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# On the Cauchy problem for integro-differential operators in Sobolev classes and the martingale problem

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**Abstract**

The existence and uniqueness in Sobolev spaces of solutions of the Cauchy problem to parabolic integro-differential equation with variable coefficients of the order $\alpha \in (0, 2)$ is investigated. The principal part of the operator has kernel $m(t, x, y)/|y|^{d+\alpha}$ with a bounded nondegenerate $m$, Hölder in $x$ and measurable in $y$. The lower order part has bounded and measurable coefficients. The result is applied to prove the existence and uniqueness of the corresponding martingale problem.


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*Keywords:* Non-local parabolic integro-differential equations; Lévy processes; Martingale problem

## 1. Introduction

In this paper we consider the Cauchy problem

$$\partial_t u(t, x) = Lu(t, x) + f(t, x), \quad (t, x) \in E = [0, T] \times \mathbf{R}^d, \tag{1.1}$$
$$u(0, x) = 0$$

in fractional Sobolev spaces for a class of integro-differential operators $L = A + B$ with variable coefficients of the order $\alpha \in (0, 2)$ whose principal part $A$ is of the form $Av(t, x) = A_{t,x}v(x) = A_{t,z}v(x)|_{z=x}$ with

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$$A_{t,z}v(x) = \int \big[v(x+y) - v(x) - \chi_\alpha(y)\big(\nabla v(x), y\big)\big] m(t,z,y) \frac{dy}{|y|^{d+\alpha}}, \tag{1.2}$$

$(t,z) \in E$, $x \in \mathbf{R}^d$, with $\chi_\alpha(y) = 1_{\alpha>1} + 1_{\alpha=1} 1_{\{|y|\leqslant 1\}}$. If $m = 1$, then $A = c_\alpha(-\Delta)^{\alpha/2}$ (fractional Laplacian) is the generator of a spherically symmetric $\alpha$-stable process. The part $B$ is a perturbing, lower order operator of the form $Bv(t,x) = B_{t,x}v(x) = B_{t,z}v(x)|_{z=x}$, $(t,x) \in E$, with

$$\begin{aligned} B_{t,z}v(x) = {} & \int_{\mathbf{R}_0^d} \big[v(x+y) - v(x) - \tilde{\chi}_\alpha(y)\big(\nabla v(x), y\big)\big] \pi(t,z,dy) \\ & + \big(b(t,z), \nabla v(x)\big) 1_{1<\alpha<2}, \end{aligned} \tag{1.3}$$

where $(\pi(t,z,dy))$ is a measurable family of nonnegative measures on $\mathbf{R}_0^d$, $\tilde{\chi}_\alpha(y) = 1_{|y|\leqslant 1} \times 1_{1<\alpha<2}$, and

$$\int |y|^\alpha \wedge 1 \, \pi(\cdot, dy), \quad b = \big(b^i\big)_{1\leqslant i\leqslant d}$$

are bounded.

In [11], the problem was considered assuming that $m$ is Hölder continuous in $x$, homogeneous of order zero and smooth in $y$ and for some $\eta > 0$

$$\int_{S^{d-1}} \big|(w,\xi)\big|^\alpha m(t,x,w)\,\mu_{d-1}(dw) \geqslant \eta, \quad (t,x) \in E, \ |\xi| = 1, \tag{1.4}$$

where $\mu_{d-1}$ is the Lebesgue measure on the unit sphere $S^{d-1}$ in $\mathbf{R}^d$. In [1], the existence and uniqueness of a solution to (1.1) in Hölder spaces was proved analytically for $m$ Hölder continuous in $x$, smooth in $y$ and such that for some constant $\eta > 0$

$$K \geqslant m \geqslant \eta > 0 \tag{1.5}$$

without assumption of homogeneity in $y$. The elliptic problem $(L - \lambda)u = f$ with $B = 0$ and $m$ independent of $x$ in $\mathbf{R}^d$ was considered in [4]. Eq. (1.1) with $\alpha = 1$ can be regarded as a linearization of the quasigeostrophic equation (see [2]).

In this note, we consider the problem (1.1), assuming that $m$ is measurable, Hölder continuous in $x$ and

$$K \geqslant m \geqslant m_0, \tag{1.6}$$

where the function $m_0 = m_0(t,x,y)$ is smooth and homogeneous in $y$ and satisfies (1.4). The density $m$ can degenerate on a substantial set and is only measurable in $y$. On the other hand, with $m_0$ as a positive constant, (1.6) includes the case of a uniformly nondegenerate $m$.

A certain aspect of the problem is that the symbol of the main part $A$,

$$\psi(t,x,\xi) = \int \big[e^{i(\xi,y)} - 1 - \chi_\alpha(y) i(\xi,y)\big] m(t,x,y) \frac{dy}{|y|^{d+\alpha}}$$

is not smooth in $\xi$ and the standard Fourier multiplier results (for example, used in [11]) do not apply in this case. We start with Eq. (1.1) assuming that $B=0$, the input function $f$ is smooth and the function $m=m(t,x,y)=m(t,y)$ does not depend on $x$. In [13], the existence and uniqueness of a weak solution in Sobolev spaces was derived. The Hölder norm estimates for the elliptic problem in [4], show that the main part $A: H_p^\alpha \to L_p$ is bounded. We give a direct proof of the continuity of $A$ based on the classical theory of singular integrals (see Lemmas 14, 4 below). The main result of the paper is the existence and uniqueness for the operators with variable coefficients. It is based on the a priori estimates using Sobolev embedding theorem and the method in [9].

As an application, we construct the Markov process associated to $L$ by proving the existence and uniqueness of the corresponding martingale problem (see [18]). The lower part of $L$ has only measurable coefficients and we do not assume any smoothness or homogeneity conditions of $m(t,x,y)$ in $y$. The kernel $m(t,x,y)$ can be zero on a substantial set. A similar martingale problem with all Hölder continuous coefficients was considered in [15] and [1] (see references therein as well). The case of a smooth and homogeneous in $y$ kernel $m$ was studied in e.g. [7] and [12]. The methods of [1] or [15] do not allow any extension to $L$ with a bounded measurable $B$ and $m$ only measurable in $y$.

The note is organized as follows. In Section 2, the main theorem is stated. In Section 3, the essential technical results are presented. The main theorem is proved in Section 4. In Section 5 we discuss the embedding of the solution space. In Section 6 the existence and uniqueness of the associated martingale problem is considered.

## 2. Notation and main results

Denote $E=[0,T]\times\mathbf{R}^d$, $\mathbf{N}=\{0,1,2,\dots\}$, $\mathbf{R}_0^d=\mathbf{R}^d\backslash\{0\}$. If $x,y\in\mathbf{R}^d$, we write

$$(x,y)=\sum_{i=1}^d x_i y_i,$$

$$|x|=(x,x)^{1/2}.$$

For a function $u=u(t,x)$ on $E$, we denote its partial derivatives by $\partial_t u=\partial u/\partial t$, $\partial_i u=\partial u/\partial x_i$, $\partial_{ij}^2 u=\partial^2 u/\partial x_i\partial x_j$ and $D^\gamma u=\partial^{|\gamma|}u/\partial x_i^{\gamma_1}\dots\partial x_d^{\gamma_d}$, where multiindex $\gamma=(\gamma_1,\dots,\gamma_d)\in\mathbf{N}^d$, $\nabla u=(\partial_1 u,\dots,\partial_d u)$ denotes the gradient of $u$ with respect to $x$.

Let $L_p(T)=L_p(E)$ be the space of $p$-integrable functions with norm

$$|f|_{L_p(T)}=\left(\int_0^T\int\left|f(t,x)\right|^p dx\,dt\right)^{1/p}.$$

Similar space of functions on $\mathbf{R}^d$ is denoted $L_p=L_p(\mathbf{R}^d)$.

Let $\mathcal{S}(\mathbf{R}^d)$ be the Schwartz space of smooth real-valued rapidly decreasing functions. We introduce the Sobolev space $H_p^\beta=H_p^\beta(\mathbf{R}^d)$ of $f\in\mathcal{S}'(\mathbf{R}^d)$ with finite norm

$$|f|_{H_p^\beta}=\left|\mathcal{F}^{-1}\left(\left(1+|\xi|^2\right)^{\beta/2}\mathcal{F}f\right)\right|_{L_p},$$

where $\mathcal{F}$ denotes the Fourier transform. We also introduce the corresponding spaces of generalized functions on $E=[0,T]\times\mathbf{R}^d$: $H_p^\beta(T)=H_p^\beta(E)$ consist of all measurable $S'(\mathbf{R}^d)$-valued functions $f$ on $[0,T]$ with finite norm

$$|f|_{H_p^\beta(T)}=\left\{\int_0^T\left|f(t)\right|_{H_p^\beta}^p dt\right\}^{\frac{1}{p}}.$$

For $\alpha\in(0,2)$ and $u\in\mathcal{S}(\mathbf{R}^d)$, we define the fractional Laplacian

$$\partial^\alpha u(x)=\int\nabla_y^\alpha u(x)\frac{dy}{|y|^{d+\alpha}}, \tag{2.1}$$

where

$$\nabla_y^\alpha u(x)=u(x+y)-u(x)-\big(\nabla u(x),y\big)\chi_\alpha(y)$$

with $\chi^{(\alpha)}(y)=1_{\{|y|\leqslant 1\}}1_{\{\alpha=1\}}+1_{\{\alpha\in(1,2)\}}$.

We denote $C_b^\infty(E)$ the space of bounded infinitely differentiable in $x$ functions whose derivatives are bounded.

$C=C(\cdot,\dots,\cdot)$ denotes constants depending only on quantities appearing in parentheses. In a given context the same letter is (generally) used to denote different constants depending on the same set of arguments.

Let $\alpha\in(0,2)$ be fixed. Let $m:E\times\mathbf{R}_0^d\to[0,\infty)$, $b:E\to\mathbf{R}^d$ be measurable functions. We also introduce an auxiliary function $m_0:[0,T]\times\mathbf{R}_0^d\to[0,\infty)$ and fix positive constants $K$ and $\eta$. Throughout the paper we assume that the function $m_0$ satisfies the following conditions.

**Assumption $\mathbf{A_0}$.** (i) The function $m_0=m_0(t,y)\geqslant 0$ is measurable, homogeneous in $y$ with index zero, differentiable in $y$ up to the order $d_0=[\frac{d}{2}]+1$ and

$$\left|D_y^\gamma m_0^{(\alpha)}(t,y)\right|\leqslant K$$

for all $t\in[0,T]$, $y\in\mathbf{R}_0^d$ and multiindices $\gamma\in\mathbf{N}_0^d$ such that $|\gamma|\leqslant d_0$.

(ii) If $\alpha=1$, then for all $t\in[0,T]$

$$\int_{S^{d-1}} w m_0(t,w)\,\mu_{d-1}(dw)=0,$$

where $S^{d-1}$ is the unit sphere in $\mathbf{R}^d$ and $\mu_{d-1}$ is the Lebesgue measure on it.

(iii) For all $t\in[0,T]$

$$\inf_{|\xi|=1}\int_{S^{d-1}}\left|(w,\xi)\right|^\alpha m_0(t,w)\,\mu_{d-1}(dw)\geqslant\eta>0.$$

**Remark 1.** Obviously, Assumption $A_0$ is satisfied if $m_0$ is a positive constant. Also, the nondegenerateness Assumption $A_0$(iii) holds with certain $\eta > 0$ if, e.g.

$$\inf_{t\in[0,T],\, w\in\Gamma} m_0(t,w) > 0$$

for a measurable subset $\Gamma \subset S^{d-1}$ of positive Lebesgue measure. Therefore $m_0$ could be zero on a substantial set.

Further we will use the following assumptions.

**Assumption A.** (i) For all $(t,x) \in E$, $y \in \mathbf{R}_0^d$,

$$K \geqslant m(t,x,y) \geqslant m_0(t,y),$$

where the function $m_0$ satisfies Assumption $A_0$.

(ii) There is $\beta \in (0,1)$ and a continuous increasing function $w(\delta)$ such that

$$\left|m(t,x,y) - m\left(t,x',y\right)\right| \leqslant w\left(\left|x-x'\right|\right), \quad t \in [0,T],\ x,x',y \in \mathbf{R}^d,$$

and

$$\int_{|y|\leqslant 1} w\left(|y|\right)\frac{dy}{|y|^{d+\beta}} < \infty, \qquad \lim_{\delta\to 0} w(\delta)\delta^{-\beta} = 0.$$

(iii) If $\alpha = 1$, then for all $(t,x) \in E$ and $r \in (0,1)$,

$$\int_{r<|y|\leqslant 1/r} y m(t,x,y)\frac{dy}{|y|^{d+\alpha}} = 0.$$

For the lower order operator $B$ we need the following assumptions.

**Assumption B.** (i) For all $(t,x) \in E$,

$$\left|b(t,x)\right| + \int |\upsilon|^\alpha \wedge 1\, \pi(t,x,d\upsilon) \leqslant K.$$

(ii)

$$\lim_{\varepsilon\to 0} \sup_{t,x} \int_{|\upsilon|\leqslant\varepsilon} |\upsilon|^\alpha\, \pi(t,x,d\upsilon) = 0.$$

(iii) For each $\varepsilon > 0$,

$$\int_0^T \int \pi\left(t,x,\left\{|\upsilon| > \varepsilon\right\}\right) dx\, dt < \infty.$$

We write

$$Lu(t,x) = L_t u(x) = L_{t,x}u(x), \qquad L = A + B.$$

According to Assumptions A, B, the operator $A$ represents the principal part of $L$ and the operator $B$ is a lower order operator.

We consider the following Cauchy problem

$$\begin{aligned} \partial_t u(t,x) &= (L-\lambda)u(t,x) + f(t,x), \quad (t,x) \in H, \\ u(0,x) &= 0, \quad x \in \mathbf{R}^d, \end{aligned} \tag{2.2}$$

in Sobolev classes $H_p^\alpha(E)$, where $\lambda \geqslant 0$ and $f \in L_p(E)$. More precisely, let $\mathcal{H}_p^\alpha = \mathcal{H}_p^\alpha(E)$ be the space of all functions $u \in H_p^\alpha(T) = H_p^\alpha(E)$ such that $u(t,x) = \int_0^t F(s,x)\,ds$, $0 \leqslant t \leqslant T$, with $F \in L_p(E)$. It is a Banach space with respect to the norm

$$|u|_{\mathcal{H}_p^\alpha} = |u|_{H_p^\alpha(T)} + |F|_{L_p(T)}.$$

**Definition 1.** Let $f \in L_p(E)$. We say that $u \in \mathcal{H}_p^\alpha(E)$ is a solution to (2.2) if $Lu \in L_p(E)$ and

$$u(t) = \int_0^t \big((L-\lambda)u(s) + f(s)\big)\,dt, \quad 0 \leqslant t \leqslant T, \tag{2.3}$$

in $L_p(\mathbf{R}^d)$.

If Assumptions A and B are satisfied, $p > \frac{d}{\alpha} \vee \frac{d}{\beta}$, then $Lu \in L_p(E)$ (see Corollary 2 below and Lemma 7 in [11]). So, (2.3) is well defined.

The main result of the paper is the following theorem.

**Theorem 1.** *Let $\beta \in (0,1)$, $p > \frac{d}{\beta}$ and Assumption A be satisfied.*

*Then for any $f \in L_p(E)$ there exists a unique strong solution $u \in \mathcal{H}_p^\alpha(E)$ to (2.2) with $B = 0$. Moreover, there is a constant $N = N(T,\alpha,\beta,d,K,w,\eta)$ and a positive number $\lambda_1 = \lambda_1(T,\alpha,\beta,d,K,w,\eta) \geqslant 1$ such that*

$$|\partial_t u|_{L_p(T)} + |u|_{H_p^\alpha(T)} \leqslant N|f|_{L_p(T)},$$
$$|u|_{L_p(T)} \leqslant \frac{N}{\lambda}|f|_{L_p(T)} \quad \text{if } \lambda \geqslant \lambda_1.$$

We prove this theorem in Section 3 below.

In order to handle (2.2) with the lower order part $Bu$, the following estimate is needed.

**Lemma 1.** *(See Lemma 3.5 in [11].) Let $p > d/\alpha \vee 1$. There is a constant $N_1 = N_1(p,\alpha,d)$ such that*

$$\left|\sup_{y\neq 0} \frac{|\nabla_y^\alpha v(\cdot)|}{|y|^\alpha}\right|_{L_p} \leqslant N_1\left|\partial^\alpha v\right|_{L_p}, \quad v \in C_0^\infty(\mathbf{R}^d).$$

Consider (2.2) with $Bv(t,x) = B^{\varepsilon_0}v(t,x) = B^{\varepsilon_0}_{t,x}v(x)$, $(t,x) \in E$, where

$$B^{\varepsilon_0}_{t,z}v(x) = \int_{|y|\leqslant\varepsilon_0} \nabla^\alpha_y v(x)\,\pi(t,z,dy), \quad (t,z)\in E,\ x\in\mathbf{R}^d, \tag{2.4}$$

with some $\varepsilon_0 \in (0,1]$.

First we consider a special case of the lower order term.

**Theorem 2.** *Let $\beta\in(0,1)$, $p > \frac{d}{\beta}\vee\frac{d}{\alpha}$ and Assumption A be satisfied. Let*

$$\int_{|y|\leqslant\varepsilon_0} |y|^\alpha\,\pi(t,x,dy)\leqslant\delta_0, \quad (t,x)\in E,$$

*and $\delta_0 N N_1 \leqslant 1/2$, where $N$ is a constant of Theorem 1. Then for any $f\in L_p(E)$ there exists a unique solution $u\in\mathcal{H}^\alpha_p(E)$ to (2.2) with $B = B^{\varepsilon_0}$. Moreover,*

$$|\partial_t u|_{L_p(T)} + |u|_{H^\alpha_p(T)} \leqslant 2N|f|_{L_p(T)},$$
$$|u|_{L_p(T)} \leqslant \frac{2N}{\lambda}|f|_{L_p(T)} \quad \text{if } \lambda\geqslant\lambda_1.$$

For the derivation of $L_p$-estimates in the associated martingale problem (see Section 6 below), we need to consider (2.2) with $B_t v(x) = \tilde{B}^{\varepsilon_0}_{t,x}v(x)$, where

$$\tilde{B}^{\varepsilon_0}_{t,z}v(x) = 1_{\alpha\in(1,2)}\Bigl(b(t,z) - \int_{1\geqslant|y|>\varepsilon_0} y\,\pi(t,z,dy), \nabla v(x)\Bigr) + B^{\varepsilon_0}_{t,z}v(x), \quad (t,z)\in E,\ x\in\mathbf{R}^d, \tag{2.5}$$

with some $\varepsilon_0\in(0,1]$ from Theorem 2.

**Corollary 1.** *Let $\beta\in(0,1)$, $p > \frac{d}{\beta}\vee\frac{d}{\alpha}$ and Assumptions A and B(i) be satisfied. Let*

$$\int_{|y|\leqslant\varepsilon_0} |y|^\alpha\,\pi(t,x,dy)\leqslant\delta_0, \quad (t,x)\in E,$$

*and $\delta_0 N N_1\leqslant 1/2$, where $N$ is a constant of Theorem 1. Then for any $f\in L_p(E)$ there exists a unique solution $u\in\mathcal{H}^\alpha_p(E)$ to (2.2) with $B=\tilde{B}^{\varepsilon_0}$. Moreover, there is a positive number $\lambda_2 = \lambda_2(T,\alpha,\beta,d,K,w,\eta,\varepsilon_0)\geqslant 1$ such that*

$$|\partial_t u|_{L_p(T)} + |u|_{H^\alpha_p(T)}\leqslant 8N|f|_{L_p(T)}, \qquad |u|_{L_p(T)}\leqslant\frac{8N}{\lambda}|f|_{L_p(T)} \quad \text{if } \lambda\geqslant\lambda_2.$$

Finally, the results can be extended to

**Theorem 3.** *Let $\beta \in (0,1)$, $p > \frac{d}{\beta} \vee \frac{d}{\alpha}$ and Assumptions A, B be satisfied.*

*Then for any $f \in L_p(E)$ there exists a unique solution $u \in \mathcal{H}_p^\alpha(E)$ to (2.2). Moreover, there is a constant $N_3$ independent of $u$ such that*

$$|\partial_t u|_{L_p(T)} + |u|_{H_p^\alpha(T)} \leqslant N_3 |f|_{L_p(T)}.$$

## 3. Auxiliary results

In this section we present some auxiliary results.

### *3.1. Continuity of the principal part*

First we prove the continuity of the operator $A$ in $L_p$-norm.

We will use the following equality for Sobolev norm estimates.

**Lemma 2.** *(See Lemma 2.1 in [7].) For $\alpha \in (0,1)$ and $u \in \mathcal{S}(\mathbf{R}^d)$,*

$$u(x+y) - u(x) = C \int k^{(\alpha)}(y,z) \partial^\alpha u(x-z)\,dz, \tag{3.1}$$

*where the constant $C = C(\alpha, d)$ and*

$$k^{(\alpha)}(z,y) = |z+y|^{-d+\alpha} - |z|^{-d+\alpha}.$$

*Moreover, there is a constant $C = C(\alpha, d)$ such that for each $y \in \mathbf{R}^d$*

$$\int \left|k^{(\alpha)}(z,y)\right| dz \leqslant C|y|^\alpha.$$

The following estimate will be used as well.

**Lemma 3.** *Let $u, v \in \mathcal{S}(\mathbf{R}^d)$ and*

$$\begin{aligned} J(x,l) &= \int \left[v(x+y) - v(x)\right]\left[u(x+y-l) - u(x-l)\right] m(x,y) \frac{dy}{|y|^{d+\alpha}}, \quad x, l \in \mathbf{R}^d, \\ &= \int_{|y|>1} \ldots + \int_{|y|\leqslant 1} \ldots = A(x,l) + B(x,l), \end{aligned}$$

*where $m(x,y)$ is a measurable bounded function such that*

$$\left|m(x,y)\right| \leqslant K, \quad x, y \in \mathbf{R}^d.$$

*Then for some $\alpha' < \alpha$ and a constant $C = C(\alpha, \alpha', p, d)$*

$$\int\int \left|J(x,l)\right|^p dx\,dl \leqslant C K^p |v|_{H_p^\alpha}^p |u|_{H_p^1}^p.$$

**Proof.** We split

$$J(x,l)=\int_{|y|>1}\ldots+\int_{|y|\leqslant 1}\ldots=A(x,l)+B(x,l),\quad x,l\in\mathbf{R}^d.$$

By Hölder inequality and Fubini theorem,

$$\begin{aligned}\int\left|A(x,l)\right|^p dx\,dl&\leqslant CK^p\int_{|y|>1}\left|v(x+y)-v(x)\right|^p\left|u(x+y-l)-u(x-l)\right|^p\frac{dy\,dl\,dx}{|y|^{d+\alpha}}\\&\leqslant CK^p|v|_{L_p}^p|u|_{L_p}^p.\end{aligned}$$

If $\alpha-1<\alpha'<1\leqslant\alpha<2$, then by Lemma 2,

$$\left|B(x,l)\right|^p\leqslant CK^p\int_0^1\int_{|y|\leqslant 1}\int\left|\partial^{\alpha'}v(x-z)\right|^p\left|k^{(\alpha')}(y,z)\right|\frac{dz}{|y|^{\alpha'}}\left|\nabla\zeta(x+sy-l)\right|^p\frac{ds\,dy}{|y|^{d+\alpha-1-\alpha'}}$$

and

$$\int\int\left|B(x,l)\right|^p dx\,dl\leqslant CK^p\left|\partial^{\alpha'}v\right|_{L_p}^p|\nabla\zeta|_{L_p}^p.$$

If $\alpha\in(0,1)$, then by Hölder inequality and Fubini theorem directly,

$$\begin{aligned}&\int\int\left|B(x,l)\right|^p dx\,dl\\&\quad\leqslant CK^p\int_0^1\int_{|y|\leqslant 1}\int\int\left|v(x+y)-v(x)\right|^p\left|\nabla\zeta(x+sy-l)\right|^p dl\,dx\,\frac{ds\,dy}{|y|^{d+\alpha-1}}\\&\quad\leqslant CK^p|v|_{L_p}^p|\nabla\zeta|_{L_p}^p.\end{aligned}$$

The statement follows. □

For a bounded measurable $m(y)$, $y\in\mathbf{R}^d$, and $\alpha\in(0,2)$, set for $v\in\mathcal{S}(\mathbf{R}^d)$, $x\in\mathbf{R}^d$,

$$\mathcal{L}v(x)=\int\left[v(x+y)-v(x)-\chi_\alpha(y)\big(\nabla v(x),y\big)\right]m(y)\frac{dy}{|y|^{d+\alpha}}.$$

The following estimate shows that $\mathcal{L}:H_p^\alpha(\mathbf{R}^d)\to L_p(\mathbf{R}^d)$ is continuous.

**Lemma 4.** *Let $|m(y)|\leqslant K$, $y\in\mathbf{R}^d$, $p>1$, and $\alpha\in(0,2)$. Assume*

$$\int_{r\leqslant|y|\leqslant R} ym(y)\frac{dy}{|y|^{d+\alpha}}=0$$

*for any* $0 < r < R$ *if* $\alpha = 1$. *Then there is a constant* $C$ *such that*

$$|\mathcal{L}v|_{L_p} \leqslant CK\left|\partial^\alpha v\right|_{L_p}, \quad v \in L_p = L_p\left(\mathbf{R}^d\right).$$

In [4], this estimate follows as a consequence of some fine Hölder norm estimates (dependence on $K$ not specified). In Appendix A below, we give a direct proof based on the classical theory of singular integrals (see Lemmas 14, 15 below).

**Remark 2.** If $\phi \in S(\mathbf{R}^d)$, then

$$\begin{aligned}
B_\phi(x) &= \int \left|\phi(x+y) - \phi(x) - \chi_\alpha(y)\left(\nabla\phi(x), y\right)\right| \frac{dy}{|y|^{d+\alpha}} \\
&\leqslant \int_{|y|>1} \nabla_y^\alpha \phi(x) \frac{dy}{|y|^{d+\alpha}} + 1_{\alpha\in(0,1)} \int_{|y|\leqslant 1} \int_0^1 \left|\nabla\phi(x+sy)\right| \frac{ds\,dy}{|y|^{d+\alpha-1}} \\
&\quad + 1_{\alpha\in[1,2)} \int_{|y|\leqslant 1} \int_0^1 (1-s) \max_{ij} \left|\partial_{ij}^2 \phi(x+sy)\right| \frac{ds\,dy}{|y|^{d+\alpha-2}},
\end{aligned}$$

and we have an obvious estimate

$$|B_\phi|_{L_p}^p \leqslant C|\phi|_{H_p^2}^p$$

with some $C = C(d, \alpha, p)$.

Now we investigate the continuity of the main part $A$ with $m$ depending on $x$. For a bounded measurable $m(x, y)$, $x, y \in \mathbf{R}^d$, consider the operator $\mathcal{A}v(x) = \mathcal{A}_z v(x)|_{z=x}$, $x \in \mathbf{R}^d$, with $v \in \mathcal{S}(\mathbf{R}^d)$ and

$$\begin{aligned}
\mathcal{A}_z v(x) &= \mathcal{A}_z^m v(x) \\
&= \int \left[v(x+y) - v(x) - \chi_\alpha(y)\left(\nabla v(x), y\right)\right] m(z, y) \frac{dy}{|y|^{d+\alpha}}.
\end{aligned}$$

**Lemma 5.** *Assume* $\beta \in (0, 1)$, $p > d/\beta$. *Let for each* $y \in \mathbf{R}^d$, $m(\cdot, y) \in H_p^\beta(\mathbf{R}^d)$ *and*

$$\left|m(z, y)\right| + \left|\partial_z^\beta m(z, y)\right| < \infty, \quad z \in \mathbf{R}^d.$$

*Then*

$$|\mathcal{A}v|_{L_p}^p \leqslant C\left|\partial^\alpha v\right|_{L_p}^p \int \sup_y \left[\left|m(z, y)\right|^p + \left|\partial^\beta m(z, y)\right|^p\right] dz.$$

**Proof.** By Sobolev embedding theorem, there is a constant $C$ such that

$$\begin{aligned}\left|\mathcal{A}_x v(x)\right|^p &\leqslant \sup_z\left|\mathcal{A}_z v(x)\right|^p \\ &\leqslant C\int\left[\left|\mathcal{A}_z v(x)\right|^p+\left|\partial_z^\beta\mathcal{A}_z v(x)\right|^p\right]dz \\ &= C\int\left[\left|\mathcal{A}_z^m v(x)\right|^p+\left|\mathcal{A}_z^{\partial_z^\beta m} v(x)\right|^p\right]dz, \quad x\in\mathbf{R}^d,\end{aligned}$$

and by Lemma 4

$$\begin{aligned}|\mathcal{A}v|_{L_p}^p &\leqslant C\int\left[|\mathcal{A}_z v|_{L_p}^p+\left|\mathcal{A}_z^{\partial_z^\beta m} v\right|_{L_p}^p\right]dz \\ &\leqslant C\left|\partial^\alpha v\right|_{L_p}^p\int\sup_y\left[\left|m(z,y)\right|^p+\left|\partial^\beta m(z,y)\right|^p\right]dz.\end{aligned}$$

The statement follows. □

The following statement holds.

**Lemma 6.** *Let $\phi, v\in S(\mathbf{R}^d)$. Assume $\beta\in(0,1)$, $p>d/\beta$.*

(a) *If*

$$\left|m(z,y)\right|+\left|\partial_z^\beta m(z,y)\right|\leqslant K, \quad z,y\in\mathbf{R}^d,$$

*then there is a constant $C=C(\alpha,p,\beta,d)$ such that*

$$|\phi\mathcal{A}v|_{L_p}^p\leqslant CK^p|v|_{H_p^\alpha}^p\left[|\phi|_{L_p}^p+|\nabla\phi|_{L_p}^p\right].$$

(b) *If $m(z,y)$ is bounded and for a continuous increasing function $w(\delta)$, $\delta>0$,*

$$\left|m(z,y)-m\left(z',y\right)\right|\leqslant w\left(\left|z-z'\right|\right), \quad z,z',y\in\mathbf{R}^d,$$

*with*

$$\int_{|y|\leqslant 1} w\left(|y|\right)\frac{dy}{|y|^{d+\beta}}<\infty,$$

*then there is a constant $C=C(\alpha,p,\beta,d)$ such that for each $\varepsilon\in(0,1]$,*

$$|\phi\mathcal{A}v|_p^p\leqslant CK(\varepsilon,\phi)|v|_{\alpha,p},$$

*where*

$$
\begin{aligned}
K(\varepsilon,\phi) &= \varepsilon^{-\beta p}\Big|\phi\sup_y\big|m(\cdot,y)\big|\Big|_{L_p}^p + \varepsilon^{(1-\beta)p}\Big|\sup_y\big|m(\cdot,y)\big|\nabla\phi\Big|_{L_p}^p \\
&\quad + w(\varepsilon)^p\varepsilon^{(1-\beta)p}|\nabla\phi|_p^p + \kappa(\varepsilon)^p|\phi|_{L_p}^p + \varepsilon^p\kappa(\varepsilon)^p|\nabla\phi|_{L_p}^p, \\
\kappa(\varepsilon) &= \int_{|\upsilon|\leqslant\varepsilon} w(\upsilon)\frac{d\upsilon}{|\upsilon|^{d+\beta}}.
\end{aligned}
$$

**Proof.** Since $\phi(x)\mathcal{A}_x v(x) = \mathcal{A}_x^{\phi m} v(x)$, we will apply Lemma 5 for $\phi(z)m(z,y)$. First, obviously,

$$
\int \sup_y\big|\phi(z)m(z,y)\big|^p\,dx \leqslant \sup_{z,y}\big|m(z,y)\big||\phi|_{L_p}^p.
$$

For each $\varepsilon\in(0,1]$,

$$
\begin{aligned}
\partial_z^\beta(\phi m) &= \int_{|\upsilon|>\varepsilon}\big[\phi(z+\upsilon)m(z+\upsilon,y)-\phi(z)m(z,y)\big]\frac{d\upsilon}{|\upsilon|^{d+\beta}} \\
&\quad + \int_{|\upsilon|\leqslant\varepsilon}\big[\phi(z+\upsilon)m(z+\upsilon,y)-\phi(z)m(z,y)\big]\frac{d\upsilon}{|\upsilon|^{d+\beta}} \\
&= A(z,y)+B(z,y),\quad z,y\in\mathbf{R}^d.
\end{aligned}
$$

By Hölder inequality,

$$
\begin{aligned}
\int\sup_y\big|A(z,y)\big|^p\,dz &\leqslant \varepsilon^{-\beta p}\int\big|\phi(z)\big|^p\sup_y\big|m(z,y)\big|^p\,dz \\
&= C\varepsilon^{-\beta p}\Big|\phi\sup_y\big|m(\cdot,y)\big|\Big|_{L_p}^p \\
&\leqslant C\varepsilon^{-\beta p}|\phi|_{L_p}^p\sup_{y,z}\big|m(z,y)\big|^p.
\end{aligned}
$$

We split $B = B_1 + B_2 + B_3$ with

$$
\begin{aligned}
B_1(z,y) &= m(z,y)\int_{|\upsilon|\leqslant\varepsilon}\big[\phi(z+\upsilon)-\phi(z)\big]\frac{d\upsilon}{|\upsilon|^{d+\beta}}, \\
B_2(z,y) &= \phi(z)\int_{|\upsilon|\leqslant\varepsilon}\big[m(z+\upsilon,y)-m(z,y)\big]\frac{d\upsilon}{|\upsilon|^{d+\beta}}, \\
B_3(z,y) &= \int_{|\upsilon|\leqslant\varepsilon}\big[m(z+\upsilon,y)-m(z,y)\big]\big[\phi(z+\upsilon)-\phi(z)\big]\frac{d\upsilon}{|\upsilon|^{d+\beta}}.
\end{aligned}
$$

We have

$$\begin{aligned}
\sup_y \left| B_1(z,y)\right| &\leqslant \sup_y \left| m(z,y)\right| \int_{|\upsilon|\leqslant\varepsilon} \int_0^1 \left|\nabla\phi(z+s\upsilon)\right| ds \frac{d\upsilon}{|\upsilon|^{d+\beta-1}} \\
&\leqslant w(\varepsilon) \int_{|\upsilon|\leqslant\varepsilon} \int_0^1 \left|\nabla\phi(z+s\upsilon)\right| ds \frac{d\upsilon}{|\upsilon|^{d+\beta-1}} \\
&\quad + \int_{|\upsilon|\leqslant\varepsilon} \int_0^1 \sup_y \left| m(z+s\upsilon,y)\right| \left|\nabla\phi(z+s\upsilon)\right| ds \frac{d\upsilon}{|\upsilon|^{d+\beta-1}}
\end{aligned}$$

and

$$\begin{aligned}
\int \sup_y \left| B_1(z,y)\right|^p dz &\leqslant C\Big[ w(\varepsilon)^p \varepsilon^{(1-\beta)p} |\nabla\phi|^p_{L_p} + \varepsilon^{(1-\beta)p} \Big| \sup_y \left| m(\cdot,y)\right| \nabla\phi\Big|^p_{L_p}\Big] \\
&\leqslant C\varepsilon^{(1-\beta)p} \sup_{x,y} \left| m(z,y)\right|^p |\nabla\phi|^p_{L_p}.
\end{aligned}$$

Since

$$\int_{|\upsilon|\leqslant\varepsilon} \left[ m(z+\upsilon,y) - m(z,y)\right] \frac{d\upsilon}{|\upsilon|^{d+\beta}} = \partial_z^\beta m(z,y) - \int_{|\upsilon|>\varepsilon} \left[ m(z+\upsilon,y) - m(z,y)\right] \frac{d\upsilon}{|\upsilon|^{d+\beta}},$$

it follows that

$$\left| B_2(z,y)\right| \leqslant \left|\phi(z)\right| \Big( \sup_{z,y} \left|\partial_z^\beta m(z,y)\right| + \varepsilon^{-\beta} \sup_{z,y} \left| m(z,y)\right| \Big)$$

and

$$\int \sup_y \left| B_2(z,y)\right|^p dz \leqslant C|\phi|^p_{L_p} \Big( \sup_{z,y} \left|\partial_z^\beta m(z,y)\right|^p + \varepsilon^{-\beta p} \sup_{z,y} \left| m(z,y)\right|^p \Big).$$

On the other hand,

$$\left| B_2(z,y)\right| \leqslant \left|\phi(z)\right| \int_{|\upsilon|\leqslant\varepsilon} w(\upsilon) \frac{d\upsilon}{|\upsilon|^{d+\beta}}$$

and

$$\int \sup_y \left| B_2(z,y)\right|^p dz \leqslant C\Big( \int_{|\upsilon|\leqslant\varepsilon} w(\upsilon) \frac{d\upsilon}{|\upsilon|^{d+\beta}} \Big)^p |\phi|^p_p.$$

Now,

$$\left|B_3(z,y)\right| \leqslant \int_{|\upsilon|\leqslant\varepsilon} w(\upsilon)|\upsilon| \int_0^1 \left|\nabla\phi(z+s\upsilon)\right| \frac{ds\,d\upsilon}{|\upsilon|^{d+\beta}}$$

and

$$\int \sup_y \left|B_3(z,y)\right|^p dz \leqslant C|\nabla\phi|_{L_p}^p \left(\int_{|\upsilon|\leqslant\varepsilon} |\upsilon| w(\upsilon) \frac{d\upsilon}{|\upsilon|^{d+\beta}}\right)^p$$

$$\leqslant C \sup_{z,y} \left|m(z,y)\right|^p \varepsilon^{(1-\beta)p} |\nabla\phi|_{L_p}^p$$

and the statement follows. □

The following is a consequence of Lemma 6.

**Corollary 2.** *Let $v \in S(\mathbf{R}^d)$. Assume $\beta \in (0,1)$, $p > d/\beta$, and*

$$\left|m(z,y)\right| + \left|\partial_z^\beta m(z,y)\right| \leqslant K, \quad z, y \in \mathbf{R}^d.$$

*Then there is a constant $C = C(\alpha, p, \beta, d)$ independent of $v$ such that*

$$|\mathcal{A}v|_{L_p}^p \leqslant CK^p |v|_{H_p^\alpha}^p.$$

**Proof.** We split

$$\mathcal{A}v(x) = \int_{|y|\leqslant 1} \ldots m(x,y) \frac{dy}{|y|^{d+\alpha}} + \int_{|y|>1} \ldots m(x,y) \frac{dy}{|y|^{d+\alpha}}$$

$$= D_1(x) + D_2(x), \quad x \in \mathbf{R}^d.$$

Obviously,

$$|D_2|_{L_p}^p \leqslant C|v|_{H_p^\alpha}^p \sup_{z,y} \left|m(z,y)\right|^p.$$

Assume first that the support of $v$ is in the ball centered at some $x_0 \in \mathbf{R}^d$ with radius 1. Let $\varphi \in C_0^\infty(\mathbf{R}^d)$, $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ if $|x| \leqslant 1$, and $\varphi(x) = 0$ if $|x| > 2$. Then

$$D_1(x) = \varphi\left(\frac{x - x_0}{2}\right) D_1(x) = \int_{|y|\leqslant 1} \ldots \varphi\left(\frac{x-x_0}{2}\right) m(x,y) \frac{dy}{|y|^{d+\alpha}}$$

and by Lemma 6 there is a constant $C = C(\alpha, p, \beta, d)$ such that

$$|D_1|_{L_p}^p \leqslant CK^p |v|_{H_p^\alpha}^p \left[|\varphi|_{L_p}^p + |\nabla\varphi|_{L_p}^p\right]. \tag{3.2}$$

Let $\zeta \in C_0^\infty(\mathbf{R}^d)$ be such that $\zeta(x) = \zeta(-x)$, $x \in \mathbf{R}^d$, $\int |\zeta|^p\, dx = 1$ and has its support in the unit ball centered at the origin. Then for each $x \in \mathbf{R}^d$,

$$\left|\mathcal{A}v(x)\right|^p = \int \left|\zeta(x-l)\mathcal{A}v(x)\right|^p dl.$$

Obviously,

$$\begin{aligned}\zeta(x-l)\mathcal{A}v(x) &= \mathcal{A}\big(v(\cdot)\zeta(\cdot-l)\big) - u(x)\mathcal{A}_x\zeta(x-l)\\ &\quad + \int \big(v(x+y)-v(x)\big)\big(\zeta(x+y-l)-\zeta(x-l)\big)m(x,y)\frac{dy}{|y|^{d+\alpha}}.\end{aligned} \tag{3.3}$$

According to Remark 2,

$$\int \left|\mathcal{A}_x\zeta(x-l)\right|^p dl \leqslant CK^p|\zeta|^p_{H^2_p}.$$

Denoting

$$D(x,l) = \int \big(v(x+y)-v(x)\big)\big(\zeta(x+y-l)-\zeta(x-l)\big)m(x,y)\frac{dy}{|y|^{d+\alpha}},$$

we have by Lemma 3

$$\int\int \left|D(x,l)\right|^p dx\, dl \leqslant CK^p|v|^p_{H^{\alpha'}_p}$$

for some $\alpha' < \alpha$. Therefore, by (3.2)

$$|\mathcal{A}v|^p_{L_p} \leqslant CK^p\left[\int \left|\big(v(\cdot)\zeta(\cdot-l)\big)\right|^p_{H^\alpha_p} dl + |v|^p_{L_p} + |v|^p_{H^{\alpha'}_p}\right].$$

Since as in (3.3)

$$\begin{aligned}\partial^\alpha_x\big(v(x)\zeta(x-l)\big) &= \partial^\alpha v(x)\zeta(x-l) + v(x)\partial^\alpha\zeta(x-l)\\ &\quad + \int \big(v(x+y)-v(x)\big)\big(\zeta(x+y-l)-\zeta(x-l)\big)\frac{dy}{|y|^{d+\alpha}},\end{aligned}$$

we derive in a similar way,

$$\begin{aligned}\int \left|\big(v(\cdot)\zeta(\cdot-l)\big)\right|^p_{H^\alpha_p} dl &= \int \left|\big(v(\cdot)\zeta(\cdot-l)\big)\right|^p_{L_p} dl + \int \left|\partial^\alpha\big(v(\cdot)\zeta(\cdot-l)\big)\right|^p_{L_p} dl\\ &\leqslant C|v|^p_{H^\alpha_p}.\end{aligned}$$

The statement follows. □

### 3.2. Solution for $m$ independent of space variable

In this section, we consider the following partial case of Eq. (2.2):

$$\begin{aligned}\partial_t u(t,x) &= A^m u(t,x) - \lambda u(t,x) + f(t,x), \\ u(0,x) &= 0,\end{aligned} \tag{3.4}$$

where $m(t,x,y) = m(t,y)$ does not depend on the space variable.

First we solve the problem for smooth input functions. We denote by $\mathfrak{D}_p(E)$, $p \geqslant 1$, the space of all measurable functions $f$ on $E$ such that $f \in \bigcap_{\kappa>0} H_p^\kappa(E)$ and for every multiindex $\gamma \in \mathbf{N}_0^d$

$$\sup_{(t,x)\in E} \left| D_x^\gamma f(t,x) \right| < \infty.$$

Obviously, $\mathfrak{D}_p(E) \subseteq C_b^\infty(E)$.

**Lemma 7.** *Let $p > 1$, $f \in \mathfrak{D}_p(E)$ and Assumption A be satisfied.*

*Then there is a unique solution $u \in \mathfrak{D}_p(E) \cap \mathcal{H}_p^\alpha(E)$ of (3.4). Moreover, there are constants $C = C(d,p)$, $N = N(\alpha,p,d,K,\eta)$ such that for any multiindex $\gamma \in \mathbf{N}^d$,*

$$\left| D^\gamma u \right|_{L_p(T)} \leqslant C\rho_\lambda \left| D^\gamma f \right|_{L_p(T)}, \qquad |u|_{H_p^\alpha(T)} \leqslant N |f|_{L_p(T)}, \tag{3.5}$$

*where $\rho_\lambda = T \wedge \frac{1}{\lambda}$.*

**Proof.** *Uniqueness*. Let $u^1, u^2 \in \mathfrak{D}_p(E) \cap \mathcal{H}_p^\alpha(E)$ be two solutions to (3.4). Then the function $u = u^1 - u^2 \in \mathfrak{D}_p(E) \cap \mathcal{H}_p^\alpha(E)$ satisfies (3.4) with $f = 0$ and is continuous in $t$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration of $\sigma$-algebras $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions. We fix $t_0 \in (0,T)$ and introduce an $\mathbb{F}$-adapted Poisson point measure $p(dt,dy)$ on $[0,t_0] \times \mathbf{R}_0^d$ with a compensator $m(t_0 - t, y)\,dt\,dy/|y|^{d+\alpha}$. Let

$$q(dt,dy) = p(dt,dy) - m(t_0 - t, y)\frac{dt\,dy}{|y|^{d+\alpha}}$$

be the corresponding martingale measure and

$$X_t = \int_{s_0}^{t}\int \chi_\alpha(y) y\, q(ds,dy) + \int_0^t \int \bigl(1 - \chi_\alpha(y)\bigr) y\, p(ds,dy)$$

for $0 \leqslant t \leqslant t_0$. By Ito's formula

$$\begin{aligned} u(t_0,x) &= u(t_0,x) - \mathbf{E}u(0, x + X_{t_0})e^{-\lambda t_0} \\ &= \mathbf{E}\int_0^{t_0} e^{-\lambda t}\left[\frac{\partial u}{\partial t} - Au + \lambda u\right](t - t_0, x + X_t)\,dt = 0. \end{aligned}$$

Since $t_0$ and $x$ are arbitrary, we have $u = 0$.

*Existence*. We introduce an $\mathbb{F}$-adapted Poisson measure $\bar{p}(dt, dz)$ on $[0, \infty) \times \mathbf{R}_0$ with a compensator $dt\,dz/z^2$. Let

$$\bar{q}(dt, dz) = \bar{p}(dt, dz) - \frac{dt\,dz}{z^2}$$

be the corresponding martingale measure. According to Lemma 14.50 in [5], there is a measurable function $\bar{c} : [0, T] \times \mathbf{R}_0 \to \mathbf{R}^d$ such that for every Borel $\Gamma \subseteq \mathbf{R}_0^d$

$$\int_\Gamma \bigl(m(t, y) - m_0(t, y)\bigr) \frac{dy}{|y|^{d+\alpha}} = \int 1_\Gamma\bigl(\bar{c}(t, z)\bigr) \frac{dz}{z^2}.$$

Let

$$Y_t = \int_0^t \int \bigl(1 - \chi_\alpha\bigl(\bar{c}(s, z)\bigr)\bigr)\bar{c}(s, z)\, \bar{p}(ds, dz) + \int_0^t \int \chi_\alpha\bigl(\bar{c}(s, z)\bigr)\bar{c}(s, z)\, \bar{q}(ds, dz).$$

For $f \in \mathfrak{D}_p(E)$, we consider the equation

$$\begin{aligned} &\partial_t u(t, x) = A^0 u(t, x) + f(t, x - Y_t), \quad (t, x) \in E, \\ &u(0, x) = 0, \quad x \in \mathbf{R}^d, \end{aligned} \tag{3.6}$$

where $A^0 = A^{m_0}$ ($m_0$ is from Assumption A). In terms of Fourier transforms, for $v \in \mathcal{S}(\mathbf{R}^d)$,

$$\mathcal{F}\bigl(A^0 v\bigr)(t, \xi) = \psi_0(t, \xi)\mathcal{F}v(\xi)$$

with

$$\begin{aligned} \psi_0(t, \xi) = -C \int_{S^{d-1}} \bigl|(w, \xi)\bigr|^\alpha \biggl[1 - i\biggl(\tan\frac{\alpha\pi}{2} \operatorname{sgn}(w, \xi) 1_{\alpha \neq 1} \\ - \frac{2}{\pi} \operatorname{sgn}(w, \xi) \ln\bigl|(w, \xi)\bigr| 1_{\alpha=1}\biggr)\biggr] m_0(t, w)\, \mu_{d-1}(dw) \end{aligned}$$

and the constant $C = C(\alpha) > 0$. By Lemma 7 in [14], there is a unique $u \in C_b^\infty(E)$ continuous in $t$ and smooth in $x$ solving (3.6) in classical sense. Moreover,

$$u(t, x) = \int_0^t \int G_{s,t}^\lambda(y) f(s, x - y - Y_s)\, dy\, ds, \quad (t, x) \in E,$$

where

$$G_{s,t}^\lambda(x) = e^{-\lambda(t-s)} G_{s,t}(x), \qquad G_{s,t}(x) = \mathcal{F}^{-1} K_{s,t},$$

$$K_{s,t}(\xi) = \exp\biggl\{\int_s^t \psi_0(r, \xi)\, dr\biggr\}, \quad s \leqslant t.$$

According to Assumption $A_0$, $\int |K_{s,t}(\xi)|\,d\xi < \infty$, $s < t$, and $G_{s,t}$ is the density function of a random variable whose characteristic function is $K_{s,t}$. Hence,

$$G_{s,t} \geqslant 0, \qquad \int G_{s,t}(y)\,dy = 1, \quad s < t. \tag{3.7}$$

By (3.7), we have for any multiindex $\gamma$,

$$D^\gamma u(t,x) = \int_0^t \int G^\lambda_{s,t}(y) D^\gamma_x f(s, x - y - Y_s)\,dy\,ds, \quad (t,x) \in E. \tag{3.8}$$

Therefore (3.7) and (3.8) imply that

$$\operatorname*{essup}_{\omega\in\Omega} \sup_{t,x} \left| D^\gamma_x u(t,x) \right| < \infty, \tag{3.9}$$

and

$$\operatorname*{essup}_{\omega\in\Omega} \left| D^\gamma u \right|_{L_p(T)} \leqslant C\rho_\lambda \left| D^\gamma f \right|_{L_p(T)} \tag{3.10}$$

with $C = C(p,d)$. By Theorem 2.1 in [11], there is a constant $N = N(\alpha, p, d, K, \eta)$ such that

$$\operatorname*{essup}_{\omega\in\Omega} |u|_{H^\alpha_p(T)} \leqslant N|f|_{L_p(T)}. \tag{3.11}$$

Let $\bar{A} = A^{m-m_0}$. According to (3.6) and the Ito–Wentzell formula (see [10]),

$$\begin{aligned} u(t, x + Y_t) - u(0,x) &= \int_0^t \left[\partial_s u(s, x + Y_s) + \bar{A} u(s, x + Y_s)\right] ds + M_t \\ &= \int_0^t \left[A u(s, x + Y_s) - \lambda u(s, x + Y_s) + f(s,x)\right] ds + M_t, \end{aligned} \tag{3.12}$$

where

$$M_t = \int_0^t \int \left[u\big(s, x + Y_{s-} + \bar{c}(t,z)\big) - u(s, x + Y_{s-})\right] \bar{q}(ds, dz).$$

Taking expectation on both sides of (3.12) and using (3.9)–(3.11), we conclude that the function $v(t,x) = \mathbf{E} u(t, x + Y_t)$ belongs to $\mathfrak{D}_p(E) \cap \mathcal{H}^\alpha_p(E)$ and solves (3.4). Moreover, (3.10), (3.11) imply that $v$ satisfies (3.5). $\square$

Passing to the limit we prove the following statement.

**Proposition 1.** *Let $p>1$, $f\in L_p(E)$ and Assumption A be satisfied.*

*Then there is a unique solution $u\in\mathcal{H}_p^\alpha(E)$ of (3.4). Moreover, there are constants $C_0=C_0(\alpha,p,d,T,K,\eta)$ and $C_{00}=C_{00}(p,d,K)$ such that*

$$|u|_{H_p^\alpha(T)}\leqslant C_0|f|_{L_p(T)}$$

*and*

$$|u|_{L_p(T)}\leqslant C_{00}\rho_\lambda|f|_{L_p(T)}$$

*where $\rho_\lambda=T\wedge\frac{1}{\lambda}$.*

**Proof.** *Existence*. There is a sequence of input functions $f_n\in\mathfrak{D}_p(E)$, $n=1,2,\dots$, such that

$$|f-f_n|_{L_p(T)}\to 0 \tag{3.13}$$

as $n\to\infty$. By Lemma 7, for every $n$ there is a unique solution $u_n\in\mathfrak{D}_p(E)\cap\mathcal{H}_p^\alpha(E)$ of (3.4) with the input function $f_n$. Since (3.4) is a linear equation, using the estimate (3.5) of Lemma 7 we derive that $(u_n)$ is a Cauchy sequence in $H_p^\alpha(E)$. Hence, there is a function $u\in H_p^\alpha(E)$ such that $|u_n-u|_{H_p^\alpha(T)}\to 0$ as $n\to\infty$.

Passing to the limit in (3.5) of Lemma 7 with $u$, $f$ replaced by $u_n$, $f_n$, we get the corresponding estimates for $u$.

Denoting $\langle f,g\rangle=\int fg\,dx$ and using Lemma 4, we pass to the limit in the definition equality (2.3)

$$\langle u_n(t,\cdot),\varphi\rangle=\int_0^t\bigl[\langle (A-\lambda)u_n(s,\cdot)+f(s,\cdot),\varphi\rangle\bigr]\,ds,\quad \varphi\in\mathcal{S}(\mathbf{R}^d),$$

as $n\to\infty$ and see that the function $u$ is a solution of (3.4).

*Uniqueness*. Let $u\in\mathcal{H}_p^\alpha(E)$ be a solution of (3.4) with zero input function $f$. Hence, for every $\varphi\in\mathcal{S}(\mathbf{R}^d)$ and $t\in[0,T]$

$$\langle u(t,\cdot),\varphi\rangle=\int_0^t\langle u(s,\cdot),A^{(\alpha)*}\varphi-\lambda\varphi\rangle\,ds. \tag{3.14}$$

Let $\zeta_\varepsilon=\zeta_\varepsilon(x)$, $x\in\mathbf{R}^d$, $\varepsilon\in(0,1)$, be a standard mollifier. Inserting $\varphi(\cdot)=\zeta_\varepsilon(x-\cdot)$ into (3.14), we get that the function

$$v_\varepsilon(t,x)=u(t,\cdot)*\zeta_\varepsilon(x)$$

belongs to $\mathfrak{D}_p(E)\cap\mathcal{H}_p^\alpha(E)$ and

$$v_\varepsilon(t,x)=\int_0^t (A-\lambda)v_\varepsilon(s,x)\,ds,\qquad (t,x)\in E.$$

By Lemma 7, $v_\varepsilon=0$ in $E$ for all $\varepsilon\in(0,1)$. Hence, $u(t,\cdot)=0$ and the statement holds. □

## 4. Proofs of main theorems

We follow the proof of Theorem 1.6.4 in [9]. In order to use the method of continuity, we derive the a priori estimates first.

**Lemma 8.** *Suppose Assumption A holds, $\beta\in(0,1)$, $p>d/\beta$. There are $\varepsilon=\varepsilon(d,\alpha,\beta,K,w,T,\eta)\in(0,1]$, $C=C(d,\alpha,\beta,p,K,w,T,\eta)$ and $\lambda_0=\lambda_0(d,\alpha,\beta,p,K,w,T,\eta)\geqslant 1$ such that for any $u\in\mathfrak{D}_p(E)\cap\mathcal{H}_p^\alpha(E)$ satisfying (2.2) with $B=0$ and with support in a ball of radius $\varepsilon$ ($u(t,x)=0$ for all $t$ if $x$ does not belong to a ball of radius $\varepsilon$),*

$$|u|_{H_p^\alpha(T)}\leqslant C|f|_{L_p(T)},$$
$$|u|_{L_p(T)}\leqslant\frac{C}{\lambda}|f|_{L_p(T)}\quad \textit{if } \lambda\geqslant\lambda_0.$$

**Proof.** Let the support of $u$ be a subset of the ball centered at $x_0$ with radius $\varepsilon\in(0,1]$. Then

$$\partial_t u=A_{t,x_0}u(t,x)+A_{t,x}u(t,x)-A_{t,x_0}u(t,x)-\lambda u+f,$$
$$u(0)=0.$$

Let $\varphi\in C_0^\infty(\mathbf{R}^d)$, $0\leqslant\varphi\leqslant 1$, $\varphi(x)=1$ if $|x|\leqslant 1$, and $\varphi(x)=0$ if $|x|>2$. Denote

$$\varphi_\varepsilon(x)=\varphi\left(\frac{x-x_0}{2\varepsilon}\right),\qquad x\in\mathbf{R}^d,$$
$$\tilde D(t,x)=\varphi_\varepsilon(x)\big[A_{t,x}u(t,x)-A_{t,x_0}u(t,x)\big],$$
$$m_0(t,x,y)=m(t,x,y)-m(t,x_0,y),\qquad (t,x)\in E,\ \ y\in\mathbf{R}^d.$$

According to Lemma 6(b) with $\phi=\varphi_\varepsilon$, $m=m_0(t,x,y)$, there is a constant $C=C(\alpha,p,\beta,d)$ such that

$$\big|\tilde D(t,\cdot)\big|_{L_p}^p\leqslant CK(\varepsilon)\big|u(t,\cdot)\big|_{H_p^\alpha}^p$$

with

$$K(\varepsilon)=\varepsilon^{-\beta p}\Big|\varphi_\varepsilon\sup_y\big|m_0(\cdot,y)\big|\Big|_p^p+\varepsilon^{(1-\beta)p}\Big|\sup_y\big|m_0(\cdot,y)\big|\nabla\varphi_\varepsilon\Big|_p^p$$
$$+w(\varepsilon)^p\varepsilon^{(1-\beta)p}|\nabla\varphi_\varepsilon|_p^p+\gamma(\varepsilon)^p|\varphi_\varepsilon|_p^p+\varepsilon^p\gamma(\varepsilon)^p|\nabla\varphi_\varepsilon|_p^p,$$
$$\gamma(\varepsilon)=\int_{|\upsilon|\leqslant\varepsilon}w(\upsilon)\frac{d\upsilon}{|\upsilon|^{d+\beta}}.$$

Thus

$$|\tilde{D}|^p_{L_p(T)} \leqslant C|u|^p_{H^\alpha_p(T)} K(\varepsilon). \tag{4.1}$$

Obviously,

$$|A_{t,x}u - A_{t,x_0}u - \tilde{D}|_{H^\alpha_p(T)} \leqslant C \int\limits_{|y|>\varepsilon} \left|\nabla^\alpha_y u(\cdot)\right|_{L_p(T)} \frac{dy}{y^{d+\alpha}} \leqslant C\varepsilon^{-\alpha}\left[|u|_{L_p(T)} + 1_{\alpha>1}|\nabla u|_{L_p(T)}\right].$$

So, by Proposition 1 and (4.1), there are constants $C_1 = C_1(\alpha, p, d, T, K, \eta)$ and $C_{11} = C_{11}(\alpha, p, d, K)$ such that

$$|u|_{H^\alpha_p(T)} \leqslant C_1\left[|f|_{L_p(T)} + K(\varepsilon)|u|_{H^\alpha_p(T)} + \varepsilon^{-\alpha}\left(|u|_{L_p(T)} + 1_{\alpha>1}|\nabla u|_{L_p(T)}\right)\right]$$

with $K(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

$$|u|_{L_p(T)} \leqslant C_{11}\rho_\lambda\left[|f|_{L_p(T)} + K_\varepsilon|u|_{H^\alpha_p(T)} + \varepsilon^{-\alpha}\left(|u|_{L_p(T)} + 1_{\alpha>1}|\nabla u|_{L_p(T)}\right)\right],$$

where $\rho_\lambda = \frac{1}{\lambda} \wedge T$. We choose $\varepsilon$ so that $C_1K(\varepsilon) \leqslant 1/2$, $K(\varepsilon) \leqslant 1$. In this case,

$$|u|_{H^\alpha_p(T)} \leqslant 2C_1\left[|f|_{L_p(T)} + \varepsilon^{-\alpha}\left(|u|_{L_p(T)} + 1_{\alpha>1}|\nabla u|_{L_p(T)}\right)\right],$$
$$|u|_{L_p(T)} \leqslant C_{11}(1+2C_1)\rho_\lambda\left[|f|_{L_p(T)} + \varepsilon^{-\alpha}\left(|u|_{L_p(T)} + 1_{\alpha>1}|\nabla u|_{L_p(T)}\right)\right].$$

By interpolation inequality, for $\alpha > 1$ and each $\kappa \in (0, 1)$ there is a constant $C = C(\alpha, p, d)$ such that

$$|\nabla u|_{L_p(T)} \leqslant \kappa|u|_{H^\alpha_p(T)} + C\kappa^{-\frac{1}{\alpha-1}}|u|_{L_p(T)}.$$

Therefore choosing $\kappa$ so that $2C_1\varepsilon^{-\alpha}\kappa \leqslant \frac{1}{2}$ (if $\alpha > 1$), one can see that there is $\tilde{C}_1 = \tilde{C}_1(\alpha, \beta, p, d, T, K, w, \eta)$ such that

$$|u|_{H^\alpha_p(T)} \leqslant \tilde{C}_1\left[|f|_{L_p(T)} + |u|_{L_p(T)}\right], \qquad |u|_{L_p(T)} \leqslant \tilde{C}_1\rho_\lambda\left[|f|_{L_p(T)} + |u|_{L_p(T)}\right].$$

The statement follows by choosing $\lambda$ so that $\tilde{C}_1\lambda^{-1} \leqslant \frac{1}{2}$ ($\lambda_0 = (2\tilde{C}_1)^{-1}$). $\square$

Now we extend the estimates.

**Lemma 9.** *Suppose Assumption A holds and $p > \frac{d}{\beta}$. There is a constant $C = C(d, \alpha, \beta, p, K, w, T, \eta)$ and a number $\lambda_1 = \lambda_1(\alpha, \beta, d, K, w, \eta, T) > 1$ such that for any $u \in \mathfrak{D}_p(E) \cap \mathcal{H}^\alpha_p(E)$ satisfying (2.2) with $B = 0$ and $\lambda \geqslant \lambda_1$,*

$$|u|_{H^\alpha_p(T)} \leqslant C|f|_{L_p(T)},$$
$$|u|_{L_p(T)} \leqslant \frac{C}{\lambda}|f|_{L_p(T)} \quad \text{if } \lambda \geqslant \lambda_1.$$

**Proof.** Take $\zeta \in C_0^\infty(\mathbf{R}^d)$ such that $\int |\zeta|^p\,dx = 1$ and whose support is in a ball of radius $\varepsilon$ from Lemma 8 centered at 0. Then

$$\left|\partial^\alpha u(t,x)\right|^p = \int \left|\partial^\alpha u(t,x)\zeta(x-\upsilon)\right|^p d\upsilon \tag{4.2}$$

and

$$\begin{aligned}\partial^\alpha u(t,x)\zeta(x-\upsilon) &= \partial^\alpha\big(u(t,x)\zeta(x-\upsilon)\big) - u(t,x)\partial_x^\alpha \zeta(x-\upsilon)\\ &\quad + \int \big[u(t,x+y)-u(t,x)\big]\big[\zeta(x+y-\upsilon)-\zeta(x-\upsilon)\big]\frac{dy}{|y|^{d+\alpha}}.\end{aligned} \tag{4.3}$$

Since

$$\begin{aligned}&\partial_t\big(u(t,x)\zeta(x-\upsilon)\big)\\ &\quad = \zeta(x-\upsilon)Au(t,x) - \lambda\zeta(x-\upsilon)u(t,x) + \zeta(x-\upsilon)f(t,x)\\ &\quad = A\big(\zeta(x-\upsilon)u(t,x)\big) - \lambda\zeta(x-\upsilon)u(t,x) + \zeta(x-\upsilon)f(t,x) - u(t,x)A\zeta(x-\upsilon)\\ &\qquad - \int \big[u(t,x+y)-u(t,x)\big]\big[\zeta(x+y-\upsilon)-\zeta(x-\upsilon)\big]m(t,x,y)\frac{dy}{|y|^{d+\alpha}},\end{aligned}$$

it follows by Lemmas 8 and 3 and Remark 2 that there is $C = C(d,\alpha,\beta,p,K,w,T,\eta)$ and $\lambda_0 = \lambda_0(d,\alpha,\beta,p,K,w,T,\eta)$ such that

$$\begin{aligned}&\int |u\zeta(\cdot-\upsilon)|^p_{H_p^\alpha(T)}\,d\upsilon \leqslant C\big[|f|^p_{L_p(T)} + |u|^p_{L_p(T)} + |u|^p_{H_p^{\alpha'}(T)}\big],\\ &\int |u\zeta(\cdot-\upsilon)|^p_{L_p(T)}\,d\upsilon \leqslant \frac{C}{\lambda^p}\big[|f|^p_{L_p(T)} + |u|^p_{L_p(T)} + |u|^p_{H_p^{\alpha'}(T)}\big] \quad \text{if } \lambda \geqslant \lambda_0,\end{aligned}$$

for some $\alpha' < \alpha$. According to (4.3) and (4.2),

$$\begin{aligned}\left|\partial^\alpha u\right|^p_{L_p(T)} &\leqslant C\big[|f|^p_{L_p(T)} + |u|^p_{L_p(T)} + |u|^p_{H_p^{\alpha'}(T)}\big],\\ |u|^p_{L_p(T)} &\leqslant \frac{C}{\lambda^p}\big[|f|^p_{L_p(T)} + |u|^p_{L_p(T)} + |u|^p_{H_p^{\alpha'}(T)}\big] \quad \text{if } \lambda \geqslant \lambda_0.\end{aligned} \tag{4.4}$$

By interpolation inequality, for each $\kappa \in (0,1)$ there is a constant $K_1 = K_1(\kappa,\alpha',\alpha,p,d)$ such that

$$|u|_{H_p^{\alpha'}(T)} \leqslant \kappa|u|_{\alpha,p} + K_1|u|_{L_p(T)}.$$

Therefore, choosing $\kappa$ so that $C\kappa \leqslant 1/2$, we get by (4.4) that there is a constant $C_1 = C_1(d,\alpha,\beta,p,K,w,T,\eta)$ such that

$$\begin{aligned}|u|^p_{H_p^\alpha(T)} &\leqslant C_1\big[|f|^p_{L_p(T)} + |u|_{L_p(T)}\big],\\ |u|^p_p &\leqslant \frac{C_1}{\lambda^p}\big[|f|^p_{L_p(T)} + |u|^p_{L_p(T)}\big].\end{aligned} \tag{4.5}$$

We finish the proof by choosing $\lambda$ so that $\frac{C_1}{\lambda^p} \leqslant \frac{1}{2}$ or $\lambda \geqslant (2C_1)^{1/p} = \lambda_1$. Thus by (4.5),

$$|u|^p_{L_p(T)} \leqslant \frac{2C_1}{\lambda^p} |f|^p_{L_p(T)}, \quad \lambda \geqslant \lambda_1; \qquad |u|^p_{H^\alpha_p(T)} \leqslant C_1\left(1 + \frac{2C_1}{\lambda_1^p}\right)|f|^p_{L_p(T)}.$$

The statement follows. □

**Corollary 3.** *Suppose Assumption A holds, $p > \frac{d}{\beta}$ and $u \in \mathfrak{D}_p(E) \cap \mathcal{H}^\alpha_p(E)$ satisfies (2.2) with $B = 0$. Then there is $C = C(d, \alpha, \beta, p, K, w, T, \eta)$ such that*

$$|u|_{H^\alpha_p(T)} \leqslant C|f|_{L_p(T)}.$$

**Proof.** For $\lambda \geqslant \lambda_1$ ($\lambda_1$ is from Lemma 9), the estimate is proved in Lemma 9. If $u \in \mathcal{H}^\alpha_p(E)$ solves (2.2) with $\lambda \leqslant \lambda_1$, then $\tilde{u}(t,x) = e^{(\lambda_1-\lambda)t}u(t,x)$ solves the same equation with $\lambda = \lambda_1$ and $f$ replaced by $e^{(\lambda_1-\lambda)t} f$. Hence

$$|u|_{H^\alpha_p(T)} \leqslant |\tilde{u}|_{H^\alpha_p(T)} \leqslant Ce^{(\lambda_1-\lambda)T}|f|_{L_p(T)}$$

with $C = C(d, \alpha, \beta, p, K, w, T, \eta)$ from Lemma 9. So, the estimate holds for all $\lambda \geqslant 0$. □

### *4.1. Proof of Theorem 1*

We use the a priori estimate and the continuation by parameter argument. Let

$$M_\tau u = \tau L u + (1-\tau)\partial^\alpha u, \quad \tau \in [0, 1].$$

The space $\mathcal{H}^\alpha_p(E)$ is a Banach space of all functions $u \in H^\alpha_p(E)$ such that $u(t) = \int_0^t F(s)\,ds$, $0 \leqslant t \leqslant T$, with $F \in L_p(E)$ and finite norm

$$|u|_{\mathcal{H}^\alpha_p} = |u|_{H^\alpha_p(T)} + |F|_{L_p(T)}.$$

Consider the mappings $T_\tau : \mathcal{H}^\alpha_p(E) \to L_p(E)$ defined by

$$u(t,x) = \int_0^t F(s,x)\,ds \mapsto F - M_\tau u.$$

Obviously, for some constant $C$ not depending on $\tau$,

$$|T_\tau u|_{L_p(T)} \leqslant C|u|_{\mathcal{H}^\alpha_p}.$$

On the other hand, there is a constant $C$ not depending on $\tau$ such that for all $u \in \mathcal{H}^\alpha_p(E)$

$$|u|_{\mathcal{H}^\alpha_p} \leqslant C|T_\tau u|_{L_p(T)}. \tag{4.6}$$

Indeed,

$$u(t,x)=\int_0^t F(s,x)\,ds=\int_0^t \bigl(M_\tau u+(F-M_\tau u)\bigr)(s,x)\,ds,$$

and, according to Corollary 3, there is a constant $C$ not depending on $\tau$ such that

$$|u|_{H_p^\alpha(T)}\leqslant C|T_\tau u|_{L_p(T)}=C|F-M_\tau u|_{L_p(T)}. \tag{4.7}$$

Thus,

$$\begin{aligned}|u|_{\mathcal{H}_p^\alpha}&=|u|_{H_p^\alpha(T)}+|F|_{L_p(T)}\leqslant |u|_{H_p^\alpha(T)}+|F-M_\tau u|_{L_p(T)}+|M_\tau u|_{L_p(T)}\\&\leqslant C\bigl(|u|_{H_p^\alpha(T)}+|F-M_\tau u|_{L_p(T)}\bigr)\leqslant C|F-M_\tau u|_{L_p(T)}=C|T_\tau u|_{L_p(T)},\end{aligned}$$

and (4.6) follows. Since $T_0$ is an onto map, by Theorem 5.2 in [3] all the $T_\tau$ are onto maps and Theorem 1 follows.

### 4.2. *Proof of Theorem 2*

Let $u\in\mathfrak{D}_p(E)\cap\mathcal{H}_p^\alpha(E)$ satisfy (2.2) with $B=B^{\varepsilon_0}$. By Theorem 1,

$$\begin{aligned}|\partial_t u|_{L_p(T)}+|u|_{H_p^\alpha(T)}&\leqslant N\bigl[|f|_{L_p(T)}+\bigl|B^{\varepsilon_0}u\bigr|_{L_p(T)}\bigr],\\|u|_{L_p(T)}&\leqslant \frac{N}{\lambda}\bigl[|f|_{L_p(T)}+\bigl|B^{\varepsilon_0}u\bigr|_{L_p(T)}\bigr]\quad \text{if } \lambda\geqslant\lambda_1,\end{aligned}$$

where $\lambda_1=\lambda_1(T,\alpha,\beta,d,K,w,\eta)\geqslant 1$. According to Lemma 1, for each $t$,

$$\begin{aligned}\bigl|B_t^{\varepsilon_0}u\bigr|_{L_p}^p&\leqslant\int\biggl(\int_{|y|\leqslant\varepsilon_0}\bigl|\nabla_y^\alpha u(t,x)\bigr|\pi(t,x,dy)\biggr)^p dx\\&\leqslant\delta_0^p\int\biggl(\sup_{y\neq 0}\frac{|\nabla_y^\alpha u(t,x)|}{|y|^\alpha}\biggr)^p dx\leqslant\delta_0^p N_1^p\bigl|\partial^\alpha u(t,\cdot)\bigr|_{L_p}^p.\end{aligned}$$

Therefore

$$\begin{aligned}|\partial_t u|_{L_p(T)}+|u|_{H_p^\alpha(T)}&\leqslant N|f|_{L_p(T)}+\frac12\bigl|\partial^\alpha u\bigr|_{L_p(T)},\\|u|_{L_p(T)}&\leqslant\frac{N}{\lambda}|f|_{L_p(T)}+\frac{1}{2\lambda}\bigl|\partial^\alpha u\bigr|_{L_p(T)}\quad\text{if }\lambda\geqslant\lambda_1,\end{aligned}$$

and

$$\begin{aligned}|\partial_t u|_{L_p(T)}+|u|_{H_p^\alpha(T)}&\leqslant 2N|f|_{L_p(T)},\\|u|_{L_p(T)}&\leqslant\frac{2N}{\lambda}|f|_{L_p(T)}\quad\text{if }\lambda\geqslant\lambda_1.\end{aligned}\tag{4.8}$$

If $u \in \mathfrak{D}_p(E) \cap \mathcal{H}_p^\alpha(E)$ satisfies (2.2) with $B = B^{\varepsilon_0}$, $\lambda \leqslant \lambda_1$, then $\tilde{u}(t,x) = e^{(\lambda_1-\lambda)t}u(t,x)$ satisfies the same equation with $\lambda_1$ and $f$ replaced by $e^{(\lambda_1-\lambda)t}f$. By (4.8),

$$|u|_{H_p^\alpha(T)} \leqslant |\tilde{u}|_{H_p^\alpha(T)} \leqslant 2Ne^{(\lambda_1-\lambda)T}|f|_{L_p(T)}. \tag{4.9}$$

The statement follows by the a priori estimates (4.8)–(4.9) and the continuation by parameter argument, repeating the proof of Theorem 1 for the operators

$$M_\tau = A + \tau B^{\varepsilon_0}, \quad 0 \leqslant \tau \leqslant 1.$$

*4.2.1. Proof of Corollary 1*

Let $u \in \mathfrak{D}_p(E) \cap \mathcal{H}_p^\alpha(E)$ satisfy (2.2) with $B = \tilde{B}^{\varepsilon_0}$. By Theorem 2,

$$\begin{aligned}
|\partial_t u|_{L_p(T)} + |u|_{H_p^\alpha(T)} &\leqslant 2N\big[|f|_{L_p(T)} + \big|\big(\tilde{B}^{\varepsilon_0} - B^{\varepsilon_0}\big)u\big|_{L_p(T)}\big], \\
|u|_{L_p(T)} &\leqslant \frac{2N}{\lambda}\big[|f|_{L_p(T)} + \big|\big(\tilde{B}^{\varepsilon_0} - B^{\varepsilon_0}\big)u\big|_{L_p(T)}\big] \quad \text{if } \lambda \geqslant \lambda_1.
\end{aligned}$$

By interpolation inequality, for each $\kappa \in (0,1)$, there is a constant $C_\kappa = C_\kappa(K, \varepsilon_0)$ such that

$$\big|\big(\tilde{B}^{\varepsilon_0} - B^{\varepsilon_0}\big)u\big|_{L_p(T)} \leqslant \kappa|u|_{H_p^\alpha(T)} + C_\kappa|u|_{L_p(T)}.$$

Choose $\kappa$ so that $2N\kappa \leqslant 1/2$. Then

$$\begin{aligned}
|\partial_t u|_{L_p(T)} + |u|_{H_p^\alpha(T)} &\leqslant 4N\big[|f|_{L_p(T)} + C_\kappa|u|_{L_p(T)}\big], \\
|u|_{L_p(T)} &\leqslant \frac{4N}{\lambda}\big[|f|_{L_p(T)} + C_\kappa|u|_{L_p(T)}\big] \quad \text{if } \lambda \geqslant \lambda_1
\end{aligned}$$

and for $4NC_\kappa/\lambda_2 = 1/2$ we have

$$|u|_{L_p(T)} \leqslant \frac{8N}{\lambda}|f|_{L_p(T)} \quad \text{if } \lambda \geqslant \lambda_2, \qquad |\partial_t u|_{L_p(T)} + |u|_{H_p^\alpha(T)} \leqslant 8N|f|_{L_p(T)}. \tag{4.10}$$

If $u \in \mathfrak{D}_p(E) \cap \mathcal{H}_p^\alpha(E)$ satisfies (2.2) with $B = \tilde{B}^{\varepsilon_0}$, $\lambda \leqslant \lambda_2$, then $\tilde{u}(t,x) = e^{(\lambda_1-\lambda)t}u(t,x)$ satisfies the same equation with $\lambda_2$ and $f$ replaced by $e^{(\lambda_1-\lambda)t}f$. By (4.10),

$$|u|_{H_p^\alpha(T)} \leqslant |\tilde{u}|_{H_p^\alpha(T)} \leqslant 8Ne^{(\lambda_1-\lambda)T}|f|_{L_p(T)}. \tag{4.11}$$

The statement follows by the a priori estimates (4.10)–(4.11) and the continuation by parameter argument, repeating the proof of Theorem 1 for the operators

$$M_\tau = A + \tau\tilde{B}^{\varepsilon_0}, \quad 0 \leqslant \tau \leqslant 1.$$

*4.3. Proof of Theorem 3*

Again we derive the a priori estimates first and use the continuation by parameter argument. There is $\varepsilon_0 \in (0, 1)$ such that

$$\int\limits_{|y|\leqslant\varepsilon_0} |y|^{\alpha}\,\pi(t,x,dy) \leqslant \delta_0, \quad (t,x) \in E,$$

where $\delta_0$ is a number in Theorem 2. Let $u \in \mathfrak{D}_p(E) \cap \mathcal{H}_p^{\alpha}(E)$ satisfy (2.2). Let

$$\tilde{L}v(t,x) = Av(t,x) + B^{\varepsilon_0}v(t,x), \quad (t,x) \in E,$$

where $B^{\varepsilon_0}v$ is defined in (2.4). Applying Theorem 2 to $\tilde{L}$, we have

$$|\partial_t u|_{L_p(T)} + |u|_{H_p^{\alpha}(T)} \leqslant 2N\big[|f|_{L_p(T)} + \big|(L-\tilde{L})u\big|_{L_p(T)}\big],$$
$$|u|_{L_p(T)} \leqslant \frac{2N}{\lambda}\big[|f|_{L_p(T)} + \big|(L-\tilde{L})u\big|_{L_p(T)}\big] \quad \text{if } \lambda \geqslant \lambda_1,$$

where $N$ and $\lambda_1$ are the constants in Theorems 1, 2. There is $\alpha' < \alpha$ such that $p > d/\alpha'$ and by Sobolev embedding theorem there is a constant $C$ such that

$$\big|(L-\tilde{L})u\big|_{L_p(T)} \leqslant C|u|_{H_p^{\alpha'}(T)}\left(\int_0^T\!\!\int \pi\big(t,x,\{|y|>\varepsilon_0\}\big)^p\,dx\,dt\right)^{1/p} + C\varepsilon_0^{-\alpha}1_{\alpha\in(1,2)}|\nabla u|_{L_p(T)}.$$

By interpolation inequality, for each $\kappa > 0$ there is a constant $\tilde{N} = \tilde{N}(\kappa, \varepsilon_0, \alpha, \alpha', p, d)$ such that

$$\big|(L-\tilde{L})u\big|_{L_p(T)} \leqslant \kappa|u|_{H_p^{\alpha}(T)} + \tilde{N}|u|_{L_p(T)}.$$

Choose $\kappa$ so that $2N\kappa \leqslant 1/2$. Then

$$|\partial_t u|_{L_p(T)} + |u|_{H_p^{\alpha}(T)} \leqslant 4N\big[|f|_{L_p(T)} + \tilde{N}|u|_{L_p(T)}\big],$$
$$|u|_{L_p(T)} \leqslant \frac{4N}{\lambda}\big[|f|_{L_p(T)} + \tilde{N}|u|_{L_p(T)}\big] \quad \text{if } \lambda \geqslant \lambda_1.$$

Choosing $\lambda \geqslant \lambda_2 = 8N\tilde{N}$, we derive

$$|u|_{L_p(T)} \leqslant \frac{8N}{\lambda}|f|_{L_p(T)}, \qquad |\partial_t u|_{L_p(T)} + |u|_{H_p^{\alpha}(T)} \leqslant 8N|f|_{L_p(T)}.$$

Multiplying $u$ by $e^{(\lambda-\lambda_2)t}$, we obtain the a priori estimate for all $\lambda \geqslant 0$ as in the proof of Corollary 3 above.

The statement follows by the a priori estimates and the continuation by parameter argument, repeating the proof of Theorem 1 for the operators

$$M_\tau v = \tilde{L}v + \tau(L - \tilde{L})v, \quad 0 \leqslant \tau \leqslant 1.$$

## 5. Embedding of the solution space

The following Hölder norm estimates of the solutions should be known in one form or another. Following the main steps of Section 7 in [8], we provide the proofs for the sake of completeness. Since the solution of (2.2) $u \in \mathcal{H}_p^\alpha(E)$, we derive an embedding theorem for $\mathcal{H}_p^\alpha(E)$.

**Remark 3.** If $u \in \mathcal{H}_p^\alpha(E)$, then $u \in H_p^\alpha(E)$ and

$$u(t) = \int_0^t F(s)\,ds, \quad 0 \leqslant t \leqslant T,$$

with $F \in L_p(E)$. It is the $\mathcal{H}_p^\alpha$-solution to the equation

$$\begin{aligned} \partial_t u &= \partial^\alpha u + f, \\ u(0) &= 0, \end{aligned} \tag{5.1}$$

where $f = F - \partial^\alpha u \in L_p(E)$ with $|f|_{L_p(T)} \leqslant |F|_{L_p(T)} + |\partial^\alpha u|_{L_p(T)} \leqslant \|u\|_{\alpha,p}$. In addition (e.g., see [13]),

$$u(t,x) = \int_0^t G_{t-s}(x-y) f(s,y)\,dy\,ds, \quad 0 \leqslant t \leqslant T,\ x \in \mathbf{R}^d, \tag{5.2}$$

where

$$G_t = \mathcal{F}^{-1}\big[e^{-t|\xi|^\alpha}\big], \quad t > 0 \tag{5.3}$$

(here $\mathcal{F}^{-1}$ is the inverse Fourier transform). The function $G_t$ is the probability density function of a spherically symmetric $\alpha$-stable process whose generator is the fractional Laplacian $\partial^\alpha$:

$$\int G_t\,dx = 1, \quad t > 0. \tag{5.4}$$

**Remark 4.** Note that for any multiindex $\gamma \in \mathbf{N}_0^d$ there is a constant $C = C(\alpha, \gamma, d)$ such that

$$\big|D_\xi^\gamma e^{-|\xi|^\alpha}\big| \leqslant C e^{-|\xi|^\alpha} \sum_{1 \leqslant k \leqslant |\gamma|} |\xi|^{k\alpha - |\gamma|}. \tag{5.5}$$

**Lemma 10.** *Let $K(x) = G_1(x)$, $x \in \mathbf{R}^d$. Then*:

(i) *$K$ is smooth and for all multiindices $\gamma \in \mathbf{N}_0^d$, $\kappa \in (0,2)$,*

$$\int \left|\partial^\kappa D^\gamma K(x)\right| dx < \infty.$$

(ii) *For $t > 0$, $x \in \mathbf{R}^d$,*

$$G_t(x) = t^{-d/\alpha} K\left(x/t^{1/\alpha}\right)$$

*and for any multiindex $\gamma \in \mathbf{N}_0^d$ and $\kappa \in (0,2)$, there is a constant $C$ such that*

$$\left|\partial^\kappa D^\gamma G_t * v\right|_{L_p} \leqslant C t^{-(|\gamma|+\kappa)/\alpha} |v|_{L_p}, \quad t > 0,\ v \in L_p\left(\mathbf{R}^d\right).$$

(iii) *Let $\kappa \in (0,1)$. There is a constant $C$ such that for $v \in \mathcal{S}(\mathbf{R}^d)$, $t > 0$,*

$$|G_t * v - v|_{L_p} \leqslant C t^\kappa \left|\partial^{\alpha\kappa} v\right|_{L_p}.$$

**Proof.** (i) For any multiindex $\gamma \in \mathbf{N}_0^d$,

$$\sup_x \left|D^\gamma K(x)\right| \leqslant \int \left|(i\xi)^\gamma e^{-|\xi|^\alpha}\right| d\xi < \infty.$$

Let $\varphi \in C_0^\infty(\mathbf{R}^d)$, $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ if $|x| \leqslant 1$, $\varphi(x) = 0$ if $|x| \geqslant 2$. Then $K(x) = K_1(x) + K_2(x)$ with

$$K_1 = \mathcal{F}^{-1}\left(e^{-|\xi|^\alpha}\varphi(\xi)\right), \qquad K_2 = \mathcal{F}^{-1}\left(\left[1 - \varphi(\xi)\right]e^{-|\xi|^\alpha}\right).$$

Since $\psi = \mathcal{F}^{-1}\varphi \in \mathcal{S}(\mathbf{R}^d)$, we have $K_1(x) = K * \psi(x)$. Therefore, by (5.4), for any multiindex $\gamma \in \mathbf{N}_0^d$, $\kappa \in (0,2)$,

$$\sup_x \left|\partial^\kappa D^\gamma K_1(x)\right| \leqslant \sup_x \left|\partial^\kappa D^\gamma \psi(x)\right| < \infty,$$
$$\int \left|\partial^\kappa D^\gamma K_1(x)\right| dx \leqslant \int \left|\partial^\kappa D^\gamma \psi(x)\right| dx < \infty.$$

By Parseval's equality and (5.5), for any multiindices $\gamma, \mu, \kappa \in (0,2)$,

$$\int \left|\partial^\kappa D^\gamma K_2(x)\right|^2 dx = \int \left|(i\xi)^\gamma |\xi|^\kappa \left[1 - \varphi(\xi)\right] e^{-|\xi|^\alpha}\right|^2 d\xi < \infty,$$
$$\int \left|(ix)^\mu \partial^\kappa D^\gamma K_2(x)\right|^2 dx = \int \left|D^\mu\left((i\xi)^\gamma |\xi|^\kappa e^{-|\xi|^\alpha}\right)\right|^2 \left[1 - \varphi(\xi)\right] d\xi < \infty.$$

Therefore, by Cauchy–Schwartz inequality with $d_1 = [\frac{d}{4}] + 1$,

$$\int \left|\partial^\kappa D^\gamma K_2(x)\right| dx \leqslant \left\{ \int \left(1+|x|^2\right)^{2d_1} \left|D^\gamma K_2(x)\right|^2 dx \right\}^{1/2} \times \left\{ \int \left(1+|x|^2\right)^{-2d_1} dx \right\}^{1/2}.$$

(ii) Changing the variable of integration in (5.3) we get $G_t(x) = t^{-d/\alpha} K(x/t^{1/\alpha})$, $x \in \mathbf{R}^d$, $t > 0$. For any $v \in \mathcal{S}(\mathbf{R}^d)$, $\gamma \in \mathbf{N}_0^d$, $\kappa \in (0,2)$,

$$\begin{aligned} \partial^\kappa D^\gamma G_t * v(x) &= \int \partial^\kappa D^\gamma G_t(x-y) v(y)\, dy \\ &= t^{-d/\alpha - (|\gamma|+\kappa)/\alpha} \int \partial^\kappa D^\gamma K\left((x-y)/t^{1/\alpha}\right) v(y)\, dy \\ &= t^{-d/\alpha - (|\gamma|+\kappa)/\alpha} \int \partial^\kappa D^\gamma K\left(y/t^{1/\alpha}\right) v(x-y)\, dy \end{aligned}$$

and the statement follows.

(iii) Since for $v \in \mathcal{S}(\mathbf{R}^d)$

$$G_t * v - v = \int_0^t \partial^\alpha G_s * v\, ds, \quad 0 \leqslant t,$$

it follows by part (ii),

$$\begin{aligned} |G_t * v - v|_{L_p} &\leqslant \int_0^t \left|\partial^\alpha G_s * v\right|_{L_p} ds = \int_0^t \left|\partial^{\alpha(1-\kappa)} G_s * \partial^{\alpha\kappa} v\right|_{L_p} ds \\ &\leqslant \int_0^t s^{\kappa-1}\, ds \left|\partial^{\alpha\kappa} v\right|_{L_p} \leqslant C t^\kappa \left|\partial^{\alpha\kappa} v\right|_{L_p}. \qquad \square \end{aligned}$$

We will need the following embedding estimate as well.

**Lemma 11.** *(See Lemma 7.4 in [8].) Let $\mu \in (0,1)$, $\mu p > 1$, $p \geqslant 1$, $\kappa \in (0,1]$. Let $h(t)$ be a continuous $H_p^{\alpha(1-\kappa)}(\mathbf{R}^d)$-valued function. Then there is a constant $C = C(d,\mu)$ such that for $s \leqslant t$,*

$$\left|\partial^{\alpha(1-\kappa)}\left[h(t)-h(s)\right]\right|_{L_p}^p \leqslant C(t-s)^{\mu p - 1} \int_0^{t-s} \frac{dr}{r^{1+\mu p}} \int_s^{t-r} \left|\partial^{\alpha(1-\kappa)}\left[h(\upsilon+r)-h(\upsilon)\right]\right|_{L_p}^p d\upsilon.$$

We apply Lemma 11 to $u$.

**Proposition 2.** *Let $\mu p > 1$, $\kappa \in (0,1]$, $\kappa p > 1$, $\kappa > \mu$. Assume $f \in \mathfrak{D}_p(E)$, and*

$$u(t) = \int_0^t G_{t-s} * f(s)\,ds, \quad 0 \leqslant t \leqslant T.$$

*Then there is a constant $C$ such that for all $0 \leqslant s \leqslant t \leqslant T$,*

$$\left|\partial^{\alpha(1-\kappa)}\left[u(t)-u(s)\right]\right|_{L_p}^p \leqslant C(t-s)^{(\kappa-\mu)p}\left[|f|_{L_p(T)}^p + \left|\partial^\alpha u\right|_{L_p(T)}^p\right].$$

**Proof.** We apply Lemma 11 to $u(t) = \int_0^t G_{t-s} * f(s)\,ds$, $0 \leqslant t \leqslant T$. Since $G_{t+s} = G_t * G_s$, it follows for $\upsilon, r \geqslant 0$,

$$\begin{aligned} u(\upsilon+r)-u(\upsilon) &= \int_0^{\upsilon+r} G_{\upsilon+r-\tau} * f(\tau)\,d\tau - \int_0^{\upsilon} G_{\upsilon-\tau} * f(\tau)\,d\tau \\ &= \int_\upsilon^{\upsilon+r} G_{\upsilon+r-\tau} * f(\tau)\,d\tau + \int_0^{\upsilon} G_{\upsilon+r-\tau} * f(\tau)\,d\tau - \int_0^{\upsilon} G_{\upsilon-\tau} * f(\tau)\,d\tau \\ &= \int_0^{r} G_{r-\tau} * f(\upsilon+\tau)\,d\tau + G_r * u(\upsilon) - u(\upsilon). \end{aligned}$$

By Hölder inequality and Lemma 10 for $r > 0$, and any $1-\kappa < \theta < 1-1/p$,

$$\begin{aligned} &\left|\partial^{\alpha(1-\kappa)} \int_0^r G_{r-\tau} * f(\upsilon+\tau)\,d\tau\right|_{L_p}^p \\ &\quad\leqslant \left(\int_0^r \tau^\theta \tau^{-\theta}\left|\partial^{\alpha(1-\kappa)} G_{r-\tau} * f(\upsilon+\tau)\right|_{L_p}d\tau\right)^p \\ &\quad\leqslant \left(\int_0^r \tau^{-\theta q}\,d\tau\right)^{p/q} \int_0^r \tau^{\theta p}\left|\partial^{\alpha(1-\kappa)} G_\tau * f(\upsilon+r-\tau)\right|_{L_p}^p d\tau \\ &\quad\leqslant C r^{(1-\theta)p-1}\int_0^r \tau^{\theta p-(1-\kappa)p}\left|f(\upsilon+r-\tau)\right|_{L_p}^p d\tau \\ &\quad\leqslant r^{(1-\theta)p-1} r^{\theta p-(1-\kappa)p}\int_0^r \left|f(\upsilon+r-\tau)\right|_{L_p}^p d\tau \\ &\quad= r^{\kappa p-1}\int_0^r \left|f(\upsilon+r-\tau)\right|_{L_p}^p d\tau. \end{aligned}$$

By Lemma 10, for $r, \upsilon > 0$,

$$\left|\partial^{\alpha(1-\kappa)}\left[G_r * u(\upsilon) - u(\upsilon)\right]\right|_{L_p}^p \leqslant C r^{p\kappa}\left|\partial^\alpha u(\upsilon)\right|_{L_p}^p.$$

Therefore for a fixed $\mu p > 1$, $\kappa \in (0, 1]$, $\kappa p > 1$, $\kappa > \mu$,

$$\begin{aligned}
&\int_0^{t-s} \frac{dr}{r^{1+\mu p}} \int_s^{t-r} \left|\partial^{\alpha(1-\kappa)}\left[u(\upsilon + r) - u(\upsilon)\right]\right|_{L_p}^p d\upsilon \\
&\quad\leqslant C\left[\int_0^{t-s} \frac{dr}{r^{1+\mu p}} \int_s^{t-r} r^{\kappa p-1} \int_0^r \left|f(\upsilon + r - \tau)\right|_{L_p}^p d\tau\, d\upsilon + \int_0^{t-s} \frac{dr}{r^{1+\mu p}} \int_s^{t-r} r^{\kappa p}\left|\partial^\alpha u(\upsilon)\right|_{L_p}^p d\upsilon\right] \\
&\quad\leqslant C\left[\int_0^{t-s} \frac{dr}{r^{2+(\mu-\kappa)p}} \int_0^r \int_{s+\tau}^{t-r+\tau} \left|f(\upsilon)\right|_{L_p}^p d\upsilon\, d\tau + \int_0^{t-s} \frac{dr}{r^{1+(\mu-\kappa)p}} \int_s^{t-r} \left|\partial^\alpha u(\upsilon)\right|_{L_p}^p d\upsilon\right] \\
&\quad\leqslant C\left[|f|_{L_p(T)}^p + \left|\partial^\alpha u\right|_{L_p(T)}^p\right](t-s)^{(\kappa-\mu)p},
\end{aligned}$$

and by Lemma 11,

$$\left|\partial^{\alpha(1-\kappa)}\left[u(t) - u(s)\right]\right|_{L_p}^p \leqslant C(t-s)^{(\kappa-\mu)p}\left[|f|_{L_p(T)}^p + \left|\partial^\alpha u\right|_{L_p(T)}^p\right]. \qquad \square$$

**Corollary 4.** *Let $u \in \mathcal{H}_p^\alpha$, $0 < \gamma < \delta = \alpha(1 - \frac{1}{p}) - \frac{d}{p} > 0$. Then there is a Hölder continuous modification of $u$ on $E$ and a constant $C$ independent of $u$ such that*

$$\sup_{(t,x)\in E}\left|u(t,x)\right| + \sup_{s\neq t,x} \frac{|u(s,x) - u(t,x)|}{|s-t|^{\gamma/\alpha}} + \sup_{s,x\neq x'} \frac{|u(s,x) - u(s,x')|}{|x-x'|^\gamma} \leqslant C|u|_{\mathcal{H}_p^\alpha}. \tag{5.6}$$

**Proof.** By Proposition 2 with $\kappa = 1$ and $\kappa = 1 - \gamma/\alpha - d/(p\alpha) > \mu > 1/p$, Remark 3 and Sobolev embedding theorem,

$$\sup_{0\leqslant s\leqslant T}\left|u(s,x)\right| + \sup_{s,x\neq x'} \frac{|u(s,x) - u(s,x')|}{|x-x'|^\gamma} \leqslant C|u|_{\mathcal{H}_p^\alpha}.$$

For $0 < \frac{\gamma}{\alpha} < 1 - \frac{1}{p} - \frac{1}{\alpha p}$ there are $1/p < \mu < \kappa < 1 - \frac{d}{\alpha p}$ such that $\kappa = \mu + \frac{\gamma}{\alpha}$. Since $\alpha(1 - \kappa) - \frac{d}{p} > 0$, by Sobolev embedding theorem and Proposition 2,

$$\begin{aligned}
\left|u(s,x) - u(t,x)\right| &\leqslant C\left|u(s,\cdot) - u(t,\cdot)\right|_{H_p^{\alpha(1-\kappa)}} \\
&\leqslant C|t-s|^{\gamma/\alpha}\left[|f|_{L_p(T)}^p + \left|\partial^\alpha u\right|_{L_p(T)}^p\right]
\end{aligned}$$

and the statement follows. $\square$

**Remark 5.** Following the proof of Morrey's lemma we could show with $\delta = \alpha(1 - \frac{1}{p}) - \frac{d}{p} > 0$ that

$$\sup_{(t,x)\in E}\left|u(t,x)\right| + \sup_{s\neq t,x}\frac{|u(t,y)-u(s,x)|}{(|t-s|+|y-x|^{\alpha})^{1-\frac{1}{p}-\frac{d}{\alpha p}}} \leqslant C|u|_{\mathcal{W}_p^{\alpha}},$$

where

$$|u|^p_{\mathcal{W}_p^{\alpha}} = \int_0^T\int_0^T\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\frac{|u(t,y)-u(s,x)|^p}{(|t-s|+|y-x|^{\alpha})^{p+1+\frac{d}{\alpha}}}\,dt\,ds\,dy\,dx + |u|^p_{L_p(T)}.$$

Since the embedding $\mathcal{W}_p^{\alpha} \supseteq \mathcal{H}_p^{\alpha}$ is continuous for $p \geqslant 2$, inequality (5.6) holds for $\gamma = \delta$ if $p \geqslant 2$.

## 6. Martingale problem

In this section, we construct the Markov process associated to $L = A + B$ by proving the existence and uniqueness of the corresponding martingale problem (see [18]). A similar martingale problem with all Hölder continuous coefficients was considered in [15]. Proposition 3 below shows the existence and uniqueness of the Markov process corresponding to $L$ with Hölder continuous $A$ and measurable $B$ ($m(t,x,y)$ is only measurable in $y$).

Let $D = D([0,T],\mathbf{R}^d)$ be the Skorokhod space of càdlàg $\mathbf{R}^d$-valued trajectories and let $X_t = X_t(w) = w_t$, $w \in D$, be the canonical process on it.

Let

$$\mathcal{D}_t = \sigma(X_s,\ s \leqslant t), \qquad \mathcal{D} = \bigvee_t \mathcal{D}_t, \qquad \mathbb{D} = (\mathcal{D}_{t+}), \quad t \in [0,T].$$

We say that a probability measure $\mathbf{P}$ on $(D,\mathcal{D})$ is a solution to the $(s,x,L)$-martingale problem (see [18,12]) if $\mathbf{P}(X_r = x,\ 0 \leqslant r \leqslant s) = 1$ and for all $v \in C_0^{\infty}(\mathbf{R}^d)$ the process

$$M_t(v) = v(X_t) - \int_s^t Lv(r,X_r)\,dr \tag{6.1}$$

is a $(\mathbb{D},\mathbf{P})$-martingale. We denote $S(s,x,L)$ the set of all solutions to the $(s,x,L)$-martingale problem.

A modification of Theorem 5 in [12] is the following statement.

**Proposition 3.** *Let Assumptions* A *and* B(i)–(ii) *hold. Then for each* $(s,x) \in E$ *there is a unique solution* $\mathbf{P}_{s,x}$ *to the martingale problem* $(s,x,L)$*, and the process* $(X_t,\mathbb{D},(\mathbf{P}_{s,x}))$ *is strong Markov.*

*If, in addition,*

$$\lim_{l\to\infty}\int_0^T \sup_x \pi\left(t,x,\left\{|\upsilon|>l\right\}\right)dt = 0, \tag{6.2}$$

*then the function* $\mathbf{P}_{s,x}$ *is weakly continuous in* $(s,x)$.

### 6.1. Auxiliary results

We will need the following $L_p$-estimate.

**Lemma 12.** *(Cf. Lemma 3.6 in [12].) Let Assumptions A and B(i)–(ii) hold. Let $p > \frac{d}{\beta} \vee \frac{2d}{\alpha} \vee 2$, $(s_0, x_0) \in E$, $\mathbf{P} \in S(s_0, x_0, L)$.*

*Then there is a constant $C = C(R, T, K, \eta, \beta, w, p)$ such that for any $f \in C_0^\infty(E)$,*

$$\mathbf{E}^{\mathbf{P}} \int_{s_0}^{\tau} f(r, X_r)\, dr \leqslant C|f|_{L_p(T)}.$$

**Proof.** Let $\zeta \in C_0^\infty(\mathbf{R}^d)$, $\zeta \geqslant 0$, $\zeta(x) = \zeta(|x|)$, $\zeta(x) = 0$ if $|x| \geqslant 1$, and $\int \zeta^p\, dx = 1$. For $\delta > 0$ denote $\zeta_\delta(x) = \varepsilon^{-d/p} \zeta(x/\delta)$, $x \in \mathbf{R}^d$. Let

$$u_\delta(t, x) = \int u(t, x - y) \zeta_\delta^p(y)\, dy, \quad (t, x) \in E.$$

Let

$$\tilde{L} v = A v + \tilde{B}^{\varepsilon_0} v,$$

where $\tilde{B}^{\varepsilon_0}$ is defined by (2.5) with $\varepsilon_0$ so that the assumptions of Corollary 1 hold. Then

$$L v = \tilde{L} v + R v$$

with

$$R v(t, x) = \int_{|y| > \varepsilon_0} \left[ v(x + y) - v(x) \right] \pi(t, x, dy).$$

Define $\tilde{L}^\delta v = A v + B^{\varepsilon_0, \delta} v$, where

$$B^{\varepsilon_0, \delta} v(t, x) = \frac{\mathbf{E}^{\mathbf{P}} [\zeta_\delta^p (X_t - x) \tilde{B}_{t, X_t}^{\varepsilon_0} v(x)]}{\mathbf{E}^{\mathbf{P}} \zeta_\delta^p (X_t - x)}$$

(here we assume $\frac{0}{0} = 0$). Since for $\tilde{L}^\delta$ the assumptions of Corollary 1 hold uniformly in $\delta$, there is $u = u^\delta \in \mathcal{H}_p^\alpha(E)$ solving

$$\begin{aligned} \partial_t u(t, x) + \tilde{L}^\delta u(t, x) &= f(t, x), \quad (t, x) \in E, \\ u(T, x) &= 0, \quad x \in \mathbf{R}^d. \end{aligned}$$

Moreover, there is a constant $C$ independent of $\delta$ such that

$$\left| u^\delta \right|_{H_p^\alpha(T)} \leqslant C|f|_{L_p(T)}. \tag{6.3}$$

In addition, by Corollary 4 and (6.3), there is a constant independent of $\delta$ such that

$$\sup_{s,x}\left|u^{\delta}(s,x)\right| \leqslant C|f|_{L_p(T)}. \tag{6.4}$$

Applying Ito formula to $u_{\delta}^{\delta}(t,x)=\int \zeta_{\delta}^{p}(x-z)u^{\delta}(t,z)\,dz=\int \zeta_{\delta}^{p}(z)u^{\delta}(t,x-z)\,dz$, we have

$$\begin{aligned}
-u_{\delta}^{\delta}(s_0,x_0) &= \int_{s_0}^{T}\left[\partial_t u^{\delta}(r,z)+\left(A+B^{\varepsilon_0,\delta}\right)u^{\delta}(r,z)\right]\kappa_{\delta}(t,z)\,dz \\
&\quad + \mathbf{E}^{\mathbf{P}}\int_{s_0}^{T} Ru_{\delta}^{\delta}(r,X_r)\,dr+\int_{s_0}^{T} R_2(r)\,dr \\
&= \int_{s_0}^{T}\mathbf{E}^{\mathbf{P}} f_{\delta}(r,X_r)\,dr+\mathbf{E}^{\mathbf{P}}\int_{s_0}^{T} Ru_{\delta}^{\delta}(r,X_r)\,dr+\int_{s_0}^{T} R_2(r)\,dr,
\end{aligned} \tag{6.5}$$

where $\kappa_{\delta}(t,z)=\mathbf{E}^{\mathbf{P}}\zeta_{\delta}^{p}(X_t-z)$ and for $s_0\leqslant r\leqslant T$,

$$\begin{aligned}
R_2(r) &= \int \mathbf{E}^{\mathbf{P}}\left[\zeta_{\delta}^{p}(X_r-z)A_{r,X_r}u^{\delta}(r,z)-\zeta_{\delta}^{p}(X_r-z)A_{r,z}u^{\delta}(r,z)\right]dz \\
&= \mathbf{E}^{\mathbf{P}}\int\int \nabla_y^{\alpha}u^{\delta}(r,z)\left[m(r,X_r,y)-m(r,z,y)\right]\frac{dy}{|y|^{d+\alpha}}\zeta_{\delta}^{p}(X_r-z)\,dz.
\end{aligned}$$

By Hölder inequality, for any $r\in[s_0,T]$,

$$\left|R_2(r)\right|^p \leqslant \mathbf{E}^{\mathbf{P}}\int\left|\int \nabla_y^{\alpha}u^{\delta}(r,z)\left[m(r,X_r,y)-m(r,z,y)\right]\zeta_{\delta}(X_r-z)\frac{dy}{|y|^{d+\alpha}}\right|^p dz.$$

According to Lemma 6(b) with $\varepsilon=\delta$ and $\phi=\zeta_{\delta}(X_t-\cdot)$, $m=m(r,X_r,y)-m(r,z,y)$,

$$\left|R_2(r)\right|^p \leqslant C\left|\partial^{\alpha}u^{\delta}(r)\right|_{L_p(T)}^{p}K(\delta),$$

with

$$K(\delta)\leqslant C\left[\delta^{-\beta p}w(\delta)^p+\left(\int_{|\upsilon|\leqslant\delta} w(\upsilon)\frac{d\upsilon}{|\upsilon|^{d+\beta}}\right)^p\right].$$

Therefore by (6.3),

$$\int_{s_0}^{T}\left|R_2(r)\right|^p dr \to 0 \quad \text{as } \delta\to 0. \tag{6.6}$$

Since

$$\int_{s_0}^{T} \mathbf{E}^{\mathbf{P}} f_\delta(r, X_r)\,dr = -\left(u_\delta^\delta(s_0, x_0) + \mathbf{E}^{\mathbf{P}} \int_{s_0}^{T} R u_\delta^\delta(r, X_r)\,dr + \int_{s_0}^{T} R_2(r)\,dr\right)$$

(see (6.5)), the statement follows by (6.4) and (6.6) passing to the limit as $\delta \to 0$. $\square$

**Corollary 5.** *Let Assumptions A and B hold, $(s_0, x_0) \in E$. Then the set $S(s_0, x_0, L)$ consists of at most one probability measure.*

**Proof.** Let $f \in C_0^\infty(E)$, $p > \frac{d}{\beta} \vee \frac{2d}{\alpha} \vee 2$. By Theorem 3, there is $u \in \mathcal{H}_p^\alpha(E)$ solving

$$\partial_t u(t,x) + Lu(t,x) = f(t,x), \quad (t,x) \in E,$$
$$u(T,x) = 0, \quad x \in \mathbf{R}^d.$$

Let $\varphi \in C_0^\infty(\mathbf{R}^d)$, $\varphi \geqslant 0$, $\int \varphi\,dx = 1$, $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$, $x \in \mathbf{R}^d$, and

$$u_\varepsilon(t,x) = \int u(t, x-y)\varphi_\varepsilon(y)\,dy, \quad (t,x) \in \mathbf{R}^d.$$

Let $\mathbf{P} \in S(s, x, L)$. Applying Ito formula,

$$-u_\varepsilon(s,x) = \mathbf{E}^{\mathbf{P}} \int_s^T \left[\partial_t u_\varepsilon(r, X_r) + Lu_\varepsilon(r, X_r)\right] dr.$$

Using Lemma 12 and Corollary 4 to pass to the limit we derive that

$$-u(s,x) = \mathbf{E}^{\mathbf{P}} \int_s^T f(r, X_r)\,dr. \tag{6.7}$$

Suppose $\mathbf{P}_1, \mathbf{P}_2 \in S(s_0, x_0, L)$. We show that

$$\mathbf{P}_1(X_{t_1} \in \Gamma_1, \ldots, X_{t_n} \in \Gamma_n) = \mathbf{P}_2(X_{t_1} \in \Gamma_1, \ldots, X_{t_n} \in \Gamma_n)$$

for any $s_0 < t_1 < \cdots < t_n \leqslant T$ and Borel $\Gamma_i \subseteq \mathbf{R}^d$, $1 \leqslant i \leqslant d$. For $n = 1$, the equality is an obvious consequence of (6.7). If it is true for $n$, then by Theorem 1.2 in [18], the regular conditional probabilities $\mathbf{P}_1^w$ and $\mathbf{P}_2^w$ of $\mathbf{P}_1$ and $\mathbf{P}_2$ with respect to $\mathcal{D}_{t_n}$ belong to $S(t_n, X_{t_n}(w), L)$. Since for $i = 1, 2$,

$$\begin{aligned}
&\mathbf{P}_i(X_{t_1} \in \Gamma_1, \ldots, X_{t_n} \in \Gamma_n, X_{t_{n+1}} \in \Gamma_{n+1}) \\
&\quad = \int \chi_{\Gamma_1}\big(X_{t_1}(w)\big) \cdots \chi_{\Gamma_n}\big(X_{t_n}(w)\big) \mathbf{E}^{\mathbf{P}_i^w} \chi_{\Gamma_{n+1}}(X_{t_{n+1}})\,\mathbf{P}_i(dw)
\end{aligned}$$

the equality follows for $n + 1$ because of the induction assumption and the proved case with $n = 1$. $\square$

Now we can construct a “local” solution of the martingale problem.

**Lemma 13.** *Let Assumptions A, B(i)–(ii) hold, $\pi(t,x,d\upsilon)=\chi_{\{|x|\leqslant R\}}\pi(t,x,d\upsilon)$, $(t,x)\in E$, for some $R>0$.*

*Then for each $(s,x)\in E$ there is a unique solution $\mathbf{P}_{s,x}\in S(s,x,L)$ and $\mathbf{P}_{s,x}$ is weakly continuous in $(s,x)$.*

**Proof.** Let $\varphi\in C_0^\infty(\mathbf{R}^d)$, $\varphi\geqslant 0$, $\int\varphi\,dx=1$, $\varphi_\varepsilon(x)=\varepsilon^{-d}\varphi(x/\varepsilon)$, $x\in\mathbf{R}^d$, and

$$\pi_\varepsilon(t,x,d\upsilon)=\int\pi(t,x-z,d\upsilon)\varphi_\varepsilon(z)\,dz,\quad (t,x)\in E.$$

Let $\varepsilon_n\to 0$ and let $L^n$ be an operator defined as $L$ with $\pi$ replaced by $\pi_{\varepsilon_n}$. It follows by Theorem IX.2.31 in [6] that the set $\mathcal{S}(s,x,L^n)\neq\emptyset$. Since by Lemma 12, for $\mathbf{P}^n_{s,x}\in S(s,x,L^n)$, denoting $\mathbf{E}^n_{s,x}$ the expectation with respect to $\mathbf{P}^n_{s,x}$,

$$\mathbf{E}^n_{s,x}\int_s^T\pi_{\varepsilon_n}\big(r,X_r,\big\{|\upsilon|>l\big\}\big)\,dr\leqslant C\int_s^T\int\pi\big(r,x,\big\{|\upsilon|>l\big\}\big)\,dx\,dr\to 0\quad\text{as } l\to\infty,$$

the sequence $\{\mathbf{P}^n_{s,x}\}$ is tight (see Theorem VI.4.18 in [6]). Obviously, for each $v\in C_0^\infty(\mathbf{R}^d)$ and $p>1$, $|L^n v-Lv|_p\to 0$ and $L^n v$ is uniformly bounded. Suppose $\mathbf{P}^n_{s,x}\to\mathbf{P}$ weakly. Let $s<q<t\leqslant T$,

$$M^n_t(v)=v(X_t)-\int_s^t L^n v(r,X_r)\,dr,\qquad M_t(v)=v(X_t)-\int_s^t Lv(r,X_r)\,dr$$

and $\phi$ be $\mathcal{D}_q$-measurable $\mathbf{P}$-a.s. continuous. Then for each $m>1$

$$\begin{aligned}0&=\mathbf{E}^n_{s,x}\big[\phi\big(M^n_t(v)-M^n_q(v)\big)\big]\\&=\mathbf{E}^n_{s,x}\big[\phi\big(M^m_t(v)-M^m_q(v)\big)\big]+\mathbf{E}^n_{s,x}\phi\int_q^t\big(L^n v-L^m v\big)(r,X_r)\,dr.\end{aligned}$$

Obviously,

$$\mathbf{E}^n_{s,x}\big[\phi\big(M^m_t(v)-M^m_q(v)\big)\big]\to\mathbf{E}^{\mathbf{P}}\big[\phi\big(M^m_t(v)-M^m_q(v)\big)\big]$$

as $n\to\infty$ and

$$\mathbf{E}^{\mathbf{P}}\big[\phi\big(M^m_t(v)-M^m_q(v)\big)\big]=\mathbf{E}^{\mathbf{P}}\big[\phi\big(M_t(v)-M_q(v)\big)\big]+\mathbf{E}^{\mathbf{P}}\phi\int_q^t\big(Lv-L^m v\big)(r,X_r)\,dr.$$

By Lemma 12 for large $p$,

$$\left|\mathbf{E}^{\mathbf{P}}\phi\int_q^t\big(Lv-L^m v\big)(r,X_r)\,dr\right|\leqslant C\big|Lv-L^m v\big|_{L_p(T)},$$

$$\left|\mathbf{E}^n_{s,x}\phi\int_q^t\big(L^n v-L^m v\big)(r,X_r)\,dr\right|\leqslant C\big|L^n v-L^m v\big|_{L_p(T)}\to C\big|Lv-L^m v\big|_{L_p(T)}$$

as $n\to\infty$. Since $m,q,t,v$ are arbitrary, $\mathbf{P}\in S(s,x,L)$. By Lemma 12, it is unique. Similarly, we see that $\mathbf{P}_{s,x}\in S(s,x,L)$ is continuous in $(s,x)$. $\square$

**Corollary 6.** *Let Assumptions A, B*(i)–(ii) *hold. Then for each $(s,x)\in E$, there is at most one solution $\mathbf{P}_{s,x}\in S(s,x,L)$.*

**Proof.** The statement is immediate consequence of Lemma 13 and Theorem 1.6(b) in [12]. $\square$

### 6.2. *Proof of Proposition 3*

The uniqueness follows by Corollary 6. In the first part of the proof we assume that (6.2) holds and use weak convergence arguments. In the second part, we cover the general case by putting together measurable families of probability measures.

(i) Assume (6.2) holds. Let $L^n$ be an operator defined as $L$ with $\pi$ replaced by $\chi_{\{|x|\leqslant n\}}\pi$. According to Lemma 13, for each $(s,x)\in E$ there is a unique $\mathbf{P}^n_{s,x}\in S(s,x,L^n)$ and $\mathbf{P}^n_{s,x}$ is weakly continuous in $(s,x)$. By Theorem VI.4.18 in [6], $\{\mathbf{P}^n_{s,x}\}$ is tight. Since $L^n v\to Lv$ pointwise and $L^n v$ is uniformly bounded for any $v\in C_0^\infty(\mathbf{R}^d)$, by Lemma 3.7 in [12], the sequence $\mathbf{P}^n_{s,x}\to\mathbf{P}_{s,x}\in S(s,x,L)$ weakly ($\mathbf{P}_{s,x}$ is unique by Corollary 6). The same Lemma 3.7, [12], implies that $\mathbf{P}_{s,x}$ is weakly continuous in $(s,x)$.

(ii) In the general case (without assuming (6.2)), we split the operator $Lu=\tilde{L}u+\tilde{B}u$, where $\tilde{L}$ is defined as $L$ with $\pi(t,x,d\upsilon)$ replaced by $\chi_{\{|\upsilon|<1\}}\pi(t,x,d\upsilon)$, and

$$\tilde{B}_{t,x}u(x)=\int_{|\upsilon|\geqslant 1}\big[u(x+\upsilon)-u(x)\big]\pi(t,x,d\upsilon),\quad (t,x)\in E,\ u\in C_0^\infty\big(\mathbf{R}^d\big).$$

Let $(\Omega_2,\mathcal{F}_2,\mathbf{P}_2)$ be a probability space with a Poisson point measure $\tilde{p}(dt,dz)$ on $[0,\infty)\times(R\backslash\{0\})$ with

$$\mathbf{E}\,\tilde{p}(dt,dz)=\frac{dz\,dt}{|z|^2}.$$

According to Lemma 14.50 in [5], there is a measurable $\mathbf{R}^d\cap\{|\upsilon|\geqslant 1\}$-valued function $c(t,x,z)$ such that for any Borel $\Gamma$

$$\int_\Gamma\chi_{\{|\upsilon|\geqslant 1\}}\pi(t,x,d\upsilon)=\int\chi_\Gamma\big(c(t,x,z)\big)\frac{dz}{z^2},\quad (t,x)\in E.$$

Consider the probability space

$$\left(\Omega, \mathcal{F}, \mathbf{P}'_{s,x}\right) = (\Omega_2 \times D, \mathcal{F}_2 \otimes \mathcal{D}, \mathbf{P}_2 \otimes \mathbf{P}_{s,x}).$$

Let

$$\begin{aligned} H_t &= \int_{s\wedge t}^{t} \int c(r, X_{r-}, z)\, \tilde{p}(dr, dz), \quad s \leqslant t \leqslant T, \\ \tau &= \inf(t > s\colon\ \Delta H_t = H_t - H_{t-} \neq 0) \wedge T, \\ K_t &= \chi_{\{\tau \leqslant t\}}, \\ Y_t &= X_{t\wedge\tau} + H_{t\wedge\tau}, \quad 0 \leqslant t \leqslant T. \end{aligned}$$

Note that $\tau = \inf(t > s\colon\ \Delta H_t \neq 0) \wedge T = \tau = \inf(t > s\colon\ |\Delta H_t| \geqslant 1) \wedge T$. Let $\hat{D} = D([0,T], \mathbf{R}^d \times [0,\infty))$ be the Skorokhod space of càdlàg $\mathbf{R}^d \times [0,\infty)$-valued trajectories and let $Z_t = Z_t(w) = (y_t(w), k_t(w)) = w_t \in \mathbf{R}^d \times [0,\infty)$, $w \in \hat{D}$ be the canonical process on it. Let

$$\hat{\mathcal{D}}_t = \sigma(Z_s,\ s \leqslant t), \qquad \hat{\mathcal{D}} = \bigvee_t \hat{\mathcal{D}}_t, \qquad \hat{\mathbb{D}} = (\hat{\mathcal{D}}_{t+}), \quad t \in [0,T].$$

Denote $\hat{\mathbf{P}}^1_{s,x}$ the measure on $\hat{D}$ induced by $(Y_t, K_t)$, $0 \leqslant t \leqslant T$. Let

$$\begin{aligned} \tau_1 &= \inf(t > s\colon\ \Delta k_t \geqslant 1) \wedge T, \quad \ldots, \\ \tau_{n+1} &= \inf(t > \tau_n\colon\ \Delta k_t \geqslant 1) \wedge T, \\ \hat{\mathcal{D}}_{\tau_n} &= \sigma(Z_{t\wedge\tau_n},\ 0 \leqslant t \leqslant T), \quad n \geqslant 1. \end{aligned}$$

Then $(\hat{\mathbf{P}}^1_{s,x})$ is a measurable family of measures on $(\hat{D}, \hat{\mathcal{D}})$ and for each $v \in C_0^\infty(\mathbf{R}^d)$,

$$\hat{M}_{t\wedge\tau_n}(v) = v(y_{t\wedge\tau_n}) - \int_s^{t\wedge\tau_n} Lv(r, y_r)\, dr, \quad s \leqslant t \leqslant T, \tag{6.8}$$

is $(\hat{\mathbf{P}}^1_{s,x}, \hat{\mathbb{D}})$-martingale with $n = 1$. Let us introduce the mappings

$$\mathcal{J}_{\tau_1}\left(w, w'\right)_t = \begin{cases} w_t & \text{if } t < \tau_1(w), \\ w'_t & \text{if } t \geqslant \tau_1(w), \end{cases}$$

and let

$$Q\left(dw, dw'\right) = \hat{\mathbf{P}}^1_{\tau_1(w), X_{\tau_1(w)}(w)}\left(dw'\right) \hat{\mathbf{P}}^1_{s,x}(dw).$$

Then $\mathbf{P}^2_{s,x} = \mathcal{J}_{\tau_1}(Q)$, the image of $Q$ under $\mathcal{J}_{\tau_1}$, is a measurable family of measures on $\hat{D}$, $\hat{M}_{t\wedge\tau_2}$ is a $(\hat{\mathbf{P}}^2_{s,x}, \hat{\mathbb{D}})$-martingale and $\hat{\mathbf{P}}^2_{s,x}|_{\hat{\mathcal{D}}_{\tau_1}} = \hat{\mathbf{P}}^1_{s,x}|_{\hat{\mathcal{D}}_{\tau_1}}$. Continuing, we construct a sequence of measures $\hat{\mathbf{P}}^n_{s,x}$ such that

$$\hat{\mathbf{P}}^{n+1}_{s,x}\big|_{\hat{\mathcal{D}}_{\tau_n}} = \hat{\mathbf{P}}^{n}_{s,x}\big|_{\hat{\mathcal{D}}_{\tau_n}}$$

and $\hat{M}_{t\wedge\tau_n}$ is $(\hat{\mathbf{P}}^n_{s,x}, \hat{\mathbb{D}})$-martingale. Since

$$\hat{\mathbf{P}}^n_{s,x}(\tau_n < T) = \hat{\mathbf{P}}^n_{s,x}(k_{T\wedge\tau_n} \geqslant n) \leqslant n^{-1} \int_s^{T\wedge\tau_n} \pi\big(r, y_r, \{|\upsilon| \geqslant 1\}\big)\, dr$$
$$\leqslant n^{-1} KT \to 0 \quad \text{as } n \to \infty,$$

there is a measurable family $(\hat{\mathbf{P}}_{s,x})$ on $\hat{D}$ such that

$$\hat{\mathbf{P}}_{s,x}|_{\hat{\mathcal{D}}_{\tau_n}} = \hat{\mathbf{P}}^n_{s,x}\big|_{\hat{\mathcal{D}}_{\tau_n}}, \quad n \geqslant 1,$$

and $\hat{M}_{t\wedge\tau_n}$ is $(\hat{\mathbf{P}}_{s,x}, \hat{\mathbb{D}})$-martingale for every $n$. Obviously, $y_\cdot$ under $\hat{\mathbf{P}}_{s,x}$ gives a measurable family $\mathbf{P}_{s,x} \in S(s, x, L)$. The strong Markov property is a consequence of Theorem 1.2 in [18]. The statement of Proposition 3 follows.

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**Appendix A**

For $\alpha \in (0,1)$ and a bounded measurable function $m(y)$, set for $v \in \mathcal{S}(\mathbf{R}^d)$, $x \in \mathbf{R}^d$,

$$\mathcal{L}v(x) = \int \big[v(x+y) - v(x)\big] m(y) \frac{dy}{|y|^{d+\alpha}}.$$

By Lemma 2,

$$\mathcal{L}v(x) = \lim_{\varepsilon\to 0} \int \int k^{(\alpha)}(z,y) \partial^\alpha v(x-z)\, dz\, m_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}}$$
$$= \lim_{\varepsilon\to 0} \int \int k^{(\alpha)}(z,y) m_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}} \partial^\alpha v(x-z)\, dz,$$

where $m_\varepsilon(y) = \chi_{\{\varepsilon \leqslant |y| \leqslant \varepsilon^{-1}\}} m(y)$. For $\varepsilon \in (0,1)$, $v \in \mathcal{S}(\mathbf{R}^d)$, consider

$$\mathcal{K}^\varepsilon v(x) = \int k_\varepsilon(z) v(x-z)\, dz = \int k_\varepsilon(x-z) v(z)\, dz$$

with

$$k_\varepsilon(x) = \int k^{(\alpha)}(x,y) m_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}}, \quad x \in \mathbf{R}^d.$$

Obviously,

$$\mathcal{L}v(x) = \lim_{\varepsilon\to 0} \mathcal{K}^{\varepsilon}\partial^{\alpha}v(x), \quad x \in \mathbf{R}^d,$$

and to prove the continuity $\mathcal{L} : H_p^{\alpha}(\mathbf{R}^d) \to L_p(\mathbf{R}^d)$, we will show that there is a constant $C$, independent of $\varepsilon \in (0,1)$ and $v \in \mathcal{S}(\mathbf{R}^d)$, such that

$$\left|\mathcal{K}^{\varepsilon}v\right|_{L_p} \leqslant C|v|_{L_p}. \tag{A.1}$$

According to Theorem 3 of Chapter I in [17], (A.1) will follow provided

$$\left|\mathcal{K}^{\varepsilon}v\right|_{L_2} \leqslant C|v|_{L_2}, \tag{A.2}$$

and

$$\int\limits_{|x|>4|s|} \left|k_{\varepsilon}(x-s) - k_{\varepsilon}(x)\right| dx \leqslant C \quad \text{for all } s \in \mathbf{R}^d. \tag{A.3}$$

**Remark 6.** For any $t > 0$, we have $k^{(\alpha)}(tx, ty) = t^{\alpha-d}k^{(\alpha)}(x,y)$. Therefore

$$k_{\varepsilon}(tx) = t^{-d}\int k^{(\alpha)}(x,y)m_{\varepsilon}(ty)\frac{dy}{|y|^{d+\alpha}} = t^{-d}k_{\varepsilon}(t,x)$$

with

$$k_{\varepsilon}(t,x) = \int k^{(\alpha)}(x,y)m_{\varepsilon}(ty)\frac{dy}{|y|^{d+\alpha}}.$$

Note that for $x \neq 0$,

$$k_{\varepsilon}(x) = k_{\varepsilon}\big(|x|\hat{x}\big) = |x|^{-d}\int k^{(\alpha)}(\hat{x},y)m_{\varepsilon}\big(|x|y\big)\frac{dy}{|y|^{d+\alpha}} = |x|^{-d}k_{\varepsilon}\big(|x|,\hat{x}\big),$$

where $\hat{x} = x/|x|$.

**Lemma 14.** *Let $\alpha \in (0,1)$, $|m(y)| \leqslant 1$, $y \in \mathbf{R}^d$. Then for each $p > 1$ there is a constant $C$ independent of $u$ and $\varepsilon$ such that,*

$$\left|\mathcal{K}^{\varepsilon}v\right|_{L_p} \leqslant C|v|_{L_p}, \quad v \in L_p\big(\mathbf{R}^d\big).$$

**Proof.** By Theorem 3 of Chapter I in [17] it is enough to show that (A.2) and (A.3) hold. By Lemma 1 of Chapter 5.1 in [16], it follows

$$\begin{aligned}\hat{k}_{\varepsilon}(\xi) &= \int e^{-i(x,\xi)}k_{\varepsilon}(x)\,dx = C|\xi|^{-\alpha}\int \big[e^{-i(\xi,y)} - 1\big]m_{\varepsilon}(y)\frac{dy}{|y|^{d+\alpha}} \\ &= \int \big[e^{-i(\hat{\xi},y)} - 1\big]m_{\varepsilon}\big(y/|\xi|\big)\frac{dy}{|y|^{d+\alpha}},\end{aligned}$$

where $\hat{\xi} = \xi/|\xi|$. Therefore, by Parseval's equality, (A.2) holds for $v \in \mathcal{S}(\mathbf{R}^d)$. The key estimate is (A.3). By Remark 6, denoting $\hat{s} = s/|s|$, we have

$$\begin{aligned}
\int_{|x|>4|s|} \left|k_\varepsilon(x-s) - k_\varepsilon(x)\right| dx &= \int_{|x|/|s|>4} \left|k_\varepsilon\left(|s|\left(\frac{x}{|s|} - \hat{s}\right)\right) - k_\varepsilon\left(|s|\frac{x}{|s|}\right)\right| dx \\
&= |s|^d \int_{|x|>4} \left|k_\varepsilon\big(|s|(x-\hat{s})\big) - k_\varepsilon\big(|s|x\big)\right| dx \\
&= \int_{|x|>4} \left|k_\varepsilon\big(|s|, x-\hat{s}\big) - k_\varepsilon\big(|s|, x\big)\right| dx
\end{aligned}$$

and it is enough to prove that

$$\int_{|x|>4} \left|k_\varepsilon\big(|s|, x-\hat{s}\big) - k_\varepsilon\big(|s|, x\big)\right| dx \leqslant M \quad \text{for all } s \in \mathbf{R}^d,\ \hat{s} = s/|s|. \tag{A.4}$$

We will estimate for $|x| \geqslant 4$, $s \in \mathbf{R}^d$, $\hat{s} = s/|s|$, the difference

$$\begin{aligned}
\left|k\big(|s|, x-\hat{s}\big) - k\big(|s|, x\big)\right| &= \int \left[k^{(\alpha)}(x-\hat{s}, y) - k^{(\alpha)}(x, y)\right] m_\varepsilon\big(|s|y\big) \frac{dy}{|y|^{d+\alpha}} \\
&= \int_{|y|\leqslant|x|/2} \ldots + \int_{|y|>|x|/2} \ldots = A_1 + A_2.
\end{aligned}$$

Let

$$F(t) = \frac{1}{|x - t\hat{s} + y|^{d-\alpha}} - \frac{1}{|x - t\hat{s}|^{d-\alpha}}, \quad 0 \leqslant t \leqslant 1.$$

If a segment connecting $x$ and $x - \hat{s}$ does not contain zero, then

$$\left|k^{(\alpha)}(x-\hat{s}, y) - k^{(\alpha)}(x, y)\right| = \left|F(1) - F(0)\right| \leqslant \int_0^1 \left|F'(t)\right| dt \tag{A.5}$$

with

$$F'(t) = (\alpha - d)\left[-\frac{1}{|x - t\hat{s} + y|^{d-\alpha+1}} \frac{(x - t\hat{s} + y, \hat{s})}{|x - t\hat{s} + y|} + \frac{1}{|x - t\hat{s}|^{d-\alpha+1}} \frac{(x - t\hat{s}, \hat{s})}{|x - t\hat{s}|}\right],$$

and

$$\begin{aligned}
\left|F'(t)\right| \leqslant C &\left|\frac{1}{|x - t\hat{s} + y|^{d-\alpha+1}} - \frac{1}{|x - t\hat{s}|^{d-\alpha+1}}\right| \\
&+ \frac{1}{|x - t\hat{s}|^{d-\alpha+1}}\left(1 + \frac{1}{|x - t\hat{s}|}\right)\big(|y| \wedge 1\big).
\end{aligned} \tag{A.6}$$

*Estimate of* $A_1$. Let $|x| \geqslant 4$, $z = x - t\hat{s}$, $t \in [0, 1]$, and $|y| \leqslant |x|/2$. In this case, $|x + y| \geqslant |x| - |y| \geqslant |x|/2 \geqslant 2$,

$$C|x| \geqslant |z + y| \geqslant |x|/4 \geqslant 1,$$
$$C|x| \geqslant |z| \geqslant |x| - 1 \geqslant 3|x|/4 \geqslant 3$$

and (A.5) holds. Since

$$\int\limits_{|y| \leqslant |x|/2} \left| \frac{1}{|z + y|^{d-\alpha+1}} - \frac{1}{|z|^{d-\alpha+1}} \right| \frac{dy}{|y|^{d+\alpha}} \leqslant \frac{1}{|z|^{d+1}} \int\limits_{|y| \leqslant 2/3} \left| \frac{1}{|\hat{z} + y|^{d-\alpha+1}} - 1 \right| \frac{dy}{|y|^{d+\alpha}} \leqslant \frac{C}{|x|^{d+1}},$$

and

$$\int\limits_{|y| \leqslant |x|/2} \big(|y| \wedge 1\big) \frac{dy}{|y|^{d+\alpha}} \leqslant C,$$

it follows by (A.6), that

$$|A_1| \leqslant C \int\limits_{|y| \leqslant |x|/2} \left[ \left| \frac{1}{|z + y|^{d-\alpha+1}} - \frac{1}{|z|^{d-\alpha+1}} \right| + \frac{1}{|z|^{d-\alpha+1}} \big(|y| \wedge 1\big) \right] \frac{dy}{|y|^{d+\alpha}} \leqslant C \left[ \frac{1}{|x|^{d+1}} + \frac{1}{|x|^{d-\alpha+1}} \right].$$

*Estimate of* $A_2$. Let $|x| \geqslant 4$, $|y| > |x|/2$. In this case we split

$$A_2 = \int\limits_{|y| > |x|/2} \big[k^{(\alpha)}(x - \hat{s}, y) - k^{(\alpha)}(x, y)\big] m\big(|s|y\big) \frac{dy}{|y|^{d+\alpha}} = \int\limits_{\{|x|-3/2 \geqslant |y| > |x|/2\} \cup \{|y| > |x|+3/2\}} \ldots + \int\limits_{\{|x|-3/2 \leqslant |y| \leqslant |x|+3/2\}} \ldots = B_1 + B_2.$$

If $|x| - 3/2 \geqslant |y| > |x|/2$ or $|y| > |x| + 3/2$, then we can apply (A.5) and (A.6). For $z = x - t\hat{s}$ we have $|z + y| \geqslant \frac{1}{2}$ and

$$\big|F'(t)\big| \leqslant C \left[ \frac{1}{|z + y|^{d-\alpha+1}} + \frac{1}{|z|^{d-\alpha+1}} \right].$$

Therefore

$$|B_1| \leqslant C\Bigg[\int\limits_{\{\frac{|x|}{2}\leqslant|y|\}} \frac{1}{|z|^{d-\alpha+1}}\frac{dy}{|y|^{d+\alpha}} + \int\limits_{\{\frac{|x|}{2}\leqslant|y|\leqslant|x|-\frac{3}{2}\}} \frac{1}{|z+y|^{d-\alpha+1}}\frac{dy}{|y|^{d+\alpha}}$$
$$+ \int\limits_{\{|x|+\frac{3}{2}\leqslant|y|\}} \frac{1}{|z+y|^{d-\alpha+1}}\frac{dy}{|y|^{d+\alpha}}\Bigg]$$
$$= B_{11}+B_{12}+B_{13}.$$

Now

$$B_{11} = C\int\limits_{\{\frac{|x|}{2}\leqslant|y|\}} \frac{1}{|z|^{d-\alpha+1}}\frac{dy}{|y|^{d+\alpha}} \leqslant \frac{C}{|x|^{d+1}},$$

and

$$B_{12} \leqslant C|z|^{-d-1}\int\limits_{\{\frac{|x|}{2|z|}\leqslant|y|\leqslant(|x|-\frac{3}{2})/|z|\}} \frac{1}{|\hat{z}+y|^{d-\alpha+1}}\frac{dy}{|y|^{d+\alpha}},$$
$$B_{13} \leqslant C|z|^{-d-1}\int\limits_{\{\frac{|x|}{|z|}+\frac{3}{2|z|}\leqslant|y|\}} \frac{1}{|\hat{z}+y|^{d-\alpha+1}}\frac{dy}{|y|^{d+\alpha}}$$

with $\hat{z}=z/|z|$. If $\frac{|x|}{2|z|}\leqslant|y|\leqslant(|x|-\frac{3}{2})/|z|$ or $\frac{|x|}{|z|}+\frac{3}{2|z|}\leqslant|y|$, then $|\hat{z}+y|\geqslant\frac{1}{2|z|}$ and $|y|\geqslant 1/3$. Therefore

$$B_{12} \leqslant C|z|^{-d-1}\int\limits_{\{|\hat{z}+y|\geqslant\frac{1}{2|z|}\}} \frac{dy}{|\hat{z}+y|^{d-\alpha+1}}$$
$$\leqslant C|z|^{-d-\alpha} \leqslant C|x|^{-d-\alpha}$$

and

$$B_{13} \leqslant C|z|^{-d-1}\int\limits_{|\hat{z}+y|\geqslant\frac{1}{2|z|}} \frac{1}{|\hat{z}+y|^{d-\alpha+1}}\,dy$$
$$= C|z|^{-d-\alpha} \leqslant C|x|^{-d-\alpha}.$$

Now we estimate $B_2$. If $|x|-\frac{3}{2}\leqslant|y|\leqslant|x|+\frac{3}{2}$, then we estimate directly. First we have

$$\int\limits_{\{|x|-\frac{3}{2}\leqslant|y|\leqslant|x|+\frac{3}{2}\}} \frac{1}{|z|^{d-\alpha}}\frac{dy}{|y|^{d+\alpha}} \leqslant C|x|^{\alpha-d}\left|\frac{1}{(|x|-\frac{3}{2})^{\alpha}} - \frac{1}{(|x|+\frac{3}{2})^{\alpha}}\right|$$
$$\leqslant C|x|^{-d-\alpha}.$$

Then, for $z = x - t\hat{s}$ with $t \in [0,1]$, we have $\frac{2}{3} \leqslant 1 - \frac{1}{|z|} \leqslant \frac{|x|}{|z|} \leqslant 1 + \frac{1}{|z|} \leqslant \frac{4}{3}$ and

$$\begin{aligned}
&\int_{\{|x|-\frac{3}{2}\leqslant|y|\leqslant|x|+\frac{3}{2}\}} \frac{1}{|z+y|^{d-\alpha}} \frac{dy}{|y|^{d+\alpha}} \\
&\leqslant |z|^{-d} \int_{\{1-\frac{5}{2|z|}\leqslant|y|\leqslant 1+\frac{5}{2|z|}\}} \frac{1}{|\hat{z}+y|^{d-\alpha}} \frac{dy}{|y|^{d+\alpha}} \\
&\leqslant |z|^{-d} \int_{\{1-\frac{5}{2|z|}\leqslant|y|\leqslant 1+\frac{5}{2|z|}, |\hat{z}+y|>|z|^{-\alpha/d}\}} \ldots + \int_{\{1-\frac{5}{2|z|}\leqslant|y|\leqslant 1+\frac{5}{2|z|}, |\hat{z}+y|\leqslant|z|^{-\alpha/d}\}} \ldots \\
&\leqslant C\left[|z|^{-d}|z|^{\frac{\alpha}{d}(d-\alpha)} \int_{\{1-\frac{5}{2|z|}\leqslant|y|\leqslant 1+\frac{5}{2|z|}\}} \frac{dy}{|y|^{d+\alpha}} + |z|^{-d} \int_{|\hat{z}+y|\leqslant|z|^{-\alpha/d}} \frac{dy}{|\hat{z}+y|^{d-\alpha}}\right] \\
&\leqslant C|z|^{-\alpha^2/d-d} \leqslant C|x|^{-\alpha^2/d-d}
\end{aligned}$$

with $\hat{z} = z/|z|$. Therefore,

$$|B_2| \leqslant C\big[|x|^{-\alpha^2/d-d} + |x|^{-d-1}\big].$$

The statement follows. $\square$

For a bounded measurable $m(y)$, $y \in \mathbf{R}^d$, and $\alpha \in (0,2)$, set for $v \in \mathcal{S}(\mathbf{R}^d)$, $x \in \mathbf{R}^d$,

$$\mathcal{L}v(x) = \int \big[v(x+y) - v(x) - \chi_\alpha(y)\big(\nabla v(x), y\big)\big] m(y) \frac{dy}{|y|^{d+\alpha}}.$$

**Lemma 15.** *Let $|m(y)| \leqslant K$, $y \in \mathbf{R}^d$, $p > 1$, and $\alpha \in (0,2)$. Assume*

$$\int_{r\leqslant|y|\leqslant R} y m(y) \frac{dy}{|y|^{d+\alpha}} = 0$$

*for any $0 < r < R$ if $\alpha = 1$. Then there is a constant $C$ such that*

$$|\mathcal{L}v|_{L_p} \leqslant CK\big|\partial^\alpha v\big|_{L_p}, \quad v \in L_p\big(\mathbf{R}^d\big).$$

**Proof.** If $\alpha \in (0,1)$, then for $v \in \mathcal{S}(\mathbf{R}^d)$ we have

$$\mathcal{L}v(x) = \lim_{\varepsilon\to 0} \mathcal{K}^\varepsilon \partial^\alpha v(x), \quad x \in \mathbf{R}^d,$$

and by Lemma 14 there is a constant $C$ independent of $u$ such that

$$\big|K^{-1}\mathcal{L}v\big|_{L_p} \leqslant C\big|\partial^\alpha v\big|_{L_p}$$

or

$$|\mathcal{L}v|_{L_p} \leqslant CK\left|\partial^\alpha v\right|_{L_p}, \quad v \in \mathcal{S}\big(\mathbf{R}^d\big).$$

If $\alpha \in (1,2)$, then it follows by Lemma 2 that for $u \in \mathcal{S}(\mathbf{R}^d)$,

$$\begin{aligned}\mathcal{L}v(x) &= \int\int_0^1 \big(\nabla v(x+sy) - \nabla v(x), y\big) m(y)\frac{ds\,dy}{|y|^{d+\alpha}} \\ &= \int\left(\nabla v(x+y) - \nabla v(x), \frac{y}{|y|}\right) M(y)\frac{dy}{|y|^{d+\alpha-1}}\end{aligned}$$

with

$$M(y) = \int_0^1 m(y/s)s^{-1+\alpha}\,ds, \quad y \in \mathbf{R}^d.$$

Therefore, the estimate reduces to the case of $\alpha \in (0,1)$: there is a constant $C$ independent of $v \in \mathcal{S}(\mathbf{R}^d)$ such that

$$|\mathcal{L}v|_{L_p} \leqslant CK\left|\partial^{\alpha-1}\nabla v\right|_{L_p} \leqslant CK\left|\partial^\alpha v\right|_{L_p}.$$

If $\alpha = 1$, then for $v \in \mathcal{S}(\mathbf{R}^d)$,

$$\mathcal{L}v(x) = \lim_{\varepsilon\to 0}\int \big[v(x+y) - v(x)\big] m_\varepsilon(y)\frac{dy}{|y|^{d+1}}$$

with $m_\varepsilon(y) = m(y)1_{\varepsilon^{-1}\geqslant|y|>\varepsilon}$, $y \in \mathbf{R}^d$. Since for $v \in \mathcal{S}(\mathbf{R}^d)$, $x \in \mathbf{R}^d$,

$$\begin{aligned}&\int \big[v(x+y) - v(x)\big] m_\varepsilon(y)\frac{dy}{|y|^{d+1}} \\ &\quad= \int\int k^{(1/2)}(z,y)\partial^{1/2}v(x-z)\,dz\,m_\varepsilon(y)\frac{dy}{|y|^{d+1}} \\ &\quad= \int\int k^{(1/2)}(z,y)\big[\partial^{1/2}v(x-z) - \partial^{1/2}v(x)\big]\,dz\,m_\varepsilon(y)\frac{dy}{|y|^{d+1}}\end{aligned}$$

(see Lemma 2), it follows that

$$\mathcal{L}u(x) = \lim_{\varepsilon\to 0}\int\int k^{(1/2)}(z,y)m_\varepsilon(y)\frac{dy}{|y|^{d+1}}\big[\partial^{1/2}u(x-z) - \partial^{1/2}u(x)\big]\,dz.$$

Obviously,

$$\int k^{(1/2)}(z,y)m_\varepsilon(y)\frac{dy}{|y|^{d+1}} = \frac{1}{|z|^{d+\frac{1}{2}}}\int\left(\frac{1}{|\hat{z}+y|^{d-\frac{1}{2}}} - 1\right)m_\varepsilon\big(|z|y\big)\frac{dy}{|y|^{d+1}} = \frac{1}{|z|^{d+\frac{1}{2}}}M_\varepsilon(z),$$

where $\hat{z} = z/|z|$. For $\varepsilon \in (0, 1/2)$, we have $|M_\varepsilon(z)| \leqslant CK$ and $\lim_{\varepsilon \to 0} M_\varepsilon(z) = M(z)$, $z \in \mathbf{R}^d$, where

$$M(z) = \int_{\{|y| \leqslant \frac{1}{2}\}} \left( \frac{1}{|\hat{z} + y|^{d-\frac{1}{2}}} - 1 + \left(d - \frac{1}{2}\right)(\hat{z}, y) \right) m\big(|z|y\big) \frac{dy}{|y|^{d+1}} + \int_{\{|y| > \frac{1}{2}\}} \left( \frac{1}{|\hat{z} + y|^{d-\frac{1}{2}}} - 1 \right) m\big(|z|y\big) \frac{dy}{|y|^{d+1}}.$$

Therefore, for $\alpha = 1$,

$$\mathcal{L}v(x) = \int \big[\partial^{1/2} v(x+z) - \partial^{1/2} v(x)\big] M(-z) \frac{dz}{|z|^{d+\frac{1}{2}}}$$

with $|M(z)| \leqslant CK$, $z \in \mathbf{R}^d$ and the estimate follows from the case $\alpha = 1/2$. $\square$

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