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Optimal $W^{2,2}_{\mathrm{loc}}$-regularity, Pohozhaev's identity, and nonexistence of weak solutions to some quasilinear elliptic equations ☆

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ABSTRACT

We begin by establishing a sharp (optimal) $W^{2,2}_{\mathrm{loc}}$-regularity result for bounded weak solutions to a nonlinear elliptic equation with the p-Laplacian, $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$. We develop very precise, optimal regularity estimates on the ellipticity of this degenerate (for $2 < p < \infty$) or singular (for $1 < p < 2$) problem. We apply this regularity result to prove Pohozhaev's identity for a weak solution $u \in W^{1,p}(\Omega)$ of the elliptic Neumann problem

$$-\Delta_p u + W'(u) = f(x) \quad \text{in } \Omega; \qquad \partial u/\partial \nu = 0 \quad \text{on } \partial\Omega. \tag{P}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a C^2-manifold, $\nu \equiv \nu(x_0)$ denotes the outer unit normal to $\partial\Omega$ at $x_0 \in \partial\Omega$, $x = (x_1, \dots, x_N)$ is a generic point in Ω, and $f \in L^\infty(\Omega) \cap W^{1,1}(\Omega)$. The potential $W : \mathbb{R} \to \mathbb{R}$ is assumed to be of class C^1 and of the typical double-well shape of type $W(s) = |1 - |s|^\beta|^\alpha$ for $s \in \mathbb{R}$, where $\alpha, \beta > 1$ are some constants. Finally, we take an advantage of the Pohozhaev identity to show that problem (P) with $f \equiv 0$ in Ω has no *phase transition solution* $u \in W^{1,p}(\Omega)$ ($1 < p \leqslant N$), such that $-1 \leqslant u \leqslant 1$ in Ω with $u \equiv -1$ in Ω_{-1} and $u \equiv 1$ in Ω_1, where both Ω_{-1} and Ω_1 are some nonempty

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subdomains of Ω. Such a scenario for u is possible only if $N = 1$ and Ω_{-1}, Ω_1 are finite unions of suitable subintervals of the open interval $\Omega \subset \mathbb{R}^1$.

1. Introduction

We consider the following quasilinear elliptic problem with the zero Neumann boundary conditions,

$$-\Delta_p u + W'(u) = 0 \quad \text{in } \Omega; \qquad \partial u/\partial \nu = 0 \quad \text{on } \partial\Omega. \tag{1}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a C^2-manifold, $\nu(x_0)$ denotes the exterior unit normal to $\partial\Omega$ at $x_0 \in \partial\Omega$, $x = (x_1, \dots, x_N)$ is a generic point in Ω, and $u \in W^{1,p}(\Omega)$ is an unknown function, $p \in (1, \infty)$. The quasilinear elliptic operator $\Delta_p : W^{1,p}(\Omega) \to W_{\mathrm{N}}^{-1,p'}(\Omega)$, called the p-Laplacian, is defined for $u \in W^{1,p}(\Omega)$ by

$$(\Delta_p u)(x) \stackrel{\text{def}}{=} \operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) \equiv \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right), \quad x \in \Omega, \tag{2}$$

with values in $W_{\mathrm{N}}^{-1,p'}(\Omega)$, the dual space of $W^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Typical forms of the potential function $W(u)$ are (i) $W(u) = (\lambda/p)|u|^p + (\mu/\gamma)|u|^\gamma$, where $\lambda \in \mathbb{R}$ is a spectral parameter and $\mu, \gamma \in \mathbb{R}$ are some constants with $\gamma > 1$, and (ii) $W(s) = |1 - |s|^\beta|^\alpha$ for $s \in \mathbb{R}$, where $\alpha, \beta \in \mathbb{R}$ are constants, $\alpha, \beta > 1$. In order to present our ideas in a tractable manner, we restrict ourselves to the case of the p-Laplacian $\Delta_p u$. We discuss a more general quasilinear elliptic operator in Section 6 and leave the details to an interested reader.

Our main objective in this work is to establish the **nonexistence** of certain types of *weak solutions u* to the boundary value problem (1) for $N \geqslant p > 1$ that are constant $= 1$ on the boundary $\partial\Omega$ and constant $= -1$ on a (nonempty) subdomain Ω_1 with compact closure $\overline{\Omega_1} \subset \Omega$. We term such a solution "*phase transition solution*" (see Definitions 5.2 and 5.7). Physically, such a solution u would correspond to a phase transition in $\Omega \setminus \overline{\Omega_1}$ from one pure phase in Ω_1 ($u \equiv -1$) to another pure phase in $\mathbb{R}^N \setminus \Omega$ ($u \equiv 1$). Solutions of this kind in the space dimension one ($N = 1$) have been obtained in a number of papers; see e.g. J.I. Díaz and J. Hernández [8], J.I. Díaz, J. Hernández, and F.J. Mancebo [9], P. Drábek, R.F. Manásevich, and P. Takáč [11], and Ph. Rosenau and E. Kashdan [30] to mention only a few. Related existence results have been obtained recently in C. Cortázar, M. Elgueta, and P. Felmer [3,4] and Y.Sh. Il'yasov and Y.V. Egorov [19, Theorem 1.1] for sufficiently high space dimension $N \geqslant 3$ and in [30] for $N = 2$ (only numerically).

We will derive our nonexistence results (Theorems 5.3 and 5.8) for $N \geqslant p > 1$ in Section 5 as an application to problem (1) of a Pohozhaev-type identity (Theorem 4.2) established in Section 4. The first type of this identity has been discovered in S.I. Pohozhaev [23]. Moreover, in the proof of this identity we take advantage of a new optimal $W^{2,2}_{\mathrm{loc}}$-regularity result (Theorems 3.1 and 3.2) for the Neumann problem

$$-\Delta_p u = f(x) \quad \text{in } \Omega; \qquad \partial u/\partial \nu = 0 \quad \text{on } \partial\Omega, \tag{3}$$

with $f \in L^\infty(\Omega) \cap W^{1,1}(\Omega)$ derived in Section 3. Also this new regularity result is of independent interest; it improves well-known results from [21,27,28,31,32,36]. Practically all work on Pohozhaev-type identities and inequalities that we know about makes essential use of some regularity of the partial derivatives of highest order that appear in the equation; typically, they should be at least in the local Lebesgue space $L^1_{\mathrm{loc}}(\Omega)$ but often even smoother (continuous), see, e.g., S.I. Pohozhaev [23,24] and P. Pucci and J. Serrin [25].

This article is organized as follows. Important notation and hypotheses are introduced in the next section (Section 2). We first establish a new sharp (optimal) $W^{2,2}_{\mathrm{loc}}$-regularity result for bounded weak solutions to the nonlinear elliptic problem (3) in Section 3, Theorems 3.1 and 3.2. This new and similar known regularity results help us to establish Pohozhaev's identity for weak solutions to problem (1) in Section 4, Theorem 4.2. In Section 5 we take advantage of this identity (for $N \geqslant p > 1$) in order to obtain the nonexistence of a "*phase transition solution*" to problem (1) (Theorems 5.3 and 5.8) in various bounded and unbounded domains in $\mathbb{R}^N$; see Sections 5.1 and 5.2, respectively. Possible generalizations of some of these results are discussed in Section 6. Finally, Appendices A, B, and C contain some known, but important results to keep the present work self-contained.

2. Notation and hypotheses

The vector field $\mathbf{a}(\nabla u) \stackrel{\text{def}}{=} |\nabla u|^{p-2}\nabla u = (a_1, \dots, a_N)$ in (1) and (2) has the components $a_i(\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{p-2}\eta_i$ $(i = 1, 2, \dots, N)$ that are functions of $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N) = \nabla u \in \mathbb{R}^N$. Of course, $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_N)$ denotes the gradient. Clearly, each function $a_i(\boldsymbol{\eta})$ satisfies $a_i \in C^0(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{\mathbf{0}\})$. In addition, we easily deduce that $\mathbf{a}$ verifies the following *ellipticity* and *growth conditions*: There exist some constants $\gamma, \Gamma \in (0, \infty)$ such that

$$a_i(\mathbf{0}) = 0; \quad i = 1, 2, \dots, N, \tag{4}$$

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(\boldsymbol{\eta}) \cdot \xi_i \xi_j \geqslant \gamma |\boldsymbol{\eta}|^{p-2} |\boldsymbol{\xi}|^2, \tag{5}$$

$$\sum_{i,j=1}^{N} \left| \frac{\partial a_i}{\partial \eta_j}(\boldsymbol{\eta}) \right| \leqslant \Gamma |\boldsymbol{\eta}|^{p-2}, \tag{6}$$

for all $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ and for all $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$.

Conditions (4), (5), and (6) are characteristic for our treatment of the elliptic boundary value problem (1). An interested reader is referred to Section 6 where we discuss a generalization of our results to an arbitrary vector field $\mathbf{a} : \mathbb{R}^N \to \mathbb{R}^N$ with a potential $\mathcal{A} \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{\mathbf{0}\})$, i.e., $\mathbf{a} = \mathcal{A}'$, that satisfies just these conditions. Of course, $a_i(\boldsymbol{\eta}) = \partial \mathcal{A}/\partial \eta_i$ for $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$; $i = 1, 2, \dots, N$.

Notice that the entries $A_{ij} = \partial a_i/\partial \eta_j$ of the Jacobian matrix $\mathbf{A} = (A_{ij})_{i,j=1}^N$ of the mapping $\boldsymbol{\eta} \mapsto \mathbf{a}(\boldsymbol{\eta}) \stackrel{\text{def}}{=} |\boldsymbol{\eta}|^{p-2}\boldsymbol{\eta} : \mathbb{R}^N \to \mathbb{R}^N$,

$$\mathbf{A}(\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{p-2} \left(\mathbf{I} + (p-2) \frac{\boldsymbol{\eta} \otimes \boldsymbol{\eta}}{|\boldsymbol{\eta}|^2} \right) \quad \text{for } \boldsymbol{\eta} \in \mathbb{R}^N \setminus \{\mathbf{0}\}, \tag{7}$$

satisfy $A_{ij} \in C^0(\mathbb{R}^N \setminus \{\mathbf{0}\})$ together with the following ellipticity and growth inequalities,

$$\gamma |\boldsymbol{\eta}|^{p-2} |\boldsymbol{\xi}|^2 \leqslant \langle \mathbf{A}(\boldsymbol{\eta})\boldsymbol{\xi}, \boldsymbol{\xi} \rangle = \sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(\boldsymbol{\eta}) \cdot \xi_i \xi_j \leqslant \Gamma |\boldsymbol{\eta}|^{p-2} |\boldsymbol{\xi}|^2 \tag{8}$$

for all $\boldsymbol{\eta} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ and all $\boldsymbol{\xi} \in \mathbb{R}^N$. Here, $\mathbf{I}$ is the identity matrix in $\mathbb{R}^{N \times N}$ and $\boldsymbol{\xi} \otimes \boldsymbol{\eta}$ is the $(N \times N)$-matrix $\mathbf{T} = (\xi_i \eta_j)_{i,j=1}^N$ for $\boldsymbol{\xi} = (\xi_i)_{i=1}^N$, $\boldsymbol{\eta} = (\eta_i)_{i=1}^N \in \mathbb{R}^N$. The symbol $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \operatorname{trace}(\boldsymbol{\xi} \otimes \boldsymbol{\eta}) = \sum_{i=1}^N \xi_i \eta_i$ stands for the Euclidean inner product of $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$.

Finally, we assume that $f : \Omega \to \mathbb{R}$ satisfies $f \in L^\infty(\Omega) \cap W^{1,1}(\Omega)$, i.e., $f \in L^\infty(\Omega)$ and all its distributional derivatives $\partial f/\partial x_i$ $(i = 1, 2, \dots, N)$ belong to $L^1(\Omega)$.

3. Optimal weighted $W^{2,2}_{\mathrm{loc}}$ regularity

Our goal in this section is to establish the regularity result below, stated in Theorems 3.1 (for $p \geqslant 2$) and 3.2 (for $p < 2$). We use the method of difference quotients, cf. [21,27,28,31,32,36]. Throughout this section we assume that Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact C^2-manifold.

Let $d(x) \stackrel{\text{def}}{=} \operatorname{dist}(x, \partial\Omega)$ denote the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. Given $\delta > 0$ small enough, we denote by Ω_δ the open δ-neighborhood of the boundary $\partial\Omega$ in Ω,

$$\Omega_\delta = \{x \in \Omega\colon\ d(x) < \delta\}.$$

The linear space of all continuously differentiable functions $u : \Omega \to \mathbb{R}$ with compact support is denoted by $C^1_0(\Omega)$. If $\varphi \in C^1_0(\Omega)$ is a nonnegative function supported in the subdomain

$$\Omega'_\delta = \Omega \setminus \overline{\Omega_\delta} = \{x \in \Omega\colon\ d(x) > \delta\},$$

i.e., $\operatorname{supp}\varphi \subset \Omega'_\delta$, then for every $h \in \mathbb{R}^N$ with $0 < |h| < \delta$, the difference quotient

$$\delta_h\varphi(x) \stackrel{\text{def}}{=} \frac{\varphi(x+h) - \varphi(x)}{|h|}, \quad x \in \Omega, \tag{9}$$

satisfies $\delta_h\varphi \in C^1_0(\Omega)$.

Recalling $\Omega'_\delta = \Omega \setminus \overline{\Omega_\delta}$ and assuming that $\delta > 0$ is small enough, we denote

$$d_\delta(x) = \begin{cases} 0 & \text{if } x \in \Omega_\delta; \\ \operatorname{dist}(x, \Omega_\delta) & \text{if } x \in \Omega \setminus \Omega_\delta, \end{cases}$$

and for $\sigma \in (0, \delta)$ introduce a thin open set

$$\mathcal{O}^\delta_\sigma = \{x \in \Omega'_\delta\colon\ d_\delta(x) < \sigma\}$$

near the boundary $\partial\Omega$ of the domain Ω with closure $\overline{\mathcal{O}^\delta_\sigma} \subset \Omega$. Next, we make a special choice of the test function φ; this test function, φ^δ_σ, will be constant $= 1$ in $\Omega \setminus (\overline{\Omega_\delta} \cup \mathcal{O}^\delta_\sigma)$ and $= 0$ in $\overline{\Omega_\delta}$, except for the thin open set $\mathcal{O}^\delta_\sigma$ that separates the sets $\{x \in \Omega\colon\ \varphi^\delta_\sigma(x) = 1\}$ and $\{x \in \Omega\colon\ \varphi^\delta_\sigma(x) = 0\}$ from one another. The purpose of this choice is to guarantee that $\nabla\varphi^\delta_\sigma = \mathbf{0}$ holds in $\Omega \setminus \overline{\mathcal{O}^\delta_\sigma}$. We construct φ^δ_σ as follows: For $\delta > 0$ sufficiently small, the function d_δ is Lipschitz-continuous in $\overline{\Omega_\delta} \cup \overline{\mathcal{O}^\delta_\sigma}$; moreover, $(d_\delta(\cdot))^2$ is of class C^1 in $\overline{\Omega_\delta} \cup \overline{\mathcal{O}^\delta_\sigma}$. Consequently, for $0 < \sigma < \delta$, the Urysohn-type test function

$$\varphi^\delta_\sigma(x) = \begin{cases} 0 & \text{if } x \in \overline{\Omega_\delta}; \\ (\sigma^{-1} d_\delta(x))^2 & \text{if } x \in \mathcal{O}^\delta_\sigma; \\ 1 & \text{if } x \in \Omega \setminus (\overline{\Omega_\delta} \cup \mathcal{O}^\delta_\sigma), \end{cases} \tag{10}$$

satisfies $0 \leqslant \varphi^\delta_\sigma \leqslant 1$ in $\overline{\Omega}$, $\varphi^\delta_\sigma \in W^{1,q}_0(\Omega)$ for $N < q < \infty$, and

$$\nabla\varphi^\delta_\sigma(x) = \begin{cases} 2\sigma^{-2} d_\delta(x)\nabla d_\delta(x) & \text{if } x \in \mathcal{O}^\delta_\sigma; \\ \mathbf{0} & \text{if } x \in \Omega \setminus \overline{\mathcal{O}^\delta_\sigma}, \end{cases}$$

by Gilbarg and Trudinger [14, Theorem 7.8, p. 153]. Hence, there is a constant $C > 0$, which is independent from $0 < \sigma < \delta$ ($\delta, \sigma > 0$ small enough), such that

$$\left|\nabla\varphi^\delta_\sigma(x)\right|^2 \leqslant C\sigma^{-2}\varphi^\delta_\sigma(x) \quad \text{for all } x \in \Omega \setminus \partial\mathcal{O}^\delta_\sigma. \tag{11}$$

Given an open set $U \subset \mathbb{R}^N$ and an integer $m \geqslant 1$, we denote by $W^{m,2}_{\mathrm{loc}}(U)$ the "local" Sobolev space of all functions $u \in L^2_{\mathrm{loc}}(U)$ whose all distributional partial derivatives of order $\leqslant m$ belong to $L^2_{\mathrm{loc}}(U)$ as well.

Now we are ready to state the main results of this section, our optimal weighted $W^{2,2}_{\mathrm{loc}}$ regularity theorems, first for $p \leqslant 2 < \infty$, then for $1 < p < 2$, respectively, keeping the notation introduced above with $\delta > 0$ small enough and $0 < \sigma < \delta$.

Theorem 3.1 *(Weighted $W^{2,2}_{\mathrm{loc}}$ regularity for $p \geqslant 2$). Assume that $2 \leqslant p < \infty$, Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a C^2-manifold, $f \in L^\infty(\Omega) \cap W^{1,1}(\Omega)$, and $u \in W^{1,p}(\Omega)$ satisfies the equation*

$$-\Delta_p u \equiv -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x) \quad \text{in } \Omega, \tag{12}$$

supplemented by standard (i.e., Dirichlet or Neumann, respectively) boundary conditions, $u = 0$ on $\partial\Omega$ or $\partial u/\partial\nu = 0$ on $\partial\Omega$.

(a) *Then the vector field $\mathbf{b}(\nabla u) \stackrel{\text{def}}{=} |\nabla u|^{(p/2)-1}\nabla u$ belongs to $[W^{1,2}_{\mathrm{loc}}(\Omega'_\delta)]^N$ with $\Omega'_\delta = \Omega \setminus \overline{\Omega_\delta}$ and it satisfies*

$$\int_{\Omega'_\delta} \big|\nabla\big(|\nabla u|^{(p/2)-1}\nabla u\big)\big|^2 \varphi^\delta_\sigma \,\mathrm{d}x \leqslant C(\sigma) < \infty,$$

where $C(\sigma) \geqslant 0$ is a constant depending on $\sigma \in (0,\delta)$ with $C(\sigma)\sigma^2 \leqslant \mathrm{const} < \infty$.

(b) *Moreover, u belongs to $W^{2,2}_{\mathrm{loc}}(U)$ over the open set*

$$U = \Omega'_\delta \cap \big\{x \in \Omega\colon\ \nabla u(x) \neq \mathbf{0}\big\}$$

and the Hessian matrix $\nabla^2 u \stackrel{\text{def}}{=} (\frac{\partial^2 u}{\partial x_i \partial x_j})^N_{i,j=1} = \nabla(\nabla u) \in \mathbb{R}^{N\times N}$ (at $x \in U$) satisfies

$$\int_U |\nabla u|^{p-2}\big|\nabla^2 u\big|^2 \varphi^\delta_\sigma \,\mathrm{d}x \leqslant C(\sigma) < \infty.$$

Theorem 3.2 *(Weighted $W^{2,2}_{\mathrm{loc}}$ regularity for $p < 2$). Assume that $1 < p < 2$, Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a C^2-manifold, $f \in L^\infty(\Omega) \cap W^{1,1}(\Omega)$, and $u \in W^{1,p}(\Omega)$ satisfies Eq.* (12) *supplemented by standard (i.e., Dirichlet or Neumann, respectively) boundary conditions, $u = 0$ on $\partial\Omega$ or $\partial u/\partial\nu = 0$ on $\partial\Omega$. In addition, assume also the following* ***hypothesis***:

There are numbers $0 < \sigma < \delta < \infty$ (which both may be chosen sufficiently small) such that $\nabla u \neq \mathbf{0}$ a.e. in $\mathcal{O}^\delta_\sigma$ and

$$\int_{\mathcal{O}^\delta_\sigma} |\nabla u|^{-(2-p)} \,\mathrm{d}x < \infty. \tag{13}$$

Then Parts (a) and (b) *of* Theorem 3.1 *are valid and, moreover, we have also*

(c) *u belongs to $W^{2,2}_{\mathrm{loc}}(\Omega'_\delta)$ and $(\nabla^2 u)(x) = \mathbf{0} \in \mathbb{R}^{N\times N}$ holds for almost every*

$$x \in U' = \Omega'_\delta \cap \big\{x \in \Omega\colon\ \nabla u(x) = \mathbf{0}\big\}.$$

More precisely, we have

$$\int_{\Omega'_\delta} |\nabla u|^{p-2} \left|\nabla^2 u\right|^2 \varphi_\sigma^\delta \, \mathrm{d}x \leqslant C(\sigma) < \infty.$$

Remark 3.3 *(Concerning Hypothesis (13) for $p < 2$).* For $p = 2$, Hypothesis (13) is trivial, whereas for $2 < p < \infty$, it follows from $|\nabla u| \in L^p(\Omega) \subset L^{p-2}(\Omega)$, by Hölder's inequality. So let us assume $1 < p < 2$ in Parts (i) and (ii) below.

(i) Hypothesis (13) is trivially satisfied in any smooth subdomain (e.g., in a ball) $\Sigma \subset \Omega$ with (compact) closure

$$\overline{\Sigma} \subset U_0 = \left\{x \in \Omega:\ \nabla u(x) \neq 0\right\},$$

whence, in Part (b) of Theorem 3.2, $u \in W^{2,2}_{\mathrm{loc}}(U_0)$ holds over the open set U_0 ($U \subset U_0$), for any $p \in (1, 2)$.

(ii) Hypothesis (13) is easily satisfied if Eq. (12) for $u \in W^{1,p}(\Omega)$ is supplemented by the Dirichlet boundary condition $u = 0$ on $\partial\Omega$, i.e., $u \in W^{1,p}_0(\Omega)$, together with $0 \leqslant f \in L^\infty(\Omega)$. Namely, the Hopf maximum principle [38, Propositions 3.2.1 and 3.2.2, p. 801] or [40, Theorem 5, p. 200] can be applied to obtain the exterior normal derivative $\partial u/\partial\nu < 0$ on $\partial\Omega$. At the beginning of our proof of Theorem 3.1 below we will briefly mention the facts that render $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, \alpha)$, cf. G.M. Lieberman [20, Theorem 1, p. 1203], for any $p \in (1, \infty)$. Consequently, Hypothesis (13) holds for any pair of sufficiently small numbers $0 < \sigma < \delta < 1$. In contrast, for the corresponding Neumann problem (3), Hypothesis (13) may pose a *rather severe* restriction. Namely, notice that the trivial example of problem (3) with $f \equiv 0$ in Ω has only constant solutions for which $\nabla u \equiv \mathbf{0}$ in Ω; thus, (13) obviously fails to hold.

Remark 3.4. (i) For $1 < p < 2$, a weaker result than Part (c) of our Theorem 3.2, claiming $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, is established in P. Pucci and R. Servadei [28, Theorem 2.5, p. 3351] (first announced in [27, Theorem 1, p. 257]). However, the latter (in [27,28]) does not impose our Hypothesis (13) which is rather strong especially for the zero Neumann boundary conditions $\partial u/\partial\nu = 0$ on $\partial\Omega$ combined with $u \equiv \pm 1$ on $\partial\Omega$, cf. Theorems 5.3 and 5.8 in Section 5 below. Indeed, by Hölder's inequality we have

$$\begin{aligned}\int_{\Omega'_\delta} \left|\nabla^2 u\right|^p \varphi_\sigma^\delta \, \mathrm{d}x &= \int_{\Omega'_\delta} |\nabla u|^{p(2-p)/2} \cdot |\nabla u|^{p(p-2)/2} \left|\nabla^2 u\right|^p \varphi_\sigma^\delta \, \mathrm{d}x \\ &\leqslant \left(\int_{\Omega'_\delta} |\nabla u|^p \varphi_\sigma^\delta \, \mathrm{d}x\right)^{(2-p)/2} \left(\int_{\Omega'_\delta} |\nabla u|^{p-2} \left|\nabla^2 u\right|^2 \varphi_\sigma^\delta \, \mathrm{d}x\right)^{p/2} \\ &\leqslant \left(\int_{\Omega} |\nabla u|^p \, \mathrm{d}x\right)^{(2-p)/2} \left(\int_{\Omega'_\delta} |\nabla u|^{p-2} \left|\nabla^2 u\right|^2 \varphi_\sigma^\delta \, \mathrm{d}x\right)^{p/2}\end{aligned}$$

which renders [28, Theorem 2.5, p. 3351].

(ii) Another analogous result to our Theorems 3.1 and 3.2 has been obtained in H.-W. Lou [21, Lemma 2.1, p. 522] for the power length $|\mathbf{b}(\nabla u)|^{2(p-1)/p} = |\mathbf{b}(\nabla u)|^{2/p'} = |\nabla u|^{p-1}$ of our vector field $\mathbf{b}(\nabla u) = |\nabla u|^{(p/2)-1} \nabla u$, that is, $|\nabla u|^{p-1} \in W^{1,2}_{\mathrm{loc}}(\Omega'_\delta)$ for any $p \in (1, \infty)$. As usual, $p' = p/(p-1) > 1$. If $2 < p < \infty$, it is easy to derive this result from our Theorem 3.1 with the help from $2/p' > 1$ and

$|\nabla u| \leqslant \text{const} < \infty$ in Ω. However, the latter (our Theorem 3.1) does not seem to follow from [21, Lemma 2.1], for instance, due to the fact that the statement

$$\nabla\big(|\nabla u|^{p-1}\big) = (p-1)|\nabla u|^{p-3}\big(\nabla u \cdot \nabla^2 u\big) \in \big[L^2_{\mathrm{loc}}(\Omega)\big]^N$$

clearly does not imply (and is even weaker than)

$$|\nabla u|^{p-2}\nabla^2 u \in \big[L^2_{\mathrm{loc}}(\Omega)\big]^{N\times N} \quad \text{or} \quad \nabla\big(|\nabla u|^{p-2}\nabla u\big) \in \big[L^2_{\mathrm{loc}}(\Omega)\big]^{N\times N}.$$

Proof of Theorems 3.1 and 3.2. Let $1 < p < \infty$ be arbitrary. We begin with some well-known classical regularity results for Eq. (12). In all these results it suffices to assume that $f \in L^\infty(\Omega)$ and $u \in W^{1,p}(\Omega)$ satisfies Eq. (12) with standard boundary conditions. A regularity result from A. Anane's thesis [1, Théorème A.1, p. 96] or from M. Ôtani [22, Theorem II, p. 142] guarantees $u \in L^\infty(\Omega)$. To be more precise, in [1,22] this result is proved for zero Dirichlet boundary conditions only, i.e., for $u \in W_0^{1,p}(\Omega)$. However, if u satisfies Eq. (12) with zero Neumann boundary conditions, the bootstrapping method (Moser's iteration scheme) in [1,22] works also in this case without any change. The only difference in the proof is that one has to apply Poincaré's inequality not to the function $u(x)$ directly, but rather to the zero average function

$$x \longmapsto u(x) - \frac{1}{|\Omega|_N}\int_\Omega u(y)\,\mathrm{d}y : \Omega \longrightarrow \mathbb{R}.$$

Then $C^{1,\beta}$-regularity results of E. DiBenedetto [10, Theorem 2, p. 829] and P. Tolksdorf [39, Theorem 1, p. 127] (interior regularity) combined with G.M. Lieberman [20, Theorem 1, p. 1203] (regularity up to the boundary) yield $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,\alpha)$. Consequently, one has $|\nabla u| \leqslant \text{const} < \infty$ in Ω.

Take $h \in \mathbb{R}^N$ arbitrary with $0 < |h| < \delta$. Multiplying Eq. (3) by $\delta_h\varphi$ (see (9)) and integrating over Ω, we arrive at

$$\int_\Omega \mathbf{a}(\nabla u) \cdot \nabla(\delta_h\varphi)\,\mathrm{d}x = \int_\Omega f(\delta_h\varphi)\,\mathrm{d}x$$

with the notation $\mathbf{a}(\nabla u) = |\nabla u|^{p-2}\nabla u$. A simple substitution $x-h$ for x on both sides, combined with $\operatorname{supp}\varphi \subset \Omega'_\delta = \{x \in \Omega:\ d(x) > \delta\}$, yields

$$\int_{\Omega'_\delta} \frac{\mathbf{a}(\nabla u(x-h)) - \mathbf{a}(\nabla u(x))}{|h|} \cdot \nabla\varphi(x)\,\mathrm{d}x = \int_{\Omega'_\delta} \frac{f(x-h) - f(x)}{|h|}\varphi\,\mathrm{d}x.$$

Here, we may substitute $-h$ for $h \in \mathbb{R}^N$, $0 < |h| < \delta$, thus arriving at

$$\int_{\Omega'_\delta} \frac{\mathbf{a}(\nabla u(x+h)) - \mathbf{a}(\nabla u(x))}{|h|} \cdot \nabla\varphi(x)\,\mathrm{d}x = \int_{\Omega'_\delta} \frac{f(x+h) - f(x)}{|h|}\varphi\,\mathrm{d}x. \tag{14}$$

Now we use the Taylor formula

$$\mathbf{a}\big(\nabla u(x+h)\big) - \mathbf{a}\big(\nabla u(x)\big) = \tilde{\mathbf{A}}(x;h)\big(\nabla u(x+h) - \nabla u(x)\big) \tag{15}$$

with the abbreviation

$$\tilde{\mathbf{A}}(x;h) \stackrel{\text{def}}{=} \int_0^1 \mathbf{A}\big((1-s)\nabla u(x+h) + s\nabla u(x)\big)\,\mathrm{d}s \in \mathbb{R}^{N\times N}$$

and replace the function φ by $(\delta_h u)\varphi$ in (14), thus arriving at

$$\int_{\Omega'_\delta} \big\langle \tilde{\mathbf{A}}(x;h)\delta_h(\nabla u), \delta_h(\nabla u)\big\rangle \varphi\,\mathrm{d}x + \int_{\Omega'_\delta} \big\langle \tilde{\mathbf{A}}(x;h)\delta_h(\nabla u), \nabla\varphi\big\rangle (\delta_h u)\,\mathrm{d}x = \int_{\Omega'_\delta} (\delta_h f)(\delta_h u)\varphi\,\mathrm{d}x. \tag{16}$$

Here, we have used the identity $\nabla[(\delta_h u)\varphi] = \varphi \cdot \delta_h(\nabla u) + (\delta_h u)\cdot \nabla\varphi$. We estimate the integrals on the left-hand side in (16) by inequalities (8), combined with Cauchy's inequality, and introduce the abbreviation

$$\tilde{a}(x;h) \stackrel{\text{def}}{=} \int_0^1 \big|(1-s)\nabla u(x+h) + s\nabla u(x)\big|^{p-2}\,\mathrm{d}s, \tag{17}$$

in order to get, with the special choice of the test function $\varphi = \varphi_\sigma^\delta$ from (10),

$$\gamma \int_{\Omega'_\delta} \tilde{a}(x;h)\big|\delta_h(\nabla u)\big|^2 \varphi_\sigma^\delta\,\mathrm{d}x \leqslant \Gamma \int_{\mathcal{O}_\sigma^\delta} \tilde{a}(x;h)\big|\delta_h(\nabla u)\big|\big|\nabla\varphi_\sigma^\delta\big||\delta_h u|\,\mathrm{d}x + \int_{\Omega'_\delta} |\delta_h f||\delta_h u|\varphi_\sigma^\delta\,\mathrm{d}x. \tag{18}$$

Let us recall that the constants $0 < \gamma \leqslant \Gamma < \infty$ originate in inequalities (8).

We use (11) to estimate the first integral on the right-hand side in inequality (18), then apply Cauchy's inequality (using the measure $\tilde{a}(x;h)\,\mathrm{d}x$), thus arriving at

$$\begin{aligned}
&\gamma \int_{\Omega'_\delta} \tilde{a}(x;h)\big|\delta_h(\nabla u)\big|^2 \varphi_\sigma^\delta\,\mathrm{d}x \quad \big(\text{by } (11)\big) \\
&\leqslant \Gamma C^{1/2}\sigma^{-1} \int_{\mathcal{O}_\sigma^\delta} \tilde{a}(x;h)\big|\delta_h(\nabla u)\big| \cdot \big(\varphi_\sigma^\delta\big)^{1/2} \cdot |\delta_h u|\,\mathrm{d}x + \int_{\Omega'_\delta} |\delta_h f||\delta_h u|\varphi_\sigma^\delta\,\mathrm{d}x \\
&\leqslant \Gamma C^{1/2}\sigma^{-1} \bigg(\int_{\mathcal{O}_\sigma^\delta} \tilde{a}(x;h)\big|\delta_h(\nabla u)\big|^2 \varphi_\sigma^\delta\,\mathrm{d}x \bigg)^{1/2} \bigg(\int_{\mathcal{O}_\sigma^\delta} \tilde{a}(x;h)|\delta_h u|^2\,\mathrm{d}x \bigg)^{1/2} + \int_{\Omega'_\delta} |\delta_h f||\delta_h u|\varphi_\sigma^\delta\,\mathrm{d}x \\
&\leqslant \frac{\gamma}{2} \int_{\mathcal{O}_\sigma^\delta} \tilde{a}(x;h)\big|\delta_h(\nabla u)\big|^2 \varphi_\sigma^\delta\,\mathrm{d}x + \frac{1}{2}\gamma^{-1}\Gamma^2 C\sigma^{-2} \int_{\mathcal{O}_\sigma^\delta} \tilde{a}(x;h)|\delta_h u|^2\,\mathrm{d}x + \int_{\Omega'_\delta} |\delta_h f||\delta_h u|\varphi_\sigma^\delta\,\mathrm{d}x
\end{aligned}$$

which yields

$$\int_{\Omega'_\delta} \tilde{a}(x;h)\big|\delta_h(\nabla u)\big|^2 \varphi_\sigma^\delta\,\mathrm{d}x \leqslant C\bigg(\frac{\Gamma}{\gamma\sigma}\bigg)^2 \int_{\mathcal{O}_\sigma^\delta} \tilde{a}(x;h)|\delta_h u|^2\,\mathrm{d}x + \frac{2}{\gamma} \int_{\Omega'_\delta} |\delta_h f||\delta_h u|\varphi_\sigma^\delta\,\mathrm{d}x.$$

Now we apply inequalities (82) and (83) (from Appendix B) to the expression $\tilde{a}(x;h)$ (defined in (17)) and abbreviate

$$\hat{a}(x;h) \stackrel{\text{def}}{=} \Big(\max_{0\leqslant s\leqslant 1}\big|(1-s)\nabla u(x+h)+s\nabla u(x)\big|\Big)^{p-2} \tag{19}$$

in order to conclude that

$$\begin{aligned}
&\int_{\Omega_\delta'} \hat{a}(x;h)\big|\delta_h(\nabla u)\big|^2 \varphi_\sigma^\delta \,\mathrm{d}x \\
&\quad\leqslant C_1' \int_{\mathcal{O}_\sigma^\delta} \hat{a}(x;h)|\delta_h u|^2 \,\mathrm{d}x + C_2' \int_{\Omega_\delta'} |\delta_h f||\delta_h u|\varphi_\sigma^\delta \,\mathrm{d}x \\
&\quad\leqslant C_1'\bigg(\int_{\mathcal{O}_\sigma^\delta} \hat{a}(x;h)\,\mathrm{d}x\bigg)\cdot \|\nabla u\|_{L^\infty(\Omega)}^2 + C_2'\bigg(\int_{\Omega_\delta'} |\delta_h f|\varphi_\sigma^\delta \,\mathrm{d}x\bigg)\cdot \|\nabla u\|_{L^\infty(\Omega)}
\end{aligned} \tag{20}$$

with some constants $C_1' > 0$, $C_2' > 0$, and $\|\nabla u\|_{L^\infty(\Omega)} \stackrel{\text{def}}{=} \operatorname{ess\,sup}_\Omega |\nabla u|$. Let us remark that the constant $C_1' = C_1'(\sigma) > 0$ depends on $\sigma \in (0,\delta)$ and satisfies $C_1'(\sigma)\sigma^2 \leqslant \text{const} < \infty$, with a help from (82) and (83).

Now we recall our hypothesis $\int_\Omega |\nabla f|\,\mathrm{d}x < \infty$ and the regularity result $|\nabla u| \leqslant \text{const} < \infty$ in Ω, together with Hypothesis (13) if $1 < p < 2$. Applying these inequalities to the right-hand side of (20), we arrive at

$$\int_{\Omega_\delta'} \hat{a}(x;h)\big|\delta_h(\nabla u)\big|^2 \varphi_\sigma^\delta \,\mathrm{d}x \leqslant C' < \infty \tag{21}$$

where the constant $C' > 0$ is independent from $h \in \mathbb{R}^N$ with $0 < |h| < \delta$. In analogy with C_1', also $C' = C'(\sigma)$ depends on $\sigma \in (0,\delta)$ and satisfies $C'(\sigma)\sigma^2 \leqslant \text{const} < \infty$. More precisely, if $2 < p < \infty$, we apply $|\nabla u| \leqslant \text{const} < \infty$ in Ω to get an upper bound on the right-hand side of (19), whereas if $1 < p < 2$, we need to combine $\hat{a}(x;h) \leqslant |\nabla u(x)|^{p-2}$ with Hypothesis (13) and $|\nabla u| \leqslant \text{const} < \infty$ in Ω again.

Finally, we deduce from (7) that the Jacobian matrix $\mathbf{B} = (B_{ij})_{i,j=1}^N$ of the mapping $\boldsymbol{\eta} \mapsto \mathbf{b}(\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{(p/2)-1}\boldsymbol{\eta} : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\mathbf{B}(\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{(p/2)-1}\bigg(\mathbf{I} + \frac{p-2}{2}\,\frac{\boldsymbol{\eta}\otimes\boldsymbol{\eta}}{|\boldsymbol{\eta}|^2}\bigg) \quad \text{for } \boldsymbol{\eta} \in \mathbb{R}^N \setminus \{\mathbf{0}\}. \tag{22}$$

In analogy with the Taylor formula in (15) we have

$$\mathbf{b}\big(\nabla u(x+h)\big) - \mathbf{b}\big(\nabla u(x)\big) = \tilde{\mathbf{B}}(x;h)\big(\nabla u(x+h) - \nabla u(x)\big) \tag{23}$$

with the abbreviation

$$\tilde{\mathbf{B}}(x;h) \stackrel{\text{def}}{=} \int_0^1 \mathbf{B}\big((1-s)\nabla u(x+h) + s\nabla u(x)\big)\,\mathrm{d}s \in \mathbb{R}^{N\times N}.$$

Now we treat the L^2-norm

$$\int_{\Omega'_\delta} \left|\delta_h\big(\mathbf{b}(\nabla u)\big)\right|^2 \varphi_\sigma^\delta \,\mathrm{d}x = \int_{\Omega'_\delta} \big\langle \tilde{\mathbf{B}}(x;h)\delta_h(\nabla u), \tilde{\mathbf{B}}(x;h)\delta_h(\nabla u)\big\rangle \varphi_\sigma^\delta \,\mathrm{d}x$$
$$= \int_{\Omega'_\delta} \big\langle \tilde{\mathbf{B}}(x;h)^2\delta_h(\nabla u), \delta_h(\nabla u)\big\rangle \varphi_\sigma^\delta \,\mathrm{d}x \tag{24}$$

where, using the abbreviation

$$\boldsymbol{\eta}(s) = (1-s)\nabla u(x+h) + s\nabla u(x) \in \mathbb{R}^N \quad \text{for } 0 \leqslant s \leqslant 1,$$

we have

$$\tilde{\mathbf{B}}(x;h)^2 = \int_0^1 \int_0^1 \mathbf{B}\big(\boldsymbol{\eta}(s)\big)\mathbf{B}\big(\boldsymbol{\eta}(t)\big)\,\mathrm{d}s\,\mathrm{d}t$$
$$= \int_0^1 \int_0^1 \left|\boldsymbol{\eta}(s)\right|^{(p/2)-1} \left|\boldsymbol{\eta}(t)\right|^{(p/2)-1} \mathbf{C}\big(\boldsymbol{\eta}(s), \boldsymbol{\eta}(t)\big)\,\mathrm{d}s\,\mathrm{d}t \tag{25}$$

with the $(N \times N)$-matrix

$$\mathbf{C}(\mathbf{a},\mathbf{b}) = \left(\mathbf{I} + \frac{p-2}{2}\frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2}\right)\left(\mathbf{I} + \frac{p-2}{2}\frac{\mathbf{b}\otimes\mathbf{b}}{|\mathbf{b}|^2}\right) \in \mathbb{R}^{N\times N}$$

being uniformly bounded for $\mathbf{a},\mathbf{b} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, cf. (22). Furthermore, we have $\mathbf{C}(\mathbf{a},\mathbf{b}) = \mathbf{C}(\mathbf{a})\mathbf{C}(\mathbf{b})$ where

$$\mathbf{C}(\mathbf{a}) \stackrel{\text{def}}{=} \left(\mathbf{I} + \frac{p-2}{2}\frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2}\right) \in \mathbb{R}^{N\times N}$$

is a symmetric matrix with the eigenvalues 1 and $p/2$ (if $N \geqslant 2$). Consequently, there is a constant $\Gamma' > 0$ such that the kernel $\mathbf{B}(\boldsymbol{\eta}(s))\mathbf{B}(\boldsymbol{\eta}(t))$ of the quadratic form contained in the integrand on the right-hand side of Eq. (24) satisfies

$$\big\langle \mathbf{B}\big(\boldsymbol{\eta}(s)\big)\mathbf{B}\big(\boldsymbol{\eta}(t)\big)\boldsymbol{\xi}, \boldsymbol{\xi}\big\rangle = \big\langle \mathbf{B}\big(\boldsymbol{\eta}(t)\big)\boldsymbol{\xi}, \mathbf{B}\big(\boldsymbol{\eta}(s)\big)\boldsymbol{\xi}\big\rangle \leqslant \left|\mathbf{B}\big(\boldsymbol{\eta}(t)\big)\boldsymbol{\xi}\right| \cdot \left|\mathbf{B}\big(\boldsymbol{\eta}(s)\big)\boldsymbol{\xi}\right|$$
$$\leqslant \left|\mathbf{B}\big(\boldsymbol{\eta}(t)\big)\right| \cdot \left|\mathbf{B}\big(\boldsymbol{\eta}(s)\big)\right| \cdot |\boldsymbol{\xi}|^2 \leqslant \Gamma' \left|\boldsymbol{\eta}(t)\right|^{(p/2)-1} \left|\boldsymbol{\eta}(s)\right|^{(p/2)-1} |\boldsymbol{\xi}|^2$$

for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and for all $s,t \in [0,1]$ such that $\boldsymbol{\eta}(s), \boldsymbol{\eta}(t) \neq \mathbf{0}$. We first integrate this inequality with respect to $s,t \in [0,1]$, then apply it to (25) to get

$$\big\langle \tilde{\mathbf{B}}(x;h)^2\boldsymbol{\xi}, \boldsymbol{\xi}\big\rangle \leqslant \Gamma' \tilde{b}(x;h)^2 |\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^N, \tag{26}$$

with the abbreviation

$$\tilde{b}(x;h) \stackrel{\text{def}}{=} \int_0^1 \left|\boldsymbol{\eta}(s)\right|^{(p/2)-1} \mathrm{d}s$$
$$= \int_0^1 \left|(1-s)\nabla u(x+h) + s\nabla u(x)\right|^{(p/2)-1} \mathrm{d}s,$$

in analogy with (17). This integral is estimated from above by inequalities (82) and (83),

$$\tilde{b}(x;h) \leqslant C_p \hat{b}(x;h), \tag{27}$$

where $C_p > 0$ is a numerical constant depending only on p $(1 < p < \infty)$ and

$$\hat{b}(x;h) \stackrel{\text{def}}{=} \Big(\max_{0 \leqslant s \leqslant 1} \big| (1-s)\nabla u(x+h) + s\nabla u(x) \big| \Big)^{(p/2)-1}.$$

We combine inequalities (26) and (27) and apply them to the quadratic form contained in the integrand on the right-hand side of Eq. (24), thus obtaining

$$\begin{aligned} \int\limits_{\Omega'_\delta} \big| \delta_h\big(\mathbf{b}(\nabla u)\big)\big|^2 \varphi_\sigma^\delta \, \mathrm{d}x &\leqslant \Gamma' \int\limits_{\Omega'_\delta} \tilde{b}(x;h)^2 \big|\delta_h(\nabla u)\big|^2 \varphi_\sigma^\delta \, \mathrm{d}x \\ &\leqslant \Gamma'' \int\limits_{\Omega'_\delta} \hat{b}(x;h)^2 \big|\delta_h(\nabla u)\big|^2 \varphi_\sigma^\delta \, \mathrm{d}x, \end{aligned} \tag{28}$$

with the constant $\Gamma'' = \Gamma' C_p^2 > 0$. As $\hat{a}(x;h) = \hat{b}(x;h)^2$ by (19), inequality (21) implies

$$\int\limits_{\Omega'_\delta} \big| \delta_h\big(\mathbf{b}(\nabla u)\big)\big|^2 \varphi_\sigma^\delta \, \mathrm{d}x \leqslant C'' < \infty \tag{29}$$

where the constant $C'' = C''(\sigma) > 0$ is independent from $h \in \mathbb{R}^N$ with $0 < |h| < \delta$, but it depends on $\sigma \in (0,\delta)$ and satisfies $C''(\sigma)\sigma^2 \leqslant \text{const} < \infty$. We are now ready to derive *all* conclusions of our theorem from this estimate.

Part (a) of both Theorems 3.1 and 3.2, follows immediately from inequality (29), by Ziemer [41, Theorem 2.1.6, pp. 45–46].

Part (b) follows from an easy combination of the chain rule [41, Theorem 2.1.11, p. 48] with Part (a). It can be derived also directly from inequality (29).

Finally, as for Part (c) of Theorem 3.2, for $1 < p < 2$, the desired claims follow again directly from a combination of inequality (29) with [41, Theorem 2.1.6, pp. 45–46].

Both theorems are proved. □

We remark that, for $1 < p < 2$, our Hypothesis (13) is clearly indispensable in inequalities (20), where we need

$$\int\limits_{\mathcal{O}_\sigma^\delta} \hat{a}(x;h) \, \mathrm{d}x \leqslant \int\limits_{\mathcal{O}_\sigma^\delta} \big|\nabla u(x)\big|^{p-2} \, \mathrm{d}x < \infty.$$

For $2 \leqslant p < \infty$, the last inequality holds trivially, by $|\nabla u| \leqslant \text{const} < \infty$ in Ω.

4. Pohozhaev's identity

We consider the degenerate (or singular) elliptic equation with the p-Laplacian (12) which is assumed to hold in the sense of distributions in Ω, i.e.,

$$-\int\limits_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla\varphi \, \mathrm{d}x = \int\limits_{\Omega} f\varphi \, \mathrm{d}x \quad \text{for every } \varphi \in C_0^1(\Omega). \tag{30}$$

We assume that $f \in L^1(\Omega)$ is a given function and $u \in C^1(\overline{\Omega})$ satisfies Eq. (12). As usual, $C^1(\overline{\Omega})$ denotes the linear space of all continuously differentiable functions $u : \overline{\Omega} \to \mathbb{R}$. Furthermore, Eq. (30) remains valid for every $\varphi \in W_0^{1,q}(\Omega)$, $N < q < \infty$, as $W_0^{1,q}(\Omega)$ is the closure of $C_0^1(\Omega)$ in the Sobolev space $W^{1,q}(\Omega)$ and the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$ is continuous.

Remark 4.1. Our hypothesis $u \in C^1(\overline{\Omega})$ above is *satisfied* typically in the following two cases:

(i) If Ω' is another domain in $\mathbb{R}^N$, such that $\overline{\Omega} \subset \Omega'$, $f \in L^\infty_{\mathrm{loc}}(\Omega')$, and $u \in W^{1,p}_{\mathrm{loc}}(\Omega')$ satisfies Eq. (12) in the sense of distributions in Ω', then we have $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega')$ for some $\beta \in (0,1)$, by a local (interior) regularity result due to E. DiBenedetto [10, Theorem 2, p. 829] and P. Tolksdorf [39, Theorem 1, p. 127]. The constant β depends solely on N, p, and Γ/γ. Hence $u \in C^{1,\beta}(\overline{\Omega})$.

(ii) If Eq. (12) is supplemented by standard (Dirichlet or Neumann) boundary conditions on the boundary $\partial\Omega$, which is assumed to be of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, then $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,\alpha)$, by a global (up to the boundary) regularity result of G.M. Lieberman [20, Theorem 1, p. 1203]. Again, the constant β depends solely on α, N, p, and Γ/γ.

The Pohozhaev identity, proved next, is an important application of our regularity results (Theorem 3.1 for $p \geqslant 2$) and those of P. Pucci and R. Servadei [28, Theorem 2.5, p. 3351] (for $1 < p < 2$, first announced in [27, Theorem 1, p. 257]). We refer to M. Ôtani [22, Section 4, §4.1, pp. 150–157] and F. de Thélin [37, Section III, pp. 384–388] for additional versions of Pohozhaev's identity and inequality and their applications to elliptic problems with the p-Laplacian. Several interesting generalizations of the original Pohozhaev identity [23] for classical solutions of quasilinear elliptic equations are studied in the classical works of S.I. Pohozhaev [24] and P. Pucci and J. Serrin [25].

Recall that $\nu \equiv \nu(x_0) \in \mathbb{R}^N$ denotes the exterior unit normal to $\partial\Omega$ at $x_0 \in \partial\Omega$. As usual, we denote by $\mathrm{d}\sigma(x_0)$ the surface measure on $\partial\Omega$.

Theorem 4.2 *(Pohozhaev's identity). Let $1 < p < \infty$ and assume that $f \in L^1(\Omega)$ possesses distributional derivatives $\partial f/\partial x_i \in L^1_{\mathrm{loc}}(\Omega)$; $i = 1, 2, \dots, N$. Finally, assume that $u \in C^1(\overline{\Omega})$ satisfies Eq.* (12) *in the sense of distributions in Ω and $|\nabla u|^q \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for some $q \in (1,p)$. Then we have the* Pohozhaev identity

$$\frac{N-p}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega f(x)(x\cdot\nabla u)\,\mathrm{d}x$$
$$= \frac{1}{p}\int_{\partial\Omega} |\nabla u|^p\big(x\cdot\nu(x)\big)\,\mathrm{d}\sigma(x) - \int_{\partial\Omega} |\nabla u|^{p-2}(x\cdot\nabla u)\big(\nu(x)\cdot\nabla u\big)\,\mathrm{d}\sigma(x). \tag{31}$$

Before giving the proof of this theorem, let us consider the special case when Ω is a ball, $\Omega = B_R(\mathbf{0}) \stackrel{\mathrm{def}}{=} \{x \in \mathbb{R}^N\colon\ |x| < R\}$ with $0 < R < \infty$.

Corollary 4.3. *Let $\Omega = B_R(\mathbf{0})$ with $0 < R < \infty$. Then Eq.* (31) *simplifies to*

$$\frac{N-p}{p}\int_{B_R(\mathbf{0})} |\nabla u|^p\,\mathrm{d}x + \int_{B_R(\mathbf{0})} f(x)(x\cdot\nabla u)\,\mathrm{d}x = R\int_{\partial B_R(\mathbf{0})} |\nabla u|^{p-2}\left(\frac{1}{p}|\nabla u|^2 - \left|\frac{\partial u}{\partial\nu}\right|^2\right)\mathrm{d}\sigma(x). \tag{32}$$

Proof. For every $x \in \partial B_R(\mathbf{0}) = \{x \in \mathbb{R}^N\colon\ |x| = R\}$ we have $\nu(x) = x/R$; hence,

$$x\cdot\nu(x) = |x| = R, \qquad \nu(x)\cdot\nabla u = \frac{\partial u}{\partial\nu}, \quad \text{and} \quad x\cdot\nabla u = R\frac{\partial u}{\partial\nu}.$$

Consequently, Eq. (32) follows from (31) as claimed. □

Remark 4.4. (i) Our regularity hypothesis $u \in C^1(\overline{\Omega})$ (cf. Remark 4.1) in Theorem 4.2 and Corollary 4.3 is justified by the following arguments from the beginning of the proof of Theorems 3.1 and 3.2: If $f \in L^\infty(\Omega)$ and $u \in W^{1,p}(\Omega)$ satisfies the Dirichlet or Neumann problem for Eq. (12) in the weak sense (30), which is then supposed to hold for all $\varphi \in C_0^1(\Omega)$ or for all $\varphi \in C^1(\overline{\Omega})$, respectively, then we have $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,\alpha)$, by G.M. Lieberman [20, Theorem 1, p. 1203] (Remark 4.1).

However, we prefer to impose the weaker, but direct hypothesis $u \in C^1(\overline{\Omega})$ as it is satisfied even if we do not require $f \in L^\infty(\Omega)$, cf. J. Giacomoni, I. Schindler, and P. Takáč [13], Hypothesis (H2), inequality (B.8), and Theorem B.1, pp. 147–148, or D.D. Hai [16, Lemma 3.1, p. 620].

(ii) Our second regularity hypothesis, $|\nabla u|^q \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for some $q \in (1,p)$, is satisfied by a result due to H.-W. Lou [21, Lemma 2.1, p. 522] with $q = p-1$, provided $f \in L^r_{\mathrm{loc}}(\Omega)$ for some $r > N/p$.

In the case of *Dirichlet* boundary conditions, $u = 0$ on $\partial\Omega$, our Theorem 4.2, Eq. (31), follows from a more general result in M. Degiovanni, A. Musesti, and M. Squassina [6, Theorem 2, p. 318]. Their result is derived from [6, Lemma 1, p. 319]. The proof of [6, Theorem 2] is based essentially on the fact that $u \in C^1(\overline{\Omega})$ can be approximated in the norm of $W^{1,p}(\Omega)$ by a sequence of functions $u_k \in C_c^\infty(\Omega)$ ($k = 1, 2, \ldots$), i.e., C^∞-functions with compact support in Ω. A special case of [6, Theorem 2] has been established earlier in M. Guedda and L. Véron [15, Theorem 1.1, p. 884] by similar approximation methods. Quite similar ideas and tools are applied by P. Pucci and R. Servadei [29, Lemma 3.1, p. 5] in order to establish an analogue of Pohozhaev's identity (31) for a function $u \in D^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$ that satisfies Eq. (12) in the sense of distributions in $\mathbb{R}^N \setminus \{0\}$. Their setting is somewhat different from ours: Eq. (12) is considered in the unbounded punctured domain $\mathbb{R}^N \setminus \{0\}$ with singular weights. It might be of considerable interest to extend the approximation procedures from [6,15,29] to our setting with *Neumann* boundary conditions.

We will derive Pohozhaev's identity (31) from its *local* version proved below:

Lemma 4.5 *(Local Pohozhaev identity). Let $1 < p < \infty$ and $f \in L^1_{\mathrm{loc}}(\Omega)$. Assume that $u \in C^1(\Omega)$ satisfies Eq.* (12) *in the sense of distributions in Ω and $|\nabla u|^q \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for some $q \in (1,p)$. Then the* local Pohozhaev identity

$$\operatorname{div}\left((x \cdot \nabla u)|\nabla u|^{p-2}\nabla u - \frac{1}{p}x|\nabla u|^p\right) = -f(x)(x \cdot \nabla u) - \frac{N-p}{p}|\nabla u|^p \tag{33}$$

holds in the sense of distributions in Ω.

We remark that formally (which we will justify in the proof Lemma 4.5)

$$\operatorname{div}\left((x \cdot \nabla u)|\nabla u|^{p-2}\nabla u - \frac{1}{p}x|\nabla u|^p\right) = (x \cdot \nabla u)\Delta_p u + \left(1 - \frac{N}{p}\right)|\nabla u|^p$$

holds in the sense of distributions in Ω.

Proof of Lemma 4.5. We begin with the formal calculations

$$\begin{aligned} \operatorname{div}\big((x \cdot \nabla u)|\nabla u|^{p-2}\nabla u\big) &= (x \cdot \nabla u)\Delta_p u + \nabla(x \cdot \nabla u) \cdot |\nabla u|^{p-2}\nabla u \\ &= (x \cdot \nabla u)\Delta_p u + |\nabla u|^p + \frac{1}{p}x \cdot \nabla|\nabla u|^p, \end{aligned} \tag{34}$$

$$\operatorname{div}\big(x|\nabla u|^p\big) = N|\nabla u|^p + x \cdot \nabla|\nabla u|^p \tag{35}$$

which are certainly valid pointwise in the open subset

$$U_0 = \big\{x \in \Omega\colon\ \nabla u(x) \neq \mathbf{0}\big\}$$

of Ω, owing to $u \in C^1(U_0) \cap W^{2,2}_{\mathrm{loc}}(U_0)$. Namely, by Part (b) of our Theorems 3.1 and 3.2, and by Remark 3.3(i), we have $u \in W^{2,2}(U_\eta)$ for every $\eta > 0$, where

$$U_\eta = \{x \in \Omega\colon\ |\nabla u(x)| > \eta\}.$$

Thus, Lemma A.3 (Appendix A) provides the correct product rule in U_η for our calculations above. This lemma can be easily applied to justify (34) and (35) in the open set U_η. It follows that both (34) and (35) are valid in the sense of distributions in

$$U_0 = \bigcup_{\eta>0} U_\eta = \{x \in \Omega\colon\ \nabla u(x) \neq \mathbf{0}\}.$$

Consequently, from Eq. (34) we may subtract $(1/p)$-multiple of Eq. (35), thus arriving at Eq. (33) valid in the sense of distributions in U_0.

Our goal is to verify that Eq. (33) holds in the sense of distributions in Ω, that is,

$$\int_\Omega \mathbf{v} \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f(x)(x \cdot \nabla u)\varphi \,\mathrm{d}x + \frac{N-p}{p} \int_\Omega |\nabla u|^p \varphi \,\mathrm{d}x \tag{36}$$

for every test function $\varphi \in C_0^1(\Omega)$, where $\mathbf{v}\colon \Omega \to \mathbb{R}^N$ stands for the vector field

$$\mathbf{v}(x) \stackrel{\text{def}}{=} (x \cdot \nabla u)|\nabla u|^{p-2}\nabla u - \frac{1}{p} x|\nabla u|^p, \quad x \in \overline{\Omega}, \tag{37}$$

which is continuous in $\overline{\Omega}$, i.e., $\mathbf{v} \in [C(\overline{\Omega})]^N$, by $u \in C^1(\overline{\Omega})$. The divergence of the vector field $\mathbf{v}$ is obtained by subtracting the left-hand side of $(1/p)$-multiple of Eq. (35) from the left-hand side of Eq. (34). Since we already know that Eq. (36) holds for every $\varphi \in C_0^1(U_0)$, it suffices to verify that it holds also for every $\varphi \in C_0^1(U'_\eta)$, where

$$U'_\eta = \{x \in \Omega\colon\ |\nabla u(x)| < \eta\} = \Omega \setminus \overline{U_\eta}$$

denotes the complement in Ω of the closure $\overline{U_\eta}$ of U_η for $\eta > 0$. All set closures in this proof are taken in $\mathbb{R}^N$. To this end, let us first fix a test function $\varphi \in C_0^1(\Omega)$ whose (compact) support we denote by

$$\operatorname{supp}\varphi = \overline{\{x \in \Omega\colon\ \varphi(x) \neq 0\}}^{\mathbb{R}^N} \subset \Omega.$$

For each $\eta > 0$, we need an Urysohn-type $W^{1,1}_{\mathrm{loc}}$-function $\psi_\eta : \Omega \to [0,1]$ that separates the (compact) sets $U'_0 \cap \operatorname{supp}\varphi$ and $\overline{U_\eta} \cap \operatorname{supp}\varphi$ from each other, where

$$U'_0 = \Omega \setminus U_0 = \bigcap_{\eta>0} U'_\eta = \{x \in \Omega\colon\ \nabla u(x) = \mathbf{0}\}.$$

Such a function ψ_η is constructed as follows:

$$\psi_\eta(x) \stackrel{\text{def}}{=} \left(\min\{\eta^{-1}|\nabla u|, 1\}\right)^q \quad \text{for } x \in \Omega.$$

Then, clearly, $\psi_\eta(x) = 0$ if $x \in U_0'$, and $\psi_\eta(x) = 1$ if $x \in \overline{U_\eta}$. Consequently, we have $\psi_\eta(x) \to \psi_0(x)$ as $\eta \to 0+$, for every $x \in \Omega$, where

$$\psi_0(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in U_0' = \{x \in \Omega\colon \nabla u(x) = \mathbf{0}\}; \\ 1 & \text{if } x \in U_0 = \{x \in \Omega\colon \nabla u(x) \neq \mathbf{0}\}. \end{cases}$$

Our hypothesis $|\nabla u|^q \in W^{1,1}_{\mathrm{loc}}(\Omega)$ guarantees $\psi_\eta \in W^{1,1}_{\mathrm{loc}}(\Omega)$ as desired. In addition, for a.e. $x \in \Omega = U_\eta' \cup \overline{U_\eta}$ we have

$$\nabla \psi_\eta(x) = \begin{cases} q\eta^{-q} |\nabla u|^{q-2} (\nabla u \cdot \nabla^2 u) & \text{if } x \in U_\eta'; \\ \mathbf{0} & \text{if } x \in \overline{U_\eta}, \end{cases} \tag{38}$$

provided we know that $|\nabla u|^{q-1} |\nabla^2 u| \in L^1_{\mathrm{loc}}(\Omega)$, cf. Parts (b) and (c) of Theorems 3.1 and 3.2.

Hence, we can decompose $\varphi \in C_0^1(\Omega)$ as the sum $\varphi = \psi_\eta \varphi + (1 - \psi_\eta)\varphi$ with $\psi_\eta \varphi \in W_0^{1,1}(U_0)$ and $(1 - \psi_\eta)\varphi \in W_0^{1,1}(U_\eta')$. Since Eq. (36) holds for $\psi_\eta \varphi$ in place of φ, by the density of $C_0^1(\Omega)$ in $W_0^{1,1}(U_0)$, it suffices to verify that for $(1 - \psi_\eta)\varphi$ in place of φ all integrals in Eq. (36) tend to zero as $\eta \to 0+$. Of course, the domain of integration Ω may be replaced by U_η'. This claim is obvious for both integrals on the right-hand side of Eq. (36), by Lebesgue's dominated convergence theorem:

$$\int_{U_\eta'} f(x)(x \cdot \nabla u)(1 - \psi_\eta)\varphi \, \mathrm{d}x \longrightarrow \int_{U_0'} f(x)(x \cdot \nabla u)(1 - \psi_0)\varphi \, \mathrm{d}x = 0,$$

$$\int_{U_\eta'} |\nabla u|^p (1 - \psi_\eta)\varphi \, \mathrm{d}x \longrightarrow \int_{U_0'} |\nabla u|^p (1 - \psi_0)\varphi \, \mathrm{d}x = 0,$$

as $\eta \to 0+$, thanks to $\nabla u = \mathbf{0}$ in U_0'. Recall that $\psi_\eta(x) \to \psi_0(x)$ as $\eta \to 0+$, for every $x \in \Omega$, and $\psi_0 = 0$ in U_0'.

The integral on the left-hand side of Eq. (36), with $(1 - \psi_\eta)\varphi$ in place of φ, for $\eta \in (0, 1)$, is first estimated by

$$\begin{aligned} \left| \int_\Omega \mathbf{v} \cdot \nabla \big[(1 - \psi_\eta)\varphi \big] \, \mathrm{d}x \right| &\leqslant \left(1 + \frac{1}{p}\right) \int_{U_\eta'} |x| |\nabla u|^p \big(|\varphi| |\nabla \psi_\eta| + |\nabla \varphi| \big) \, \mathrm{d}x \\ &= \left(1 + \frac{1}{p}\right) \eta^{p-q} \int_{U_\eta'} |x| \big(\eta^{-1} |\nabla u|\big)^p \big(|\varphi| |\nabla \hat{\psi}_\eta| + \eta^q |\nabla \varphi| \big) \, \mathrm{d}x \\ &\leqslant \left(1 + \frac{1}{p}\right) \eta^{p-q} \int_{U_\eta'} |x| \big(|\varphi| |\nabla \hat{\psi}_\eta| + |\nabla \varphi| \big) \, \mathrm{d}x, \end{aligned} \tag{39}$$

where

$$\hat{\psi}_\eta(x) \stackrel{\text{def}}{=} \eta^q \psi_\eta(x) = \big(\min\big\{|\nabla u|, \eta\big\}\big)^q \quad \text{for } x \in \Omega. \tag{40}$$

Now we observe that all functions in the family $\nabla \hat{\psi}_\eta$, for $\eta \in (0, 1)$, have uniformly bounded L^1-norms over the compact set $\overline{U_\eta'} \cap \operatorname{supp} \varphi \subset \Omega$. Applying this fact to the last integral in inequality (39) above, we arrive at

$$\left| \int_{\Omega} \mathbf{v} \cdot \nabla \big[(1 - \psi_\eta) \varphi \big] \, \mathrm{d}x \right| \leqslant C \eta^{p-q} \longrightarrow 0 \quad \text{as } \eta \longrightarrow 0+,$$

thanks to $1 < q < p$, where $C > 0$ is a constant independent from $\eta \in (0, 1)$.

Finally, recalling that Eq. (36) holds for $\psi_\eta \varphi$ in place of φ, from our estimates above we deduce that it holds also for $\varphi = \psi_\eta \varphi + (1 - \psi_\eta)\varphi$, by letting $\eta \to 0+$. This concludes our proof of Eq. (36) and, thus, of the local Pohozhaev identity (33). □

Proof of Theorem 4.2. The vector field $\mathbf{v} : \overline{\Omega} \to \mathbb{R}^N$ under the divergence on the left-hand side of Eq. (33), defined in (37), is continuous, by $u \in C^1(\overline{\Omega})$. We further observe that the right-hand side of Eq. (33) belongs to $L^1(\Omega)$, by $f \in L^1(\Omega)$ and $\nabla u \in [C(\overline{\Omega})]^N$. We complete the proof of (31) by applying the divergence theorem (Lemma A.1) to Eq. (33) in Lemma 4.5. □

5. Some applications

Let us consider the so-called bi-stable equation

$$-\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) + W'(u) = 0 \quad \text{in } \Omega. \tag{41}$$

Here, $1 < p \leqslant N$ and W is typically a double-well potential of type $W(s) = |1 - |s|^\beta|^\alpha$ for $s \in \mathbb{R}$, where $\alpha, \beta > 1$ are some constants. We remark that the restriction $p \leqslant N$ is forced by our use of Pohozhaev's identity; it might not be essential for our results below. Also observe that the function $s \mapsto -sW'(s) : \mathbb{R} \to \mathbb{R}$ is bounded above (by a positive constant), owing to

$$-sW'(s) = \alpha\beta \big| 1 - |s|^\beta \big|^{\alpha - 2} \big(1 - |s|^\beta\big) |s|^\beta < \alpha\beta \quad \text{for all } s \in \mathbb{R}.$$

More generally, we assume that the potential W satisfies only the following

Hypotheses.

(W1) $W : \mathbb{R} \to \mathbb{R}$ is a nonnegative, continuously differentiable function that attains its global minimum $W(1) = W(-1) = 0$ precisely at the two "wells" $s = \pm 1$, and a local maximum $W(0)$ at $s = 0$, i.e., $W(s) > W(\pm 1) = 0$ for all $s \in \mathbb{R} \setminus \{-1, +1\}$.

(W2) The function $s \mapsto -sW'(s) : \mathbb{R} \to \mathbb{R}$ is bounded above (by a positive constant).

Remark 5.1. It is worth of mentioning that the function $W(s)$ is allowed to grow with even Sobolev-*supercritical* growth as $s \to \pm\infty$, i.e., we do *not* require that there be a constant $C \geqslant 0$ such that

$$\big| W(s) \big| \leqslant C\big(1 + |s|^{p^*}\big) \quad \text{holds for all } s \in \mathbb{R},$$

where $p^* = \frac{Np}{N-p}$ if $1 < p < N$, and $p^* \in (1, \infty)$ is arbitrary (sufficiently large) if $p = N$. Only the *variation* of $W(s)$ is limited by Hypothesis (W2).

Nonexistence results for elliptic problems with the p-Laplacian and *supercritical* growth have been obtained, by virtue of Pohozhaev's identity and inequality, in M. Ôtani [22, Theorem III, p. 142] and F. de Thélin [37, Théorème 4, p. 384].

In our examples below we consider two types of domain $\Omega \subset \mathbb{R}^N$: a bounded domain Ω with C^2-boundary and the exterior domain Ω of (the closure of) such a domain ($\Omega_0 = \mathbb{R}^N \setminus \overline{\Omega}$ is assumed to be bounded and simply connected in the latter case).

5.1. A bounded domain Ω

Let Ω and Ω_1 ($\neq \emptyset$) be bounded domains in $\mathbb{R}^N$ with C^2-boundaries, such that $\overline{\Omega_1} \subset \Omega$ and the open set $\Omega_1' = \Omega \setminus \overline{\Omega_1}$ is connected.

Definition 5.2. We say that a function $u : \overline{\Omega_1'} \to \mathbb{R}$ is a **phase transition solution** of

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + W'(u) = 0 \quad \text{in } \Omega_1' \tag{42}$$

if $u \in W^{1,p}(\Omega_1')$ verifies Eq. (42) above in the weak sense (cf. Eq. (30)) with the Neumann boundary conditions

$$\partial u/\partial \nu = 0 \quad \text{on } \partial\Omega_1' = \partial\Omega \cup \partial\Omega_1, \tag{43}$$

and the following additional "phase transition" property holds

$$u = -1 \quad \text{on } \partial\Omega \quad \text{and} \quad u = 1 \quad \text{on } \partial\Omega_1. \tag{44}$$

Of course, one may replace Eq. (44) by

$$u = 1 \quad \text{on } \partial\Omega \quad \text{and} \quad u = -1 \quad \text{on } \partial\Omega_1.$$

Obviously, Eq. (42) with the boundary conditions (43) and (44) pose an overdetermined boundary value problem. In case $N = 1 < p < \infty$ and $W(s) = |1 - |s|^\beta|^\alpha$ for $s \in \mathbb{R}$, such phase transition solutions have been obtained in P. Drábek, R.F. Manásevich, and P. Takáč [11, Proposition 4.3, p. 104] and, in a special case closely related to $\beta = p = 2$ above, also in Ph. Rosenau and E. Kashdan [30]. On the other hand, mathematically somewhat related "*compacton*" *solutions* (i.e., nonnegative solutions with compact support in a given domain $\subset \mathbb{R}^N$) have been obtained in Y.Sh. Il'yasov and Y.V. Egorov [19, Theorem 1.1] for sufficiently high space dimension $N \geqslant 3$ and in [30] for $N = 2$ (only numerically). Also related "*dead core*" *solutions* to quasilinear elliptic problems are treated in the work of J.I. Díaz and J. Hernández [8] and J.I. Díaz, J. Hernández, and F.J. Mancebo [9] (for $N = 1$) and, in a higher space dimension ($N \geqslant 1$), in S.N. Antontsev, J.I. Díaz, and S.I. Shmarev [2], C. Cortázar, M. Elgueta, and P. Felmer [3,4], J.I. Díaz [7], S. Kamin and L. Véron [18], and P. Pucci and J. Serrin [26]. Especially the last of these works, [26] (in Theorems 1.1 and 1.2 on p. 261) provides necessary and sufficient conditions on the existence of "dead core" solutions in *any* space dimension N; their method is based on a necessary and sufficient condition for the validity of the strong maximum principle.

The next theorem implies the nonexistence of a phase transition solution to Eq. (42) for $p \leqslant N$ or, in other words, for *sufficiently high* space dimension $N \geqslant p$.

Theorem 5.3. *Let Ω and Ω_1 ($\emptyset \neq \overline{\Omega_1} \subset \Omega \subset \mathbb{R}^N$) be as in* Definition 5.2 *above and $1 < p \leqslant N$. Furthermore, assume that Ω_1 is star-shaped with respect to the origin $\mathbf{0} \in \mathbb{R}^N$. Then any weak solution $u \in W^{1,p}(\Omega_1')$ of Eq.* (42) *with the Neumann boundary conditions* (43)*, such that $|u| = 1$ on $\partial\Omega$, must be constant, i.e., $u \equiv \pm 1$ throughout Ω_1'.*

Proof. Suppose that $u \in W^{1,p}(\Omega_1')$ is such a solution. Since the function $s \mapsto -sW'(s) : \mathbb{R} \to \mathbb{R}$ is bounded above (by a positive constant), by Hypothesis (W2), a regularity result from A. Anane's thesis [1, Théorème A.1, p. 96] or M. Ôtani [22, Theorem II, p. 142] guarantees $u \in L^\infty(\Omega_1')$. This claim is justified at the beginning of the proof of Theorems 3.1 and 3.2. Again, $C^{1,\beta}$-regularity results of E. DiBenedetto [10, Theorem 2, p. 829] and P. Tolksdorf [39, Theorem 1, p. 127] (interior regularity) combined with G.M. Lieberman [20, Theorem 1, p. 1203] (regularity up to the boundary) yield $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, \alpha)$. In particular, we have $W'(u) \in L^\infty(\Omega_1')$ together with $\nabla W(u) = W'(u)\nabla u \in [L^\infty(\Omega_1')]^N$.

Consequently, we are allowed to apply Pohozhaev's identity (31) to Eq. (42) to get

$$\begin{aligned}
&\frac{N-p}{p}\int_{\Omega_1'}|\nabla u|^p\,\mathrm{d}x-\int_{\Omega_1'}W'(u)(x\cdot\nabla u)\,\mathrm{d}x\\
&=\frac{1}{p}\int_{\partial\Omega_1'}|\nabla u|^p\big(x\cdot\nu(x)\big)\,\mathrm{d}\sigma(x)-\int_{\partial\Omega_1'}|\nabla u|^{p-2}(x\cdot\nabla u)\big(\nu(x)\cdot\nabla u\big)\,\mathrm{d}\sigma(x)\\
&=\frac{1}{p}\int_{\partial\Omega_1'}|\nabla u|^p\big(x\cdot\nu(x)\big)\,\mathrm{d}\sigma(x),
\end{aligned}\tag{45}$$

the last equality being a consequence of the Neumann boundary conditions (43). In the Pohozhaev identity we have

$$\begin{aligned}
\int_{\Omega_1'}W'(u)(x\cdot\nabla u)\,\mathrm{d}x&=\int_{\Omega_1'}x\cdot\nabla W(u)\,\mathrm{d}x\\
&=-N\int_{\Omega_1'}W(u)\,\mathrm{d}x+\int_{\partial\Omega_1'}W(u)\big(x\cdot\nu(x)\big)\,\mathrm{d}\sigma(x),
\end{aligned}\tag{46}$$

by the divergence theorem (Lemma A.1) applied to the vector field $x\mapsto x\cdot W(u(x))$ which is continuous in $\overline{\Omega_1'}$ and satisfies

$$\operatorname{div}\big(x\cdot W\big(u(x)\big)\big)=NW\big(u(x)\big)+x\cdot\nabla W(u).$$

Furthermore, we have

$$\int_{\partial\Omega_1'}\ldots\mathrm{d}\sigma(x)=\int_{\partial\Omega}\ldots\mathrm{d}\sigma(x)-\int_{\partial\Omega_1}\ldots\mathrm{d}\sigma(x).\tag{47}$$

Next, from our assumption $|u|=1$ on $\partial\Omega$ we get $W(u(x))=0$ for all $x\in\partial\Omega$. As Ω_1 is assumed to be star-shaped, we have also $x\cdot\nu(x)\geqslant 0$ for every $x\in\partial\Omega_1$. Combining these facts with Eq. (46) we arrive at

$$\int_{\Omega_1'}W'(u)(x\cdot\nabla u)\,\mathrm{d}x\leqslant -N\int_{\Omega_1'}W(u)\,\mathrm{d}x.\tag{48}$$

We first add (45) and (48), then take advantage of (47) to conclude that

$$\begin{aligned}
\frac{N-p}{p}\int_{\Omega_1'}|\nabla u|^p\,\mathrm{d}x&\leqslant -N\int_{\Omega_1'}W(u)\,\mathrm{d}x+\frac{1}{p}\int_{\partial\Omega_1'}|\nabla u|^p\big(x\cdot\nu(x)\big)\,\mathrm{d}\sigma(x)\\
&\leqslant -N\int_{\Omega_1'}W(u)\,\mathrm{d}x+\frac{1}{p}\int_{\partial\Omega}|\nabla u|^p\big(x\cdot\nu(x)\big)\,\mathrm{d}\sigma(x).
\end{aligned}\tag{49}$$

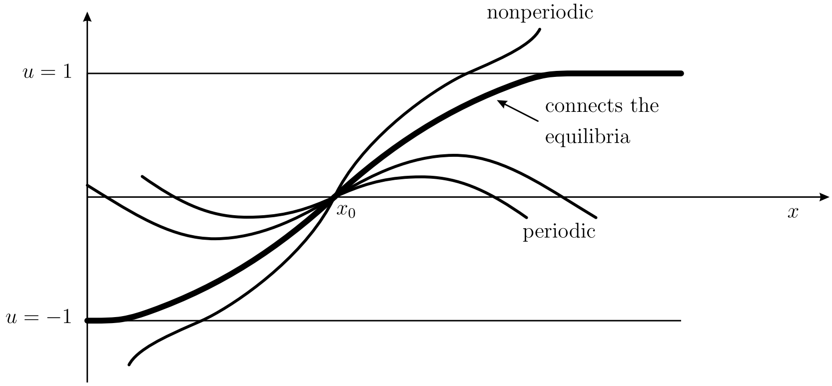

Fig. 1. $1 < \alpha < p < \infty$, $\beta = 2$, $W(s) = |1 - s^2|^\alpha$.

The boundary integral on the right-hand side vanishes, thanks to $|\nabla u| = 0$ on $\partial\Omega$ which follows from our assumptions $\partial u/\partial\nu = 0$ and $|u| = 1$ on $\partial\Omega$. Thus, (49) yields

$$\frac{N-p}{p} \int_{\Omega_1'} |\nabla u|^p \,\mathrm{d}x + N \int_{\Omega_1'} W(u)\,\mathrm{d}x \leqslant 0.$$

Since $p \leqslant N$, this is possible only if $W(u) = 0$ in Ω_1', that is, $|u| = 1$ in Ω_1', with regard to $|u| = 1$ on $\partial\Omega$ and the continuity of u in $\overline{\Omega_1'}$. Again, the continuity of u forces $u \equiv \mathrm{const} = \pm 1$ throughout $\overline{\Omega_1'}$.

The theorem is proved. □

From Theorem 5.3 we easily deduce the **nonexistence** of a phase transition solution to Eq. (42):

Corollary 5.4. *Let Ω and Ω_1 be as in* Definition 5.2 *and* Theorem 5.3 *above, and* $1 < p \leqslant N$. *Furthermore, assume that Ω_1 is star-shaped with respect to the origin $\mathbf{0} \in \mathbb{R}^N$. Then Eq.* (42) *has* no *phase transition solution $u \in W^{1,p}(\Omega_1')$ in the sense of* Definition 5.2.

In particular, we may take the double-well potential $W(s) = |1 - |s|^\beta|^\alpha$ for $s \in \mathbb{R}$, where $\alpha, \beta \in \mathbb{R}$ are arbitrary constants, $1 < \alpha < p$ *and* $\beta > 1$.

Remark 5.5. In the recent works, P. Drábek, R.F. Manásevich, and P. Takáč [11, Proposition 4.3, p. 104] and P. Takáč [33, p. 235], Theorem 3.5, Part (III), have shown that, if $N = 1 < \alpha < p < \infty$, $1 < \beta < \infty$, and $\varepsilon > 0$ is a suitable, sufficiently small number, then Eq. (42), rewritten in the form

$$-\varepsilon^p\left(\left|u'\right|^{p-2}u'\right)' + W'(u) = 0 \quad \text{in } \Omega_1' = (0,1) \subset \mathbb{R} \tag{50}$$

with the Neumann boundary conditions (43), i.e. $u'(0) = u'(1) = 0$, may have a *phase transition solution* $u \in W^{1,p}(0,1)$ (depending on the size of $\varepsilon > 0$) that connects two distinct equilibrium points $u \equiv -1$ and $u \equiv 1$ by a smooth C^1-transition function in a compact interval $\subset (0,1)$; cf. Fig. 1 above.

In the radial case of Eq. (42) one can prove the following similar result under weaker assumptions on the boundary conditions. This is a very simple application of Theorem 5.3 and its proof to the radial case. Let us recall that, for $0 < R < \infty$,

$$B_R \equiv B_R(\mathbf{0}) \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^N \colon |x| < R\right\} \quad \text{and} \quad S_R \equiv \partial B_R(\mathbf{0}) = \left\{x \in \mathbb{R}^N \colon |x| = R\right\}.$$

Corollary 5.6. *Let $\Omega_1' = B_R \setminus \overline{B_\varrho}$ with $0 < \varrho < R < \infty$ and $1 < p \leqslant N$. Then any radially symmetric weak solution $u \in W^{1,p}(\Omega_1')$ of Eq. (42), $u(x) \equiv u(r)$ for all $r \stackrel{\text{def}}{=} |x| \in (\varrho, R)$, with the "phase transition" boundary conditions*

$$\begin{cases} \dfrac{\partial u}{\partial \nu} = \left.\dfrac{\mathrm{d}u}{\mathrm{d}r}\right|_{r=\varrho} = 0 & \text{on } S_\varrho \ (\text{i.e., } r = \varrho) \quad \text{and} \\ |u| = 1 & \text{on } S_R \ (\text{i.e., } r = R), \end{cases} \tag{51}$$

must be constant, i.e., $u \equiv \pm 1$ throughout Ω_1'.

Notice that the current boundary conditions (51) are *weaker* than (43) and (44) assumed in Theorem 5.3 above; in particular, here, we do not need to assume $|u| = 1$ on S_ϱ. Closely related results that prohibit the existence of a phase transition solution to Eq. (42) with the boundary conditions (51) are proved in P. Takáč [34], Theorem 3.1, Part (I), on p. 233, and Theorem 3.5, Part (I), on p. 235, under some additional hypotheses on the potential $W(s)$.

Proof of Corollary 5.6. Suppose that $u \in W^{1,p}(\Omega_1')$ is such a solution. By the same arguments that we have used at the beginning of our proof of Theorem 5.3, we arrive at identity (45). In our present "radial" situation, with $r = |x|$, $u_r \stackrel{\text{def}}{=} \mathrm{d}u/\mathrm{d}r$, and $\nabla u(x) = (x/r)u_r(r)$, this identity becomes

$$\frac{N-p}{p}\int_\varrho^R |u_r|^p r^{N-1}\,\mathrm{d}r - \int_\varrho^R W'(u)\frac{\mathrm{d}u}{\mathrm{d}r} r^N\,\mathrm{d}r = \left(\frac{1}{p} - 1\right)\left(|u_r|^p r^N\big|_{r=R} - |u_r|^p r^N\big|_{r=\varrho}\right), \tag{52}$$

by Corollary 4.3, cf. Eq. (32). In this identity we have the term (cf. Eq. (46))

$$\begin{aligned} \int_\varrho^R W'(u)\frac{\mathrm{d}u}{\mathrm{d}r} r^N\,\mathrm{d}r &= \int_\varrho^R \frac{\mathrm{d}W(u)}{\mathrm{d}r} r^N\,\mathrm{d}r \\ &= -N\int_\varrho^R W(u) r^{N-1}\,\mathrm{d}r + W(u)r^N\big|_{r=R} - W(u)r^N\big|_{r=\varrho}, \end{aligned} \tag{53}$$

by integration-by-parts. Here, we have $u_r(\varrho) = 0$ and $W(u(R)) = 0$, by the boundary conditions (51). Thus, adding Eqs. (52) and (53) we arrive at

$$\frac{N-p}{p}\int_\varrho^R |u_r|^p r^{N-1}\,\mathrm{d}r + N\int_\varrho^R W(u) r^{N-1}\,\mathrm{d}r = -\left(1 - \frac{1}{p}\right)|u_r|^p r^N\big|_{r=R} - W(u)r^N\big|_{r=\varrho}, \tag{54}$$

cf. inequality (49). The function $W(u(r)) \geqslant 0$ being nonnegative for every $r \in [\varrho, R]$, we deduce from Eq. (54) that all terms (summands) there must vanish, that is, $u_r(R) = W(u(\varrho)) = 0$ together with both integrals ($= 0$) on the left-hand side of Eq. (54). Both functions

$$r \longmapsto u(r), r \longmapsto W\big(u(r)\big) : [\varrho, R] \longrightarrow \mathbb{R}$$

being continuous, we thus conclude that the latter must vanish identically on $[\varrho, R]$, $W(u) = W \circ u \equiv 0$. This is possible only if $u(x) \equiv u(r) = \pm 1$ for all $r \in [\varrho, R]$. It follows that $u(x) = u(r)$ must be constant, i.e., $u \equiv \pm 1$ throughout $[\varrho, R]$, as desired.

The corollary is proved. □

5.2. An unbounded exterior domain $\Omega = \mathbb{R}^N \setminus \overline{\Omega_0}$

Let Ω_0 be a simply connected, bounded domain in $\mathbb{R}^N$ with C^2-boundary $\partial\Omega_0$, such that $\mathbf{0} \in \Omega_0$. We consider the exterior domain $\Omega \overset{\text{def}}{=} \mathbb{R}^N \setminus \overline{\Omega_0}$ which is open and connected, and $\partial\Omega = \partial\Omega_0$.

Definition 5.7. We say that a function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is an **asymptotic phase transition solution** of

$$-\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) + W'(u) = 0 \quad \text{in } \Omega \tag{55}$$

if it has the following properties:

(i) $u \in C^1(\overline{\Omega})$, $\nabla u \in [L^p(\Omega)]^N$, and $W(u) \equiv W \circ u \in L^1(\Omega)$;
(ii) u verifies Eq. (55) in Ω in the weak sense;
(iii) $u = -1$ and $\partial u/\partial\nu = 0$ on $\partial\Omega$;
(iv) $\lim_{|x|\to\infty} |u(x) - 1| = 0$.

Of course, the signs of ±1 in the last two conditions, (iii) and (iv), may be interchanged correspondingly:

(iii′) $u = 1$ and $\partial u/\partial\nu = 0$ on $\partial\Omega$;
(iv′) $\lim_{|x|\to\infty} |u(x) - (-1)| = 0$.

Observe that the conditions $u \in C^1(\overline{\Omega})$ and $W(u) \in L^1(\Omega)$, supplemented by $\nabla W(u) = W'(u)\nabla u \in [L^p(\Omega)]^N$ in (i) above, imply $W(u(x)) \to 0$ as $|x| \to \infty$. Then $u(x) \to +1$ or $u(x) \to -1$ as $|x| \to \infty$ is forced by the facts that $W(s) > 0$ whenever $s \in \mathbb{R} \setminus \{-1, +1\}$, and $W(s) = 0$ if and only if $s = \pm1$.

The next theorem implies the nonexistence of an asymptotic phase transition solution to Eq. (55) for $p \leqslant N$.

Theorem 5.8. *Let Ω and Ω_0 ($\Omega = \mathbb{R}^N \setminus \overline{\Omega_0}$) be as in* Definition 5.7 *above and $1 < p \leqslant N$. Furthermore, assume that Ω_0 is star-shaped with respect to the origin $\mathbf{0} \in \mathbb{R}^N$. Then any weak solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ of Eq.* (55) *with the following properties*:

(a) $\nabla u \in [L^p(\Omega)]^N$ *and* $W(u) \equiv W \circ u \in L^1(\Omega)$;
(b) $\partial u/\partial\nu = 0$ *on* $\partial\Omega = \partial\Omega_0$; *and*
(c) $|u| = 1$ *on* $\partial\Omega$ *and* $\lim_{|x|\to\infty} ||u(x)| - 1| = 0$,

must be constant, i.e., $u \equiv \pm1$ throughout Ω.

Proof. Let $\operatorname{diam}(\Omega_0)$ denote the diameter of the (bounded) set $\Omega_0 \subset \mathbb{R}^N$. For any $R > \operatorname{diam}(\Omega_0)$, let us abbreviate $\Omega_R \overset{\text{def}}{=} B_R(\mathbf{0}) \setminus \overline{\Omega_0}$; hence, $\Omega_R = \Omega \cap B_R(\mathbf{0})$. Notice that the set Ω_R has the same properties as Ω_1' in Theorem 5.3. We recall that $S_R = \partial B_R(\mathbf{0}) = \{x \in \mathbb{R}^N\colon |x| = R\}$ and $\nu(x) = x/R$ and $x \cdot \nu(x) = |x| = R$ for every $x \in S_R$.

Suppose that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution of Eq. (55) specified in the text of Theorem 5.8. In analogy with our proof of Theorem 5.3, since the function $s \mapsto -sW'(s)\colon \mathbb{R} \to \mathbb{R}$ is bounded above (by a positive constant), by Hypothesis (W2), a regularity result in A. Anane [1, Théorème A.1, p. 96] or M. Ôtani [22, Theorem II, p. 142] guarantees $u \in L^\infty(\Omega_R)$. Moreover, the $C^{1,\beta}$-regularity results of E. DiBenedetto [10, Theorem 2, p. 829] and P. Tolksdorf [39, Theorem 1, p. 127] (interior regularity) combined with G.M. Lieberman [20, Theorem 1, p. 1203] (regularity up to the boundary) render $u \in C^{1,\beta}(\overline{\Omega_R})$ for some $\beta \in (0, \alpha)$. In particular, we have $W'(u) \in L^\infty(\Omega_R)$ together with $\nabla W(u) = W'(u)\nabla u \in [L^\infty(\Omega_R)]^N$.

Substituting Ω_R for Ω_1' in the proof of Theorem 5.3, Eq. (45), from Pohozhaev's identity (31) applied to Eq. (55) we derive

$$
\begin{aligned}
&\frac{N-p}{p}\int\limits_{\Omega_R}|\nabla u|^p\,\mathrm{d}x-\int\limits_{\Omega_R}W'(u)(x\cdot\nabla u)\,\mathrm{d}x\\
&=\frac{1}{p}\int\limits_{\partial\Omega_R}|\nabla u|^p\big(x\cdot\nu(x)\big)\,\mathrm{d}\sigma(x)-\int\limits_{\partial\Omega_R}|\nabla u|^{p-2}(x\cdot\nabla u)\big(\nu(x)\cdot\nabla u\big)\,\mathrm{d}\sigma(x)\\
&=\frac{R}{p}\int\limits_{S_R}|\nabla u|^p\,\mathrm{d}\sigma(x)-\frac{1}{p}\int\limits_{\partial\Omega_0}|\nabla u|^p\big(x\cdot\nu(x)\big)\,\mathrm{d}\sigma(x)-\frac{1}{R}\int\limits_{S_R}|\nabla u|^{p-2}\big|(x\cdot\nabla u)\big|^2\,\mathrm{d}\sigma(x),
\end{aligned}
$$

the last equality being a consequence of $\nu(x)=x/R$ and $x\cdot\nu(x)=|x|=R$ for $x\in S_R$, and the Neumann boundary conditions (b) on $\partial\Omega=\partial\Omega_0$. As Ω_0 is assumed to be star-shaped, we have also $x\cdot\nu(x)\geqslant 0$ for every $x\in\partial\Omega_0$. Applying this fact to the equation above we arrive at

$$
\frac{N-p}{p}\int\limits_{\Omega_R}|\nabla u|^p\,\mathrm{d}x-\int\limits_{\Omega_R}W'(u)(x\cdot\nabla u)\,\mathrm{d}x\leqslant\frac{R}{p}\int\limits_{S_R}|\nabla u|^p\,\mathrm{d}\sigma(x). \tag{56}
$$

By similar arguments, from (46) and (47) we derive, in analogy with inequality (48),

$$
\int\limits_{\Omega_R}W'(u)(x\cdot\nabla u)\,\mathrm{d}x\leqslant -N\int\limits_{\Omega_R}W(u)\,\mathrm{d}x+R\int\limits_{S_R}W(u)\,\mathrm{d}\sigma(x). \tag{57}
$$

Now we add inequalities (56) and (57), thus arriving at (cf. (49))

$$
\frac{N-p}{p}\int\limits_{\Omega_R}|\nabla u|^p\,\mathrm{d}x+N\int\limits_{\Omega_R}W(u)\,\mathrm{d}x\leqslant\frac{R}{p}\int\limits_{S_R}|\nabla u|^p\,\mathrm{d}\sigma(x)+R\int\limits_{S_R}W(u)\,\mathrm{d}\sigma(x). \tag{58}
$$

Next, from the facts that $\nabla u\in[L^p(\Omega)]^N$, $\nabla W(u)=W'(u)\nabla u\in[L^p(\Omega)]^N$, and $W(u)\in L^1(\Omega)$ combined with Lemma C.1 (Appendix C) we deduce that there is a monotone increasing sequence $\{R_n\}_{n=1}^{\infty}\subset(0,\infty)$ with $R_n\nearrow+\infty$ as $n\nearrow\infty$, such that both summands on the right-hand side of inequality (58) with $R=R_n$ tend to zero as $n\to\infty$,

$$
R_n\int\limits_{S_{R_n}}|\nabla u|^p\,\mathrm{d}\sigma(x)=R_n^N\int\limits_{S_1}\big|\nabla u(R_ny)\big|^p\,\mathrm{d}\sigma(y)\longrightarrow 0\quad\text{and} \tag{59}
$$

$$
R_n\int\limits_{S_{R_n}}W(u)\,\mathrm{d}\sigma(x)=R_n^N\int\limits_{S_1}W\big(u(R_ny)\big)\,\mathrm{d}\sigma(y)\longrightarrow 0. \tag{60}
$$

Taking $R=R_n$ in inequality (58) and passing to the limit as $n\to\infty$, we obtain the following inequality,

$$
\frac{N-p}{p}\int\limits_{\Omega}|\nabla u|^p\,\mathrm{d}x+N\int\limits_{\Omega}W(u)\,\mathrm{d}x\leqslant 0. \tag{61}
$$

Note that $N-p\geqslant 0$ and $W(u)\geqslant 0$, by hypotheses. Hence, the last inequality is possible only if $W(u)\equiv 0$ throughout Ω, i.e., $|u|\equiv 1$ in Ω which entails $u\equiv -1$ or $u\equiv 1$, by $\nabla u\in[L^p(\Omega)]^N$. However, an asymptotic phase transition solution cannot be constant throughout Ω, by definition. We have thus reached a contradiction.

The theorem is proved. $\square$

From Theorem 5.8 we easily deduce the **nonexistence** of an asymptotic phase transition solution to Eq. (55):

Corollary 5.9. *Let Ω and Ω_0 ($\Omega = \mathbb{R}^N \setminus \overline{\Omega_0}$) be as in* Definition 5.7 *and* Theorem 5.8 *above, and $1 < p \leqslant N$. Furthermore, assume that Ω_0 is star-shaped with respect to the origin $\mathbf{0} \in \mathbb{R}^N$. Then Eq.* (55) *has* no *asymptotic phase transition solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ in the sense of* Definition 5.7.

In particular, we may take the double-well potential $W(s) = |1 - |s|^\beta|^\alpha$ for $s \in \mathbb{R}$, where $\alpha, \beta \in \mathbb{R}$ are arbitrary constants, $1 < \alpha < p$ and $\beta > 1$.

For radially symmetric solutions in the exterior of a ball we have the following consequence of Theorem 5.8:

Corollary 5.10. *Let $\Omega = \mathbb{R}^N \setminus \overline{B_\varrho}$ with $0 < \varrho < \infty$ and $1 < p \leqslant N$. Then any radially symmetric weak solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of Eq.* (42)*, $u(x) \equiv u(r)$ for all $r \stackrel{\mathrm{def}}{=} |x| \in (\varrho, \infty)$, with the Neumann boundary conditions*

$$\frac{\partial u}{\partial \nu} = \frac{\mathrm{d}u}{\mathrm{d}r}\bigg|_{r=\varrho} = 0 \quad \text{on } S_\varrho \ (\text{i.e., } r = \varrho) \tag{62}$$

and the integrability conditions

$$\nabla u \in \left[L^p(\Omega)\right]^N \quad \text{and} \quad W(u) \equiv W \circ u \in L^1(\Omega), \tag{63}$$

must be constant, i.e., $u \equiv \pm 1$ throughout Ω.

Notice that the current boundary conditions (62) are *weaker* than those assumed in Theorem 5.8 above; in particular, here, we do not assume $|u| = 1$ on S_ϱ.

Proof of Corollary 5.10. Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is such a solution. In analogy with our proof of Corollary 5.6, using the same notation, from Eqs. (52) and (53) we derive

$$\begin{aligned}
&\frac{N-p}{p} \int_\varrho^R |u_r|^p r^{N-1}\,\mathrm{d}r + N \int_\varrho^R W(u) r^{N-1}\,\mathrm{d}r \\
&\quad = -\left(1 - \frac{1}{p}\right)\left(|u_r|^p r^N\big|_{r=R} - |u_r|^p r^N\big|_{r=\varrho}\right) + W(u) r^N\big|_{r=R} - W(u) r^N\big|_{r=\varrho}
\end{aligned}$$

for every $R \geqslant \varrho$, cf. Eq. (54). Here, we have $u_r(\varrho) = 0$, by the boundary conditions (62), and $W(u(r)) \geqslant 0$ for every $r \geqslant \varrho$. Hence, the last equation above yields

$$\begin{aligned}
&\frac{N-p}{p} \int_\varrho^R |u_r|^p r^{N-1}\,\mathrm{d}r + N \int_\varrho^R W(u) r^{N-1}\,\mathrm{d}r \\
&\quad = -\left(1 - \frac{1}{p}\right)|u_r|^p r^N\big|_{r=R} + W(u) r^N\big|_{r=R} - W(u) r^N\big|_{r=\varrho} \\
&\quad \leqslant W(u) r^N\big|_{r=R}
\end{aligned} \tag{64}$$

for every $R \geqslant \varrho$. From (63) we deduce

$$0 \leqslant \int_{\varrho}^{+\infty} W(u) r^{N-1}\,\mathrm{d}r = \int_{\Omega} W(u)\,\mathrm{d}x < \infty$$

and, thus, we may apply Lemma C.1 (Appendix C) to obtain a monotone increasing sequence $\{R_n\}_{n=1}^{\infty} \subset (0,\infty)$ with $R_n \nearrow +\infty$ as $n \nearrow \infty$, such that $W(u)r^N|_{r=R_n} \to 0$ as $n \to \infty$. We apply this result to inequality (64), thus arriving at

$$\frac{N-p}{p} \int_{\varrho}^{R} |u_r|^p r^{N-1}\,\mathrm{d}r + N \int_{\varrho}^{R} W(u) r^{N-1}\,\mathrm{d}r \leqslant 0,$$

by the Lebesgue monotone convergence theorem. The reasoning similar to that following Eq. (54) in the proof of Corollary 5.6 now forces $u \equiv \pm 1$ throughout $[\varrho, \infty)$, as desired.

The corollary is proved. $\square$

6. Discussion and generalization

Very general forms of Pohozhaev's identity [23] for *classical solutions* (in $C^2(\Omega) \cap C^1(\overline{\Omega})$) of quasilinear elliptic equations are treated in the classical works of S.I. Pohozhaev [24] and P. Pucci and J. Serrin [25]. Our local version of Pohozhaev's identity (33) (Lemma 4.5) corresponds to [25, Proposition 1, p. 683], while our Pohozhaev's identity (31) (Theorem 4.2) corresponds to [24, §2, Lemma 3, pp. 207–208] and [25, Eq. (4), p. 683]. However, in contrast with [23–25], we consider *weak solutions* of a type close to $W^{2,2}_{\mathrm{loc}}(\Omega) \cap C^1(\overline{\Omega})$; cf. Theorems 3.1 and 3.2 (Section 3). A careful inspection of our results in all previous sections shows that, after perhaps some minor modifications in notation, our results remain valid also for the following nonlinear elliptic problem with the zero Neumann boundary conditions,

$$-\operatorname{div}\big(\mathbf{a}(\nabla u)\big) = b(x,u) + f(x) \quad \text{in } \Omega; \qquad \partial u/\partial \nu = 0 \quad \text{on } \partial\Omega. \tag{65}$$

The quasilinear elliptic operator $u \mapsto \operatorname{div}(\mathbf{a}(\nabla u)) : W^{1,p}(\Omega) \to W_{\mathrm{N}}^{-1,p'}(\Omega)$, $\mathbf{a} = (a_1, \dots, a_N)$, is defined for $u \in W^{1,p}(\Omega)$ by

$$\operatorname{div}\big(\mathbf{a}(\nabla u)\big) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\big(\nabla u(x)\big), \quad x \in \Omega. \tag{66}$$

Recall that $W_{\mathrm{N}}^{-1,p'}(\Omega)$ stands for the dual space of $W^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. A typical example is the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ used throughout all previous sections. Typical forms of the function $b(x,u)$ are (i) $b(x,u) = \lambda |u|^{p-2}u$, where $\lambda \in \mathbb{R}$ is a spectral parameter, and (ii) $b(x,u) = -W'(u)$ with a potential $W(s) = |1 - |s|^{\beta}|^{\alpha}$ for $s \in \mathbb{R}$, where $\alpha, \beta \in \mathbb{R}$ are constants, $\alpha, \beta > 1$.

Each component $a_i(\boldsymbol{\eta})$ ($i = 1, 2, \dots, N$; $\boldsymbol{\eta} \in \mathbb{R}^N$) of the vector field $\mathbf{a} = (a_1, \dots, a_N)$ in (65) and (66) is assumed to satisfy $a_i \in C^0(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{\mathbf{0}\})$. In addition, we assume that $\mathbf{a}$ verifies the *ellipticity* and *growth conditions* (4), (5), and (6) with some constants $\gamma, \Gamma \in (0,\infty)$. These conditions imply that the entries $A_{ij} = \partial a_i/\partial \eta_j$ of the Jacobian matrix $\mathbf{A} = (A_{ij})_{i,j=1}^N$ of the mapping $\boldsymbol{\eta} \mapsto \mathbf{a}(\boldsymbol{\eta})$ satisfy $A_{ij} \in C^0(\mathbb{R}^N \setminus \{\mathbf{0}\})$ together with the following ellipticity and growth inequalities,

$$\gamma |\boldsymbol{\eta}|^{p-2} |\boldsymbol{\xi}|^2 \leqslant \big\langle \mathbf{A}(\boldsymbol{\eta})\boldsymbol{\xi}, \boldsymbol{\xi} \big\rangle = \sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(\boldsymbol{\eta}) \cdot \xi_i \xi_j \leqslant \Gamma |\boldsymbol{\eta}|^{p-2} |\boldsymbol{\xi}|^2 \tag{67}$$

for all $\boldsymbol{\eta} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ and all $\boldsymbol{\xi} \in \mathbb{R}^N$.

The function $b:\Omega\times\mathbb{R}\to\mathbb{R}$ is assumed to be a *Carathéodory function*, i.e.,

(i) for every $s\in\mathbb{R}$, the function $b(\cdot,s):\Omega\to\mathbb{R}$ is Lebesgue measurable;
(ii) for almost every $x\in\Omega$, the function $b(x,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous.

Further hypotheses on the smoothness of $b(x,\cdot)$ and its behavior near the points $s=\pm 1$ formulated in the previous section for $b(x,s)=W'(s)$ in the bi-stable equation (41), cf. Section 5, Hypotheses (W1) and (W2), can be easily deduced from the properties of the function $W(s)=|1-|s|^{\beta}|^{\alpha}$ for $s\in\mathbb{R}$, where $\alpha,\beta\in\mathbb{R}$ are constants, $\alpha,\beta>1$.

Finally, we assume that $f:\Omega\to\mathbb{R}$ satisfies $f\in L^{\infty}(\Omega)$ and all its distributional derivatives $\partial f/\partial x_i$ $(i=1,2,\dots,N)$ belong to $L^1(\Omega)$.

Conditions (4), (5), and (6) hold automatically for the elliptic boundary value problem

$$-\Delta_p u=\lambda|u|^{p-2}u+f(x)\quad\text{in }\Omega;\qquad \partial u/\partial\nu=0\quad\text{on }\partial\Omega,\tag{68}$$

for the p-Laplacian $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1<p<\infty$ and the spectral parameter $\lambda\in\mathbb{R}$.

For instance, if the vector field $\mathbf{a}=(a_1,\dots,a_N):\mathbb{R}^N\to\mathbb{R}^N$ possesses a potential $\mathcal{A}\in C^1(\mathbb{R}^N)\cap C^2(\mathbb{R}^N\setminus\{\mathbf{0}\})$, i.e., $\mathbf{a}=\mathcal{A}'$ with $a_i(\boldsymbol{\eta})=\partial\mathcal{A}/\partial\eta_i$ for $\boldsymbol{\eta}=(\eta_1,\dots,\eta_N)\in\mathbb{R}^N$; $i=1,2,\dots,N$, then Pohozhaev's identity (31) (Theorem 4.2) takes the following more general form:

$$\begin{aligned}&\int_{\Omega}\big[N\mathcal{A}(\nabla u)-\mathbf{a}(\nabla u)\cdot\nabla u\big]\,\mathrm{d}x+\int_{\Omega}f(x)(x\cdot\nabla u)\,\mathrm{d}x\\&\quad=\int_{\partial\Omega}\mathcal{A}(\nabla u)\big(x\cdot\nu(x)\big)\,\mathrm{d}\sigma(x)-\int_{\partial\Omega}(x\cdot\nabla u)\big[\mathbf{a}(\nabla u)\cdot\nu(x)\big]\,\mathrm{d}\sigma(x),\end{aligned}\tag{69}$$

provided $u\in C^1(\overline{\Omega})$ satisfies the following equation in the sense of distributions in Ω:

$$-\operatorname{div}\big(\mathbf{a}(\nabla u)\big)=f(x)\quad\text{in }\Omega;\qquad \partial u/\partial\nu=0\quad\text{on }\partial\Omega.\tag{70}$$

The proof of the identity in (69) is analogous with that of (31), where Eqs. (34) and (35), respectively, need to be replaced by

$$\begin{aligned}\operatorname{div}\big((x\cdot\nabla u)\mathbf{a}(\nabla u)\big)&=(x\cdot\nabla u)\operatorname{div}\mathbf{a}(\nabla u)+\nabla(x\cdot\nabla u)\cdot\mathbf{a}(\nabla u)\\&=(x\cdot\nabla u)\operatorname{div}\mathbf{a}(\nabla u)+\nabla u\cdot\mathbf{a}(\nabla u)+x\cdot\nabla^2u\cdot\mathbf{a}(\nabla u)\\&=(x\cdot\nabla u)\operatorname{div}\mathbf{a}(\nabla u)+\mathbf{a}(\nabla u)\cdot\nabla u+x\cdot\nabla\mathcal{A}(\nabla u),\end{aligned}\tag{71}$$

$$\operatorname{div}\big(x\mathcal{A}(\nabla u)\big)=N\mathcal{A}(\nabla u)+x\cdot\nabla\mathcal{A}(\nabla u),\tag{72}$$

where the gradient of the scalar function $x\mapsto\mathcal{A}((\nabla u)(x))$, equal to $\nabla\mathcal{A}(\nabla u)=\mathcal{A}'(\nabla u)\cdot\nabla^2u=\mathbf{a}(\nabla u)\cdot\nabla^2u$, is not to be confused with the Fréchet derivative $\mathbf{a}=\mathcal{A}'$ of $\mathcal{A}$. From Eq. (71) we subtract Eq. (72) to obtain the difference

$$\begin{aligned}\operatorname{div}\mathbf{v}(x)&=\operatorname{div}\big((x\cdot\nabla u)\mathbf{a}(\nabla u)\big)-\operatorname{div}\big(x\mathcal{A}(\nabla u)\big)\\&=(x\cdot\nabla u)\operatorname{div}\mathbf{a}(\nabla u)+\mathbf{a}(\nabla u)\cdot\nabla u-N\mathcal{A}(\nabla u)\\&=-f(x)(x\cdot\nabla u)+\mathbf{a}(\nabla u)\cdot\nabla u-N\mathcal{A}(\nabla u)\end{aligned}\tag{73}$$

which belongs to $L^1(\Omega)$, by Eq. (70), with the vector field

$$\mathbf{v}(x) \stackrel{\text{def}}{=} (x \cdot \nabla u)\mathbf{a}(\nabla u) - x\mathcal{A}(\nabla u)$$

being continuous in $\overline{\Omega}$, i.e., $\mathbf{v} \in [C(\overline{\Omega})]^N$. One completes the proof of (69) exactly as in the proof of Theorem 4.2.

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Appendix A. Divergence and Green's theorems

We begin with the divergence theorem which, in this form, is proved in Cuesta and Takáč [5, Lemma A.1, p. 742]. Although a number of various versions of the *divergence theorem* for strongly or weakly differentiable vector fields appear in the literature, see for instance Evans and Gariepy [12, Section 5.8, Theorem 1, p. 209], Temam [35, Chapter I, Theorem 1.2, p. 9], and Ziemer [41, Theorem 5.8.2, p. 248], we have been unable to find the following one for merely continuous vector fields:

Lemma A.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$. Assume that $\mathbf{a} : \overline{\Omega} \to \mathbb{R}^N$ satisfies $\mathbf{a} \in [C^0(\overline{\Omega})]^N$ and $\operatorname{div} \mathbf{a} = f \in L^1(\Omega)$ in the sense of distributions in Ω. Then we have*

$$\int_{\partial\Omega} \mathbf{a}(x) \cdot \nu(x)\, \mathrm{d}\sigma(x) = \int_{\Omega} f(x)\, \mathrm{d}x. \tag{74}$$

As usual, we denote by $\nu \equiv \nu(x_0) \in \mathbb{R}^N$ the exterior unit normal to $\partial\Omega$ at $x_0 \in \partial\Omega$, and by $\mathrm{d}\sigma(x_0)$ the surface measure on $\partial\Omega$. Notice that the relation $\operatorname{div} \mathbf{a} = f$ with $f \in L^1(\Omega)$ means

$$-\int_{\Omega} \mathbf{a} \cdot \nabla\varphi\, \mathrm{d}x = \int_{\Omega} f\varphi\, \mathrm{d}x \quad \text{for every } \varphi \in C_0^1(\Omega). \tag{75}$$

Here, $C_0^1(\Omega)$ denotes the set of all functions from $C^1(\Omega)$ that have compact support contained in Ω. Furthermore, the equality (75) remains valid for every $\varphi \in W_0^{1,q}(\Omega)$, $N < q < \infty$, as $W_0^{1,q}(\Omega)$ is the closure of $C_0^1(\Omega)$ in the Sobolev space $W^{1,q}(\Omega)$.

A closely related version of the *divergence theorem* (called the *generalized Gauss–Green theorem*) is established in the monograph by L.C. Evans and R.F. Gariepy [12, Section 5.8], Theorem 1 on p. 209. However, they assume $\mathbf{a} \in [C^1(\overline{\Omega})]^N$.

Proof of Lemma A.1. First, let us consider $d(x) \stackrel{\text{def}}{=} \operatorname{dist}(x, \partial\Omega)$, the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. We denote by Ω_δ the open δ-neighborhood of the boundary $\partial\Omega$ in Ω,

$$\Omega_\delta = \{x \in \Omega \colon d(x) < \delta\} \quad \text{for } \delta > 0 \text{ small enough.}$$

Since $\partial\Omega$ is a compact manifold of class C^2, making use of [14, Lemma 14.16, p. 355] and its proof, we obtain $d \in C^2(\overline{\Omega_\delta})$, and $\overline{\Omega_\delta}$ is C^1-diffeomorphic to $\partial\Omega \times [0, \delta]$ with $x \mapsto (x, 0)$ for all $x \in \partial\Omega$.

This diffeomorphism is considered between manifolds with boundary of class C^2. It can be replaced by a C^2-diffeomorphism, see M.W. Hirsch [17, Theorem 3.5, p. 57]. Observe that the restriction $\nu = -(\nabla d)|_{\partial\Omega}$ of the C^1-vector field $-\nabla d$ to $\partial\Omega$ yields the exterior unit normal ν on $\partial\Omega$; we have $|\nabla d(x_0)| = 1$ for all $x_0 \in \partial\Omega$.

Next, given any $\eta \in (0, \delta)$, define the test function

$$\varphi_\eta(x) = \begin{cases} \eta^{-1} d(x) & \text{if } x \in \Omega_\eta \cup \partial\Omega; \\ 1 & \text{if } x \in \Omega \setminus \Omega_\eta. \end{cases}$$

Hence $0 \leqslant \varphi_\eta \leqslant 1$ in $\overline{\Omega}$, $\varphi_\eta \in W_0^{1,q}(\Omega)$ for $N < q < \infty$, and

$$\nabla\varphi_\eta(x) = \begin{cases} \eta^{-1} \nabla d(x) & \text{if } x \in \Omega_\eta \cup \partial\Omega; \\ \mathbf{0} & \text{if } x \in \Omega \setminus \overline{\Omega_\eta}, \end{cases}$$

by Gilbarg and Trudinger [14, Theorem 7.8, p. 153]. Inserting $\varphi = \varphi_\eta$ into Eq. (75), we arrive at

$$-\eta^{-1} \int_{\Omega_\eta} \mathbf{a}(x) \cdot \nabla d(x)\, \mathrm{d}x = -\int_{\Omega_\eta} f(x)\bigl(1 - \eta^{-1} d(x)\bigr)\, \mathrm{d}x + \int_{\Omega} f(x)\, \mathrm{d}x \tag{76}$$

whenever $0 < \eta < \delta$.

In order to compute the limit of the integral on the left-hand side in Eq. (76) as $\eta \to 0+$, we introduce the mapping $\mathbf{h} : \partial\Omega \times [0, \delta] \to \overline{\Omega_\delta}$ defined by

$$\mathbf{h}(x_0, t) = x_0 - t\nu(x_0) \quad \text{for } x_0 \in \partial\Omega \text{ and } t \in [0, \delta].$$

From the proof of [14, Lemma 14.16, p. 355] we deduce that $\mathbf{h}$ is a C^1-diffeomorphism of $\partial\Omega \times [0, \delta]$ onto $\overline{\Omega_\delta}$ with the Jacobian determinant $J(x_0, t)$ satisfying

$$\bigl|J(x_0, t)\bigr| \longrightarrow 1 \quad \text{as } t \longrightarrow 0+, \text{ uniformly for } x_0 \in \partial\Omega.$$

Consequently, we can perform a substitution of variables in Eq. (76) followed by Fubini's theorem, thus arriving at

$$\begin{aligned} &-\eta^{-1} \int_0^\eta \biggl[\int_{\partial\Omega} \mathbf{a}\bigl(\mathbf{h}(x_0, t)\bigr) \cdot \nabla d\bigl(\mathbf{h}(x_0, t)\bigr) \bigl|J(x_0, t)\bigr|\, \mathrm{d}\sigma(x_0) \biggr] \mathrm{d}t \\ &\quad = -\int_{\Omega_\eta} f(x)\bigl(1 - \eta^{-1} d(x)\bigr)\, \mathrm{d}x + \int_{\Omega} f(x)\, \mathrm{d}x \end{aligned} \tag{77}$$

whenever $0 < \eta < \delta$. Finally, letting $\eta \to 0+$ and using the mean value theorem for continuous functions, we obtain the divergence formula (74) as desired. □

Our next auxiliary result treats the difference quotients for weakly differentiable functions; it is an easy variation of Ziemer [41, Theorem 2.1.6, pp. 45–46] adapted to the case when the partial derivatives are merely in $L^1(\Omega)$ (with no smoothness assumption on the boundary $\partial\Omega$).

Lemma A.2. *Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that $u : \Omega \to \mathbb{R}^N$ is a continuous function whose all distributional derivatives $\partial u/\partial x_i$ ($i = 1, 2, \ldots, N$) belong to $L^1(\Omega)$. Then, given any $\eta > 0$ small enough, for every $h \in \mathbb{R}^N$ with $0 < |h| < \eta$ we have*

$$\int\limits_{\Omega\setminus\Omega_\eta} \left|\frac{u(x+h)-u(x)}{|h|}\right| \mathrm{d}x \leqslant \int\limits_{\Omega} |\nabla u|\,\mathrm{d}x \quad (<\infty). \tag{78}$$

Moreover, given any vector $\mathbf{e} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, for $t \in \mathbb{R}$ with $0 < |t| < \eta$ we have also

$$\int\limits_{\Omega} \frac{u(x+t\mathbf{e})-u(x)}{t}\varphi(x)\,\mathrm{d}x \longrightarrow \int\limits_{\Omega} (\nabla u\cdot\mathbf{e})\varphi\,\mathrm{d}x \tag{79}$$

as $t \to 0$, for every function $\varphi \in C_0^0(\Omega)$.

Clearly, (79) is equivalent with saying that, as $t \to 0$, the difference quotients $t^{-1}(u(x+t\mathbf{e}) - u(x))$ converge to $(\nabla u(x)\cdot\mathbf{e})$ in the weak-star topology on the Banach space $M(\Omega)$ of all bounded regular Borel measures on Ω.

The *proof* of Lemma A.2 is a straightforward modification of that given in Ziemer [41], proof of Theorem 2.1.6, p. 46.

The following product rule is an easy consequence of Lemma A.2; it subsequently implies Green's formula (integration-by-parts), by the divergence theorem (Lemma A.1).

Lemma A.3. *Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that $u, v : \Omega \to \mathbb{R}^N$ are continuous functions whose all first-order distributional derivatives belong to $L^1_{\mathrm{loc}}(\Omega)$. Then also their product uv has the same properties and the product rule*

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i}v + u\frac{\partial v}{\partial x_i} \quad (i = 1, 2, \ldots, N) \tag{80}$$

holds in the sense of distributions in Ω. Furthermore, if both u and v are continuous on $\overline{\Omega}$ (up to the boundary) and all their first-order distributional derivatives belong to $L^1(\Omega)$, then (80) holds in $L^1(\Omega)$. If, in addition, $\partial\Omega$ is a C^2-manifold then one has also Green's formula,

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_i} v\,\mathrm{d}x + \int\limits_{\Omega} u\frac{\partial v}{\partial x_i}\,\mathrm{d}x = \int\limits_{\partial\Omega} uv\nu_i\,\mathrm{d}\sigma(x). \tag{81}$$

Here, $\nu = (\nu_1, \nu_2, \ldots, \nu_N)$.

Appendix B. Some geometric inequalities

We state a few geometric inequalities proved in Takáč [33, Lemma A.1, p. 233]. Let $1 < p < \infty$ and $p \neq 2$. Assume that $\Theta \in L^\infty(0,1)$ satisfies $\Theta \geqslant 0$ in $(0,1)$ and $T = \int_0^1 \Theta(s)\,\mathrm{d}s > 0$. Then there exists a constant $c_p \equiv c_p(\Theta) > 0$ such that the following inequalities hold true for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$: If $p > 2$ then

$$\begin{aligned} c_p(\Theta)^{p-2}\Big(\max_{0\leqslant s\leqslant 1} |\mathbf{a}+s\mathbf{b}|\Big)^{p-2} &\leqslant \int\limits_0^1 |\mathbf{a}+s\mathbf{b}|^{p-2}\Theta(s)\,\mathrm{d}s \\ &\leqslant T\cdot\Big(\max_{0\leqslant s\leqslant 1} |\mathbf{a}+s\mathbf{b}|\Big)^{p-2}, \end{aligned} \tag{82}$$

and if $1 < p < 2$ and $|\mathbf{a}| + |\mathbf{b}| > 0$ then

$$T \cdot \Bigl(\max_{0\leqslant s\leqslant 1} |\mathbf{a} + s\mathbf{b}|\Bigr)^{p-2} \leqslant \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2} \Theta(s)\,\mathrm{d}s \leqslant c_p(\Theta)^{p-2}\Bigl(\max_{0\leqslant s\leqslant 1} |\mathbf{a} + s\mathbf{b}|\Bigr)^{p-2}. \tag{83}$$

Equivalently, in both cases ($p \neq 2$), the ratio

$$\int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2} \Theta(s)\,\mathrm{d}x \Big/ \Bigl(\max_{0\leqslant s\leqslant 1} |\mathbf{a} + s\mathbf{b}|\Bigr)^{p-2}$$

is bounded below and above by positive constants, whenever $|\mathbf{a}| + |\mathbf{b}| > 0$.

Appendix C. An integrability lemma

The following lemma is a simple property of Lebesgue-integrable functions on $\mathbb{R}^N$.

Lemma C.1. *Let Ω and Ω_0 be as in* Definition 5.7 *and* Theorem 5.8. *Assume that $0 \leqslant h \in L^1(\Omega)$ is a non-negative function such that*

$$H : [R, \infty) \longrightarrow \mathbb{R} : r \longmapsto H(r) \stackrel{\text{def}}{=} \int_{S_r} h(y)\,\mathrm{d}\sigma(y)$$

is a continuous function, for some $R > 0$ large enough, where $S_r = \partial B_r(\mathbf{0}) = \{x \in \mathbb{R}^N \colon |x| = r\}$ for $r > 0$. Then we have $\liminf_{r\to+\infty}(rH(r)) = 0$. More generally, $\liminf_{r\to+\infty}(\zeta(r)H(r)) = 0$ holds for every monotone increasing function $\zeta : [R, \infty) \to (0, \infty)$ such that $\int_R^{+\infty} \zeta(r)^{-1}\,\mathrm{d}r = +\infty$.

Proof. Let $\zeta : [R, \infty) \to (0, \infty)$ be monotone increasing with $\int_R^{+\infty} \zeta(r)^{-1}\,\mathrm{d}r = +\infty$. On the contrary, suppose that $c \stackrel{\text{def}}{=} \liminf_{r\to+\infty}(\zeta(r)H(r)) > 0$. Set $c' = c/2 > 0$ if $c < \infty$ and take $c' \in (0, \infty)$ arbitrarily large if $c = \infty$.

Then there is a number $R' > R$ such that $\zeta(r)H(r) \geqslant c'$ for every $r \geqslant R'$. It follows by Fubini's theorem in $B^{\mathrm{c}}_{R'} = B^{\mathrm{c}}_{R'}(\mathbf{0}) = \{x \in \mathbb{R}^N \colon |x| \geqslant R'\} \subset \Omega = \mathbb{R}^N \setminus \overline{\Omega_0}$ that

$$\int_{B^{\mathrm{c}}_{R'}} h(x)\,\mathrm{d}x = \int_{R'}^{+\infty} \Bigl(\int_{S_r} h(y)\,\mathrm{d}\sigma(y)\Bigr)\mathrm{d}r = \int_{R'}^{+\infty} H(r)\,\mathrm{d}r \geqslant c' \int_{R'}^{+\infty} \frac{\mathrm{d}r}{\zeta(r)} = +\infty,$$

a contradiction to $0 \leqslant h \in L^1(B^{\mathrm{c}}_{R'})$. The lemma is proved. □

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