag{5.34}$$

for $|\xi'| \leqslant 1$.

We further rewrite the second term $\gamma^2\langle \rho_s i\xi' \cdot \widehat{T_{j,k}w'_\infty}\rangle$ on the left of (5.34). We introduce $(n-1) \times (n+1)$ matrix operators $\tilde{A}'$, $\tilde{B}'$ and $\tilde{C}'_0$:

$$\tilde{A}' = \begin{pmatrix} 0 & -\frac{\nu}{\rho_s}\Delta' I_{n-1} - \frac{\tilde{\nu}}{\rho_s}\nabla'^{\top}\nabla' & -\frac{\tilde{\nu}}{\rho_s}\nabla'\partial_{x_n} \end{pmatrix},$$
$$\tilde{B}' = \begin{pmatrix} \frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\nabla' & v_s^1\partial_{x_1} I_{n-1} & 0 \end{pmatrix},$$
$$\tilde{C}'_0 = \begin{pmatrix} \frac{\nu\Delta v_s^1}{\gamma^2\rho_s^2}\boldsymbol{e}'_1 & 0 & (\partial_{x_n} v_s^1)\boldsymbol{e}'_1 \end{pmatrix}.$$

Since $\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)} = \alpha_0$ (see Lemma 4.1(ii)), we have

$$\tilde{B}'\big(\sigma_1 u^{(0)}\big) = \tilde{B}'\big(\sigma_1 u_0^{(0)}\big) + \tilde{B}'\big(\sigma_1 u_1^{(0)}\big) = \alpha_0\nabla'\sigma_1 + \tilde{B}'\big(\sigma_1 u_1^{(0)}\big). \tag{5.35}$$

Furthermore,

$$\tilde{A}'\big(\sigma_1 u^{(0)}\big) = \tilde{A}'\big(\sigma_1 u_1^{(0)}\big). \tag{5.36}$$

Therefore, by (4.11), (5.35) and (5.36), we have

$$\begin{aligned} &\partial_t w'_\infty - \frac{\nu}{\rho_s}\Delta w'_\infty - \frac{\tilde{\nu}}{\rho_s}\nabla' \operatorname{div} w_\infty + \tilde{B}' u_\infty + \tilde{C}'_0 u_\infty \\ &\quad + \tilde{A}'\big(\sigma_1 u_1^{(0)}\big) + \alpha_0\nabla'\sigma_1 + \tilde{B}'\big(\sigma_1 u_1^{(0)}\big) - \big\langle Q_0\tilde{B}\big(\sigma_1 u^{(0)} + u_\infty\big)\big\rangle_1 w^{(0),1}\boldsymbol{e}'_1 \\ &= \boldsymbol{f}'_\infty. \end{aligned}$$

This gives

$$-\Delta T_{j,k} w'_\infty = -\frac{\alpha_0}{\nu}\rho_s\nabla' T_{j,k}\sigma_1 + T_{j,k}\tilde{\boldsymbol{f}}'_\infty + \frac{\rho_s}{\nu}T_{j,k}\boldsymbol{f}'_\infty, \tag{5.37}$$

where

$$\begin{aligned} \tilde{\boldsymbol{f}}'_\infty = -\frac{\rho_s}{\nu}\bigg\{ &\partial_t w'_\infty - \frac{\tilde{\nu}}{\rho_s}\nabla' \operatorname{div} w_\infty + \tilde{B}' u_\infty + \tilde{C}'_0 u_\infty \\ &+ \tilde{A}'\big(\sigma_1 u_1^{(0)}\big) + \tilde{B}'\big(\sigma_1 u_1^{(0)}\big) - \big\langle Q_0\tilde{B}\big(\sigma_1 u^{(0)} + u_\infty\big)\big\rangle_1 w^{(0),1}\boldsymbol{e}'_1 \bigg\}. \end{aligned}$$

We take the Fourier transform of (5.37) to obtain

$$\big(|\xi'|^2 + \mathscr{A}\big)\widehat{T_{j,k}w'_\infty} = -\frac{\alpha_0}{\nu}\big(i\xi'\widehat{T_{j,k}\sigma_1}\big)\rho_s + \mathscr{F}\big(T_{j,k}\tilde{\boldsymbol{f}}'_\infty\big) + \frac{\rho_s}{\nu}\mathscr{F}\big(T_{j,k}\boldsymbol{f}'_\infty\big). \tag{5.38}$$

Here $\mathscr{A}$ is an operator on $L^2(0,1)$ with domain $D(\mathscr{A}) = H^2(0,1) \cap H_0^1(0,1)$ defined by

$$\mathscr{A} v = -\partial_{x_n}^2 v \quad \text{for } v \in D(\mathscr{A}).$$

It follows from (5.38) that

$$\widehat{T_{j,k} w'_\infty} = -\frac{\alpha_0}{\nu}\big(i\xi' \widehat{T_{j,k}\sigma_1}\big)\big(|\xi'|^2 + \mathscr{A}\big)^{-1}\rho_s + \big(|\xi'|^2 + \mathscr{A}\big)^{-1}\tilde{\boldsymbol{h}}'_{j,k} \tag{5.39}$$

for $|\xi'| \leqslant 1$, where

$$\tilde{\boldsymbol{h}}'_{j,k} = \mathscr{F}\big(T_{j,k}\tilde{\boldsymbol{f}}'_\infty\big) + \frac{\rho_s}{\nu}\mathscr{F}\big(T_{j,k}\boldsymbol{f}'_\infty\big).$$

Substituting (5.39) into (5.34), we obtain

$$\begin{aligned}
&\partial_t(\widehat{T_{j,k}\sigma_1}) + \frac{\alpha_0\gamma^2}{\nu}\big\langle \rho_s\big(|\xi'|^2 + \mathscr{A}\big)^{-1}\rho_s\big\rangle |\xi'|^2 \widehat{T_{j,k}\sigma_1} + \big\langle Q_0 \tilde{B}_{\xi'}\big(\widehat{T_{j,k}\sigma_1} u^{(0)}\big)\big\rangle \\
&\quad = \langle Q_0 \widehat{T_{j,k}\boldsymbol{F}}\rangle - \big\langle i\xi_1 v_s^1 \widehat{T_{j,k}\phi_\infty}\big\rangle - \gamma^2 \big\langle \rho_s i\xi' \cdot \big(|\xi'|^2 + \mathscr{A}\big)^{-1}\tilde{\boldsymbol{h}}'_{j,k}\big\rangle
\end{aligned} \tag{5.40}$$

for $|\xi'| \leqslant 1$.

We multiply (5.40) by $\overline{\widehat{T_{j,k}\sigma_1}}$. Here $\bar{z}$ denotes the complex conjugate of $z$. Since

$$\big\langle Q_0 \tilde{B}_{\xi'} u^{(0)}\big\rangle = i\xi_1 \big\langle v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0),1}\big\rangle \in i\mathbf{R},$$

and $\operatorname{supp}(\widehat{T_{j,k}\sigma_1}) \subset \{|\xi'| \leqslant 1\}$, integrating the resulting equation in $\xi'$ and taking its real part, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{T_{j,k}\sigma_1}\|_2^2 + \frac{\alpha_0\gamma^2}{\nu}\Big\|\sqrt{\big\langle \rho_s\big(|\xi'|^2 + \mathscr{A}\big)^{-1}\rho_s\big\rangle |\xi'|^2}\,\widehat{T_{j,k}\sigma_1}\Big\|_2^2 \\
&\quad = \operatorname{Re}\big\{(Q_0 \widehat{T_{j,k}\boldsymbol{F}}, \widehat{T_{j,k}\sigma_1}) - \big(i\xi_1 v_s^1 \widehat{T_{j,k}\phi_\infty}, \widehat{T_{j,k}\sigma_1}\big) \\
&\qquad - \gamma^2\big(\rho_s i\xi' \cdot \big(|\xi'|^2 + \mathscr{A}\big)^{-1}\widehat{\boldsymbol{h}'_{j,k}}, \widehat{T_{j,k}\sigma_1}\big)\big\}.
\end{aligned} \tag{5.41}$$

We here note that there is a constant $c > 0$ such that

$$\big\langle \rho_s\big(|\xi'|^2 + \mathscr{A}\big)^{-1}\rho_s\big\rangle = \big|\big(|\xi'|^2 + \mathscr{A}\big)^{-1/2}\rho_s\big|_2^2 \geqslant c \quad \text{for } |\xi'| \leqslant 1.$$

This implies

$$\frac{\alpha_0\gamma^2}{\nu}\Big\|\sqrt{\big\langle \rho_s\big(|\xi'|^2 + \mathscr{A}\big)^{-1}\rho_s\big\rangle |\xi'|^2}\,\widehat{T_{j,k}\sigma_1}\Big\|_2^2 \geqslant \frac{\alpha_1\gamma^2}{\nu}\big\|i\xi' \widehat{T_{j,k}\sigma_1}\big\|_2^2 \tag{5.42}$$

for some constant $\alpha_1 > 0$. As for the second term on the right of (5.41), since $\hat{\chi}_1\langle \widehat{T_{j,k}\phi_\infty}\rangle = 0$ and $\hat{\chi}_1\hat{\sigma}_1 = \hat{\sigma}_1$, we see that for $0 \leqslant p \leqslant k$,

$$\begin{aligned}
\left|\left(i\xi_1 v_s^1 \widehat{T_{j,k}\phi_\infty}, \widehat{T_{j,k}\sigma_1}\right)\right| &= \left|\left(v_s^1 \hat{\chi}_1 \widehat{T_{j,k}\phi_\infty}, i\xi_1 \widehat{T_{j,k}\sigma_1}\right)\right| \\
&\leqslant C\|\hat{\chi}_1 \widehat{T_{j,p}\phi_\infty}\|_2 \|i\xi_1 \widehat{T_{j,k}\sigma_1}\|_2 \\
&\leqslant C\|\hat{\chi}_1 \partial_{x_n} \widehat{T_{j,p}\phi_\infty}\|_2 \|i\xi_1 \widehat{T_{j,k}\sigma_1}\|_2 \\
&\leqslant \frac{\alpha_1\gamma^2}{4\nu}\|i\xi_1 \widehat{T_{j,k}\sigma_1}\|_2^2 + \frac{C\nu}{\gamma^2}\|\partial_{x_n}\widehat{T_{j,p}\phi_\infty}\|_2^2. \qquad (5.43)
\end{aligned}$$

As for the last term on the right of (5.41), we have

$$\begin{aligned}
&\left|\gamma^2\left(\rho_s i\xi' \cdot \left(|\xi'|^2 + \mathscr{A}\right)^{-1}\widehat{\boldsymbol{h}'_{j,k}}, \widehat{T_{j,k}\sigma_1}\right)\right| \\
&\quad = \left| -\gamma^2\left(\rho_s\left(|\xi'|^2 + \mathscr{A}\right)^{-1}\widehat{T_{j,k}\tilde{\boldsymbol{f}}'_\infty}, i\xi'\widehat{T_{j,k}\sigma_1}\right) \right. \\
&\qquad \left. + \frac{\gamma^2}{\nu}\left(\rho_s i\xi' \cdot \left(|\xi'|^2 + \mathscr{A}\right)^{-1}\left(\rho_s\widehat{T_{j,k}\boldsymbol{f}'_\infty}\right), \widehat{T_{j,k}\sigma_1}\right)\right|, \\
&\quad \leqslant \frac{\alpha_1\gamma^2}{4\nu}\|i\xi_1\widehat{T_{j,k}\sigma_1}\|_2^2 + C\nu\gamma^2\left\|\rho_s\hat{\chi}_1\left(|\xi'|^2 + \mathscr{A}\right)^{-1}\widehat{T_{j,k}\tilde{\boldsymbol{f}}'_\infty}\right\|_2^2 \\
&\qquad + \left|\frac{\gamma^2}{\nu}\left(\rho_s i\xi' \cdot \left(|\xi'|^2 + \mathscr{A}\right)^{-1}\left(\rho_s\widehat{T_{j,k}\boldsymbol{f}'_\infty}\right), \widehat{T_{j,k}\sigma_1}\right)\right| \\
&\quad \leqslant \frac{\alpha_1\gamma^2}{4\nu}\|i\xi_1\widehat{T_{j,k}\sigma_1}\|_2^2 + C\frac{\gamma^2}{\nu}\left\{\|\partial_t\widehat{T_{j,k}w_\infty}\|_2^2 + \tilde{\nu}^2\|\mathrm{div}\,\widehat{T_{j,k}w_\infty}\|_2^2\right. \\
&\qquad + \left\|i\xi'\widehat{T_{j,k}w_\infty}\right\|_2^2 + \left(1 + \frac{1}{\gamma^4} + \frac{\nu^2}{\gamma^4}\right)\|\hat{\chi}_1\partial_{x_n}\widehat{T_{j,p}\phi_\infty}\|_2^2 \\
&\qquad \left. + \left(\frac{(\nu+\tilde{\nu})^2}{\gamma^4} + \frac{1}{\gamma^4}\right)\left\|i\xi'\widehat{T_{j,k}\sigma_1}\right\|_2^2\right\} \\
&\qquad + \left|\frac{\gamma^2}{\nu}\left(\rho_s i\xi' \cdot \left(|\xi'|^2 + \mathscr{A}\right)^{-1}\left(\rho_s\widehat{T_{j,k}\boldsymbol{f}'_\infty}\right), \widehat{T_{j,k}\sigma_1}\right)\right|. \qquad (5.44)
\end{aligned}$$

The desired estimate (5.33) now follows from (5.41)–(5.44) through the Plancherel theorem, provided that $\nu \geqslant \nu_0$ and $\frac{\gamma^2}{\nu+\tilde{\nu}} \geqslant \gamma_0^2$ for some $\nu_0 > 0$ and $\gamma_0 > 0$. This completes the proof. □

We next apply estimates for the Stokes system to obtain the estimates for higher order derivatives.

**Proposition 5.7.** *If $\nu \geqslant \nu_0$, $\gamma^2 \geqslant 1$ and $\tilde{\omega}$ is sufficiently small, then there holds the following estimate for $0 \leqslant 2j + k + l \leqslant m - 1$:*

$$\begin{aligned}
&\frac{\nu^2}{\nu+\tilde{\nu}}\left\|\partial_x^{l+2}T_{j,k}w_\infty\right\|_2^2 + \frac{1}{\nu+\tilde{\nu}}\left\|\partial_x^{l+1}T_{j,k}\phi_\infty\right\|_2^2 \\
&\quad \leqslant CR_{j,k,l}^{(6)} + C\left\{\left(\frac{\nu+\tilde{\nu}}{\gamma^4} + \frac{1}{\nu+\tilde{\nu}}\right)\|\partial_{x'}T_{j,k}\sigma_1\|_2^2\right. \\
&\qquad + \left(\frac{\nu+\tilde{\nu}}{\gamma^4} + \frac{\tilde{\omega}^2}{\nu+\tilde{\nu}}\right)\|T_{j,k}\phi_\infty\|_{H^l}^2 + \frac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\dot{\phi}_\infty\|_{H^{l+1}}^2 + \frac{1}{\nu+\tilde{\nu}}\|\partial_t T_{j,k}w_\infty\|_{H^l}^2 \\
&\qquad \left. + \left(\frac{1}{\nu+\tilde{\nu}} + (\nu+\tilde{\nu})\tilde{\omega}^2\right)\|T_{j,k}w_\infty\|_{H^{l+1}}^2 + D^{(0)}[T_{j,k}w_\infty]\right\}, \qquad (5.45)
\end{aligned}$$

*where*

$$R^{(6)}_{j,k,l} = \frac{\nu+\tilde{\nu}}{\gamma^4}\big\|T_{j,k}\tilde{f}^0_\infty\big\|^2_{H^{l+1}} + \frac{1}{\nu+\tilde{\nu}}\|T_{j,k}\boldsymbol{f}_\infty\|^2_{H^l}.$$

**Proof.** To prove Proposition 5.7, we employ estimates for the Stokes system. Let ${}^\top(\tilde{\phi},\tilde{w})$ be the solution of the Stokes system

$$\operatorname{div}\tilde{w} = F \quad \text{in } \Omega,$$

$$-\Delta\tilde{w} + \nabla\tilde{\phi} = G \quad \text{in } \Omega,$$

$$\tilde{w}|_{\partial\Omega} = 0.$$

Then for any $l \in \mathbf{Z}$, $l \geqslant 0$, there exists a constant $C > 0$ such that

$$\big\|\partial_x^{l+2}\tilde{w}\big\|_2^2 + \big\|\partial_x^{l+1}\tilde{\phi}\big\|_2^2 \leqslant C\big\{\|F\|^2_{H^{l+1}} + \|G\|^2_{H^l} + \|\partial_x\tilde{w}\|_2^2\big\}. \tag{5.46}$$

(See, e.g., [1], [2, Appendix].)

By (4.11), we have

$$\operatorname{div}(T_{j,k}w_\infty) = F_{j,k} \quad \text{in } \Omega,$$

$$-\Delta(T_{j,k}w_\infty) + \nabla\left(\frac{\tilde{P}'(\rho_s)}{\nu\gamma^2}T_{j,k}\phi_\infty\right) = G_{j,k} \quad \text{in } \Omega,$$

$$T_{j,k}w_\infty|_{\partial\Omega} = 0,$$

where

$$F_{j,k} = \frac{1}{\gamma^2\rho_s}T_{j,k}\tilde{f}^0_\infty - \frac{\partial_{x_n}\rho_s}{\rho_s}T_{j,k}w^n_\infty - \frac{1}{\gamma^2\rho_s}\big\{T_{j,k}\dot{\phi}_\infty + \big(v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1}\big)\partial_{x_1}T_{j,k}\sigma_1$$
$$- \big\langle Q_0\tilde{B}\big(T_{j,k}\sigma_1 u^{(0)} + T_{j,k}u_\infty\big)\big\rangle_1\phi^{(0)}\big\},$$

$$G_{j,k} = \frac{\rho_s}{\nu}T_{j,k}\boldsymbol{f}_\infty + \frac{\tilde{P}'(\rho_s)\nabla\rho_s}{\nu\gamma^2\rho_s}T_{j,k}\phi_\infty$$
$$- \frac{\rho_s}{\nu}\left\{\partial_t T_{j,k}w_\infty - \frac{\tilde{\nu}}{\rho_s}\nabla F_{j,k} + v_s^1\partial_{x_1}T_{j,k}w_\infty + \frac{\nu\Delta v_s^1}{\gamma^2\rho_s^2}T_{j,k}\phi_\infty\boldsymbol{e}_1\right.$$
$$+ \big(\partial_{x_n}v_s^1\big)T_{j,k}w^n_\infty\boldsymbol{e}_1 - \frac{\nu}{\rho_s}w^{(0)}\Delta' T_{j,k}\sigma_1 - \frac{\tilde{\nu}}{\rho_s}\nabla\big(w^{(0),1}\partial_{x_1}T_{j,k}\sigma_1\big)$$
$$\left. + \alpha_0\nabla T_{j,k}\sigma_1 + v_s^1 w^{(0)}\partial_{x_1}T_{j,k}\sigma_1 - \big\langle Q_0\tilde{B}\big(T_{j,k}\sigma_1 u^{(0)} + T_{j,k}u_\infty\big)\big\rangle_1 w^{(0)}\right\}.$$

Here and in what follows we denote $w^{(0)} = w^{(0),1}\boldsymbol{e}_1$.

We now apply (5.46) to obtain

$$\big\|\partial_x^{l+2}T_{j,k}w_\infty\big\|_2^2 + \frac{1}{\nu^2}\left\|\partial_x^{l+1}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\phi_\infty\right)\right\|_2^2$$
$$\leqslant C\big\{\|F_{j,k}\|^2_{H^{l+1}} + \|G_{j,k}\|^2_{H^l} + \|\partial_x T_{j,k}w_\infty\|_2^2\big\}. \tag{5.47}$$

Since

$$\begin{aligned}\left\|\partial_x^{l+1}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\phi_\infty\right)\right\|_2^2 &\geqslant \left\|\frac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_x^{l+1}T_{j,k}\phi_\infty\right\|_2^2 - \left\|\left[\partial_x^{l+1}, \frac{\tilde{P}'(\rho_s)}{\gamma^2}\right]T_{j,k}\phi_\infty\right\|_2^2 \\ &\geqslant \left\|\partial_x^{l+1}T_{j,k}\phi_\infty\right\|_2^2 - C\tilde{\omega}^2\|T_{j,k}\phi_\infty\|_{H^{l+1}}^2 \\ &\geqslant \frac{1}{2}\left\|\partial_x^{l+1}T_{j,k}\phi_\infty\right\|_2^2 - C\tilde{\omega}^2\|T_{j,k}\phi_\infty\|_{H^l}^2,\end{aligned}$$

the desired estimate (5.45) now follows from $\frac{\nu^2}{\nu+\tilde{\nu}} \times$ (5.47) by using Lemmas 4.3–4.5. This completes the proof. □

We finally estimate the time derivatives of $\sigma_1$ and $\phi_\infty$.

**Proposition 5.8.**

(i) *If* $0 \leqslant 2j+k \leqslant m-1$, *then there holds the following estimate*:

$$\|\partial_t T_{j,k}\sigma_1\|_2^2 \leqslant C\left\{R_{j,k}^{(7)} + \|\partial_{x'}T_{j,k}\sigma_1\|_2^2 + \|\partial_{x'}T_{j,k}\phi_\infty\|_2^2 + \gamma^4\|\partial_{x'}T_{j,k}w_\infty\|_2^2\right\}. \tag{5.48}$$

*Here* $R_{j,k}^{(7)} = \|\langle Q_0 T_{j,k}\boldsymbol{F}\rangle_1\|_2^2$.

(ii) *If* $0 \leqslant 2j \leqslant m-1$ *and* $\tilde{\omega}^2 \leqslant 1$, *then there holds the following estimate*:

$$\left\|\partial_t^{j+1}\phi_\infty\right\|_2^2 \leqslant C\left\{R_j^{(8)} + \left\|\partial_{x'}\partial_t^j\phi_\infty\right\|_2^2 + \gamma^4\left\|\partial_x\partial_t^j w_\infty\right\|_2^2 + \left\|\partial_{x_1}\partial_t^j\sigma_1\right\|_2^2\right\}. \tag{5.49}$$

*Here* $R_j^{(8)} = \|\partial_t^j Q_0\boldsymbol{F}_\infty\|_2^2$.

**Proof.** We see from (4.10)

$$\partial_t T_{j,k}\sigma_1 = \langle Q_0 T_{j,k}\boldsymbol{F}\rangle_1 - \left\langle Q_0\tilde{B}\left(T_{j,k}\sigma_1 u^{(0)} + T_{j,k}u_\infty\right)\right\rangle_1,$$

which gives the estimate in (i).

As for (ii), we see from (4.11)

$$\begin{aligned}\partial_t^{j+1}\phi_\infty = \partial_t^j f_\infty^0 - \big\{&v_s^1\partial_{x_1}\partial_t^j\phi_\infty + \gamma^2\operatorname{div}\left(\rho_s\partial_t^j w_\infty\right) \\ &+ \left(v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1}\right)\partial_{x_1}\partial_t^j\sigma_1 - \left\langle Q_0\tilde{B}\left(\partial_t^j\sigma_1 u^{(0)} + \partial_t^j u_\infty\right)\right\rangle_1\phi^{(0)}\big\},\end{aligned}$$

which gives the estimate in (ii). This completes the proof. □

### 5.2. $H^m$-energy bound: induction argument

In this subsection we establish $H^m$-energy bound by using Propositions 5.2–5.8.

We will show the following estimate.

**Proposition 5.9.** *There are positive constants* $\nu_0$, $\gamma_0$ *and* $\tilde{\omega}$ *such that if* $\nu \geqslant \nu_0$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geqslant \gamma_0^2$ *and* $\|\tilde{g}\|_{C^m[0,1]} \leqslant \tilde{\omega}$, *then*

$$
\begin{aligned}
&[\![u(t)]\!]_m^2 + \int_0^t \left( |||D\sigma_1|||_{m-1}^2 + |||D\phi_\infty|||_{m-1}^2 + |||Dw_\infty|||_m^2 \right) d\tau \\
&\quad \leqslant C \left\{ \|u_0\|_{H^m}^2 + [\![u(t)]\!]_m^4 + \sum_{j=0}^{8} \int_0^t R^{(j)}(\tau)\, d\tau \right\},
\end{aligned}
$$

*where*

$$
R^{(1)} = \sum_{2j+k \leqslant m} R_{j,k}^{(1)}, \qquad R^{(p)} = \sum_{2j+k \leqslant m-1} R_{j,k}^{(p)} \quad (p = 2, 5, 7), \qquad R^{(4)} = \sum_{\substack{2j+k \leqslant m \\ 2j \neq m}} R_{j,k}^{(4)},
$$

$$
R^{(p)} = \sum_{2j+k+l \leqslant m-1} R_{j,k,l}^{(p)} \quad (p = 3, 6), \qquad R^{(8)} = \sum_{2j \leqslant m-1} R_j^{(8)}.
$$

**Proof.** We set

$$
\tilde{E}^{(0)}(t) = \sum_{\substack{2j+k \leqslant m \\ 2j \neq m}} E^{(0)}\left[T_{j,k} u(t)\right], \qquad E^{(1)}(t) = \sum_{2j+k \leqslant m-1} E^{(1)}\left[T_{j,k} u(t)\right],
$$

$$
\tilde{D}^{(0)}(t) = \sum_{\substack{2j+k \leqslant m \\ 2j \neq m}} D^{(0)}\left[T_{j,k} w_\infty(t)\right]
$$

and

$$
D^{(1)}(t) = \sum_{2j+k \leqslant m-1} \left( \frac{2b_1\gamma^2}{\nu(\nu+\tilde{\nu})} D^{(0)}\left[T_{j,k} w_\infty(t)\right] + \frac{1}{\nu+\tilde{\nu}} \left\| \sqrt{\rho_s}\, \partial_t T_{j,k} w_\infty(t) \right\|_2^2 \right).
$$

By (5.1) with $2j + k \leqslant m$, $2j \neq m$, and $\frac{1}{\nu+\tilde{\nu}} \times$ (5.11), we have

$$
\begin{aligned}
&\frac{d}{dt}\left( \tilde{E}^{(0)}(t) + \frac{1}{\nu+\tilde{\nu}} E^{(1)}(t) \right) + \frac{3}{2}\tilde{D}^{(0)}(t) + D^{(1)}(t) \\
&\quad \leqslant C\left(R^{(1)} + R^{(2)}\right) + C \sum_{\substack{2j+k \leqslant m \\ 2j \neq m}} \left\{ \left( \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) \|\partial_{x'} T_{j,k}\sigma_1\|_2^2 + \left( \frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) \|\tilde{T}_{j,k}\phi_\infty\|_2^2 \right\} \\
&\qquad + C \sum_{2j+k \leqslant m-1} \left\{ \left( \frac{1}{\nu(\nu+\tilde{\nu})} + \frac{1}{\gamma^2} \right) \|\partial_{x'} T_{j,k}\sigma_1\|_2^2 \right. \\
&\qquad \left. + \left( \frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) \frac{1}{\nu+\tilde{\nu}} \|\tilde{T}_{j,k}\phi_\infty\|_2^2 + \frac{1}{\gamma^2(\nu+\tilde{\nu})} \|\partial_{x'} T_{j,k}\phi_\infty\|_2^2 \right\}. \qquad (5.50)
\end{aligned}
$$

We set

$$
E^{(2)}(t) = \sum_{2j+k \leqslant m-1} \frac{1}{\gamma^2} \left\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}}\, \partial_{x_n} T_{j,k}\phi_\infty(t) \right\|_2^2,
$$

$$D^{(2)}(t) = \sum_{2j+k\leqslant m-1}\left(\frac{1}{\nu+\tilde{\nu}}\left\|\frac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}T_{j,k}\phi_\infty(t)\right\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^4}\|T_{j,k}\dot{\phi}_\infty\|_{H^1}^2\right).$$

By (5.30) with $l=0$ and (5.31) with $2j+k\leqslant m$, $2j\neq m$, we have

$$\begin{aligned}\frac{d}{dt}E^{(2)}(t)+D^{(2)}(t) &\leqslant C\big(R^{(3)}+R^{(4)}\big)+C\sum_{2j+k\leqslant m-1}\frac{\nu+\tilde{\nu}}{\gamma^4}\|K_{j,k,0}\|_2^2\\ &\quad + C\sum_{\substack{2j+k\leqslant m\\ 2j\neq m}}\bigg\{D^{(0)}[T_{j,k}w_\infty]+(\nu+\tilde{\nu})\tilde{\omega}^2\|T_{j,k}w_\infty\|_2^2\\ &\quad + \frac{\nu+\tilde{\nu}}{\gamma^4}\|\partial_{x_1}T_{j,k}\sigma_1\|_2^2+\frac{\nu+\tilde{\nu}}{\gamma^4}\|\tilde{T}_{j,k}\phi_\infty\|_2^2\bigg\}. \end{aligned} \tag{5.51}$$

Let $b_2$ be a positive number with $b_2\leqslant 1$ to be determined later. It then follows from (5.50) and $b_2\times$ (5.51) that

$$\begin{aligned}&\frac{d}{dt}\left(\tilde{E}^{(0)}(t)+\frac{1}{\nu+\tilde{\nu}}E^{(1)}(t)+b_2E^{(2)}(t)\right)+\frac{3}{2}\tilde{D}^{(0)}(t)+D^{(1)}(t)+b_2D^{(2)}(t)\\ &\leqslant C\sum_{j=1}^{4}R^{(j)}+C\sum_{\substack{2j+k\leqslant m\\ 2j\neq m}}\left(\frac{1}{\gamma^2}+\frac{\nu+\tilde{\nu}}{\gamma^4}\right)\big(\|\partial_{x'}T_{j,k}\sigma_1\|_2^2+\|\tilde{T}_{j,k}\phi_\infty\|_2^2\big)\\ &\quad + C\sum_{2j+k\leqslant m-1}\bigg\{\left(\frac{1}{\nu(\nu+\tilde{\nu})}+\frac{1}{\gamma^2}\right)\|\partial_{x'}T_{j,k}\sigma_1\|_2^2+\left(\frac{1}{\nu}+\frac{1}{\gamma^2}+\frac{\nu^2}{\gamma^4}\right)\frac{1}{\nu+\tilde{\nu}}\|\tilde{T}_{j,k}\phi_\infty\|_2^2\\ &\quad + \frac{1}{\gamma^2(\nu+\tilde{\nu})}\|\partial_{x'}T_{j,k}\phi_\infty\|_2^2\bigg\}+Cb_2\bigg\{\sum_{2j+k\leqslant m-1}\frac{\nu+\tilde{\nu}}{\gamma^4}\|K_{j,k,0}\|_2^2\\ &\quad + \sum_{\substack{2j+k\leqslant m\\ 2j\neq m}}\big(D^{(0)}[T_{j,k}w_\infty]+(\nu+\tilde{\nu})\tilde{\omega}^2\|T_{j,k}w_\infty\|_2^2\big)\bigg\}. \end{aligned}\tag{5.52}$$

We take $b_2>0$ so small that $2Cb_2(1+\frac{1}{\nu(\nu+\tilde{\nu})})\leqslant\frac{1}{2}$. Furthermore, $\tilde{\omega}$ is taken so small that $2Cb_2(\nu+\tilde{\nu})\tilde{\omega}^2\leqslant\frac{1}{2}$. Then the terms $\|T_{j,k+1}\partial_x w_\infty\|_2^2$, $\|\sqrt{\rho_s}\partial_t T_{j,k}w_\infty\|_2^2$ and $\|T_{j,k}\partial_x w_\infty\|_2^2$ in $\frac{\nu+\tilde{\nu}}{\gamma^4}\|K_{j,k,0}\|_2^2$ on the right of (5.52) are absorbed in the left. We thus arrive at

$$\begin{aligned}&\frac{d}{dt}\left(\tilde{E}^{(0)}(t)+\frac{1}{\nu+\tilde{\nu}}E^{(1)}(t)+b_2E^{(2)}(t)\right)+\frac{3}{2}\tilde{D}^{(0)}(t)+\frac{3}{4}D^{(1)}(t)+b_2D^{(2)}(t)\\ &\quad\leqslant C\sum_{j=1}^{4}R^{(j)}+C\sum_{\substack{2j+k\leqslant m\\ 2j\neq m}}\left(\frac{1}{\nu(\nu+\tilde{\nu})}+\frac{1}{\gamma^2}+\frac{\nu+\tilde{\nu}}{\gamma^4}\right)\big(\|\partial_{x'}T_{j,k}\sigma_1\|_2^2+\|\tilde{T}_{j,k}\phi_\infty\|_2^2\big)\\ &\quad\quad + C\sum_{2j+k\leqslant m-1}\left(\frac{1}{\gamma^2(\nu+\tilde{\nu})}+\frac{\nu+\tilde{\nu}}{\gamma^4}\right)\|\partial_x T_{j,k}\phi_\infty\|_2^2, \end{aligned}\tag{5.53}$$

provided that $\nu+\tilde{\nu}\geqslant 1$.

We set

$$E^{(3)}(t) = \sum_{2j+k \leqslant m-1} \frac{\nu}{\gamma^2(\nu+\tilde{\nu})} \|T_{j,k}\sigma_1\|_2^2$$

and

$$D^{(3)}(t) = \sum_{2j+k \leqslant m-1} \frac{\alpha_1}{\nu+\tilde{\nu}} \|\partial_{x'} T_{j,k}\sigma_1\|_2^2.$$

Let $b_3$ be a positive number to be determined later. We add $b_3 \times \sum_{2j+k \leqslant m-1}(5.33)$ to (5.53). It then follows

$$\begin{aligned}
&\frac{d}{dt}\left(\tilde{E}^{(0)}(t) + \frac{1}{\nu+\tilde{\nu}} E^{(1)}(t) + b_2 E^{(2)}(t) + b_3 E^{(3)}(t)\right) \\
&\quad + \frac{3}{2}\tilde{D}^{(0)}(t) + \frac{3}{4} D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t) \\
&\quad \leqslant C\sum_{j=1}^{5} R^{(j)} + Cb_3\big\{D^{(1)}(t) + \tilde{D}^{(0)}(t) + D^{(2)}(t)\big\} \\
&\quad + C\left(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\gamma^2} + \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\right) D^{(3)}(t) \\
&\quad + C\sum_{\substack{2j+k \leqslant m \\ 2j \neq m}} \left(\frac{1}{\nu(\nu+\tilde{\nu})} + \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4}\right) \|\tilde{T}_{j,k}\phi_\infty\|_2^2 \\
&\quad + C\sum_{2j+k \leqslant m-1} \left(\frac{1}{\gamma^2(\nu+\tilde{\nu})} + \frac{\nu+\tilde{\nu}}{\gamma^4}\right) \|\partial_x T_{j,k}\phi_\infty\|_2^2.
\end{aligned} \tag{5.54}$$

We take $b_3 > 0$ so small that $Cb_3 \leqslant \frac{1}{4}$. We then take $\nu$, $\tilde{\nu}$ and $\gamma^2$ so large that $C(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\gamma^2} + \frac{(\nu+\tilde{\nu})^2}{\gamma^4}) \leqslant \frac{b_3}{2}$. The terms $\tilde{D}^{(0)}(t)$, $D^{(j)}(t)$ $(j = 1, 2, 3)$ on the right of (5.54) are absorbed in the left. We thus arrive at

$$\begin{aligned}
&\frac{d}{dt}\tilde{E}^{(4)}(t) + \frac{1}{2}\big(\tilde{D}^{(0)}(t) + D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t)\big) \\
&\quad \leqslant C\sum_{\substack{2j+k \leqslant m \\ 2j \neq m}} \left(\frac{1}{\nu(\nu+\tilde{\nu})} + \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4}\right) \|\tilde{T}_{j,k}\phi_\infty\|_2^2 \\
&\quad + C\sum_{2j+k \leqslant m-1} \left(\frac{1}{\gamma^2(\nu+\tilde{\nu})} + \frac{\nu+\tilde{\nu}}{\gamma^4}\right) \|\partial_x T_{j,k}\phi_\infty\|_2^2.
\end{aligned} \tag{5.55}$$

Here

$$\tilde{E}^{(4)}(t) = \tilde{E}^{(0)}(t) + \frac{1}{\nu+\tilde{\nu}} E^{(1)}(t) + b_2 E^{(2)}(t) + b_3 E^{(3)}(t).$$

Let $b_4$ be a positive number to be determined later and set

$$\tilde{D}^{(4)}(t) = \frac{1}{2}\big(\tilde{D}^{(0)}(t) + D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t)\big) + b_4 \sum_{2j+k\leqslant m-1} \left(\frac{\nu^2}{\nu+\tilde{\nu}}\big\|\partial_x^2 T_{j,k} w_\infty\big\|_2^2 + \frac{1}{\nu+\tilde{\nu}}\|\partial_x T_{j,k}\phi_\infty\|_2^2\right).$$

We may assume that $\nu \geqslant 1$ and $(\nu+\tilde{\nu})\tilde{\omega}^2 \leqslant 1$. It then follows from $b_4 \times \{(5.45)$ with $l=0\}$ and (5.55) that

$$\begin{aligned}
\frac{d}{dt}\tilde{E}^{(4)}(t) + \tilde{D}^{(4)}(t) &\leqslant C\sum_{j=1}^{6} R^{(j)}(t) + Cb_4\left\{\left(1 + \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\right)D^{(3)}(t) + D^{(2)}(t) + D^{(1)}(t) + \tilde{D}^{(0)}(t)\right\} \\
&\quad + Cb_4 \sum_{2j+k\leqslant m-1}\left(\frac{\nu+\tilde{\nu}}{\gamma^4} + \frac{\tilde{\omega}^2}{\nu+\tilde{\nu}}\right)\|T_{j,k}\phi_\infty\|_2^2 \\
&\quad + C\sum_{\substack{2j+k\leqslant m \\ 2j\neq m}}\left(\frac{1}{\nu(\nu+\tilde{\nu})} + \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4}\right)\|\tilde{T}_{j,k}\phi_\infty\|_2^2 \\
&\quad + C\sum_{2j+k\leqslant m-1}\left(\frac{1}{\gamma^2(\nu+\tilde{\nu})} + \frac{\nu+\tilde{\nu}}{\gamma^4}\right)\|\partial_x T_{j,k}\phi_\infty\|_2^2. \qquad (5.56)
\end{aligned}$$

We may also assume that $\frac{\nu+\tilde{\nu}}{\gamma^2} \leqslant 1$. We take $b_4 > 0$ so small that $Cb_4 \leqslant \frac{1}{4}$. We also take $\nu$, $\tilde{\nu}$ and $\gamma^2$ so large that $Cb_4\frac{(\nu+\tilde{\nu})^2}{\gamma^4} \leqslant \frac{1}{8}$ and $C(\frac{\nu+\tilde{\nu}}{\gamma^2} + \frac{(\nu+\tilde{\nu})^2}{\gamma^4} + \frac{1}{\nu}) \leqslant \frac{1}{8}$; and then, take $\tilde{\omega}$ so small that $Cb_4\tilde{\omega}^2 \leqslant \frac{1}{4}$. It then follows that the term on the right of (5.56) except $R^{(j)}$ are absorbed in the left since $\|T_{j,k}\phi_\infty\|_2^2 \leqslant \|T_{j,k}\tilde{\phi}_\infty\|_2^2$ by Lemma 4.3(iv). We thus arrive at

$$\frac{d}{dt}\tilde{E}^{(4)}(t) + \tilde{D}^{(4)}(t) \leqslant C\sum_{j=1}^{6} R^{(j)}(t). \qquad (5.57)$$

By (5.48) and (5.49) we see

$$\begin{aligned}
&\sum_{2j\leqslant m-2}\big(\big\|\partial_t^{j+1}\sigma_1(t)\big\|_2^2 + \big\|\partial_t^{j+1}\phi_\infty(t)\big\|_2^2\big) \\
&\quad \leqslant C\big(R^{(7)} + R^{(8)}\big) + C\left\{(\nu+\tilde{\nu})\tilde{D}^{(4)}(t) + \frac{\gamma^4}{\nu}\tilde{D}^{(0)}(t)\right\}. \qquad (5.58)
\end{aligned}$$

We add $\frac{\nu}{\gamma^4}b_5 \times (5.58)$ to (5.57) to obtain

$$\begin{aligned}
&\frac{d}{dt}\tilde{E}^{(4)}(t) + \tilde{D}^{(4)}(t) + \frac{\nu b_5}{\gamma^4}\sum_{2j\leqslant m-2}\big(\big\|\partial_t^{j+1}\sigma_1(t)\big\|_2^2 + \big\|\partial_t^{j+1}\phi_\infty(t)\big\|_2^2\big) \\
&\quad \leqslant C\sum_{j=1}^{8} R^{(j)}(t) + Cb_5\left\{\frac{(\nu+\tilde{\nu})^2}{\gamma^4}\tilde{D}^{(4)}(t) + \tilde{D}^{(0)}(t)\right\}.
\end{aligned}$$

Take $b_5 > 0$ so small that $Cb_5(\frac{(\nu+\tilde{\nu})^2}{\gamma^4} + 1) \leqslant \frac{1}{2}$. We then obtain

$$\frac{d}{dt}\tilde{E}^{(4)}(t)+\frac{1}{2}\tilde{D}^{(4)}(t)+\frac{\nu b_5}{\gamma^4}\sum_{2j\leqslant m-2}\left(\left\|\partial_t^{j+1}\sigma_1(t)\right\|_2^2+\left\|\partial_t^{j+1}\phi_\infty(t)\right\|_2^2\right)\leqslant C\sum_{j=1}^{8}R^{(j)}(t). \tag{5.59}$$

Let $b_6$ be a positive number to be determined later. We set

$$E^{(4)}(t)=\tilde{E}^{(4)}(t)+\frac{b_6\nu}{\gamma^2}E^{(0)}\left[\partial_t^{\frac{m}{2}}u(t)\right]$$

and

$$D^{(4)}(t)=\tilde{D}^{(4)}(t)+\frac{b_6\nu}{\gamma^2}D^{(0)}\left[\partial_t^{\frac{m}{2}}w_\infty(t)\right]+\frac{\nu b_5}{\gamma^4}\sum_{2j\leqslant m-2}\left(\left\|\partial_t^{j+1}\sigma_1(t)\right\|_2^2+\left\|\partial_t^{j+1}\phi_\infty(t)\right\|_2^2\right).$$

Since $\|\partial_{x'}T_{j,0}\sigma_1\|_2^2\leqslant\|T_{j,0}\sigma_1\|_2^2$, it follows from (5.1) with $2j=m$ and (5.59) that

$$\begin{aligned}&\frac{d}{dt}E^{(4)}(t)+D^{(4)}(t)\\&\quad\leqslant C\sum_{j=1}^{8}R^{(j)}(t)+Cb_6\frac{\nu}{\gamma^4}\left(1+\frac{\nu+\tilde{\nu}}{\gamma^2}\right)\left(\left\|\partial_t^{\frac{m}{2}}\sigma_1\right\|_2^2+\left\|\partial_t^{\frac{m}{2}}\phi_\infty\right\|_2^2\right).\end{aligned}$$

Taking $b_6>0$ so small that $2Cb_6\leqslant\frac{b_5}{2}$, we see that

$$\frac{d}{dt}E^{(4)}(t)+D^{(4)}(t)\leqslant C\sum_{j=1}^{8}R^{(j)}(t). \tag{5.60}$$

We next bound higher order derivatives in $x_n$.

Let $b_7$ be a positive number to be determined later. For $1\leqslant l\leqslant m-1$, we set

$$E_l^{(4)}(t)=\sum_{2j+k\leqslant m-1-l}\frac{1}{\gamma^2}\left\|\sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty(t)\right\|_2^2$$

and

$$\begin{aligned}D_l^{(4)}(t)=\frac{1}{2}\sum_{2j+k\leqslant m-1-l}&\left(\frac{1}{\nu+\tilde{\nu}}\left\|\frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty(t)\right\|_2^2+\frac{c_0(\nu+\tilde{\nu})}{\gamma^4}\left\|T_{j,k}\partial_{x_n}^{l+1}\dot{\phi}_\infty(t)\right\|_2^2\right)\\&+\frac{b_7}{2}\sum_{2j+k\leqslant m-1-l}\left(\frac{\nu^2}{\nu+\tilde{\nu}}\left\|\partial_x^{l+2}T_{j,k}w_\infty(t)\right\|_2^2+\frac{1}{\nu+\tilde{\nu}}\left\|\partial_x^{l+1}T_{j,k}\phi_\infty(t)\right\|_2^2\right).\end{aligned}$$

We will show that there exists a number $b_8>0$ such that the following estimate holds for $1\leqslant l\leqslant m-1$:

$$\frac{d}{dt}\left(2^lE^{(4)}(t)+\sum_{p=1}^{l}2^{l+1-p}b_8^pE_p^{(4)}(t)\right)+\sum_{p=0}^{l}b_8^pD_p^{(4)}(t)\leqslant\left(\sum_{p=0}^{l}2^p\right)C_1\sum_{j=1}^{8}R^{(j)}(t), \tag{5.61}$$

where $D_0^{(4)}(t)=D^{(4)}(t)$.

By (5.30) and (5.45), we have

$$\frac{d}{dt}E_l^{(4)}(t)+2D_l^{(4)}(t)\leqslant C_1\left\{R^{(3)}(t)+R^{(5)}(t)\right\}+C_2F_l^{(1)}(t)+C_3b_7F_l^{(2)}(t)$$
$$+\frac{\nu+\tilde{\nu}}{\gamma^4}\sum_{2j+k\leqslant m-1-l}\left(C_2\left\|T_{j,k}\partial_{x'}^{l+1}\phi_\infty\right\|_2^2+C_3b_7\left\|T_{j,k}\partial_{x_n}^{l+1}\dot{\phi}_\infty\right\|_2^2\right),$$

where

$$F_l^{(1)}=\sum_{2j+k\leqslant m-1-l}\left\{\frac{\nu^2}{\nu+\tilde{\nu}}\left\|T_{j,k+1}\partial_{x'}^{l+1}w_\infty\right\|_2^2+\frac{1}{\nu+\tilde{\nu}}\left\|\sqrt{\rho_s}\partial_tT_{j,k}\partial_{x_n}^{l}w_\infty\right\|_2^2\right.$$
$$+\left(\frac{1}{\nu+\tilde{\nu}}+(\nu+\tilde{\nu})\tilde{\omega}^2\right)\|T_{j,k}w_\infty\|_{H^{l+1}}^2$$
$$\left.+\frac{\nu+\tilde{\nu}}{\gamma^4}\left(\sum_{p=0}^{l-1}\left\|T_{j,k}\partial_{x'}^{p+1}\phi_\infty\right\|_2^2+\|\partial_{x'}T_{j,k}\sigma_1\|_2^2\right)\right\},$$

$$F_l^{(2)}=\sum_{2j+k\leqslant m-1-l}\left\{\left(\frac{\nu+\tilde{\nu}}{\gamma^4}+\frac{\tilde{\omega}^2}{\nu+\tilde{\nu}}\right)\|T_{j,k}\phi_\infty\|_{H^l}^2+\frac{1}{\nu+\tilde{\nu}}\|\partial_tT_{j,k}w_\infty\|_{H^l}^2\right.$$
$$\left.+\left(\frac{1}{\nu+\tilde{\nu}}+(\nu+\tilde{\nu})\tilde{\omega}^2\right)\|T_{j,k}w_\infty\|_{H^{l+1}}^2+\frac{\nu+\tilde{\nu}}{\gamma^4}\left(\|T_{j,k+1}\dot{\phi}_\infty\|_{H^l}^2+\|T_{j,k}\dot{\phi}_\infty\|_{H^l}^2\right)\right\}$$
$$+\left(1+\frac{(\nu+\tilde{\nu})^2}{\gamma^4}\right)D_0^{(4)}(t).$$

We take $b_7>0$ so that $C_3b_7\leqslant\frac{c_0}{2}$, and then, take $\nu$, $\tilde{\nu}$ and $\gamma^2$ so that $C_2\frac{\nu+\tilde{\nu}}{\gamma^4}\leqslant\frac{1}{2(\nu+\tilde{\nu})}$. It then follows

$$\frac{d}{dt}E_l^{(4)}(t)+D_l^{(4)}(t)\leqslant C_1\left(R^{(4)}+R^{(6)}\right)+C_2F_l^{(1)}+C_3b_7F_l^{(2)}. \tag{5.62}$$

We here note that if $\nu$, $\tilde{\nu}$, $\gamma^2$ and $\tilde{\omega}$ are in the range restricted above, then there exists a constant $C_4>0$ such that

$$C_2F_l^{(1)}(t)+C_3b_7F^{(2)}(t)\leqslant C_4\sum_{p=0}^{l-1}D_p^{(4)}(t),$$

which, together with (5.62), yields

$$\frac{d}{dt}E_l^{(4)}(t)+D_l^{(4)}(t)\leqslant C_1\sum_{j=1}^{8}R^{(j)}+C_4\sum_{p=0}^{l-1}D_p^{(4)}(t). \tag{5.63}$$

Let $b_8$ be a positive number satisfying $C_4b_8\leqslant\frac{1}{2}$ and $b_8\leqslant\frac{1}{2}$. We now prove (5.61) by induction on $l$. Consider first $l=1$. By adding (5.60) to $b_8\times\{$(5.63) with $l=1\}$, we have

$$\frac{d}{dt}\left(E^{(4)}(t)+b_8E_1^{(4)}(t)\right)+\sum_{p=0}^{1}b_8^pD_p^{(4)}(t)\leqslant(1+b_8)C_1\sum_{j=1}^{8}R^{(j)}+b_8C_4D_0^{(4)}(t).$$

Since $C_4 b_8 \leqslant \frac{1}{2}$, we have

$$\frac{d}{dt}\big(E^{(4)}(t)+b_8E_1^{(4)}(t)\big)+\frac{1}{2}\sum_{p=0}^{1} b_8^p D_p^{(4)}(t)\leqslant (1+b_8)C_1\sum_{j=1}^{8} R^{(j)},$$

and, since $2b_8 \leqslant 1$, we arrive at

$$\frac{d}{dt}\big(2E^{(4)}(t)+2b_8E_1^{(4)}(t)\big)+\sum_{p=0}^{1} b_8^p D_p^{(4)}(t)\leqslant (2+1)C_1\sum_{j=1}^{8} R^{(j)}.$$

This shows that (5.61) holds for $l=1$.

Let $1\leqslant l\leqslant m-2$ and suppose that (5.61) holds up to $l$. We will prove that (5.61) holds with $l$ replaced by $l+1$. By adding (5.61) to $b_8^{l+1}\times\{$(5.63) with $l$ replaced by $l+1\}$, we have

$$\begin{aligned}&\frac{d}{dt}\left(2^lE^{(4)}(t)+\sum_{p=1}^{l+1}2^{l+1-p}b_8^pE_p^{(4)}(t)\right)+\sum_{p=0}^{l+1}b_8^pD_p^{(4)}(t)\\&\quad\leqslant\left(\sum_{p=0}^{l}2^p+b_8^{l+1}\right)C_1\sum_{j=1}^{8}R^{(j)}(t)+b_8^{l+1}C_4\sum_{p=0}^{l}D_p^{(4)}(t).\end{aligned}$$

Since $b_8^{l+1}C_4\leqslant\frac{1}{2}b_8^l\leqslant\frac{1}{2}b_8^p$ for $0\leqslant p\leqslant l$, we have

$$\begin{aligned}&\frac{d}{dt}\left(2^lE^{(4)}(t)+\sum_{p=1}^{l+1}2^{l+1-p}b_8^pE_p^{(4)}(t)\right)+\frac{1}{2}\sum_{p=0}^{l+1}b_8^pD_p^{(4)}(t)\\&\quad\leqslant\left(\sum_{p=0}^{l}2^p+b_8^{l+1}\right)C_1\sum_{j=1}^{8}R^{(j)}(t).\end{aligned}$$

We thus obtain

$$\frac{d}{dt}\left(2^{l+1}E^{(4)}(t)+\sum_{p=1}^{l+1}2^{l+2-p}b_8^pE_p^{(4)}(t)\right)+\sum_{p=0}^{l+1}b_8^pD_p^{(4)}(t)\leqslant\left(\sum_{p=0}^{l+1}2^p\right)C_1\sum_{j=1}^{8}R^{(j)}(t).$$

This proves (5.61) with $l$ replaced by $l+1$. Therefore, we conclude that (5.61) holds for all $1\leqslant l\leqslant m-1$. We thus arrive at

$$\frac{d}{dt}\tilde{E}(t)+2D(t)\leqslant C\tilde{R}(t), \tag{5.64}$$

where

$$\tilde{E}(t)=2^{m-1}E^{(4)}(t)+\sum_{p=1}^{m-1}2^{m-p}b_8^pE_p^{(4)}(t),\qquad D(t)=\frac{1}{2}\sum_{p=0}^{l+1}b_8^pD_p^{(4)}(t),$$

$$\tilde{R}(t)=\sum_{j=1}^{8}R^{(j)}(t).$$

Integrating (5.64) in $t$, we obtain

$$\tilde{E}(t) + 2\int_0^t D(\tau)\,d\tau \leqslant \tilde{E}(0) + C\int_0^t \tilde{R}(\tau)\,d\tau. \tag{5.65}$$

To complete the $H^m$-energy estimate, it remains to estimate $[\![\partial_{x_n}^2 w_\infty(t)]\!]_{m-2}^2$. This can be estimated by the equations

$$\begin{aligned}
-\nu\partial_{x_n}^2 w_\infty' = \rho_s \boldsymbol{f}_\infty' - \rho_s\bigg\{&\partial_t w_\infty' - \frac{\nu}{\rho_s}\Delta' w_\infty' - \frac{\tilde{\nu}}{\rho_s}\nabla' \operatorname{div} w_\infty + \nabla'\bigg(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi_\infty\bigg) \\
&+ v_s^1\partial_{x_1} w_\infty' + \frac{\nu\Delta v_s^1}{\gamma^2\rho_s^2}\phi_\infty \boldsymbol{e}_1' + \big(\partial_{x_n} v_s^1\big)\partial_{x_1} w_\infty^n \boldsymbol{e}_1' \\
&- \frac{\nu}{\rho_s} w^{(0)'}\Delta'\sigma_1 - \frac{\tilde{\nu}}{\rho_s}\nabla'\big(w^{(0),1}\partial_{x_1}\sigma_1\big) + \alpha_0\nabla'\sigma_1 + v_s^1 w^{(0)'}\partial_{x_1}\sigma_1 \\
&- \big\langle Q_0\tilde{B}\big(\sigma_1 u^{(0)} + u_\infty\big)\big\rangle_1 w^{(0),1}\boldsymbol{e}_1'\bigg\}, \\
-(\nu+\tilde{\nu})\partial_{x_n}^2 w_\infty^n = \rho_s f_\infty^n - \rho_s\bigg\{&\partial_t w_\infty^n - \frac{\nu}{\rho_s}\Delta' w_\infty^n - \frac{\tilde{\nu}}{\rho_s}\partial_{x_n}\nabla'\cdot w_\infty' + \partial_{x_n}\bigg(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi_\infty\bigg) \\
&+ v_s^1\partial_{x_1} w_\infty^n - \frac{\tilde{\nu}}{\rho_s}\big(\partial_{x_n} w^{(0),1}\big)\partial_{x_1}\sigma_1\bigg\},
\end{aligned}$$

which follows from (4.11). Applying $\partial_{x_n}^l T_{j,k}$ with $2j+k+l \leqslant m-2$ to these relations, one can show the following estimate by a direct computation:

$$\frac{\nu^2}{\nu+\tilde{\nu}}\big[\!\big[\partial_{x_n}^2 w_\infty(t)\big]\!\big]_{m-2}^2 \leqslant C\tilde{E}(t) + C\big[\!\big[\boldsymbol{f}_\infty(t)\big]\!\big]_{m-2}^2. \tag{5.66}$$

Furthermore, by using Lemmas 5.10–5.13 below, one can show

$$\big[\!\big[\boldsymbol{f}_\infty(t)\big]\!\big]_{m-2}^2 \leqslant C\big[\!\big[u(t)\big]\!\big]_m^4.$$

It then follows from (5.65) and (5.66) that

$$E(t) + 2\int_0^t D(\tau)\,d\tau \leqslant C\bigg\{\tilde{E}(0) + \big[\!\big[u(t)\big]\!\big]_m^4 + \int_0^t \tilde{R}(\tau)\,d\tau\bigg\}, \tag{5.67}$$

where

$$E(t) = \tilde{E}(t) + \frac{\nu^2}{\nu+\tilde{\nu}}\big[\!\big[\partial_{x_n}^2 w_\infty(t)\big]\!\big]_{m-2}^2.$$

Clearly,

$$|\!|\!|D\sigma_1|\!|\!|_{m-1}^2 + |\!|\!|D\phi_\infty|\!|\!|_{m-1}^2 + |\!|\!|Dw_\infty|\!|\!|_m^2 \leqslant CD(t). \tag{5.68}$$

Since $E^{(1)}[u]$ is equivalent to $E^{(0)}[u] + D^{(0)}[w]$ with our choice of $b_1$ in Proposition 5.3, we also have

$$[\![u(t)]\!]_m^2 \leqslant CE(t). \tag{5.69}$$

Therefore, by (5.67)–(5.69) we obtain the estimate in Proposition 5.9. This completes the proof of Proposition 5.9. $\square$

### 5.3. *Estimates on the nonlinearities*

It remains to estimate $R(t)$. The points in the estimates on the nonlinearities are as follows. Compared with the standard Matsumura–Nishida energy method, we have more terms which involve $\sigma_1$. By Lemma 4.4(i), we have

$$\|\partial_{x'}^k \sigma_1\|_2 \leqslant \|\sigma_1\|_2 \quad \text{for } k = 1, 2, \ldots. \tag{5.70}$$

This, together with the Gagliardo–Nirenberg–Sobolev inequality, implies

$$\|\sigma_1\|_\infty \leqslant C\|\sigma_1\|_2. \tag{5.71}$$

Also, since $n \geqslant 3$, we have

$$\|\sigma_1\|_4 \leqslant C\|\sigma_1\|_2^{1/2} \|\partial_{x'}\sigma_1\|_2^{1/2}. \tag{5.72}$$

See Lemma 5.12 below. As for $u_\infty$-part, by Lemma 4.3, we have the Poincaré inequality

$$\|u_\infty\|_2 \leqslant C\|\partial_x u_\infty\|_2. \tag{5.73}$$

Using these inequalities, one can control the terms involving $\sigma_1$, which are classified as

$$\begin{aligned}
\text{in } \langle Q_0 \boldsymbol{F}\rangle_1: \quad & O(\sigma_1 \partial_{x_1}\sigma_1), \quad O(\sigma_1 \partial_{x'} u_\infty), \quad O(u_\infty \partial_{x'}\sigma_1), \\
\text{in } \boldsymbol{F}_\infty: \quad & O(\sigma_1 \partial_{x_1}\sigma_1), \quad O(\sigma_1 \partial_x u_\infty), \quad O(u_\infty \partial_{x'}\sigma_1), \quad O\big(\sigma_1 \partial_{x'}^2 \sigma_1\big), \\
& O\big(\phi_\infty \partial_{x'}^2 \sigma_1\big), \quad O\big(\sigma_1 \partial_x^2 w_\infty\big), \quad O(\sigma_1 u_\infty), \quad O\big(\sigma_1^2\big), \\
& \text{and terms in } O\big(\phi^3\big),\ O\big(\phi^2 \nabla\phi\big).
\end{aligned}$$

We note that in the computations of the nonlinearities, we use the relations

$$-\frac{\nu}{\rho_s}\partial_{x_n}^2 w^{(0),1} + \frac{\nu \partial_{x_n}^2 v_s^1}{\gamma^2 \rho_s^2}\phi^{(0)} = 0 \quad \text{and} \quad \frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}\phi^{(0)} = \alpha_0.$$

See the proof of Lemma 4.1.

Using the inequalities (5.70)–(5.73), one can estimate the terms involving $\sigma_1$ as classified above. For example, as for the term $O(\sigma_1^2)$ in $\boldsymbol{F}_\infty$, which comes from the term $-\frac{1}{2\gamma^4 \rho_s}\partial_{x_n}(P''(\rho_s)\sigma_1^2\{\phi^{(0)}\}^2)$, we see from (5.72) and (5.73)

$$\begin{aligned}
\left| \left( -\frac{1}{2\gamma^4 \rho_s}\partial_{x_n}\big(P''(\rho_s)\sigma_1^2 \big\{\phi^{(0)}\big\}^2\big), w_\infty \rho_s \right) \right| &\leqslant C\|\sigma_1\|_4^2 \|w_\infty\|_2 \\
&\leqslant C\|\sigma_1\|_2 \|\partial_{x'}\sigma_1\|_2 \|\partial_x w_\infty\|_2 \\
&\leqslant C[\![u]\!]_m \big(|\!|\!| D\sigma_1 |\!|\!|_{m-1}^2 + |\!|\!| D w_\infty |\!|\!|_m^2\big).
\end{aligned}$$

As for the term $O(\sigma_1 \partial_{x_1} \sigma_1)$ in $\langle Q_0 \boldsymbol{F}\rangle_1$, we have, by integration by parts,

$$\big(\big\langle \sigma_1 \partial_{x_1} \sigma_1 \phi^{(0)} w^{(0),1}\big\rangle_1, \sigma_1\big) = \big(\sigma_1 \partial_{x_1} \sigma_1 \phi^{(0)} w^{(0),1}, \sigma_1\big) = -\frac{1}{2}\big(\sigma_1^2 \phi^{(0)} w^{(0),1}, \partial_{x_1} \sigma_1\big),$$

and thus,

$$\big(\big\langle \sigma_1 \partial_{x_1} \sigma_1 \phi^{(0)} w^{(0),1}\big\rangle_1, \sigma_1\big) = 0.$$

The terms involving only $u_\infty$ can be treated similarly to the standard Matsumura–Nishida energy method. We also note that the Poincaré inequality (5.73) is effectively used in the estimates on the terms involving only $u_\infty$.

We summarize inequalities to estimate the nonlinearities.

**Lemma 5.10.** *Let $2 \leqslant p \leqslant \infty$ and let $j$ and $k$ be integers satisfying*

$$0 \leqslant j < k, \qquad k > j + n\left(\frac{1}{2} - \frac{1}{p}\right).$$

*Then there exists a constant $C > 0$ such that*

$$\big\|\partial_x^j f\big\|_{L^p(\mathbf{R}^n)} \leqslant C \|f\|_{L^2(\mathbf{R}^n)}^{1-a} \big\|\partial_x^k f\big\|_{L^2(\mathbf{R}^n)}^a,$$

*where $a = \frac{1}{k}(j + \frac{n}{2} - \frac{n}{p})$.*

Lemma 5.10 can be proved by using Fourier transform.

Combining Lemma 5.10, the extension argument and the Poincaré inequality, we have the following inequalities for $u_\infty$-part.

**Lemma 5.11.** *Let $p$, $k$ and $j$ be as in Lemma 5.10 and let $u_\infty = {}^\top(\phi_\infty, w_\infty)$ be in* $\mathrm{Range}(P_\infty)$. *Then*

$$\|\phi_\infty\|_p \leqslant C \|\partial_x \phi_\infty\|_{H^{k-1}}.$$

*The same inequality also holds for $w_\infty$ if $w_\infty|_{x_n=0,1} = 0$.*

As for (5.71) and (5.72), Lemma 4.4 and Lemma 5.10 with $n$ replaced by $n-1$ yields the following inequalities.

**Lemma 5.12.**

(i) $\|\langle f\rangle_1\|_\infty \leqslant C\|\langle f\rangle_1\|_2$.
(ii) $\|\langle f\rangle_1\|_4 \leqslant C\|\langle f\rangle_1\|_2^{1/2}\|\partial_{x'}\langle f\rangle_1\|_2^{1/2}$.

**Proof.** We first note that $\langle f\rangle_1$ is a function of $x' \in \mathbf{R}^{n-1}$. Let $k$ be an integer satisfying $k > \frac{n-1}{2}$. Then, Lemma 5.10 with $n$ replaced by $n-1$ implies

$$\big\|\langle f\rangle_1\big\|_\infty \leqslant C\big\|\langle f\rangle_1\big\|_{H^k}.$$

Lemma 4.4 thus gives (i).

As for (ii), let $k$ be an integer satisfying $k > \frac{n-1}{4}$. Then we have

$$\big\|\langle f\rangle_1\big\|_4 \leqslant C\big\|\langle f\rangle_1\big\|_2^{1-a}\big\|\big\langle \partial_{x'}^k f\big\rangle_1\big\|_2^a \tag{5.74}$$

with $a = \frac{n-1}{4k}$. We see that $\frac{1}{2} \leqslant a < 1$ if and only if $\frac{n-1}{4} < k \leqslant \frac{n-1}{2}$. Since $\frac{n-1}{2} \geqslant 1$ for $n \geqslant 3$, we find an integer $k \geqslant 1$ for which (5.74) holds with some $a$ satisfying $\frac{1}{2} \leqslant a < 1$. With this $k$, we see from (5.74) and Lemma 4.4

$$\begin{aligned}\big\|\langle f\rangle_1\big\|_4 &\leqslant C\big\|\langle f\rangle_1\big\|_2^{1-a}\big\|\big\langle\partial_{x'}^k f\big\rangle_1\big\|_2^{(a-\frac{1}{2})+\frac{1}{2}} \\ &\leqslant C\big\|\langle f\rangle_1\big\|_{H^k}^{\frac{1}{2}}\big\|\big\langle\partial_{x'}^k f\big\rangle_1\big\|_2^{\frac{1}{2}} \\ &\leqslant C\big\|\langle f\rangle_1\big\|_2^{\frac{1}{2}}\big\|\langle\partial_{x'} f\big\rangle_1\big\|_2^{\frac{1}{2}}.\end{aligned}$$

This completes the proof. $\square$

We next state estimate on composite functions.

**Lemma 5.13.**

(i) *Let $m$ and $m_k$ $(k = 1, \dots, \ell)$ be nonnegative integers and let $\alpha_k$ $(k = 1, \dots, \ell)$ be multi-indices. Suppose that*

$$m \geqslant \left[\frac{n}{2}\right] + 1, \qquad 0 \leqslant |\alpha_k| \leqslant m_k \leqslant m + |\alpha_k| \quad (k = 1, \dots, \ell)$$

*and*

$$m_1 + \dots + m_\ell \geqslant (\ell - 1)m + |\alpha_1| + \dots + |\alpha_\ell|.$$

*Then there exists a constant $C > 0$ such that*

$$\big\|\partial_x^{\alpha_1} f_1 \cdots \partial_x^{\alpha_\ell} f_\ell\big\|_2 \leqslant C \prod_{1\leqslant k\leqslant\ell} \|f_k\|_{H^{m_k}}.$$

(ii) *Let $1 \leqslant k \leqslant m$. Suppose that $F(x, t, y)$ is a smooth function on $\Omega \times [0, \infty) \times I$, where $I$ is a compact interval in $\mathbf{R}$. Then for $|\alpha| + 2j = k$ there hold*

$$\begin{aligned}&\big\|\big[\partial_x^\alpha \partial_t^j, F(x, t, f_1)\big] f_2\big\|_2 \\ &\quad\leqslant \begin{cases} C_0\big(t, f_1(t)\big)[\![f_2]\!]_{k-1} + C_1\big(t, f_1(t)\big)\big\{1 + |\!|\!|Df_1|\!|\!|_{m-1}^{|\alpha|+j-1}\big\}|\!|\!|Df_1|\!|\!|_{m-1}[\![f_2]\!]_k, \\ C_0\big(t, f_1(t)\big)[\![f_2]\!]_{k-1} + C_1\big(t, f_1(t)\big)\big\{1 + |\!|\!|Df_1|\!|\!|_{m-1}^{|\alpha|+j-1}\big\}|\!|\!|Df_1|\!|\!|_m[\![f_2]\!]_{k-1}. \end{cases}\end{aligned}$$

*Here*

$$C_0\big(t, f_1(t)\big) = \sum_{\substack{(\beta,l)\leqslant(\alpha,j)\\(\beta,l)\neq(0,0)}} \sup_x\big|\big(\partial_x^\beta \partial_t^l F\big)\big(x, t, f_1(t)\big)\big|$$

*and*

$$C_1\big(t, f_1(t)\big) = \sum_{\substack{(\beta,l)\leqslant(\alpha,j)\\1\leqslant p\leqslant j+|\alpha|}} \sup_x\big|\big(\partial_x^\beta \partial_t^l \partial_y^p F\big)\big(x, t, f_1(t)\big)\big|.$$

The proof of Lemma 5.13 can be found in [5,6].

Using inequalities mentioned above, one can obtain the following estimates for the nonlinearities.

Before stating the estimates for the nonlinearities, we observe that since $m \geqslant [n/2]+1$ we have the Sobolev inequality

$$\|f\|_\infty \leqslant C_2\|f\|_{H^m}.$$

Let $\varepsilon_2 > 0$ be a number such that $C_2\varepsilon_2 \leqslant \frac{\gamma^2\rho_1}{2}$. Then whenever $[\![u(t)]\!]_m \leqslant \varepsilon_2$, we have

$$\big\|\phi(t)\big\|_\infty \leqslant C_2\,\big[\!\big[u(t)\big]\!\big]_m \leqslant C_2\varepsilon_2 \leqslant \frac{\gamma^2\rho_1}{2},$$

and hence,

$$\rho(x,t) = \rho_s(x_n) + \gamma^{-2}\phi(x,t) \geqslant \rho_1 - \gamma^{-2}\big\|\phi(t)\big\|_\infty \geqslant \frac{\rho_1}{2}\ (>0).$$

We thus see that $\boldsymbol{F}(t)$ is smooth, whenever $[\![u(t)]\!]_m \leqslant \varepsilon_2$. So, we assume that $[\![u(t)]\!]_m \leqslant \varepsilon_2$ for $t \in [0,T]$.

**Proposition 5.14.** *Let $j$ be an integer satisfying $0 \leqslant j \leqslant m-1$ and let $u(t)$ be a solution of* (4.1) *in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,T];H^{m-2j})$ with $\int_0^T |\!|\!|Dw_\infty|\!|\!|_m^2\,d\tau < \infty$. Assume that $[\![u(t)]\!]_m \leqslant \min\{1,\varepsilon_2\}$ for $t \in [0,T]$. Then the following estimates hold for $t \in [0,T]$ with $C > 0$ independent of $T$.*

$$\text{(i)}\qquad \big\|\partial_t^j(\phi\,\mathrm{div}\,w)\big\|_{H^{m-2j}} + \big\|\partial_t^j\big(w\cdot\nabla\big(\sigma_1\phi^{(0)}\big)\big)\big\|_{H^{m-2j}} + \big\|\partial_t^j\langle Q_0\boldsymbol{F}\rangle_1\big\|_{H^{m-2j}} \leqslant CE(t)^{1/2}D(t)^{1/2},$$

$$\text{(ii)}\qquad \big\|\partial_t^j(w\cdot\nabla\phi_\infty)\big\|_{H^{m-1-2j}} + \big\|\partial_t^j\boldsymbol{f}\big\|_{H^{m-1-2j}} \leqslant CE(t)^{1/2}D(t)^{1/2}.$$

**Proposition 5.15.** *Let $u(t)$ be a solution of* (4.1) *in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,T];H^{m-2j})$ with $\int_0^T |\!|\!|Dw_\infty|\!|\!|_m^2\,d\tau < \infty$. Assume that $[\![u(t)]\!]_m \leqslant \min\{1,\varepsilon_2\}$ for $t \in [0,T]$. Then there hold the following estimates for $t \in [0,T]$ with $C > 0$ independent of $T$.*

(i) *Let $0 \leqslant 2j+k \leqslant m$. Then*

$$\big|\big(T_{j,k}\langle Q_0\boldsymbol{F}\rangle_1, T_{j,k}\sigma_1\big)\big| + \big|\langle\!\langle T_{j,k}\boldsymbol{F}_\infty, T_{j,k}\rangle\!\rangle\big| \leqslant CE(t)^{1/2}D(t).$$

(ii) *Let $0 \leqslant 2j+k \leqslant m-1$. Then*

$$\big|\big(\big\langle\rho_s(-\Delta)^{-1}\big(\rho_s\nabla'\cdot T_{j,k}\boldsymbol{f}'_\infty\big)\big\rangle_1, T_{j,k}\sigma_1\big)\big| \leqslant CE(t)^{1/2}D(t).$$

(iii) *Let $0 \leqslant 2j+k+l \leqslant m-1$. Then*

$$\big\|\big[T_{j,k}\partial_{x_n}^{l+1}, w\big]\cdot\nabla\phi_\infty\big\|_2 \leqslant CE(t)^{1/2}D(t)^{1/2},$$

$$\left|\left(\mathrm{div}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}w\right), \big|T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\big|^2\right)\right| \leqslant CE(t)^{1/2}D(t).$$

We are now in a position to derive the a priori estimate. We deduce from Propositions 5.9, 5.14 and 5.15 that

$$
\begin{aligned}
&[\![u(t)]\!]_m^2 + \int_0^t \left( |\!|\!| D\sigma_1 |\!|\!|_{m-1}^2 + |\!|\!| D\phi_\infty |\!|\!|_{m-1}^2 + |\!|\!| D w_\infty |\!|\!|_m^2 \right) d\tau \\
&\quad \leqslant C_3 \left\{ \|u_0\|_{H^m}^2 + [\![u(t)]\!]_m \int_0^t \left( |\!|\!| D\sigma_1 |\!|\!|_{m-1}^2 + |\!|\!| D\phi_\infty |\!|\!|_{m-1}^2 + |\!|\!| D w_\infty |\!|\!|_m^2 \right) d\tau \right\},
\end{aligned}
$$

provided that $[\![u(t)]\!]_m \leqslant \min\{1, \varepsilon_2\}$. The desired a priori estimate in Proposition 5.1 now follows by taking $\varepsilon_1 = \min\{1, \varepsilon_2, C_3/2\}$.

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