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Dynamics of the 3-D fractional complex Ginzburg–Landau equation

Hong Lu [a,b], Peter W. Bates [b,*], Shujuan Lü [a], Mingji Zhang [b,c]

[a] *School of Mathematics and Systems Science & LMIB, Beihang University, Beijing, 100191, China*

[b] *Department of Mathematics, Michigan State University, 619 Red Cedar Road, East Lansing, MI 48824, USA*

[c] *Department of Mathematics, New Mexico Institute of Mining and Technology, 801 Leroy Place, Socorro, NM 87801, USA*

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Abstract

We study the initial boundary value problem of the fractional complex Ginzburg–Landau equation in three spatial dimensions with the dissipative effect given by a fractional Laplacian. *A priori* estimates are derived when the nonlinearity satisfies certain growth conditions. Using Galerkin's method, the existence of a global smooth solution is established. Uniqueness is also proved. Furthermore, the existence of a global attractor is proved, and estimates of the Hausdorff and fractal dimensions for the global attractor are obtained.

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* Corresponding author.

E-mail addresses: ljwenling@163.com (H. Lu), bates@math.msu.edu (P.W. Bates), lsj@buaa.edu.cn (S. Lü), mzhang@math.msu.edu, mzhang0129@gmail.com (M. Zhang).

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1. Introduction

The Ginzburg–Landau equation [14,15] is one of the most-studied nonlinear equations in physics. It describes a vast variety of phenomena from nonlinear waves to second-order phase transitions, from superconductivity, superfluidity, and Bose–Einstein condensation to liquid crystals and strings in field theory. The Ginzburg–Landau equation with *fractional derivatives* was suggested in [41] and studied in [39] and [40], where it is used to describe processes in media with fractal dispersion or long-range interaction.

This new type of problem has rapidly become a focus of interest since the fractional derivative and fractional integral have a wide range of applications in finance, fluid dynamics, physics, biology, chemistry and other fields of science. One encounters them in the theory of systems with chaotic dynamics [35,42]; pseudochaotic dynamics [44]; dynamics in a complex or porous medium [10,30,37]; random walks with a memory and flights [28,36,43]; obstacle problems [6,34] and many other situations. Recently, fractional partial differential equation versions of some of the classical equations of mathematical physics have been studied, including the fractional Schrödinger equation [9,12,13,16,18–20,29], the fractional Landau–Lifshitz equation [17], the fractional Landau–Lifshitz–Maxwell equation [31] and the fractional Ginzburg–Landau equation [25,27,39]. Furthermore, many recent studies of fractional derivative problems arise from probabilistic or purely mathematical considerations (see [1–3,5,11], for instance). Unfortunately, the fractional derivative (nonlocal) term sometimes makes it necessary to apply tools that are non-traditional when only dealing with local smooth equations.

Here, we consider the following fractional complex Ginzburg–Landau equation [21,32]:

$$u_t = \rho u - (1+\mathrm{i}\nu)(-\triangle)^{\alpha} u + f(u), \qquad x \in \mathbb{R}^n, \quad t > 0, \tag{1.1}$$

with the initial condition and the periodic boundary condition:

$$u(x,0) = u_0, \qquad x \in \mathbb{R}^n, \tag{1.2}$$

$$u(x+2\pi \mathbf{e}_i, t) = u(x,t), \qquad x \in \mathbb{R}^n, \quad t > 0, \quad i = 1, 2, \cdots n \tag{1.3}$$

where $\mathbf{e}_i$ $(i = 1, 2, \cdots, n)$ is an orthonormal basis of $\mathbb{R}^n$. Here, α, ν, and ρ are real constants with $\rho > 0$ and $\alpha \in (1/2, 1)$, and f is a nonlinear function, for instance $f(u) = (1 + i\mu)|u|^{2\sigma} u$ with $\sigma > 0$. For convenience, we sometimes write it as $f = f(u, \bar{u})$, and in the various lemmas that follow we assume f satisfies some of the following conditions:

$$\mathrm{Re} f(u,\bar{u})\bar{u} \leq -\beta_1 |u|^{2\sigma+2} + \gamma_1, \tag{1.4}$$

$$\mathrm{Re} f_u |\mathbf{V}|^2 + \mathrm{Re} f_{\bar{u}} (\bar{\mathbf{V}})^2 \leq -\beta_\sigma |u|^{2\sigma} |\mathbf{V}|^2 + |u|^{2\sigma-2} (\lambda_\sigma (u\bar{\mathbf{V}})^2 + \bar{\lambda}_\sigma (\bar{u}\mathbf{V})^2), \tag{1.5}$$

$$\max\{|f_u|, |f_{\bar{u}}|\} \leq \beta_2 |u|^{2\sigma} + \gamma_2, \tag{1.6}$$

$$\max\{|f_{uu}|, |f_{u\bar{u}}|, |f_{\bar{u}\bar{u}}|\} \leq \beta_3 |u|^{2\sigma-1} + \gamma_3, \tag{1.7}$$

$$|f(u,\bar{u})| \leq \beta_4 |u|^{2\sigma+1} + \gamma_4, \tag{1.8}$$

for $u \in \mathbb{C}$ and $\mathbf{V} \in \mathbb{C}^{\mathbf{n}}$, where σ, β_i, and γ_i $(i = 1, 2, 3, 4)$ are positive constants, β_σ is a positive constant depending on σ, λ_σ is a complex constant depending on σ, and $(\mathbf{V})^2 = \mathbf{V} \cdot \mathbf{V} = \sum_{i=1}^n V_i^2$, (which is not an inner product on $\mathbb{C}^n$).

In particular, we include the special but important case with $f(u) = -(1 + \mathrm{i}\mu)|u|^{2\sigma} u$.

The condition we place on α is from the derivation of the fractional Ginzburg–Landau equation (see [39]). Mathematically, it is a necessary condition for our analysis to derive inequalities (3.15) and (3.22) (using the Gagliardo–Nirenberg inequality) in the proofs of Lemmas 3.3 and 3.4, which establish our *a priori* estimates for the solution. We would like to point out that the standard complex Ginzburg–Landau equation (for $\alpha = 1$ in (1.1))

$$u_t = \rho u + (1 + \mathrm{i}\nu)\triangle u - (1 + \mathrm{i}\mu)|u|^{2\sigma} u, \tag{1.9}$$

which has been the object of intense study (see [4,7,8,12,22–24,33]), satisfies the conditions (1.4)–(1.8).

In a recent paper [21], the authors studied (1.1)–(1.3) with spatial dimension *two* and with the special pure power nonlinearity. They proved the well-posedness and studied the asymptotic behavior of the solutions, proving the existence of the global attractor. Estimates of the Hausdorff and fractal dimensions for the global attractor were also obtained.

In this work, we study (1.1)–(1.3) with the more physically relevant spatial dimension *three*, again proving well-posedness and examining the long-time behavior of solutions. In three dimensions the constraints in the Gagliardo–Nirenberg inequality (Sobolev interpolation inequalities), make the L^2-norm estimate of higher order derivatives of the solution more challenging.

Usually to derive the necessary *a priori* estimates, one proves the boundedness of $\|u\|$, $\|\nabla u\|$, and $\|\triangle u\|$ successively (e.g. [21], etc.). Here, to obtain the boundless of $\|\triangle u\|$, a technical step (see Lemma 3.3) is to show that $\|(-\triangle)^{\frac{1+\alpha}{2}} u\|$ is bounded under a more restrictive condition on σ, namely,

$$\sigma < 3/(2 - \alpha) - 1.$$

Once we obtain these estimates, the existence of global smooth solutions is established through Galerkin's method. We also prove the existence of the global attractor and obtain estimates for its Hausdorff and fractal dimension.

The rest of this paper is organized as follows. In Section 2, some notation and preliminary results are introduced. In Section 3, *a priori* estimates are derived, and a bounded absorbing set is obtained. In Section 4, using Galerkin's method, the existence of global smooth solutions is established, uniqueness of solutions and the continuity of the semigroup of solutions are proved. In Section 5, the existence of the global attractor and estimates for its Hausdorff dimension and fractal dimension are obtained.

2. Preliminaries and notations

If u is smooth and 2π-periodic in each of the three coordinates, it can be expressed by a Fourier series $u = \sum_{k\in\mathbb{Z}^3} u_k e^{\mathrm{i}\langle k,x\rangle}$. It follows that $u_{x_j} = \sum_{k\in\mathbb{Z}^3} \mathrm{i}k_j u_k e^{\mathrm{i}\langle k,x\rangle}$ $(j = 1, 2, 3)$, and $(-\triangle)^\alpha$ is defined by

$$(-\triangle)^\alpha u = \sum_{k\in\mathbb{Z}^3} |k|^{2\alpha} u_k e^{\mathrm{i}\langle k,x\rangle}.$$

Let $\Omega = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi] \subset \mathbb{R}^3$ and let $H^\beta = H^\beta(\Omega)$ denote the Sobolev space of order β equipped with the norm:

$$\|u\|_{H^\beta} = \left(\sum_{k\in\mathbb{Z}^3} |k|^{2\beta}|u_k|^2 + \sum_{k\in\mathbb{Z}^3} |u_k|^2 \right)^{\frac{1}{2}}.$$

We denote by H_p^β those functions that are 2π-periodic in all the coordinate variables and when restricted to Ω, lie in $H^\beta(\Omega)$. Throughout this paper, we denote by $(\cdot,\cdot)$ the usual inner product in $L^2 = L^2(\Omega;\mathbb{C})$, $\|\cdot\|_{H^m}$ the norm of Sobolev space $H^m(\Omega)$, and $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$, $1 \le q \le \infty$. In the forthcoming discussion, we use T to denote an arbitrary positive constant, and use c_j ($j = 1, 2, \cdots$) to denote different positive constants which depend on the constants ρ, ν, α, and the constants appearing in the constraints on f. In addition, the following embedding theorem [26] is frequently used.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having the cone property and let $u \in L^q(\Omega)$ and its derivatives of order m, $D^m u$, belong to $L^r(\Omega)$, $1 \le q, r \le \infty$. For the derivatives $D^j u$, $0 \le j < m$, the following inequality holds*

$$\|D^j u\|_{L^p} \le c(\|D^m u\|_{L^r} + \|u\|_{L^q})^\theta \|u\|_{L^q}^{1-\theta}, \tag{2.1}$$

where

$$\frac{1}{p} = \frac{j}{n} + \theta(\frac{1}{r} - \frac{m}{n}) + (1-\theta)\frac{1}{q}$$

for all θ in the interval

$$\frac{j}{m} \le \theta \le 1$$

(the constant c depending only on n, m, j, q, r, and θ), with the following exceptional case:

- ⋆ *If $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer then (2.1) holds only for θ satisfying $j/m \le \theta < 1$.*

Of course, we will be concerned with the case when $n = 3$. The following lemma (Uniform Gronwall Inequality [38]) will also be used.

Lemma 2.1. *Let $t_0 \in \mathbb{R}$ and let $y(t)$, $g(t)$, and $h(t)$ be three nonnegative locally integrable functions on (t_0, ∞). Suppose that y' is locally integrable on (t_0, ∞) and satisfies*

$$y'(t) \le g(t)y(t) + h(t) \qquad \textit{for all } t \ge t_0$$

and

$$\int_t^{t+r} y(s)ds \le \alpha_0, \quad \int_t^{t+r} h(s)ds \le \alpha_1, \quad \int_t^{t+r} g(s)ds \le \alpha_2 \quad \textit{for all } t \ge t_0,$$

where r, α_0, α_1, and α_2 are positive constants. Then we have

$$y(t+r) \le (\frac{\alpha_0}{r} + \alpha_1)e^{\alpha_2}, \quad \textit{for all } t \ge t_0.$$

Finally, the following result [38] will be used to establish the existence of the global attractor of (1.1)–(1.3).

Theorem 2.2. *Suppose that E is a Banach space and $\{S(t)\}_{t\ge 0}$ is a semigroup of continuous operators that map E into itself and enjoy the usual semigroup properties:*

$$S(t)\cdot S(\tau)=S(t+\tau), \qquad S(0)=I,$$

where I is the identity operator. Also suppose that the operator $S(t)$ satisfies

(i) *$S(t)$ is bounded, i.e., for any given $R>0$, if $\|u_0\|_E\le R$, then there exists a constant $C(R)$ such that*

$$\|S(t)u_0\|_E\le C(R), \qquad \text{for } t\in[0,\infty);$$

(ii) *There is a bounded absorbing set $\mathcal{B}_1\subset E$, i.e., for any given bounded set $\mathcal{B}\subset E$, there exists a constant $T=T(\mathcal{B})$ such that*

$$S(t)\mathcal{B}\subset\mathcal{B}_1, \quad \text{for } t\ge T;$$

(iii) *$S(t)$ is a completely continuous operator for $t>0$ sufficiently large.*

Then the semigroup $\{S(t)\}_{t\ge 0}$ of operators has a compact global attractor $\mathcal{A}\subset E$.

3. A priori estimates

In this section the main goal is to obtain *a priori* estimates of the solution to problem (1.1)–(1.3). In what follows, for brevity, we write $\int F$ to denote $\int_\Omega F dx$.

Lemma 3.1. *Suppose that $u_0\in L^2(\Omega)$ and f satisfies (1.4). For the solution $u(t)$ of (1.1)–(1.3), we have*

$$\|u\|^2+\int_0^t e^{\rho(s-t)}\left(2\|(-\triangle)^{\frac{\alpha}{2}}u\|^2+\beta_1\|u\|_{2\sigma+2}^{2\sigma+2}\right)ds\le e^{-\rho t}\|u_0\|^2+C_0'\le E_0, \ \text{for all } t>0,$$

$$\varlimsup_{t\to\infty}\left(\|u\|^2+\int_0^t e^{\rho(s-t)}\left(2\|(-\triangle)^{\frac{\alpha}{2}}u\|^2+\beta_1\|u\|_{2\sigma+2}^{2\sigma+2}\right)ds\right)\le C_0', \tag{3.1}$$

where

$$E_0=\|u_0\|^2+C_0' \ \text{ and } \ C_0'=\frac{3\sigma}{\sigma+1}\left(\frac{3\rho}{\beta_1(\sigma+1)}\right)^{\frac{1}{\sigma}}+\frac{2\gamma_1}{\rho}.$$

Proof. Taking the inner product in L^2 of (1.1) with u and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|(-\triangle)^{\frac{\alpha}{2}}u\|^2 = \rho\|u\|^2 + \mathrm{Re}\int f(u)\bar{u}. \tag{3.2}$$

Applying (1.4), we see that

$$\mathrm{Re}\int f(u)\bar{u} \le -\beta_1\int |u|^{2\sigma+2} + \gamma_1|\Omega|.$$

By Young's inequality,

$$3\rho\|u\|^2 = 3\rho\int |u|^2 \le \beta_1\int |u|^{2\sigma+2} + \rho\frac{3\sigma}{\sigma+1}\left(\frac{3\rho}{\beta_1(\sigma+1)}\right)^{\frac{1}{\sigma}}|\Omega|.$$

Then (3.2) implies

$$\begin{aligned}\frac{d}{dt}\|u\|^2 + 2\|(-\triangle)^{\frac{\alpha}{2}}u\|^2 + \beta_1\int |u|^{2\sigma+2} + \rho\|u\|^2 &\le \rho\left(\frac{3\sigma}{\sigma+1}\left(\frac{3\rho}{\beta_1(\sigma+1)}\right)^{\frac{1}{\sigma}} + \frac{2\gamma_1}{\rho}\right)|\Omega| \\ &= \rho C_0', \end{aligned} \tag{3.3}$$

and so

$$\|u\|^2 + \int_0^t e^{\rho(s-t)}\left(2\|(-\triangle)^{\frac{\alpha}{2}}u\|^2 + \beta_1\|u\|_{2\sigma+2}^{2\sigma+2}\right)ds \le e^{-\rho t}\|u_0\|^2 + C_0' \le E_0.$$

Therefore,

$$\overline{\lim_{t\to\infty}}\left(\|u\|^2 + \int_0^t e^{\rho(s-t)}\left(2\|(-\triangle)^{\frac{\alpha}{2}}u\|^2 + \beta_1\|u\|_{2\sigma+2}^{2\sigma+2}\right)ds\right) \le C_0'. \qquad \square$$

Remark 3.1. In particular, for $f(u) = -(1+\mathrm{i}\mu)|u|^{2\sigma}u$, one obtains the same result.

Corollary 3.1. *For any given $C_0 > C_0'$ and $R > 0$, if $\|u_0\|^2 \le R$ and f satisfies (1.4), then there exists $t_0 = t_0(R)$ such that*

$$\|u\|^2 \le C_0, \quad \text{for all } t \ge t_0 = \frac{1}{\rho}\ln\frac{R}{C_0 - C_0'}.$$

Lemma 3.2. *Suppose that $u_0 \in H_p^1(\Omega)$. If f satisfies (1.4) and (1.5), and if $\beta_\sigma \le 2|\lambda_\sigma|$, then the solution $u(t)$ to (1.1)–(1.3) satisfies*

$$\|\nabla u\|^2 + \int_0^t e^{\rho(s-t)}\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2 \le \|\nabla u_0\|^2 + \left(c_1 + \frac{1}{\rho}\right)(\|u_0\|^2 + C_0') \triangleq E_1,$$

and

$$\overline{\lim_{t\to\infty}}\left(\|\nabla u\|^2+\int_0^t e^{\rho(s-t)}\left(\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2\right)ds\right)\le 2\left(c_1+\frac{1}{\rho}\right)C_0'\triangleq C_1'.$$

Proof. Taking the inner product in L^2 of (1.1) with $-\triangle u$ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2=\rho\|\nabla u\|^2-\mathrm{Re}(f(u),\triangle u). \tag{3.4}$$

Integrating by parts and using (1.5) gives

$$\begin{aligned}-\mathrm{Re}(f(u),\triangle u)&=\mathrm{Re}\int\left(f_u(u)|\nabla u|^2+f_{\bar u}(u)\nabla\bar u\cdot\nabla\bar u\right)\\ &\le\int\left(-\beta_\sigma|u|^{2\sigma}|\nabla u|^2+|u|^{2\sigma-2}(\lambda_\sigma(u\nabla\bar u)^2+\bar\lambda_\sigma(\bar u\nabla u)^2)\right)\\ &\le\int|u|^{2(\sigma-1)}\left(-\beta_\sigma|u|^2|\nabla u|^2+\lambda_\sigma(u\nabla\bar u)^2+\bar\lambda_\sigma(\bar u\nabla u)^2\right)\\ &=\int|u|^{2(\sigma-1)}\mathrm{trace}(YMY^H),\end{aligned} \tag{3.5}$$

where

$$Y=\begin{pmatrix}\bar u\nabla u\\ u\nabla\bar u\end{pmatrix}^H,\quad M=\begin{pmatrix}-\frac{\beta_\sigma}{2} & \lambda_\sigma\\ \bar\lambda_\sigma & -\frac{\beta_\sigma}{2}\end{pmatrix},$$

and Y^H is the conjugate transpose of the matrix Y. We observe that the condition $\beta_\sigma\le 2|\lambda_\sigma|$ implies that the matrix M is nonpositive definite. From (3.4), we get

$$\frac{d}{dt}\|\nabla u\|^2+2\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2\le 2\rho\|\nabla u\|^2. \tag{3.6}$$

Using (2.1) and Young's inequality, we have

$$3\rho\|\nabla u\|^2\le\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2+(\rho c_1+1)\|u\|^2, \tag{3.7}$$

where

$$c_1=\frac{3c\alpha}{\alpha+1}\left(\frac{6\rho}{\alpha+1}\right)^{\frac{1}{\alpha}}.$$

Combining (3.6) and (3.7), we infer that

$$\frac{d}{dt}\|\nabla u\|^2+\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2+\rho\|\nabla u\|^2\le(\rho c_1+1)\|u\|^2. \tag{3.8}$$

Multiplying (3.8) by $e^{\rho t}$, integrating with respect to t and applying Lemma 3.1, we deduce that

$$\|\nabla u\|^2 + \int_0^t e^{\rho(s-t)} \|(-\triangle)^{\frac{\alpha+1}{2}} u\|^2 \le e^{-\rho t}\|\nabla u_0\|^2 + \left(c_1 + \frac{1}{\rho}\right)(\|u_0\|^2 + C_0')(1-e^{-\rho t})$$

$$\le \|\nabla u_0\|^2 + \left(c_1 + \frac{1}{\rho}\right)(\|u_0\|^2 + C_0') \triangleq E_1. \tag{3.9}$$

Therefore,

$$\overline{\lim_{t\to\infty}} \left(\|\nabla u\|^2 + \int_0^t e^{\rho(s-t)} \left(\|(-\triangle)^{\frac{\alpha+1}{2}} u\|^2\right) ds \right) \le 2\left(c_1 + \frac{1}{\rho}\right) C_0' \triangleq C_1'. \qquad \square$$

Remark 3.2. For $f(u) = -(1+\mathrm{i}\mu)|u|^{2\sigma}u$, taking $\beta_\sigma = (\sigma+1)$ and $\lambda_\sigma = \sigma(1+\mathrm{i}\mu)/2$, we obtain the same result under the condition

$$\sigma \ge \frac{1}{\sqrt{1+\mu^2}-1}.$$

Corollary 3.2. *For any given $C_1 > C_1'$ and $R > 0$, if $\|u_0\|^2_{H^1_p} \le R$, if f satisfies (1.4) and (1.5), and if $\beta_\sigma \le 2|\lambda_\sigma|$, then there exists $t_1 = t_1(R) > t_0$ such that*

$$\|\nabla u\|^2 \le C_1, \quad \text{for all } t \ge t_1.$$

Lemma 3.3. *Suppose that $u_0 \in H_p^{1+\alpha}(\Omega)$. If f satisfies (1.4)–(1.6), and if $\sigma < 3/(2-\alpha) - 1$, then the solution $u(t)$ of (1.1)–(1.3) satisfies*

$$\|(-\triangle)^{\frac{\alpha+1}{2}} u\|^2 \le \max\{\varrho_1, \widetilde{E}_1\} \triangleq \widehat{E} \text{ for all } t \ge 0,$$

where

$$\widetilde{E}_1 = e^{\alpha_1'}(\|(-\triangle)^{\frac{1+\alpha}{2}} u_0\|^2 + \alpha_2'),$$

for some constants α_j' $(j = 1, 2)$ and ϱ_1, defined in the proof below.

Proof. First, we estimate $\|u\|^{2\sigma+2}_{2\sigma+2}$. Taking the inner product in L^2 of (1.1) with $2(\sigma+1)|u|^{2\sigma}u$ and taking the real part, we obtain

$$\frac{d}{dt}\|u\|^{2\sigma+2}_{2\sigma+2} = 2(\sigma+1)\left(\rho\|u\|^{2\sigma+2}_{2\sigma+2} - \mathrm{Re}(1+\mathrm{i}\nu)\left((-\triangle)^\alpha u, |u|^{2\sigma}u\right) + \mathrm{Re}\left(f(u), |u|^{2\sigma}u\right)\right). \tag{3.10}$$

By Young's inequality, one has

$$-2\mathrm{Re}(1+\mathrm{i}\nu)\left((-\triangle)^\alpha u, |u|^{2\sigma}u\right) \le \frac{(1+\nu^2)}{\beta_1}\|(-\triangle)^\alpha u\|^2 + \beta_1\|u\|^{4\sigma+2}_{4\sigma+2}.$$

Applying (1.4), we see that

$$2\mathrm{Re}\left(f(u), |u|^{2\sigma}u\right) \le -2\beta_1\|u\|_{4\sigma+2}^{4\sigma+2} + 2\gamma_1\|u\|_{2\sigma}^{2\sigma}.$$

Then, (3.10) implies

$$\begin{aligned}
&\frac{d}{dt}\|u\|_{2\sigma+2}^{2\sigma+2} + \rho\|u\|_{2\sigma+2}^{2\sigma+2} + \beta_1(\sigma+1)\|u\|_{4\sigma+2}^{4\sigma+2} \\
&\le \frac{(\sigma+1)(1+\nu^2)}{\beta_1}\|(-\triangle)^{\alpha}u\|^2 + \rho(2\sigma+3)\|u\|_{2\sigma+2}^{2\sigma+2} + 2\gamma_1(\sigma+1)\|u\|_{2\sigma}^{2\sigma} \\
&\le \frac{(\sigma+1)(1+\nu^2)}{\beta_1}\|(-\triangle)^{\alpha}u\|^2 + (\rho(2\sigma+3)+2\gamma_1\sigma)\|u\|_{2\sigma+2}^{2\sigma+2} + 2\gamma_1|\Omega|.
\end{aligned}$$

Multiplying the above inequality by $e^{\rho t}$ and integrating with respect to t, we have

$$\begin{aligned}
&\|u\|_{2\sigma+2}^{2\sigma+2} + \beta_1(\sigma+1)\int_0^t e^{\rho(s-t)}\|u\|_{4\sigma+2}^{4\sigma+2}ds \\
&\le e^{-\rho t}\|u(0)\|_{2\sigma+2}^{2\sigma+2} + \frac{(\sigma+1)(1+\nu^2)}{\beta_1}\int_0^t e^{\rho(s-t)}\|(-\triangle)^{\alpha}u\|^2 ds \\
&\quad + (\rho(2\sigma+3)+2\gamma_1\sigma)\int_0^t e^{\rho(s-t)}\|u\|_{2\sigma+2}^{2\sigma+2}ds + \frac{2\gamma_1|\Omega|}{\rho}.
\end{aligned} \tag{3.11}$$

For the second term on the right-hand side of (3.11), by (2.1) and the Young inequality, we have

$$\|(-\triangle)^{\alpha}u\|^2 \le c(\|(-\triangle)^{\frac{\alpha+1}{2}}u\| + \|u\|)^{\frac{4\alpha}{\alpha+1}}\|u\|^{\frac{2(1-\alpha)}{\alpha+1}} \le c(\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2 + \|u\|^2).$$

Applying (3.1) and (3.9), we have

$$\int_0^t e^{\rho(s-t)}\|(-\triangle)^{\alpha}u\|^2 ds \le c(E_1 + \frac{E_0}{\rho}). \tag{3.12}$$

So, by (3.1) and (3.12), (3.11) implies

$$\begin{aligned}
&\|u\|_{2\sigma+2}^{2\sigma+2} + \beta_1(\sigma+1)\int_0^t e^{\rho(s-t)}\|u\|_{4\sigma+2}^{4\sigma+2}ds \\
&\le e^{-\rho t}\|u(0)\|_{2\sigma+2}^{2\sigma+2} + \frac{c(\sigma+1)(1+\nu^2)}{\beta_1}(E_1+\frac{E_0}{\rho}) + (\rho(2\sigma+3)+2\gamma_1\sigma)E_0 + \frac{2\gamma_1|\Omega|}{\rho} \\
&\le \|u(0)\|_{2\sigma+2}^{2\sigma+2} + \frac{c(\sigma+1)(1+\nu^2)}{\beta_1}(E_1+\frac{E_0}{\rho}) + (\rho(2\sigma+3)+2\gamma_1\sigma)E_0 + \frac{2\gamma_1|\Omega|}{\rho} \triangleq E_2'.
\end{aligned} \tag{3.13}$$

Next, we estimate $\|(-\triangle)^{\frac{1+\alpha}{2}}u\|$. Taking the inner product in L^2 of (1.1) with $(-\triangle)^{1+\alpha}u$ and taking the real part, we obtain

$$\frac{d}{dt}\|(-\triangle)^{\frac{1+\alpha}{2}}u\|^2 - 2\rho\|(-\triangle)^{\frac{1+\alpha}{2}}u\|^2 + 2\|(-\triangle)^{\alpha+\frac{1}{2}}u\|^2 = 2\mathrm{Re}\left(f(u), (-\triangle)^{1+\alpha}u\right). \tag{3.14}$$

Integrating by parts, applying (1.6) and (2.1), and using the Hölder, Young, and Gagliardo–Nirenberg inequalities, we infer that, when

$$\sigma < 3/(2-\alpha) - 1,$$

$$\begin{aligned}
&2\mathrm{Re}\left(f(u), (-\triangle)^{1+\alpha}u\right)\\
&\leq 2|(f_u(u)\nabla u + f_{\bar{u}}(u)\nabla\bar{u}, (-\triangle)^{\frac{1}{2}+\alpha}u)|\\
&\leq 4\beta_2\int |u|^{2\sigma}|\nabla u||(-\triangle)^{\frac{1}{2}+\alpha}u| + 4\gamma_2\int |\nabla u||(-\triangle)^{\frac{1}{2}+\alpha}u|\\
&\leq 4(\beta_2\|u\|_\infty^{2\sigma} + \gamma_2)\|(-\triangle)^{\frac{1}{2}+\alpha}u\|\|\nabla u\|\\
&\leq \frac{1}{2}\|(-\triangle)^{\frac{1}{2}+\alpha}u\|^2 + 16(\beta_2^2\|u\|_\infty^{4\sigma} + \gamma_2^2)\|\nabla u\|^2\\
&\leq \frac{1}{2}\|(-\triangle)^{\frac{1}{2}+\alpha}u\|^2 + 16c\beta_2^2\|\nabla u\|^2\left(\|(-\triangle)^{\frac{\alpha+1}{2}}u\| + \|u\|_{2\sigma+2}\right)^{4\sigma\theta_1}\|u\|_{2\sigma+2}^{4\sigma(1-\theta_1)} + 16\gamma_2^2\|\nabla u\|^2\\
&\leq \frac{1}{2}\|(-\triangle)^{\frac{1}{2}+\alpha}u\|^2 + 16c\beta_2^2\|\nabla u\|^2\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^4\\
&\quad + 16c(\beta_2^2+\gamma_2^2)\|\nabla u\|^2\left(1 + \left(1 + (1-\sigma\theta_1)(8\sigma\theta_1)^{\frac{\sigma\theta_1}{1-\sigma\theta_1}}\right)\|u\|_{2\sigma+2}^{\frac{4\sigma(1-\theta_1)}{1-\sigma\theta_1}}\right),
\end{aligned} \tag{3.15}$$

where

$$\theta_1 = \frac{3}{(2\alpha-1)\sigma + 2(\alpha+1)}.$$

We also obtain

$$2\rho\|(-\triangle)^{\frac{1+\alpha}{2}}u\|^2 \leq 2c\rho\left(\|(-\triangle)^{\frac{1}{2}+\alpha}u\| + \|u\|\right)^{\frac{2(1+\alpha)}{2\alpha+1}}\|u\|^{\frac{2\alpha}{2\alpha+1}} \leq \frac{1}{2}\|(-\triangle)^{\frac{1}{2}+\alpha}u\|^2 + c_2\|u\|^2, \tag{3.16}$$

where

$$c_2 = \frac{1}{2} + (2c\rho)^{\frac{2\alpha+1}{\alpha}}\frac{\alpha}{2\alpha+1}\left(\frac{4(\alpha+1)}{2\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}.$$

Combining (3.14)–(3.16), we deduce that

$$\frac{d}{dt}\|(-\triangle)^{\frac{1+\alpha}{2}}u\|^2+\|(-\triangle)^{\alpha+\frac{1}{2}}u\|^2\le h(t)\|(-\triangle)^{\frac{1+\alpha}{2}}u\|^2+g(t), \tag{3.17}$$

where

$$h(t)=16c\beta_2^2\|\nabla u\|^2\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2,$$

and

$$g(t)=16c(\beta_2^2+\gamma_2^2)\|\nabla u\|^2\left(1+\left(1+(1-\sigma\theta_1)(8\sigma\theta_1)^{\frac{\sigma\theta_1}{1-\sigma\theta_1}}\right)\|u\|_{2\sigma+2}^{\frac{4\sigma(1-\theta_1)}{1-\sigma\theta_1}}\right)+c_2\|u\|^2.$$

Dropping the third term on the left-hand side of (3.8), integrating from t to $t+1$, and applying Corollaries 3.1 and 3.2, we infer that

$$\int_t^{t+1}\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2ds\le\|\nabla u\|^2+\int_t^{t+1}(\rho c_1+1)\|u\|^2ds\le C_1+(\rho c_1+1)C_0\triangleq\alpha_0\quad\text{for } t\ge t_1.$$

So, by Corollaries 3.1 and 3.2 and (3.13), we obtain, for any $t\ge t_1$,

$$\int_t^{t+1}h(s)ds\le 16c\beta_2^2C_1\alpha_0\triangleq\alpha_1$$

and

$$\int_t^{t+1}g(s)ds\le 16c(\beta_2^2+\gamma_2^2)C_1\left(1+\left(1+(1-\sigma\theta_1)(8\sigma\theta_1)^{\frac{\sigma\theta_1}{1-\sigma\theta_1}}\right)E_2'^{\frac{2\sigma(1-\theta_1)}{(1-\sigma\theta_1)(\sigma+1)}}\right)+c_2C_0\triangleq\alpha_2.$$

Then using the Uniform Gronwall inequality (Lemma 2.1), we obtain

$$\|(-\triangle)^{\frac{1+\alpha}{2}}u\|^2\le(\alpha_0+\alpha_1)e^{\alpha_2}\triangleq\varrho_1,\quad\text{for any } t\ge t_1'=t_1+1. \tag{3.18}$$

For $t\le t_1'$, integrating (3.8) with respect to t from 0 to t, and applying Lemmas 3.1 and 3.2, we obtain

$$\int_0^t\|(-\triangle)^{\frac{\alpha+1}{2}}u\|^2ds\le\|\nabla u_0\|^2+(\rho c_1+1)t_1'\|u\|^2\le E_1+(\rho c_1+1)t_1'E_0\triangleq\alpha_0'.$$

So, by Lemmas 3.1 and 3.2 and (3.13), we obtain, for any $t\le t_1'$,

$$\int_0^t h(s)ds\le 16c\beta_2^2E_0\alpha_0'\triangleq\alpha_1',$$

and

$$\int_0^t g(s)ds \le 16c(\beta_2^2+\gamma_2^2)t_1' E_1\left(1+\left(1+(1-\sigma\theta_1)(8\sigma\theta_1)^{\frac{\sigma\theta_1}{1-\sigma\theta_1}}\right)E_2'^{\frac{2\sigma(1-\theta_1)}{(1-\sigma\theta_1)(\sigma+1)}}\right)$$
$$+c_2 t_1' E_0 \triangleq \alpha_2'.$$

Applying Gronwall's inequality, for any $t \le t_1'$, we infer

$$\begin{aligned}\|(-\triangle)^{\frac{1+\alpha}{2}}u\|^2 &\le e^{\int_0^t h(s)ds}\left(\|(-\triangle)^{\frac{1+\alpha}{2}}u_0\|^2+\int_0^t g(s)ds\right)\\ &\le e^{\alpha_1'}(\|(-\triangle)^{\frac{1+\alpha}{2}}u_0\|^2+\alpha_2')\triangleq \widetilde{E}_1.\end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19), we have

$$\|(-\triangle)^{\frac{1+\alpha}{2}}u\|^2 \le \widehat{E}=\max\{\varrho_1,\widetilde{E}_1\},\quad \text{for any } t\ge 0.$$

This completes the proof. □

Remark 3.3. For $f(u)=-(1+\mathrm{i}\mu)|u|^{2\sigma}u$, the same result can be established under the condition

$$\frac{1}{\sqrt{1+\mu^2}-1}\le\sigma<\frac{3}{2-\alpha}-1.$$

Corollary 3.3. *Suppose that $u_0\in H_p^{1+\alpha}(\Omega)$. If f satisfies (1.4)–(1.6), and if $\sigma<3/(2-\alpha)-1$, with c_3 the constant associated with the embedding of $H^{1+\alpha}$ into L^∞, then one has*

$$\|u\|_\infty^2\le c_3(E_0+\widehat{E})\triangleq E_\infty \textit{ for all } t\ge 0.$$

Therefore, if $\|u_0\|^2_{H^{1+\alpha}}\le R$, then

$$\|u\|_\infty^2\le c_3(C_0+\varrho_1)\triangleq C_\infty,\qquad \textit{for all } t\ge t_1'.$$

We would like to point out that, in Corollary 3.3, one has $C_\infty\le E_\infty$, which follows from $C_0 \backsimeq C_0'\le E_0$ and $\varrho_1\le\widehat{E}$ that can be derived from Lemma 3.1, Corollary 3.1 and Lemma 3.3.

Lemma 3.4. *Suppose that $u_0\in H_p^2(\Omega)$. If f satisfies (1.4)–(1.7) and if $\sigma\ge 1/2$, then the solution $u(t)$ of (1.1)–(1.3) satisfies*

$$\|\triangle u\|^2+\int_0^t e^{\rho(s-t)}\|(-\triangle)^{1+\frac{\alpha}{2}}u\|^2\le\|\triangle u_0\|^2+\frac{2B+c_4E_0}{\rho}\triangleq E_2,$$

and

$$\varlimsup_{t\to\infty}\left(\|\Delta u\|^2+\int_0^t e^{\rho(s-t)}\left(\|(-\Delta)^{1+\frac{\alpha}{2}}u\|^2\right)ds\right)\le\frac{2B'+c_5C_0'}{\rho}\triangleq C_2',$$

where the constants B, B', c_4 and c_5 are given in the proof of this lemma.

Proof. Taking the inner product in L^2 of (1.1) with $\Delta^2 u$ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2+\|(-\Delta)^{1+\frac{\alpha}{2}}u\|^2=\rho\|\Delta u\|^2+\mathrm{Re}\left(f(u),\Delta^2u\right). \tag{3.20}$$

Integrating by parts and applying (1.6) and (1.7), we deduce that

$$\begin{aligned}&\mathrm{Re}\left(f(u),\Delta^2u\right)\\&=\mathrm{Re}\left(f_{uu}(u)\nabla u\nabla u+2f_{u\bar u}(u)|\nabla u|^2+f_{\bar u}(u)\nabla\bar u\nabla\bar u+f_u(u)\Delta u+f_{\bar u}(u)\Delta\bar u,\Delta u\right)\\&\le 4\beta_3\int|u|^{2\sigma-1}|\nabla u|^2|\Delta u|+4\gamma_3\int|\nabla u|^2|\Delta u|+2\beta_2\int|u|^{2\sigma}||\Delta u|^2+2\gamma_2\int|\Delta u|^2.\end{aligned} \tag{3.21}$$

Applying Hölder, Gagliardo–Nirenberg, and Young inequalities, we obtain the following estimates when $\sigma\ge 1/2$,

$$\begin{aligned}&4\beta_3\int|u|^{2\sigma-1}|\nabla u|^2|\Delta u|+4\gamma_3\int|\nabla u|^2|\Delta u|\\&\le 4\beta_3\|\Delta u\|\|\nabla u\|_4^2\|u\|_\infty^{2\sigma-1}+4\gamma_3\|\nabla u\|_4^2\|\Delta u\|\\&\le\|\Delta u\|^2+8\beta_3^2\|\nabla u\|_4^4\|u\|_\infty^{2(2\sigma-1)}+8\gamma_3^2\|\nabla u\|_4^4\\&\le\|\Delta u\|^2+\left(\|(-\Delta)^{1+\frac{\alpha}{2}}u\|+\|\nabla u\|\right)^{\frac{3}{\alpha+1}}\|\nabla u\|^{\frac{4\alpha+1}{\alpha+1}}\left(8c\beta_3^2\|u\|_\infty^{2(2\sigma-1)}+8c\gamma_3^2\right)\\&\le\|\Delta u\|^2+\frac14\|(-\Delta)^{1+\frac{\alpha}{2}}u\|^2+A,\end{aligned} \tag{3.22}$$

where

$$A=\frac{2\alpha-1}{2(\alpha+1)}\left(\frac{12}{\alpha+1}\right)^{\frac{3}{2\alpha-1}}\left(8c\beta_3^2\|u\|_\infty^{2(2\sigma-1)}+8c\gamma_3^2\right)^{\frac{2(\alpha+1)}{2\alpha-1}}\|\nabla u\|^{\frac{2(4\alpha+1)}{2\alpha-1}}+\frac14\|\nabla u\|^2.$$

A direct application of Lemma 3.2 and Corollary 3.3 gives

$$A\le\frac{2\alpha-1}{2(\alpha+1)}\left(\frac{12}{\alpha+1}\right)^{\frac{3}{2\alpha-1}}\left(8c\beta_3^2E_\infty^{2\sigma-1}+8c\gamma_3^2\right)^{\frac{2(\alpha+1)}{2\alpha-1}}E_1^{\frac{4\alpha+1}{2\alpha-1}}+\frac14E_1\triangleq B.$$

For the last two terms on the right-hand side of (3.21), we obtain

$$2\beta_2 \int |u|^{2\sigma} |\triangle u|^2 + 2\gamma_2 \int |\triangle u|^2 \leq 2\left(\gamma_2 + \beta_2 E_\infty^\sigma\right) \|\triangle u\|^2. \tag{3.23}$$

By (3.21)–(3.23), we have

$$\mathrm{Re}\left(f(u), \triangle^2 u\right) \leq 2\left(\frac{1}{2} + \gamma_2 + \beta_2 E_\infty^\sigma\right) \|\triangle u\|^2 + \frac{1}{4}\|(-\triangle)^{1+\frac{\alpha}{2}} u\|^2 + B. \tag{3.24}$$

Putting (3.24) into (3.20), we obtain

$$\frac{d}{dt}\|\triangle u\|^2 + \frac{3}{2}\|(-\triangle)^{1+\frac{\alpha}{2}} u\|^2 \leq 2(1+\rho+2\gamma_2+2\beta_2 E_\infty^\sigma)\|\triangle u\|^2 + 2B. \tag{3.25}$$

Applying the Gagliardo–Nirenberg and Young inequalities, we deduce that

$$\begin{aligned}
&(2+3\rho+4\gamma_2+4\beta_2 E_\infty^\sigma)\|\triangle u\|^2 \\
&\leq c(2+3\rho+4\gamma_2+4\beta_2 E_\infty^\sigma)\left(\|(-\triangle)^{1+\frac{\alpha}{2}} u\| + \|u\|\right)^{\frac{4}{2+\alpha}} \|u\|^{\frac{2\alpha}{2+\alpha}} \\
&\leq \frac{1}{2}\|(-\triangle)^{1+\frac{\alpha}{2}} u\|^2 + c_4\|u\|^2,
\end{aligned}$$

where

$$c_4 = \frac{c\alpha}{2+\alpha}\left(\frac{8}{2+\alpha}\right)^{\frac{2}{\alpha}} (2+3\rho+4\gamma_2+4\beta_2 E_\infty^\sigma)^{\frac{2+\alpha}{\alpha}} + \frac{1}{2}.$$

Then (3.25) can be rewritten as

$$\frac{d}{dt}\|\triangle u\|^2 + \|(-\triangle)^{1+\frac{\alpha}{2}} u\|^2 + \rho\|\triangle u\|^2 \leq 2B + c_4\|u\|^2 \leq 2B + c_4 E_0. \tag{3.26}$$

Multiplying (3.26) by $e^{\rho t}$ and integrating with respect to t, one has

$$\begin{aligned}
\|\triangle u\|^2 + \int_0^t e^{\rho(s-t)}\|(-\triangle)^{1+\frac{\alpha}{2}} u\|^2 &\leq e^{-\rho t}\|\triangle u_0\|^2 + \frac{2B + c_4 E_0}{\rho} \\
&\leq \|\triangle u_0\|^2 + \frac{2B + c_4 E_0}{\rho} \triangleq E_2,
\end{aligned}$$

and

$$\varlimsup_{t\to\infty}\left(\|\triangle u\|^2 + \int_0^t e^{\rho(s-t)}\left(\|(-\triangle)^{1+\frac{\alpha}{2}} u\|^2\right) ds\right) \leq \frac{2B' + c_5 C_0'}{\rho} \triangleq C_2',$$

where

$$B' = \frac{2\alpha-1}{2(\alpha+1)}\left(\frac{12}{\alpha+1}\right)^{\frac{3}{2\alpha-1}}\left(8c\beta_3^2 C_\infty^{2\sigma-1} + 8c\gamma_3^2\right)^{\frac{2(\alpha+1)}{2\alpha-1}} C'^{\frac{4\alpha+1}{2\alpha-1}}_1 + \frac{1}{4}C_1'$$

and

$$c_5 = \frac{c\alpha}{2+\alpha}\left(\frac{8}{2+\alpha}\right)^{\frac{2}{\alpha}}(2+3\rho+4\gamma_2+4\beta_2 C_\infty^\sigma)^{\frac{2+\alpha}{\alpha}} + \frac{1}{2}. \qquad \square$$

Remark 3.4. For $f(u) = -(1+\mathrm{i}\mu)|u|^{2\sigma}u$, we have the same result under the condition

$$\max\left\{\frac{1}{2}, \frac{1}{\sqrt{1+\mu^2}-1}\right\} \le \sigma < \frac{3}{2-\alpha} - 1.$$

Corollary 3.4. *For any given $C_2 > C_2'$ and $R > 0$, if $\|u_0\|_{H_p^2}^2 \le R$, if f satisfies (1.4)–(1.7), and if $\sigma \ge 1/2$, then there exists $t_2 = t_2(R) > t_1'$ such that*

$$\|\triangle u\|^2 \le C_2, \quad \textit{for all } t \ge t_2.$$

Lemma 3.5. *Suppose that $u_0 \in H_p^2(\Omega)$ with $\|u_0\|_{H_p^2}^2 \le R$. If f satisfies (1.4)–(1.7), then the solution to (1.1)–(1.3) satisfies*

$$\|(-\triangle)^{1+\frac{\alpha}{2}}u\|^2 \le (\alpha_3' + B_1)e^{2\rho} \triangleq C_3 \quad \textit{for all } t \ge t_2(R)+1,$$

where constants α_3' and B_1 are given in the proof of this lemma.

Proof. Taking the inner product in L^2 of (1.1) with $(-\triangle)^{2+\alpha}u$ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(-\triangle)^{1+\frac{\alpha}{2}}u\|^2 + \|(-\triangle)^{1+\alpha}u\|^2 = \rho\|(-\triangle)^{1+\frac{\alpha}{2}}u\|^2 + \mathrm{Re}\left(f(u), (-\triangle)^{2+\alpha}u\right). \tag{3.27}$$

Integrating by parts and applying (1.6) and (1.7), we deduce that

$$\begin{aligned} &\mathrm{Re}\left(f(u), (-\triangle)^{2+\alpha}u\right) \\ &= \mathrm{Re}\left(f_u''(u)\nabla u\nabla u + 2f_{u\bar u}''(u)|\nabla u|^2 + f_{\bar u}''(u)\nabla\bar u\nabla\bar u + f_u'(u)\triangle u + f_{\bar u}'(u)\triangle\bar u, (-\triangle)^{1+\alpha}u\right) \\ &\le 4\beta_3\int |u|^{2\sigma-1}|\nabla u|^2|(-\triangle)^{1+\alpha}u| + 4\gamma_3\int |\nabla u|^2|(-\triangle)^{1+\alpha}u| \\ &\quad + 2\beta_2\int |u|^{2\sigma}||\triangle u||(-\triangle)^{1+\alpha}u| + 2\gamma_2\int |\triangle u||(-\triangle)^{1+\alpha}u|. \end{aligned} \tag{3.28}$$

Applying the Hölder, Gagliardo–Nirenberg, and Young inequalities, one has

$$
\begin{aligned}
&4\beta_3 \int |u|^{2\sigma-1}|\nabla u|^2|(-\triangle)^{1+\alpha}u| + 4\gamma_3 \int |\nabla u|^2|(-\triangle)^{1+\alpha}u| \\
&\le 4\beta_3\|u\|_\infty^{2\sigma-1}\|\nabla u\|_4^2\|(-\triangle)^{1+\alpha}u\| + 4\gamma_3\|\nabla u\|_4^2\|(-\triangle)^{1+\alpha}u\| \\
&\le \frac{1}{4}\|(-\triangle)^{1+\alpha}u\|^2 + 32\left(\beta_3^2\|u\|_\infty^{2(2\sigma-1)} + \gamma_3^2\right)\|\nabla u\|_4^4 \\
&\le \frac{1}{4}\|(-\triangle)^{1+\alpha}u\|^2 + 32c\left(\beta_3^2\|u\|_\infty^{2(2\sigma-1)} + \gamma_3^2\right)\|u\|_{H_p^2}^4 \\
&\le \frac{1}{4}\|(-\triangle)^{1+\alpha}u\|^2 + 32c\left(\beta_3^2 C_\infty^{2\sigma-1} + \gamma_3^2\right)(C_0+C_2)^2, \qquad \text{for all } t \ge t_2.
\end{aligned} \tag{3.29}
$$

For the last two terms on the right-hand side of (3.28), we obtain

$$
\begin{aligned}
&2\beta_2 \int |u|^{2\sigma}||\triangle u||(-\triangle)^{1+\alpha}u| + 2\gamma_2 \int |\triangle u||(-\triangle)^{1+\alpha}u| \\
&\le 2\left(\beta_2\|u\|_\infty^{2\sigma} + \gamma_2\right)\|\triangle u\|\|(-\triangle)^{1+\alpha}u\| \\
&\le \frac{1}{4}\|(-\triangle)^{1+\alpha}u\|^2 + 8\left(\beta_2^2\|u\|_\infty^{4\sigma)} + \gamma_2^2\right)\|\triangle u\|^2 \\
&\le \frac{1}{4}\|(-\triangle)^{1+\alpha}u\|^2 + 8\left(\beta_2^2 C_\infty^{2\sigma} + \gamma_2^2\right)C_2^2, \qquad \text{for all } t \ge t_2.
\end{aligned} \tag{3.30}
$$

Then the second term on the right-hand side of (3.27) is controlled by

$$
\mathrm{Re}\left(f(u), (-\triangle)^{2+\alpha}u\right) \le \frac{1}{2}\|(-\triangle)^{1+\alpha}u\|^2 + \frac{B_1}{2}, \tag{3.31}
$$

where

$$
B_1 = 64c\left(\beta_3^2 C_\infty^{2\sigma-1} + \gamma_3^2\right)(C_0+C_2)^2 + 16\left(\beta_2^2 C_\infty^{2\sigma} + \gamma_2^2\right)C_2^2.
$$

Putting (3.31) into (3.27), we obtain that

$$
\frac{d}{dt}\|(-\triangle)^{1+\frac{\alpha}{2}}u\|^2 + \|(-\triangle)^{1+\alpha}u\|^2 \le 2\rho\|(-\triangle)^{1+\frac{\alpha}{2}}u\|^2 + B_1. \tag{3.32}
$$

By Lemma 3.4, one has

$$
\int_t^{t+1} \|(-\triangle)^{1+\frac{\alpha}{2}}u\|^2 \le C_2 + 2B + c_4E_0 \triangleq \alpha_3'. \tag{3.33}
$$

Using the Uniform Gronwall inequality (Lemma 2.1), we obtain

$$
\|(-\triangle)^{1+\frac{\alpha}{2}}u\|^2 \le (\alpha_3' + B_1)e^{2\rho} \triangleq C_3, \qquad \text{for all } t \ge t_2 + 1. \qquad \square
$$

4. Existence and uniqueness of solution

In this section, our aim is to establish the existence of a weak solution to (1.1)–(1.3) by using the Faedo–Galerkin method. Throughout this section $T > 0$ is a fixed time. The following lemmas [13] are crucial for our main result.

Lemma 4.1. *Let X_0, X and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$ and X_i, $i = 0, 1$ reflexive. Assume that X_0 is compactly embedded in X. Let*

$$W = \{\omega \in L^{p_0}(0, T; X_0) : \omega' = \frac{d\omega}{dt} \in L^{p_1}(0, T; X_1)\},$$

where $1 < p_i < \infty$, $i = 0, 1$. Then with the norm

$$\|\omega\|_{L^{p_0}(0,T;X_0)} + \|\omega'\|_{L^{p_1}(0,T;X_1)},$$

W is compactly embedded in $L^{p_0}(0, T; X)$.

Lemma 4.2. *Let X be a Banach space. Assume that $\psi \in L^p(0, T; X)$ and $d\psi/dt \in L^p(0, T; X)$ for some $p \in [1, \infty]$. Then $\psi \in C([0, T]; X)$, after possibly being redefined on a set of measure zero.*

Theorem 4.1. *Suppose that $u_0 \in H^2_p(\Omega)$. If f satisfies (1.4)–(1.8), if $\beta_\sigma \le 2|\lambda_\sigma|$, and if $1/2 \le \sigma < 3/(2 - \alpha) - 1$, then there exists a unique global smooth solution $u = u(x, t)$ to (1.1)–(1.3) such that*

$$u \in L^\infty\left(0, T; H^2_p(\Omega)\right) \text{ and } u_t \in L^\infty\left(0, T; L^q(\Omega)\right), \tag{4.1}$$

with $q = 1 + \frac{1}{2\sigma+1}$. Moreover, (1.1) and (1.3) define a continuous dynamical system $S(t)$ on $H^2_p(\Omega)$, and has a bounded absorbing set $\mathcal{B}_1 \subset H^2_p(\Omega)$.

Proof. There are two parts for the proof. In the first part, the existence of a solution to problem (1.1)–(1.3) is established. In the second part, we prove uniqueness.

Part I: Existence of solutions

Using the Faedo–Galerkin method, we construct approximate solutions to the problem. Given a positive integer m, one can find a function $u_m = u_m(t)$ of the form

$$u_m(t) = \sum_{|k|=0}^{m} d^k_m(t)\omega_k, \qquad \omega_k = e^{\mathrm{i}\langle k,x\rangle}, \tag{4.2}$$

where the coefficients $d^k_m(t)$, for $0 \le t \le T$, $|k| = 0, 1, 2, \cdots, m$, are chosen so that

$$(u'_m, \omega_k) = \rho(u_m, \omega_k) - (1 + \mathrm{i}\nu)((-\triangle)^\alpha u_m, \omega_k) + (f(u_m), \omega_k), \tag{4.3}$$

and

$$u_m(0) = u_{0m} \in X_m \equiv \operatorname{span}\{\omega_k\}_{|k|=0}^m, \quad u_{0m} \to u_0 \quad \text{in } H_p^2 \quad \text{as}\, m \to \infty. \tag{4.4}$$

Note that (4.3) is a system of nonlinear ODEs subject to the initial condition (4.4). Since we are working in $H_p^2 \subset C(\Omega)$, the nonlinearity is locally Lipschitz in X_m. According to standard existence theory for nonlinear ordinary differential equations, there exists a unique solution of (4.3)–(4.4) for $0 \le t \le t_m$. The *a priori* estimates for u_m are similar to those for u, following the same proof, and so we can take $t_m = T$, independent of m. To illustrate this we follow the proof of Lemma 3.1 to obtain uniform estimates.

First, we show that u_m is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$.

Multiply equality (4.3) by $d_m^k(t)$, sum $|k| = 0, 1, 2, \dots, m$, and recall (4.2) to discover

$$(u_m', u_m) = \rho(u_m, u_m) - (1 + \mathrm{i}\nu)((-\triangle)^\alpha u_m, u_m) + (f(u_m), u_m)\,. \tag{4.5}$$

Taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \|(-\triangle)^{\frac{\alpha}{2}} u_m\|^2 = \rho\|u_m\|^2 + \operatorname{Re}\int f(u_m)\bar{u}_m. \tag{4.6}$$

Applying (1.4), we deduce

$$\operatorname{Re}\int f(u_m)\bar{u}_m \le -\beta_1 \int |u_m|^{2\sigma+2} + \gamma_1|\Omega|.$$

By Young's inequality, one has

$$3\rho\|u_m\|^2 = 3\rho\int |u_m|^2 \le \beta_1 \int |u_m|^{2\sigma+2} + \rho\frac{3\sigma}{\sigma+1}\left(\frac{3\rho}{\beta_1(\sigma+1)}\right)^{\frac{1}{\sigma}} |\Omega|.$$

Then (4.6) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\|u_m\|^2 + 2\|(-\triangle)^{\frac{\alpha}{2}} u_m\|^2 + \beta_1\int |u_m|^{2\sigma+2} + \rho\|u_m\|^2 &\le \rho\left(\frac{3\sigma}{\sigma+1}\left(\frac{3\rho}{\beta_1(\sigma+1)}\right)^{\frac{1}{\sigma}} + \frac{2\gamma_1}{\rho}\right)|\Omega| \\ &= \rho C_0',\end{aligned} \tag{4.7}$$

which implies

$$\|u_m\|^2 \le e^{-\rho t}\|u_0\|^2 + C_0'.$$

Therefore, one has that u_m is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$.

Following exactly the same argument as in the proofs of Lemmas 3.2–3.4, we obtain

$$\|\nabla u_m\|^2 \le \|\nabla u_0\|^2 + \left(c_1 + \frac{1}{\rho}\right)(\|u_0\|^2 + C_0') \triangleq E_1,$$

and

$$\|\triangle u_m\|^2 \leq \|\triangle u_0\|^2 + \frac{2B + c_4 E_0}{\rho} \triangleq E_2.$$

So we infer that u_m is uniformly bounded in $L^\infty(0, T; H^2(\Omega))$.

Secondly, we prove that $u'_m \in L^\infty(0, T; L^q(\Omega))$, with $q = 1 + \frac{1}{2\sigma+1}$.

For for all $v \in L^{2\sigma+2}(\Omega)$, we have

$$(u'_m, v) = \rho(u_m, v) - (1 + \mathrm{i}\nu)((-\triangle)^\alpha u_m, v) + (f(u_m), v). \tag{4.8}$$

Applying (1.8), together with Hölder inequality and Gagliardo–Nirenberg inequality, we have

$$\begin{aligned} |(u'_m, v)| &\leq |\rho(u_m, v)| + |(1 + \mathrm{i}\nu)((-\triangle)^\alpha u_m, v)| + |(f(u_m), v)| \\ &\leq C(\|u_m\|\|v\| + \|(-\triangle)^\alpha u_m\|\|v\| + \|u_m\|_{2\sigma+2}^{2\sigma+1}\|v\|_{2\sigma+2} + \|v\|) \\ &\leq C\|v\|_{2\sigma+2}. \end{aligned} \tag{4.9}$$

Therefore, u'_m is uniformly bounded in $L^\infty(0, T; L^q(\Omega))$.

With these estimates, we pass to limits as $m \to \infty$, to build a weak solution of our problem (1.1)–(1.3).

Using our uniform bound on u_m, there exists a subsequence $\{u_{ml}\}_{l=1}^\infty \subset \{u_m\}_{m=1}^\infty$ and a function $u \in L^\infty(0, T; H_p^2(\Omega))$ with $u' \in L^\infty(0, T; L^q(\Omega))$, such that

$$u_{ml} \rightharpoonup u \qquad * -\text{weakly in } L^\infty(0, T; H_p^2(\Omega)), \tag{4.10}$$

$$u'_{ml} \rightharpoonup u' = u_t \qquad * -\text{weakly in } L^\infty(0, T; L^q(\Omega)). \tag{4.11}$$

And we obtain

$$\{u_m\}_{m=1}^\infty \text{ is bounded in } L^2(0, T; H_p^2(\Omega)), \tag{4.12}$$

$$\{u'_m\}_{m=1}^\infty \text{ is bounded in } L^2(0, T; L^q(\Omega)). \tag{4.13}$$

By the Sobolev embedding theorem, we infer

$$\{u_m\}_{m=1}^\infty \text{ is bounded in } L^2(0, T; H^{\frac{3}{2}}(\Omega)). \tag{4.14}$$

Let $W = \{v : v \in L^2(0, T; H^{\frac{3}{2}}(\Omega)), v' \in L^2(0, T; L^q(\Omega))\}$. Since $H^{\frac{3}{2}}(\Omega)$ is compactly embedded in $L^2(\Omega)$, W is compactly embedded in $L^2(0, T; L^2(\Omega))$ by Lemma 4.1. By (4.13) and (4.14), $u_m \in W$. Then, there exists a subsequence u_{ml} which satisfies

$$u_{ml} \longrightarrow u \qquad \text{strongly in } L^2(0, T; L^2(\Omega)) \text{ and a.e.} \tag{4.15}$$

Applying (1.8), (4.15) and Lemma 3.3, we infer that

$$f(u_{ml}) \rightharpoonup f(u) \qquad * -\text{weakly in } L^\infty(0, T; L^q(\Omega)). \tag{4.16}$$

By (4.3), fixing k, we deduce that

$$(u'_{ml}, \omega_k) = \rho(u_{ml}, \omega_k) - (1 + i\nu)((-\triangle)^\alpha u_{ml}, \omega_k) + (f(u_{ml}), \omega_k)\,. \tag{4.17}$$

Applying (4.10), (4.11) and (4.16), we obtain that there exists a subsequence $\{u_{ml}\}_{l=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ such that

$$\begin{aligned} ((-\triangle)^\alpha u_{ml}, \omega_k) &\rightharpoonup ((-\triangle)^\alpha u, \omega_k) && * \text{-weakly in } L^\infty(0,T), \\ (u'_{ml}, \omega_k) &\rightharpoonup (u', \omega_k) && * \text{-weakly in } L^\infty(0,T), \\ (f(u_{ml}), \omega_k) &\rightharpoonup (f(u), \omega_k) && * \text{-weakly in } L^\infty(0,T). \end{aligned}$$

Then from (4.17), we have

$$(u_t, \omega_k) = \rho(u, \omega_k) - (1 + i\nu)((-\triangle)^\alpha u, \omega_k) + (f(u), \omega_k)\,.$$

The above equality holds for any fixed k. By the density of the basis ω_k, we obtain

$$(u_t, \varphi) = \rho(u, \varphi) - (1 + i\nu)((-\triangle)^\alpha u, \varphi) + (f(u), \varphi)\,, \quad \text{for all } \varphi \in H^2(\Omega).$$

Hence, u satisfies (1.1) and (4.1). By the uniform bound on u_m and Lemma 4.2, we obtain

$$u_{ml} \in C([0,T], L^q(\Omega)).$$

Then, we have

$$u_{ml}(0) \rightharpoonup u(0) \qquad \text{weakly in } L^q(\Omega).$$

But from (4.4), we obtain $u_{ml}(0) \to u_0$ in $H^2(\Omega)$. Hence, we deduce that $u(0) = u_0$.

Part II: Uniqueness

Assume that there are two solutions u and v to the problem (1.1)–(1.3). Let $\psi = u - v$. Then

$$\psi_t = \rho\psi - (1 + i\nu)(-\triangle)^\alpha \psi + (f(u) - f(v))\,, \tag{4.18}$$

with the initial condition $\psi(0) = 0$. Taking the inner product in L^2 of (4.18) with ψ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi\|^2 + \|(-\triangle)^{\frac{\alpha}{2}}\psi\|^2 = \rho\|\psi\|^2 + \mathrm{Re}\,(f(u) - f(v), u - v)\,. \tag{4.19}$$

Applying (1.6), we deduce that

$$\begin{aligned} f(u) - f(v) &= \int_0^1 \frac{d}{d\theta} f(v + \theta(u - v))d\theta \\ &\le \Big(\beta_2(|v + \theta_1(u - v)|^{2\sigma} + |v + \theta_2(u - v)|^{2\sigma}) + 2\gamma_2\Big)\,|u - v|, \end{aligned}$$

for some $\theta_1, \theta_2 \in (0,1)$. Then by Corollary 3.3 we have

$$\operatorname{Re}(f(u) - f(v), u - v) \le C\|\psi\|^2, \tag{4.20}$$

and so

$$\frac{d}{dt}\|\psi\|^2 \le C\|\psi\|^2. \tag{4.21}$$

Since $\psi(0) = 0$, we obtain $\psi \equiv 0$. □

5. The global attractor and its dimensions

Next, we establish the existence of the global attractor and estimate its Hausdorff and fractal dimensions.

Theorem 5.1. *Under the conditions of Theorem 4.1, there exists a global attractor $\mathcal{A}$ of the semigroup $\{S(t)\}_{t\ge 0}$ of operators generated by problem (1.1)–(1.3), i.e., there is a set $\mathcal{A}$ such that*

(i) $S(t)\mathcal{A} = \mathcal{A}$ $t \in \mathbb{R}^+$,
(ii) $\lim\limits_{t\to\infty} dist(S(t)\mathcal{B}, \mathcal{A}) = 0$, *for any bounded* $\mathcal{B} \subset H_p^2(\Omega)$, *where*

$$dist(X, Y) = \sup_{x\in X}\inf_{y\in Y}\|x - y\|_E.$$

Proof. It suffices to check conditions (i)–(iii) in Theorem 2.2. Define the Banach space $E = H_p^2(\Omega)$ and $S(t) : H_p^2(\Omega) \to H_p^2(\Omega)$ the solution operator.

Setting $\mathcal{B} = \{u_0 \in H_p^2(\Omega) : \|u_0\|^2_{H^2} \le R\}$, by the results of Lemmas 3.1–3.4 and their corollaries, we deduce that

$$\|S(t)u_0\|^2_{H_p^2} = \|u\|^2_{H_p^2} \le E_0 + E_1 + E_2, \qquad \text{for all } t \ge 0,$$

which implies that $S(t)$ is uniformly bounded in $H_p^2(\Omega)$.

By the results stated in the corollaries of Lemmas 3.1–3.4, we infer that

$$\|S(t)u_0\|^2_{H_p^2} = \|u\|^2_{H_p^2} \le C_0 + C_1 + C_2, \qquad \text{for all } t \ge t_2(R).$$

Therefore, the set $\mathcal{B}_1 = \{u \in H_p^2(\Omega) : \|u\|^2_{H_p^2} \le C_0 + C_1 + C_2\}$ is a bounded absorbing set for the semigroup of operators $S(t)$.

Using Lemma 3.5 and the compact embedding $H_p^{2+\alpha}(\Omega) \hookrightarrow H_p^2(\Omega)$, we infer that the semigroup of operators $S(t) : H_p^2 \to H_p^2$ is completely continuous for t sufficiently large.

Applying Theorem 2.2 completes the proof. □

We now show that both the Hausdorff and fractal dimensions of the maximal attractor $\mathcal{A}$ are finite. To the end, we first rewrite (1.1) in the abstract form:

$$\frac{du}{dt} = F(u), \tag{5.1}$$

where

$$F(u) = \rho u - (1 + \mathrm{i}\nu)(-\triangle)^{\alpha} u + f(u).$$

Let $u(t) = S(t)u_0$ be the solution to (1.1)–(1.3) with $u_0 \in \mathcal{A}$. Then we consider the first variation equation of (1.1)–(1.3),

$$v_t = F'(u(t))v, \tag{5.2}$$

with the initial condition

$$v(0) = v_0 \in L^2(\Omega), \tag{5.3}$$

and require v to be Ω-periodic. Here,

$$F'(u(t))v = \rho v - (1 + \mathrm{i}\nu)(-\triangle)^{\alpha} v + f'(u)v.$$

We know that given $u_0 \in \mathcal{A} \subset H_p^2(\Omega)$, $S(t)u_0 \in \mathcal{A} \subset H_p^2(\Omega)$. Hence, applying standard methods, we can show that for all $v_0 \in L^2(\Omega)$, the linear initial–boundary value problem (5.2) and (5.3) possesses a unique solution

$$v(t) \in L^2(0, T; H_p^2(\Omega)) \cap L^{\infty}(0, T; L^2(\Omega)), \qquad \text{for all } T > 0. \tag{5.4}$$

This then defines a solution operator $S_1(t, u_0)$ such that $v(t) = S_1(t, u_0)v_0$. In addition, we can prove that the semigroup operator $S(t)$ is differentiable in $L^2(\Omega)$, and its Fréchet derivative $\delta S(t)u_0 = S_1(t, u_0)$. In fact, if we define $u_1(t) = S(t)(u_0 + v_0)$ and

$$w(t) = S(t)(u_0 + v_0) - S(t)u_0 - S_1(t, u_0)v_0 = u_1(t) - u(t) - v(t),$$

then $w(t)$ satisfies

$$\begin{cases} w_t = \rho w - (1 + \mathrm{i}\nu)(-\triangle)^{\alpha} w + \Phi + \Psi, \\ w(0) = 0 \end{cases} \tag{5.5}$$

where

$$\Phi = f(u_1) - f(u) - f'(u)(u_1 - u),$$

and

$$\Psi = f'(u)w.$$

If $\sigma \geq 1/2$, applying Taylor's formula for the function f at the point u and using Corollary 3.3, we deduce that

$$|\Phi| \le C|u_1 - u|^2.$$

Hence, taking the inner product of (5.5) with w and taking the real part, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 &= \rho\|w\|^2 - \|(-\triangle)^{\frac{\alpha}{2}}w\|^2 + \mathrm{Re}(\Phi, w) + \mathrm{Re}(\Psi, w)\\ &\le C\|w\|^2 + C\|u_1 - u\|^4.\end{aligned}$$

Using an integrating factor, we infer that

$$\|w\|^2 \le C(t)\int_0^t \|u_1 - u\|^4 dt \quad \text{for} \quad t > 0,$$

for a continuous function $C(t)$. By an argument similar to that in the proof of uniqueness of the solution, we deduce that for some function $C_1(t)$

$$\|u_1 - u\|^2 \le C_1(t)\|u_1(0) - u(0)\|^2 = C_1(t)\|v(0)\|^2.$$

Therefore, we have

$$\|w\|^2 \le C_2(t)\|v(0)\|^4.$$

Hence,

$$\frac{\|S(t)(u_0 + v_0) - S(t)u_0 - S_1(t)v_0\|^2}{\|v(0)\|^2} \le C_2(T)\|v(0)\|^2, \quad \text{for all } 0 \le t \le T,$$

where $C_2(T)$ also depends on the data $\rho, \nu, \alpha, \sigma$, etc., and C_0 where $\|\mathcal{A}\|_{H^2} \le C_0$. This inequality shows that the semigroup operator $S(t)$ is uniformly differentiable on $\mathcal{A}$. Furthermore, the differential in $L^2(\Omega)$ of $S(t)$ at $u_0 \in \mathcal{A}$ is $S_1(t, u_0) : v_0 \in L^2(\Omega) \to L^2(\Omega)$, $v_0 \mapsto S_1(t, u_0)v_0$.

For any given positive integer m, we consider $\{v_{01}, v_{02}, \cdots, v_{0m}\} \subset L^2(\Omega)$ being linearly independent, and the corresponding solutions $v_1(t) = S_1(t, u_0)v_{01}$, $v_2(t) = S_1(t, u_0)v_{02}, \cdots$, $v_m(t) = S_1(t, u_0)v_{0m}$, of (5.2) and (5.3). Then it follows from (5.2) that

$$|v_1(t) \wedge \cdots \wedge v_m(t)|_{\wedge^m L^2} = |v_{(01)} \wedge \cdots \wedge v_{(0m)}|_{\wedge^m L^2} \exp\left(\int_0^t \mathrm{Re}Tr(F'(u(\tau)) \circ Q_m(\tau))d\tau\right),$$

where $Q_m(\tau) = Q_m(\tau, u_0, v_{01}, v_{02}, \cdots, v_{0m})$ is the orthogonal projection in $L^2(\Omega)$ onto the space spanned by $v_1(\tau), v_2(\tau), \cdots, v_m(\tau)$. At a given time τ, let $\{\psi_j(\tau)\} \in L^2(\Omega)$, $j \in \mathbb{N}$, be an orthonormal basis of $L^2(\Omega)$ with $\psi_1(\tau), \psi_2(\tau), \cdots, \psi_m(\tau)$ spanning the subspace

$$Q_m(\tau)L^2 = span\{v_1(\tau), v_2(\tau), \cdots, v_m(\tau)\}.$$

We see that $v_j(\tau) \in H^1(\Omega)$ for a.e. τ from (5.4), so $\psi_j(\tau) \in H^1(\Omega)$ for a.e. τ. Hence, we obtain

$$\begin{aligned}\mathrm{Re}Tr(F'(u(\tau))\circ Q_m(\tau)) &= \sum_{j=1}^{\infty}\mathrm{Re}\left(F'(u(\tau))\circ Q_m(\tau)\psi_j(\tau),\psi_j(\tau)\right)\\ &= \sum_{j=1}^{m}\mathrm{Re}\left(F'(u(\tau))\psi_j(\tau),\psi_j(\tau)\right).\end{aligned}$$

We deduce that

$$\begin{aligned}\mathrm{Re}TrF'(u(\tau))\circ\mathrm{Re}\left(F'(u(\tau))\psi_j(\tau),\psi_j(\tau)\right) &= \rho\|\psi_j\|^2-\|(-\triangle)^{\frac{\alpha}{2}}\psi_j\|^2+\mathrm{Re}\int f'(u)|\psi_j|^2\\ &\le(\rho+\gamma_2+\beta_2\|u\|_\infty^{2\sigma})\|\psi_j\|^2-\|(-\triangle)^{\frac{\alpha}{2}}\psi_j\|^2.\end{aligned}$$

Hence, we have

$$\mathrm{Re}TrF'(u(\tau))\circ Q_m(\tau)\le(\rho+\gamma_2+\beta_2\|u\|_\infty^{2\sigma})\sum_{j=1}^{m}\|\psi_j\|^2-\sum_{j=1}^{m}\|(-\triangle)^{\frac{\alpha}{2}}\psi_j\|^2.$$

Since $\{\psi_j\}_{j=1}^m$ is an orthonormal set in $L^2(\Omega)$,

$$\sum_{j=1}^{m}\|\psi_j\|^2=m.$$

It follows from the Sobolev–Lieb–Thirring inequality [38] that

$$\sum_{j=1}^{m}\|(-\triangle)^{\frac{\alpha}{2}}\psi_j\|^2\ge\kappa|\Omega|^{\alpha}m^{1+\alpha}-m,$$

where constant κ is independent of the family $\{\psi_j\}_{j=1}^m$ and the parameters of the equation. So we infer that

$$\mathrm{Re}Tr(F'(u(\tau))\circ Q_m(\tau))\le(\rho+\gamma_2+\beta_2\|u\|_\infty^{2\sigma})m-\kappa|\Omega|^{\alpha}m^{1+\alpha}.$$

For $i=1,2,\cdots,m$ and $v_{0i}\in L^2$, we define

$$\begin{aligned}q_m(t) &= \sup_{u_0\in\mathcal{A}}\sup_{\|v_{0i}\|\le 1}\left(\frac{1}{t}\int_0^t\mathrm{Re}Tr(F'(u(\tau))\circ Q_m(\tau))d\tau\right),\\ q_m &= \limsup_{t\to\infty}q_m(t).\end{aligned}$$

Then by Corollary 3.3, we obtain

$$q_m\le\kappa_1 m-\kappa_2 m^{1+\alpha},$$

where $\kappa_1=\rho+\gamma_2+\beta_2C_\infty^{\sigma}$, $\kappa_2=\kappa|\Omega|^{\alpha}$. This shows that if m is defined by

$$m = \left[\left(\frac{\kappa_1}{\kappa_2} \right)^{\frac{1}{\alpha}} \right] + 1, \tag{5.6}$$

then $q_m < 0$. Hence, by the results of [38], we obtain

Theorem 5.2. *Assume the conditions of Theorem 4.1 and let $\mathcal{A}$ be the global attractor of (1.1)–(1.3). Then the Hausdorff dimension of $\mathcal{A}$ is less than or equal to m, and the fractal dimension is less than or equal to $2m$, where m is given in (5.6).*

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