Attractors for processes on time-dependent spaces. Applications to wave equations

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A R T I C L E   I N F O

A B S T R A C T

For a process $U(t, \tau): X_\tau \to X_t$ acting on a one-parameter family of normed spaces, we present a notion of time-dependent attractor based only on the minimality with respect to the pullback attraction property. Such an attractor is shown to be invariant whenever $U(t, \tau)$ is T-closed for some $T > 0$, a much weaker property than continuity (defined in the text). As a byproduct, we generalize the recent theory of attractors in time-dependent spaces developed in Di Plinio et al. (2011) [13]. Finally, we exploit the new framework to study the longterm behavior of wave equations with time-dependent speed of propagation.

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1. Introduction

The evolution of systems arising from mechanics and physics is described in many instances by differential equations of the form

$$\begin{cases} u_t = A(u, t), \quad t > \tau, \\ u(\tau) = u_\tau \in X, \end{cases}$$

where X is a normed space and, for every fixed t, $A(\cdot, t)$ is a densely defined operator on X. Assuming the Cauchy problem is well posed and calling $u(t)$ the solution at time t, we can construct the family of solving operators

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$$U(t, \tau): X \to X, \quad t \geqslant \tau \in \mathbb{R},$$

by setting

$$U(t, \tau)u_\tau = u(t).$$

Such a family is called a *process*, characterized by the properties that $U(\tau, \tau) = I$ and

$$U(t, s)U(s, \tau) = U(t, \tau), \quad \forall t \geqslant s \geqslant \tau \in \mathbb{R}.$$

The issue of understanding the longtime behavior of solutions to dynamical systems is thus translated into studying the dissipative properties of the operators $U(t, \tau)$. A well-established theory of attractors provides nowadays a full description of many important autonomous systems from mathematical physics, including nonautonomous models with time-dependent external forces (see e.g. the classical textbooks [3,18,19,24] and the more recent references [7,10,22]). A paradigmatic example is given by the nonlinear damped wave equation in a smooth bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} \varepsilon u_{tt}(\boldsymbol{x}, t) + u_t(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) + f\big(u(\boldsymbol{x}, t)\big) = g(\boldsymbol{x}, t), \\ u(\boldsymbol{x}, t)|_{\boldsymbol{x} \in \partial \Omega} = 0, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, f is a nonlinear term and g an external given force. If g is independent of time, the system is autonomous and the problem is completely understood within the framework of semigroups, whereas the dependence of g on time requires further integrability assumptions and the theory of attractors for processes, suitable to deal with nonautonomous situations.

On the contrary, the standard theory generally fails to capture the dissipation mechanism involved in evolution problems where the coefficients of the differential operator depend explicitly on time, leading to time-dependent terms at a *functional level*. This can be seen in the model equation (1.1), assuming that ε is not a constant, but rather a positive decreasing function of time $\varepsilon(t)$ vanishing at infinity. In such a case, (twice) the natural energy associated with the system reads

$$\mathcal{E}(t) = \int_\Omega \big|\nabla u(\boldsymbol{x}, t)\big|^2 \, \mathrm{d}\boldsymbol{x} + \varepsilon(t) \int_\Omega \big|u_t(\boldsymbol{x}, t)\big|^2 \, \mathrm{d}\boldsymbol{x},$$

which exhibits a structural dependence on time. Furthermore, it is easy to realize that the vanishing character of ε at infinity alters the dissipativity of the system and prevents the existence of absorbing sets in the usual sense, namely, bounded sets of the phase space $X = H_0^1(\Omega) \times L^2(\Omega)$ absorbing all the trajectories after a certain period of time.

A progress in this direction has been made recently in [13], where the authors adopt the point of view of describing the solution operator as a family of maps

$$U(t, \tau): X_\tau \to X_t, \quad t \geqslant \tau \in \mathbb{R},$$

acting on a time-dependent family of spaces X_t. For instance, in the model problem (1.1) all the spaces coincide with the linear space X, but the X_t-norm is dictated by the time-dependent energy $\mathcal{E}(t)$ of the solution at time t. Based on this idea, the paper [13] provides a suitable modification of the classical notion of pullback attractor, establishing a new theory of pullback flavor for dynamical systems acting on time-dependent spaces, which are not embedded in a universal space; for the classical theory of pullback attractors see e.g. [7,20,22].

Of course the utilization of a scale of spaces for studying nonautonomous systems is not new, and a natural one. For example, when studying flows in time-dependent domains $\Omega = \Omega(t)$, it is natural to introduce such a scale of spaces, see e.g. [21]. Another typical situation appears for the study of

stochastic dynamical systems which are nonautonomous by nature; besides the books quoted above, see e.g. [5] where a one-parameter family of Banach spaces is introduced in the context of cocycles for nonautonomous and random dynamical systems; in [6] time-dependent spaces are used in the context of stochastic partial differential equations and filtration; finally in e.g. [15] concerning the stochastic Navier–Stokes equations the authors introduce a family of metric spaces depending on a parameter (the stochastic variable). Compared to these previous publications, the specificity of our article (as well as of [13]) is that we apply the general (abstract) results to a specific partial differential equation of mathematical physics, and, furthermore, the scale of spaces that we introduce is adapted to the structure of the equation, more precisely, to the properties of its time-dependent coefficients.

Plan of the paper

Our aim in this article is twofold: first, in the spirit of [9] (and [13,14]), we give new insights on attractors on time-dependent spaces (see Sections 2–6). The main idea is to define the basic objects of the theory (such as pullback absorbing and attracting sets, time-dependent attractors) only in terms of their attraction properties. In particular, the time-dependent attractor will be the smallest (pullback) attracting set, which in turn implies its uniqueness. Quite interestingly, here neither the process is required to be continuous, nor the attractor to be invariant by definition. Indeed, we prove that the invariance property is automatically satisfied by the attractor whenever the process $U(t, \tau)$ is T-closed for some $T > 0$, a much weaker condition than continuity (see Definition 5.4 below). As a byproduct, we recover and improve the results of [13]. The second goal, discussed in Sections 7 to 11, is to study the longtime dynamics of the model problem (1.1) with a time-dependent coefficient $\varepsilon(t)$. This is done in the last part of the paper, where, by handling the system within the new framework, we show the existence of a time-dependent attractor of optimal regularity.

2. The abstract framework

For $t \in \mathbb{R}$, let X_t be a family of normed spaces without, so far, any other hypotheses on these spaces. We consider a two-parameter family of operators

$$U(t, \tau): X_\tau \to X_t,$$

depending on $t \geqslant \tau \in \mathbb{R}$, and satisfying the following properties:

(i) $U(\tau, \tau)$ is the identity map on X_τ;
(ii) $U(t, \tau)U(\tau, \sigma) = U(t, \sigma)$ for every $\sigma \in \mathbb{R}$ and every $t \geqslant \tau \geqslant \sigma$.

The family $U(t, \tau)$ will still be called a process.

Remark 2.1. We stress that the spaces X_t can be in principle completely unrelated. Besides, no continuity property is assumed in the definition of the process.

In the next four sections we will provide an abstract setting in order to study the asymptotic behavior of the operators $U(t, \tau)$ when $t \to +\infty$ and/or $\tau \to -\infty$. The goal is to define a suitably "thin" object $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}}$, where each $A_t \subset X_t$ is able to attract (at time t) all the solutions of the system originating sufficiently far in the past. This will be done in the spirit of [9], leading to the notion of time-dependent attractor in Definition 4.1 below. Then, in Section 3, we state the main existence result for time-dependent attractors, and in the subsequent Section 5 we discuss the issue of their invariance. In Section 6, we complete the presentation of the abstract setting with some comments and a comparison with the theory of [13].

Notation

For every $t \in \mathbb{R}$, we introduce the R-ball of X_t

$$\mathbb{B}_t(R) = \{z \in X_t \colon \|z\|_{X_t} \leqslant R\}.$$

For any given $\varepsilon > 0$, the ε-neighborhood of a set $B \subset X_t$ is defined as

$$\mathcal{O}_t^{\varepsilon}(B) = \bigcup_{x \in B} \{y \in X_t \colon \|x - y\|_{X_t} < \varepsilon\} = \bigcup_{x \in B} \{x + \mathbb{B}_t(\varepsilon)\}.$$

We denote the Hausdorff semidistance of two (nonempty) sets $B, C \subset X_t$ by

$$\delta_t(B, C) = \sup_{x \in B} \mathrm{dist}_{X_t}(x, C) = \sup_{x \in B} \inf_{y \in C} \|x - y\|_{X_t}.$$

Finally, given any set $B \subset X_t$, the symbol $\overline{B}$ stands for the closure of B in X_t.

3. Pullback attracting sets

We begin with some definitions.

Definition 3.1. A family $\mathfrak{C} = \{C_t\}_{t \in \mathbb{R}}$ of bounded sets $C_t \subset X_t$ is called *uniformly bounded* if there exists $R > 0$ such that

$$C_t \subset \mathbb{B}_t(R), \quad \forall t \in \mathbb{R}.$$

Definition 3.2. A family $\mathfrak{B} = \{B_t\}_{t \in \mathbb{R}}$ is called *pullback absorbing* if it is uniformly bounded and, for every $R > 0$, there exists $t_0 = t_0(t, R) \leqslant t$ such that

$$\tau \leqslant t_0 \quad \Rightarrow \quad U(t, \tau)\mathbb{B}_\tau(R) \subset B_t. \tag{3.1}$$

The process $U(t, \tau)$ is called *dissipative* whenever it admits a pullback absorbing family.

Definition 3.3. A (uniformly bounded) family $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ is called *pullback attracting* if for all $\varepsilon > 0$ the family $\{\mathcal{O}_t^{\varepsilon}(K_t)\}_{t \in \mathbb{R}}$ is pullback absorbing.

Remark 3.4. The attracting property can be equivalently stated in terms of the Hausdorff semidistance: $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ is pullback attracting if and only if it is uniformly bounded and the limit

$$\lim_{\tau \to -\infty} \delta_t \big(U(t, \tau)C_\tau, K_t\big) = 0$$

holds for every uniformly bounded family $\mathfrak{C} = \{C_t\}_{t \in \mathbb{R}}$ and every $t \in \mathbb{R}$.

Remark 3.5. If the spaces X_t are all the same for every t, then we recover the classical definition of pullback attraction (see e.g. the recent treatises [7,20,22]).

We can describe the pullback attraction in terms of sequences. To this aim, let Σ_t denote the collection of all possible sequences of the form

$$y_n = U(t, \tau_n)x_n,$$

where $\tau_n \to -\infty$ and $x_n \in X_{\tau_n}$ is any uniformly bounded sequence. For any $y_n \in \Sigma_t$ we denote

$$\mathcal{L}_t(y_n) = \left\{ x \in X_t \colon \|y_{n_\iota} - x\|_{X_t} \to 0 \text{ for some subsequence } n_\iota \to \infty \right\}.$$

It is immediately seen from the definitions that a uniformly bounded family $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ is pullback attracting if and only if

$$\mathrm{dist}_{X_t}(y_n, K_t) \to 0, \quad \forall y_n \in \Sigma_t, \tag{3.2}$$

for all $t \in \mathbb{R}$. In particular, each element of $\mathcal{L}_t(y_n)$ belongs to the closure of K_t. Therefore, setting

$$A_t^\star = \bigcup_{y_n \in \Sigma_t} \mathcal{L}_t(y_n),$$

we have proved

Lemma 3.6. *Assume that there exists a pullback attracting family of closed sets $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$. Then*

$$A_t^\star \subset K_t, \quad \forall t \in \mathbb{R}.$$

Lemma 3.7. *If the process $U(t, \tau)$ is dissipative, then $\mathfrak{A}^\star = \{A_t^\star\}_{t \in \mathbb{R}}$ coincides with the time-dependent ω-limit of any pullback absorbing set $\mathfrak{B} = \{B_t\}_{t \in \mathbb{R}}$, that is:*

$$A_t^\star = \bigcap_{y \leqslant t} \overline{\bigcup_{\tau \leqslant y} U(t, \tau)B_\tau}. \tag{3.3}$$

In particular, $A_t^\star$ is closed and contained in $\overline{B_t}$ for all $t \in \mathbb{R}$; hence $\mathfrak{A}^\star$ is uniformly bounded.

Proof. The validity of (3.3) is a direct consequence of the definitions, so $A_t^\star$ is obviously closed. Besides, since $\mathfrak{B}$ is uniformly bounded, it absorbs itself and

$$U(t, \tau)B_\tau \subset B_t, \quad \forall \tau \leqslant t_0,$$

for some $t_0 = t_0(t, \mathfrak{B}) \leqslant t$, implying the inclusion $A_t^\star \subset \overline{B_t}$. $\square$

The next lemma characterizes the attraction property for compact sets.

Lemma 3.8. *Let $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ be a uniformly bounded family of compact sets. Then $\mathfrak{K}$ is pullback attracting if and only if for all $t \in \mathbb{R}$*

$$\emptyset \neq \mathcal{L}_t(y_n) \subset K_t, \quad \forall y_n \in \Sigma_t.$$

Proof. Let $\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}}$ be a family of compact sets. If $\mathfrak{K}$ is pullback attracting, then given $y_n \in \Sigma_t$ we have

$$\mathcal{L}_t(y_n) \subset K_t \quad \text{and} \quad \text{dist}_{X_t}(y_n, \xi_n) \to 0,$$

for some $\xi_n \in K_t$. Since K_t is compact, there exists $\xi \in K_t$ such that (up to a subsequence)

$$\xi_n \to \xi \in K_t \quad \Rightarrow \quad y_n \to \xi \quad \Rightarrow \quad \mathcal{L}_t(y_n) \neq \emptyset.$$

Conversely, if $\mathfrak{K}$ is not pullback attracting, we deduce from (3.2) that

$$\text{dist}_{X_t}(y_n, K_t) > \varepsilon,$$

for some $t \in \mathbb{R}$, $\varepsilon > 0$ and $y_n \in \Sigma_t$. Therefore, $\mathcal{L}_t(y_n) \cap K_t = \emptyset$. $\quad\square$

4. Time-dependent global attractors

It is clear from the earlier discussion that a pullback attracting family of compact sets is capable of controlling the regime of the system at any time $t \in \mathbb{R}$. This leads quite naturally to the definition of an attractor as the smallest set possessing such a property. To this aim we consider the collection

$$\mathbb{K} = \big\{\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}}: K_t \subset X_t \text{ compact, } \mathfrak{K} \text{ pullback attracting}\big\}. \tag{4.1}$$

When $\mathbb{K} \neq \emptyset$ we say that the process is *asymptotically compact*.

Definition 4.1. We call a *time-dependent global attractor* the smallest element of $\mathbb{K}$, i.e. the family $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}} \in \mathbb{K}$ such that

$$A_t \subset K_t, \quad \forall t \in \mathbb{R},$$

for any element $\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}} \in \mathbb{K}$.

The next result tells that the definition is consistent: the minimal element of $\mathbb{K}$ exists (and it is unique) if and only if $\mathbb{K}$ is not empty.

Theorem 4.2. *If $U(t, \tau)$ is asymptotically compact, then the time-dependent attractor $\mathfrak{A}$ exists and coincides with the set $\mathfrak{A}^\star = \{A_t^\star\}_{t\in\mathbb{R}}$. In particular, it is unique.*

Proof. Let $\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}}$ be an element of $\mathbb{K}$. Then, by Lemmas 3.6 and 3.8,

$$\emptyset \neq A_t^\star \subset K_t, \quad \forall t \in \mathbb{R}.$$

Since $U(t, \tau)$ is dissipative, we know by Lemma 3.7 that $\mathfrak{A}^\star$ is uniformly bounded and $A_t^\star$ is closed for all $t \in \mathbb{R}$. Since $A_t^\star$ is contained in the compact set K_t, then $A_t^\star$ is compact as well. The attraction property is contained in (3.2), saying that $\mathfrak{A}^*$ is an element of $\mathbb{K}$. Thanks to Lemma 3.6 it is also the smallest element of $\mathbb{K}$, hence it is the (unique) time-dependent attractor by the very definition. $\quad\square$

We now provide a necessary condition for $\mathbb{K}$ to be nonempty, which turns out to be sufficient as well when the spaces X_t are complete.

Definition 4.3. A process $U(t, \tau)$ is ε-*dissipative* if for every $t \in \mathbb{R}$ there exists a set $F_t \subset X_t$ made of a finite number of points such that the family $\{\mathcal{O}_t^\varepsilon(F_t)\}_{t \in \mathbb{R}}$ is pullback absorbing (cf. Definition 4 in [9]). The process is called *totally dissipative* whenever it is ε-dissipative for every $\varepsilon > 0$. Note that the sets F_t need not be the same for all ε.

Theorem 4.4. *Assume that X_t is a Banach space for all $t \in \mathbb{R}$. Then $U(t, \tau)$ is totally dissipative if and only if $\mathbb{K} \neq \emptyset$.*

Proof. If $\mathbb{K} \neq \emptyset$, then $U(t, \tau)$ is totally dissipative. Indeed, if $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ belongs to $\mathbb{K}$, it follows that any K_t can be covered by finitely many ε-balls, and calling F_t the union of the centers of those balls, the family $\{\mathcal{O}_t^\varepsilon(F_t)\}_{t \in \mathbb{R}}$ is pullback absorbing. Conversely, if $U(t, \tau)$ is totally dissipative, for any arbitrarily fixed $\varepsilon > 0$, we can choose a finite set F_t^ε such that the family $\{\mathcal{O}_t^\varepsilon(F_t^\varepsilon)\}_{t \in \mathbb{R}}$ is uniformly bounded and absorbing. If we select any $y_n \in \Sigma_t$, then y_n eventually falls into

$$V_t^\varepsilon = \overline{\mathcal{O}_t^\varepsilon(F_t^\varepsilon)}.$$

Set

$$K_t = \bigcap_{\varepsilon > 0} V_t^\varepsilon.$$

Accordingly, the family $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ is uniformly bounded. Furthermore, both K_t and $\{y_n\}$ are coverable by finitely many balls of arbitrarily small radius, which, in Banach spaces, means precompactness. In particular, K_t being closed, it is compact in X_t. Since the sequence y_n is precompact, then $\mathcal{L}_t(y_n)$ is nonempty. Also, it is contained in every closed set V_t^ε and hence in their intersection K_t. In other words,

$$\mathrm{dist}_{X_t}(y_n, K_t) \to 0,$$

meaning that $\mathfrak{K}$ is pullback attracting. Hence $\mathfrak{K} \in \mathbb{K}$. □

Collecting Theorems 4.2 and 4.4 we draw a corollary.

Corollary 4.5. *If the family $U(t, \tau)$ is totally dissipative, then the time-dependent attractor $\mathfrak{A}$ exists and coincides with the set $\mathfrak{A}^\star$. In particular, it is unique and uniformly bounded.*

Remark 4.6. A less direct characterization of a totally dissipative process is also possible, based on the *Kuratowski measure of noncompactness* of a bounded set $B \subset X_t$ (see [18])

$$\alpha_t(B) = \inf\{d > 0 : B \text{ has a finite covering by balls of } X_t \text{ of diameter less than } d\}.$$

Indeed, it is easily seen that the family $U(t, \tau)$ is totally dissipative if and only if there exists a pullback absorbing set $\mathfrak{B} = \{B_t\}_{t \in \mathbb{R}}$ for which

$$\lim_{\tau \to -\infty} \alpha_t\big(U(t, \tau)B_\tau\big) = 0, \quad \forall t \in \mathbb{R}.$$

5. Invariance of the attractor

A further question is the invariance of the time-dependent global attractor.

Definition 5.1. We say that $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}}$ is *invariant* if

$$U(t, \tau) A_\tau = A_t, \quad \forall t \geqslant \tau.$$

This property is usually a priori postulated in the literature. In particular, in [13] the time-dependent attractor is by definition a family of compact sets which is at the same time pullback attracting and invariant, and its existence is proved by exploiting the continuity of the process $U(t, \tau)$ (see Theorem 2.1 and Remark 2.4 therein).

Remark 5.2. On the other hand, if we know that $\mathfrak{K}$ is an invariant pullback attracting family of compact sets, it is clear that $\mathfrak{K}$ is the smallest element of $\mathbb{K}$, hence it coincides with the time-dependent attractor $\mathfrak{A}$.

Our purpose here is to show that the time-dependent global attractor provided by Theorem 4.2 is automatically invariant whenever the process $U(t, \tau)$ is T-closed for some $T > 0$ in the sense of Definition 5.4 below, a very mild continuity-like assumption. We start with a sufficient condition.

Proposition 5.3. *If there exists $T > 0$ such that*

$$A_t \subset U(t, t - T) A_{t-T}, \quad \forall t \in \mathbb{R},$$

then $\mathfrak{A}$ is invariant.

Proof. Let $t \in \mathbb{R}$ be arbitrarily fixed. For any $s \geqslant t$ and any $n \in \mathbb{N}$, we have by induction

$$U(s, t) A_t \subset U(s, t - T) A_{t-T} \subset \cdots \subset U(s, t - nT) A_{t-nT}. \tag{5.1}$$

Consequently,

$$\delta_s \big(U(s, t) A_t, A_s \big) \leqslant \delta_s \big(U(s, t - nT) A_{t-nT}, A_s \big).$$

Since $\mathfrak{A}$ is attracting, letting $n \to \infty$ we obtain

$$\delta_s \big(U(s, t) A_t, A_s \big) = 0,$$

implying in turn, since A_s is closed, that

$$U(s, t) A_t \subset A_s, \quad \forall s \geqslant t. \tag{5.2}$$

In particular, (5.1)–(5.2) for $s = t$ entail

$$A_t \subset U(t, t - nT) A_{t-nT} \subset A_t,$$

that is,

$$A_t = U(t, t - nT) A_{t-nT}. \tag{5.3}$$

Let now $\tau \leqslant t$. Taking n large enough, we infer from (5.2)–(5.3) that

$$A_t = U(t, t - nT)A_{t-nT} = U(t, \tau)U(\tau, t - nT)A_{t-nT} \subset U(t, \tau)A_\tau \subset A_t,$$

proving the equality $U(t, \tau)A_\tau = A_t$. $\quad\square$

In order to establish an invariance criterion, we need one more definition. Recall that, for any pair of fixed times $t \geqslant \tau$, the map $U(t, \tau): X_\tau \to X_t$ is said to be closed if

$$\begin{cases} x_n \to x & \text{in } X_\tau \\ U(t, \tau)x_n \to \zeta & \text{in } X_t \end{cases} \quad \Rightarrow \quad U(t, \tau)x = \zeta.$$

Definition 5.4. The process $U(t, \tau)$ is called

- *closed* if $U(t, \tau)$ is a closed map for any pair of fixed times $t \geqslant \tau$;
- *T-closed* for some $T > 0$ if $U(t, t - T)$ is a closed map for all t.

Remark 5.5. Of course if the process $U(t, \tau)$ is closed it is T-closed, for any $T > 0$. Note also that if the process $U(t, \tau)$ is a continuous (or even norm-to-weak continuous) map for all $t \geqslant \tau$, then the process is closed.

Theorem 5.6. *If $U(t, \tau)$ is a T-closed process for some $T > 0$, which possesses a time-dependent global attractor $\mathfrak{A}$, then $\mathfrak{A}$ is invariant.*

Proof. In view of Proposition 5.3, it is enough to prove the inclusion

$$A_t \subset U(t, t - T)A_{t-T}, \quad \forall t \in \mathbb{R}.$$

To this end, select an arbitrary $y \in A_t$. By Theorem 4.2,

$$y_n \to y \quad \text{for some } y_n = U(t, \tau_n)x_n \in \boldsymbol{\Sigma}_t.$$

Define the sequence

$$w_n = U(t - T, \tau_n)x_n.$$

On account of Lemma 3.8,

$$w_n \to w \quad \text{for some } w \in A_{t-T}.$$

On the other hand,

$$U(t, t - T)w_n = U(t, \tau_n)x_n = y_n \to y,$$

and since $U(t, t - T)$ is closed we conclude that

$$U(t, t - T)w = y.$$

Therefore

$$y \in U(t, t - T)A_{t-T},$$

yielding the desired inclusion. □

Remark 5.7. In fact, Theorem 5.6 holds under a weaker continuity condition on $U(t, \tau)$. It suffices to require the existence of a sequence

$$0 = T_0 < T_1 < T_2 < T_3 \cdots \to \infty$$

with the following property: for all $k \in \mathbb{N}$

$$\begin{cases} x_n^k \to \xi_0^k & \text{in } X_{t-T_k} \\ U(t, t - T_k)x_n^k \to \xi^k & \text{in } X_t \end{cases} \implies U(t, t - T_k)\xi_0^k = \xi^k.$$

If so, we call the process *asymptotically closed*, in analogy to the semigroup case (see [9]).

6. Further remarks

I. The notion of pullback absorber given in [13] looks apparently different from ours, and is based on the notion of a *pullback-bounded family*, namely, a family $\mathfrak{B} = \{B_t\}_{t \in \mathbb{R}}$ satisfying

$$R(t) = \sup_{\tau \leqslant t} \|B_\tau\|_{X_t} < \infty, \quad \forall t \in \mathbb{R}. \tag{6.1}$$

Accordingly, $\mathfrak{B}$ is called a *pullback absorber* if it is a pullback-bounded family with the following property: for every $t \in \mathbb{R}$ and every pullback-bounded family $\mathfrak{C} = \{C_t\}_{t \in \mathbb{R}}$ there exists $t_0 = t_0(t, \mathfrak{C}) \leqslant t$ such that

$$\tau \leqslant t_0 \implies U(t, \tau)C_\tau \subset B_t. \tag{6.2}$$

Since any family of balls $\{\mathbb{B}_t(R)\}_{t \in \mathbb{R}}$ is pullback-bounded, (3.1) obviously follows from (6.2). As a matter of fact, the two notions of absorption are equivalent. Indeed, if $\mathfrak{C}$ is any pullback-bounded family with maximal size $R(t)$ on $(-\infty, t]$, then

$$U(t, \tau)C_\tau \subset U(t, \tau)\mathbb{B}_\tau\big(R(t)\big), \quad \forall \tau \leqslant t.$$

Hence, if $\mathfrak{B}$ is a pullback absorbing family in the sense of Definition 3.2 and $t \in \mathbb{R}$ is any fixed time, we have

$$\tau \leqslant t_0 \implies U(t, \tau)C_\tau \subset U(t, \tau)\mathbb{B}_\tau\big(R(t)\big) \subset B_t$$

for some $t \geqslant t_0 = t_0(t, R(t))$, where $R(t)$ depends only on $\mathfrak{C}$. But this is exactly the absorption property (6.2).

In the present work, we decided to postulate in the definition of absorbing family the stronger property of being uniformly bounded, instead of merely pullback-bounded. Such a notion seems to reflect more closely the dissipation mechanism of most equations of mathematical physics, where the dynamics at time t is confined in bounded sets B_t (the pullback absorbing family) whose size in the phase space X_t remains bounded as $t \to +\infty$ (whereas the boundedness as $t \to -\infty$ is a consequence of (6.1)). This happens, for instance, for the so-called *oscillon* equation arising in recent cosmological theories that motivated the authors of [13] to develop this novel theory (see also [14]), as well as for the wave equation (1.1) studied in this paper. Conversely, having a pullback-bounded absorbing family

does not prevent the possibility of B_t becoming larger and larger as time increases, in contrast with the common intuition of dissipation.

II. Similarly, the notion of pullback attracting set in Definition 3.3 (or Remark 3.4) can be rephrased in the following way: a uniformly bounded family $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ is pullback attracting if and only if

$$\lim_{\tau \to -\infty} \delta_t \big(U(t, \tau) C_\tau, K_t \big) = 0 \qquad (6.3)$$

for every pullback-bounded family $\mathfrak{C} = \{C_t\}_{t \in \mathbb{R}}$ and $t \in \mathbb{R}$. Observe that this is exactly the pullback attraction property defined in [13].

III. An interesting question is whether property (6.3) holds uniformly with respect to intervals of time. This is not true in general. In particular, it cannot happen on unbounded intervals. The next result shows that, if the process is sufficiently smooth, then the attraction exerted by any invariant pullback attracting family (such as the time-dependent attractor) is uniform on compact intervals.

Proposition 6.1. *Let $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ be an invariant pullback attracting family. Assume that*

$$\big\| U(t, \tau) z_1 - U(t, \tau) z_2 \big\|_{X_t} \leqslant \mathcal{Q}(t - \tau, r) \| z_1 - z_2 \|_{X_\tau}, \qquad (6.4)$$

for all $t \geqslant \tau \in \mathbb{R}$ and $\| z_i \|_{X_t} \leqslant r$, where $\mathcal{Q}$ is a positive function, increasing in each of its arguments. Then, for all $R > 0$,

$$\lim_{\tau \to -\infty} \delta_t \big(U(t, \tau) \mathbb{B}_\tau(R), K_t \big) = 0,$$

uniformly for t belonging to a compact set.

Proof. Let $[a, b]$ with $-\infty < a < b < \infty$ be given. Let $R_0 > 0$ be such that

$$\mathcal{O}_a^1(K_a) \subset \mathbb{B}_a(R_0).$$

For every $\varrho > 0$ small enough, set

$$\varepsilon = \frac{\varrho}{\mathcal{Q}(b - a, R_0)} < 1.$$

Since $\mathfrak{K}$ is pullback attracting, for any given $R > 0$ there exists

$$\tau_0 = \tau_0(R, \varepsilon) < a$$

such that

$$U(a, \tau) \mathbb{B}_\tau(R) \subset \mathcal{O}_\varepsilon(K_a), \quad \forall \tau < \tau_0.$$

Let now $\tau < \tau_0$ be fixed, and select any $x \in \mathbb{B}_\tau(R)$. Calling $z = U(a, \tau) x$, choose $k \in K_a$ for which

$$\| z - k \|_{X_a} < \varepsilon.$$

Then, in light of (6.4), for all $t \in [a, b]$ we have

$$\big\| U(t,a)z - U(t,a)k \big\|_{X_t} \leqslant \mathcal{Q}(t-a, R_0)\|z-k\|_{X_a} \leqslant \varepsilon \mathcal{Q}(b-a, R_0) = \varrho.$$

Observe that, from the invariance of $\mathfrak{K}$,

$$U(t,\tau)x = U(t,a)U(a,\tau)x = U(t,a)z \quad \text{and} \quad U(t,a)k \subset K_t.$$

Thus,

$$\text{dist}_{X_t}\big(U(t,\tau)x, K_t\big) \leqslant \big\| U(t,a)z - U(t,a)k \big\|_{X_t} \leqslant \varrho.$$

In conclusion, we proved that for all $\varrho > 0$ small there exists $\tau_0 < a$ such that

$$\delta_t\big(U(t,\tau)\mathbb{B}_\tau(R), K_t\big) \leqslant \varrho, \quad \forall \tau < \tau_0.$$

Since τ_0 is independent of $t \in [a,b]$, the proof is finished. $\quad\square$

7. Wave equations with time-dependent speed of propagation

We now want to apply the theory above to the nonautonomous wave equations (7.1)–(7.3) below. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. For any $\tau \in \mathbb{R}$, we consider the evolution equation for the unknown variable $u = u(\boldsymbol{x}, t) : \Omega \times [\tau, \infty) \to \mathbb{R}$

$$\varepsilon u_{tt} + \alpha u_t - \Delta u + f(u) = g, \quad t > \tau, \tag{7.1}$$

subject to Dirichlet boundary condition

$$u_{|\partial\Omega} = 0, \tag{7.2}$$

and to the initial conditions

$$u(\boldsymbol{x}, \tau) = a(\boldsymbol{x}) \quad \text{and} \quad u_t(\boldsymbol{x}, \tau) = b(\boldsymbol{x}), \tag{7.3}$$

where $a, b : \Omega \to \mathbb{R}$ are assigned data. Here $\varepsilon = \varepsilon(t)$ is a function of t and we postulate the following assumptions.

7.1. Conditions on ε

We let $\varepsilon \in \mathcal{C}^1(\mathbb{R})$ be a decreasing bounded function satisfying

$$\lim_{t \to +\infty} \varepsilon(t) = 0. \tag{7.4}$$

In particular, there exists $L > 0$ such that

$$\sup_{t \in \mathbb{R}} \big[\big| \varepsilon(t) \big| + \big| \varepsilon'(t) \big| \big] \leqslant L. \tag{7.5}$$

7.2. Conditions on f

We let $f \in \mathcal{C}^2(\mathbb{R})$ with $f(0) = 0$ satisfying, for every $s \in \mathbb{R}$, the growth bound

$$\left| f''(s) \right| \leqslant c \left(1 + |s| \right), \quad \text{for some } c \geqslant 0, \tag{7.6}$$

along with the dissipation condition

$$\liminf_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1, \tag{7.7}$$

where $\lambda_1 > 0$ is the first eigenvalue of the strictly positive Dirichlet operator

$$A = -\Delta \quad \text{with domain } \operatorname{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega) \Subset L^2(\Omega).$$

Finally, the damping coefficient α is a positive constant and the time-independent external source $g = g(\boldsymbol{x})$ is taken in $L^2(\Omega)$.

Eq. (7.1) can be seen as a nonlinear damped wave equation with time-dependent speed of propagation $1/\varepsilon(t)$. Besides, it can also be interpreted as a model for the thermal evolution in a homogeneous isotropic (rigid) heat conductor according to the Maxwell–Cattaneo law [8] (see also [12, Appendix B]), with $\varepsilon(t)$ representing a time-dependent relaxation parameter.

In the case when ε is a positive constant, the asymptotic behavior of solutions to Eqs. (7.1)–(7.3) has been the object of extensive studies since the eighties (see, e.g. [1–3,16]), and it is well-known to generate a strongly continuous semigroup $S(t)$ on the phase space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega).$$

We refer the reader to the recent reference [23] for a review on the subject and a discussion on the assumptions on f which are suitable to prove the existence of the compact global attractor of optimal regularity.

The aim of the subsequent sections is to study the longtime behavior of the solutions to (7.1)–(7.3) with ε depending on time, according to the abstract framework developed in the first part of this article. Our main result is Theorem 11.1 below, proving the existence of a time-dependent global attractor for the process associated with (7.1) acting on a suitable time-dependent family of spaces. Besides, the attractor turns out to be invariant and of optimal regularity, in a sense explained below.

8. Preliminaries

8.1. The functional setting

We set $\mathrm{H} = L^2(\Omega)$, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. For $0 \leqslant \sigma \leqslant 2$, we define the hierarchy of (compactly) nested Hilbert spaces

$$\mathrm{H}_\sigma = \operatorname{dom}\left(A^{\sigma/2} \right), \qquad \langle w, v \rangle_\sigma = \left\langle A^{\sigma/2} w, A^{\sigma/2} v \right\rangle, \qquad \| w \|_\sigma = \left\| A^{\sigma/2} w \right\|.$$

Then, for $t \in \mathbb{R}$ and $0 \leqslant \sigma \leqslant 2$, we introduce the time-dependent spaces

$$\mathcal{H}_t^\sigma = \mathrm{H}_{\sigma+1} \times \mathrm{H}_\sigma$$

endowed with the time-dependent product norms

$$\big\|\{a,b\}\big\|_{\mathcal{H}_t^\sigma}^2 = \|a\|_{\sigma+1}^2 + \varepsilon(t)\|b\|_\sigma^2.$$

The symbol σ is always omitted whenever zero. In particular, the time-dependent phase space where we settle the problem is

$$\mathcal{H}_t = H_1 \times H \quad \text{with} \ \big\|\{a,b\}\big\|_{\mathcal{H}_t}^2 = \|a\|_1^2 + \varepsilon(t)\|b\|^2.$$

Then, we have the compact embeddings

$$\mathcal{H}_t^\sigma \Subset \mathcal{H}_t, \quad 0 < \sigma \leqslant 2,$$

with injection constants independent of $t \in \mathbb{R}$. Note that the spaces $\mathcal{H}_t$ are all the same as linear spaces; besides, since $\varepsilon(\cdot)$ is a decreasing function of t, for every $z \in H_1 \times H$ and $t \geqslant \tau \in \mathbb{R}$ there holds

$$\|z\|_{\mathcal{H}_t}^2 \leqslant \|z\|_{\mathcal{H}_\tau}^2 \leqslant \max\left\{1, \frac{\varepsilon(\tau)}{\varepsilon(t)}\right\} \|z\|_{\mathcal{H}_t}^2.$$

Hence the norms $\|\cdot\|_{\mathcal{H}_t}^2$ and $\|\cdot\|_{\mathcal{H}_\tau}^2$ are equivalent for any fixed $t, \tau \in \mathbb{R}$, but the equivalence constant blows up when $t \to +\infty$.

Along the paper, we will perform a number of formal energy-type estimates, which are rigorously justified in a Galerkin approximation scheme. Moreover, the Hölder, Young and Poincaré inequalities will be tacitly used.

8.2. Technical lemmas

We shall exploit the following Gronwall-type lemma, whose proof can be found in [11].

Lemma 8.1. *Let $\Lambda : [\tau, \infty) \to \mathbb{R}^+$ be an absolutely continuous function satisfying the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + 2\omega\Lambda(t) \leqslant q(t)\Lambda(t) + k$$

for some $\omega > 0$, $k \geqslant 0$ and where $q : [\tau, \infty) \to \mathbb{R}^+$ fulfills

$$\int_\tau^\infty q(y)\,\mathrm{d}y \leqslant m,$$

with $m \geqslant 0$. Then,

$$\Lambda(t) \leqslant \Lambda(\tau)\mathrm{e}^m \mathrm{e}^{-\omega(t-\tau)} + k\omega^{-1}\mathrm{e}^m.$$

Calling

$$F(s) = \int_0^s f(y)\,\mathrm{d}y,$$

in light of (7.7) it is a standard matter to verify that

Lemma 8.2. *The following inequalities hold for some $0 < v < 1$ and $c_1 \geqslant 0$:*

$$2\langle F(u), 1 \rangle \geqslant -(1 - v)\|u\|_1^2 - c_1, \tag{8.1}$$

$$\langle f(u), u \rangle \geqslant -(1 - v)\|u\|_1^2 - c_1, \quad \forall u \in H_1. \tag{8.2}$$

8.3. A word of warning

Similarly to the classical damped wave equation with constant coefficients, proving the dissipativity of the system with $f(u)$ satisfying (7.6) and (7.7) is quite technical and requires several steps (see e.g. [1,3], see also [4] for a different strategy).

Since the main focus in this paper is the presence of the time-dependent coefficient $\varepsilon(t)$, in order to avoid technical complications only due to the nonlinear term $f(u)$, we require the additional assumption

$$2\langle f(u), u \rangle \geqslant 2\langle F(u), 1 \rangle - (1 - v)\|u\|_1^2 - c_1, \tag{8.3}$$

that will be used for providing a simple and direct proof of Theorem 10.2 below.[1] Condition (8.3) is ensured by asking, for instance, that

$$\liminf_{|s| \to \infty} f'(s) > -\lambda_1,$$

which is slightly less general than (7.7) but still widely used in the literature.

9. Well-posedness

For any $\tau \in \mathbb{R}$, we rewrite problem (7.1)–(7.3) as

$$\begin{cases} \varepsilon u_{tt} + \alpha u_t + Au + f(u) = g, & t > \tau, \\ u(\tau) = a, \\ u_t(\tau) = b. \end{cases} \tag{9.1}$$

Theorem 9.1. *Problem (9.1) generates a strongly continuous[2] process $U(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t$, $t \geqslant \tau \in \mathbb{R}$, satisfying the following continuous dependence property: for every pair of initial data $z_i = \{a_i, b_i\} \in \mathcal{H}_\tau$ such that $\|z_i\|_{\mathcal{H}_\tau} \leqslant R$, $i = 1, 2$, the difference of the corresponding solutions satisfies*

$$\left\| U(t, \tau)z_1 - U(t, \tau)z_2 \right\|_{\mathcal{H}_t} \leqslant e^{K(t - \tau)} \|z_1 - z_2\|_{\mathcal{H}_\tau}, \quad \forall t \geqslant \tau, \tag{9.2}$$

for some constant $K = K(R) \geqslant 0$.

Global existence of (weak) solutions u to (9.1) is classical, and can be obtained by means of a standard Galerkin scheme, based on the subsequent Lemma 10.3. Such solutions satisfy, on any interval $[\tau, t]$ with $t \geqslant \tau$,

$$u \in \mathcal{C}([\tau, t], H_1), \qquad u_t \in \mathcal{C}([\tau, t], H),$$

---

[1] The results are true even without (8.3), but the proofs require the introduction of a suitable Gronwall-like lemma which renders the calculations much more complicated.

[2] Recall that $U(t, \tau)$ is strongly continuous if $U(t, \tau) \in \mathcal{C}(\mathcal{H}_\tau, \mathcal{H}_t)$ for all $t \geqslant \tau$.

see e.g. [24]. Uniqueness of solutions will then follow by the continuous dependence estimate (9.2). As a consequence, the family of maps with $t \geqslant \tau \in \mathbb{R}$

$$U(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t \quad \text{acting as} \quad U(t, \tau)z = \{u(t), u_t(t)\},$$

where u is the unique solution to (9.1) with initial time τ and initial condition $z = \{a, b\} \in \mathcal{H}_\tau$, defines a strongly continuous process on the family $\{\mathcal{H}_t\}_{t \in \mathbb{R}}$.

Proof of estimate (9.2). Let $z_1, z_2 \in \mathcal{H}_\tau$ be such that $\|z_i\|_{\mathcal{H}_\tau} \leqslant R$, $i = 1, 2$, and denote by C a generic positive constant depending on R but independent of z_i. We first observe that the energy estimate in Lemma 10.3 below ensures

$$\big\| U(t, \tau)z_i \big\|_{\mathcal{H}_t} \leqslant C. \tag{9.3}$$

We call $\{u_i(t), \partial_t u_i(t)\} = U(t, \tau)z_i$ and denote $\bar{z}(t) = \{\bar{u}(t), \bar{u}_t(t)\} = U(t, \tau)z_1 - U(t, \tau)z_2$. Then, the difference between the two solutions satisfies

$$\varepsilon \bar{u}_{tt} + \alpha \bar{u}_t - \Delta \bar{u} + f(u_1) - f(u_2) = 0,$$

with initial datum $z(\tau) = z_1 - z_2$. Multiplying by $2\bar{u}_t$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\bar{z}\|_{\mathcal{H}_t}^2 + \big[2\alpha - \varepsilon'\big] \|\bar{u}_t\|^2 = -2\big\langle f(u_1) - f(u_2), \bar{u}_t \big\rangle.$$

Estimating the right-hand side in light of (7.6) and (9.3)

$$-2\big\langle f(u_1) - f(u_2), \bar{u}_t \big\rangle \leqslant C\|\bar{u}\|_1 \|\bar{u}_t\| \leqslant 2\alpha \|\bar{u}_t\|^2 + C\|\bar{u}\|_1^2,$$

we end up with the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\|\bar{z}(t)\big\|_{\mathcal{H}_t}^2 \leqslant C \big\|\bar{z}(t)\big\|_{\mathcal{H}_t}^2,$$

and an application of the Gronwall lemma on $[\tau, t]$ completes the proof. □

10. Absorbing sets

This section is devoted to studying the dissipation properties of the process $U(t, \tau)$ associated with (9.1). We start with a new notion of absorption, which is *stronger* than the pullback dissipativity of Definition 3.2.

Definition 10.1. A *time-dependent absorbing set* for the process $U(t, \tau)$ is a uniformly bounded family $\mathfrak{B} = \{B_t\}_{t \in \mathbb{R}}$ with the following property: for every $R \geqslant 0$ there exists $\theta_e = \theta_e(R) \geqslant 0$ such that

$$\tau \leqslant t - \theta_e \quad \Rightarrow \quad U(t, \tau)\mathbb{B}_\tau(R) \subset B_t.$$

The existence of a time-dependent absorbing set (hence pullback absorbing) for $U(t, \tau)$ is witnessed by

Theorem 10.2. *There exists $R_0 > 0$ such that the family $\mathfrak{B} = \{\mathbb{B}_t(R_0)\}_{t \in \mathbb{R}}$ is a time-dependent absorbing set for $U(t, \tau)$. Besides,*

$$\sup_{z \in \mathbb{B}_\tau(R_0)} \left[\|U(t,\tau)z\|_{\mathcal{H}_t} + \int_\tau^\infty \|u_t(y)\|^2 \, \mathrm{d}y \right] \leqslant I_0, \quad \forall \tau \in \mathbb{R}, \tag{10.1}$$

for some $I_0 \geqslant R_0$.

As already discussed, we propose an easy and direct proof of this result, based on the extra assumption (8.3). The crucial ingredient is the following dissipation estimate.

Lemma 10.3. *Let $t \geqslant \tau$. For $z \in \mathcal{H}_\tau$, let $U(t, \tau)z$ be the solution of (9.1) with initial time τ and datum $z = \{a, b\}$. Then, if (8.3) holds, there exist $\omega = \omega(\alpha, \|\varepsilon\|_{L^\infty}, \|\varepsilon'\|_{L^\infty}) > 0$, $K_1 \geqslant 0$ and an increasing positive function $\mathcal{Q}$ such that*

$$\|U(t,\tau)z\|_{\mathcal{H}_t} \leqslant \mathcal{Q}\big(\|z\|_{\mathcal{H}_\tau}\big) e^{-\omega(t-\tau)} + K_1, \quad \forall \tau \leqslant t.$$

Proof. Let $C \geqslant 0$ be a *generic* constant independent of the initial datum z and denote

$$E(t) = \|U(t,\tau)z\|_{\mathcal{H}_t}^2$$

(double) the *energy* associated with problem (9.1). Due to (7.5), (7.6) and (8.1), the functional

$$\mathcal{E} = E + \delta\alpha\|u\|^2 + 2\delta\varepsilon\langle u_t, u\rangle + 2\langle F(u), 1\rangle - 2\langle g, u\rangle$$

fulfills, for $\delta > 0$ small and some $0 < \nu < 1$ provided by Lemma 8.2,

$$\nu E(t) - C \leqslant \mathcal{E}(t) \leqslant C E^2(t) + C. \tag{10.2}$$

Indeed, in light of (7.5), if δ is small enough we have

$$2\delta\varepsilon|\langle u_t, u\rangle| \leqslant \frac{\varepsilon}{2}\|u_t\|^2 + CL\delta^2\|u\|^2 \leqslant \frac{\varepsilon}{2}\|u_t\|^2 + \frac{\delta\alpha}{2}\|u\|^2.$$

Multiplying (7.1) by $2u_t + 2\delta u$, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \big[2\alpha - \varepsilon' - 2\delta\varepsilon\big]\|u_t\|^2 + 2\delta\|u\|_1^2 + 2\delta\langle f(u), u\rangle - 2\delta\langle g, u\rangle = 2\delta\varepsilon'\langle u_t, u\rangle,$$

and estimating

$$2\delta|\varepsilon'\langle u_t, u\rangle| \leqslant 2\delta L\|u_t\|\|u\| \leqslant \frac{\alpha}{2}\|u_t\|^2 + \frac{\delta\nu}{2}\|u\|_1^2$$

for δ small, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \left[\frac{3}{2}\alpha - \varepsilon' - 2\delta\varepsilon\right]\|u_t\|^2 + \delta\left[2 - \frac{\nu}{2}\right]\|u\|_1^2 + 2\delta\langle f(u), u\rangle - 2\delta\langle g, u\rangle \leqslant 0. \tag{10.3}$$

In light of (8.3) we can reconstruct the functional $\mathcal{E}$, which provides

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \delta\mathcal{E} + \alpha\|u_t\|^2 + \Gamma \leqslant \delta c_1,$$

where

$$\Gamma = \left[\frac{\alpha}{2} - \varepsilon' - 3\delta\varepsilon\right]\|u_t\|^2 + \frac{\delta\nu}{2}\|u\|_1^2 - \delta^2\alpha\|u\|^2 - 2\delta^2\varepsilon\langle u_t, u\rangle.$$

Therefore, setting δ small enough so that $\Gamma \geqslant 0$, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \delta\mathcal{E} + \alpha\|u_t\|^2 \leqslant \delta c_1. \tag{10.4}$$

Applying the Gronwall lemma, together with (10.2), we have proved Lemma 10.3. $\quad\square$

Proof of Theorem 10.2. Let $R_0 = 1 + 2K_1$. An application of Lemma 10.3 for $z \in \mathbb{B}_\tau(R)$ yields

$$\|U(t,\tau)z\|_{\mathcal{H}_t} \leqslant \mathcal{Q}(R)\mathrm{e}^{-\omega(t-\tau)} + K_1 \leqslant 1 + 2K_1 = R_0,$$

provided that $t - \tau \geqslant \theta_\mathrm{e}$, where

$$\theta_\mathrm{e} = \max\left\{0, \omega^{-1}\log\frac{\mathcal{Q}(R)}{1 + K_1}\right\}.$$

This concludes the proof of the existence of the time-dependent absorbing set. In order to prove the integral estimate for $\|u_t\|$, it is enough to integrate (10.4) with $\delta = 0$ on $[\tau, \infty)$. $\quad\square$

Remark 10.4. We can assume that the time-dependent absorbing set $B_t = \mathbb{B}_t(R_0)$ is positively invariant (namely $U(t,\tau)B_\tau \subset B_t$ for all $t \geqslant \tau$). Indeed, calling θ_e the entering time of B_t such that

$$U(t,\tau)B_\tau \subset B_t, \quad \forall \tau \leqslant t - \theta_\mathrm{e},$$

we can substitute B_t with the invariant absorbing family

$$\bigcup_{\tau \leqslant t - \theta_\mathrm{e}} U(t,\tau)B_\tau \subset B_t.$$

11. Existence of the time-dependent global attractor

The main result concerning the asymptotic behavior of problem (9.1) is contained in the following theorem.

Theorem 11.1. *The process $U(t,\tau) : \mathcal{H}_\tau \to \mathcal{H}_t$ generated by problem (9.1) admits an invariant time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$. Besides, A_t is bounded in $\mathcal{H}_t^1$, with a bound independent of t.*

The existence of the attractor, according to Definition 4.1, will be proved by a direct application of the abstract Theorem 4.2. Precisely, in order to show that the process is asymptotically compact, we shall exhibit a pullback attracting family of (nonvoid) compact sets. To this aim, the strategy classically consists in finding a suitable decomposition of the process in the sum of a *decaying* part and of a *compact* one.

11.1. The decomposition

We write $f = f_0 + f_1$, where $f_0, f_1 \in C^2(\mathbb{R})$ fulfill, for some $k \geqslant 0$,

$$\left| f_1'(s) \right| \leqslant k, \quad \forall s \in \mathbb{R}, \tag{11.1}$$

$$\left| f_0''(s) \right| \leqslant k\left(1 + |s| \right), \quad \forall s \in \mathbb{R}, \tag{11.2}$$

$$f_0(0) = f_0'(0) = 0, \tag{11.3}$$

$$f_0(s)s \geqslant 0, \quad \forall s \in \mathbb{R}. \tag{11.4}$$

This is possible owing to assumptions (7.6) and (7.7) (cf. [1,17]).

Let $\mathfrak{B} = \{\mathbb{B}_t(R_0)\}_{t \in \mathbb{R}}$ be a time-dependent absorbing set according to Theorem 10.2 and let $\tau \in \mathbb{R}$ be fixed. Then, for any $z \in \mathbb{B}_\tau(R_0)$, we split $U(t, \tau)z$ into the sum

$$U(t, \tau)z = \left\{ u(t), u_t(t) \right\} = U_0(t, \tau)z + U_1(t, \tau)z,$$

where

$$U_0(t, \tau)z = \left\{ v(t), v_t(t) \right\} \quad \text{and} \quad U_1(t, \tau)z = \left\{ w(t), w_t(t) \right\}$$

solve the systems

$$\begin{cases} \varepsilon v_{tt} + \alpha v_t + Av + f_0(v) = 0, \\ U_0(\tau, \tau) = z, \end{cases} \tag{11.5}$$

and

$$\begin{cases} \varepsilon w_{tt} + \alpha w_t + Aw + f(u) - f_0(v) = g, \\ U_1(\tau, \tau) = 0. \end{cases} \tag{11.6}$$

In what follows, the *generic* constant $C \geqslant 0$ depends only on $\mathfrak{B}$.

Lemma 11.2. *There exists* $\delta = \delta(\mathfrak{B}) > 0$ *such that*

$$\left\| U_0(t, \tau)z \right\|_{\mathcal{H}_t} \leqslant C e^{-\delta(t-\tau)}, \quad \forall t \geqslant \tau.$$

Proof. Repeating word by word the proof of Lemma 10.3 with f_0 instead of f we immediately get the bound

$$\left\| U_0(t, \tau)z \right\|_{\mathcal{H}_t} \leqslant C. \tag{11.7}$$

Then, denoting

$$\mathcal{E}_0 = \left\| U_0(t, \tau)z \right\|_{\mathcal{H}_t}^2 + \delta\alpha \|v\|^2 + 2\delta\varepsilon \langle v_t, v \rangle + 2\langle F_0(v), 1 \rangle,$$

with

$$F_0(s) = \int_0^s f_0(y)\,\mathrm{d}y,$$

we multiply (11.5) by $2v_t + 2\delta v$. In view of (11.4) and since $g = 0$, the analogous of the differential inequality (10.3) now reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0 + \delta\|U_0(t,\tau)z\|_{\mathcal{H}_t}^2 \leqslant 0.$$

Exploiting (11.7) we have

$$\frac{1}{2}\|U_0(t,\tau)z\|_{\mathcal{H}_t}^2 \leqslant \mathcal{E}_0(t) \leqslant C\|U_0(t,\tau)z\|_{\mathcal{H}_t}^2,$$

and the Gronwall lemma completes the argument. □

Summing up, the following uniform bound holds

$$\sup_{t \geqslant \tau}\big[\|U(t,\tau)z\|_{\mathcal{H}_t} + \|U_0(t,\tau)z\|_{\mathcal{H}_t} + \|U_1(t,\tau)z\|_{\mathcal{H}_t}\big] \leqslant C. \tag{11.8}$$

Lemma 11.3. *There exists* $M = M(\mathfrak{B}) > 0$ *such that*

$$\sup_{t \geqslant \tau}\|U_1(t,\tau)z\|_{\mathcal{H}_t^{1/3}} \leqslant M.$$

Proof. We choose $\delta > 0$ small and $C > 0$ large enough such that, calling

$$\Lambda = \|U_1(t,\tau)z\|_{\mathcal{H}_t^{1/3}}^2 + \delta\alpha\|w\|_{1/3}^2 + 2\delta\varepsilon\langle w_t, A^{1/3}w\rangle + 2\langle f(u) - f_0(v) - g, A^{1/3}w\rangle + C,$$

we have

$$\frac{1}{2}\|U_1(t,\tau)z\|_{\mathcal{H}_t^{1/3}}^2 \leqslant \Lambda(t) \leqslant 2\|U_1(t,\tau)z\|_{\mathcal{H}_t^{1/3}}^2 + 2C. \tag{11.9}$$

Indeed, in view of (11.8) and the growth of f,

$$2\langle f(u) - f_0(v), A^{1/3}w\rangle \leqslant 2\|f(u) - f_0(v)\|\|A^{1/3}w\| \leqslant C\|w\|_{2/3} \leqslant \frac{1}{4}\|w\|_{4/3}^2 + C.$$

Besides, by (7.5), for δ small we can estimate

$$2\delta\varepsilon\big|\langle w_t, A^{1/3}w\rangle\big| \leqslant \frac{\varepsilon}{2}\|w_t\|_{1/3}^2 + \frac{\delta\alpha}{2}\|w\|_{1/3}^2.$$

By multiplying (11.6) with $2A^{1/3}w_t + 2\delta A^{1/3}w$, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \big[2\alpha - \varepsilon' - 2\delta\varepsilon\big]\|w_t\|_{1/3}^2 + 2\delta\|w\|_{4/3}^2 + 2\delta\langle f(u) - f_0(v) - g, A^{1/3}w\rangle$$

$$= 2\delta\varepsilon'\langle w_t, A^{1/3}w\rangle + I_1 + I_2 + I_3,$$

where

$$I_1 = 2\langle [f_0'(u) - f_0'(v)]u_t, A^{1/3}w\rangle,$$
$$I_2 = 2\langle f_0'(v)w_t, A^{1/3}w\rangle,$$
$$I_3 = 2\langle f_1'(u)u_t, A^{1/3}w\rangle.$$

Then, for any fixed $\delta > 0$ small enough, we easily get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \delta\Lambda + \alpha\|w_t\|_{1/3}^2 \leqslant I_1 + I_2 + I_3 + \delta C. \tag{11.10}$$

By exploiting conditions (11.2)–(11.3) for f_0 and the embeddings $\mathrm{H}_{(3p-6)/2p} \subset L^p(\Omega)$ $(p > 2)$, we draw from (11.8)–(11.9) the estimates

$$I_1 \leqslant C\big(1 + \|u\|_{L^6} + \|v\|_{L^6}\big)\|u_t\|\|w\|_{L^{18}}\big\|A^{1/3}w\big\|_{L^{18/5}} \leqslant C\|u_t\|\|w\|_{4/3}^2$$
$$\leqslant \frac{\delta}{2}\Lambda + C\|u_t\|^2\|w\|_{4/3}^2,$$
$$I_2 \leqslant C\big(\|v\|_{L^6} + \|v\|_{L^6}^2\big)\|w_t\|_{L^{18/7}}\big\|A^{1/3}w\big\|_{L^{18/5}} \leqslant C\|v\|_1\|w_t\|_{1/3}\|w\|_{4/3}$$
$$\leqslant \frac{\alpha}{2}\|w_t\|_{1/3}^2 + C\|v\|_1^2\|w\|_{4/3}^2.$$

Besides, in view of (11.1), we have

$$I_3 \leqslant k\|u_t\|\big\|A^{1/3}w\big\| \leqslant \|u_t\|^2\|w\|_{4/3}^2 + C.$$

As a consequence, inequality (11.10) improves to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \frac{\delta}{2}\Lambda \leqslant q\Lambda + C,$$

with $q = C\|u_t\|^2 + C\|v\|_1^2$ satisfying

$$\int_\tau^\infty q(y)\,\mathrm{d}y \leqslant C,$$

by virtue of the dissipation integral (10.1) and Lemma 11.2. By Lemma 8.1,

$$\Lambda(t) \leqslant C\Lambda(\tau)e^{-\frac{\delta}{4}(t-\tau)} + C \leqslant C.$$

In turn, (11.9) yields the boundedness of $U_1(t, \tau)z$ in $\mathcal{H}_t^{1/3}$. $\quad\square$

11.2. Existence of the invariant attractor

According to Lemma 11.3, we consider the family $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ where

$$K_t = \big\{ z \in \mathcal{H}_t^{1/3} : \|z\|_{\mathcal{H}_t^{1/3}} \leqslant M \big\}.$$

K_t is compact by the compact embedding $\mathcal{H}_t^{1/3} \Subset \mathcal{H}_t$; besides, since the injection constants are independent of t, $\mathfrak{K}$ is uniformly bounded. Finally, Theorem 10.2, Lemma 11.2 and Lemma 11.3 show that $\mathfrak{K}$ is pullback attracting; indeed,

$$\delta_t \big(U(t, \tau) \mathbb{B}_\tau(R_0), K_t \big) \leqslant C e^{-\delta(t - \tau)}, \quad \forall t \geqslant \tau.$$

Hence the process $U(t, \tau)$ is asymptotically compact, which allows the application of Theorem 4.2 and proves the existence of the unique time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}}$. The invariance of $\mathfrak{A}$ follows by the abstract Theorem 5.6, due to the strong continuity of the process stated in Theorem 9.1.

Remark 11.4. The attraction exerted by the attractor is uniform on compact intervals of time by virtue of Proposition 6.1, due to the continuous dependence estimate (9.2).

11.3. Regularity of the attractor

The minimality of $\mathfrak{A}$ in $\mathbb{K}$ establishes that $A_t \subset K_t$ for all $t \in \mathbb{R}$. Therefore, we immediately have the following regularity result.

Corollary 11.5. *A_t is bounded in $\mathcal{H}_t^{1/3}$ (with a bound independent of t).*

To prove that A_t is bounded in $\mathcal{H}_t^1$, as claimed in Theorem 11.1, we argue as follows. We fix $\tau \in \mathbb{R}$ and, for $z \in A_\tau$, we split the solution $U(t, \tau)z$ into the sum $U_0(t, \tau)z + U_1(t, \tau)z$, where $U_0(t, \tau)z = \{v(t), v_t(t)\}$ and $U_1(t, \tau)z = \{w(t), w_t(t)\}$, instead of (11.5)–(11.6), now solve

$$\begin{cases} \varepsilon v_{tt} + \alpha v_t + A v = 0, \\ U_0(\tau, \tau) = z, \end{cases} \qquad \begin{cases} \varepsilon w_{tt} + \alpha w_t + A w + f(u) = g, \\ U_1(\tau, \tau) = 0. \end{cases}$$

As a particular case of Lemma 11.2, we learn that

$$\big\| U_0(t, \tau)z \big\|_{\mathcal{H}_t} \leqslant C e^{-\delta(t - \tau)}, \quad \forall t \geqslant \tau. \tag{11.11}$$

Lemma 11.6. *We have the uniform bound*

$$\sup_{t \geqslant \tau} \big\| U_1(t, \tau)z \big\|_{\mathcal{H}_t^1} \leqslant M_1,$$

for some $M_1 = M_1(\mathfrak{A}) > 0$.

Proof. We set

$$\mathcal{E}_1 = \big\| U_1(t, \tau)z \big\|_{\mathcal{H}_t^1}^2 + \delta\alpha \|w\|_1^2 + 2\delta\varepsilon \langle w_t, A w \rangle - 2 \langle g, A w \rangle + c,$$

for $\delta > 0$ small and some $c \geqslant 0$ (depending on $\|g\|$) large enough to ensure

$$\frac{1}{4}\big\|U_1(t,\tau)z\big\|_{\mathcal{H}_t^1}^2 \leqslant \mathcal{E}_1(t) \leqslant 2\big\|U_1(t,\tau)z\big\|_{\mathcal{H}_t^1}^2 + 2c. \tag{11.12}$$

A multiplication by $2Aw_t + 2\delta Aw$ leads to the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1 + \big[2\alpha - \varepsilon' - 2\delta\varepsilon\big]\|w_t\|_1^2 + 2\delta\|w\|_2^2 - 2\delta\langle g, Aw\rangle$$
$$= 2\delta\varepsilon'\langle w_t, Aw\rangle - 2\big\langle f(u), Aw_t\big\rangle - 2\delta\big\langle f(u), Aw\big\rangle,$$

and after standard computations we get, for δ small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1 + \delta\mathcal{E}_1 \leqslant -2\big\langle f(u), Aw_t\big\rangle - 2\delta\big\langle f(u), Aw\big\rangle + \delta c.$$

Denoting by $C > 0$ a *generic* constant depending on the size of A_t in $\mathcal{H}_t^{1/3}$, we find, using the invariance of the attractor,

$$\big\|U(t,\tau)z\big\|_{\mathcal{H}_t^{1/3}} \leqslant C.$$

Hence, exploiting the embeddings $H_{4/3} \subset L^{18}(\Omega)$ and $H_{1/3} \subset L^{18/7}(\Omega)$, we deduce the bound

$$\big\|f(u)\big\|_1 \leqslant \big\|f'(u)\big\|_{L^9}\big\|A^{1/2}u\big\|_{L^{18/7}} \leqslant C\big(1 + \|u\|_{L^{18}}^2\big) \leqslant C,$$

yielding

$$-2\big\langle f(u), Aw_t\big\rangle - 2\delta\big\langle f(u), Aw\big\rangle \leqslant 2\big\|f(u)\big\|_1\big(\|w_t\|_1 + \|w\|_1\big) \leqslant \frac{\delta}{2}\mathcal{E}_1 + C.$$

We finally end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1 + \frac{\delta}{2}\mathcal{E}_1 \leqslant C,$$

and an application of the standard Gronwall lemma, recalling (11.12), provides the uniform boundedness of $\|U_1(t,\tau)z\|_{\mathcal{H}_t^1}$, as claimed. □

We are now in position to conclude the proof of Theorem 11.1. Indeed, inequality (11.11) and Lemma 11.6 imply that, for all $t \in \mathbb{R}$,

$$\lim_{\tau\to-\infty} \delta_t\big(U(t,\tau)A_\tau, K_t^1\big) = 0,$$

having defined

$$K_t^1 = \big\{z \in \mathcal{H}_t^1 \colon \|z\|_{\mathcal{H}_t^1} \leqslant M_1\big\}.$$

Since $\mathfrak{A}$ is invariant, this means

$$\delta_t\big(A_t, K_t^1\big) = 0.$$

Hence, $A_t \subset \overline{K_t^1} = K_t^1$, proving that A_t is bounded in $\mathcal{H}_t^1$ with a bound independent of $t \in \mathbb{R}$.

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