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L^p-theory for Stokes and Navier–Stokes equations with Navier boundary condition

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Abstract

This work was intended as an attempt at studying stationary Stokes and Navier–Stokes problem with Navier boundary conditions (1.3). We wish to investigate some results of existence, uniqueness and regularity of solutions in Hilbert case and in L^p-theory.

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1. Introduction

Throughout this work, if we do not say otherwise, we assume that Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ of class $C^{2,1}$. We consider the stationary Stokes equations:

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega \tag{1.1}$$

and the stationary Navier–Stokes equations:

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$$-\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

where $\boldsymbol{u}$ denotes a velocity, π a pressure and $\boldsymbol{f}$ are the external forces.

To study these problems, it is necessary to add some boundary conditions. Note that these systems are often studied with Dirichlet boundary condition, also called no-slip boundary condition, which is applicable in the case where the boundary of the flow is solid. Yet, in the physical applications, we are often encounter situations where this condition does not quite feasible. In this case, it is really important to introduce another boundary conditions to describe the behavior of fluid on the wall. For example, when a part of flow boundary is the air, it is convenient to use a slip boundary condition.

In the literature, in 1827, Navier [10] was the first to propose a Navier with friction boundary condition, in which there is the stagnant layer of fluid close to the wall allowing a fluid to slip, and the tangential component of the strain tensor should be proportional to the tangential component of the fluid velocity on the boundary:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0}, \tag{1.3}$$

where $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top})$ denotes the deformation tensor associated with the velocity field $\boldsymbol{u}$, α the scalar friction function, $\boldsymbol{n}$ the exterior unit normal and $\boldsymbol{\tau}$ the corresponding tangent vector. Systems (1.1) and (1.2) with (1.3) have been studied by many authors. Note that, the first paper is due to V.A. Solonnikov and V.E. Ščadilov [12] in 1973 without friction function ($\alpha = 0$) in the Hilbert case and only for external force $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. We also refer to the paper of H.B. da Veiga [5]. We can cite the work of Clopeau, Mikelic and Robert in two dimension [8] or the paper of Verfürth [13], who has studied the mathematical formulation of Newtonian fluid flow with slip boundary condition. In the case of special boundary, such that periodic rough boundary we can mention the works of D. Bucur et al. [6] and the paper of J. Casado-Díaz et al. [7].

The purpose of this work is to study some results of existence, uniqueness and regularity of solution for the stationary Stokes problem (1.1) and also Navier–Stokes problem (1.2) with the boundary condition (1.3) with $\alpha = 0$.

To study the Stokes problem in the Hilbert case we use the Lax–Milgram theorem. For the case of $\boldsymbol{L}^p$ theory, $1 < p < \infty$, we prove the existence and uniqueness of weak solution by duality arguments. The regularity results are obtained by exploiting the relationship between slip-Navier boundary condition and the following condition:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}, \tag{1.4}$$

to prove the regularity. To study Navier–Stokes equation, we use Galerkin method and some compactness results.

The outline of this paper is as follows: In Section 2, we will recall some preliminaries results, introduce an important frameworks and announce a crucial result which gives the relationship between (1.3) and (1.4) in the weak sense on the conventional spaces.

In Section 3, we investigate the existence and uniqueness of weak solution for system (1.1), (1.3). The main idea is to begin by proving the existence and uniqueness in the Hilbert ($p = 2$) case for Ω only of class $C^{1,1}$ then generalizing this result in $\boldsymbol{L}^p$-theory for $1 < p < \infty$ for Ω of class $C^{2,1}$.

In Section 4, we establish the existence and uniqueness of strong solution when we impose more regularity to data.

The purpose of Section 5 is to proof the existence of another class of solutions called very weak solution for less regular data. The idea is based on using the regularity of solution and a duality argument.

In Section 6, we prove the existence of a similar basis of eigenfunctions of Stokes operator to that given by T. Clopeau et al. in [8].

The purpose of Section 7 is to study the Navier–Stokes problem with Navier boundary condition in the Hilbert case. The idea here is to use the Galerkin method and the Brouwer theorem.

2. Preliminaries and functionals spaces

In this section we review such basic notations and functional frameworks. Let us first recall some elementary properties. We note that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta \boldsymbol{v} = 2\,\mathrm{div}\,\mathbf{D}(\boldsymbol{v}) - \mathbf{grad}(\mathrm{div}\,\boldsymbol{v}) \tag{2.1}$$

or by

$$\Delta \boldsymbol{v} = \mathbf{grad}(\mathrm{div}\,\boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}. \tag{2.2}$$

Second, we define the following spaces: For all $1 \leqslant p < \infty$,

$$\boldsymbol{L}^p_0(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \int_\Omega \boldsymbol{v}\, d\boldsymbol{x} = 0 \right\},$$

is equipped with the norm of $\boldsymbol{L}^p(\Omega)$,

$$\boldsymbol{H}^p(\mathrm{div}, \Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} \in L^p(\Omega) \right\},$$

which is equipped with the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathrm{div},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

and

$$\boldsymbol{H}^p(\mathbf{curl}, \Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \right\},$$

which is equipped with the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathbf{curl},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} \right)^{\frac{1}{p}}.$$

According to [3], the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense both in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ and $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$. The closure of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ and in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ is denoted respectively by $\boldsymbol{H}^p_0(\mathrm{div}, \Omega)$ and $\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)$ and can be characterized respectively by

$$\boldsymbol{H}_0^p(\mathrm{div},\Omega)=\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\Gamma\},$$
$$\boldsymbol{H}_0^p(\mathbf{curl},\Omega)=\{\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n}=0\text{ on }\Gamma\}.$$

Let now introduce some notation to describe a boundary. Let us consider any point P on Γ and choose an open neighborhood W of P in Γ small enough to allow the existence of 2 families of $\mathcal{C}^2$ curves on W with these properties: a curve of each family passes through every point of W and the unit tangent vectors to these curves form an orthonormal system (which we assume to have the direct orientation) at every point of W. The lengths s_1, s_2 along each family of curves, respectively, are a possible system of coordinates in W. We denote by $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ the unit tangent vectors to each family of curves, respectively.

With this notation, we have $\boldsymbol{v}=\sum_{k=1}^2 v_k\boldsymbol{\tau}_k+(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ where $\boldsymbol{\tau}_k^T=(\tau_{k1},\tau_{k2},\tau_{k3})$ and $v_k=\boldsymbol{v}\cdot\boldsymbol{\tau}_k$.

In the sequel, for simplicity of notation, we write

$$\boldsymbol{\Lambda}\boldsymbol{w}=\sum_{k=1}^{2}\left(\boldsymbol{w}_{\boldsymbol{\tau}}\cdot\frac{\partial\boldsymbol{n}}{\partial s_k}\right)\boldsymbol{\tau}_k. \tag{2.3}$$

In this paper we need a relationship between slip-Navier boundary conditions (1.3) and (1.4), for this reason we state the following result, where the proof is given in Appendix A.

Lemma 2.1. *For any $\boldsymbol{v}\in\boldsymbol{W}^{2,p}(\Omega)$, we have the following equalities*:

$$\left[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}=\nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n})+\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\right)_{\boldsymbol{\tau}}-\boldsymbol{\Lambda}\boldsymbol{v}, \tag{2.4}$$
$$\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}=\nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n})-\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\right)_{\boldsymbol{\tau}}-\boldsymbol{\Lambda}\boldsymbol{v}. \tag{2.5}$$

Remark 2.2. In the particular case $\boldsymbol{v}\cdot\boldsymbol{n}=0$, we have the following equality: For all $\boldsymbol{v}\in\boldsymbol{W}^{2,p}(\Omega)$,

$$\left[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}=\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\right)_{\boldsymbol{\tau}}-\boldsymbol{\Lambda}\boldsymbol{v},$$
$$\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}=-\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\right)_{\boldsymbol{\tau}}-\boldsymbol{\Lambda}\boldsymbol{v},$$

which implies that

$$\left[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}=-\,\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}-2\boldsymbol{\Lambda}\boldsymbol{v}\quad\text{in }\boldsymbol{W}^{\frac{1}{p},p}(\Gamma). \tag{2.6}$$

Note that we can obtain the equality (2.6) in weak sense. To do this, we must give a sense to $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ and to $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$ when $\boldsymbol{v}$ belongs to an appropriate space. We need the following space:

$$\boldsymbol{V}^p(\Omega)=\{\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega);\ \mathrm{div}\,\boldsymbol{v}=0\text{ in }\Omega,\ \boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\Gamma\},$$

with the norm of $\boldsymbol{W}^{1,p}(\Omega)$ and

$$\boldsymbol{E}^p(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta \boldsymbol{v} \in \left[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)\right]'\right\},$$

where $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'$ is the dual space of $\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)$. $\boldsymbol{E}^p(\Omega)$ is equipped with the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

We note that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{E}^p(\Omega)$ (see [11, Lemma 4.2.1]). We need also the following two lemmas. To prove the first one we refer to [11, Corollary 4.2.2].

Lemma 2.3. *We suppose that Ω is of class $C^{1,1}$. The linear mapping $\gamma : \boldsymbol{v} \to \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear and continuous mapping*

$$\gamma : \boldsymbol{E}^p(\Omega) \to \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: For any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$,

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_\Omega = \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, d\boldsymbol{x} - \langle \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma, \tag{2.7}$$

where $\langle .,.\rangle_\Gamma$ denotes the duality between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ and $\langle .,.\rangle_\Omega$ denotes the duality between $(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$ and $\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)$.

Lemma 2.4. *Suppose that Ω is of class $C^{1,1}$. The linear mapping $\Theta : \boldsymbol{v} \to [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping*

$$\Theta : \boldsymbol{E}^p(\Omega) \to \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: For any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$,

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_\Omega = 2\int_\Omega \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} - 2\left\langle\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\right\rangle_\Gamma. \tag{2.8}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ with $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on Γ, then, by using (2.1), we have

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_\Omega = 2\int_\Omega \mathbf{D}(\boldsymbol{v}) : \nabla \boldsymbol{\varphi}\, d\boldsymbol{x} - \left\langle 2\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\right\rangle_\Gamma - \int_\Omega \mathrm{div}\, \boldsymbol{v}\, \mathrm{div}\, \boldsymbol{\varphi}\, d\boldsymbol{x}.$$

Then, for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ and for any $\boldsymbol{\varphi}$ in $\boldsymbol{V}^{p'}(\Omega)$ we have

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_\Omega = 2\int_\Omega \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} - \left\langle 2\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\right\rangle_\Gamma. \tag{2.9}$$

Now, let $\boldsymbol{\mu}$ be any element of $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$, then there exists an element $\boldsymbol{\varphi}$ in $\boldsymbol{W}^{1,p'}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ with

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leqslant C\|\boldsymbol{\mu}_{\boldsymbol{\tau}}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} \leqslant C\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}. \tag{2.10}$$

Consequently,

$$\begin{aligned}
\left|\left\langle 2\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\mu}\right\rangle_\Gamma\right| &= \left|\left\langle 2\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\mu}_{\boldsymbol{\tau}}\right\rangle_\Gamma\right| \\
&= \left|\left\langle 2\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\right\rangle_\Gamma\right| \\
&\leqslant \|\Delta\boldsymbol{v}\|_{(\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega))'}\|\boldsymbol{\varphi}\|_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} + 2\left\|\mathbf{D}(\boldsymbol{v})\right\|_{\boldsymbol{L}^p(\Omega)}\left\|\mathbf{D}(\boldsymbol{\varphi})\right\|_{\boldsymbol{L}^{p'}(\Omega)},
\end{aligned}$$

$$\begin{aligned}
\left|\left\langle 2\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\mu}\right\rangle_\Gamma\right| &\leqslant \left(\|\Delta\boldsymbol{v}\|^p_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'} + 2^{p/2}\left\|\mathbf{D}(\boldsymbol{v})\right\|^p_{\boldsymbol{L}^p(\Omega)}\right)^{\frac{1}{p}} \\
&\quad \times \left(\|\boldsymbol{\varphi}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} + 2^{p'/2}\left\|\mathbf{D}(\boldsymbol{\varphi})\right\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}\right)^{\frac{1}{p'}}.
\end{aligned} \tag{2.11}$$

It follows that from Korn's inequality, we have

$$\left|\left\langle\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\mu}\right\rangle_\Gamma\right| \leqslant C_p\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)}.$$

Thus, by using (2.10), we deduce that

$$\left\|\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}\right\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)}.$$

Therefore, the linear mapping $\Theta : \boldsymbol{v} \to [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is continuous for the norm of $\boldsymbol{E}^p(\Omega)$. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{E}^p(\Omega)$, Θ can be extended by continuity to a mapping still called $\Theta \in \mathfrak{L}(\boldsymbol{E}^p(\Omega), \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma))$ and formula (2.9) holds for all $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$. □

Owing the previous result, it is possible to extend (2.6) in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and we have the following corollary.

Corollary 2.5. *For any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ such that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, we have*

$$\left[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = -\operatorname{\mathbf{curl}}\boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{in } \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma). \tag{2.12}$$

Remark 2.6. Note that, if Ω is of class $C^{2,1}$, then the slip-Navier boundary condition (1.3) differs from (1.4) only by the term $-2\boldsymbol{\Lambda}\boldsymbol{v}$. This term is equal to zero in the case of the flat boundary, consequently we have (1.3) and (1.4) are identical, in this context we cite the paper of H.B. da Veiga et al. [5].

In the sequel, we need the result of existence and uniqueness for the following intermediate problem:

$$(\mathcal{S}_1)\quad \begin{cases} -\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g & \text{on } \Gamma, \\ \operatorname{\mathbf{curl}} \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h} & \text{on } \Gamma. \end{cases}$$

We introduce the following spaces:

$$\boldsymbol{K}_N^p(\Omega) = \big\{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v} = 0,\ \operatorname{\mathbf{curl}}\boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\big\},$$
$$\boldsymbol{K}_T^p(\Omega) = \big\{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v} = 0,\ \operatorname{\mathbf{curl}}\boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\big\}.$$

We now recall a result of existence and uniqueness of weak solution for the problem $(\mathcal{S}_1)$ (see [2] and [11]):

Theorem 2.7. *Suppose that Ω is of class $C^{1,1}$. Let*

$$\boldsymbol{f}\in\big[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\big]',\qquad \chi\in \boldsymbol{L}^p(\Omega),\qquad g\in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma),\qquad \boldsymbol{h}\times\boldsymbol{n}\in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

and verifying the following compatibility conditions: For any $\boldsymbol{\varphi}\in \boldsymbol{K}_T^{p'}(\Omega)$,

$$\langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h},\boldsymbol{\varphi}\rangle_\Gamma = 0, \tag{2.13}$$
$$\int_\Omega \chi\, d\boldsymbol{x} = \int_\Gamma g\, d\boldsymbol{\sigma}. \tag{2.14}$$

Then, Stokes problem $(\mathcal{S}_1)$ has a unique solution $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p}(\Omega)\times \boldsymbol{L}^p(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{\boldsymbol{L}^p(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\chi\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

We need also the following result (we refer to [11]):

Theorem 2.8. *Let $\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\operatorname{\mathbf{curl}},\Omega)]'$ with $\operatorname{div}\boldsymbol{f} = 0$ in Ω satisfying the following compatibility condition:*

$$\forall \boldsymbol{v}\in \boldsymbol{K}_N^{p'}(\Omega),\quad \langle \boldsymbol{f},\boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{p'}(\operatorname{\mathbf{curl}},\Omega)]'\times \boldsymbol{H}_0^{p'}(\operatorname{\mathbf{curl}},\Omega)} = 0, \tag{2.15}$$

and $\boldsymbol{h}$ be such that $\boldsymbol{h}\times\boldsymbol{n}\in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Then the following problem

$$\begin{cases} -\Delta\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma, \end{cases}$$

has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}_N^p(\Omega)$ *and we have*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}_N^p(\Omega)} \leqslant C\big(\|f\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big).$$

3. Weak solutions of $(\mathcal{S}_T)$

In this paper we are interested in the following problem:

$$(\mathcal{S}_T)\quad \begin{cases} -\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u}=\chi & \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n}=g & \text{on } \Gamma,\\ 2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}=\boldsymbol{h} & \text{on } \Gamma.\end{cases}$$

The aim of this section is to study the existence and uniqueness of weak solution for problem $(\mathcal{S}_T)$. First, we consider the Hilbert case. Then, we will study the case of $\boldsymbol{L}^p$-theory, $1<p<\infty$. We start by the following proposition:

Proposition 3.1. *We suppose that* $\chi=0$ *and* $g=0$. *Let* $\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$, $\boldsymbol{h}\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ *such that*

$$\boldsymbol{h}\cdot\boldsymbol{n}=0\quad \text{on } \Gamma. \tag{3.1}$$

The problem: Find that $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ *satisfying* $(\mathcal{S}_T)$*, in the distribution sense, is equivalent to*

$$\begin{cases} \text{Find } \boldsymbol{u}\in\boldsymbol{V}^p(\Omega) \text{ such that,}\\ \forall\boldsymbol{\varphi}\in\boldsymbol{V}^{p'}(\Omega),\quad 2\displaystyle\int_\Omega \mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega+\langle\boldsymbol{h},\boldsymbol{\varphi}\rangle_\Gamma.\end{cases} \tag{3.2}$$

Proof. First, we note that if $\boldsymbol{h}\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, then we have $\boldsymbol{h}\cdot\boldsymbol{n}\in W^{-\frac{1}{p},p}(\Gamma)$. That means that the relation (3.1) holds in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. In fact, let a be in $W^{\frac{1}{p},p'}(\Gamma)$, or Ω is of class $C^{1,1}$, then $a\boldsymbol{n}\in\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ since $\boldsymbol{n}\in\boldsymbol{W}^{1,\infty}(\Gamma)$. The regularity $W^{-\frac{1}{p},p}(\Gamma)$ of $\boldsymbol{h}\cdot\boldsymbol{n}$ is then consequence of the relation

$$\langle\boldsymbol{h}\cdot\boldsymbol{n},a\rangle_\Gamma=\langle\boldsymbol{h},a\boldsymbol{n}\rangle_\Gamma.$$

Now, using Green formula (2.8), we deduce that every solution of $(\mathcal{S}_T)$ also solves (3.2). Conversely, let $\boldsymbol{u}$ be a solution of the problem (3.2). Let us take a function $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi}=0$ as a test function in (3.2). Then we have

$$2\int_\Omega \mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}=\langle-\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}.$$

As a consequence,

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \quad \langle -\Delta \boldsymbol{u} - \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

So, by the De Rham theorem, there exists a distribution π in $\mathcal{D}'(\Omega)$ defined uniquely up to an additive constant such that

$$-\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f}. \tag{3.3}$$

As $\nabla\pi \in [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]' \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$, we deduce that $\pi \in L^p(\Omega)$ (we refer to [4]). Moreover, by the fact that $\boldsymbol{u}$ belongs to the space $\boldsymbol{V}^p(\Omega)$, we have $\mathrm{div}\, \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ. It remains to prove the Navier boundary condition $2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h}$ on Γ. We multiply Eq. (3.3) by $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$ and we integrate on Ω:

$$2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} - 2\big\langle \big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\big\rangle_\Gamma = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega. \tag{3.4}$$

Using (3.2) and (3.4), we deduce that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega), \quad \big\langle 2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\big\rangle_\Gamma = \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma.$$

Let now $\boldsymbol{\mu}$ be any element of the space $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)$. So, there exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that $\mathrm{div}\, \boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ. Its clear that $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$ and

$$\begin{aligned}
\big\langle 2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu}\big\rangle_\Gamma &= \big\langle 2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu}_{\boldsymbol{\tau}}\big\rangle_\Gamma \\
&= \big\langle 2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\varphi}\big\rangle_\Gamma \\
&= 0.
\end{aligned}$$

This implies that

$$2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}} = \boldsymbol{h} \quad \text{on } \Gamma,$$

it is that end of the proof. □

First, we denote by $\boldsymbol{a}$ the bilinear form defined on $\boldsymbol{V}^p(\Omega) \times \boldsymbol{V}^{p'}(\Omega)$ by

$$\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{\varphi}) = \int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x}. \tag{3.5}$$

We introduce the kernel $\boldsymbol{\mathcal{T}}(\Omega)$ for any $1 < p < \infty$:

$$\boldsymbol{\mathcal{T}}^p(\Omega) = \big\{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \mathbf{D}(\boldsymbol{v}) = \mathbf{0} \text{ in } \Omega,\ \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\big\}.$$

Observe that, if Ω is obtained by rotation around a constant vector $\boldsymbol{b}$ of $\mathbb{R}^3$, then

$$\boldsymbol{\mathcal{T}}^p(\Omega) = \mathrm{Span}\{\boldsymbol{\beta}\},$$

where

$$\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{b} \times \boldsymbol{x} \quad \text{for } \boldsymbol{x} \in \Omega.$$

Else, the kernel $\boldsymbol{\mathcal{T}}^p(\Omega)$ is equal to zero (see [14] for more details). We denote

$$\boldsymbol{\mathcal{N}}^p(\Omega) = \{(\boldsymbol{u}, c);\ \boldsymbol{u} \in \boldsymbol{\mathcal{T}}^p(\Omega),\ c \in \mathbb{R}\}.$$

Since $\boldsymbol{\mathcal{T}}^p(\Omega)$ and $\boldsymbol{\mathcal{N}}^p(\Omega)$ do not depend on p, we note $\boldsymbol{\mathcal{T}}(\Omega)$ instead of $\boldsymbol{\mathcal{T}}^p(\Omega)$ and $\boldsymbol{\mathcal{N}}(\Omega)$ instead $\boldsymbol{\mathcal{N}}^p(\Omega)$.

Remark 3.2. By this characterization, we deduce the following compatibility condition:

$$\langle \boldsymbol{f}, \boldsymbol{\beta}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\beta}\rangle_\Gamma = 0$$

(with $\boldsymbol{h} \cdot \boldsymbol{n} = 0$) that is necessary to solve problem (3.2) and then also to solve $(\mathcal{S}_T)$ with $\chi = 0$ and $g = 0$.

3.1. The case $p = 2$

In this section we prove the existence and uniqueness of weak solution for problem $(\mathcal{S}_T)$ in the Hilbert case. Our proof is based on the use of Lax–Milgram theorem. Before that, we introduce the following spaces:

$$X(\Omega) = \boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega). \tag{3.6}$$

To prove the coercivity of the bilinear form $\boldsymbol{a}$, we are now going to give the proof of the following lemma which was cited in [13] without proof.

Lemma 3.3 *(Poincaré–Morrey inequality). Let Ω be a Lipschitz bounded domain. Then, we have*

$$\inf_{\boldsymbol{v} \in \boldsymbol{\mathcal{T}}(\Omega)} \|\boldsymbol{u} + \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \leqslant C\left(\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u} \cdot \boldsymbol{n}|^2 \, d\boldsymbol{\sigma}\right), \quad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \tag{3.7}$$

where the constant C only depends on Ω. In particular, the semi-norm $\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}$ is a norm which is equivalent to the norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ if $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ and $\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta} \, d\boldsymbol{x} = 0$.

Proof. Let us P denote the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{\mathcal{T}}(\Omega)$ with the product scalar of $\boldsymbol{L}^2(\Omega)$. Then

$$\inf_{\boldsymbol{\rho} \in \boldsymbol{\mathcal{T}}(\Omega)} \|\boldsymbol{u} + \boldsymbol{\rho}\|^2_{\boldsymbol{L}^2(\Omega)} = \|\boldsymbol{u} - P\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Therefore, it suffices to show that, there exits $C > 0$ such that

$$\|\boldsymbol{u} - P\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leqslant C\left(\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u}\cdot\boldsymbol{n}|^2\,d\boldsymbol{\sigma}\right) \quad \text{for any } \boldsymbol{u} \in \boldsymbol{H}^1(\Omega). \tag{3.8}$$

To establish the existence of such constant, assume the contrary. Then, for any $k \geqslant 1$, there exists $\boldsymbol{u}_k$, such that

$$\|\boldsymbol{u}_k - P\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)} > k\left(\|\mathbf{D}(\boldsymbol{u}_k)\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u}_k\cdot\boldsymbol{n}|^2\,d\boldsymbol{\sigma}\right).$$

We can suppose that

$$\|\boldsymbol{u}_k - P\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)} = 1.$$

As consequence,

$$\frac{1}{k} > \|\mathbf{D}(\boldsymbol{u}_k)\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u}_k\cdot\boldsymbol{n}|^2\,d\boldsymbol{\sigma}, \quad \forall k \in \mathbb{N}^*.$$

Setting $\boldsymbol{w}_k = \boldsymbol{u}_k - P\boldsymbol{u}_k$ and using the Korn inequality, we deduce that $\boldsymbol{w}_k$ is bounded in $\boldsymbol{H}^1(\Omega)$. Then, by Rellich theorem, there exists a subsequence still called $\boldsymbol{w}_k$ that converges to $\boldsymbol{w}$ in $\boldsymbol{L}^2(\Omega)$ and weakly in $\boldsymbol{H}^1(\Omega)$. Since $\|\mathbf{D}(\boldsymbol{w})\|_{\boldsymbol{L}^2(\Omega)} = 0$ and $\boldsymbol{w}\cdot\boldsymbol{n} = 0$, we deduce that $\boldsymbol{w}$ belongs to $\boldsymbol{\mathcal{T}}(\Omega)$. Moreover, we have $\boldsymbol{w} \in \boldsymbol{\mathcal{T}}(\Omega)^\perp$, where $\boldsymbol{\mathcal{T}}(\Omega)^\perp$ is the orthogonal complement of $\boldsymbol{\mathcal{T}}(\Omega)$ in $\boldsymbol{L}^2(\Omega)$. Hence $\boldsymbol{w} = \boldsymbol{0}$, in contradiction with the relation $\|\boldsymbol{w}_k\|_{\boldsymbol{L}^2(\Omega)} = 1$, $\forall k \geqslant 1$. The proof of lemma is completed. □

Now, we can solve the Stokes problem.

Theorem 3.4. *Suppose that $\chi = 0$ and $g = 0$. Let $\boldsymbol{f} \in [\boldsymbol{H}^2_0(\mathrm{div},\Omega)]'$ and $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, satisfying (3.1) and*

$$\langle \boldsymbol{f}, \boldsymbol{\beta}\rangle_{[\boldsymbol{H}^2_0(\mathrm{div},\Omega)]'\times \boldsymbol{H}^2_0(\mathrm{div},\Omega)} + \langle \boldsymbol{h}, \boldsymbol{\beta}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)} = 0. \tag{3.9}$$

Then, Stokes problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u},\pi) \in (\boldsymbol{H}^1(\Omega)\times L^2(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$. In addition, we have the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^2_0(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big). \tag{3.10}$$

Proof. It is clear that the bilinear form $\boldsymbol{a}(.,.)$ given by (3.5) is continuous on $\boldsymbol{H}^1(\Omega)$ and using Poincaré–Morrey inequality (3.7) we deduce that it is also coercive on $\boldsymbol{X}(\Omega)$. Moreover, we have the linear form $\ell : \boldsymbol{X}(\Omega) \to \mathbb{R}$, which is defined by

$$\ell(\boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{H}^2_0(\mathrm{div},\Omega)]'\times \boldsymbol{H}^2_0(\mathrm{div},\Omega)} + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}$$

is continuous over $\boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)$. Thus, we deduce by Lax–Milgram's theorem that problem (3.2) given in Proposition 3.1 has a unique solution $\boldsymbol{u} \in \boldsymbol{X}(\Omega)$ and $\pi \in L^2(\Omega)/\mathbb{R}$.

Using the variational problem and Poincaré–Morrey inequality, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{X}(\Omega)} \leqslant C_0\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big). \tag{3.11}$$

On the other hand,

$$\|\nabla\pi\|_{\boldsymbol{H}^{-1}(\Omega)} \leqslant \|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)}.$$

We know that

$$\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} \leqslant C_1\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'}$$

and

$$\|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.$$

Therefore,

$$\inf_{k\in\mathbb{R}}\|\pi + k\|_{L^2(\Omega)} \leqslant C_2\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big). \tag{3.12}$$

The estimation (3.10) follows easily from (3.11) and (3.12). □

We can also solve the Stokes problem when the divergence does not vanish and we have the following corollary the proof of which is given later in Corollary 3.8.

Corollary 3.5. *Let $\boldsymbol{f}$, χ, g and $\boldsymbol{h}$ be such that*

$$\boldsymbol{f} \in \big[\boldsymbol{H}_0^2(\mathrm{div},\Omega)\big]', \qquad \chi \in L^2(\Omega), \qquad g \in H^{\frac{1}{2}}(\Omega) \quad \text{and} \quad \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma),$$

satisfying the compatibility conditions (3.1), (3.9) and

$$\int_\Omega \chi\, d\boldsymbol{x} = \int_\Gamma g\, d\boldsymbol{\sigma}. \tag{3.13}$$

Then, problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u},\pi) \in (\boldsymbol{H}^1(\Omega) \times L^2(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$. Moreover we have the following estimate:

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \\ &\quad\leqslant C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} + \|\chi\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big).\end{aligned} \tag{3.14}$$

Remark 3.6. In the case $g = 0$, we can write problem $(\mathcal{S}_T)$ as follows:

$$(\mathcal{S}'_T) \quad \begin{cases} -\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = -2\boldsymbol{\Lambda}\boldsymbol{u} - \boldsymbol{h} & \text{on } \Gamma. \end{cases}$$

3.2. The general case $1 < p < \infty$

We now investigate the case $1 < p < \infty$. We start by showing the existence and uniqueness of weak solution for $(\mathcal{S}_T)$. We start by studying the case $p \geqslant 2$.

Theorem 3.7. *Assume that* $p \geqslant 2$. *Suppose that* $\chi = 0$ *and* $g = 0$. *Then, for any* $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega))'$ *and* $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ *satisfying* (3.1) *the following compatibility conditions are satisfied*

$$\langle \boldsymbol{f}, \boldsymbol{\beta}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\beta}\rangle_\Gamma = 0, \tag{3.15}$$

where $\langle .,.\rangle_\Gamma = \langle .,.\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}$ *and* $\langle .,.\rangle_\Omega = \langle .,.\rangle_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'\times \boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)}$.

Then, problem $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$. *In addition, we have the following estimate*:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

Proof. We know that for all $p \geqslant 2$ we have

$$\big(\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)\big)' \hookrightarrow \big(\boldsymbol{H}_0^{2}(\operatorname{div}, \Omega)\big)' \quad \text{and} \quad \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma).$$

Then according to the Hilbert case the problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi) \in (\boldsymbol{H}^1(\Omega) \times L^2(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$. Applying Corollary 2.5, we have

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} + 2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}} = -2\boldsymbol{\Lambda}\boldsymbol{u} \quad \text{on } \boldsymbol{H}^{-\frac{1}{2}}(\Gamma),$$

because $\boldsymbol{u} \in \boldsymbol{E}^2(\Omega)$ and Ω is of class $C^{2,1}$. As consequence, $(\boldsymbol{u}, \pi)$ is solution of $(\mathcal{S}'_T)$. Therefore, $(\boldsymbol{u}, \pi)$ verifying the following problem: For all $\boldsymbol{\varphi} \in \boldsymbol{V}^2(\Omega)$,

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{H}_0^2(\operatorname{div},\Omega))'\times \boldsymbol{H}_0^2(\operatorname{div},\Omega)} + \langle 2\boldsymbol{\Lambda}\boldsymbol{u} + \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

In particular, we have

$$\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{H}_0^2(\operatorname{div},\Omega))'\times \boldsymbol{H}_0^2(\operatorname{div},\Omega)} + \langle 2\boldsymbol{\Lambda}\boldsymbol{u} + \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)} = 0 \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{K}_T^2(\Omega).$$

Or more generally, for all $p \geqslant 2$ and $\boldsymbol{\varphi} \in \boldsymbol{K}_T^{p'}(\Omega)$

$$\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega))'\times \boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} + \langle 2\boldsymbol{\Lambda}\boldsymbol{u} + \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma = 0. \tag{3.16}$$

In the other hand, we have $\boldsymbol{u}_\tau \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{6},6}(\Gamma)$. Therefore, by (3.16) and Theorem 2.7, we have $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)$, for $2 \leqslant p \leqslant 6$. Now, suppose that $p \geqslant 6$, then repeated application of Theorem 2.7 enables us to assume that $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{6},6}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ for all $p \geqslant 6$. Clearly, $-2\boldsymbol{\Lambda u} - \boldsymbol{h}$ belongs to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Consequently, by the same reasoning, we deduce that the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)$. $\square$

We can also solve the Stokes problem when the divergence does not vanish.

Corollary 3.8. *Let $p \geqslant 2$. Let $\boldsymbol{f}$, χ, g and $\boldsymbol{h}$ such that*

$$\boldsymbol{f} \in \big[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)\big]', \qquad \chi \in \boldsymbol{L}^p(\Omega), \qquad g \in W^{1-\frac{1}{p},p}(\Gamma) \quad \textit{and} \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma),$$

satisfying (3.1), (3.13) and (3.15). Then, Stokes problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$. Moreover we have the following estimate:

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \\ &\quad \leqslant C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\chi\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big). \end{aligned}$$

Proof. We solve the following Neumann problem:

$$\Delta\theta = \chi \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial\theta}{\partial\boldsymbol{n}} = g \quad \text{on } \Gamma. \tag{3.17}$$

For $\chi \in \boldsymbol{L}^p(\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\Gamma)$, problem (3.17) has a unique solution $\theta \in W^{2,p}(\Omega)/\mathbb{R}$ satisfying the following estimate:

$$\|\theta\|_{W^{2,p}(\Omega)/\mathbb{R}} \leqslant C\big(\|\chi\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Omega)}\big).$$

Setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$, then $(\mathcal{S}_T)$ becomes: Find $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)$ solution of problem

$$\begin{cases} -\Delta\boldsymbol{z} + \nabla\pi = \boldsymbol{f} + \nabla\chi & \text{in } \Omega, \\ \mathrm{div}\, \boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ 2\big[\mathbf{D}(\boldsymbol{z})\boldsymbol{n}\big]_{\boldsymbol{\tau}} = \boldsymbol{H} & \text{on } \Gamma \end{cases} \tag{3.18}$$

with $\boldsymbol{H} = \boldsymbol{h} - 2[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}}$. Observe that $\boldsymbol{f} + \nabla\chi$ belongs to $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'$ and $\boldsymbol{H}$ belongs to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and satisfies $\boldsymbol{H} \cdot \boldsymbol{n} = 0$. Using Remark 3.2, to prove that problem (3.18) has a solution, it is necessary to prove the following compatibility condition:

$$\langle \boldsymbol{f} + \nabla\chi, \boldsymbol{\beta}\rangle_\Omega + \langle \boldsymbol{H}, \boldsymbol{\beta}\rangle_\Gamma = 0, \tag{3.19}$$

which is equivalent by (3.15) to the condition

$$\langle \nabla \chi, \boldsymbol{\beta} \rangle_\Omega - \big\langle 2\big[\mathbf{D}(\nabla\theta)\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\beta} \big\rangle_\Gamma = 0.$$

In fact, it is clear that $\nabla\theta$ belongs to $\boldsymbol{E}^p(\Omega)$. Then applying Green formula (2.8) with $\boldsymbol{v} = \nabla\theta$ and $\boldsymbol{\varphi} = \boldsymbol{\beta}$ we obtain

$$\begin{aligned} -\big\langle 2\big[\mathbf{D}(\nabla\theta)\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\beta} \big\rangle_\Gamma &= -\big\langle \Delta(\nabla\theta), \boldsymbol{\beta} \big\rangle_\Omega \\ &= -\langle \nabla\chi, \boldsymbol{\beta} \rangle_\Omega. \end{aligned}$$

Thus, due to Theorem 3.7, problem (3.18) has a unique solution $(z, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\boldsymbol{\mathcal{N}}(\Omega)$. $\square$

The following theorem will be proved by duality argument.

Theorem 3.9. *Assume that* $1 < p < 2$. *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$, $\chi \in \boldsymbol{L}^p(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ *and* $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, *satisfying the compatibility conditions* (3.1), (3.13) *and* (3.15). *Then, problem* $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$. *In addition, we have the following estimate*:

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \\ &\quad \leqslant C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big). \end{aligned}$$

Proof. The proof will be divided into two steps.

First step: We suppose that $g = 0$. Using Green formula (2.8), we deduce that problem $(\mathcal{S}_T)$ has the following equivalent variational formulation: Find $(\boldsymbol{u}, \pi)$ in $(\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$ satisfying $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ such that: $\forall \eta \in L^{p'}(\Omega)$, $\forall \boldsymbol{w} \in \boldsymbol{E}^{p'}(\Omega)$ satisfying $\boldsymbol{w} \cdot \boldsymbol{n} = 0$ and $[\boldsymbol{D}(\boldsymbol{w})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0}$ on Γ

$$\begin{aligned} &\langle \boldsymbol{u}, -\Delta\boldsymbol{w} + \nabla\eta \rangle_{\boldsymbol{H}_0^p(\mathrm{div},\Omega)\times(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} - \int_\Omega \pi \,\mathrm{div}\,\boldsymbol{w}\,d\boldsymbol{x} \\ &\quad = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \langle \boldsymbol{h}, \boldsymbol{w} \rangle_\Gamma - \int_\Omega \chi\eta\,d\boldsymbol{x}. \end{aligned} \tag{3.20}$$

According to Corollary 3.8 for any pair $(\boldsymbol{F}, \varphi)$ in $((\boldsymbol{H}_0^p(\mathrm{div}, \Omega))' \perp \boldsymbol{\mathcal{T}}(\Omega)) \times L_0^{p'}(\Omega)$ there exists a unique solution $(\boldsymbol{w}, \eta) \in (\boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$ such that

$$-\Delta\boldsymbol{w} + \nabla\eta = \boldsymbol{F} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{w} = \varphi \quad \text{in } \Omega, \qquad \boldsymbol{w}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \big[\boldsymbol{D}(\boldsymbol{w})\boldsymbol{n}\big]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on } \Gamma$$

and

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}(\Omega)}\big(\|\boldsymbol{w}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{1,p'}(\Omega)}+\|\eta+\mu\|_{\boldsymbol{L}^{p'}(\Omega)}\big)$$
$$\leqslant C\big(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'}+\|\varphi\|_{\boldsymbol{L}^{p'}(\Omega)}\big). \tag{3.21}$$

Let T be a linear form defined from $((\boldsymbol{H}_0^p(\mathrm{div},\Omega))'\bot\mathcal{T}(\Omega))\times L_0^{p'}(\Omega)$ onto $\mathbb{R}$ by

$$T:(\boldsymbol{F},\varphi)\mapsto\langle\boldsymbol{f},\boldsymbol{w}\rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}+\langle\boldsymbol{h},\boldsymbol{w}\rangle_\Gamma-\int_\Omega\chi\eta\,d\boldsymbol{x}.$$

Note that, using (3.21), for any $(\boldsymbol{\lambda},k)\in\mathcal{N}(\Omega)$ we have

$$\big|T(\boldsymbol{F},\varphi)\big|\leqslant\Big|\langle\boldsymbol{f},\boldsymbol{w}+\boldsymbol{\lambda}\rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}+\langle\boldsymbol{h},\boldsymbol{w}+\boldsymbol{\lambda}\rangle_\Gamma-\int_\Omega\chi(\eta+k)\,d\boldsymbol{x}\Big|$$
$$\leqslant C\big(\|f\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\chi\|_{\boldsymbol{L}^{p'}(\Omega)}+\|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big)\big(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'}+\|\varphi\|_{\boldsymbol{L}^{p'}(\Omega)}\big).$$

Thus the linear form T is continuous on $((\boldsymbol{H}_0^p(\mathrm{div},\Omega))'\bot\mathcal{T}(\Omega))\times L_0^{p'}(\Omega)$ and we deduce that there exists a unique $(\boldsymbol{u},\pi)$ in $\boldsymbol{H}_0^p(\mathrm{div},\Omega)_{/\mathcal{T}(\Omega)}\times\boldsymbol{L}^p(\Omega)_{/\mathbb{R}}$ such that

$$T(\boldsymbol{F},\varphi)=\langle\boldsymbol{u},\boldsymbol{F}\rangle_{\boldsymbol{H}_0^p(\mathrm{div},\Omega)\times(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'}+\int_\Omega\pi\varphi\,d\boldsymbol{x}.$$

To finish, we shall prove that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. We recall that $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ and $\Delta\boldsymbol{u}=\nabla\pi-\boldsymbol{f}\in(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\hookrightarrow\boldsymbol{W}^{-1,p}(\Omega)$. Now, we introduce the following spaces:

$$\boldsymbol{X}_p(\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{W}_0^{1,p}(\Omega);\ \mathrm{div}\,\boldsymbol{v}\in W_0^{1,p}(\Omega)\big\}$$

and

$$\boldsymbol{T}_p(\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \Delta\boldsymbol{v}\in\big(\boldsymbol{X}_{p'}(\Omega)\big)'\big\}.$$

It is clear that $\boldsymbol{W}^{-1,p}(\Omega)\hookrightarrow(\boldsymbol{X}_{p'}(\Omega))'$ and thus we have $\boldsymbol{u}\in\boldsymbol{T}_p(\Omega)$. Therefore, according to Lemma 12 of [1], we have $\boldsymbol{u}_\tau\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and also $-2\boldsymbol{\Lambda}\boldsymbol{u}-\boldsymbol{h}$ belongs to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Finally, using Remark 3.6 and Theorem 2.7, we deduce that $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times\boldsymbol{L}^p(\Omega)$.

Second step: We solve Neumann problem (3.17) with $\chi\in\boldsymbol{L}^p(\Omega)$, $g\in\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ satisfying (3.13). There exists a unique $\theta\in W^{2,p}(\Omega)/\mathbb{R}$ solution of (3.17). Setting $\boldsymbol{z}=\boldsymbol{u}-\nabla\theta$ the rest of the proof runs as in proof of Corollary 3.8. □

Remark 3.10. We are so far interested in the boundary conditions on the stress tensor. However, it is also interesting to consider the boundary conditions on the tangential components of the normal derivative:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{and} \quad \left(\dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{3.22}$$

As in Corollary 2.5, using the relation (2.5), we can write

$$\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{u} \cdot \boldsymbol{n}) - \operatorname{\mathbf{curl}} \boldsymbol{u} \times \boldsymbol{n} - \boldsymbol{\Lambda} \boldsymbol{u} \quad \text{in } \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma). \tag{3.23}$$

As consequence, in the same way as for problem $(\mathcal{S}_T)$ we can solve the problem (3.22) and we have the following theorem:

Theorem 3.11. *Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$, $\chi \in \boldsymbol{L}^p(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, satisfying the compatibility conditions (3.1) and (3.13). Then, problem (3.22) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)/\mathbb{R}$. In addition, we have the following estimate*:

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{\boldsymbol{L}^p(\Omega)/\mathbb{R}} \\ &\quad \leqslant C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\chi\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big). \end{aligned}$$

4. Strong solutions of $(\mathcal{S}_T)$

We prove the existence of strong solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{L}^p(\Omega)$ for the Stokes problem.

Theorem 4.1. *Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $\chi \in W^{1,p}(\Omega)$, $g \in W^{2-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, satisfying the compatibility conditions (3.1), (3.13) and (3.15). Then, problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi)$ which belongs to $(\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$ and satisfies the estimate*:

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \\ &\quad \leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{2-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{\frac{1}{p},p}(\Gamma)}\big). \end{aligned} \tag{4.1}$$

Proof. Before, we note that under the hypothesis of Theorem 4.1, the problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)/\boldsymbol{\mathcal{N}}(\Omega)$.

To prove the regularity of the velocity, we set $\boldsymbol{z} = \operatorname{\mathbf{curl}} \boldsymbol{u}$. Observe that $-\Delta \boldsymbol{z} = \operatorname{\mathbf{curl}} \operatorname{\mathbf{curl}} \boldsymbol{z}$ and $\operatorname{\mathbf{curl}} \boldsymbol{z} = -\Delta \boldsymbol{u} + \nabla \chi = \boldsymbol{f} + \nabla(\chi - \pi)$. Using Remark 2.2, we deduce that $\boldsymbol{z}$ satisfies the following problem:

$$\begin{cases} -\Delta \boldsymbol{z} = \operatorname{\mathbf{curl}} \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{H} & \text{on } \Gamma, \end{cases}$$

where $\boldsymbol{H} = -2\boldsymbol{\Lambda}\boldsymbol{u} - \boldsymbol{h}$. Since, Ω is of class $\mathcal{C}^{2,1}$ and $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, we have $\boldsymbol{\Lambda}\boldsymbol{u} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Consequently, $\boldsymbol{H} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ and satisfies the compatibility condition (3.1). Since, $\operatorname{\mathbf{curl}} \boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^{p'}(\operatorname{\mathbf{curl}}, \Omega)]'$ and satisfies the following compatibility condition

$$\langle \mathbf{curl}\, \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)} = 0 \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{K}_N^{p'}(\Omega),$$

we deduce from Theorem 2.8 that $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$. Then $\boldsymbol{u} \in \boldsymbol{X}^{2,p}(\Omega)$, where

$$\boldsymbol{X}^{2,p}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} \in W^{1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} \in W^{2-\frac{1}{p},\frac{1}{p}}(\Gamma)\right\}.$$

As a consequence, thanks to the imbedding of $\boldsymbol{X}^{2,p}(\Omega)$ in $\boldsymbol{W}^{2,p}(\Omega)$ (see [3]), the solution $\boldsymbol{u}$ of the problem $(\mathcal{S}_T)$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$. Finally, since

$$\nabla \pi = \Delta \boldsymbol{u} + \boldsymbol{f} \in \boldsymbol{L}^p(\Omega),$$

we deduce that $\pi \in W^{1,p}(\Omega)$. □

5. Very weak solutions of $(\mathcal{S}_T)$

In this section we want to prove the existence of a very weak solution for the Stokes problem $(\mathcal{S}_T)$. To prove this, we shall apply a technique used in [1] for the Dirichlet boundary conditions and in [2] for the Navier boundary conditions. First, we start by introducing the following space:

$$\boldsymbol{T}^p(\Omega) = \left\{\boldsymbol{\varphi} \in \boldsymbol{H}_0^p(\mathrm{div}, \Omega);\ \operatorname{div} \boldsymbol{\varphi} \in W_0^{1,p}(\Omega)\right\}.$$

We recall now some preliminary results which we shall use in the sequel (for instance see [2]).

Lemma 5.1. *The space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and for all $\chi \in W^{-1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{T}^{p'}(\Omega)$, we have*

$$\langle \nabla \chi, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} = -\langle \chi, \operatorname{div} \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p}(\Omega)}. \tag{5.1}$$

Lemma 5.2. *A distribution $\boldsymbol{f}$ belongs to $(\boldsymbol{T}^p(\Omega))'$ if and only if there exist $\boldsymbol{\Psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $f_0 \in W^{-1,p'}(\Omega)$, such that*

$$\boldsymbol{f} = \boldsymbol{\Psi} + \nabla f_0.$$

Moreover, we have

$$\|\boldsymbol{f}\|_{(\boldsymbol{T}^p(\Omega))'} = \mathrm{Max}\left\{\|\boldsymbol{\Psi}\|_{\boldsymbol{L}^{p'}(\Omega)}, \|f_0\|_{W^{-1,p'}(\Omega)}\right\}.$$

Giving a meaning to Navier boundary condition of a very weak solution of a Stokes problem is not easy. For this reason, we need to introduce the space

$$\boldsymbol{H}_p(\Delta; \Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \Delta \boldsymbol{v} \in \left(\boldsymbol{T}^{p'}(\Omega)\right)'\right\},$$

which is a Banach space for the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_p(\Delta;\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta \boldsymbol{v}\|_{(\boldsymbol{T}^{p'}(\Omega))'}.$$

The following lemma (see [2]) is important to prove a trace result.

Lemma 5.3. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}_p(\Delta;\Omega)$.*

Let us introduce the space

$$\boldsymbol{S}^p(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0,\ \operatorname{div}\boldsymbol{\varphi} = 0,\ [\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \text{ on } \Gamma\}$$

and recall the following formula (see [4]):

$$\operatorname{div}\boldsymbol{v} = \operatorname{div}_\Gamma \boldsymbol{v}_{\boldsymbol{\tau}} + \boldsymbol{K}\boldsymbol{v}\cdot\boldsymbol{n} + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\cdot\boldsymbol{n} \quad \text{on } \Gamma. \tag{5.2}$$

The following lemma proves that for any $\boldsymbol{v}$ which belongs to $\boldsymbol{H}_p(\Delta;\Omega)$, we have $[\boldsymbol{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$ is well defined in $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$.

Lemma 5.4. *The mapping $\Upsilon : \boldsymbol{u} \mapsto [\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear mapping still denoted by Υ, from $\boldsymbol{H}_p(\Delta;\Omega)$ into $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ and we have the following Green formula: For any $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta;\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$,*

$$\begin{aligned}&\langle \Delta\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} \\ &\quad = \int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,d\boldsymbol{x} + \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}.\end{aligned} \tag{5.3}$$

Proof. First, let $\boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$. Then for any $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$, we have on one hand

$$\begin{aligned}\langle \Delta\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} &= \int_\Omega \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{\varphi}\,d\boldsymbol{x} - 2\int_\Omega \boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} \\ &\quad + \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}\end{aligned}$$

and on the other hand

$$\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,d\boldsymbol{x} = \int_\Omega \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{\varphi}\,d\boldsymbol{x} - 2\int_\Omega \boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}.$$

Thus, for any $\boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ and $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$

$$\langle \Delta\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} = \int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,d\boldsymbol{x} + \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}.$$

Now, let $\boldsymbol{\mu} \in \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)$. We know that there exits $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}(\Omega)$ such that

$$\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}} \quad \text{and} \quad \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} = \boldsymbol{\Lambda}\boldsymbol{\mu} - \boldsymbol{n}\operatorname{div}_\Gamma \boldsymbol{\mu}_{\boldsymbol{\tau}} \quad \text{on } \Gamma.$$

In addition, we have the following estimate

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)} \leqslant C\|\boldsymbol{\mu}_{\boldsymbol{\tau}}\|_{\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} \leqslant C\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}. \tag{5.4}$$

As $\boldsymbol{\Lambda}\boldsymbol{\mu} \cdot \boldsymbol{n} = 0$ on Γ (see (2.3)), we have

$$\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} = -\operatorname{div}_\Gamma \boldsymbol{\mu}_{\boldsymbol{\tau}} \quad \text{on } \Gamma.$$

Using identity (5.2), we deduce that

$$\operatorname{div} \boldsymbol{\varphi} = 0 \quad \text{on } \Gamma.$$

Also, using (2.4), we have

$$\begin{aligned}
\big[\boldsymbol{D}(\boldsymbol{\varphi})\boldsymbol{n}\big]_{\boldsymbol{\tau}} &= \left(\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} - \boldsymbol{\Lambda}\boldsymbol{\varphi} \\
&= \boldsymbol{\Lambda}\boldsymbol{\mu} - \boldsymbol{\Lambda}\boldsymbol{\mu} \\
&= 0.
\end{aligned}$$

To recapitulate, $\boldsymbol{\varphi}$ belongs to $\boldsymbol{S}^{p'}(\Omega)$ and satisfies

$$\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}} \quad \text{and} \quad \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} = \boldsymbol{\Lambda}\boldsymbol{\mu} - \boldsymbol{n}\operatorname{div}_\Gamma \boldsymbol{\mu}_{\boldsymbol{\tau}} \quad \text{on } \Gamma. \tag{5.5}$$

Therefore, we can bound the boundary term as follows: For functions $\boldsymbol{\varphi}$ belonging to $\boldsymbol{S}^{p'}(\Omega)$

$$\begin{aligned}
\big|\big\langle\big[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\mu}_{\boldsymbol{\tau}}\big\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}\big| &= \big|\big\langle\big[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\big\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}\big| \\
&\leqslant \left|\langle \Delta \boldsymbol{u}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} - \int_\Omega \boldsymbol{u}\cdot \Delta\boldsymbol{\varphi}\, d\boldsymbol{x}\right| \\
&\leqslant \|\boldsymbol{\varphi}\|_{\boldsymbol{T}^{p'}(\Omega)}\|\Delta \boldsymbol{u}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)} \\
&\leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}_p(\Delta;\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)} \\
&\leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}_p(\Delta;\Omega)}\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}.
\end{aligned}$$

Consequently, we obtain for any $\boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$:

$$\big\|\big[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}\big\|_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}_p(\Delta;\Omega)}.$$

It follows that, the linear mapping $\Upsilon : \boldsymbol{u} \mapsto [\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}$ defined in $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is continuous for the norm of $\boldsymbol{H}_p(\Delta; \Omega)$. Finally, by density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{H}_p(\Delta; \Omega)$, we can extend continuously this mapping from $\boldsymbol{H}_p(\Delta; \Omega)$ into $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ and the Green formula (5.3) holds. □

Now, we are in position to prove the existence and uniqueness of a very weak solution for Stokes problem $(\mathcal{S}_T)$. The proof of the following theorem is similar to Theorem 4.15 in [2].

Theorem 5.5. *Given any $\boldsymbol{f}$, χ, g and $\boldsymbol{h}$ with*

$$\boldsymbol{f} \in \left(\boldsymbol{T}^{p'}(\Omega)\right)', \qquad \chi \in \boldsymbol{L}^p(\Omega), \qquad g \in W^{-\frac{1}{p},p}(\Gamma), \qquad \boldsymbol{h} \in W^{-1-\frac{1}{p},p}(\Gamma),$$

and satisfying the compatibility conditions (3.1), (3.15)

$$\int_\Omega \chi \, d\boldsymbol{x} = \langle g, 1\rangle_{W^{-\frac{1}{p},p}(\Gamma)\times W^{\frac{1}{p},p'}(\Gamma)}. \tag{5.6}$$

Then, problem $(\mathcal{S}_T)$ has a unique solution

$$\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)/\boldsymbol{\mathcal{T}}(\Omega) \quad \text{and} \quad \pi \in W^{-1,p}(\Omega)/\mathbb{R}.$$

Moreover, we have the estimate:

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \\ &\quad \leqslant C\left(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\chi\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{W^{-1-\frac{1}{p},p}(\Gamma)}\right).\end{aligned}$$

Proof. The basic idea of this proof is to use the duality argument and the strong solution of the adjoint problem with Navier boundary condition. The proof falls into three parts: in the first part we will write the variational formulation, after that we prove the existence of a very weak solution when $g = 0$ and, in the third part, we will finish with the case that g is not vanish.

First step: Observe that if $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta; \Omega)$ and $\pi \in W^{1,p}(\Omega)$, then according to (5.1) and (5.3) we have for any $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$ and for any $q \in W^{1,p'}(\Omega)$,

$$\begin{aligned}\langle -\Delta \boldsymbol{u} + \nabla \pi, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} &= -\int_\Omega \boldsymbol{u}\cdot \Delta\boldsymbol{\varphi}\, d\boldsymbol{x} - \left\langle 2\left[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\right\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} \\ &\quad - \langle \pi, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)}, \\ \int_\Omega \boldsymbol{u}\cdot\nabla q\, d\boldsymbol{x} &= -\int_\Omega q \operatorname{div}\boldsymbol{u}\, d\boldsymbol{x} + \langle \boldsymbol{u}\cdot\boldsymbol{n}, q\rangle_\Gamma.\end{aligned}$$

Thus, if $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is solution of $(\mathcal{S}_T)$, then $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta; \Omega)$ and for any $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$ and for any $q \in W^{1,p'}(\Omega)$ we have

$$
\begin{aligned}
&-\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,d\boldsymbol{x} - \langle \pi, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} \\
&\quad = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}, \qquad (5.7)\\
&\int_\Omega \boldsymbol{u}\cdot\nabla q\,d\boldsymbol{x} = -\int_\Omega \chi q\,d\boldsymbol{x} + \langle g, q\rangle_\Gamma.
\end{aligned}
$$

Conversely, let $(\boldsymbol{u},\pi)\in \boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)$ be a solution of (5.7). Therefore, it follows immediately that,

$$-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u}=\chi \quad \text{in } \Omega.$$

Now, writing

$$\Delta\boldsymbol{u}=\nabla\pi-\boldsymbol{f}$$

and using Lemma 5.2 we deduce that $\boldsymbol{u}$ belongs to $\boldsymbol{H}_p(\Delta;\Omega)$. As consequence, applying Lemma 5.4 we have $2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}\in \boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$. We repeat application of (5.3) and using Lemma 5.1 enables us to write: for any $\boldsymbol{\varphi}\in \boldsymbol{S}^{p'}(\Omega)$,

$$
\begin{aligned}
&\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,d\boldsymbol{x} + \big\langle 2\big[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\big\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} \\
&\quad = -\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} - \langle \pi, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)}. \qquad (5.8)
\end{aligned}
$$

Comparing (5.7) with (5.8), we get

$$\big\langle 2\big[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\big\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} = \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}.$$

Let $\boldsymbol{\mu}\in \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)$. Analysis similar to that in the proof of Lemma 5.4, there exists a function $\boldsymbol{\varphi}\in \boldsymbol{S}^{p'}(\Omega)$ such that $\boldsymbol{\varphi}=\boldsymbol{\mu}_{\boldsymbol{\tau}}$.

Consequently,

$$\big\langle 2\big[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\mu}\big\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} = \langle \boldsymbol{h}, \boldsymbol{\mu}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}.$$

Accounting on the above equality, we have

$$2\big[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}}=\boldsymbol{h} \quad \text{on } \Gamma.$$

It still remains to prove that $\boldsymbol{u}\cdot\boldsymbol{n}=g$ on Γ. For this, we consider the equation $\operatorname{div}\boldsymbol{u}=\chi$ in Ω. We multiply this equation by $q\in W^{1,p'}(\Omega)$, do the integration by parts and compare with (5.7). Then we get

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}, q\rangle_\Gamma = \langle g, q\rangle_\Gamma.$$

As consequence,

$$\boldsymbol{u}\cdot\boldsymbol{n}=g \quad \text{in } W^{-\frac{1}{p},p}(\Gamma).$$

This finishes the first step.

Second step: We suppose that

$$g=0 \quad \text{on } \Gamma \quad \text{and} \quad \int_\Omega \chi\,d\boldsymbol{x}=0.$$

According to Theorem 4.1, for any $(\boldsymbol{F},\xi)\in(\boldsymbol{L}^{p'}(\Omega)\perp\boldsymbol{\mathcal{T}}(\Omega))\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ there exists a unique solution $(\boldsymbol{\varphi},q)\in(\boldsymbol{W}^{2,p'}(\Omega)\times W^{1,p'}(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$ such that

$$-\Delta\boldsymbol{\varphi}+\nabla q=\boldsymbol{F} \quad \text{and} \quad \operatorname{div}\boldsymbol{\varphi}=\xi \quad \text{in } \Omega, \qquad \boldsymbol{\varphi}\cdot\boldsymbol{n}=0, \quad \text{and} \quad \big[\boldsymbol{D}(\boldsymbol{\varphi})\boldsymbol{n}\big]_{\boldsymbol{\tau}}=\boldsymbol{0} \quad \text{on } \Gamma$$

and

$$\inf_{(\boldsymbol{\lambda},k)\in\boldsymbol{\mathcal{N}}(\Omega)}\big(\|\boldsymbol{\varphi}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{2,p'}(\Omega)}+\|q+k\|_{W^{1,p'}(\Omega)}\big)\leqslant C\big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}+\|\varphi\|_{W^{1,p'}(\Omega)}\big). \tag{5.9}$$

Let T be a linear form defined from $(\boldsymbol{L}^{p'}(\Omega)\perp\boldsymbol{\mathcal{T}}(\Omega))\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ onto $\mathbb{R}$ by

$$T:(\boldsymbol{F},\xi)\mapsto\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}+\langle\boldsymbol{h},\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}-\int_\Omega\chi q\,d\boldsymbol{x}.$$

Note that for any $(\boldsymbol{\lambda},k)\in\boldsymbol{\mathcal{N}}(\Omega)$,

$$\begin{aligned}\big|T(\boldsymbol{F},\xi)\big|&\leqslant\left|\langle\boldsymbol{f},\boldsymbol{\varphi}+\boldsymbol{\lambda}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}+\langle\boldsymbol{h},\boldsymbol{\varphi}+\boldsymbol{\lambda}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}-\int_\Omega\chi(q+k)\right|\\&\leqslant C\big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}+\|\chi\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{h}\|_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)}\big)\big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}+\|\xi\|_{W^{1,p'}(\Omega)}\big).\end{aligned}$$

Thus, the linear form T is continuous on $(\boldsymbol{L}^{p'}(\Omega)\perp\boldsymbol{\mathcal{T}}(\Omega))\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ and we deduce that there exists a unique $(\boldsymbol{u},\pi)$ in $(\boldsymbol{L}^p(\Omega)_{/\boldsymbol{\mathcal{T}}(\Omega)}\times W^{-1,p}(\Omega)/_{\mathbb{R}})$ solution of problem (5.7).

It remains to prove the existence and uniqueness of very weak solution when g is not vanish.

Third step: Now, we suppose that $g\neq0$ satisfying the compatibility condition (5.6). Then, there exists a unique $\theta\in W^{1,p}(\Omega)/\mathbb{R}$ solution of Neumann problem (3.17). Now, setting $\boldsymbol{z}=\boldsymbol{u}-\nabla\theta$, $(\mathcal{S}_T)$ becomes: Find $(z,\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times\boldsymbol{L}^p(\Omega)$ satisfying (3.18), with $\boldsymbol{H}=\boldsymbol{h}-2[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}}$. Since $\nabla\theta\in\boldsymbol{H}_p(\Delta;\Omega)$, it is clear that $\boldsymbol{H}$ belongs to $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ and satisfies $\boldsymbol{H}\cdot\boldsymbol{n}=0$. On the other hand, we have $\boldsymbol{f}+\nabla\chi$ belongs to $(\boldsymbol{T}^{p'}\Omega)'$. Thus, due to the second step, there exists a $(z,\pi)\in(\boldsymbol{L}^p(\Omega)\times W^{1,p}(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$ solution of (3.18), and the proof is complete. □

6. Eigenfunctions of the Stokes problem

In [8] T. Clopeau et al. showed, in two dimensions, the existence of an orthonormal basis formed by the eigenfunctions of Stokes operator with Navier boundary condition. It was the idea to invest in a relationship between $\mathbf{curl}\,\boldsymbol{u}$ and $[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau$ on the boundary and reduce to the resolution of bi-Laplacian problem. By this method they were able to show the existence of the basis without solving the stationary Stokes problem. Yet, this technique is not valid in the case of three dimension.

Our objective in this section is to show the existence of the Hilbertian basis of $\boldsymbol{V}^2(\Omega)$ formed by a sequence of eigenfunctions $(\boldsymbol{v}_k)_k \in \boldsymbol{H}^1(\Omega)$ of problem $(\mathcal{S}_T)$. Let us introduce the following problem:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{0} \qquad \text{on } \Gamma, \\ \displaystyle\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta}\,d\boldsymbol{x} = 0. \end{cases} \tag{6.1}$$

We start by the following lemma:

Lemma 6.1. *Let $\boldsymbol{f} \in (\boldsymbol{H}_0^2(\operatorname{div},\Omega))'$ satisfying the following compatibility condition*:

$$\langle \boldsymbol{f}, \boldsymbol{\beta}\rangle_{(\boldsymbol{H}_0^2(\operatorname{div},\Omega))'\times \boldsymbol{H}_0^2(\operatorname{div},\Omega)} = 0. \tag{6.2}$$

Then, problem (6.1) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\pi \in L^2(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leqslant C\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^2(\operatorname{div},\Omega))'}.$$

Proof. Let $(\boldsymbol{u},\pi)$ be one of solutions given by Theorem 3.4. We know that $\boldsymbol{u}$ can be written as $\boldsymbol{u} = P\boldsymbol{u} + c\boldsymbol{\beta}$, where $P\boldsymbol{u}$ is the orthogonal projection of $\boldsymbol{u}$ onto $\boldsymbol{L}^2(\Omega)\perp\boldsymbol{\mathcal{T}}(\Omega)$ and $c \in \mathbb{R}$. It is immediate that $P\boldsymbol{u}$ is the unique solution of (6.1). This finishes the proof. □

We introduce the following spaces:

$$\boldsymbol{Z}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,\ \int_\Omega \boldsymbol{\beta}\cdot\boldsymbol{v}\,d\boldsymbol{x} = 0\right\}$$

and

$$\boldsymbol{H}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,\ \int_\Omega \boldsymbol{\beta}\cdot\boldsymbol{v}\,d\boldsymbol{x} = 0\right\}.$$

We have the following theorem:

Theorem 6.2. *There exists a sequence $(\boldsymbol{v}_j)_j$ of $\boldsymbol{V}^2(\Omega)$ and $(\lambda_j)_j \subset \mathbb{R}$ such that*: *The sequence $(\boldsymbol{v}_j)_j$ is a Hilbert basis of $\boldsymbol{H}(\Omega)$. Moreover we have*

(i) $\lambda_j > C(\Omega) > 0$ *and* $\lim_{j\to+\infty}\lambda_j = +\infty$;
(ii) $2\int_\Omega \mathbf{D}(\boldsymbol{v}_j):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} = \lambda_j\int_\Omega \boldsymbol{v}_j\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}$, $\forall\boldsymbol{\varphi}\in \boldsymbol{V}^2(\Omega)$;
(iii) $\int_\Omega \mathbf{D}(\boldsymbol{v}_j):\mathbf{D}(\boldsymbol{v}_k)\,d\boldsymbol{x} = \lambda_j\delta_{jk}$.

Proof. We introduce the operator

$$\Lambda : \boldsymbol{H}(\Omega) \to \mathbf{Z}(\Omega) \to \boldsymbol{H}(\Omega),$$
$$\boldsymbol{f} \mapsto \boldsymbol{u} \mapsto \boldsymbol{u},$$

where $\boldsymbol{u}$ is the solution given by Lemma 6.1. Note that, $\boldsymbol{H}(\Omega)$ is a Hilbert separable space and Λ is compact. Moreover, Λ is self-adjoint operator, indeed

$$\int_\Omega \Lambda\boldsymbol{f}_1\cdot\boldsymbol{f}_2\,d\boldsymbol{x} = 2\int_\Omega \mathbf{D}(\boldsymbol{u}_1):\mathbf{D}(\boldsymbol{u}_2)\,d\boldsymbol{x} = \int_\Omega \boldsymbol{f}_1\cdot\Lambda\boldsymbol{f}_2\,d\boldsymbol{x},$$

where $\Lambda\boldsymbol{f}_i = \boldsymbol{u}_i$, $i=1,2$. To summarize, Λ is a compact and self-adjoint operator, consequently $\boldsymbol{H}(\Omega)$ possesses a Hilbert basis formed by a sequence of eigenfunctions $\boldsymbol{u}_k$:

$$\Lambda\boldsymbol{u}_j = \mu_j\boldsymbol{u}_j, \qquad \mu_j > 0,\ j\geqslant 1 \quad\text{and}\quad \lim_{j\to+\infty}\mu_j = 0.$$

Particularly,

$$\int_\Omega \boldsymbol{u}_j\cdot\boldsymbol{u}_k\,d\boldsymbol{x} = \delta_{jk} \quad\text{and}\quad \int_\Omega \Lambda\boldsymbol{u}_j\cdot\boldsymbol{v}\,d\boldsymbol{x} = \int_\Omega \mu_j\boldsymbol{u}_j\cdot\boldsymbol{v}\,d\boldsymbol{x}, \quad \forall\boldsymbol{v}\in\boldsymbol{V}^2(\Omega).$$

So,

$$\forall\boldsymbol{v}\in\boldsymbol{V}^2(\Omega), \quad \int_\Omega \mu_j\mathbf{D}(\boldsymbol{u}_j):\mathbf{D}(\boldsymbol{v})\,d\boldsymbol{x} = \int_\Omega \boldsymbol{u}_j\cdot\boldsymbol{v}\,d\boldsymbol{x}.$$

We set $\lambda_j = \frac{1}{\mu_j}$, $\forall j\geqslant 1$. Then we have $\lim_{j\to+\infty}\lambda_j = +\infty$ and

$$\forall\boldsymbol{v}\in\boldsymbol{V}^2(\Omega), \quad \int_\Omega \mathbf{D}(\boldsymbol{u}_j):\mathbf{D}(\boldsymbol{v})\,d\boldsymbol{x} = \lambda_j\int_\Omega \boldsymbol{u}_j\cdot\boldsymbol{v}\,d\boldsymbol{x}.$$

As consequence, by applying Lemma 3.3 we have

$$\|\mathbf{D}(\boldsymbol{u}_j)\|^2_{\boldsymbol{L}^2(\Omega)} = \lambda_j \geqslant \frac{1}{C^2}\|\boldsymbol{u}_j\|^2_{\boldsymbol{L}^2(\Omega)} = \frac{1}{C^2}. \qquad \square$$

This basis will be very useful in the study of problem of existence of weak solution for Navier–Stokes equation in the Hilbert case.

7. Navier–Stokes equation

We consider the following Navier–Stokes problem:

$$(\mathcal{NS}_T)\quad \begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma,\\ 2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases}$$

The aim of this section is to prove the existence of weak solution and strong solution of problem $(\mathcal{NS}_T)$. To prove the existence of weak solution, we need to introduce some spaces. First, we introduce the following space:

$$\boldsymbol{H}_0^{\frac{6}{5},2}(\operatorname{div},\Omega) = \big\{\boldsymbol{v}\in \boldsymbol{L}^{\frac{6}{5}}(\Omega);\ \operatorname{div}\boldsymbol{v}\in \boldsymbol{L}^2(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\big\},$$

which is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_0^{\frac{6}{5},2}(\operatorname{div},\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}.$$

We define also the space

$$\boldsymbol{E}^{\frac{6}{5},2}(\Omega) = \big\{\boldsymbol{v}\in \boldsymbol{H}^1(\Omega);\ \Delta\boldsymbol{v}\in \big[\boldsymbol{H}_0^{6,2}(\operatorname{div},\Omega)\big]'\big\}.$$

We note that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}_0^{\frac{6}{5},2}(\operatorname{div},\Omega)$. As consequence, we have the following lemma, which has a similar proof as Lemma 2.4.

Lemma 7.1. *Suppose that Ω is of class $C^{1,1}$. The linear mapping $\Upsilon : \boldsymbol{v} \to [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping*

$$\Upsilon : \boldsymbol{E}^{\frac{6}{5},2}(\Omega) \to \boldsymbol{H}^{-\frac{1}{2}}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v}\in \boldsymbol{E}^{\frac{6}{5},2}(\Omega)$ and $\boldsymbol{\varphi}\in \boldsymbol{V}^2(\Omega)$,

$$-\langle \Delta\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{H}_0^{6,2}(\operatorname{div},\Omega)]'\times \boldsymbol{H}_0^{6,2}(\operatorname{div},\Omega)} = 2\int_\Omega \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} - 2\big\langle\big[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\big]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\big\rangle_\Gamma. \tag{7.1}$$

Thanks to this lemma, we can show that the Navier–Stokes problem $(\mathcal{NS}_T)$ is equivalent to the following formulation:

$$(\mathcal{FNS})\quad \begin{cases} \text{Find } \boldsymbol{u}\in \boldsymbol{V}^2(\Omega) \text{ such that,}\\ \forall\boldsymbol{\varphi}\in \boldsymbol{V}^2(\Omega),\quad 2\displaystyle\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} + \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\varphi}\,d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma, \end{cases}$$

where $\langle .,.\rangle_\Omega = \langle .,.\rangle_{[\boldsymbol{H}_0^{6,2}(\operatorname{div},\Omega)]'\times \boldsymbol{H}_0^{6,2}(\operatorname{div},\Omega)}$ and $\langle .,.\rangle_\Gamma = \langle .,.\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}$.

In the sequel we write $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,d\boldsymbol{x}$. To facilitate the work, we give some properties of the operator b.

Lemma 7.2. *For any $\boldsymbol{u}\in\boldsymbol{V}^2(\Omega)$ and $\boldsymbol{\eta}\in\boldsymbol{\mathcal{T}}(\Omega)$, the following identities hold*

$$b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\eta})=b(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\eta})=b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{u})=0.$$

Proof. We know that $b(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\eta})=0$. Since, $\boldsymbol{D}(\boldsymbol{\eta})=0$, we have $\frac{\partial\eta_i}{\partial x_j}=-\frac{\partial\eta_j}{\partial x_i}$, then, for any $\boldsymbol{u}\in\boldsymbol{V}^2(\Omega)$

$$\begin{aligned}
b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\eta})&=\sum_{i,j=1}^{3}\int_\Omega \boldsymbol{u}_i\frac{\partial\boldsymbol{u}_j}{\partial\boldsymbol{x}_i}\boldsymbol{\eta}_j\,d\boldsymbol{x}=-\sum_{i,j=1}^{3}\int_\Omega \boldsymbol{u}_i\frac{\partial\boldsymbol{\eta}_j}{\partial\boldsymbol{x}_i}\boldsymbol{u}_j\,d\boldsymbol{x}\\
&=\sum_{i,j=1}^{3}\int_\Omega \boldsymbol{u}_i\frac{\partial\boldsymbol{\eta}_i}{\partial\boldsymbol{x}_j}\boldsymbol{u}_j\,d\boldsymbol{x}=-\sum_{i,j=1}^{3}\int_\Omega \boldsymbol{u}_j\frac{\partial\boldsymbol{u}_i}{\partial\boldsymbol{x}_j}\boldsymbol{\eta}_i\,d\boldsymbol{x}\\
&=-b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\eta}).
\end{aligned}$$

Consequently, $b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\eta})=0$. Finally, because $\boldsymbol{D}(\boldsymbol{\eta})=0$, we have

$$b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{u})=\frac{1}{2}\int_\Omega \boldsymbol{u}\cdot\nabla|\boldsymbol{\eta}|^2\,d\boldsymbol{x}=0. \qquad \square$$

Using Lemma 7.2, we deduce that the following compatibility condition is necessary to solve problem $(\mathcal{FNS})$:

$$\langle\boldsymbol{f},\boldsymbol{\beta}\rangle_\Omega+\langle\boldsymbol{h},\boldsymbol{\beta}\rangle_\Gamma=0. \tag{7.2}$$

Due to all these results we can now solve the problem $(\mathcal{FNS})$.

Theorem 7.3. *Let $\boldsymbol{f}\in(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$ and $\boldsymbol{h}\in\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, satisfying $\boldsymbol{h}\cdot\boldsymbol{n}=0$ on Γ and the compatibility condition (7.2). Then problem $(\mathcal{FNS})$ has solution $(\boldsymbol{u},\pi)\in\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$. Moreover, we have the following estimate*:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}+\|\pi\|_{L^2(\Omega)}\leqslant C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'}+\|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big).$$

Proof. To show the existence of $\boldsymbol{u}$, we start by constructing the approximate solutions of the problem $(\mathcal{FNS})$ by Galerkin method and then thanks to compactness arguments, we prove, by passing to the limit, some convergence properties. We note that, if $\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{\eta}$, with $\boldsymbol{w}\in\boldsymbol{Z}(\Omega)$ and $\boldsymbol{\eta}\in\boldsymbol{\mathcal{T}}(\Omega)$, is a solution of $(\mathcal{FNS})$, then for any $\boldsymbol{\psi}\in\boldsymbol{Z}(\Omega)$,

$$2\int_\Omega \mathbf{D}(\boldsymbol{w}):\mathbf{D}(\boldsymbol{\psi})\,d\boldsymbol{x}+b(\boldsymbol{w}+\boldsymbol{\eta},\boldsymbol{w},\boldsymbol{\psi})+b(\boldsymbol{w},\boldsymbol{\eta},\boldsymbol{\psi})=\langle\boldsymbol{f},\boldsymbol{\psi}\rangle_\Omega+\langle\boldsymbol{h},\boldsymbol{\psi}\rangle_\Gamma. \tag{7.3}$$

Note that for any $\boldsymbol{\eta}' \in \mathcal{T}(\Omega)$, we have

$$b\big(\boldsymbol{w}+\boldsymbol{\eta}, \boldsymbol{w}+\boldsymbol{\eta}, \boldsymbol{\psi}+\boldsymbol{\eta}'\big) = b(\boldsymbol{w}+\boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{\psi}) + b(\boldsymbol{w}, \boldsymbol{\eta}, \boldsymbol{\psi}).$$

It follows that problem $(\mathcal{FNS})$ is reduced to finding $\boldsymbol{w} \in \boldsymbol{Z}(\Omega)$ verifying (7.3).

Now, for each fixed integer $m \geqslant 1$, we define an approximate solution $\boldsymbol{w}_m$ of (7.3) by

$$\begin{cases} \boldsymbol{w}_m \in \boldsymbol{V}_m, \\ 2\displaystyle\int_\Omega \mathbf{D}(\boldsymbol{w}_m) : \mathbf{D}(\boldsymbol{v}_k)\, d\boldsymbol{x} + b(\boldsymbol{w}_m+\boldsymbol{\eta}, \boldsymbol{w}_m, \boldsymbol{v}_k) + b(\boldsymbol{w}_m, \boldsymbol{\eta}, \boldsymbol{v}_k) \\ \quad = \langle \boldsymbol{f}, \boldsymbol{v}_k\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{v}_k\rangle_\Gamma, \quad k = 1, \dots, m \end{cases} \tag{7.4}$$

where $\boldsymbol{V}_m = \langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_m\rangle$ is the space spanned by the vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ and $\{\boldsymbol{v}_i\}_i$ is the Hilbertian basis of $\boldsymbol{H}(\Omega)$ given by eigenfunctions of Stokes problem. We define a scalar product on $\boldsymbol{Z}(\Omega)$ by

$$\forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{Z}(\Omega), \quad \big((\boldsymbol{u}_1, \boldsymbol{u}_2)\big) = \int_\Omega \mathbf{D}(\boldsymbol{u}_1) : \mathbf{D}(\boldsymbol{u}_2)\, d\boldsymbol{x}.$$

With an aim to establish the existence of the solutions of the problem (7.4), we consider the following operator

$$\begin{aligned} \boldsymbol{P}_m : \boldsymbol{V}_m &\to \boldsymbol{V}_m, \\ \boldsymbol{w} &\to \boldsymbol{P}_m(\boldsymbol{w}) \end{aligned}$$

defined, for each $\boldsymbol{\eta} \in \mathcal{T}(\Omega)$ fixed, by

$$\big(\big(\boldsymbol{P}_m(\boldsymbol{w}), \boldsymbol{z}\big)\big) = 2\big((\boldsymbol{w}, \boldsymbol{z})\big) + b(\boldsymbol{w}+\boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{z}) + b(\boldsymbol{w}, \boldsymbol{\eta}, \boldsymbol{z}) - \langle \boldsymbol{f}, \boldsymbol{z}\rangle_\Omega - \langle \boldsymbol{h}, \boldsymbol{z}\rangle_\Gamma.$$

Let us note that

$$\forall \boldsymbol{w} \in \boldsymbol{Z}(\Omega), \quad b(\boldsymbol{w}+\boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{w}) = b(\boldsymbol{w}, \boldsymbol{\eta}, \boldsymbol{w}) = 0.$$

Thus, using inequality (3.7), we show that

$$\big(\big(\boldsymbol{P}_m(\boldsymbol{w}), \boldsymbol{w}\big)\big) \geqslant \|\boldsymbol{w}\|_{\boldsymbol{V}_m}\big(2\|\boldsymbol{w}\|_{\boldsymbol{V}_m} - C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big)\big),$$

where the norm on $\boldsymbol{V}_m$ is induced by the norm on $\boldsymbol{Z}(\Omega)$.

As consequence, $((\boldsymbol{P}_m(\boldsymbol{w}), \boldsymbol{w})) > 0$ for $\|\boldsymbol{w}\|_{\boldsymbol{V}_m} > \frac{C}{2}(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)})$. We know that $\boldsymbol{P}_m : \boldsymbol{V}_m \to \boldsymbol{V}_m$ is continuous. Therefore, the hypothesis of Brouwer theorem is satisfied and there exists a solution $\boldsymbol{w}_m$ of (7.4).

Passage to the limit: Since $\boldsymbol{w}_m$ is a solution of problem (7.4), we have

$$2\big\|\mathbf{D}(\boldsymbol{w}_m)\big\|^2_{L^2(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{w}_m\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{w}_m\rangle_\Gamma.$$

Using compatibility condition (7.2) and Lemma 3.3, we obtain the *a priori* estimate:

$$\|\boldsymbol{w}_m\|_{\boldsymbol{V}_m} \leqslant C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big).$$

Since the sequence $\boldsymbol{w}_m$ remains bounded in $\boldsymbol{Z}(\Omega)$, we can extract a subsequence $\boldsymbol{w}_k$ such that

$$\boldsymbol{w}_k \rightharpoonup \boldsymbol{w} \quad \text{weakly in } \boldsymbol{Z}(\Omega).$$

The injection of $\boldsymbol{Z}(\Omega)$ into $\boldsymbol{H}(\Omega)$ is compact, so we have

$$\boldsymbol{w}_k \to \boldsymbol{w} \quad \text{in } \boldsymbol{H}(\Omega).$$

Therefore, we can pass to the limit in (7.4) and we obtain

$$2\int_\Omega \mathbf{D}(\boldsymbol{w}) : \mathbf{D}(\boldsymbol{\psi})\, d\boldsymbol{x} + b(\boldsymbol{w}+\boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{\psi}) + b(\boldsymbol{w}, \boldsymbol{\eta}, \boldsymbol{\psi}) = \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\psi}\rangle_\Gamma$$

for any $\boldsymbol{\psi} \in \boldsymbol{Z}(\Omega)$.

Finally, we conclude that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{\eta}$ is a solution of $(\mathcal{NS}_T)$. $\square$

Remark 7.4. We proved that, in the case of symmetric domain, the Navier–Stokes problem $(\mathcal{NS}_T)$ has an infinity of solutions for all data satisfying the compatibility condition (7.2). Then, we have a situation where we don't have a uniqueness of solution even if the data is sufficiently small. Unlike the Navier–Stokes problem with Dirichlet boundary condition where we have a uniqueness of solution for data sufficiently small.

Theorem 7.5. *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, satisfying $\boldsymbol{h}\cdot\boldsymbol{n} = 0$ on Γ and the compatibility condition (7.2). Then problem $(\mathcal{NS}_T)$ has solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$. Moreover, we have the following estimate*:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\big).$$

Proof. According to Theorem 7.3, there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ solution of $(\mathcal{NS}_T)$. So, $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ belongs to $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$. Thus, $\boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and satisfies the following condition:

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\beta}\, d\boldsymbol{x} - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\beta}) - \int_\Gamma \boldsymbol{h}\cdot\boldsymbol{\beta}\, d\boldsymbol{\sigma} = 0. \tag{7.5}$$

Furthermore, $\boldsymbol{u}$ satisfies the following problem:

$$\begin{cases} -\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad [2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{7.6}$$

Thanks to Theorem 4.1 we deduce that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$. Thus, $\nabla \boldsymbol{u} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Therefore $\boldsymbol{f} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ satisfying again the condition (7.5). As consequence, we apply again Theorem 4.1, we deduce that $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ solves (7.6). □

Appendix A

In this appendix we give the proof of Lemma 2.1. The main idea is to use the local coordinates (similar to Theorem 3.1.1.1 in [9]) and the density of $\mathcal{D}(\overline{\Omega})$ in $\boldsymbol{W}^{2,p}(\Omega)$. Let $\boldsymbol{v} \in \mathcal{D}(\overline{\Omega})$ and let us start by calculating a gradient of $\boldsymbol{v}$ on the boundary.

$$\begin{aligned}
\nabla \boldsymbol{v} &= \sum_{\ell=1}^{2} \frac{\partial \boldsymbol{v}}{\partial s_\ell} \boldsymbol{\tau}_\ell^T + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \boldsymbol{n}^T \\
&= \sum_{\ell,k} \left(v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} + \frac{\partial v_k}{\partial s_\ell} \boldsymbol{\tau}_k \right) \boldsymbol{\tau}_\ell^T + \sum_{\ell=1}^{2} \left(\frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_\ell} \boldsymbol{n} + \boldsymbol{v} \cdot \boldsymbol{n} \frac{\partial \boldsymbol{n}}{\partial s_\ell} \right) \boldsymbol{\tau}_\ell^T \\
&\quad + \sum_{k=1}^{2} \left(\frac{\partial v_k}{\partial \boldsymbol{n}} \boldsymbol{\tau}_k + v_k \frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}} \right) \boldsymbol{n}^T + \left(\frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial \boldsymbol{n}} \boldsymbol{n} + \boldsymbol{v} \cdot \boldsymbol{n} \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}} \right) \boldsymbol{n}^T.
\end{aligned}$$

As a consequence,

$$(\nabla \boldsymbol{v})\boldsymbol{n} = \sum_{k=1}^{2} \left(\frac{\partial v_k}{\partial \boldsymbol{n}} \boldsymbol{\tau}_k + v_k \frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}} \right) + \frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial \boldsymbol{n}} \boldsymbol{n} + \boldsymbol{v} \cdot \boldsymbol{n} \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}}. \tag{A.1}$$

Therefore,

$$\big[(\nabla \boldsymbol{v})\boldsymbol{n}\big]_{\boldsymbol{\tau}} = \sum_{k=1}^{2} \left(\frac{\partial v_k}{\partial \boldsymbol{n}} \boldsymbol{\tau}_k + v_k \left(\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}} \right)_{\boldsymbol{\tau}} \right) + \boldsymbol{v} \cdot \boldsymbol{n} \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}}. \tag{A.2}$$

Note that $\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} = 0$, then we have

$$\begin{aligned}
(\nabla \boldsymbol{v})^T &= \sum_{\ell,k=1}^{2} \boldsymbol{\tau}_\ell \left(v_k \left(\frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \right)^T + \frac{\partial v_k}{\partial s_\ell} \boldsymbol{\tau}_k^T \right) + \sum_{\ell=1}^{2} \boldsymbol{\tau}_\ell \left(\frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_\ell} \boldsymbol{n}^T + \boldsymbol{v} \cdot \boldsymbol{n} \left(\frac{\partial \boldsymbol{n}}{\partial s_\ell} \right)^T \right) \\
&\quad + \sum_{k=1}^{2} \boldsymbol{n} \left(\frac{\partial v_k}{\partial \boldsymbol{n}} \boldsymbol{\tau}_k^T + v_k \left(\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}} \right)^T \right) + \boldsymbol{n} \left(\frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial \boldsymbol{n}} \boldsymbol{n}^T + \boldsymbol{v} \cdot \boldsymbol{n} \left(\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}} \right)^T \right).
\end{aligned}$$

Since $\frac{\partial \boldsymbol{n}}{\partial s_k} \cdot \boldsymbol{n} = 0$, we deduce that

$$\begin{aligned}
(\nabla \boldsymbol{v})^T \boldsymbol{n} &= \sum_{\ell,k} \boldsymbol{\tau}_\ell v_k \left(\frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \right)^T \boldsymbol{n} + \sum_{\ell=1}^{2} \boldsymbol{\tau}_\ell \frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_\ell} \\
&\quad + \sum_{k=1}^{2} \boldsymbol{n} v_k \left(\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}} \right)^T \boldsymbol{n} + \boldsymbol{n} \frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial \boldsymbol{n}}.
\end{aligned} \tag{A.3}$$

Then,

$$\big[(\nabla \boldsymbol{v})^T \boldsymbol{n}\big]_{\boldsymbol{\tau}} = \sum_{\ell,k=1}^{2} v_k \left(\left(\frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \right)^T \boldsymbol{n} \right) \boldsymbol{\tau}_\ell + \sum_{\ell=1}^{2} \frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_\ell} \boldsymbol{\tau}_\ell. \tag{A.4}$$

Adding up equality (A.2) to (A.4) we obtain

$$\begin{aligned} \big[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\big]_{\boldsymbol{\tau}} = {} & \sum_{k=1}^{2} \frac{\partial v_k}{\partial \boldsymbol{n}} \boldsymbol{\tau}_k + v_k \left(\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}} \right)_{\boldsymbol{\tau}} + \boldsymbol{v} \cdot \boldsymbol{n} \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}} \\ & + \sum_{\ell,k=1}^{2} \left(v_k \left(\frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \right)^T \boldsymbol{n} \right) \boldsymbol{\tau}_\ell + \sum_{\ell=1}^{2} \frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_\ell} \boldsymbol{\tau}_\ell. \end{aligned} \tag{A.5}$$

But we know that

$$\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \right)_{\boldsymbol{\tau}} = \sum_{k=1}^{2} \frac{\partial v_k}{\partial \boldsymbol{n}} \boldsymbol{\tau}_k + v_k \left(\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}} \right)_{\boldsymbol{\tau}} + \boldsymbol{v} \cdot \boldsymbol{n} \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}},$$

and

$$\nabla_{\boldsymbol{\tau}} (\boldsymbol{v} \cdot \boldsymbol{n}) = \sum_{\ell=1}^{2} \frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_\ell} \boldsymbol{\tau}_\ell,$$

which implies that

$$\big[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\big]_{\boldsymbol{\tau}} = \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \right)_{\boldsymbol{\tau}} + \nabla_{\boldsymbol{\tau}} (\boldsymbol{v} \cdot \boldsymbol{n}) + \sum_{\ell,k} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \cdot \boldsymbol{n} \boldsymbol{\tau}_\ell. \tag{A.6}$$

In addition, because $\boldsymbol{\tau}_k^T \cdot \boldsymbol{n} = 0$ on Γ, we have

$$\begin{aligned} \sum_{\ell,k} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \cdot \boldsymbol{n} \boldsymbol{\tau}_\ell &= -\sum_{k=1}^{2} v_k \sum_{\ell=1}^{2} \boldsymbol{\tau}_k \cdot \frac{\partial \boldsymbol{n}}{\partial s_\ell} \boldsymbol{\tau}_\ell \\ &= -\sum_{\ell=1}^{2} \left(\sum_{k=1}^{2} v_k \boldsymbol{\tau}_k \cdot \frac{\partial \boldsymbol{n}}{\partial s_\ell} \right) \boldsymbol{\tau}_\ell \\ &= -\sum_{\ell=1}^{2} \left(\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_\ell} \right) \boldsymbol{\tau}_\ell = -\boldsymbol{\Lambda} \boldsymbol{v}. \end{aligned}$$

Therefore, we have

$$\sum_{\ell,k} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \cdot \boldsymbol{n} \boldsymbol{\tau}_\ell = -\boldsymbol{\Lambda} \boldsymbol{v}.$$

Since Ω is of class $C^{2,1}$, we see that $\frac{\partial \boldsymbol{n}}{\partial s_\ell} \in \boldsymbol{W}^{1,\infty}(\Gamma)$, $\ell = 1, 2$. Hence, using (A.6), (A.7) and the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{W}^{2,p}(\Omega)$, we obtain (2.4).

In the other hand, we know that

$$\mathbf{curl}\,\boldsymbol{v} = \sum_{k=1}^{2} \frac{\partial \boldsymbol{v}}{\partial s_k} \times \boldsymbol{\tau}_k + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \times \boldsymbol{n}.$$

As consequence, we have

$$\mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n} = \sum_{k=1}^{2} \left(\frac{\partial \boldsymbol{v}}{\partial s_k} \times \boldsymbol{\tau}_k \right) \times \boldsymbol{n} + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{v}} \times \boldsymbol{n} \right) \times \boldsymbol{n}.$$

Now, in general

$$(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}.$$

Using this equality, we have

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n} &= \sum_{k=1}^{2} \left(\frac{\partial \boldsymbol{v}}{\partial s_k} \cdot \boldsymbol{n} \right) \boldsymbol{\tau}_k - (\boldsymbol{\tau}_k \cdot \boldsymbol{n}) \frac{\partial \boldsymbol{v}}{\partial s_k} + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} \right) \boldsymbol{n} - \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \\ &= \sum_{k=1}^{2} \left(\frac{\partial \boldsymbol{v}}{\partial s_k} \cdot \boldsymbol{n} \right) \boldsymbol{\tau}_k + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} \right) \boldsymbol{n} - \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}. \end{aligned}$$

On other hand, we have

$$\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} \right) \boldsymbol{n} - \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = -\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \right)_{\boldsymbol{\tau}}.$$

Consequently,

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n} &= \sum_{j=1}^{2} \left(\frac{\partial \boldsymbol{v}}{\partial s_j} \cdot \boldsymbol{n} \right) \boldsymbol{\tau}_j - \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \right)_{\boldsymbol{\tau}} \\ &= \sum_{j=1}^{2} \frac{(\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_j} \boldsymbol{\tau}_j - \boldsymbol{\Lambda} \boldsymbol{v} - \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \right)_{\boldsymbol{\tau}}. \end{aligned} \tag{A.7}$$

In conclusion, since Ω is of class $C^{2,1}$, we can pass to the limit in (A.7) and we obtain equality (2.5) in $\boldsymbol{W}^{2,p}(\Omega)$.

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