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# Strong solutions to the density-dependent incompressible nematic liquid crystal flows

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**Abstract**

In this paper, we investigate the density-dependent incompressible nematic liquid crystal flows in $n$-dimensional ($n = 2$ or 3) bounded domain. The local existence and uniqueness of strong solutions are obtained when the viscosity coefficient of fluid depends on density. Furthermore, one establishes blowup criterions for the regularity of the strong solutions in dimensions two and three respectively. In particular, we build a blowup criterion just in terms of the gradient of density if the initial direction field satisfies some geometric configuration. For these results, the initial density need not be strictly positive.


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*Keywords:* Incompressible nematic liquid crystal flows; Density-dependent; Strong solutions; Blowup criterion

## 1. Introduction

Nematic liquid crystals contain a large number of elongated, rod-like molecules and possess the same orientational order. The continuum theory of liquid crystals due to Ericksen [1] and Leslie [2] was developed around 1960s, see also [3]. Since then, numerous researchers have obtained some important developments for liquid crystals not only in theory but also in the

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application. When the fluid containing nematic liquid crystal materials is at rest, we have the well-known Ossen–Frank theory for static nematic liquid crystals, see the pioneering work by Hardt et al. [4] on the analysis of energy minimal configurations of nematic liquid crystals. Generally speaking, the motion of fluid always takes place. The so-called Ericksen–Leslie system is a macroscopic description of the time evolution of the materials under the influence of both the flow velocity field and the macroscopic description of the microscopic orientation configuration of rod-like liquid crystals. In this paper, we investigate the motion of incompressible nematic liquid crystal flows, which are described by the following simplified version of the Ericksen–Leslie equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) D(u)) + \nabla P = -\lambda \operatorname{div}(\nabla d \odot \nabla d), \\ \operatorname{div} u = 0, \\ d_t + u \cdot \nabla d = \theta(\Delta d + |\nabla d|^2 d), \end{cases} \tag{1.1}$$

in $\Omega \times (0, +\infty)$, where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^n$ ($n = 2$ or 3). Here $\rho$, $u$, $P$ and $d$ denote the unknown density, velocity, pressure and macroscopic average of the nematic liquid crystal orientation respectively. $D(u) = \frac{\nabla u + \nabla^T u}{2}$ is the deformation tensor, where $\nabla u$ presents the gradient matrix of $u$ and $\nabla^T u$ is its transpose. $\mu > 0$, $\lambda > 0$, $\theta > 0$ are viscosity of fluid, competition between kinetic and potential energy, and microscopic elastic relaxation time respectively. The viscosity coefficient $\mu = \mu(\rho)$ is a general function of density, which is assumed to satisfy

$$\mu \in C^1[0, \infty) \quad \text{and} \quad \mu \geq \underline{\mu} > 0 \quad \text{on} \quad [0, \infty), \tag{1.2}$$

for some positive constant $\underline{\mu}$. Without loss of generality, both $\lambda$ and $\theta$ are normalized to 1. The symbol $\nabla d \odot \nabla d$ denotes the $n \times n$ matrix whose $(i, j)$-th entry is given by $\nabla d_i \cdot \nabla d_j$, for $i, j = 1, 2, \ldots, n$. To complete the equations (1.1), we consider an initial boundary value problem for (1.1) with the following initial and boundary conditions

$$(\rho, u, d)|_{t=0} = (\rho_0, u_0, d_0), \ |d_0| = 1, \ \text{in } \Omega; \tag{1.3}$$

$$u = 0, \quad \frac{\partial d}{\partial \nu} = 0 \quad \text{on } \partial\Omega; \tag{1.4}$$

where $\nu$ is the unit outward normal vector to $\partial\Omega$.

When the fluid is the homogeneous case, the systems (1.1)–(1.4) are the simplified model of nematic liquid crystals with constant density. If the term $|\nabla d|^2 d$ be replaced by the Ginzburg–Laudan type approximation term $\frac{1-|d|^2}{\varepsilon^2} d$, Lin [5] first derived a simplified Ericksen–Leslie equations modeling the liquid crystal flows in 1989. Later, Lin and Liu [6,7] made some important analytic studies, such as the existence of weak/strong solutions and the partial regularity of suitable solutions. Recently, Dai et al. [8] studied the large time behavior of solutions and gave the decay rate with small initial data in the three dimensional whole space $\mathbb{R}^3$. On the other hand, Grasselli and Wu [9] considered the long time behavior of solutions and obtained the estimates on the convergence rates with external force. They also showed the existence of global strong solutions provided that either the viscosity is large enough or the initial datum is closed to a given equilibrium. For the case of $|\nabla d|^2 d$, Huang and Wang [10] established a blowup criterion for the short time classical solutions in dimensions two and three respectively. Recently, Li [11] proved the local well-posedness of mild solutions with $L^\infty$ initial data, in particular, that the initial energy may be infinite.

When the fluid is nonhomogeneous case, we would like to point out that the systems (1.1) include two important equations, which have attracted large number of analysts' interests:

(i) When $d$ is a constant and $\mu$ is a function depending only on the density $\rho$, then system (1.1)–(1.4) are the well-known Navier–Stokes equations with density-dependent viscosity coefficient. First, Lions [12] established the global existence of weak solutions. As for the uniqueness, Lions pointed out the fact that sufficiently smooth solutions are unique and any weak solutions must be equal to the strong one if the latter exists. Later, Cho and Kim [13] proved local existence of unique strong solutions for all initial data satisfying a natural compatibility condition for the case of vacuum. They also built following blowup criterion:

$$\sup_{0\le t\le T^*}\big(\|\nabla\rho(t)\|_{L^q}+\|\nabla u(t)\|_{L^2}\big)=\infty, \tag{1.5}$$

if $T^* < \infty$ is the maximal existence time of the local strong solutions. For more interesting blowup criterions about the Navier–Stokes equations, the readers can refer to [14–16] and references therein.

(ii) When $\mu$ is a constant, the systems (1.1)–(1.4) are density-dependent incompressible hydrodynamic flow of liquid crystals. If the term $|\nabla d|^2 d$ be replaced by the Ginzburg–Laudan type approximation term $\frac{1-|d|^2}{\varepsilon^2}d$, the global existence of weak solutions was obtained in [17–19] for each $\varepsilon > 0$. Recently, Hu and Wu [20] proved the decay of the velocity field for arbitrary large regular initial data with the initial density being away from vacuum in two dimensional bounded domain with smooth boundary. As for the case of $|\nabla d|^2 d$, Wen and Ding [21] obtained local existence and uniqueness of strong solutions to the Dirichlet problem in bounded domain with initial density being allowed to have vacuum. Since the strong solutions of a harmonic map can blow up in finite time [22], one cannot expect to get a global strong solution with general initial data. Therefore, many researchers attempt to obtain global strong solutions under some additional assumptions. Wen and Ding [21] also established the global existence and uniqueness of solutions for two dimensional case if the initial density was away from vacuum and the initial data is small. Global existence of strong solutions with small initial data to three dimensional liquid crystal equations was obtained by Li and Wang [23,24]. Recently, Li proved the global existence and uniqueness of strong solutions with initial data being of small norm for the dimensions two and three in bounded domain in [25] and the initial direction field satisfying some geometric structure for the two dimensional whole space in [26]. For more recent results about the compressible nematic liquid crystal flows, the readers can refer to [27–33] and references therein.

In this paper, one investigates the density-dependent incompressible nematic liquid crystal flows when the viscosity coefficient is a function of the density of fluid. The local unique strong solutions to the initial boundary value problem (1.1)–(1.4) are established in a bounded domain with smooth boundary. Then, we consider the possible breakdown of regularity for the strong solutions. Firstly, one builds up a blowup criterion in three dimensional bounded domain with smooth boundary. Secondly, by applying a logarithmic inequality, one improves the preceding blowup criterion by omitting the velocity in a two dimensional bounded domain. Lastly, if the initial direction satisfies some geometric configuration, we establish a blowup criterion just in terms of the gradient of the density in two dimensional space. For all these results, the initial density is allowed to be vacuum.

Before stating our main result, we first explain the notations and conventions used throughout this paper. We denote

$$\int f dx = \int_{\Omega} f dx.$$

Let

$$\dot{f} \triangleq f_t + u \cdot \nabla f$$

represents the material derivative of $f$. For $1 \le q \le \infty$ and integer $k \ge 0$, the standard Sobolev spaces are denoted by

$$\begin{cases} L^q = L^q(\Omega), \quad W^{k,q} = W^{k,q}(\Omega), \quad H^k = W^{k,2}, \\ W_0^{1,q} = \{u \in W^{1,q} | u = 0 \text{ on } \partial\Omega\}, \quad H_0^1 = W_0^{1,2}. \end{cases}$$

For two $n \times n$ matrices $M = (M_{ij})$, $N = (N_{ij})$, one denotes the scalar product between $M$ and $N$ by

$$M : N = \sum_{i,j=1}^{n} M_{ij} N_{ij}.$$

Finally, we recall the definition on the weak $L^p$-space which is defined as follows

$$L_w^p \triangleq \left\{ f \in L_{loc}^1 : \|f\|_{L_w^p} = \sup_{t>0} t\, |\{x \in \Omega : |f(x) > t|\}|^{\frac{1}{p}} < \infty \right\}.$$

Now, one states the first result involving local existence of strong solutions for the density-dependent incompressible nematic liquid crystal flows in this paper.

**Theorem 1.1.** *Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n (n = 2, 3)$ and $q \in (n, \infty)$ be a fixed constant. Suppose that the initial data $(\rho_0, u_0, d_0)$ satisfies the regularity conditions*

$$0 \le \rho_0 \in W^{1,q},\ u_0 \in H_0^1 \cap H^2,\ d_0 \in H^3 \text{ and } |d_0| = 1 \text{ in } \Omega,$$

*and the compatibility condition*

$$-\operatorname{div}(2\mu(\rho_0) D(u_0)) + \nabla P_0 + \operatorname{div}(\nabla d_0 \odot \nabla d_0) = \sqrt{\rho_0} g \quad \text{and} \quad \operatorname{div} u_0 = 0 \text{ in } \Omega \tag{1.6}$$

*for some $(P_0, g) \in H^1 \times L^2$. Then there exist a positive time $T_0 > 0$ and a unique strong solution $(\rho, u, d, P)$ for the initial boundary value problem (1.1)–(1.4) such that*

$$\begin{aligned}
&\rho \in C([0,T_0]; W^{1,q}), \quad \rho_t \in C([0,T_0]; L^q),\\
&u \in C([0,T_0]; H_0^1 \cap H^2) \cap L^2(0,T_0; W^{2,r}),\\
&u_t \in L^2(0,T_0; H_0^1),\ \sqrt{\rho}u_t \in L^\infty(0,T_0; L^2),\\
&P \in L^\infty(0,T_0; H^1) \cap L^2(0,T_0; W^{1,r}),\\
&d \in C([0,T_0]; H^3) \cap L^2(0,T_0; H^4),\ |d| = 1 \ in\ \overline{Q_{T_0}},\\
&d_t \in C([0,T_0]; H^1) \cap L^2(0,T_0; H^2),\ d_{tt} \in L^2(0,T_0; L^2),
\end{aligned}$$

*for some $r$ with $n < r < \min\{q, \frac{2n}{n-2}\}$ and $Q_{T_0} = \Omega \times [0, T_0]$.*

After having the local existence of strong solutions in Theorem 1.1 at hand, one will build the following blowup criterion of possible breakdown of local strong solutions for the initial boundary value problem (1.1)–(1.4) in three dimensional bounded domain.

**Theorem 1.2.** *Suppose the dimension $n = 3$ and all the assumptions in Theorem 1.1 are satisfied. Let $(\rho, u, d, P)$ be a strong solution of the initial boundary value problem (1.1)–(1.4) and $T^*$ be the maximal time of existence. If $0 < T^* < \infty$, then*

$$\lim_{T\to T^*}\left(\|\nabla\rho\|_{L^\infty(0,T;L^q)} + \|u\|_{L^{s_1}(0,T;L_w^{r_1})} + \|\nabla d\|_{L^{s_2}(0,T;L_w^{r_2})}\right) = \infty, \tag{1.7}$$

*where $r_i$ and $s_i$ satisfy*

$$\frac{2}{s_i} + \frac{3}{r_i} \le 1,\ 3 < r_i \le \infty,\ i = 1, 2. \tag{1.8}$$

As a corollary of Theorem 1.2, it also provides a blowup criterion for the density-dependent incompressible flows when the viscosity depends on the density in three dimensional domain. More precisely, if $d$ is a constant vector, then we have following corollary.

**Corollary 1.3.** *Suppose $d$ be a unit constant vector and all the assumptions in Theorem 1.2 are satisfied. Let $(\rho, u, d, P)$ be a strong solution of the initial boundary value problem (1.1)–(1.4) and $T^*$ be the maximal time of existence. If $0 < T^* < \infty$, then*

$$\lim_{T\to T^*}\left(\|\nabla\rho\|_{L^\infty(0,T;L^q)} + \|u\|_{L^s(0,T;L_w^r)}\right) = \infty, \tag{1.9}$$

*where $r$ and $s$ satisfy*

$$\frac{2}{s} + \frac{3}{r} \le 1,\ 3 < r \le \infty. \tag{1.10}$$

**Remark 1.1.** The criterion for $u$ in (1.9) is given by a Serrin type and is more general than the blowup criterion (1.5).

Our next work is to improve the proceeding blowup criterion (1.7) by utilizing a logarithmic inequality in two dimensional bounded domain.

**Theorem 1.4.** *Suppose the dimension $n = 2$ and all the assumptions in Theorem 1.1 are satisfied. Let $(\rho, u, d, P)$ be a strong solution of the initial boundary value problem (1.1)–(1.4) and $T^*$ be the maximal time of existence. If $0 < T^* < \infty$, then*

$$\lim_{T \to T^*} \left( \|\nabla \rho\|_{L^\infty(0,T;L^q)} + \|\nabla d\|_{L^s(0,T;L^r_w)} \right) = \infty, \tag{1.11}$$

*where $r$ and $s$ satisfy*

$$\frac{2}{s} + \frac{2}{r} \le 1, \ 2 < r \le \infty. \tag{1.12}$$

As a corollary of Theorem 1.4, it provides a blowup criterion for the density-dependent incompressible flows when the viscosity depends on the density in two dimensional bounded domain.

**Corollary 1.5.** *Suppose $d$ be a unit constant vector and all the assumptions in Theorem 1.4 are satisfied. Let $(\rho, u, d, P)$ be a strong solution of the initial boundary value problem (1.1)–(1.4) and $T^*$ be the maximal time of existence. If $0 < T^* < \infty$, then*

$$\lim_{T \to T^*} \|\nabla \rho\|_{L^\infty(0,T;L^q)} = \infty. \tag{1.13}$$

The final work in this paper concentrates on building blowup criterion the same as (1.13) for the density-dependent incompressible nematic liquid crystal flows if the initial direction field satisfies some special geometric configuration.

**Corollary 1.6.** *For any $i$ $(i = 1, 2)$, suppose the $i$-th component of initial direction field $d_{0i}$ satisfies the condition*

$$0 \le \underline{d}_{0i} \le d_{0i} \le 1 \quad \textit{or} \quad -1 \le d_{0i} \le -\underline{d}_{0i} \le 0,$$

*where $\underline{d}_{0i}$ is defined in (6.5), and all the assumptions in Theorem 1.4 are satisfied. Let $(\rho, u, d, P)$ be a strong solution of the initial boundary value problem (1.1)–(1.4) and $T^*$ be the maximal time of existence. If $0 < T^* < \infty$, then*

$$\lim_{T \to T^*} \|\nabla \rho\|_{L^\infty(0,T;L^q)} = \infty. \tag{1.14}$$

The rest of this paper is organized as follows. In Section 2, we present some useful lemmas that will play an important role in this paper. In Section 3, we prove the Theorem 1.1 by applying the method in [13]. From Section 4 to Section 6, we discuss and verify some blowup criterions of strong solutions.

## 2. Preliminaries

In this section, one collects some useful lemmas which will be used frequently in this paper. The first lemma is the regularity estimates for the stationary Stokes equations, i.e.,

**Lemma 2.1.** *(See [13].) Assume $\mu \in C^2[0,\infty)$ and $\rho \in W^{2,q}$, $0 \le \rho \le C$. Let $(u, P) \in H_0^1 \times L^2$ be the unique weak solution to the boundary value problem*

$$-\operatorname{div}(2\mu(\rho)D(u)) + \nabla P = F, \quad \operatorname{div} u = 0 \quad \text{in } \Omega; \int P dx = 0,$$

*where $D(u) = \frac{\nabla u + \nabla^T u}{2}$. Then we have the following regularity results:*

(1) *If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and*

$$\|u\|_{H^2} + \|P\|_{H^1} \le C\|F\|_{L^2}(1 + \|\nabla\rho\|_{L^q})^{\frac{q}{q-n}}. \tag{2.1}$$

(2) *If $F \in L^r$ for some $r \in (n, q)$ then $(u, P) \in W^{2,r} \times W^{1,r}$ and*

$$\|u\|_{W^{2,r}} + \|P\|_{W^{1,r}} \le C\|F\|_{L^r}(1 + \|\nabla\rho\|_{L^q})^{\frac{qr}{2(q-r)}}. \tag{2.2}$$

(3) *If $F \in H^1$, then $(u, P) \in H^3 \times H^2$ and*

$$\|u\|_{H^3} + \|P\|_{H^2} \le \widetilde{C}\|F\|_{H^1}(1 + \|\rho\|_{W^{2,q}})^N \tag{2.3}$$

*for some $N = N(n, q) > 0$. The constant $\widetilde{C}$ depends also on $\|\partial^2\mu/\partial\rho^2\|_C$.*

Next, we introduce a Hölder inequality in Lorentz space. The Lorentz space and its norm are denoted, respectively, by $L^{p,q}$ and $\|\cdot\|_{L^{p,q}}$, where $1 < p < \infty$ and $1 \le q \le \infty$. Now, one can state the following Hölder inequality in Lorentz space $L^{p,q}$.

**Lemma 2.2.** *(See [14].) Let $1 < p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and let $1 \le q_1, q_2 \le \infty$. Then for $f \in L^{p_1,q_1}$ and $g \in L^{p_2,q_2}$, it holds that*

$$\|f \cdot g\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_1,q_1}} \quad \text{with } q = \min\{q_1, q_2\},$$

*where $C$ is a positive constant depending only on $p_1, p_2, q_1$ and $q_2$.*

The following lemma has been proved in [14], we give the proof in detail for the readers' convenience.

**Lemma 2.3.** *Assume $g \in H^1$ and $f \in L_w^r$ with $r \in (n, \infty]$, then $f \cdot g \in L^2$. Furthermore, for any $\varepsilon > 0$ and $r \in (n, \infty]$, we have*

$$\|f \cdot g\|_{L^2}^2 \le \varepsilon\|g\|_{H^1}^2 + C(\varepsilon)\|f\|_{L_w^r}^{\frac{2r}{r-n}}\|g\|_{L^2}^2, \tag{2.4}$$

*where $C(\varepsilon)$ is a positive constant depending only on $\varepsilon$, $n$, $r$ and the domain $\Omega$.*

**Proof.** Applying the Lemma 2.2, it is easy to get

$$\|f \cdot g\|_{L^2} = \|f \cdot g\|_{L^{2,2}} \le C\|f\|_{L_w^r}\|g\|_{L^{\frac{2r}{r-2},2}}, \quad \text{where } r \in (n, \infty]. \tag{2.5}$$

Next, one will show that

$$\|g\|_{L^{\frac{2r}{r-2},2}} \le C\|g\|_{L^2}^{\frac{r-n}{r}} \|g\|_{H^1}^{\frac{n}{r}}, \quad \text{where } r \in (n,\infty]. \tag{2.6}$$

Indeed, if $r=\infty$, then (2.6) holds on obviously. If $r \in (n,\infty)$, then $L^{\frac{2r}{r-2},2}$ is a real interpolation space of $L^{\frac{2r_1}{r_1-2}}$ and $L^{\frac{2r_2}{r_2-2}}$, where $r_1, r_2$ and $r$ satisfy $n<r_1<r<r_2<\infty$ and $\frac{2}{r}=\frac{1}{r_1}+\frac{1}{r_2}$, then

$$\begin{aligned}
\|g\|_{L^{\frac{2r}{r-2},2}} &\le C\|g\|^{\frac{1}{2}}_{L^{\frac{2r_1}{r_1-2}}} \|g\|^{\frac{1}{2}}_{L^{\frac{2r_2}{r_2-2}}} \\
&\le C\left(\|g\|_{L^2}^{\frac{r_1-n}{r_1}} \|g\|_{H^1}^{\frac{n}{r_1}}\right)^{\frac{1}{2}} \left(\|g\|_{L^2}^{\frac{r_2-n}{r_2}} \|g\|_{H^1}^{\frac{n}{r_2}}\right)^{\frac{1}{2}} \\
&\le C\|g\|_{L^2}^{\frac{r-n}{r}} \|g\|_{H^1}^{\frac{n}{r}},
\end{aligned}$$

where we have used the Sobolev inequality. Therefore, combining (2.5) with (2.6) gives (2.4) directly if one exploits the Cauchy inequality. □

The last lemma introduced in this section will be the following logarithmic Sobolev inequality which plays an important role in the proof of the Lemma 5.2. Omitting the proof for brief, one can read [34,35].

**Lemma 2.4.** *Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^2$, and $f \in L^2(s,t;H^1 \cap W^{1,q})$ for $q \in (2,\infty)$. Then there exists a constant $C$ depending only on $q$ such that*

$$\|f\|^2_{L^2(s,t;L^\infty)} \le C\left[1+\|f\|^2_{L^2(s,t;H^1)} \ln(e+\|f\|_{L^2(s,t;W^{1,q})})\right], \tag{2.7}$$

*where $C$ depends only on $q$ and $\Omega$, but independent of $s,t$.*

## 3. Proof of Theorem 1.1

In this section, we only give the existence proof for the Theorem 1.1 since the uniqueness of the solutions is easy to obtain by a standard argument (cf. [12]). In order to solve the initial boundary problem (1.1)–(1.4), one will split the proof into three parts. In part one, one establishes the global strong solutions for some linearized systems. In part two, we prove the solutions of the linearized systems converges to the original initial problem (1.1)–(1.4) in a local time for positive initial density. In part three, one verifies Theorem 1.1 for the case of general initial density with vacuum.

### *3.1. Global existence for the linearized equations*

We consider the following linearized systems

$$\begin{cases} \rho_t + v \cdot \nabla\rho = 0, \\ \rho u_t + \rho v \cdot \nabla u - \operatorname{div}(2\mu(\rho)D(u)) + \nabla P = -\operatorname{div} f, \\ \operatorname{div} u = 0, \end{cases} \tag{3.1}$$

with initial and boundary conditions

$$(\rho, u)_{t=0} = (\rho_0, u_0), \text{ in } \Omega; \ u = 0, \text{ on } \partial\Omega. \tag{3.2}$$

Here $2D(u) = \nabla u + \nabla^T u$, $\mu = \mu(\rho)$ satisfies (1.2) and $v$ is a known divergence-free vector field.
Then, we state the main result in this subsection.

**Proposition 3.1.** *Assume that the data $(\rho_0, u_0, f)$ satisfies the regularity conditions:*

$$0 \le \rho_0 \in W^{1,q}, \ u_0 \in H_0^1 \cap H^2 \ \textit{and} \ f \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \ f_t \in L^2(0, T; H^1)$$

*for some $q$ with $n < q < \infty$, and the compatibility condition*

$$-\operatorname{div}(2\mu(\rho_0)D(u_0)) + \nabla P_0 + \operatorname{div} f_0 = \sqrt{\rho_0}g \quad \textit{and} \ \operatorname{div} u_0 = 0 \ \textit{in} \ \Omega, \tag{3.3}$$

*for some $(P_0, g) \in H^1 \times L^2$. If in addition, $v$ satisfies the regularity conditions*

$$v \in L^\infty(0, T; H_0^1 \cap H^2) \cap L^2(0, T; W^{2,r}), \ v_t \in L^2(0, T; H_0^1) \ \textit{and} \ \operatorname{div} v = 0 \ \textit{in} \ \Omega,$$

*for some $r$ with $n < r < \min\{q, \frac{2n}{n-2}\}$. Then, for any $T > 0$, there exists a unique strong solution $(\rho, u, P)$ to the initial boundary value problem (3.1), (3.2), (1.2) such that*

$$\begin{aligned} &\rho \in C([0, T]; W^{1,q}), \ \rho_t \in C([0, T]; L^q), \\ &u \in C([0, T]; H_0^1 \cap H^2) \cap L^2(0, T; W^{2,r}), \ u_t \in L^2(0, T; H_0^1), \\ &\sqrt{\rho}u_t \in L^\infty(0, T; L^2), \ P \in L^\infty(0, T; H^1) \cap L^2(0, T; W^{1,r}). \end{aligned} \tag{3.4}$$

In order to prove Proposition 3.1, we will take by three steps:
(1) In addition to the assumptions in Proposition 3.1, if suppose

$$\mu \in C^2[0, \infty), \quad \rho_0 \in W^{2,q}, \quad \rho_0 \ge \delta \text{ for some } \delta > 0, \tag{3.5}$$

then we give the proof of Proposition 3.1;

(2) To remove the additional condition (3.5), one needs to derive some uniform estimates independent of $\delta$, $\|\rho_0\|_{W^{2,q}}$ and $\|\partial^2\mu/\partial\rho^2\|_C$;

(3) Having the results of the proceeding two steps at hand, it is a standard argument to give the proof of Proposition 3.1.

Now, let us to begin our first step. Indeed, taking the method in [13], it is easy to get the following results. For the sake of brief, we only state the results and omit the proof.

**Lemma 3.2.** *In addition to the hypotheses of Proposition 3.1, we assume that the condition (3.5) are satisfied. Then, for any $T > 0$, there exists a unique strong solution $(\rho, u, P)$ to the initial boundary value problem (3.1), (3.2), (1.2) such that*

$$\begin{aligned}
&\rho \in C([0,T]; W^{1,\infty}) \cap L^2(0,T; W^{2,r}),\ \rho_t \in L^\infty(0,T; W^{1,r}),\\
&u \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; H^3),\ u_t \in L^\infty(0,T; L^2) \cap L^2(0,T; H_0^1),\\
&P \in L^\infty(0,T; H^1) \cap L^2(0,T; H^2),
\end{aligned}$$

*where $n < r < \min\{q, \frac{2n}{n-2}\}$.*

Thanks to the previous Lemma 3.2, there exists a unique strong solution $(\rho, u, P)$ satisfying the regularity (3.4). To remove the additional hypotheses (3.5), we will derive some uniform estimates independent of $\delta$, $\|\rho_0\|_{W^{2,q}}$ and $\|\partial^2\mu/\partial\rho^2\|_C$.

**Lemma 3.3.** *Suppose $(\rho, u, P)$ be the strong solution to the problem (3.1), (3.2), (1.2), then we have*

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\|\rho\|_{W^{1,q}} + \|\rho_t\|_{L^q} + \|u\|_{H^2} + \|P\|_{H^1} + \|\sqrt{\rho}u_t\|_{L^2}\right)\\
&+\int_0^T \left(\|u\|_{W^{2,r}}^2 + \|P\|_{W^{1,r}}^2 + \|\nabla u_t\|_{L^2}^2\right) dt \le C,
\end{aligned} \tag{3.6}$$

*where $C$ independent of $\delta$, $\|\rho_0\|_{W^{2,q}}$ and $\|\partial^2\mu/\partial\rho^2\|_C$ and $n < r < \min\{q, \frac{2n}{n-2}\}$.*

**Proof.** Step 1: We deduce from $(3.1)_1$ by applying the characteristic method that

$$\|\rho(t)\|_{L^s} = \|\rho_0\|_{L^s} \quad \text{for} \quad 0 \le t \le T,\ 1 \le s \le \infty. \tag{3.7}$$

Taking the gradient operator to $(3.1)_1$, multiplying by $q|\nabla\rho|^{q-2}\nabla\rho$ and integrating by parts, we obtain

$$\|\nabla\rho(t)\|_{L^q} \le \|\nabla\rho_0\|_{L^q} \exp\left(C\int_0^t \|v(s)\|_{W^{2,r}} ds\right),$$

which implies

$$\|\partial_t\rho(t)\|_{L^q} \le C\|v\|_{H^2}\|\nabla\rho_0\|_{L^q} \exp\left(C\int_0^t \|v(s)\|_{W^{2,r}} ds\right),$$

due to $(3.1)_1$. It is easy to observe from (1.2) and (3.7) that

$$C^{-1} \le \mu \le C \quad \text{and} \quad |\nabla\mu| \le C|\nabla\rho|, \tag{3.8}$$

which will be used repeatedly.

Step 2: Multiplying $(1.1)_2$ by $u_t$ and integrating over $(0,t)\times\Omega$, we have

$$
\begin{aligned}
&\int \mu(\rho)|D(u)|^2(t)dx+\int_0^t\int \rho|u_t|^2dxd\tau\\
&\le C+\underbrace{\int |f||\nabla u|dx}_{I_{11}}+\underbrace{\int_0^t\int |f_t||\nabla u|dxd\tau}_{I_{12}}+\underbrace{\int_0^t\int \rho|v||\nabla u||u_t|dxd\tau}_{I_{13}}\\
&\quad+\underbrace{\int_0^t\int |\mu'||v||\nabla\rho||D(u)|^2dxd\tau}_{I_{14}}.
\end{aligned}
\tag{3.9}
$$

To estimate the terms $I_{1i}$ $(1\le i\le 4)$, we will make use of the Young and Sobolev inequalities.

$$
\begin{aligned}
I_{11}&\le C(\varepsilon)\int |f|^2dx+\varepsilon\int|\nabla u|^2dx,\\
I_{12}&\le 2\int_0^t(\|f_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2)d\tau,\\
I_{13}&\le C(\varepsilon)\int_0^t\int\rho|v|^2|\nabla u|^2dxd\tau+\varepsilon\int_0^t\int\rho|u_t|^2dxd\tau\\
&\le C(\varepsilon)\|\rho\|_{L^\infty}\|v\|_{L^\infty}^2\int_0^t\int|\nabla u|^2dxd\tau+\varepsilon\int_0^t\int\rho|u_t|^2dxd\tau,\\
I_{14}&\le C\int_0^t\|v\|_{L^6}\|\nabla\rho\|_{L^q}\|\nabla u\|^2_{L^{\frac{12q}{5q-6}}}d\tau\\
&\le C\int_0^t\|\nabla v\|_{L^2}\|\nabla\rho\|_{L^q}\|\nabla u\|_{L^2}^{\frac{12q-n(q+6)}{6q}}\|\nabla u\|_{H^1}^{\frac{n(q+6)}{6q}}d\tau\\
&\le C\|\nabla\rho\|_{L^q}\|\nabla v\|_{L^2}\int_0^t\|\nabla u\|_{L^2}^2d\tau+\varepsilon\int_0^t\|\nabla u\|_{H^1}^2d\tau.
\end{aligned}
$$

On the other hand, we get

$$
\int\mu(\rho)|D(u)|^2dx\ge C^{-1}\int|D(u)|^2dx=\frac{1}{2C}\int|\nabla u|^2dx,
\tag{3.10}
$$

due to (3.8) and $(3.1)_3$. Substituting $I_{1i}(1 \le i \le 4)$ and (3.10) into (3.9) yields

$$\frac{1}{4C}\int |\nabla u|^2 dx + \frac{1}{2}\int_0^t \int \rho |u_t|^2 dx d\tau$$
$$\le C\left(1 + \|f\|_{L^2}^2 + \int_0^t \|f_t\|_{L^2}^2 d\tau\right) + C\int_0^t \|\nabla u\|_{L^2}^2 d\tau + \varepsilon \int_0^t \|\nabla u\|_{H^1}^2 d\tau. \tag{3.11}$$

In order to deal with the term $\int_0^t \|\nabla u\|_{H^1}^2 d\tau$, we will applying the regularity estimate for the stationary Stokes equations in Lemma 2.1. More precisely, it is easy to deduce

$$\|u\|_{H^2} + \|P\|_{H^1} \le C\|F\|_{L^2}(1 + \|\nabla\rho\|_{L^q})^{\frac{q}{q-n}} \le C\|F\|_{L^2},$$

where

$$\|F\|_{L^2} = \|-\rho u_t - \rho v \cdot \nabla u - \operatorname{div} f\|_{L^2}$$
$$\le C\|\sqrt{\rho} u_t\|_{L^2} + C\|\nabla u\|_{L^2} + \|\operatorname{div} f\|_{L^2}.$$

Then we have the following regularity estimate

$$\|u\|_{H^2} + \|P\|_{H^1} \le C(\|\operatorname{div} f\|_{L^2} + \|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2}). \tag{3.12}$$

Substituting (3.12) into (3.11) and choosing $\varepsilon$ small enough, we obtain

$$\int |\nabla u|^2 dx + \int_0^t \int \rho |u_t|^2 dx d\tau \le C + C\int_0^t \|\nabla u\|_{L^2}^2 d\tau,$$

which, if we exploit the Grönwall inequality, implies

$$\sup_{0 \le t \le T} \|\nabla u\|_{L^2}^2 + \int_0^t \|\sqrt{\rho} u_t\|_{L^2}^2 d\tau \le C. \tag{3.13}$$

Step 3: Differentiating $(1.1)_2$ with respect to $t$, multiplying by $u_t$ and integrating over $\Omega$, we have

$$\frac{1}{2}\frac{d}{dt}\int \rho |u_t|^2 dx + \frac{1}{C}\int |\nabla u_t|^2 dx$$
$$\lesssim \underbrace{\int |v||\nabla\rho||\nabla u||\nabla u_t| dx}_{I_{21}} + \underbrace{\int \rho |v||u_t||\nabla u_t| dx}_{I_{22}} + \underbrace{\int |v|^2 |\nabla\rho||\nabla u||u_t| dx}_{I_{23}}$$
$$+ \underbrace{\int \rho |v_t||\nabla u||u_t| dx}_{I_{24}} + \underbrace{\int |f_t||\nabla u_t| dx}_{I_{25}}. \tag{3.14}$$

Here the notation $a \lesssim b$ means that $a \le Cb$ for a universal constant $C > 0$ independent of $\delta$, $\|\rho_0\|_{W^{2,q}}$ and $\|\partial^2\mu/\partial\rho^2\|_C$. To estimate the terms $I_{2i}$ $(1 \le i \le 5)$, we will apply (3.12), the Gagliardo–Nirenberg and Hölder inequalities repeatedly:

$$
\begin{aligned}
I_{21} &\le \|v\|_{L^\infty}\|\nabla\rho\|_{L^q}\|\nabla u\|_{L^{\frac{2q}{q-2}}}\|\nabla u_t\|_{L^2}\\
&\le C(\varepsilon)\|v\|_{L^\infty}^2\|\nabla\rho\|_{L^q}^2\|\nabla u\|_{H^1}^2+\varepsilon\|\nabla u_t\|_{L^2}^2\\
&\le C(\varepsilon)(1+\|\mathrm{div} f\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2)+\varepsilon\|\nabla u_t\|_{L^2}^2,\\
I_{22} &\le C(\varepsilon)\|\rho\|_{L^\infty}\|v\|_{L^\infty}^2\int\rho|u_t|^2dx+\varepsilon\int|\nabla u_t|^2dx\\
&\le C(\varepsilon)\|\sqrt{\rho}u_t\|_{L^2}^2+\varepsilon\|\nabla u_t\|_{L^2}^2,\\
I_{23} &\le \|v\|_{L^\infty}^2\|\nabla\rho\|_{L^q}\|\nabla u\|_{L^{\frac{6q}{5q-6}}}\|u_t\|_{L^6}\\
&\le C\|v\|_{L^\infty}^2\|\nabla\rho\|_{L^q}\|\nabla u\|_{L^2}^{\frac{3q+nq-3n}{6q}}\|\nabla u\|_{H^1}^{\frac{3q+3n-nq}{6q}}\|\nabla u_t\|_{L^2}\\
&\le C(\varepsilon)(1+\|\mathrm{div} f\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2)+\varepsilon\|\nabla u_t\|_{L^2}^2,\\
I_{24} &\le \|\rho\|_{L^\infty}\|v_t\|_{L^3}\|\nabla u\|_{L^2}\|u_t\|_{L^6}\\
&\le C\|\rho\|_{L^\infty}\|\nabla v_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\le C(\varepsilon)\|\nabla v_t\|_{L^2}^2++\varepsilon\|\nabla u_t\|_{L^2}^2,\\
I_{25} &\le C(\varepsilon)\|f_t\|_{L^2}^2+\varepsilon\|\nabla u_t\|_{L^2}^2.
\end{aligned}
$$

For any fixed $\tau \in (0,t)$, substituting $I_{2i}$ $(1 \le i \le 5)$ into (3.14) and integrating over $(\tau,t) \subset [0,T]$ yield

$$
\begin{aligned}
&\frac{1}{2}\int\rho|u_t|^2dx+\frac{1}{2C}\int_\tau^t\int|\nabla u_t|^2dxds\\
&\le\frac{1}{2}\int\rho(\tau)|u_t(\tau)|^2dx+C\int_\tau^t\left(\|\mathrm{div} f\|_{L^2}^2+\|f_t\|_{L^2}^2+\|\nabla v_t\|_{L^2}^2\right)ds+\int_\tau^t\int\rho|u_t|^2dxds.
\end{aligned}
$$

Thanks to the compatibility condition (3.3), letting $\tau \to 0^+$ and applying the Grönwall inequality, it arrives at

$$
\sup_{0\le t\le T}\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\int_0^T\|\nabla u_t\|_{L^2}^2dt\le C. \tag{3.15}
$$

Step 4: High order estimates. Indeed, combing (3.15) with (3.12)–(3.13) yields

$$
\|u\|_{H^2}+\|P\|_{H^1}\le C. \tag{3.16}
$$

Applying the stationary Stokes regularity estimate, i.e. (2.2), we get

$$\|u\|_{W^{2,r}} + \|P\|_{W^{1,r}} \le C\|F\|_{L^r}(1+\|\nabla\rho\|_{L^q})^{\frac{qr}{2(q-r)}} \le C\|F\|_{L^r},$$

where

$$\begin{aligned}\|F\|_{L^r} &= \|-\rho u_t - \rho v\cdot\nabla u - \operatorname{div} f\|_{L^r}\\ &\le \|\rho\|_{L^\infty}\|u_t\|_{L^r} + \|\rho\|_{L^\infty}\|v\|_{L^\infty}\|\nabla u\|_{L^r} + \|\operatorname{div} f\|_{L^r}\\ &\le C\left(\|\rho\|_{L^\infty}\|\nabla u_t\|_{L^2} + \|\rho\|_{L^\infty}\|v\|_{L^\infty}\|\nabla u\|_{H^1} + \|\operatorname{div} f\|_{H^1}\right)\\ &\le C(1+\|\nabla u_t\|_{L^2} + \|\operatorname{div} f\|_{H^1}).\end{aligned}$$

Hence we obtain the following regularity estimate

$$\|u\|_{W^{2,r}} + \|P\|_{W^{1,r}} \le C(1+\|\nabla u_t\|_{L^2} + \|\operatorname{div} f\|_{H^1}). \tag{3.17}$$

Therefore, we complete the proof of lemma. □

After having the Lemmas 3.2–3.3 at hand, we turn to prove the Proposition 3.1. We only sketch the proof here since it is a standard argument (cf. [13]). Let $(\rho_0, u_0)$ be an initial data satisfying the hypotheses of Proposition 3.1. For each $\delta \in (0,1)$, choose $\rho_0^\delta \in W^{2,q}$ and $\mu^\delta \in C^2[0,\infty)$ such that

$$0<\delta\le\rho_0^\delta\le\rho_0+1, \quad \rho_0^\delta\to\rho_0 \text{ in } W^{1,q} \text{ and } \mu^\delta\to\mu \text{ in } C^1[0,\infty),$$

as $\delta\to 0$, and denote $(u_0^\delta, P_0^\delta)\in H_0^1\times L^2$ a solution to the problem

$$-\operatorname{div}(\mu^\delta(\rho_0^\delta)D(u_0^\delta)) + \nabla P_0^\delta + \operatorname{div} f_0 = \sqrt{\rho_0^\delta}g \quad \text{and} \quad \operatorname{div} u_0^\delta = 0 \quad \text{in } \Omega.$$

Then, applying the Lemma 3.3, the corresponding solution $(\rho^\delta, u^\delta, P^\delta)$ satisfies the estimate

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho^\delta\|_{W^{1,q}} + \|\rho_t^\delta\|_{L^q} + \|u^\delta\|_{H^2} + \|P^\delta\|_{H^1} + \|\sqrt{\rho^\delta}u_t^\delta\|_{L^2}\right)\\ &\quad+\int_0^T\left(\|u^\delta\|_{W^{2,r}}^2 + \|P^\delta\|_{W^{1,r}}^2 + \|\nabla u_t^\delta\|_{L^2}^2\right)dt \le C,\end{aligned}$$

where $C$ independent of $\delta$, $\|\rho_0\|_{W^{2,q}}$ and $\|\partial^2\mu/\partial\rho^2\|_C$ and $n<r<\min\{q,\frac{2n}{n-2}\}$. We choose a subsequence of solutions $(\rho^\delta, u^\delta)$ which converge to a limit $(\rho, u)$ in a weak sense. Thus, it is a strong solution to the linearized problem satisfying the regularity estimates in Lemma 3.3. Therefore, we complete the proof of Proposition 3.1.

### 3.2. Local existence for the original problem

In this subsection, we will prove a local existence result on strong solutions with positive initial density to the original problem (1.1)–(1.4) at first. Furthermore, one derives some uniform bounds which are independent of the lower bounds of the initial density. Then, these uniform bounds will be used to prove the existence of strong solutions with nonnegative initial density in the last part of this subsection.

**Proposition 3.4.** *Assume that the data $(\rho_0, u_0, d_0)$ satisfies the regularity conditions*

$$\rho_0 \in W^{1,q},\ u_0 \in H_0^1 \cap H^2,\ d_0 \in H^3 \text{ and } |d_0| = 1 \text{ in } \Omega,$$

*for some $q$ with $n < q < \infty$ and the compatibility condition*

$$-\operatorname{div}(2\mu(\rho_0)D(u_0)) + \nabla P_0 + \operatorname{div}(\nabla d_0 \odot \nabla d_0) = \sqrt{\rho_0} g \quad \text{and} \quad \operatorname{div} u_0 = 0 \quad \text{in } \Omega, \tag{3.18}$$

*for some $(P_0, g) \in H^1 \times L^2$. Assume further that $\rho_0 \ge \delta$ in $\Omega$ for some constant $\delta > 0$. Then there exist a time $T_0 \in (0, T)$ and a unique strong solution $(\rho, u, P, d)$ to the nonlinear problem (1.1)–(1.4) such that*

$$\begin{aligned}
&\rho \in C([0, T_0]; W^{1,q}), \quad \rho_t \in C([0, T_0]; L^q),\\
&u \in C([0, T_0]; H_0^1 \cap H^2) \cap L^2(0, T_0; W^{2,r}),\\
&u_t \in L^2(0, T_0; H_0^1),\ \sqrt{\rho} u_t \in L^\infty(0, T_0; L^2),\\
&P \in L^\infty(0, T_0; H^1) \cap L^2(0, T_0; W^{1,r}),\\
&d \in C([0, T_0]; H^3) \cap L^2(0, T_0; H^4),\ |d| = 1 \text{ in } \overline{Q_{T_0}},\\
&d_t \in C([0, T_0]; H^1) \cap L^2(0, T_0; H^2),\ d_{tt} \in L^2(0, T_0; L^2),
\end{aligned} \tag{3.19}$$

*for some $r$ with $n < r < \min\{q, \frac{2n}{n-2}\}$.*

To prove the Proposition 3.4, we first construct approximate solutions, as follows:

(1) first define $u^0 = 0$ and $d^0 = d_0$;

(2) assuming that $u^{k-1}$ and $d^{k-1}$ was defined for $k \ge 1$, let $(\rho^k, u^k, d^k, P^k)$ be the unique solution to the following initial boundary value problem

$$\begin{cases}
\rho_t^k + u^{k-1} \cdot \nabla \rho^k = 0,\\
\rho^k u_t^k + \rho^k (u^{k-1} \cdot \nabla) u^k - \operatorname{div}(2\mu(\rho^k) D(u^k)) + \nabla P^k = -\operatorname{div}(\nabla d^k \odot \nabla d^k),\\
\operatorname{div} u^k = 0,\\
d_t^k - \Delta d^k = |\nabla d^{k-1}|^2 d^{k-1} - (u^{k-1} \cdot \nabla) d^{k-1},
\end{cases} \tag{3.20}$$

with the initial and boundary conditions

$$(\rho^k, u^k, d^k)\Big|_{t=0} = (\rho_0, u_0, d_0) \quad x \in \Omega, \tag{3.21}$$

$$(u^k, \frac{\partial d^k}{\partial \nu}) = (0,0) \quad \text{on } \partial\Omega, \tag{3.22}$$

where $\nu$ is the unit outward normal vector to $\partial\Omega$.

*3.2.1. Uniform bounds*

Thanks to the Proposition 3.1 to $(3.20)_1$–$(3.20)_2$ and existence and uniqueness of the theory of parabolic equation to $(3.20)_4$ (cf. [36]), it is easy to get the existence of a global strong solution $(\rho^k, u^k, P^k, d^k)$ with the regularity (3.19) to the linearized problem (3.20)–(3.22).

From now on, we derive uniform bounds on the approximate solutions and then prove that the approximate solutions converge to a strong solution of the original nonlinear problem. Let $K \geq 1$ be a fixed large integer, and define a function as

$$\Phi_K(t) = \max_{1\leq k\leq K} \sup_{0\leq s\leq t} \Big(1 + \|\nabla u^k(s)\|_{L^2} + \|\nabla d^k(s)\|_{H^2} + \|\nabla \rho^k(s)\|_{L^q}\Big).$$

Observe that

$$\delta \leq \rho^k \leq C, \quad C^{-1} \leq \mu^k \leq C, \quad |\nabla \mu^k| \leq C|\nabla \rho^k|. \tag{3.23}$$

Then we will estimate each term of $\Phi_K(t)$ in terms of some integral of $\Phi_K(t)$.

**Lemma 3.5.** *There exists a positive constant $N = N(n,q)$ such that*

$$\|\nabla u^k(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\rho_t^k} u^k(s)\|_{L^2}^2 ds \leq C + C\int_0^t \Phi_K(s)^N ds \tag{3.24}$$

*for all $k$, $1 \leq k \leq K$.*

**Proof.** Multiplying $(3.20)_2$ by $u_t^k$, integrating by parts and making use of $(3.21)_1$, we have

$$\begin{aligned}
&\int \rho^k |u_t^k|^2 dx + \frac{d}{dt}\int \mu(\rho^k)|D(u^k)|^2 dx \\
&= -\int \rho^k (u^{k-1}\cdot\nabla)u^k \cdot u_t^k dx - \int \mu'(u^{k-1}\cdot\nabla\rho^k)|D(u^k)|^2 dx + \int \nabla d^k \odot \nabla d^k : \nabla u_t^k dx.
\end{aligned}$$

On account of the identity

$$\begin{aligned}
&\int \nabla d^k \odot \nabla d^k : \nabla u_t^k dx \\
&= \frac{d}{dt}\int \nabla d^k \odot \nabla d^k : \nabla u^k dx - \int \nabla d_t^k \odot \nabla d^k : \nabla u^k + \nabla d_t^k \odot \nabla d_t^k : \nabla u^k dx,
\end{aligned}$$

one integrates over $(0,t)$ and apply (3.10) and $(3.20)_3$ to deduce that

$$
\begin{aligned}
&\frac{1}{2C}\int |\nabla u^k|^2 dx - \int |\nabla d^k|^2 |\nabla u^k| dx + \frac{1}{2}\int_0^t \int \rho^k |u_t^k|^2 dx d\tau \\
&\lesssim 1 + \underbrace{\int_0^t \int \rho^k |u^{k-1}|^2 |\nabla u^k|^2 dx d\tau}_{I_{31}} + \underbrace{\int_0^t \int |u^{k-1}||\nabla \rho^k||\nabla u^k| dx d\tau}_{I_{32}} \\
&\quad + \underbrace{\int_0^t \int |\nabla d^k||\nabla d_t^k||\nabla u^k| dx d\tau}_{I_{33}} . \qquad (3.25)
\end{aligned}
$$

Here and below the notation $a \lesssim b$ means that $a \le Cb$ for a universal constant $C > 0$ independent of $k$. Applying the Gagliardo–Nirenberg, Hölder and Young inequalities repeatedly, we obtain

$$
\begin{aligned}
I_{31} &\le C \int_0^t \|u^{k-1}\|_{L^6}^2 \|\nabla u^k\|_{L^3}^2 d\tau \\
&\le C \int_0^t \|\nabla u^{k-1}\|_{L^2}^2 \|\nabla u^k\|_{L^2}^{\frac{6-n}{3}} \|\nabla u^k\|_{H^1}^{\frac{n}{3}} d\tau \\
&\le \varepsilon \int_0^t \|\sqrt{\rho^k} u_t^k\|_{L^2}^2 d\tau + C(\varepsilon) \int_0^t \Phi_K^{N_2} d\tau, \\
I_{32} &\le C \int_0^t \|\nabla \rho^k\|_{L^q} \|\nabla u^{k-1}\|_{L^2} \|\nabla u^k\|_{L^{\frac{12q}{5q-6}}}^2 d\tau \\
&\le C \int_0^t \|\nabla \rho^k\|_{L^q} \|\nabla u^{k-1}\|_{L^2} \|\nabla u^k\|_{L^2}^{\frac{12q-n(q+6)}{6q}} \|\nabla u^k\|_{H^1}^{\frac{n(q+6)}{6q}} d\tau \\
&\le \varepsilon \int_0^t \|\sqrt{\rho^k} u_t^k\|_{L^2}^2 d\tau + C(\varepsilon) \int_0^t \Phi_K^{N_3} d\tau, \\
I_{33} &\le \int_0^t \|\nabla d^k\|_{L^6} \|\nabla d_t^k\|_{L^2} \|\nabla u^k\|_{L^3} d\tau \\
&\le C \int_0^t \|\nabla d^k\|_{H^1} \|\nabla d_t^k\|_{L^2} \|\nabla u^k\|_{L^2}^{\frac{6-n}{3}} \|\nabla u^k\|_{H^1}^{\frac{n}{3}} d\tau
\end{aligned}
$$

$$\leq \varepsilon \int_0^t \|\sqrt{\rho^k} u_t^k\|_{L^2}^2 d\tau + C(\varepsilon) \int_0^t \Phi_K^{N_4} d\tau$$

for some $N_i = N_i(n, q) > 0$ $(i = 2, 3, 4)$, where we have used the following regularity estimate

$$\|u^k\|_{H^2} + \|P^k\|_{H^1} \leq C(1 + \|\sqrt{\rho^k} u_t^k\|_{L^2})\Phi_K^{N_1} \quad \text{for some } N_1 = N_1(n, q) > 0. \tag{3.26}$$

Substituting $I_{3i}$ $(i = 1, 2, 3)$ into (3.25) and choosing $\varepsilon$ small enough yield

$$\frac{1}{2C}\int |\nabla u^k|^2 dx - \int |\nabla d^k|^2 |\nabla u^k| dx + \frac{1}{4}\int_0^t \int \rho^k |u_t^k|^2 dx d\tau \leq C + C\int_0^t \Phi_K^{N_5} d\tau \tag{3.27}$$

for some $N_5 = N_5(n, q) > 0$. In order to control the term $-\int |\nabla d^k|^2 |\nabla u^k| dx$ on left hand side of (3.27), taking $\nabla$ operator to $(3.20)_4$, then one obtains

$$\nabla d_t^k - \nabla \Delta d^k = \nabla \Big[ |\nabla d^{k-1}|^2 d^{k-1} - (u^{k-1} \cdot \nabla) d^{k-1} \Big]. \tag{3.28}$$

Multiplying (3.28) by $4|\nabla d^k|^2 \nabla d^k$, integrating (by parts) over $\Omega$ and exploiting the boundary condition $\frac{\partial d^k}{\partial \nu}\Big|_{\partial\Omega} = 0$, we get

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla d^k|^4 dx + 4\int |\nabla d^k|^2 |\Delta d^k|^2 dx \\
&\lesssim \underbrace{\int |\nabla d^k|^3 |\nabla u^{k-1}| |\nabla d^{k-1}| dx}_{I_{41}} + \underbrace{\int |\nabla d^k|^3 |u^{k-1}| |\nabla^2 d^{k-1}| dx}_{I_{42}} + \underbrace{\int |\nabla d^k|^3 |\nabla d^{k-1}|^3 dx}_{I_{43}} \\
&\quad + \underbrace{\int |\nabla d^k|^3 |\nabla d^{k-1}| |\nabla^2 d^{k-1}| dx}_{I_{44}} + \underbrace{\int |\nabla d^k|^2 |\nabla^2 d^k|^2 dx}_{I_{45}},
\end{aligned} \tag{3.29}$$

where we have used the basic fact

$$-\int \nabla \Delta d^k \cdot 4|\nabla d^k|^2 \nabla d^k dx c = \int 4|\nabla d^k|^2 |\Delta d^k|^2 dx + 8\int \partial_j \partial_j d^k \partial_l \partial_i d^k \partial_l d^k \partial_i d^k dx.$$

Applying the Hölder and Gagliardo–Nirenberg inequalities, one deduces that

$$\begin{aligned}
I_{41} &\leq \|\nabla d^{k-1}\|_{L^\infty} \|\nabla d^k\|_{L^6}^3 \|\nabla u^{k-1}\|_{L^2} \leq \|\nabla d^{k-1}\|_{H^2} \|\nabla d^k\|_{H^1}^3 \|\nabla u^{k-1}\|_{L^2} \leq C\Phi_K^5, \\
I_{42} &\leq \|\nabla d^k\|_{L^6}^3 \|u^{k-1}\|_{L^6} \|\nabla^2 d^{k-1}\|_{L^3} \leq \|\nabla d^k\|_{H^1}^3 \|\nabla u^{k-1}\|_{L^2} \|\nabla^2 d^{k-1}\|_{H^1} \leq C\Phi_K^5, \\
I_{43} &\leq \|\nabla d^k\|_{L^6}^3 \|\nabla d^{k-1}\|_{L^6}^3 \leq C\|\nabla d^k\|_{H^1}^3 \|\nabla d^{k-2}\|_{H^1}^3 \leq C\Phi_K^6, \\
I_{44} &\leq \|\nabla d^k\|_{L^6}^3 \|\nabla d^{k-1}\|_{L^6} \|\nabla^2 d^{k-1}\|_{L^3} \leq \|\nabla d^k\|_{H^1}^3 \|\nabla d^{k-1}\|_{H^1} \|\nabla^2 d^{k-1}\|_{H^1} \leq C\Phi_K^5. \\
I_{45} &\leq \|\nabla d^k\|_{L^\infty}^2 \|\nabla^2 d^k\|_{L^2}^2 \leq C\|\nabla d^k\|_{H^2}^2 \|\nabla^2 d^k\|_{L^2}^2 \leq C\Phi_K^4.
\end{aligned}$$

Substituting $I_{4i}$ $(i = 1, 2, 3, 4, 5)$ into (3.29) and integrating over $(0, t)$, it arrives at

$$\int |\nabla d^k|^4 dx + 4\int_0^t \int |\nabla d^k|^2 |\Delta d^k|^2 dx d\tau \le C + C\int_0^t \Phi_K^6(s)ds. \tag{3.30}$$

Choosing a constant $C_*$ sufficiently large such that

$$\frac{1}{2C}|\nabla u^k|^2 - |\nabla u^k||\nabla d^k|^2 + C_*|\nabla d^k|^4 \ge \frac{1}{4C}|\nabla u^k|^2 + \frac{C_*}{2}|\nabla d^k|^4,$$

then (3.27) + (3.30) $\times\, C_*$ yields

$$\int |\nabla u^k|^2 dx + \int_0^t \int \rho^k |u_t^k|^2 dx d\tau \le C + C\int_0^t \Phi_K^{N_6} d\tau,$$

for some $N_6 = N_6(n, q) > 0$. Therefore, we complete the proof of lemma. □

Next, we estimate the term $\|\sqrt{\rho^k} u_t^k\|_{L^2}$ and $\|\nabla u_t^k\|_{L^2}$ to guarantee the higher regularity.

**Lemma 3.6.** *There exists a positive constant $N = N(n, q)$ such that*

$$\|\sqrt{\rho^k} u_t^k(t)\|_{L^2}^2 + \int_0^t \|\nabla u_t^k\|_{L^2}^2 ds \le C \exp\left[ C\int_0^t \Phi_K^N(s)ds \right] \tag{3.31}$$

*for any $k$, $1 \le k \le K$.*

**Proof.** Differentiating $(3.20)_2$ with respect to $t$, multiplying by $u_t^k$ and using $(3.20)_1$, then we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho^k |u_t^k|^2 dx + \frac{1}{C}\int |\nabla u_t^k|^2 dx \\
&\lesssim \underbrace{\int \rho^k |u^{k-1}||u_t^k||\nabla u_t^k| dx}_{I_{51}} + \underbrace{\int \rho^k |u^{k-1}||\nabla u^{k-1}||u_t^k| dx}_{I_{52}} + \underbrace{\int \rho^k |u^{k-1}|^2 |\nabla^2 u^k||u_t^k| dx}_{I_{53}} \\
&\quad + \underbrace{\int \rho^k |u^{k-1}|^2 |\nabla u^k||\nabla u_t^k| dx}_{I_{54}} + \underbrace{\int \rho^k |u_t^{k-1}||\nabla u^k||\nabla u_t^k| dx}_{I_{55}} + \underbrace{\int |\nabla d^k||\nabla d_t^k||\nabla u_t^k| dx}_{I_{56}} \\
&\quad + \underbrace{\int |\partial_t \rho^k||\nabla u^k||\nabla u_t^k| dx}_{I_{57}}.
\end{aligned} \tag{3.32}$$

To estimate the term $I_{5i}$ $(1 \le i \le 7)$, we make use of the Hölder, Gagliardo–Nirenberg and Young inequalities repeatedly.

$$
\begin{aligned}
I_{51} &\le \|\rho^k\|_{L^\infty}^{\frac{1}{2}} \|u^{k-1}\|_{L^6} \|\sqrt{\rho^k} u_t^k\|_{L^3} \|\nabla u_t^k\|_{L^2} \\
&\le C\|\rho^k\|_{L^\infty}^{\frac{1}{2}} \|\nabla u^{k-1}\|_{L^2} \|\sqrt{\rho^k} u_t^k\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho^k} u_t^k\|_{L^6}^{\frac{1}{2}} \|\nabla u_t^k\|_{L^2} \\
&\le C(\varepsilon)\|\sqrt{\rho^k} u_t^k\|_{L^2}^2 \Phi_K^4 + \varepsilon\|\nabla u_t^k\|_{L^2}^2, \\
I_{52} &\le \|\rho^k\|_{L^\infty} \|u^{k-1}\|_{L^6} \|\nabla u^{k-1}\|_{L^2} \|\nabla u^k\|_{L^6} \|u_t^k\|_{L^6} \\
&\le C(\varepsilon)\|\nabla u^{k-1}\|_{L^2}^4 \|u^k\|_{H^2}^2 + \varepsilon\|\nabla u_t^k\|_{L^2}^2 \\
&\le C(\varepsilon)(1+\|\sqrt{\rho^k} u_t^k\|_{L^2}^2)\Phi_K^{4+2N_1} + \varepsilon\|\nabla u_t^k\|_{L^2}^2, \\
I_{53} &\le \|\rho^k\|_{L^\infty} \|\nabla u^{k-1}\|_{L^2}^2 \|\nabla^2 u^k\|_{L^2} \|u_t^k\|_{L^6} \\
&\le C(\varepsilon)\|\nabla u^{k-1}\|_{L^2}^4 \|u^k\|_{H^2}^2 + \varepsilon\|\nabla u_t^k\|_{L^2}^2 \\
&\le C(\varepsilon)(1+\|\sqrt{\rho^k} u_t^k\|_{L^2}^2)\Phi_K^{4+2N_1} + \varepsilon\|\nabla u_t^k\|_{L^2}^2, \\
I_{54} &\le \|\rho^k\|_{L^\infty} \|\nabla u^{k-1}\|_{L^2}^2 \|\nabla u^k\|_{H^1} \|\nabla u_t^k\|_{L^2} \\
&\le C(\varepsilon)\|\nabla u^{k-1}\|_{L^2}^4 \|u^k\|_{H^2}^2 + \varepsilon\|\nabla u_t^k\|_{L^2}^2 \\
&\le C(\varepsilon)(1+\|\sqrt{\rho^k} u_t^k\|_{L^2}^2)\Phi_K^{4+2N_1} + \varepsilon\|\nabla u_t^k\|_{L^2}^2, \\
I_{55} &\le \|\rho^k\|_{L^\infty}^{\frac{1}{2}} \|u_t^{k-1}\|_{L^6} \|\nabla u^k\|_{L^2} \|\sqrt{\rho^k} u_t^k\|_{L^3} \\
&\le C\|\nabla u_t^{k-1}\|_{L^2} \|\nabla u^k\|_{L^2} \|\sqrt{\rho^k} u_t^k\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho^k} u_t^k\|_{L^2}^{\frac{1}{2}} \\
&\le C(\varepsilon,\eta)\|\sqrt{\rho^k} u_t^k\|_{L^2}^2 \Phi_K^4 + \varepsilon\|\nabla u_t^k\|_{L^2}^2 + \eta\|\nabla u_t^{k-1}\|_{L^2}^2, \\
I_{56} &\le C(\varepsilon)\int |\nabla d^k|^2 |\nabla d_t^k|^2 dx + \varepsilon\|\nabla u_t^k\|_{L^2}^2 \\
&\le C(\varepsilon)\|\nabla d^k\|_{L^\infty}^2 \|\nabla d_t^k\|_{L^2}^2 + \varepsilon\|\nabla u_t^k\|_{L^2}^2 \\
&\le C(\varepsilon)\|\nabla d^k\|_{H^2}^2 \|\nabla d_t^k\|_{L^2}^2 + \varepsilon\|\nabla u_t^k\|_{L^2}^2,
\end{aligned}
$$

where we have used (3.26). In order to control the term $\|\nabla d_t^k\|_{L^2}$, applying the $L^2$-estimate to (3.28), we obtain

$$
\begin{aligned}
\|\nabla d_t^k\|_{L^2} &\le \|\nabla\left(\Delta d^k + |\nabla d^{k-1}|^2 d^{k-1} - u^{k-1}\cdot\nabla d^{k-1}\right)\|_{L^2} \\
&\lesssim \|\nabla^3 d^k\|_{L^2} + \|\nabla d^{k-1}\|_{L^6} \|\nabla^2 d^{k-1}\|_{L^3} + \|\nabla d^{k-1}\|_{L^6}^3 \\
&\quad + \|\nabla d^{k-1}\|_{L^\infty} \|\nabla u^{k-1}\|_{L^2} + \|u^{k-1}\|_{L^6} \|\nabla^2 d^{k-1}\|_{L^3} \\
&\lesssim \|\nabla^3 d^k\|_{L^2} + \|\nabla d^{k-1}\|_{H^1} \|\nabla^2 d^{k-1}\|_{H^1} + \|\nabla d^{k-1}\|_{H^1}^3 \\
&\quad + \|\nabla d^{k-1}\|_{H^2} \|\nabla u^{k-1}\|_{L^2} + \|u^{k-1}\|_{H^1} \|\nabla^2 d^{k-1}\|_{H^1} \\
&\le C\Phi_K^3. 
\end{aligned} \tag{3.33}
$$

Substituting (3.33) into $I_{56}$ to deduce that

$$I_{56} \leq C(\varepsilon)\Phi_K^8 + \varepsilon\|\nabla u_t^k\|_{L^2}^2.$$

Since the term $I_{57}$ is somewhat complicated, we will deal with it as follows: If $n=2$, then

$$\begin{aligned} I_{57} &\leq \int |u^{k-1}||\nabla\rho^k||\nabla u^k||\nabla u_t^k|dx \\ &\leq \|\nabla\rho^k\|_{L^q}\left\||u^{k-1}||\nabla u^k|\right\|_{L^{\frac{2q}{q-2}}}\|\nabla u_t^k\|_{L^2} \\ &\leq \|\nabla\rho^k\|_{L^q}\|\nabla u^{k-1}\|_{L^2}\|\nabla u^k\|_{H^1}\|\nabla u_t^k\|_{L^2} \\ &\leq C(\varepsilon)(1+\|\sqrt{\rho^k}u_t^k\|_{L^2}^2)\Phi_K^{4+2N_1} + \varepsilon\|\nabla u_t^k\|_{L^2}^2. \end{aligned}$$

If $n=3$, then

$$\begin{aligned} I_{57} &\leq \|\nabla\rho^k\|_{L^q}\|u^{k-1}\|_{L^6}\|\nabla u^k\|_{L^{\frac{3q}{q-3}}}\|\nabla u_t^k\|_{L^2} \\ &\leq \|\nabla\rho^k\|_{L^q}\|\nabla u^{k-1}\|_{L^2}\|\nabla u^k\|_{L^2}^{\frac{2q-6}{3q}}\|\nabla u^k\|_{L^\infty}^{\frac{q+6}{3q}}\|\nabla u_t^k\|_{L^2} \\ &\leq C(\varepsilon)\Phi_K^{N_8} + \varepsilon\|\nabla u_t^k\|_{L^2}^2, \end{aligned}$$

where we have used the regularity estimate

$$\|u^k\|_{W^{2,r}} + \|P^k\|_{W^{1,r}} \leq C(1+\|\nabla u_t^k\|_{L^2})\Phi_K^{N_7} \quad \text{for some } N_7 = N_7(n,q) > 0. \tag{3.34}$$

Substituting $I_{5i}$ $(1 \leq i \leq 7)$ into (3.32) and choosing $\varepsilon$ small enough, we get

$$\frac{1}{2}\frac{d}{dt}\int \rho^k|u_t^k|^2dx + \frac{1}{2C}\int |\nabla u_t^k|^2dx \leq (1+\|\sqrt{\rho^k}u_t^k\|_{L^2}^2)\Phi_K^{N_9} + \eta\|\nabla u_t^{k-1}\|_{L^2}^2.$$

Fixing $\tau$ in $(0,T)$ and integrating over $(\tau,t) \subset (0,T)$, we have

$$\begin{aligned} &\|\sqrt{\rho^k}u_t^k(t)\|_{L^2}^2 + \int_\tau^t \|\nabla u_t^k\|_{L^2}^2ds \\ &\leq C\|\sqrt{\rho^k}u_t^k(\tau)\|_{L^2}^2 + C\int_\tau^t (1+\|\sqrt{\rho^k}u_t^k\|_{L^2}^2)\Phi_K^{N_9}ds + \frac{1}{2}\int_\tau^t \|\nabla u_t^{k-1}\|_{L^2}^2ds. \end{aligned}$$

From the recursive relation of $\|\nabla u_t^k\|_{L^2}$, one deduces

$$\int_\tau^t \|\nabla u_t^k\|_{L^2}^2 ds \le C\left(\sum_{i=1}^k \frac{1}{2^{i-1}}\right)\left(\|\sqrt{\rho^k}u_t^k(\tau)\|_{L^2}^2 + \int_\tau^t (1+\|\sqrt{\rho^k}u_t^k\|_{L^2}^2)\Phi_K^{N_9} ds\right)$$
$$\le 2C\left(\|\sqrt{\rho^k}u_t^k(\tau)\|_{L^2}^2 + \int_\tau^t (1+\|\sqrt{\rho^k}u_t^k\|_{L^2}^2)\Phi_K^{N_9} ds\right).$$

Hence, we have the following estimate

$$\|\sqrt{\rho^k}u_t^k(t)\|_{L^2}^2 + \int_\tau^t \|\nabla u_t^k\|_{L^2}^2 ds \le 2C\left(\|\sqrt{\rho^k}u_t^k(\tau)\|_{L^2}^2 + \int_\tau^t (1+\|\sqrt{\rho^k}u_t^k\|_{L^2}^2)\Phi_K^{N_9} ds\right).$$

Thanks to the compatibility condition (3.18), making use of the Grönwall inequality, we obtain

$$\|\sqrt{\rho^k}u_t^k(t)\|_{L^2}^2 + \int_0^t \|\nabla u_t^k\|_{L^2}^2 ds \le C\exp\left[C\int_0^t \Phi_K^{N_9}(s)ds\right],$$

which completes the proof of the lemma. □

As a corollary of Lemma 3.6, we can obtain the following estimate immediately.

**Lemma 3.7.** *There exists a positive constant $N = N(n,q)$ such that*

$$\|\nabla\rho^k(t)\|_{L^q} \le C\exp\left[C\exp(C\int_0^t \Phi_K^N(s)ds)\right] \tag{3.35}$$

*for any $k$, $1 \le k \le K$.*

Now we turn to the estimate the second term $\|\nabla d^k\|_{H^2}$ in $\Phi_K$. Indeed, we should obtain the following estimate first.

**Lemma 3.8.** *There exists a positive constant $N = N(n,q)$ such that*

$$\|\nabla d^k(t)\|_{H^1}^2 + \int_0^t \|\nabla d_t^k\|_{L^2}^2 ds \le C + C\int_0^t \Phi_K^N ds \tag{3.36}$$

*for any $k$, $1 \le k \le K$.*

**Proof.** Step 1: Multiplying $(3.20)_4$ by $\Delta d^k$ and integrating (by parts) over $\Omega$, we get

$$\frac{1}{2}\frac{d}{dt}\int |\nabla d^k|^2 dx + \int |\Delta d^k|^2 dx = \int \left[(u^{k-1}\cdot\nabla)d^{k-1} - |\nabla d^{k-1}|^2 d^{k-1}\right]\cdot\Delta d^k dx.$$

Applying the Hölder and Gagliardo–Nirenberg inequalities, it arrives at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla d^k|^2 dx + \int |\Delta d^k|^2 dx \\
&\le \int (|u^{k-1}||\nabla d^{k-1}||\nabla^2 d^k| + |\nabla d^{k-1}|^2 |d^{k-1}||\nabla^2 d^k|) dx \\
&\le \|u^{k-1}\|_{L^6}\|\nabla d^{k-1}\|_{L^3}\|\nabla^2 d^k\|_{L^2} + \|\nabla d^{k-1}\|_{L^3}^2 \|\nabla^2 d^{k-1}\|_{L^3} \le C\Phi_K^3.
\end{aligned}$$

Integrating the proceeding inequality over $(0, t)$, we have

$$\int |\nabla d^k|^2 dx + \int_0^t \int |\Delta d^k|^2 dx d\tau \le C + C\int_0^t \Phi_K^3 d\tau. \tag{3.37}$$

Step 2: Multiplying (3.28) by $\nabla d_t^k$ and integrating (by parts) over $\Omega$ yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta d^k|^2 dx + \int |\nabla d_t^k|^2 dx \\
&\le \varepsilon \int |\nabla d_t^k|^2 dx + C(\varepsilon)\int \left(|\nabla u^{k-1}|^2 |\nabla d^{k-1}|^2 + |u^{k-1}|^2 |\nabla^2 d^{k-1}|^2\right) dx \\
&\quad + C(\varepsilon)\int \left(|\nabla d^{k-1}|^6 + |\nabla d^{k-1}|^2 |\nabla^2 d^{k-1}|^2\right) dx,
\end{aligned}$$

where we have used the Cauchy inequality. Choosing $\varepsilon$ small enough, integrating over $(0, t)$ and applying the Hölder and Sobolev inequality, one arrives at

$$\begin{aligned}
&\int |\Delta d^k(t)|^2 dx + \int_0^t \int |\nabla d_t^k|^2 dx ds \\
&\le C + C\int_0^t \int \left(|\nabla u^{k-1}|^2 |\nabla d^{k-1}|^2 + |u^{k-1}|^2 |\nabla^2 d^{k-1}|^2\right) dx ds \\
&\quad + C\int_0^t \int \left(|\nabla d^{k-1}|^6 + |\nabla d^{k-1}|^2 |\nabla^2 d^{k-1}|^2\right) dx ds \\
&\le C + C\int_0^t \left(\|\nabla d^{k-1}\|_{H^2}^2 \|\nabla u^{k-1}\|_{L^2}^2 + \|\nabla u^{k-1}\|_{L^2}^2 \|\nabla^2 d^{k-1}\|_{H^1}^2\right) ds \\
&\quad + C\int \left(\|\nabla d^{k-1}\|_{H^1}^6 + \|\nabla d^{k-1}\|_{H^2}^2 \|\nabla^2 d^{k-1}\|_{L^2}^2\right) ds \\
&\le C + C\int_0^t \Phi_K^6 ds.
\end{aligned}$$

Hence, if applying the elliptic regularity, we obtain

$$\int |\nabla^2 d^k(t)|^2 dx + \int_0^t \int |\nabla d_t^k|^2 dx ds \le C + C\int_0^t \Phi_K^6 ds,$$

which, together with (3.37), completes the proof of this lemma. □

Now, we can derive the high order estimate for $d^k$.

**Lemma 3.9.** *There exists a positive constant $N = N(n,q)$ such that*

$$\|\nabla^3 d^k(t)\|_{L^2}^2 + \int_0^t \|\nabla^2 d_t^k\|_{L^2}^2 ds \le C\exp\left[C\int_0^t \Phi_K^N ds\right] \tag{3.38}$$

*for all $k$, $1 \le k \le K$.*

**Proof.** Taking $\nabla$ operator to (3.20)$_4$, multiplying by $\nabla\Delta d_t^k$ and integrating over $\Omega$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla\Delta d^k|^2 dx + \int |\Delta d_t^k|^2 dx\\
&= \frac{d}{dt}\int \nabla(u^{k-1}\cdot\nabla d^{k-1} - |\nabla d^{k-1}|^2 d^{k-1})\cdot\nabla\Delta d^k dx\\
&\quad - \int \frac{\partial}{\partial t}\Big[\nabla(u^{k-1}\cdot\nabla d^{k-1}) - \nabla(|\nabla d^{k-1}|^2 d^{k-1})\Big]\cdot\nabla\Delta d^k dx.
\end{aligned} \tag{3.39}$$

Now we need to estimate the second term of right hand side of (3.39) as follows:

$$\begin{aligned}
&-\int \frac{\partial}{\partial t}\Big[\nabla(u^{k-1}\cdot\nabla d^{k-1}) - \nabla(|\nabla d^{k-1}|^2 d^{k-1})\Big]\cdot\nabla\Delta d^k dx\\
&\lesssim \underbrace{\int |\nabla u_t^{k-1}||\nabla d^{k-1}||\nabla\Delta d^k| dx}_{I_{61}} + \underbrace{\int |\nabla u^{k-1}||\nabla d_t^{k-1}||\nabla\Delta d^k| dx}_{I_{62}}\\
&\quad + \underbrace{\int |u_t^{k-1}||\nabla^2 d^{k-1}||\nabla\Delta d^k| dx}_{I_{63}} + \underbrace{\int |u^{k-1}||\nabla^2 d_t^{k-1}||\nabla\Delta d^k| dx}_{I_{64}}\\
&\quad + \underbrace{\int |\nabla d_t^{k-1}||\nabla^2 d^{k-1}||\nabla\Delta d^k| dx}_{I_{65}} + \underbrace{\int |\nabla d^{k-1}||\nabla^2 d_t^{k-1}||\nabla\Delta d^k| dx}_{I_{66}}\\
&\quad + \underbrace{\int |\nabla d^{k-1}||\nabla^2 d^{k-1}||d_t^{k-1}||\nabla\Delta d^k| dx}_{I_{67}} + \underbrace{\int |\nabla d^{k-1}|^2|\nabla d_t^{k-1}||\nabla\Delta d^k| dx}_{I_{68}}.
\end{aligned} \tag{3.40}$$

To estimate each term $I_{6i}$ $(1 \le i \le 8)$, applying Hölder, Gagliardo–Nirenberg and Young inequalities, we obtain

$$
\begin{aligned}
I_{61} &\le \|\nabla d^{k-1}\|_{L^\infty}\|\nabla u_t^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2} \le C\|\nabla d^{k-1}\|_{H^2}\|\nabla u_t^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2}\\
&\le C\Phi_K^4 + C\|\nabla u_t^{k-1}\|_{L^2}^2,\\
I_{62} &\le \|\nabla u^{k-1}\|_{L^3}\|\nabla d_t^{k-1}\|_{L^6}\|\nabla^3 d^k\|_{L^2} \le C\|\nabla u^{k-1}\|_{H^1}\|\nabla^2 d_t^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2}\\
&\le C(\delta)\|\nabla u^{k-1}\|_{H^1}^2\|\nabla^3 d^k\|_{L^2}^2 + \delta\|\nabla^2 d_t^{k-1}\|_{L^2}^2\\
&\le C(\delta)(1+\|\sqrt{\rho^k}u_t^k\|_{L^2}^2)\Phi_K^{2N_1+2} + \delta\|\nabla^2 d_t^{k-1}\|_{L^2}^2,\\
I_{63} &\le \|u_t^{k-1}\|_{L^6}\|\nabla^2 d^{k-1}\|_{L^3}\|\nabla^3 d^k\|_{L^2} \le C\|\nabla u_t^{k-1}\|_{L^2}\|\nabla^2 d^{k-1}\|_{H^1}\|\nabla^3 d^k\|_{L^2}\\
&\le C\Phi_K^4 + C\|\nabla u_t^{k-1}\|_{L^2}^2,\\
I_{64} &\le \|u^{k-1}\|_{L^\infty}\|\nabla^2 d_t^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2} \le C\|u^{k-1}\|_{H^2}\|\nabla^3 d^k\|_{L^2}\|\nabla^2 d_t^{k-1}\|_{L^2}\\
&\le C(\delta)\|u^{k-1}\|_{H^2}^2\|\nabla^3 d^k\|_{L^2}^2 + \delta\|\nabla^2 d_t^{k-1}\|_{L^2}^2\\
&\le C(\delta)(1+\|\sqrt{\rho^k}u_t^k\|_{L^2}^2)\Phi_K^{2N_1+2} + \delta\|\nabla^2 d_t^{k-1}\|_{L^2}^2,\\
I_{65} &\le \|\nabla d_t^{k-1}\|_{L^6}\|\nabla^2 d^{k-1}\|_{L^3}\|\nabla^3 d^k\|_{L^2} \le C\|\nabla^2 d_t^{k-1}\|_{L^2}\|\nabla^2 d^{k-1}\|_{H^1}\|\nabla^3 d^k\|_{L^2}\\
&\le C(\delta)\Phi_K^4 + \delta\|\nabla^2 d_t^{k-1}\|_{L^2}^2,\\
I_{66} &\le \|\nabla d^{k-1}\|_{L^\infty}\|\nabla^2 d_t^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2} \le C\|\nabla d^{k-1}\|_{H^2}\|\nabla^3 d\|_{L^2}\|\nabla^2 d_t^{k-1}\|_{L^2}\\
&\le C(\delta)\Phi_K^4 + \delta\|\nabla^2 d_t^{k-1}\|_{L^2}^2,\\
I_{67} &\le \|\nabla d^{k-1}\|_{L^\infty}\|\nabla^2 d^{k-1}\|_{L^3}\|d_t^{k-1}\|_{L^6}\|\nabla^3 d^k\|_{L^2}\\
&\le C\|\nabla d^{k-1}\|_{H^2}\|\nabla^2 d^{k-1}\|_{H^1}\|d_t^{k-1}\|_{H^1}\|\nabla^3 d^k\|_{L^2} \le C\Phi_K^6,\\
I_{68} &\le \|\nabla d^{k-1}\|_{L^6}^2\|\nabla d_t^{k-1}\|_{L^6}\|\nabla^3 d^k\|_{L^2} \le \|\nabla d^{k-1}\|_{H^1}^2\|\nabla^2 d_t^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2}\\
&\le C(\delta)\Phi_K^6 + \delta\|\nabla^2 d_t^{k-1}\|_{L^2}^2,
\end{aligned}
$$

where we have used (3.26), (3.33) and (3.34). Substituting $I_{6i}(1 \le i \le 8)$ into (3.40), integrating (3.39) over $(0,t)$ and applying (3.31), then we have

$$
\begin{aligned}
&\frac{1}{2}\int |\nabla^3 d^k|^2 dx + \int_0^t \int |\nabla^2 d_t^k|^2 dx ds\\
&\le \underbrace{\int |\nabla u^{k-1}||\nabla d^{k-1}||\nabla\Delta d^k| dx}_{I_{71}} + \underbrace{\int |u^{k-1}||\nabla^2 d^{k-1}||\nabla\Delta d^k| dx}_{I_{72}}\\
&\quad + \underbrace{\int |\nabla d^{k-1}||\nabla^2 d^{k-1}||\nabla\Delta d^k| dx}_{I_{73}} + \underbrace{\int |\nabla d^{k-1}|^3|\nabla\Delta d^k| dx}_{I_{74}}
\end{aligned}
$$

$$+ C\exp\left[C\int_0^t \Phi_K^{N_{10}} ds\right] + \delta\int_0^t \|\nabla^2 d_t^{k-1}\|_{L^2}^2 ds. \tag{3.41}$$

By Hölder, Gagliardo–Nirenberg and Young inequalities, we obtain

$$\begin{aligned}
I_{71} &\le \|\nabla d^{k-1}\|_{L^\infty}\|\nabla u^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2} \\
&\le C\|\nabla d^{k-1}\|_{L^2}^{\frac{4-n}{4}}\|\nabla d^{k-1}\|_{H^2}^{\frac{n}{4}}\|\nabla u^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2} \\
&\le C\|\nabla d^{k-1}\|_{H^1}\|\nabla u^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2} \\
&\quad + C\|\nabla d^{k-1}\|_{L^2}^{\frac{4-n}{4}}\|\nabla^3 d^{k-1}\|_{L^2}^{\frac{n}{4}}\|\nabla u^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2} \\
&\le \varepsilon\|\nabla^3 d^k\|_{L^2}^2 + \eta\|\nabla^3 d^{k-1}\|_{L^2}^2 \\
&\quad + C(\varepsilon,\eta)\left(\|\nabla d^{k-1}\|_{H^1}^2\|\nabla u^{k-1}\|_{L^2}^2 + \|\nabla d^{k-1}\|_{L^2}^2\|\nabla u^{k-1}\|_{L^2}^{\frac{8}{4-n}}\right), \\
I_{72} &\le \|u^{k-1}\|_{L^6}\|\nabla^2 d^{k-1}\|_{L^3}\|\nabla^3 d^k\|_{L^2} \\
&\le C\|\nabla u^{k-1}\|_{L^2}\|\nabla^2 d^{k-1}\|_{L^2}\|\nabla^3 d^k\|_{L^2} \\
&\quad + C\|\nabla u^{k-1}\|_{L^2}\|\nabla^2 d^{k-1}\|_{L^2}^{\frac{6-n}{6}}\|\nabla^3 d^{k-1}\|_{L^2}^{\frac{n}{6}}\|\nabla^3 d^k\|_{L^2} \\
&\le \varepsilon\|\nabla^3 d^k\|_{L^2}^2 + \eta\|\nabla^3 d^{k-1}\|_{L^2}^2 \\
&\quad + C(\varepsilon,\eta)\left(\|\nabla^2 d^{k-1}\|_{L^2}^2\|\nabla u^{k-1}\|_{L^2}^2 + \|\nabla^2 d^{k-1}\|_{L^2}^2\|\nabla u^{k-1}\|_{L^2}^{\frac{12}{6-n}}\right), \\
I_{73} &\le \|\nabla d^{k-1}\|_{L^6}\|\nabla^2 d^{k-1}\|_{L^3}\|\nabla^3 d^k\|_{L^2} \\
&\le C\|\nabla d^{k-1}\|_{H^1}\|\nabla^2 d^{k-1}\|_{L^2}^{\frac{6-n}{6}}\|\nabla^2 d^{k-1}\|_{H^1}^{\frac{n}{6}}\|\nabla^3 d^k\|_{L^2} \\
&\le C\|\nabla d^{k-1}\|_{H^1}^2\|\nabla^3 d^k\|_{L^2} + C\|\nabla d^{k-1}\|_{H^1}^{\frac{12-n}{6}}\|\nabla^3 d^{k-1}\|_{L^2}^{\frac{n}{6}}\|\nabla^3 d^k\|_{L^2} \\
&\le \varepsilon\|\nabla^3 d^k\|_{L^2}^2 + \eta\|\nabla^3 d^{k-1}\|_{L^2}^2 + C(\varepsilon,\eta)\left(\|\nabla d^{k-1}\|_{H^1}^4 + \|\nabla d^{k-1}\|_{H^1}^{\frac{2(12-n)}{6-n}}\right), \\
I_{74} &\le \|\nabla d^{k-1}\|_{L^6}^3\|\nabla^3 d^k\|_{L^2} \\
&\le \|\nabla d^{k-1}\|_{H^1}^3\|\nabla^3 d^k\|_{L^2} \\
&\le \varepsilon\|\nabla^3 d^k\|_{L^2}^2 + C(\varepsilon)\|\nabla d^{k-1}\|_{H^1}^6.
\end{aligned}$$

Substituting $I_{7i}$ $(1 \le i \le 4)$ into (3.41), applying (3.24) and (3.36) and choosing $\varepsilon$ small enough, we obtain

$$\frac{1}{4}\int |\nabla^3 d^k|^2 dx + \int_0^t\int |\nabla^2 d_t^k|^2 dx ds$$

$$\leq \eta\|\nabla^3 d^{k-1}\|_{L^2}^2 + \delta\int_0^t\int |\nabla^2 d_t^{k-1}|^2 dxds + C(\delta,\eta)\exp\left[C\int_0^t \Phi_K^{N_{11}} ds\right].$$

Choosing $\delta$ and $\eta$ suitably small, we have

$$\begin{aligned}
&\|\nabla^3 d^k\|_{L^2}^2 + \int_0^t \|\nabla^2 d_t^k\|_{L^2}^2 ds \\
&\leq \frac{1}{2}\left(\|\nabla^3 d^{k-1}\|_{L^2}^2 + \int_0^t \|\nabla^2 d_t^{k-1}\|_{L^2}^2 ds\right) + C\exp\left[C\int_0^t \Phi_K^{N_{11}} ds\right].
\end{aligned}$$

By recursive relation, it arrives at

$$\|\nabla^3 d^k\|_{L^2}^2 + \int_0^t \|\nabla^2 d_t^k\|_{L^2}^2 ds \leq C\left(\sum_{i=1}^k \frac{1}{2^{i-1}}\right)\exp\left[C\int_0^t \Phi_K^{N_{11}} ds\right],$$

which completes the proof. $\square$

Now, from Lemmas 3.5–3.9, we conclude that

$$\Phi_K(t) \leq C\exp\left[C\exp\left(C\int_0^t \Phi_K^N ds\right)\right],$$

for some $N = N(n,q) > 0$. Thanks to this integral inequality, we can easily show that there exists a time $T_0 \in (0,T)$ (see Lemma 6 in [37]), depending only on the parameter of $C$ (independent of $k$) such that

$$\sup_{0\leq t\leq T_0} \Phi_K(t) \leq C.$$

Therefore, using the previous Lemmas 3.5–3.9 and other three estimates, we can derive the following uniform bounds:

$$\begin{aligned}
&\sup_{0\leq t\leq T_0}\left(\|\rho^k\|_{W^{1,q}} + \|\rho_t^k\|_{L^q} + \|\sqrt{\rho^k}u_t^k\|_{L^2} + \|u^k\|_{H^2} + \|P^k\|_{H^1} + \|d^k\|_{H^3} + \|\nabla d_t^k\|_{L^2}\right) \\
&+ \int_0^{T_0}\left(\|u^k\|_{W^{2,r}}^2 + \|P^k\|_{W^{1,r}}^2 + \|\nabla^4 d^k\|_{L^2}^2 + \|\nabla^2 d_t^k\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2\right) dt \leq C,
\end{aligned}$$

for all $k \geq 1$.

*3.2.2. Convergence*

We next show that the whole sequences $(\rho^k, u^k, d^k)$ of approximate solutions converge to a strong solution to the original problem (1.1)–(1.4) in a strong sense. To prove this, let us define

$$\bar{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \bar{u}^{k+1} = u^{k+1} - u^k \quad \text{and} \quad \bar{d}^{k+1} = d^{k+1} - d^k.$$

Step 1: It follows from the linearized momentum equation $(3.20)_2$ that

$$\begin{aligned}
&\rho^{k+1}\bar{u}_t^{k+1} + \rho^{k+1}u^k \cdot \nabla\bar{u}^{k+1} - \operatorname{div}[2\mu(\rho^{k+1})D(\bar{u}^{k+1})] + \nabla(P^{k+1} - P^k) \\
&= -\bar{\rho}^{k+1}u_t^k - \bar{\rho}^{k+1}u^k \cdot \nabla u^k - \rho^k\bar{u}^k \cdot \nabla u^k + \operatorname{div}\Big[2(\mu(\rho^{k+1}) - \mu(\rho^k))D(u^k)\Big] \\
&\quad - \operatorname{div}(\nabla\bar{d}^{k+1} \odot \nabla d^{k+1} + \nabla d^k \odot \nabla\bar{d}^{k+1}).
\end{aligned} \tag{3.42}$$

Hence multiplying (3.42) by $\bar{u}^{k+1}$ and integrating (by parts) over $\Omega$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho^{k+1}|\bar{u}^{k+1}|^2dx + \frac{1}{C}\int |\nabla\bar{u}^{k+1}|^2dx \\
&= \underbrace{-\int (\bar{\rho}^{k+1}u_t^k + \bar{\rho}^{k+1}u^k \cdot \nabla u^k) \cdot \bar{u}^{k+1}dx}_{I_{81}} \underbrace{-\int \rho^k\bar{u}^k \cdot \nabla u^k \cdot \bar{u}^{k+1}dx}_{I_{82}} \\
&\quad + \underbrace{\int \operatorname{div}\Big[2(\mu(\rho^{k+1}) - \mu(\rho^k))D(u^k)\Big] \cdot \bar{u}^{k+1}dx}_{I_{83}} \\
&\quad \underbrace{-\int \operatorname{div}(\nabla\bar{d}^{k+1} \odot \nabla d^{k+1} + \nabla d^k \odot \nabla\bar{d}^{k+1}) \cdot \bar{u}^{k+1}dx}_{I_{84}}.
\end{aligned} \tag{3.43}$$

To estimate $I_{8i}$ $(1 \le i \le 4)$, using Hölder, Sobolev and Young inequalities, we obtain

$$\begin{aligned}
I_{81} &\le \|\bar{\rho}^{k+1}\|_{L^2}\|u_t^k - u^k \cdot \nabla u^k\|_{L^3}\|\bar{u}^{k+1}\|_{L^6} \\
&\le C\|\bar{\rho}^{k+1}\|_{L^2}\|u_t^k - u^k \cdot \nabla u^k\|_{L^3}\|\nabla\bar{u}^{k+1}\|_{L^2} \\
&\le C(\varepsilon)\|\bar{\rho}^{k+1}\|_{L^2}^2\|u_t^k - u^k \cdot \nabla u^k\|_{L^3}^2 + \varepsilon\|\nabla\bar{u}^{k+1}\|_{L^2}^2, \\
I_{82} &\le \|\rho^k\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho^k}\bar{u}^k\|_{L^2}\|\nabla u^k\|_{L^3}\|\bar{u}^{k+1}\|_{L^6} \\
&\le \|\rho^k\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho^k}\bar{u}^k\|_{L^2}\|\nabla u^k\|_{L^3}\|\nabla\bar{u}^{k+1}\|_{L^2} \\
&\le C(\varepsilon)\|\rho^k\|_{L^\infty}\|\sqrt{\rho^k}\bar{u}^k\|_{L^2}^2\|\nabla u^k\|_{L^3}^2 + \varepsilon\|\nabla\bar{u}^{k+1}\|_{L^2}^2, \\
I_{83} &\le C\int |\bar{\rho}^{k+1}||\nabla u^k||\nabla\bar{u}^{k+1}|dx \\
&\le C(\varepsilon)\int |\bar{\rho}^{k+1}|^2|\nabla u^k|^2dx + \varepsilon\int |\nabla\bar{u}^{k+1}|^2dx
\end{aligned}$$

$$
\begin{aligned}
&\le C(\varepsilon)\|\nabla u^k\|_{W^{1,r}}^2\|\bar\rho^{k+1}\|_{L^2}^2+\varepsilon\|\nabla\bar u^{k+1}\|_{L^2}^2,\\
I_{84}&\le\int(|\nabla d^{k+1}|+|\nabla d^k|)|\nabla\bar d^{k+1}||\nabla\bar u^{k+1}|dx\\
&\le C(\varepsilon)\int(|\nabla d^{k+1}|^2+|\nabla d^k|^2)|\nabla\bar d^{k+1}|^2dx+\varepsilon\int|\nabla\bar u^{k+1}|^2dx\\
&\le C(\varepsilon)(\|\nabla d^{k+1}\|_{H^2}^2+\|\nabla d^k\|_{H^2}^2)\|\nabla\bar d^{k+1}\|_{L^2}^2+\varepsilon\int|\nabla\bar u^{k+1}|^2dx.
\end{aligned}
$$

Substituting $I_{8i}$ $(1\le i\le 4)$ into (3.43) and choosing $\varepsilon$ small enough, one arrives at

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int\rho^{k+1}|\bar u^{k+1}|^2dx+\frac{1}{2C}\int|\nabla\bar u^{k+1}|^2dx\\
&\le C(\|u_t^k-u^k\cdot\nabla u^k\|_{L^3}^2+\|\nabla u^k\|_{W^{1,r}}^2)\|\bar\rho^{k+1}\|_{L^2}^2+C\|\rho^k\|_{L^\infty}\|\nabla u^k\|_{L^3}^2\|\sqrt{\rho^k}\bar u^k\|_{L^2}^2\\
&\quad+C(\|\nabla d^{k+1}\|_{H^2}^2+\|\nabla d^k\|_{H^2}^2)\|\nabla\bar d^{k+1}\|_{L^2}^2.
\end{aligned}
\tag{3.44}
$$

Step 2: Since $\bar\rho_t^{k+1}+u^k\cdot\nabla\bar\rho^{k+1}+\bar u^k\cdot\nabla\rho^k=0$, then multiplying by $\bar\rho^{k+1}$ and integrating over $\Omega$ yield immediately

$$
\begin{aligned}
\frac12\frac{d}{dt}\int|\bar\rho^{k+1}|^2dx&=-\int u^k\cdot\nabla\left(\frac12|\bar\rho^{k+1}|^2\right)dx-\int\bar u^k\cdot\nabla\rho^k\cdot\bar\rho^{k+1}dx\\
&\le C\int|\bar\rho^{k+1}|^2|\nabla u^k|dx+\|\nabla\rho^k\|_{L^q}\|\bar\rho^{k+1}\|_{L^2}\|\bar u^k\|_{L^{\frac{2q}{q-2}}}\\
&\le C\|\nabla u^k\|_{L^\infty}\|\bar\rho^{k+1}\|_{L^2}^2+\|\nabla\rho^k\|_{L^q}\|\bar\rho^{k+1}\|_{L^2}\|\nabla\bar u^k\|_{L^2}\\
&\le C(\delta)(\|\nabla u^k\|_{W^{1,r}}+\|\nabla\rho^k\|_{L^q}^2)\|\bar\rho^{k+1}\|_{L^2}^2+\delta\|\nabla\bar u^k\|_{L^2}^2,
\end{aligned}
\tag{3.45}
$$

where we have used Young and Sobolev inequalities.

Step 3: Since we have

$$
\bar d_t^{k+1}-\Delta\bar d^{k+1}=\nabla\bar d^k\cdot(\nabla d^k+\nabla d^{k-1})\cdot d^k+|\nabla d^{k-1}|^2\bar d^k-(\bar u^k\cdot\nabla)d^k-(u^{k-1}\cdot\nabla)\bar d^k.
\tag{3.46}
$$

Multiplying (3.46) by $-\Delta\bar d^{k+1}$ and integrating (by parts) over $\Omega$, we obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int|\nabla\bar d^{k+1}|^2dx+\int|\Delta\bar d^{k+1}|^2dx\\
&=\underbrace{\int\left[\nabla\bar d^{k+1}:(\nabla d^k+\nabla d^{k-1})d\right]\cdot(-\Delta\bar d^{k+1})dx}_{I_{91}}+\underbrace{\int|\nabla d^{k-1}|^2\bar d^k\cdot(-\Delta\bar d^{k+1})dx}_{I_{92}}\\
&\quad+\underbrace{\int(\bar u^k\cdot\nabla)d^k\cdot\Delta\bar d^{k+1}dx}_{I_{93}}+\underbrace{\int(u^{k-1}\cdot\nabla)\bar d^k\cdot\Delta\bar d^{k+1}dx}_{I_{94}}.
\end{aligned}
\tag{3.47}
$$

To estimate $I_{9i}$ $(1 \le i \le 4)$, using Hölder, Young and Sobolev inequalities, we have

$$
\begin{aligned}
I_{91} &\le \int |\nabla \bar{d}^k|(|\nabla d^k| + |\nabla d^{k-1}|)|d^k||\Delta \bar{d}^{k+1}|dx \\
&\le C(\varepsilon)\int |\nabla \bar{d}^k|^2(|\nabla d^k| + |\nabla d^{k-1}|)^2 dx + \varepsilon\|\Delta \bar{d}^{k+1}\|_{L^2}^2 \\
&\le C(\varepsilon)(\|\nabla d^k\|_{L^\infty}^2 + \|\nabla d^{k-1}\|_{L^\infty}^2)\|\nabla \bar{d}^k\|_{L^2}^2 + \varepsilon\|\Delta \bar{d}^{k+1}\|_{L^2}^2, \\
I_{92} &\le C(\varepsilon)\int |\nabla d^{k-1}|^4|\bar{d}^k|^2 dx + \varepsilon\|\Delta \bar{d}^{k+1}\|_{L^2}^2 \\
&\le C(\varepsilon)\|\nabla d^{k-1}\|_{L^6}^4\|\bar{d}^k\|_{L^6}^2 + \varepsilon\|\Delta \bar{d}^{k+1}\|_{L^2}^2 \\
&\le C(\varepsilon)\|\nabla d^{k-1}\|_{H^1}^4\|\nabla \bar{d}^k\|_{L^2}^2 + \varepsilon\|\Delta \bar{d}^{k+1}\|_{L^2}^2, \\
I_{93} &= -\int \partial_j \bar{u}_i^k \partial_i d^k \partial_j \bar{d}^{k+1} + \bar{u}_i^k \partial_i \partial_j d^k \partial_j \bar{d}^{k+1} dx \\
&\le C\int (|\nabla \bar{u}^k||\nabla d^k||\nabla \bar{d}^{k+1}| + |\bar{u}^k||\nabla^2 d^k||\nabla \bar{d}^{k+1}|)dx \\
&\le C\|\nabla d^k\|_{L^\infty}\|\nabla \bar{u}^k\|_{L^2}\|\nabla \bar{d}^{k+1}\|_{L^2} + C\|\bar{u}^k\|_{L^6}\|\nabla^2 d^k\|_{L^3}\|\nabla \bar{d}^{k+1}\|_{L^2} \\
&\le C(\delta)\|\nabla d^k\|_{H^2}^2\|\nabla \bar{d}^{k+1}\|_{L^2}^2 + C(\delta)\|\nabla^2 d^k\|_{L^6}^2\|\nabla \bar{d}^{k+1}\|_{L^2}^2 + \delta\|\nabla \bar{u}^k\|_{L^2}^2 \\
&\le C(\delta)\|\nabla d^k\|_{H^2}^2\|\nabla \bar{d}^{k+1}\|_{L^2}^2 + \delta\|\nabla \bar{u}^k\|_{L^2}^2, \\
I_{94} &\le C(\varepsilon)\int |u^{k-1}|^2|\nabla \bar{d}^k|^2 dx + \varepsilon\|\Delta \bar{d}^{k+1}\|_{L^2}^2 \\
&\le C(\varepsilon)\|u^{k-1}\|_{H^2}^2\|\nabla \bar{d}^k\|_{L^2}^2 dx + \varepsilon\|\Delta \bar{d}^{k+1}\|_{L^2}^2.
\end{aligned}
$$

Substituting $I_{9i}$ $(1 \le i \le 4)$ into (3.47) and choosing $\varepsilon$ small enough, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla \bar{d}^{k+1}|^2 dx + \int |\Delta \bar{d}^{k+1}|^2 dx \\
&\le C(\|\nabla d^k\|_{H^2}^2 + \|\nabla d^{k-1}\|_{H^2}^2 + \|\nabla d^{k-1}\|_{H^1}^2 + \|u^{k-1}\|_{H^2}^2)\|\nabla \bar{d}^k\|_{L^2}^2 \\
&\quad + C(\delta)\|\nabla d^k\|_{H^2}^2\|\nabla \bar{d}^{k+1}\|_{L^2}^2 + \delta\|\nabla \bar{u}^k\|_{L^2}^2. 
\end{aligned} \tag{3.48}
$$

Denoting

$$
\begin{aligned}
\varphi^k(t) &= \|\bar{\rho}^k(t)\|_{L^2}^2 + \|\sqrt{\rho^k}\bar{u}^k\|_{L^2}^2 + \|\nabla \bar{d}^k\|_{L^2}^2, \\
\psi^k(t) &= \|\nabla \bar{u}^k(t)\|_{L^2}^2 + \|\Delta \bar{d}^k(t)\|_{L^2}^2, \\
F^k(t) &= \|u_t^k - u^k \cdot \nabla u^k\|_{L^3}^2 + \|\nabla u^k\|_{W^{1,r}}^2 + \|\nabla \rho^k\|_{L^q}^2 + \|\nabla d^k\|_{H^2}^2 + \|\nabla d^{k+1}\|_{H^2}^2,
\end{aligned}
$$

and

$$
G^k(t) = \|\rho^k\|_{L^\infty}^2\|\nabla u^k\|_{L^3}^2 + \|\nabla d^k\|_{H^2}^2 + \|\nabla^2 d^{k-1}\|_{L^2}^2 + \|\nabla d^{k-1}\|_{H^1}^4 + \|\nabla d^{k+1}\|_{H^2}^2.
$$

Then (3.44) + (3.45) + (3.48) yields immediately

$$\varphi^{k+1}(t)+\int_0^t \psi^{k+1}(s)ds \le \int_0^t \Big[C\varphi^k(s)+\delta\psi^k(s)\Big]ds+\int_0^t CF^k(s)\varphi^{k+1}(s)ds,$$

which implies, by virtue of Grönwall inequality, that

$$\varphi^{k+1}(t)+\int_0^t \psi^{k+1}(s)ds \le C\exp(C_\delta t)\int_0^t \Big[C\varphi^k(s)+\delta\psi^k(s)\Big]ds.$$

Choosing $\delta>0$ and $T_1$ so small that $8C(T_1+\delta)<1$ and $\exp(C_\delta T_1)<2$, then

$$\varphi^{k+1}(t)+\int_0^t \psi^{k+1}(s)ds \le \frac{1}{4}\left[\varphi^k(t)+\int_0^t \psi^k(s)ds\right],$$

for all $1\le k\le K$. By iteration, we have

$$\varphi^{k+1}(t)+\int_0^t \psi^{k+1}(s)ds \le \frac{1}{4^k}\left[\varphi^1(t)+\int_0^t \psi^1(s)ds\right],\quad k\ge 0.$$

Together with the Poincaré inequality and elliptic estimate, we get

$$\begin{aligned}&\|\bar\rho^{k+1}\|_{L^\infty(0,T;L^2)}+\|\sqrt{\rho^{k+1}}\bar u^{k+1}\|_{L^\infty(0,T;L^2)}+\|\bar d^{k+1}\|_{L^\infty(0,T;H^1)}\\&\quad+\|\bar u^{k+1}\|_{L^2(0,T;H^1)}+\|\bar d^{k+1}\|_{L^2(0,T;H^2)}\le\frac{C}{2^k}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\sum_{k=1}^\infty\|\bar\rho^{k+1}\|_{L^\infty(0,T;L^2)}<\infty,\\&\sum_{k=1}^\infty\|\bar u^{k+1}\|_{L^2(0,T;H^1)}<\infty,\\&\sum_{k=1}^\infty(\|\bar d^{k+1}\|_{L^\infty(0,T;H^1)}+\|\bar d^{k+1}\|_{L^2(0,T;H^2)})<\infty,\end{aligned}$$

which means

$$\rho^k \to \rho^1 + \sum_{i=2}^{\infty} \bar{\rho}^i \quad \text{in} \quad L^\infty(0,T;L^2),$$

$$u^k \to u^1 + \sum_{i=2}^{\infty} \bar{u}^i \quad \text{in} \quad L^2(0,T;H^1),$$

$$d^k \to d^1 + \sum_{i=2}^{\infty} \bar{d}^i \quad \text{in} \quad L^\infty(0,T;H^1) \cap L^2(0,T;H^2),$$

as $k \to \infty$.

### 3.2.3. *Conclusion*

Now it is a simple matter to check that $(\rho, u, d)$ is a weak solution to the original problem (1.1)–(1.4) with positive initial density. Then, by virtue of the lower semi-continuity of norms, we deduce from the uniform bound that $(\rho, u, P, d)$ satisfies the following regularity estimate:

$$\sup_{0\le t\le T_0} \left(\|\rho\|_{W^{1,q}} + \|\rho_t\|_{L^q} + \|\sqrt{\rho}u_t\|_{L^2} + \|u\|_{H^2} + \|P\|_{H^1} + \|d\|_{H^3} + \|\nabla d_t\|_{L^2}\right)$$
$$+ \int_0^{T_0} \left(\|u\|_{W^{2,r}}^2 + \|P\|_{W^{1,r}}^2 + \|\nabla^4 d\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2\right) dt \le C.$$

## 3.3. *Proof of Theorem 1.1*

Let $(\rho_0, u_0, d_0)$ satisfies the assumptions in Theorem 1.1. For each $\delta > 0$, let $\rho_0^\delta = \rho_0 + \delta$, satisfying

$$\rho_0^\delta \to \rho_0 \text{ in } W^{1,,q} \text{ as } \delta \to 0^+,$$

and $d_0^\delta = d_0$. Suppose $(u^\delta, P^\delta) \in H_0^1 \times L^2$ is a solution to the problem

$$-\operatorname{div}(2\mu(\rho_0^\delta)D(u_0^\delta)) + \nabla P_0^\delta + \operatorname{div}(\nabla d_0^\delta \odot \nabla d_0^\delta) = \sqrt{\rho_0^\delta} g \text{ and } \operatorname{div} u_0^\delta = 0 \text{ in } \Omega.$$

Then by the regularity estimate, it is easy to get

$$u_0^\delta \in H_0^1 \cap H^2 \quad \text{and} \quad u_0^\delta \to u_0 \text{ in } H^2 \text{ as } \delta \to 0^+.$$

Then, by Proposition 3.4, there exist a time $T_0 \in (0,T)$ and a unique strong solution $(\rho^\delta, u^\delta, P^\delta, d^\delta)$ in $[0,T_0] \times \Omega$ to the problem with the initial data replaced by $(\rho_0^\delta, u_0^\delta, d_0^\delta)$. Note that $(\rho^\delta, u^\delta, P^\delta, d^\delta)$ will satisfies the regularity estimate where $C$ independent of the parameter $\delta$. Hence, let $\delta \to 0^+$, it is a simple matter to see $(\rho, u, P, d)$ is a strong solution and have the following regularity estimates

$$\sup_{0\le t\le T_0}\left(\|\rho\|_{W^{1,q}}+\|\rho_t\|_{L^q}+\|\sqrt{\rho}u_t\|_{L^2}+\|u\|_{H^2}+\|P\|_{H^1}+\|d\|_{H^3}+\|\nabla d_t\|_{L^2}\right)$$
$$+\int_0^{T_0}\left(\|u\|^2_{W^{2,r}}+\|P\|^2_{W^{1,r}}+\|\nabla^4 d\|^2_{L^2}+\|\nabla^2 d_t\|^2_{L^2}+\|d_{tt}\|^2_{L^2}\right)dt\le C.$$

Therefore, we complete the proof of Theorem 1.1.

## 4. Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2 by contradiction. More precisely, let $0<T^*<\infty$ be the maximum time for the existence of strong solution $(\rho,u,P,d)$ to (1.1)–(1.4). Suppose that (1.7) were false, that is

$$M_0\triangleq\lim_{T\to T^*}\left(\|\nabla\rho\|_{L^\infty(0,T;L^q)}+\|u\|_{L^{s_1}(0,T;L^{r_1}_w)}+\|\nabla d\|_{L^{s_2}(0,T;L^{r_2}_w)}\right)<\infty. \tag{4.1}$$

Under the condition (4.1), one will extend existence time of the strong solutions to (1.1)–(1.4) beyond $T^*$, which contradicts with the definition of maximum existence time.

**Lemma 4.1.** *For any $0\le T<T^*$, it holds that*

$$\sup_{0\le t\le T}\left(\|\rho\|_{L^m}+\|\sqrt{\rho}u\|^2_{L^2}+\|\nabla d\|^2_{L^2}\right)+\int_0^T\int\left(|\nabla d|^2+\left||\nabla d|^2 d+\Delta d\right|^2\right)dxds\le C, \tag{4.2}$$

*where $1\le m\le\infty$ and in what follows, $C$ denotes generic constants depending only on $\Omega$, $M_0$, $T^*$ and the initial data.*

**Proof.** Multiplying $(1.1)_1$ by $m\rho^{m-1}$ $(1\le m\le\infty)$ and integrating the resulting equation over $\Omega$, then it is easy to deduce that

$$\|\rho(t)\|_{L^m}=\|\rho_0\|_{L^m}\quad(1\le m\le\infty).$$

Multiplying $(1.1)_2$ by $u$ and integrating (by parts) over $\Omega$, it is easy to deduce

$$\frac{1}{2}\frac{d}{dt}\int\rho|u|^2dx+\frac{1}{C}\int|\nabla u|^2dx\le-\int(u\cdot\nabla)d\cdot\Delta d\,dx. \tag{4.3}$$

Multiplying $(1.1)_4$ by $\Delta d+|\nabla d|^2 d$ and integrating (by parts) over $\Omega$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int|\nabla d|^2dx+\int\left||\nabla d|^2d+\Delta d\right|^2dx=\int(u\cdot\nabla)d\cdot\Delta d\,dx+\int(d_t+u\cdot\nabla d)|\nabla d|^2d\,dx. \tag{4.4}$$

By virtue of $|d| = 1$, we have the fact

$$(d_t + u \cdot \nabla d)|\nabla d|^2 d = 0. \tag{4.5}$$

Substituting (4.5) into (4.4), it arrives at

$$\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx + \int \Big||\nabla d|^2 d + \Delta d\Big|^2 dx = \int (u \cdot \nabla)d \cdot \Delta d dx,$$

which, together with (4.3), gives

$$\frac{1}{2}\frac{d}{dt}\int (\rho|u|^2 + |\nabla d|^2)dx + \frac{1}{C}\int |\nabla u|^2 dx + \int \Big||\nabla d|^2 d + \Delta d\Big|^2 dx \le 0. \tag{4.6}$$

Integrating (4.6) over $(0, t)$ yields

$$\begin{aligned}&\frac{1}{2}\int (\rho|u|^2 + |\nabla d|^2)dx + \int_0^t\int \left(\frac{1}{C}|\nabla u|^2 + \Big||\nabla d|^2 d + \Delta d\Big|^2\right) dx d\tau \\ &\le \frac{1}{2}\int \left(\rho_0|u_0|^2 + |\nabla d_0|^2\right) dx,\end{aligned}$$

which completes the proof. □

Now, we give the estimate for $\|\nabla u\|_{L^2}$ and $\|\nabla^2 d\|_{L^2}$.

**Lemma 4.2.** *Under the condition (4.1), it holds that for $0 \le T < T^*$,*

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^4}^4 + \|\nabla^2 d\|_{L^2}^2\right) \\ &\quad + \int_0^T\int \left(|\sqrt{\rho}\dot{u}|^2 + |\nabla d|^2|\nabla^2 d|^2 + |\nabla^3 d|^2 + |\nabla d_t|^2\right) dx d\tau \le C.\end{aligned} \tag{4.7}$$

**Proof.** Step 1: Multiplying $(1.1)_1$ by $u_t$ and integrating (by parts) over $\Omega$, we have

$$\begin{aligned}&\frac{d}{dt}\int \mu(\rho)|D(u)|^2 dx - \frac{d}{dt}\int \nabla d \odot \nabla d : \nabla u \, dx + \int \rho|\dot{u}|^2 dx \\ &= -\int \Big(\rho\dot{u} \cdot (u \cdot \nabla u) + \mu'(u \cdot \nabla\rho)|D(u)|^2 + \nabla d_t \odot \nabla d : \nabla u + \nabla d \odot \nabla d_t : \nabla u\Big) dx.\end{aligned} \tag{4.8}$$

For any $0 \le s < t \le T$, integrating over $(s, t)$ and applying the Cauchy inequality yield

$$\frac{1}{2C}\int|\nabla u|^2dx-\int\nabla d\odot\nabla d:\nabla u dx+\int_s^t\int\rho|\dot{u}|^2dxd\tau$$

$$\le C\int(|\nabla u|^2+|\nabla d|^2|\nabla u|)(s)dx+C(\varepsilon,\delta)\int_s^t\int(\rho|u|^2|\nabla u|^2+|u||\nabla\rho||\nabla u|^2)dxd\tau$$

$$+C(\varepsilon,\delta)\int_s^t\int|\nabla d|^2|\nabla u|^2dxd\tau+\varepsilon\int_s^t\int\rho|\dot{u}|^2dxd\tau+\delta\int_s^t\int|\nabla d_t|^2dxd\tau.$$

Choosing $\varepsilon=\frac{1}{4}$, we obtain

$$\frac{1}{2C}\int|\nabla u|^2dx-\int\nabla d\odot\nabla d:\nabla u\,dx+\frac{3}{4}\int_s^t\int\rho|\dot{u}|^2dxd\tau$$

$$\le C\int(|\nabla u|^2+|\nabla d|^4)(s)dx+C(\delta)\int_s^t\int(\rho|u|^2|\nabla u|^2+|u||\nabla\rho||\nabla u|^2)dxd\tau$$

$$+C(\delta)\int_s^t\int|\nabla d|^2|\nabla u|^2dxd\tau+\delta\int_s^t\int|\nabla d_t|^2dxd\tau. \tag{4.9}$$

Estimate for the term $\int_s^t\int|u||\nabla\rho||\nabla u|^2dxd\tau$. Indeed, by Cauchy, Hölder and Sobolev inequalities, we get

$$\int_s^t\int|u||\nabla\rho||\nabla u|^2dxd\tau$$

$$\le C(\varepsilon)\int_s^t\int|u|^2|\nabla u|^2dxd\tau+\varepsilon\int_s^t\int|\nabla\rho|^2|\nabla u|^2dxd\tau$$

$$\le C(\varepsilon)\int_s^t\int|u|^2|\nabla u|^2dxd\tau+\varepsilon\int_s^t\left\||\nabla\rho|^2\right\|_{L^{\frac{q}{2}}}\left\||\nabla u|^2\right\|_{L^{\frac{q}{q-2}}}d\tau$$

$$\le C(\varepsilon)\int_s^t\int|u|^2|\nabla u|^2dxd\tau+\varepsilon\int_s^t\|\nabla u\|_{H^1}^2d\tau.$$

In order to control the term $\int_s^t\|\nabla u\|_{H^1}^2d\tau$, by virtue of the Lemma 2.1, we have

$$\|u\|_{H^2}+\|P\|_{H^1}$$

$$\le C\|F\|_{L^2}(1+\|\nabla\rho\|_{L^q})^{\frac{q}{q-n}}$$

$$
\begin{aligned}
&\le C\| -\rho u_t - \rho u\cdot\nabla u - \operatorname{div}(\nabla d\odot\nabla d)\|_{L^2}\\
&\le C\left(\|\sqrt{\rho}\dot{u}\|_{L^2} + \left\||\nabla d||\nabla^2 d|\right\|_{L^2}\right).
\end{aligned}
\tag{4.10}
$$

Hence we get the estimate

$$
\begin{aligned}
&\int_s^t\int |u||\nabla\rho||\nabla u|^2 dxd\tau\\
&\le C(\varepsilon)\int_s^t\int |u|^2|\nabla u|^2 dxd\tau + \varepsilon\int_s^t\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \left\||\nabla d|\,|\nabla^2 d|\right\|_{L^2}^2\right)d\tau.
\end{aligned}
\tag{4.11}
$$

Substituting (4.11) into (4.9) and choosing $\varepsilon = \frac{1}{4}$, one arrives at

$$
\begin{aligned}
&\frac{1}{2C}\int|\nabla u|^2dx - \int\nabla d\odot\nabla d:\nabla u dx + \frac{1}{2}\int_s^t\int\rho|\dot{u}|^2dxd\tau\\
&\le C\int(|\nabla u|^2 + |\nabla d|^4)(s)dx + C(\delta)\int_s^t\int(|u|^2|\nabla u|^2 + |\nabla d|^2|\nabla u|^2)dxd\tau\\
&\quad + C(\delta)\int_s^t\int|\nabla d|^2|\nabla^2 d|^2dxd\tau + \delta\int_s^t\int|\nabla d_t|^2dxd\tau.
\end{aligned}
\tag{4.12}
$$

Step 2: Taking $\nabla$ operator to $(1.1)_4$, then we have

$$
\nabla d_t - \nabla\Delta d = -\nabla(u\cdot\nabla d) + \nabla(|\nabla d|^2 d).
\tag{4.13}
$$

Multiplying (4.13) by $4|\nabla d|^2\nabla d$ and integrating (by parts) over $\Omega$, it is easy to deduce

$$
\begin{aligned}
&\frac{d}{dt}\int|\nabla d|^4dx + 4\int|\nabla d|^2|\nabla^2 d|^2dx + 2\int|\nabla(|\nabla d|^2)|^2dx\\
&= \underbrace{2\int_{\partial\Omega}|\nabla d|^2 < \nabla(|\nabla d|^2),\nu > d\sigma}_{II_{11}} + \underbrace{4\int\nabla(|\nabla d|^2 d):|\nabla d|^2\nabla d\,dx}_{II_{12}}\\
&\quad \underbrace{-4\int\nabla(u\cdot\nabla d):|\nabla d|^2\nabla d\,dx}_{II_{13}},
\end{aligned}
\tag{4.14}
$$

where $\nu$ is the unite outward normal vector to $\partial\Omega$. To estimate $II_{11} = 2\int_{\partial\Omega}|\nabla d|^2 < \nabla(|\nabla d|^2),\nu > d\sigma$. Indeed, applying the Sobolev embedding inequality $W^{1,1}(\Omega)\hookrightarrow L^1(\partial\Omega)$, it is easy to get

$$
\begin{aligned}
II_{11} &\le 4\int_{\partial\Omega} |\nabla d|^3|\nabla^2 d|d\sigma \le C\left\| |\nabla d|^3|\nabla^2 d| \right\|_{W^{1,1}(\Omega)} \\
&\le C\int (|\nabla d|^3|\nabla^2 d| + |\nabla d|^2|\nabla^2 d|^2 + |\nabla d|^3|\nabla^3 d|)dx \\
&\le C(\eta)\int (|\nabla d|^2|\nabla^2 d|^2 + |\nabla d|^4 + |\nabla d|^6)dx + \eta\int |\nabla^3 d|^2 dx. \qquad (4.15)
\end{aligned}
$$

To estimate $II_{12} = 4\int \nabla(|\nabla d|^2 d) : |\nabla d|^2\nabla d\ dx$. Since $|d| = 1$, we have $d\cdot\nabla d = 0$. Then, we deduce

$$
\nabla(|\nabla d|^2 d) : |\nabla d|^2\nabla d = (\nabla(|\nabla d|^2)d + |\nabla d|^2\nabla d) : |\nabla d|^2\nabla d = |\nabla d|^6.
$$

Hence, one arrives at

$$
II_{12} = 4\int \nabla(|\nabla d|^2 d) : |\nabla d|^2\nabla d\ dx = 4\int |\nabla d|^6 dx. \qquad (4.16)
$$

By the Cauchy inequality, we have

$$
\begin{aligned}
II_{13} &= -4\int \nabla(u\cdot\nabla d) : |\nabla d|^2\nabla d\ dx \\
&\le \int (|\nabla u||\nabla d|^4 + |u||\nabla d|^3|\nabla^2 d|)dx \\
&\le C\int (|\nabla d|^2|\nabla u|^2 + |\nabla d|^6 + |u|^2|\nabla^2 d|^2)dx. \qquad (4.17)
\end{aligned}
$$

Substituting (4.15)–(4.17) into (4.14), choosing $\varepsilon$ small enough, and integrating over $(s,t)$, one arrives at

$$
\begin{aligned}
&\int |\nabla d|^4 dx + 4\int_s^t\int |\nabla d|^2|\nabla^2 d|^2 dxd\tau + 2\int_s^t\int |\nabla(|\nabla d|^2)|^2 dxd\tau \\
&\le \int |\nabla d|^4(s)dx + C(\eta)\int_s^t\int (|\nabla d|^4 + |\nabla d|^6 + |\nabla d|^2|\nabla^2 d|^2 + |\nabla d|^2|\nabla u|^2)dxd\tau \\
&\quad + C(\eta)\int_s^t\int |u|^2|\nabla^2 d|^2 dxd\tau + \eta\int_s^t\int |\nabla^3 d|^2 dxd\tau. \qquad (4.18)
\end{aligned}
$$

Step 3: Multiplying (4.13) by $\nabla\Delta d$, integrating (by parts) over $\Omega$ and applying Young inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2 dx + \int |\nabla\Delta d|^2 dx \\
&\le \int (|\nabla u||\nabla d| + |u||\nabla^2 d| + |\nabla d|^3 + |\nabla d||\nabla^2 d|)|\nabla\Delta d| dx \\
&\le C(\varepsilon)\int (|\nabla u|^2|\nabla d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla d|^6 + |\nabla d|^2|\nabla^2 d|^2) dx + \varepsilon\int |\nabla\Delta d|^2 dx.
\end{aligned}
$$

Choosing $\varepsilon = \frac{1}{2}$ and integrating over $(s, t)$, we get

$$
\begin{aligned}
&\int |\Delta d|^2 dx + \int_s^t \int |\nabla\Delta d|^2 dx d\tau \\
&\le 2\int |\Delta d|^2(s) dx + C\int_s^t \int (|\nabla d|^2|\nabla^2 d|^2 + |\nabla d|^2|\nabla u|^2 + |u|^2|\nabla^2 d|^2 + |\nabla d|^6) dx d\tau.
\end{aligned}
\tag{4.19}
$$

Step 4: Multiplying (4.13) by $\nabla d_t$, integrating (by parts) over $\Omega$ and applying Young inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2 dx + \int |\nabla d_t|^2 dx \\
&= \int [\nabla(|\nabla d|^2 d) - \nabla(u\cdot\nabla d)]\cdot\nabla d_t dx \\
&\le C(\varepsilon)\int (|\nabla u|^2|\nabla d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla d|^6 + |\nabla d|^2|\nabla^2 d|^2) dx + \varepsilon\int |\nabla d_t|^2 dx.
\end{aligned}
$$

Choosing $\varepsilon = \frac{1}{2}$ and integrating over $(s, t)$, we get

$$
\begin{aligned}
&\int |\Delta d|^2 dx + \int_s^t \int |\nabla d_t|^2 dx d\tau \\
&\le 2\int |\Delta d|^2(s) dx \\
&\quad + C\int_s^t \int (|\nabla d|^2|\nabla^2 d|^2 + |\nabla u|^2|\nabla d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla d|^6) dx d\tau.
\end{aligned}
\tag{4.20}
$$

Then, choosing $\delta$ small enough and some constant $C_{**}$ suitably large such that

$$
\frac{1}{2C}|\nabla u|^2 - \nabla d\odot\nabla d : \nabla u + C_{**}|\nabla d|^4 \ge \frac{1}{4C}(|\nabla u|^2 + |\nabla d|^4),
$$

then (4.12) + (4.18) $\times C_{**}$ + (4.19) + (4.20) and choosing $\eta$ and $\delta$ small enough yield

$$\begin{aligned}
&\int (|\nabla u|^2+|\nabla^2 d|^2+|\nabla d|^4)(t)dx \\
&\quad +\int_s^t\int (\rho|\dot u|^2+|\nabla d|^2|\nabla^2 d|^2+|\nabla(|\nabla d|^2)|^2+|\nabla^3 d|^2+|\nabla d_t|^2)dxd\tau \\
&\le C+C\int (|\nabla u|^2+|\nabla^2 d|^2+|\nabla d|^4)(s)dx+C\int_s^t\int (1+|\nabla^2 d|^2+|\nabla d|^4)dxd\tau \\
&\quad +C\int_s^t\int (|u|^2|\nabla u|^2+|\nabla d|^2|\nabla^2 d|^2+|\nabla d|^2|\nabla u|^2+|u|^2|\nabla^2 d|^2+|\nabla d|^6)dxd\tau. \qquad (4.21)
\end{aligned}$$

Choosing $s=0$ in (4.21), then we obtain

$$\begin{aligned}
&\int (|\nabla u|^2+|\nabla^2 d|^2+|\nabla d|^4)(t)dx \\
&\quad +\int_0^t\int (\rho|\dot u|^2+|\nabla d|^2|\nabla^2 d|^2+|\nabla(|\nabla d|^2)|^2+|\nabla^3 d|^2+|\nabla d_t|^2)dxd\tau \\
&\le C+C\int_0^t\int \left(1+|\nabla^2 d|^2+|\nabla d|^4\right)dxd\tau+\underbrace{C\int_0^t\int |\nabla d|^2|\nabla^2 d|^2dxd\tau}_{II_{21}} \\
&\quad +\underbrace{C\int_0^t\int |u|^2|\nabla u|^2dxd\tau}_{II_{22}}+\underbrace{C\int_0^t\int |u|^2|\nabla^2 d|^2dxd\tau}_{II_{23}} \\
&\quad +\underbrace{C\int_0^t\int |\nabla d|^2|\nabla u|^2dxd\tau}_{II_{24}}+\underbrace{C\int_0^t\int |\nabla d|^6dxd\tau}_{II_{25}}. \qquad (4.22)
\end{aligned}$$

Applying Lemma 2.3 to $II_{2i}$ $(1\le i\le 5)$ repeatedly, then one arrives at

$$\begin{aligned}
II_{21}&\le \varepsilon\int_0^t \|\nabla^2 d\|_{H^1}^2d\tau+C(\varepsilon)\int_0^t \|\nabla d\|_{L_w^{r_2}}^{\frac{2r_2}{r_2-3}}\|\nabla^2 d\|_{L^2}^2d\tau, \\
II_{22}&\le \varepsilon\int_0^t \|\nabla u\|_{H^1}^2d\tau+C(\varepsilon)\int_0^t \|u\|_{L_w^{r_1}}^{\frac{2r_1}{r_1-3}}\|\nabla u\|_{L^2}^2d\tau \\
&\le \varepsilon\int_0^t (\|\sqrt{\rho}\dot u\|_{L^2}^2+\left\||\nabla d||\nabla^2 d|\right\|_{L^2}^2)d\tau+C(\varepsilon)\int_0^t \|u\|_{L_w^{r_1}}^{\frac{2r_1}{r_1-3}}\|\nabla u\|_{L^2}^2d\tau,
\end{aligned}$$

$$\le \varepsilon \int_0^t (\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\nabla^2 d\|_{H^1}^2) d\tau + C(\varepsilon) \int_0^t \left( \|\nabla d\|_{L_w^{r_2}}^{\frac{2r_2}{r_2-3}} \|\nabla^2 d\|_{L^2}^2 + \|u\|_{L_w^{r_1}}^{\frac{2r_1}{r_1-3}} \|\nabla u\|_{L^2}^2 \right) d\tau,$$

$$II_{23} \le \varepsilon \int_0^t \|\nabla^2 d\|_{H^1}^2 d\tau + C(\varepsilon) \int_0^t \|u\|_{L_w^{r_1}}^{\frac{2r_1}{r_1-3}} \|\nabla^2 d\|_{L^2}^2 d\tau,$$

$$II_{24} \le \varepsilon \int_0^t \|\nabla u\|_{H^1}^2 d\tau + C(\varepsilon) \int_0^t \|\nabla d\|_{L_w^{r_2}}^{\frac{2r_2}{r_2-3}} \|\nabla u\|_{L^2}^2 d\tau$$

$$\le \varepsilon \int_0^t (\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\nabla^2 d\|_{H^1}^2) d\tau + C(\varepsilon) \int_0^t \|\nabla d\|_{L_w^{r_2}}^{\frac{2r_2}{r_2-3}} \left( \|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) d\tau,$$

$$II_{25} \le \varepsilon \int_0^t \||\nabla d|^2\|_{H^1}^2 d\tau + C(\varepsilon) \int_0^t \|\nabla d\|_{L_w^{r_2}}^{\frac{2r_2}{r_2-3}} \||\nabla d|^2\|_{L^2}^2 d\tau$$

$$\le \varepsilon \int_0^t \left( \|\nabla d\|_{L^4}^4 + \|\nabla(|\nabla d|^2)\|_{L^2}^2 \right) d\tau + C(\varepsilon) \int_0^t \|\nabla d\|_{L_w^{r_2}}^{\frac{2r_2}{r_2-3}} \|\nabla d\|_{L^4}^4 d\tau.$$

Substituting $II_{2i}$ $(1 \le i \le 5)$ into (4.22) and choosing $\varepsilon$ small enough, it is easy to deduce that

$$\begin{aligned} &\int (|\nabla u|^2 + |\nabla^2 d|^2 + |\nabla d|^4)(t) dx \\ &\quad + \int_0^t \int \left( \rho|\dot{u}|^2 + |\nabla d|^2 |\nabla^2 d|^2 + |\nabla(|\nabla d|^2)|^2 + |\nabla^3 d|^2 + |\nabla d_t|^2 \right) dx d\tau \\ &\le C + C \int_0^t (1 + \|u\|_{L_w^{r_1}}^{\frac{2r_1}{r_1-3}} + \|\nabla d\|_{L_w^{r_2}}^{\frac{2r_2}{r_2-3}}) \left( 1 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^4}^4 \right) d\tau, \end{aligned}$$

which, applying (4.1) and Grönwall inequality, completes the proof of the lemma. □

As a corollary of Lemma 4.2, it is a direct result from $(1.1)_4$.

**Corollary 4.3.** *Under the condition (4.1), it holds that for* $0 \le T < T^*$,

$$\sup_{0 \le t \le T} \|d_t\|_{L^2} + \int_0^T \|u\|_{H^2}^2 d\tau \le C. \tag{4.23}$$

Now, we give the second important estimate-norm of $\|\sqrt{\rho} u_t\|_{L^2}$ and $\|\nabla^3 d\|_{L^2}$.

**Lemma 4.4.** *Under the condition (4.1), it holds that for* $0 \le T < T^*$,

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)+\int_0^t(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2)d\tau \le C. \tag{4.24}$$

**Proof.** Step 1: Differentiating $(1.1)_2$ with respect to $t$, we get

$$\begin{aligned}&\rho u_{tt}+\rho u\cdot\nabla u_t-\operatorname{div}(2\mu(\rho)D(u_t))+\nabla P_t\\&=-\rho_t u_t-\rho_t u\cdot\nabla u-\rho u_t\cdot\nabla u+\operatorname{div}(2\mu'\rho_t D(u))-\operatorname{div}(\nabla d_t\odot\nabla d+\nabla d\odot\nabla d_t).\end{aligned}\tag{4.25}$$

Multiplying (4.25) by $u_t$ and integrating (by parts) over $\Omega$, one arrives at

$$\begin{aligned}&\frac12\frac{d}{dt}\int\rho|u_t|^2dx+\frac1C\int|\nabla u_t|^2dx\\&\lesssim\underbrace{\int\rho|u||u_t||\nabla u_t|dx}_{II_{31}}+\underbrace{\int\rho|u||\nabla u|^2|u_t|dx}_{II_{32}}+\underbrace{\int\rho|u|^2|\nabla^2u||u_t|dx}_{II_{33}}\\&\quad+\underbrace{\int\rho|u|^2|\nabla u||\nabla u_t|dx}_{II_{34}}+\underbrace{\int\rho|u_t|^2|\nabla u|dx}_{II_{35}}+\underbrace{\int|\nabla d||\nabla d_t||\nabla u_t|dx}_{II_{36}}\\&\quad+\int|u||\nabla\rho||\nabla u||\nabla u_t|dx.\end{aligned}\tag{4.26}$$

Using (4.7), Hölder, interpolation, Sobolev, and Young inequalities repeatedly, we get

$$\begin{aligned}II_{31}&\le\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2}\\&\le C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac12}\|\nabla u_t\|_{L^2}\\&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32}\le C(\varepsilon)\|\sqrt{\rho}u_t\|_{L^2}^2+\varepsilon\|\nabla u_t\|_{L^2}^2,\\II_{32}&\le\|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}^2\|u_t\|_{L^6}\le C\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\&\le C(\varepsilon)\|\nabla u\|_{H^1}^2+\varepsilon\|\nabla u_t\|_{L^2}^2,\\II_{33}&\le\|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla^2u\|_{L^2}\|u_t\|_{L^6}\le C\|\nabla u\|_{L^2}\|\nabla^2u\|_{L^2}\|\nabla u_t\|_{L^2}\\&\le C(\varepsilon)\|\nabla^2u\|_{L^2}^2+\varepsilon\|\nabla u_t\|_{L^2}^2,\\II_{34}&\le\|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\le C\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2}\\&\le C(\varepsilon)\|\nabla u\|_{H^1}^2+\varepsilon\|\nabla u_t\|_{L^2}^2,\\II_{35}&\le\|\sqrt{\rho}\|_{L^\infty}\|u_t\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u\|_{L^2}\le C\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac12}\\&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32}\le C(\varepsilon)\|\sqrt{\rho}u_t\|_{L^2}^2+\varepsilon\|\nabla u_t\|_{L^2}^2,\end{aligned}$$

$$
\begin{aligned}
II_{36} &\le \|\nabla d\|_{L^6}\|\nabla d_t\|_{L^3}\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla d\|_{H^1}\|\nabla d_t\|_{L^2}^{\frac{1}{2}}\|\nabla d_t\|_{L^6}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}\\
&\le C(\varepsilon,\eta)\|\nabla d_t\|_{L^2}^2+\eta\|\nabla d_t\|_{H^1}^2+\varepsilon\|\nabla u_t\|_{L^2}^2.
\end{aligned}
$$

Substituting $II_{3i}$ $(1 \le i \le 6)$ into (4.26), then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx+\frac{1}{C}\int |\nabla u_t|^2dx\\
&\le C(\varepsilon)(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u\|_{H^1}^2)+C(\varepsilon,\eta)\|\nabla d_t\|_{L^2}^2+\eta\|\nabla d_t\|_{H^1}^2+\varepsilon\|\nabla u_t\|_{L^2}^2\\
&\quad +C\int |u||\nabla\rho||\nabla u||\nabla u_t|dx. \qquad (4.27)
\end{aligned}
$$

Using (4.10), (4.7) and interpolation inequality, we deduce

$$
\begin{aligned}
\|u\|_{H^2}+\|P\|_{H^1} &\le C(\|\sqrt{\rho}u_t\|_{L^2}+\||u||\nabla u|\|_{L^2}+\||\nabla d||\nabla^2 d|\|_{L^2})\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}+\|u\|_{L^6}\|\nabla u\|_{L^3}+\|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^2})\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}+\|\nabla d\|_{H^2})\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla d\|_{H^2}+1)+\frac{1}{2}\|\nabla u\|_{H^1},
\end{aligned}
$$

which implies

$$
\|u\|_{H^2}+\|P\|_{H^1} \le C(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla d\|_{H^2}+1). \qquad (4.28)
$$

Combining (4.27) with (4.28), one arrives at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx+\frac{1}{C}\int |\nabla u_t|^2dx\\
&\le C(\varepsilon)(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2+1)+C(\varepsilon,\eta)\|\nabla d_t\|_{L^2}^2\\
&\quad +\eta\|\nabla d_t\|_{H^1}^2+\varepsilon\|\nabla u_t\|_{L^2}^2+C\int |u||\nabla\rho||\nabla u||\nabla u_t|dx. \qquad (4.29)
\end{aligned}
$$

To estimate the term $\int |u||\nabla\rho||\nabla u||\nabla u_t|dx$. Indeed, by using (4.1), Hölder, Gagliardo–Nirenberg and Young inequalities, we obtain

$$
\begin{aligned}
&\int |u||\nabla\rho||\nabla u||\nabla u_t|dx\\
&\le \|u\|_{L^6}\|\nabla\rho\|_{L^q}\|\nabla u\|_{L^{\frac{3q}{q-3}}}\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}\|\nabla\rho\|_{L^q}\|\nabla u\|_{L^2}^{\frac{2q-6}{3q}}\|\nabla u\|_{L^\infty}^{\frac{q+6}{3q}}\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla u\|_{W^{1,r}}^{\frac{q+6}{3q}}\|\nabla u_t\|_{L^2}. \qquad (4.30)
\end{aligned}
$$

By Lemma 2.1, it is easy to deduce that

$$\|u\|_{W^{2,r}} + \|P\|_{W^{1,r}} \le C(1 + \|\nabla u_t\|_{L^2} + \|\nabla^3 d\|_{L^2}^2). \tag{4.31}$$

By Young inequality, combining (4.30) with (4.31) yields

$$\begin{aligned}
&\int |u||\nabla\rho|\,|\nabla u||\nabla u_t|dx \\
&\le C(1 + \|\nabla u_t\|_{L^2} + \|\nabla^3 d\|_{L^2}^2)^{\frac{q+6}{3q}} \|\nabla u_t\|_{L^2} \\
&\le C(\varepsilon)(1 + \|\nabla^3 d\|_{L^2}^4) + \varepsilon\|\nabla u_t\|_{L^2}^2.
\end{aligned} \tag{4.32}$$

Substituting (4.32) into (4.29) and choosing $\varepsilon$ small enough, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2 dx + \frac{1}{C}\int |\nabla u_t|^2 dx \\
&\le C(1 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^4) + C(\eta)\|\nabla d_t\|_{L^2}^2 + \eta\|\nabla d_t\|_{H^1}^2.
\end{aligned}$$

Thanks to the compatibility condition and (4.7), we get

$$\begin{aligned}
&\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^t \|\nabla u_t\|_{L^2}^2 d\tau \\
&\le C(\eta) + C\int_0^t (1 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^4) d\tau + \eta\int_0^t \|\nabla^2 d_t\|_{L^2}^2 d\tau.
\end{aligned} \tag{4.33}$$

Step 2: In order to the control the term $\int_0^t \|\nabla^3 d\|_{L^2}^4 d\tau$, we should derive some estimates for $\|\nabla^3 d\|_{L^2}^2$. Multiplying (4.13) by $\nabla\Delta d_t$ and integrating (by parts) over $\Omega$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla\Delta d|^2 dx + \int |\Delta d_t|^2 dx \\
&= \frac{d}{dt}\int \nabla(u\cdot\nabla d - |\nabla d|^2 d)\cdot\nabla\Delta d\, dx - \int \frac{\partial}{\partial t}\Big[\nabla(u\cdot\nabla d - |\nabla d|^2 d)\Big]\cdot\nabla\Delta d\, dx.
\end{aligned} \tag{4.34}$$

To give the second term of the left hand side of (4.34) first. Indeed, it is easy to deduce that

$$
\begin{aligned}
&-\int \frac{\partial}{\partial t}\Big[\nabla(u\cdot\nabla d-|\nabla d|^2 d)\Big]\cdot\nabla\Delta d\,dx\\
&\lesssim \underbrace{\int |\nabla u_t||\nabla d||\nabla\Delta d|dx}_{II_{41}}+\underbrace{\int |\nabla u||\nabla d_t||\nabla\Delta d|dx}_{II_{42}}+\underbrace{\int |u_t||\nabla^2 d||\nabla\Delta d|dx}_{II_{43}}\\
&\quad+\underbrace{\int |u||\nabla^2 d_t||\nabla\Delta d|dx}_{II_{44}}+\underbrace{\int |\nabla d_t||\nabla^2 d||\nabla\Delta d|dx}_{II_{45}}+\underbrace{\int |\nabla d||\nabla^2 d_t||\nabla\Delta d|dx}_{II_{46}}\\
&\quad+\underbrace{\int |\nabla d||\nabla^2 d||d_t||\nabla\Delta d|dx}_{II_{47}}+\underbrace{\int |\nabla d|^2|\nabla d_t||\nabla\Delta d|dx}_{II_{48}}. \qquad (4.35)
\end{aligned}
$$

By using (4.7), (4.10), Hölder, Sobolev and Young inequalities repeatedly, we obtain

$$
\begin{aligned}
II_{41}&\le C\|\nabla d\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\nabla\Delta d\|_{L^2}\\
&\le C\|\nabla d\|_{H^2}\|\nabla^3 d\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\le C(\delta)(\|\nabla^3 d\|_{L^2}^2+1)\|\nabla^3 d\|_{L^2}^2+\delta\|\nabla u_t\|_{L^2}^2,\\
II_{42}&\le C\|\nabla u\|_{L^3}\|\nabla d_t\|_{L^6}\|\nabla\Delta d\|_{L^2}\\
&\le C\|\nabla u\|_{H^1}\|\nabla^3 d\|_{L^2}\|\nabla d_t\|_{H^1}\\
&\le C(\varepsilon)\|\nabla u\|_{H^1}^2\|\nabla^3 d\|_{L^2}^2+\varepsilon\|\nabla d_t\|_{H^1}^2\\
&\le C(\varepsilon)(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2+1)\|\nabla^3 d\|_{L^2}^2+\varepsilon\|\nabla d_t\|_{H^1}^2,\\
II_{43}&\le \|u_t\|_{L^6}\|\nabla^2 d\|_{L^3}\|\nabla\Delta d\|_{L^2}\\
&\le \|\nabla u_t\|_{L^2}\|\nabla^2 d\|_{H^1}\|\nabla^3 d\|_{L^2}\\
&\le C(\delta)(1+\|\nabla^3 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2}^2+\delta\|\nabla u_t\|_{L^2}^2,\\
II_{44}&\le C\|u\|_{L^\infty}\|\nabla^2 d_t\|_{L^2}\|\nabla\Delta d\|_{L^2}\\
&\le C\|u\|_{H^2}\|\nabla^3 d\|_{L^2}\|\nabla^2 d_t\|_{L^2}\\
&\le (\varepsilon)(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2+1)\|\nabla^3 d\|_{L^2}^2+\varepsilon\|\nabla^2 d_t\|_{L^2}^2,\\
II_{45}&\le C\|\nabla d_t\|_{L^6}\|\nabla^2 d\|_{L^3}\|\nabla\Delta d\|_{L^2}\\
&\le C\|\nabla d_t\|_{H^1}\|\nabla^2 d\|_{H^1}\|\nabla^3 d\|_{L^2}\\
&\le C(\varepsilon)(1+\|\nabla^3 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2}^2+\varepsilon\|\nabla d_t\|_{H^1}^2,\\
II_{46}&\le C\|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^6}\|d_t\|_{L^6}\|\nabla\Delta d\|_{L^2}\\
&\le C\|\nabla d\|_{H^1}\|\nabla^2 d\|_{H^1}\|d_t\|_{H^1}\|\nabla^3 d\|_{L^2}\\
&\le C(\varepsilon)(1+\|\nabla^3 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2}^2+\varepsilon(\|\nabla d_t\|_{L^2}^2+1),
\end{aligned}
$$

$$\begin{aligned}
II_{47} &\leq C\|\nabla d\|_{L^\infty}\|\nabla^2 d_t\|_{L^2}\|\nabla\Delta d\|_{L^2}\\
&\leq C\|\nabla d\|_{H^2}\|\nabla^3 d\|_{L^2}\|\nabla^2 d_t\|_{L^2}\\
&\leq C(\varepsilon)(1+\|\nabla^3 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2}^2+\varepsilon\|\nabla^2 d_t\|_{L^2}^2,\\
II_{48} &\leq C\|\nabla d\|_{L^6}^2\|\nabla d_t\|_{L^6}\|\nabla\Delta d\|_{L^2}\\
&\leq C\|\nabla d\|_{H^1}^2\|\nabla d_t\|_{H^1}\|\nabla^3 d\|_{L^2}\\
&\leq C(\varepsilon)\|\nabla^3 d\|_{L^2}^2+\varepsilon\|\nabla d_t\|_{H^1}^2.
\end{aligned}$$

Substituting $II_{4i}$ $(1\leq i\leq 8)$ into (4.35), then we get

$$\begin{aligned}
&-\int\frac{\partial}{\partial t}\Big[\nabla(u\cdot\nabla d-|\nabla d|^2 d)\Big]\cdot\nabla\Delta d\,dx\\
&\leq C(\varepsilon,\delta)(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2}^2+\varepsilon\|\nabla d_t\|_{H^1}^2+\delta\|\nabla u_t\|_{L^2}^2.
\end{aligned}\tag{4.36}$$

Substituting (4.36) into (4.34) and integrating the resulting inequality over $(0,t)$, one arrives at

$$\begin{aligned}
&\frac{1}{2}\int|\nabla^3 d|^2dx+\int_0^t\int|\nabla^2 d_t|^2dxd\tau\\
&\leq C+C(\varepsilon,\delta)\int_0^t(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2}^2d\tau+\varepsilon\int_0^t\|\nabla^2 d_t\|_{L^2}^2d\tau\\
&\quad+\delta\int_0^t\|\nabla u_t\|_{L^2}^2d\tau+C\int(|\nabla u||\nabla d|+|u||\nabla^2 d|+|\nabla d||\nabla^2 d|+|\nabla d|^3)|\nabla\Delta d|dx.
\end{aligned}$$

Choosing $\varepsilon=\frac{1}{2}$, we obtain

$$\begin{aligned}
&\int|\nabla^3 d|^2dx+\int_0^t\int|\nabla^2 d_t|^2dxd\tau\\
&\leq C+C(\delta)\int_0^t(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2}^2d\tau+\delta\int_0^t\|\nabla u_t\|_{L^2}^2d\tau\\
&\quad+\underbrace{C\int|\nabla u||\nabla d||\nabla\Delta d|dx}_{II_{51}}+\underbrace{C\int|u||\nabla^2 d||\nabla\Delta d|dx}_{II_{52}}\\
&\quad+\underbrace{C\int|\nabla d||\nabla^2 d||\nabla\Delta d|dx}_{II_{53}}+\underbrace{C\int|\nabla d|^3|\nabla\Delta d|dx}_{II_{54}}.
\end{aligned}\tag{4.37}$$

By using the Hölder, Gagliardo–Nirenberg and Young inequalities repeatedly, we get

$$\begin{aligned}
II_{51} &\le C\|\nabla d\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}\\
&\le C\|\nabla d\|_{L^2}^{\frac14}\|\nabla d\|_{H^2}^{\frac34}\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}\\
&\le C(1+\|\nabla^3 d\|_{L^2})^{\frac34}\|\nabla^3 d\|_{L^2}\le C(\varepsilon)+\varepsilon\|\nabla^3 d\|_{L^2}^2,\\
II_{52} &\le C\|u\|_{L^6}\|\nabla^2 d\|_{L^3}\|\nabla\Delta d\|_{L^2}\le C\|\nabla u\|_{L^2}\|\nabla^2 d\|_{L^2}^{\frac12}\|\nabla^2 d\|_{H^1}^{\frac12}\|\nabla\Delta d\|_{L^2}\\
&\le C(1+\|\nabla^3 d\|_{L^2})^{\frac12}\|\nabla^3 d\|_{L^2}\le C(\varepsilon)+\varepsilon\|\nabla^3 d\|_{L^2}^2,\\
II_{53} &\le C\|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^3}\|\nabla\Delta d\|_{L^2}\le C\|\nabla d\|_{H^1}\|\nabla^2 d\|_{L^2}^{\frac12}\|\nabla^2 d\|_{H^1}^{\frac12}\|\nabla\Delta d\|_{L^2}\\
&\le C(1+\|\nabla^3 d\|_{L^2})^{\frac12}\|\nabla^3 d\|_{L^2}\le C(\varepsilon)+\varepsilon\|\nabla^3 d\|_{L^2}^2,\\
II_{54} &\le C\|\nabla d\|_{L^6}^3\|\nabla\Delta d\|_{L^2}\le C\|\nabla d\|_{H^1}^3\|\nabla^3 d\|_{L^2}\le C(\varepsilon)+\varepsilon\|\nabla^3 d\|_{L^2}^2.
\end{aligned}$$

Substituting $II_{5i}$ $(1\le i\le 4)$ into (4.37) and choosing $\varepsilon$ small enough yield directly

$$\begin{aligned}
&\|\nabla^3 d\|_{L^2}^2+\int_0^t\|\nabla^2 d_t\|_{L^2}^2d\tau\\
&\le C+C(\delta)\int_0^t(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2}^2d\tau+\delta\int_0^t\|\nabla u_t\|_{L^2}^2d\tau. \qquad (4.38)
\end{aligned}$$

Adding (4.38) to (4.33) and choosing $\delta$ and $\eta$ suitably small, we obtain

$$\begin{aligned}
&\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2+\int_0^t(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2)d\tau\\
&\le C+C\int_0^t(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)(1+\|\nabla^3 d\|_{L^2}^2)d\tau,
\end{aligned}$$

which, together with Grönwall inequality and (4.7), completes the proof. □

Finally, we derive the high-order estimates for the strong solution $(\rho,u,P,d)$.

**Lemma 4.5.** *Under the condition (4.1), it holds that for $0\le T<T^*$,*

$$\begin{aligned}
&\sup_{0\le t\le T}(\|\rho_t\|_{L^q}+\|u\|_{H^2}+\|P\|_{H^1}+\|\nabla d_t\|_{L^2})\\
&\quad+\int_0^T\left(\|u\|_{W^{2,r}}^2+\|P\|_{W^{1,r}}^2+\|d_{tt}\|_{L^2}^2+\|\nabla^4 d\|_{L^2}^2\right)dt\le C. \qquad (4.39)
\end{aligned}$$

**Proof.** By (4.28) and (4.31), it is easy to deduce

$$\|u\|_{H^2} + \|P\|_{H^1} + \int_0^T \left(\|u\|_{W^{2,r}}^2 + \|P\|_{W^{1,r}}^2\right) dt \le C, \tag{4.40}$$

which, together with $(1.1)_1$, yields

$$\|\rho_t\|_{L^q} \le \|u\|_{L^\infty} \|\nabla\rho\|_{L^q} \le C\|u\|_{H^2}\|\nabla\rho\|_{L^q} \le C. \tag{4.41}$$

By (4.13), (4.25), (4.40) and Sobolev inequality, we obtain

$$\begin{aligned}
&\|\nabla d_t\|_{L^2} = \|\nabla\Delta d + \nabla(|\nabla d|^2 d - u\cdot\nabla d)\|_{L^2}\\
&\le C(\|\nabla^3 d\|_{L^2} + \||\nabla u||\nabla d|\|_{L^2} + \||u||\nabla^2 d|\|_{L^2} + \|\nabla d\|_{L^6}^3 + \||\nabla d||\nabla^2 d|\|_{L^2})\\
&\le C(\|\nabla^3 d\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla d\|_{H^2} + \|u\|_{H^2}\|\nabla^2 d\|_{L^2} + \|\nabla d\|_{H^1}^3 + \|\nabla d\|_{H^2}\|\nabla^2 d\|_{L^2})\\
&\le C.
\end{aligned} \tag{4.42}$$

Differentiating $(1.1)_4$ with respect to $t$, one arrives at

$$d_{tt} - \Delta d_t = (|\nabla d|^2 d - u\cdot\nabla d)_t. \tag{4.43}$$

Taking $L^2$ estimate to (4.43), by virtue of (4.24) and (4.40), we obtain

$$\begin{aligned}
\int_0^T \|d_{tt}\|_{L^2}^2 dt &\lesssim \int_0^T (\|\Delta d_t\|_{L^2}^2 + \||\nabla d_t||\nabla d|\|_{L^2}^2 + \||\nabla d|^2|d_t|\|_{L^2}^2) dt\\
&\quad + \int_0^T (\||u_t||\nabla d|\|_{L^2}^2 + \||u||\nabla d_t|\|_{L^2}^2) dt\\
&\lesssim \int_0^T \left(\|\nabla^2 d_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\|\nabla d\|_{H^2}^2 + \|d_t\|_{L^2}^2\|\nabla d\|_{H^2}^4\right) dt\\
&\quad + \int_0^T \left(\|\nabla u_t\|_{L^2}^2\|\nabla d\|_{H^1}^2 + \|u\|_{H^2}^2\|\nabla d_t\|_{L^2}^2\right) dt\\
&\lesssim \int_0^T (\|\nabla^2 d_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + 1) dt \le C.
\end{aligned} \tag{4.44}$$

Taking $\nabla$ operator to $(1.1)_1$, we get

$$-\nabla^2\Delta d = \nabla^2(|\nabla d|^2 d - u\cdot\nabla d - d_t). \tag{4.45}$$

Applying $H^4$ estimate to $d$ with boundary $\frac{\partial d}{\partial \nu}\big|_{\partial\Omega} = 0$ and (4.45), we have

$$\begin{aligned}
\|\nabla^4 d\|_{L^2} &\le C(\|\nabla^2 \Delta d\|_{L^2} + \|\nabla^2 d\|_{H^1}) \\
&\le C(\|\nabla^2(|\nabla d|^2 d - u\cdot\nabla d - d_t)\|_{L^2} + 1) \\
&\le C\Big(\|\nabla^2 d_t\|_{L^2} + \||\nabla^2 d|^2\|_{L^2} + \||\nabla d|^2|\nabla^2 d|\|_{L^2} + \||\nabla^2 u||\nabla d|\|_{L^2} \\
&\qquad + \||\nabla u||\nabla^2 d|\|_{L^2} + \||u||\nabla^3 d|\|_{L^2}\Big) \\
&\le C\Big(\|\nabla^2 d_t\|_{L^2} + \|\nabla^2 d\|_{L^4}^2 + \|\nabla d\|_{H^2}^2\|\nabla^2 d\|_{L^2} + \|\nabla d\|_{H^2}\|\nabla^2 u\|_{L^2} \\
&\qquad + \|\nabla u\|_{H^1}\|\nabla^2 d\|_{H^1} + \|u\|_{H^2}\|\nabla^3 d\|_{L^2}\Big) \\
&\le C(\|\nabla^2 d_t\|_{L^2} + 1),
\end{aligned} \tag{4.46}$$

which means

$$\int_0^T \|\nabla^4 d\|_{L^2}^2 dt \le C,$$

which completes the proof. $\square$

After having the Lemmas 4.1–4.5 at hand, it is easy to apply the Theorem 1.1 to extend the strong solution $(\rho, u, P, d)$ beyond time $T^*$. Therefore, we complete the proof of Theorem 1.2.

## 5. Proof of Theorem 1.4

In this section, we will give the proof of Theorem 1.4 by contradiction. Assume $0 < T^* < \infty$ to be the maximum time for the existence of strong solution $(\rho, u, P, d)$ to (1.1)–(1.4). Suppose that (1.10) were false, that is

$$M_1 \triangleq \lim_{T\to T^*}\left(\|\nabla\rho\|_{L^\infty(0,T;L^q)} + \|\nabla d\|_{L^s(0,T;L^r_w)}\right) < \infty. \tag{5.1}$$

**Lemma 5.1.** *Under the condition (5.1), it holds that for $0 \le T < T^*$,*

$$\int_0^T \int |\nabla^2 d|^2 dx d\tau \le C, \tag{5.2}$$

*where $C$ denotes generic constants depending only on $\Omega$, $M_1$, $T^*$ and the initial data.*

**Proof.** By virtue of $|d| = 1$, we have $d \cdot \Delta d = |\nabla d|^2$. Then, by (4.2), we obtain

$$\int_0^T \int |\Delta d|^2 dx d\tau = \int_0^T \int |\nabla d|^4 dx d\tau + \int_0^T \int \left| \Delta d + |\nabla d|^2 d \right|^2 dx d\tau$$
$$\leq \int_0^T \int |\nabla d|^4 dx d\tau + C. \tag{5.3}$$

By virtue of Lemma 2.3 and (4.2) it is easy to deduce

$$\int |\nabla d|^4 dx \leq \varepsilon \|\nabla d\|_{H^1}^2 + C(\varepsilon) \|\nabla d\|_{L_w^r}^{\frac{2r}{r-2}} \|\nabla d\|_{L^2}^2$$
$$\leq \varepsilon \|\nabla^2 d\|_{L^2}^2 + C(\varepsilon)(1 + \|\nabla d\|_{L_w^r}^{\frac{2r}{r-2}}).$$

Applying the elliptic regularity with Neumann boundary, one arrives at

$$\int_0^T \|\nabla^2 d\|_{L^2}^2 d\tau \leq C \int_0^T (\|\Delta d\|_{L^2}^2 + \|d\|_{H^1}^2) d\tau$$
$$\leq \varepsilon \int_0^T \|\nabla^2 d\|_{L^2}^2 d\tau + C(\varepsilon) \int_0^T (1 + \|\nabla d\|_{L_w^r}^{\frac{2r}{r-2}}) d\tau$$
$$\leq \varepsilon \int_0^T \|\nabla^2 d\|_{L^2}^2 d\tau + C(\varepsilon) \int_0^T (1 + \|\nabla d\|_{L_w^r}^{s}) d\tau$$
$$\leq \varepsilon \int_0^T \|\nabla^2 d\|_{L^2}^2 d\tau + C,$$

where, $r$ and $s$ satisfy the condition (1.11). Choosing $\varepsilon = \frac{1}{2}$, we get

$$\int_0^T \int |\nabla^2 d|^2 dx d\tau \leq C,$$

which completes the proof. $\square$

**Lemma 5.2.** *Under the condition (5.1), it holds that for* $0 \le T < T^*$,

$$\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|\nabla d\|_{L^4}^4+\|\nabla^2 d\|_{L^2}^2\right)$$
$$+\int_0^T\int\left(|\sqrt{\rho}\dot{u}|^2+|\nabla d|^2|\nabla^2 d|^2+|\nabla^3 d|^2+|\nabla d_t|^2\right)dxd\tau<\infty. \tag{5.4}$$

**Proof.** By virtue of (4.21), it is easy to deduce, for any $0 \le s < t \le T$,

$$\int(|\nabla u|^2+|\nabla^2 d|^2+|\nabla d|^4)(t)dx$$
$$+\int_s^t\int\left(\rho|\dot{u}|^2+|\nabla d|^2|\nabla^2 d|^2+|\nabla(|\nabla d|^2)|^2+|\nabla^3 d|^2+|\nabla d_t|^2\right)dxd\tau$$
$$\le C+C\int(|\nabla u|^2+|\nabla^2 d|^2+|\nabla d|^4)(s)dx$$
$$+C\int_s^t(1+\|u\|_{L^\infty}^2+\|\nabla d\|_{L^\infty}^2)(1+\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2+\|\nabla d\|_{L^4}^4)d\tau. \tag{5.5}$$

Let

$$A(t)\triangleq e+\int(|\nabla u|^2+|\nabla^2 d|^2+|\nabla d|^4)(t)dx$$
$$+\int_0^t\int\left(\rho|\dot{u}|^2+|\nabla d|^2|\nabla^2 d|^2+|\nabla(|\nabla d|^2)|^2+|\nabla^3 d|^2+|\nabla d_t|^2\right)dxd\tau,$$

then we deduce from (5.5) that

$$A(t)\le CA(s)+C\int_0^t(1+\|u\|_{L^\infty}^2+\|\nabla d\|_{L^\infty}^2)A(\tau)d\tau,$$

which, together with the Grönwall inequality, gives directly that

$$A(t)\le CA(s)\exp\left[C\int_s^t(1+\|u\|_{L^\infty}^2+\|\nabla d\|_{L^\infty}^2)d\tau\right].$$

Let

$$\Phi(t) \triangleq e + \sup_{0\le\tau\le t}(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^4}^4)$$
$$+ \int_0^t \int \left(\rho|\dot{u}|^2 + |\nabla d|^2|\nabla^2 d|^2 + |\nabla(|\nabla d|^2)|^2 + |\nabla^3 d|^2 + |\nabla d_t|^2\right) dx d\tau,$$

then we have

$$\Phi(T) \le C\Phi(s)\exp\left[C\int_s^T (1 + \|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) d\tau\right]. \tag{5.6}$$

Now we control the term $\int_s^T (1 + \|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) d\tau$. Indeed, by the Lemma 2.4, we have

$$\begin{aligned} &C\int_s^T (1 + \|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) d\tau \\ &\le C\Big[(T-s) + \left(\|u\|_{L^2(s,T;H^1)}^2 + \|\nabla d\|_{L^2(s,T;H^1)}^2\right)\left(\ln\left(e + \|u\|_{L^2(s,T;W^{1,3})}\right)\right. \\ &\quad \left.+ \ln\left(e + \|\nabla d\|_{L^2(s,T;W^{1,3})}\right)\right)\Big]. \end{aligned} \tag{5.7}$$

Applying the Sobolev inequality and regularity estimate, one arrives at

$$\|u\|_{W^{1,3}}^2 \le C\|u\|_{W^{2,2}}^2 \le C(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \||\nabla d||\nabla^2 d|\|_{L^2}^2), \tag{5.8}$$

$$\|\nabla d\|_{W^{1,3}}^2 \le C\|\nabla d\|_{W^{2,2}}^2 \le C(1 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2). \tag{5.9}$$

Substituting (5.8)–(5.9) into (5.7) yields

$$C\int_s^T (1 + \|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) d\tau \le C + \ln[C\Phi(T)]^{C\left(\|u\|_{L^2(s,T;H^1)}^2 + \|\nabla d\|_{L^2(s,T;H^1)}^2\right)},$$

which, together with (5.6), gives directly that

$$\Phi(T) \le C\Phi(s)\Phi(T)^{C\left(\|u\|_{L^2(s,T;H^1)}^2 + \|\nabla d\|_{L^2(s,T;H^1)}^2\right)}.$$

Thanks to (4.2) and (5.2), we can choose $s$ closes enough to $T^*$ such that

$$\lim_{T\to T^*} C\left(\|u\|_{L^2(s,T;H^1)}^2 + \|\nabla d\|_{L^2(s,T;H^1)}^2\right) \le \frac{1}{2},$$

which, means immediately

$$\Phi(T) \le C\Phi(s)^2 < \infty.$$

Thus, we complete the proof. □

**Remark 5.1.** Unfortunately, we cannot derive the bound, just depending on the initial data, for (5.4) owing to the technique used here. However, the bound is uniform with respect to time in (5.4) since $s$, which closed enough to $T^*$, is fixed in process of the proof for Lemma 5.2. Thus, we can rewrite (5.4) as

$$\sup_{t\in[0,T]} \int \left(|\nabla u|^2 + |\nabla^2 d|^2 + |\nabla d|^4\right) dx + \int_0^T \int \left(\rho|\dot{u}|^2 + |\nabla d|^2|\nabla^2 d|^2 + |\nabla d_t|^2 + |\nabla^3 d|^2\right) dxdt \le C(s),$$

where and in what follows, $C(s)$ denotes generic constants depending not only on $\Omega$, $M_1$, $T^*$ and the initial data, but also on the data that is fixed on time $s$.

Similar to Lemma 4.3, as a corollary of Lemma 4.2, we have the following estimate.

**Corollary 5.3.** *Under the condition (5.1), it holds that for* $0 \le T < T^*$*,*

$$\sup_{0\le t\le T} \|d_t\|_{L^2} + \int_0^T \|u\|_{H^2}^2 d\tau \le C(s). \tag{5.10}$$

Finally, we derive the high order estimate $\|\sqrt{\rho}u_t\|_{L^2}$ and $\|\nabla^3 d\|_{L^2}$.

**Lemma 5.4.** *Under the condition (5.1), it holds that for* $0 \le T < T^*$*,*

$$\sup_{0\le t\le T} (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) + \int_0^T (\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2)dt \le C(s). \tag{5.11}$$

**Proof.** Recall from (4.29), we have the following estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx + \frac{1}{C}\int |\nabla u_t|^2dx \\
&\le C(\varepsilon)(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + 1) + C(\varepsilon,\eta)\|\nabla d_t\|_{L^2}^2 + \eta\|\nabla d_t\|_{H^1}^2 + \varepsilon\|\nabla u_t\|_{L^2}^2 \\
&\quad + C\int |u||\nabla\rho||\nabla u||\nabla u_t|dx.
\end{aligned} \tag{5.12}$$

By (5.1), Hölder, Sobolev and Young inequalities, we obtain

$$
\begin{aligned}
\int |u||\nabla\rho|\,|\nabla u||\nabla u_t|dx &\le \|u\|_{L^{\frac{2r}{r-2}}}\|\nabla u\|_{L^{\frac{rq}{q-r}}}\|\nabla\rho\|_{L^q}\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}\|\nabla\rho\|_{L^q}\|\nabla u_t\|_{L^2}\\
&\le C(\varepsilon)\|\nabla u\|_{H^1}^2+\varepsilon\|\nabla u_t\|_{L^2}^2\\
&\le C(\|\sqrt{\rho}u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2+1)+\varepsilon\|\nabla u_t\|_{L^2}^2. \qquad (5.13)
\end{aligned}
$$

Substituting (5.13) into (5.12) yields directly

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx+\frac{1}{C}\int|\nabla u_t|^2dx\\
&\le C(\varepsilon)(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2+1)+C(\varepsilon,\eta)\|\nabla d_t\|_{L^2}^2+\eta\|\nabla d_t\|_{H^1}^2+\varepsilon\|\nabla u_t\|_{L^2}^2. \qquad (5.14)
\end{aligned}
$$

Thanks to the compatibility condition, after choosing $\varepsilon$ small enough, we get

$$
\begin{aligned}
&\|\sqrt{\rho}u_t\|_{L^2}^2+\int_0^t\|\nabla u_t\|_{L^2}^2d\tau\\
&\le C(\eta)+C\int_0^t(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)d\tau+\eta\int_0^t\|\nabla^2 d_t\|_{L^2}^2d\tau. \qquad (5.15)
\end{aligned}
$$

Recall from (4.38), we have the estimate

$$
\begin{aligned}
&\|\nabla^3 d\|_{L^2}^2+\int_0^t\|\nabla^2 d_t\|_{L^2}^2d\tau\\
&\le C+C(\delta)\int_0^t(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2}^2d\tau+\delta\int_0^t\|\nabla u_t\|_{L^2}^2d\tau. \qquad (5.16)
\end{aligned}
$$

Adding (5.15) to (5.16) and choosing $\varepsilon$ and $\delta$ small enough, one deduces that

$$
\begin{aligned}
&(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)+\int_0^t(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2)d\tau\\
&\le C+C\int_0^t(1+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)(1+\|\nabla^3 d\|_{L^2}^2)d\tau,
\end{aligned}
$$

which, together with the Grönwall inequality, completes the proof of the lemma. □

As a corollary of Lemma 5.4, it is easy to deduce the following estimate

$$\sup_{0\le t\le T}(\|\rho_t\|_{L^q}+\|u\|_{H^2}+\|P\|_{H^1}+\|\nabla d_t\|_{L^2})$$
$$+\int_0^T\left(\|u\|^2_{W^{2,r}}+\|P\|^2_{W^{1,r}}+\|d_{tt}\|^2_{L^2}+\|\nabla^4 d\|^2_{L^2}\right)dt\le C(s).$$

Therefore, having all the estimates at hand, it is easy to extend the strong solutions beyond time $T^*$. Thus, we complete the proof of Theorem 1.4.

## 6. Proof of Corollary 1.6

In this section, we will give the proof Corollary 1.6. Indeed, let $(\rho, u, P, d)$ be the solution of (1.1)–(1.4), we will derive some maximal principle for the direction field.

**Lemma 6.1.** *For some given constant $\underline{d}_{0i}$ $(i=1,2)$, we have the following maximal principle:*

(i) *If $0\le \underline{d}_{0i}\le d_{0i}\le 1$, then $0\le \underline{d}_{0i}\le d_i\le 1$ for any $i=1,2$;*
(ii) *If $-1\le d_{0i}\le -\underline{d}_{0i}\le 0$, then $-1\le d_i\le -\underline{d}_{0i}\le 0$ for any $i=1,2$.*

**Proof.** Since (i) has been proved in [32], we only give the proof of (ii). Indeed, letting $V_i=d_i+\underline{d}_{0i}$ and $V_i^+=\max\{V_i,0\}\ge 0$, then we obtain

$$\partial_t V_i-\Delta V_i=|\nabla d|^2(V_i-\underline{d}_{0i})-u\cdot\nabla V_i. \tag{6.1}$$

Multiplying (6.1) by $V_i^+$ and using the Neumann boundary condition $\frac{\partial d}{\partial \nu}|_{\partial\Omega}=0$, we get that

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\|V_i^+\|^2_{L^2}+\|\nabla V_i^+\|^2_{L^2}\\
&=\int\left[|\nabla d|^2(V_i-\underline{d}_{0i})-u\cdot\nabla V_i\right]\cdot V_i^+dx\\
&=\int|\nabla d|^2|V_i^+|^2dx-\int|\nabla d|^2\underline{d}_{0i}V_i^+dx+\frac{1}{2}\int \mathrm{div}u|V_i^+|^2dx\\
&\le\int|\nabla d|^2|V_i^+|^2dx\le\|\nabla d\|^2_{L^\infty}\|V_i^+\|^2_{L^2}.
\end{aligned}$$

Hence, if we applying Grönwall inequality to the above inequality, one deduces

$$\|V_i^+(t)\|^2_{L^2}\le\|V_i^+(0)\|^2_{L^2}\exp\left(C\int_0^t\|\nabla d\|^2_{L^\infty}d\tau\right)=0,$$

which implies $-1\le d_i\le -\underline{d}_{0i}\le 0$. □

Before starting our main result in this section, we recall the elliptic estimate

$$\|\nabla^2 f\|_{L^2}^2 \le C_1(\|\Delta f\|_{L^2}^2 + \|f\|_{H^1}^2), \tag{6.2}$$

and the Gagliardo–Nirenberg inequality

$$\|\nabla f\|_{L^4}^4 \le C_2(\|\nabla^2 f\|_{L^2}^2 \|f\|_{L^\infty}^2 + \|f\|_{L^\infty}^4), \tag{6.3}$$

where the constants $C_1$ and $C_2$ independent of the function $f$.

**Lemma 6.2.** *For any $i$ $(i = 1, 2)$, if the $i$-th component of initial direction field $d_{0i}$ satisfies the condition*

$$0 \le \underline{d}_{0i} \le d_{0i} \le 1 \quad or \quad -1 \le d_{0i} \le -\underline{d}_{0i} \le 0, \tag{6.4}$$

*where $\underline{d}_{0i}$ is a constant and is defined by*

$$\begin{cases} \underline{d}_{0i} > \sqrt{1 - \dfrac{1}{C_1 C_2}}, & if\ C_1 C_2 \ge 1; \\ \underline{d}_{0i} = 0, & if\ C_1 C_2 < 1\ ; \end{cases} \tag{6.5}$$

*where the constant $C_1$ and $C_2$ are defined in (6.2) and (6.3) respectively. Then we have the following estimate*

$$\int_0^T \|\nabla d\|_{L^4}^4 dt \le C. \tag{6.6}$$

**Proof.** We only give the proof for the case $i = 2$. Indeed, if $0 \le \underline{d}_{02} \le d_{02} \le 1$, then applying the maximal principle (see Lemma 6.1), we obtain

$$0 \le \underline{d}_{02} \le d_2 \le 1 \quad \text{for any } t > 0 \text{ and any } x \in \Omega. \tag{6.7}$$

By virtue of $|d| = 1$, then

$$\|d - e_2\|_{L^\infty}^2 \le 1 - \underline{d}_{02}^2, \tag{6.8}$$

where $e_2 = (0, 1)$. Substituting (6.4)–(6.5) into (6.8), we get

$$C_1 C_2 \|d - e_2\|_{L^\infty}^2 < 1. \tag{6.9}$$

Combining (6.2) with (6.3), we get

$$\begin{aligned} \|\nabla^2 d\|_{L^2}^2 &\le C_1(\|\Delta d\|_{L^2}^2 + \|d\|_{H^1}^2) \\ &\le C_1(\|\nabla d\|_{L^4}^4 + \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 + \|d\|_{H^1}^2) \\ &\le C_1 C_2 \|d - e_2\|_{L^\infty}^2 \|\nabla^2 d\|_{L^2}^2 + C_1\left(C_2\|d - e_2\|_{L^\infty}^4 + \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 + \|d\|_{H^1}^2\right), \end{aligned} \tag{6.10}$$

which, together with (6.5) and (6.9), gives immediately

$$\|\nabla^2 d\|_{L^2}^2 \leq C\left(\|d-e_2\|_{L^\infty}^4 + \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 + \|d\|_{H^1}^2\right). \tag{6.11}$$

In view of the basic energy inequality (4.2), we have

$$\int_0^T \|\nabla^2 d\|_{L^2}^2 dt \leq C,$$

which, together with (6.3), implies

$$\int_0^T \|\nabla d\|_{L^4}^4 dt \leq C. \tag{6.12}$$

As for the case $-1 \leq d_{0i} \leq -\underline{d}_{0i} \leq 0$, it is simple fact if we replaced the function $d - e_2$ by $d + e_2$ in (6.8)–(6.11) respectively. Hence, we complete the proof. □

Choosing $r = s = 4$ in (1.11), if the maximal existence of time $T^* < \infty$, we have

$$\lim_{T \to T^*}\left(\|\nabla\rho\|_{L^\infty(0,T;L^q)} + \|\nabla d\|_{L^4(0,T;L_w^4)}\right) = \infty,$$

which, together with (6.12), gives immediately

$$\lim_{T \to T^*}\|\nabla\rho\|_{L^\infty(0,T;L^q)} = \infty.$$

Therefore, we complete the proof of Corollary 1.6.

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