



# Random exponential attractor for stochastic reaction–diffusion equation with multiplicative noise in $\mathbb{R}^3$ <sup>☆</sup>

Shengfan Zhou

*Department of Mathematics, Zhejiang Normal University, Jinhua 321004, PR China*

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## Abstract

In this paper, we first improve the existing conditions for the existence of a random exponential attractor for a continuous cocycle on a separable Banach space. Then we consider the existence of a random exponential attractor for stochastic non-autonomous reaction–diffusion equation with multiplicative noise defined in  $\mathbb{R}^3$ , which implies the existence of a random attractor with finite fractal dimension. The essential difficulty here is the continuity of the spectrum of the linear part of the equation, which can be overcome by the “tail” estimation of solutions of equation and carefully decomposing the solution into a sum of three parts, of whose, one part is finite-dimensional and other two parts are “quickly decay” in mean sense.

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## 1. Introduction

Consider the following initial problem for stochastic non-autonomous reaction–diffusion equation with multiplicative white noise in the entire space  $\mathbb{R}^3$ :

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E-mail address: [zhoushengfan@yahoo.com](mailto:zhoushengfan@yahoo.com).

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$$\begin{cases} du + (\lambda u - \Delta u)dt = (f(x, u) + g(x, t))dt + \epsilon u \circ dW, & t > \tau, \tau \in \mathbb{R}, \\ u(x, \tau) = u_\tau(x), & x \in \mathbb{R}^3, \tau \in \mathbb{R}, \end{cases} \quad (1.1)$$

where  $u = u(x, t)$  is a real function of  $(x, t) \in \mathbb{R}^3 \times [\tau, +\infty)$ ;  $\lambda, \epsilon$  are positive constants,  $W(t)$  is a two-sided real-valued Wiener process on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ , the Borel  $\sigma$ -algebra  $\mathcal{F}$  on  $\Omega$  is generated by the compact open topology, and  $\mathbb{P}$  is the corresponding Wiener measure on  $\mathcal{F}$ . We identify  $\omega(t)$  with  $W(t)$ , i.e.,  $W(t) = W(t, \omega) = \omega(t)$  for  $\omega \in \Omega$ ,  $t \in \mathbb{R}$ ; “ $\circ$ ” in (1.1) denotes the Stratonovich sense in the stochastic term.

The functions  $g, f$  are assumed to satisfy the following conditions:

**(A1)**  $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ ,  $g(x, \cdot) \in C_b(\mathbb{R}, L^2(\mathbb{R}^3))$  with  $\|g(\cdot, t)\|^2 = \sup_{t \in \mathbb{R}} \|g(\cdot, t)\|^2 < \infty$  and for any  $\varepsilon > 0$ , there exists  $R_\varepsilon \geq 0$  such that  $\sup_{r \in \mathbb{R}} \int_{|x| \geq R_\varepsilon} g^2(x, r) dx \leq \varepsilon$ , where  $\|\cdot\|$  denotes the norm in  $L^2(\mathbb{R}^3)$ .

**(A2)** There exist real positive constants  $c_1, c_2, c_3 > 0$  and integral functions  $\beta_1 \in L^1(\mathbb{R}^3, \mathbb{R}_+)$ ,  $\beta_2, \beta_3 \in L^2(\mathbb{R}^3, \mathbb{R}_+)$ ,  $\beta_4 \in L^3(\mathbb{R}^3, \mathbb{R}_+)$  such that

$$\begin{cases} uf(x, u) \leq \beta_1(x), \\ |f(x, u)| \leq c_1|u|^3 + \beta_2(x), \\ \left| \frac{\partial f}{\partial x}(x, u) \right| \leq \beta_3(x), \\ \frac{\partial f}{\partial u}(x, u) \leq c_2, \\ \left| \frac{\partial f}{\partial u}(x, u) \right| \leq c_3u^2 + \beta_4(x), \end{cases} \quad \forall x \in \mathbb{R}^3, u \in \mathbb{R}. \quad (1.2)$$

It is known that the asymptotic behavior of a random dynamical system can be determined by a random attractor. The random attractors for stochastic reaction–diffusion equations with multiplicative and additive noise defined in bounded and unbounded domains were studied extensively, see, e.g., [6–19, 26–32, 34–42, 44, 46] and the references therein. The essential difference in unbounded domain from the case of bounded domain is that Sobolev embedding is not compact and the spectrum of the linear part of the equation is continuous, which was overcome by the “tail” estimation of solutions or the energy equation approach [3–6, 34]. In the case of bounded domain, there were some publications to estimate the upper bound of Hausdorff and fractal dimensions of the random attractor [17, 18, 27, 44, 45].

We notice that a random attractor maybe is infinite dimensional and sometimes attracts orbits at a slow rate so that it takes a long time to reach it and makes it unobservable in practical experiments and numerical simulations. To overcome this drawback, Shirikyan and Zelik in [31] introduced the concept of random exponential attractor and established some sufficient conditions for constructing a random exponential attractor for an autonomous random dynamical system, and gave an application to a parabolic partial differential equation with a random perturbation. By definition, a random exponential attractor is a positively invariant, compact random set with finite fractal dimension and attracts exponentially any trajectory in the pullback sense. Thus, the existence of a random exponential attractor implies the existence of a random attractor with finite fractal dimension and hence the asymptotical behavior of a random dynamical system can be described by finite independent parameters.

Until now, as we know, there is no result concerning the finiteness of dimension for the random attractor and the existence of random exponential attractor with finite fractal dimension for

stochastic reaction–diffusion equation (1.1) defined in  $\mathbb{R}^n$ . Even in the deterministic case of (1.1) without noise term in the entire space  $\mathbb{R}^n$ , the known results concerning the finite and infinite dimensional uniform or pullback exponential attractors are mainly aimed at those equations driven by a quasi-periodic forcing term on the time or modifying the classical definition of exponential attractor by replacing the usual condition of finite fractal dimensionality by a relation satisfied by a Kolmogorov  $\varepsilon$ -entropy, see [4,11,21–23].

It is observable that the conditions given in [31] is not easy to be verified for some stochastic partial differential equations driven by white noises. Motivated by the ideas of [20,31,44,46], we recently established a new sufficient condition for the existence and construction of a random exponential attractor for a continuous cocycle (non-autonomous random dynamical system) on a separable Banach space and gave an application to first order stochastic lattice system driven by linear multiplicative white noise, see [43].

In this article, we mainly consider the existence of a random exponential attractor for stochastic non-autonomous reaction–diffusion equation (1.1) in  $\mathbb{R}^3$  under conditions (A1)–(A2). We notice that the criterion for constructing a random exponential attractor for a continuous cocycle in [43] can not be directly applied to (1.1) in  $\mathbb{R}^3$ . So we first present a slightly improved sufficient condition for the existence of a random exponential attractor for a continuous cocycle on a separable Banach space. Then we apply our new criterion to prove the existence of a random exponential attractor for (1.1) by carefully estimating the “tail” of solutions and decomposing the difference between two solutions of equation into a sum of three parts, of whose, one part is finite-dimensional and other two parts are “quickly decay” for suitable large  $x \in \mathbb{R}^3$  and large time  $t$ . This implies the existence of a finite dimensional random attractor for (1.1) in  $\mathbb{R}^3$ .

This paper is organized as follows. In Section 2, we give some sufficient conditions to obtain a random exponential attractor for a continuous cocycle on a separable Banach space. In Section 3, we apply our conditions to prove the existence of a random exponential attractor for (1.1). Finally, we remark that we can prove the existence of a random exponential attractor for stochastic non-autonomous reaction–diffusion equation with additive white noise in  $\mathbb{R}^3$  by the same technique.

## 2. Random exponential attractor for continuous cocycle

In this section, we present some sufficient conditions for the existence of a random exponential attractor for a continuous cocycle on a separable Banach space, which is a slightly improvement to one in [43].

Let  $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t \omega\}_{t \in \mathbb{R}})$  be an ergodic metric dynamical system on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $X$  be a separable Banach space with norm  $\|\cdot\|_X$  and Borel  $\sigma$ -algebra  $\mathcal{B}(X)$ . A mapping  $\Psi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \rightarrow X$  is called a *continuous cocycle* (see [34]) on  $X$  over  $\mathbb{R}$  and  $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t \omega\}_{t \in \mathbb{R}})$  if for every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$  and  $t, s \in \mathbb{R}^+$ , (i)  $\Psi(\cdot, \tau, \cdot, \cdot): \mathbb{R}^+ \times \Omega \times X \rightarrow X$  is  $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ -measurable; (ii)  $\Psi(0, \tau, \omega, \cdot)$  is the identity on  $X$ ; (iii)  $\Psi(t + s, \tau, \omega, \cdot) = \Psi(t, \tau + s, \theta_s \omega, \Psi(s, \tau, \omega, \cdot))$ ; (iv)  $\Psi(t, \tau, \omega) = \Psi(t, \tau, \omega, \cdot): X \rightarrow X$  is continuous.

A random variable  $\xi_\omega$  is said to be *tempered* with respect to  $\{\theta_t \omega\}_{t \in \mathbb{R}}$  if for every  $\gamma > 0$  and almost every (a.e.)  $\omega \in \Omega$ ,  $\limsup_{t \rightarrow \infty} e^{-\gamma|t|} |\xi_{\theta_t \omega}| = 0$  [2]. A family  $B = \{B(\tau, \omega) \subset X : \tau \in \mathbb{R}, \omega \in \Omega\}$  of nonempty bounded subsets of  $X$  is said to be tempered with respect to  $\{\theta_t \omega\}_{t \in \mathbb{R}}$  if for every  $\gamma > 0$ ,  $\tau \in \mathbb{R}$  and a.e.  $\omega \in \Omega$ ,  $\lim_{t \rightarrow \infty} e^{-\gamma|t|} \|B(\tau + t, \theta_t \omega)\|_X = 0$  for  $B(\tau, \omega) \in B$ , where  $\|B(\tau, \omega)\|_X = \sup_{x \in B(\tau, \omega)} \|x\|_X$  [34]. The Hausdorff semidistance between two subsets is defined by  $d_h(F_1, F_2) = \sup_{u \in F_1} \inf_{v \in F_2} \|u - v\|_X$  for  $F_1, F_2 \subset X$ . Denote  $\mathcal{D}(X)$  the collection of all tempered families of nonempty subsets of  $X$  with respect to  $\{\theta_t \omega\}_{t \in \mathbb{R}}$ .

**Definition 2.1.** A family  $\{\mathcal{E}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$  of subsets of  $X$  is called a *random exponential attractor* in  $\mathcal{D}(X)$  for a continuous cocycle  $\{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$  over  $\mathbb{R}$  and  $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t \omega\}_{t \in \mathbb{R}})$  if there is a set of full measure  $\tilde{\Omega} \in \mathcal{F}$  such that for any  $\tau \in \mathbb{R}$  and  $\omega \in \tilde{\Omega}$ , it holds that

(i) Compactness:  $\mathcal{E}(\tau, \omega)$  is measurable in  $\omega$  and compact in  $X$ .

(ii) Finite-dimensionality: there exists a random variable  $\zeta_\omega$  ( $< \infty$ ) such that the fractal dimension of  $\mathcal{E}(\tau, \omega)$  is bounded by  $\zeta_\omega$ :  $\sup_{\tau \in \mathbb{R}} \dim_f \mathcal{E}(\tau, \omega) \leq \zeta_\omega < \infty$ , where  $\dim_f \mathcal{E}(\tau, \omega) = \limsup_{\varepsilon \rightarrow 0^+} \frac{\ln N_\varepsilon(\mathcal{E}(\tau, \omega))}{-\ln \varepsilon}$ ,  $N_\varepsilon(\mathcal{E}(\tau, \omega))$  is the minimal number of balls with radius  $\varepsilon$  covering  $\mathcal{E}(\tau, \omega)$  in  $X$ .

(iii) Positive invariance:  $\Psi(t, \tau - t, \theta_{-t}\omega)\mathcal{E}(\tau - t, \theta_{-t}\omega) \subseteq \mathcal{E}(\tau, \omega)$  for all  $t \geq 0$ .

(iv) Exponential attraction: there exists a constant  $\tilde{a} > 0$  such that for any  $B \in \mathcal{D}(X)$ , there exist random variables  $t_B(\tau, \omega) \geq 0$ ,  $Q(\tau, \omega, \|B\|_X) > 0$  satisfying

$$d_h(\Psi(t, \tau - t, \theta_{-t}\omega)B(\tau - t, \theta_{-t}\omega), \mathcal{E}(\tau, \omega)) \leq Q(\tau, \omega, \|B\|_X)e^{-\tilde{a}t}, \quad t \geq t_B(\tau, \omega).$$

In the following of this article, for simplicity, we identify “*a.e.*  $\omega \in \Omega$ ” and “ $\omega \in \Omega$ ”. In fact, the property holds for *a.e.*  $\omega \in \Omega$  throughout the article.

We make the following assumptions on the cocycle  $\{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ :

- (H1) there exists a family of tempered bounded random subsets  $\{\check{\chi}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$  of  $X$  such that for any  $\tau \in \mathbb{R}$  and  $\omega \in \Omega$ ,
- (h11) the diameter  $\|\check{\chi}(\tau, \omega)\|_X$  of  $\check{\chi}(\tau, \omega)$  is uniformly bounded (with respect to  $\tau \in \mathbb{R}$ ) by a tempered random variable  $R_\omega$  (independent of  $\tau$ ), i.e.,  $\sup_{\tau \in \mathbb{R}} \sup_{u, v \in \check{\chi}(\tau, \omega)} \|u - v\|_X \leq R_\omega < \infty$ , where  $R_{\theta_t \omega}$  is continuous in  $t$  for all  $t \in \mathbb{R}$ ;
- (h12)  $\check{\chi}(\tau, \omega)$  is positively invariant with respect to  $\{\theta_t\}_{t \in \mathbb{R}}$  in the sense that  $\Psi(t, \tau - t, \theta_{-t}\omega)\check{\chi}(\tau - t, \theta_{-t}\omega) \subseteq \check{\chi}(\tau, \omega)$  for all  $t \geq 0$ ;
- (h13)  $\check{\chi}(\tau, \omega)$  is pullback absorbing in the sense that for any family of sets  $B \in \mathcal{D}(X)$ , there exists  $T_B = T_B(\tau, \omega) \geq 0$  such that  $\Psi(t, \tau - t, \theta_{-t}\omega)B(\tau - t, \theta_{-t}\omega) \subseteq \check{\chi}(\tau, \omega)$  for  $t \geq T_B$ ;
- (H2) there exist positive numbers  $\hat{t}_0, \delta_1, \delta_2$ , random variables  $\hat{C}_0(\omega), \hat{C}_1(\omega), \hat{C}_2(\omega) \geq 0$  and  $N$ -dimensional projector  $P_N: X \rightarrow P_N X$  ( $\dim(P_N X) = N \in \mathbb{N}$ ) such that for every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$  and any  $u, v \in \check{\chi}(\tau, \omega)$ ,

$$\|\Psi(t, \tau, \omega)u - \Psi(t, \tau, \omega)v\|_X \leq e^{\hat{C}_1(\theta_{\hat{t}_0}\omega) + \int_0^{\hat{t}_0} \hat{C}_0(\theta_s\omega)ds} \|u - v\|_X, \quad \forall t \in [0, \hat{t}_0] \quad (2.1)$$

and

$$\begin{aligned} & \|(I - P_N)\Psi(\hat{t}_0, \tau, \omega)u - (I - P_N)\Psi(\hat{t}_0, \tau, \omega)v\|_X \\ & \leq (\delta_1 e^{\int_0^{\hat{t}_0} \hat{C}_2(\theta_s\omega)ds} + \frac{\delta_2}{2} e^{\hat{C}_1(\theta_{\hat{t}_0}\omega) + \int_0^{\hat{t}_0} \hat{C}_0(\theta_s\omega)ds}) \|u - v\|_X, \end{aligned} \quad (2.2)$$

where  $\hat{t}_0, \delta_1, \delta_2, N$  are independent of  $(\tau, \omega)$  but  $\delta_1, \delta_2, N$  maybe depend on  $\hat{t}_0$ ;

(H3)  $\hat{C}_0(\omega), \hat{C}_1(\omega), \hat{C}_2(\omega), \hat{t}_0, \delta_1, \delta_2$  satisfy:

$$\begin{cases} -\infty \leq \mathbf{E}[\hat{C}_2(\omega)] < 0, \\ \hat{t}_0 \geq 1 + \frac{2 \ln \frac{3}{16\delta_1}}{\mathbf{E}[\hat{C}_2(\omega)]} > 0, \quad \delta_1 \geq \frac{3}{16}, \\ \mathbf{E}[\hat{C}_i^2(\omega)] < \infty, \quad i = 0, 1, 2, \\ 0 < \delta_2 \leq \min \left\{ \frac{1}{8}, \delta_1 e^{-\frac{2\hat{t}_0^2}{\ln \frac{4}{3}} \left( 2\mathbf{E}[\hat{C}_0^2(\omega)] + 2\mathbf{E}[\hat{C}_1^2(\omega)] + \mathbf{E}[\hat{C}_2^2(\omega)] \right)} \right\}, \end{cases} \quad (2.3)$$

where “ $\mathbf{E}$ ” denotes the expectation;

$$(H4) \quad \forall \tau \in \mathbb{R}, \omega \in \Omega, \begin{cases} \lim_{t \searrow 0} \sup_{u \in \check{\chi}(\tau, \omega)} \|\Psi(t, \tau, \omega)u - u\|_X = 0, \\ \lim_{t \searrow 0} \sup_{u \in \check{\chi}(\tau-t, \theta_{-t}\omega)} \|\Psi(0, \tau-t, \theta_{-t}\omega)u - u\|_X = 0. \end{cases}$$

**Theorem 2.1.** Assume that conditions (H1)–(H4) are satisfied. Then there exists a random exponential attractor  $\{\mathcal{E}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$  for the continuous cocycle  $\{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$  with the following properties: for every  $\tau \in \mathbb{R}$  and  $\omega \in \Omega$ ,

(i)  $\mathcal{E}(\tau, \omega) (\subseteq \check{\chi}(\tau, \omega))$  is measurable in  $\omega$  and compact in  $X$ ;

(ii)  $\Psi(t, \tau-t, \theta_{-t}\omega)\mathcal{E}(\tau-t, \theta_{-t}\omega) \subseteq \mathcal{E}(\tau, \omega)$  for all  $t \geq 0$ ;

(iii)  $\dim_f \mathcal{E}(\tau, \omega) \leq \frac{2N \ln \left( \frac{2\sqrt{N}}{\delta_2} + 1 \right)}{\ln \frac{4}{3}} < \infty$ ;

(iv) for any family of sets  $B \in \mathcal{D}(X)$ , there exist a random variable  $\tilde{T}_{\omega, B, R_\omega} \geq 0$  and a tempered random variable  $\check{b}_{\omega, B} > 0$  such that

$$d_h(\Psi(t, \tau-t, \theta_{-t}\omega)B(\tau-t, \theta_{-t}\omega), \mathcal{E}(\tau, \omega)) \leq \check{b}_{\omega, B} e^{-\frac{\ln \frac{4}{3}}{4\hat{t}_0} t}, \quad t \geq \tilde{T}_{\omega, B, R_\omega}; \quad (2.4)$$

(v) for any  $\tau \in \mathbb{R}, \omega \in \Omega, \lim_{t \rightarrow 0} d_h(\mathcal{E}(\tau+t, \theta_t\omega), \mathcal{E}(\tau, \omega)) = 0$ .

**Proof.** Notice that  $\{\check{\chi}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$  is a family of tempered closed random subsets. By the cocycle property and continuity of  $\Psi(t, \tau, \omega)$ , it follows that (H1)–(H4) still hold replacing  $\check{\chi}(\tau, \omega)$  by  $\check{\chi}(\tau, \omega)$  for  $\tau \in \mathbb{R}$  and  $\omega \in \Omega$ . Then combining Theorems 2.2 and 2.4 of [43], the proof is completed.  $\square$

**Remark 2.1.** If there exists a tempered random variable  $q_\omega$  such that  $q_{\theta_t\omega}$  is continuous in  $t \in \mathbb{R}$  and

$$\hat{B}_0 = \{\hat{B}_0(\tau, \omega) = \hat{B}_0(\omega) \subset X \mid \sup_{u, v \in \hat{B}_0(\omega)} \|u - v\|_X \leq q_\omega < \infty\}_{\tau \in \mathbb{R}, \omega \in \Omega}$$

is a family of uniformly (with respect to  $\tau \in \mathbb{R}$ ) tempered closed measurable  $\mathcal{D}(X)$ -pullback absorbing set for  $\Psi$ , that is, for any  $\tau \in \mathbb{R}, \omega \in \Omega$  and  $B(\tau, \omega) \in B \in \mathcal{D}(X)$ , there exists a  $t_B(\tau, \omega) \geq 0$  such that  $\Psi(t, \tau-t, \omega)B(\tau-t, \theta_{-t}\omega) \subseteq \hat{B}_0(\omega)$  for all  $t \geq t_B(\tau, \omega)$ . Particularly,

there exists a  $t_{\hat{B}_0}(\omega) \geq 0$  (independent of  $\tau$ ) such that  $\Psi(t, \tau - t, \omega) \hat{B}_0(\theta_{-t}\omega) \subseteq \hat{B}_0(\omega)$  for all  $t \geq t_{\hat{B}_0}(\omega)$ . Define a family of bounded random subsets:

$$\hat{\chi}(\tau, \omega) = \cup_{t \geq t_{\hat{B}_0}(\omega)} \Psi(t, \tau - t, \theta_{-t}\omega) \hat{B}_0(\theta_{-t}\omega) \subseteq \hat{B}_0(\omega), \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.$$

Then it is easy to check that  $\{\hat{\chi}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$  satisfies condition (H1).

### 3. Stochastic reaction–diffusion equation with multiplicative noise in $\mathbb{R}^3$

In this section, we consider the stochastic reaction–diffusion equation (1.1) defined in  $\mathbb{R}^3$ . From now on, we assume that conditions (A1)–(A2) for the functions  $g, f$  in (1.1) hold and the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is defined as in section 1.

#### 3.1. Mathematical setting

For each  $t \in \mathbb{R}$ , define a mapping  $\theta_t$  on  $\Omega$ :  $\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t)$  for  $\omega \in \Omega$ , then  $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$  is an ergodic metric dynamical system [2].

Write  $z(\theta_t \omega) = -\int_{-\infty}^0 e^s (\theta_t \omega)(s) ds$  for  $t \in \mathbb{R}$ , an Ornstein–Uhlenbeck stationary process which solves the equation  $dz + zdt = dW(t)$ . It is known from [6,8] that the random variable  $|z(\omega)|$  is tempered and for every  $\omega \in \Omega$  (in fact, *a.e.*  $\omega \in \Omega$ ),  $t \mapsto z(\theta_t \omega)$  is continuous in  $t$  and  $\lim_{t \rightarrow \pm\infty} \frac{|z(\theta_t \omega)|}{|t|} = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t z(\theta_s \omega) ds = 0$ .

Let  $v(t) = e^{-\epsilon z(\theta_t \omega)} u(t)$ , where  $u$  is a solution of problem (1.1). Then the problem (1.1) can be changed into the following equivalent random equation without noise term:

$$\begin{cases} \frac{dv(t)}{dt} = \Delta v(t) - \lambda v(t) + \epsilon z(\theta_t \omega) v(t) + e^{-\epsilon z(\theta_t \omega)} f(x, e^{\epsilon z(\theta_t \omega)} v(t)) + e^{-\epsilon z(\theta_t \omega)} g(x, t), & t > \tau, \\ v(x, \tau) = v_\tau(x) = e^{-\epsilon z(\theta_\tau \omega)} u_\tau(x), & x \in \mathbb{R}^3, \tau \in \mathbb{R}. \end{cases} \quad (3.1)$$

It follows from [6,33,34,38] that for every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ , and for  $v_\tau(\omega) \in L^2(\mathbb{R}^3)$ , the (weak) solution  $v(t, \tau, \omega, v_\tau)$  of (3.1) exists globally for  $t \in [\tau, +\infty)$  and

$$v(\cdot, \tau, \omega, v_\tau) \in C([\tau, +\infty); L^2(\mathbb{R}^3)) \cap L^2_{loc}([\tau, +\infty); H^1_0(\mathbb{R}^3)) \cap L^4_{loc}([\tau, +\infty); L^4_{loc}(\mathbb{R}^3));$$

and  $v(t, \tau, \omega, v_\tau)$  is continuous in  $v_\tau$  and measurable in  $\omega$ . Thus, we can define a continuous cocycle  $\Phi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$  by

$$(t, \tau, \omega, v_\tau) \rightarrow \Phi(t, \tau, \omega, v_\tau) = \Phi(t, \tau, \omega) v_\tau = v(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau(\theta_{-\tau} \omega))$$

over  $\mathbb{R}$  and  $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ , where  $\Phi(0, \tau, \omega) v_\tau(\omega) = v_\tau(\theta_{-\tau} \omega)$ .

Denote the inner products and norms in  $L^2(\mathbb{R}^3)$ ,  $H^1(\mathbb{R}^3)$  as  $(\cdot, \cdot)$ ,  $\|\cdot\|$  and  $(\cdot, \cdot)_1$ ,  $\|\cdot\|_1$ , respectively. By the embedding  $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$  [1], there exists a positive constant  $C_0$  such that

$$\|v\|_{L^6(\mathbb{R})} \leq C_0 \|v\|_1 = C_0 (\|\nabla v\|^2 + \|v\|^2)^{\frac{1}{2}}, \quad \forall v \in H^1(\mathbb{R}^3). \quad (3.2)$$

### 3.2. Bound of solutions

Let  $\mathcal{D} = \mathcal{D}(L^2(\mathbb{R}^3))$  be the collection of all tempered families of nonempty subsets of  $L^2(\mathbb{R}^3)$ . We have the following estimation on the bounds of solutions of (3.1).

For every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$  and  $t \geq 0$ , let  $v(r) = v(r, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))$  ( $r \geq \tau - t$ ) be a solution of (3.1) with initial value  $v_{\tau-t}(\theta_{-\tau}\omega) \in L^2(\mathbb{R}^3)$ .

**Lemma 3.1.** (i) For any  $r \geq \tau - t$ ,  $v(r)$  satisfies:

$$\begin{aligned} & \|v(r)\|^2 + 2 \int_{\tau-t}^r e^{\int_s^r (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} \|\nabla v(s)\|^2 ds \\ & \leq e^{\int_{\tau-t}^r (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} \|v_{\tau-t}(\theta_{-\tau}\omega)\|^2 + c_4 \int_{\tau-t}^r e^{\int_s^r (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl - 2\epsilon z(\theta_{s-\tau}\omega)} ds, \end{aligned} \quad (3.3)$$

where  $\|\beta_1\|_{L^1} = \int_{\mathbb{R}^3} \beta_1(x) dx$ ,  $c_4 = \frac{2}{\lambda} \|g\|^2 + 2\|\beta_1\|_{L^1}$ .

(ii) Taking a tempered random variable  $M_0(\omega) \geq 0$  (independent of  $\tau$ ) as

$$M_0^2(\omega) = 1 + c_4 K_0(\omega) < \infty, \text{ where } K_0(\omega) = \int_{-\infty}^0 e^{\lambda s + 2\epsilon \int_s^0 z(\theta_l \omega) dl - 2\epsilon z(\theta_s \omega)} ds, \quad (3.4)$$

then for any  $B \in \mathcal{D}$  and  $B(\tau, \omega) \in B$ , there exists a  $T_B(\tau, \omega)$ :

$$T_B(\tau, \omega) = \min \left\{ t : e^{\int_{-t}^0 (2\epsilon z(\theta_l \omega) - \lambda) dl} \sup_{v \in B(\tau-t, \theta_{-\tau}\omega)} \|v\|^2 \leq 1 \right\} \geq 0$$

such that for any  $t \geq T_B(\tau, \omega)$ ,

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))\|^2 + 2 \int_{\tau-t}^{\tau} e^{\int_s^{\tau} (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} \|\nabla v(s)\|^2 ds \leq M_0^2(\omega). \quad (3.5)$$

**Proof.** (i) Taking the inner product ((3.1),  $v$ ), we find that for  $r \geq \tau - t$ ,

$$\begin{aligned} & \frac{d}{dt} \|v\|^2 + 2\|\nabla v\|^2 + 2\lambda \|v\|^2 \\ & = 2e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) v dx + 2\epsilon z(\theta_{r-\tau}\omega) \|v\|^2 + 2e^{-\epsilon z(\theta_{r-\tau}\omega)} (g(x, r), v), \end{aligned}$$

where

$$e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) v dx \leq e^{-2\epsilon z(\theta_{r-\tau}\omega)} \|\beta_1\|_{L^1},$$

$$2e^{-\epsilon z(\theta_{r-\tau}\omega)} (g(x, r), v) \leq \frac{2}{\lambda} e^{-2\epsilon z(\theta_{r-\tau}\omega)} \|g\|^2 + \frac{\lambda}{2} \|v\|^2.$$

Thus, for  $r \geq \tau - t$ ,

$$\frac{d}{dt} \|v\|^2 + 2\|\nabla v\|^2 \leq (2\epsilon z(\theta_{r-\tau}\omega) - \lambda) \|v\|^2 + c_4 e^{-2\epsilon z(\theta_{r-\tau}\omega)}. \quad (3.6)$$

By applying Gronwall inequality to (3.6) on  $[\tau - t, r]$  ( $r \geq \tau - t$ ), we obtain (3.3).

(ii) Let  $r = \tau$  in (3.3), we have

$$\begin{aligned} \|v(\tau)\|^2 + \int_{\tau-t}^{\tau} e^{\int_s^{\tau} (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} 2\|\nabla v(s)\|^2 ds \\ \leq e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} \|v_{\tau-t}(\theta_{-\tau}\omega)\|^2 + c_4 \int_{-\infty}^0 e^{\int_s^0 (2\epsilon z(\theta_l\omega) - \lambda) dl - 2\epsilon z(\theta_s\omega)} ds. \end{aligned}$$

For  $v_{\tau-t}(\theta_{-\tau}\omega) \in B(\tau - t, \theta_{-\tau}\omega) \in B$ , we have  $\lim_{t \rightarrow \infty} e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} \|v_{\tau-t}(\theta_{-\tau}\omega)\|^2 = 0$ . Thus, (3.5) holds.  $\square$

**Lemma 3.2.** *Let*

$$B_0(\omega) = \{u \in L^2(\mathbb{R}^3) : \|u\| \leq M_0(\omega)\} \subset L^2(\mathbb{R}^3), \quad \forall \omega \in \Omega.$$

*Then for any  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $t \geq 0$  and  $v_{\tau-t}(\theta_{-\tau}\omega) \in B_0(\theta_{-t}\omega)$ , there exist a number  $T_{1,B_0}(\omega)$ :*

$$T_{1,B_0}(\omega) = \min \left\{ 1 + t : e^{\lambda - \int_{-1}^0 2\epsilon z(\theta_l\omega) dl} e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_0^2(\theta_{-t}\omega) \leq 1 \right\} \geq 1 \quad (3.7)$$

*and a tempered random variable  $\tilde{M}_0(\omega)$ :*

$$\tilde{M}_0^2(\omega) = [1 + 2c_2 + 2\epsilon \max_{-1 \leq s \leq 0} |z(\theta_s\omega)|] \tilde{K}_0^2(\omega) + c_5 \int_{-1}^0 e^{-2\epsilon z(\theta_s\omega)} ds,$$

*where*

$$\tilde{K}_0^2(\omega) = 1 + c_4 e^{\lambda - \int_{-1}^0 2\epsilon z(\theta_l\omega) dl} K_0(\omega), \quad c_5 = \|\beta_3\|^2 + \|g\|^2$$

*such that  $v(r)$  satisfies:*

$$\|\nabla v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))\| \leq \tilde{M}_0(\omega), \quad t \geq T_{1,B_0}(\omega) \quad (3.8)$$



and

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \|\nabla v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \leq M_0^2(\omega) + \tilde{M}_0^2(\omega), \quad (3.9)$$

for  $t \geq T_{1,B_0}(\omega) + T_{B_0}(\omega)$ , where

$$T_{B_0}(\omega) = \min \left\{ t : e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_0^2(\theta_{-t}\omega) \leq 1 \right\} \geq 0.$$

**Proof.** By  $v_{\tau-t}(\theta_{-\tau}\omega) \in B_0(\theta_{-t}\omega)$  and (3.3), for  $t \geq 1$ ,

$$\|v(\tau - 1)\|^2 \leq e^{\int_{-t}^{-1} (2\epsilon z(\theta_l\omega) - \lambda) dl} M_0^2(\theta_{-t}\omega) + c_4 \int_{-t}^{-1} e^{\int_s^{-1} (2\epsilon z(\theta_l\omega) - \lambda) dl - 2\epsilon z(\theta_s\omega)} ds. \quad (3.10)$$

Thus, applying Gronwall inequality to (3.6) over  $[\tau - 1, \tau]$  and by (3.10), we have

$$\begin{aligned} \|v(\tau)\|^2 + 2 \int_{\tau-1}^{\tau} e^{\int_s^{\tau} 2\epsilon z(\theta_l\omega) dl - \lambda(\tau-s)} \|\nabla v(s)\|^2 ds \\ \leq e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_0^2(\theta_{-t}\omega) + c_4 \int_{-\infty}^0 e^{\int_s^0 (2\epsilon z(\theta_l\omega) - \lambda) dl - 2\epsilon z(\theta_s\omega)} ds. \end{aligned} \quad (3.11)$$

Note that

$$\int_{\tau-1}^{\tau} e^{\int_s^{\tau} 2\epsilon z(\theta_l\omega) dl - \lambda(\tau-s)} \|\nabla v(s)\|^2 ds \geq e^{\int_{-1}^0 2\epsilon z(\theta_l\omega) dl - \lambda} \int_{\tau-1}^{\tau} \|\nabla v(s)\|^2 ds. \quad (3.12)$$

Thus, by (3.11)–(3.12),

$$\begin{aligned} 2 \int_{\tau-1}^{\tau} \|\nabla v(s)\|^2 ds &\leq 2e^{-\int_{-1}^0 2\epsilon z(\theta_l\omega) dl + \lambda} \int_{\tau-1}^{\tau} e^{\int_s^{\tau} 2\epsilon z(\theta_l\omega) ds - \lambda(\tau-s)} \|\nabla v(s)\|^2 ds \\ &\leq e^{\lambda - \int_{-1}^0 2\epsilon z(\theta_l\omega) dl + \int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_0^2(\theta_{-t}\omega) \\ &\quad + c_4 e^{\lambda - \int_{-1}^0 2\epsilon z(\theta_l\omega) dl} \int_{-\infty}^0 e^{\int_s^0 (2\epsilon z(\theta_l\omega) - \lambda) dl - 2\epsilon z(\theta_s\omega)} ds. \end{aligned} \quad (3.13)$$

Thus, it follows from (3.7) and (3.13) that

$$2 \int_{\tau-1}^{\tau} \|\nabla v(s)\|^2 ds \leq \tilde{K}_0^2(\omega), \quad \forall t \geq T_{1,B_0}(\omega).$$

Taking the inner product of (3.1) with  $-\Delta v$  in  $L^2(\mathbb{R}^3)$ , we find that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla v\|^2 + \|\Delta v\|^2 + \lambda \|\nabla v\|^2 \\ &= -e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v(t)) \Delta v dx + \epsilon z(\theta_{r-\tau}\omega) \|\nabla v\|^2 \\ & \quad + e^{-\epsilon z(\theta_{r-\tau}\omega)} (g(x, r), -\Delta v). \end{aligned}$$

By (1.2), we have

$$\begin{aligned} & -2e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) \Delta v dx \\ &= 2e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \left( \frac{\partial f}{\partial x}(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) \nabla v + e^{\epsilon z(\theta_{r-\tau}\omega)} \frac{\partial f}{\partial u}(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) |\nabla v|^2 \right) dx \\ &\leq 2e^{-\epsilon z(\theta_{r-\tau}\omega)} \|\beta_3\| \cdot \|\nabla v\| + 2c_3 \|\nabla v\|^2 \leq e^{-2\epsilon z(\theta_{r-\tau}\omega)} \|\beta_3\|^2 + (1 + 2c_2) \|\nabla v\|^2, \\ & \quad -e^{-\epsilon z(\theta_t\omega)} (g(x, r), \Delta v) \leq \frac{1}{2} e^{-2\epsilon z(\theta_t\omega)} \|g\|^2 + \frac{1}{2} \|\Delta v\|^2. \end{aligned}$$

Thus, for  $r \geq \tau - t$ ,

$$\begin{aligned} & \frac{d}{dt} \|\nabla v\|^2 + \|\Delta v\|^2 \\ &\leq (2\epsilon z(\theta_{r-\tau}\omega) - \lambda) \|\nabla v\|^2 + (1 + 2c_2) \|\nabla v\|^2 + c_5 e^{-2\epsilon z(\theta_{r-\tau}\omega)}. \end{aligned} \quad (3.14)$$

Take  $t \geq T_{1, B_0}(\omega) \geq 1$  and  $s \in [\tau - 1, \tau]$ . Integrating (3.14) over  $[s, \tau]$ , we get

$$\begin{aligned} & \|\nabla v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))\|^2 - \|\nabla v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))\|^2 \\ &\leq c_5 \int_s^\tau e^{-2\epsilon z(\theta_{l-\tau}\omega)} dl + \int_s^\tau (1 + 2\epsilon |z(\theta_{l-\tau}\omega)| + 2c_2) \|\nabla v(l)\|^2 dl. \end{aligned} \quad (3.15)$$

Integrating (3.15) with respect to  $s$  over  $[\tau - 1, \tau]$ , we find that for  $t \geq T_{1, B_0}(\omega)$ ,

$$\begin{aligned} & \|\nabla v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))\|^2 \\ &\leq \int_{\tau-1}^\tau (1 + 2c_2 + 2\epsilon |z(\theta_{s-\tau}\omega)|) \|\nabla v(s)\|^2 ds + c_5 \int_{-1}^0 e^{-2\epsilon z(\theta_s\omega)} ds \\ &\leq (1 + 2c_2 + 2\epsilon \max_{-1 \leq s \leq 0} |z(\theta_s\omega)|) \tilde{K}_0^2(\omega) + c_5 \int_{-1}^0 e^{-2\epsilon z(\theta_s\omega)} ds \\ &= \tilde{M}_0^2(\omega). \end{aligned}$$

Thus, (3.8)–(3.9) hold.  $\square$

**Lemma 3.3.** For any  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $t \geq 0$  and  $v_{\tau-t}(\theta_{-\tau}\omega) \in B_0(\theta_{-t}\omega)$  with

$$\|\nabla v_{\tau-t}(\theta_{-\tau}\omega)\|^2 + \|v_{\tau-t}(\theta_{-\tau}\omega)\|^2 \leq M_0^2(\theta_{-t}\omega) + \tilde{M}_0^2(\theta_{-t}\omega),$$

then there exists a number  $T_{2,B_0}(\omega)$ :

$$T_{2,B_0}(\omega) = \min \left\{ t : \left( 1 + \frac{1+2c_2}{2} \right) e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} \left( M_0^2(\theta_{-t}\omega) + \tilde{M}_0^2(\theta_{-t}\omega) \right) \leq 1 \right\} \geq 0$$

and a tempered random variables  $M_1(\omega)$ :

$$M_1^2(\omega) = 1 + c_6 K_0(\omega), \quad c_6 = \frac{3}{2} c_4 + c_2 c_4 + c_5 \quad (3.16)$$

such that  $v(r)$  satisfies:

$$\begin{aligned} & \|\nabla v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))\|^2 + \|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))\|^2 \\ & \leq M_1^2(\omega), \quad t \geq T_{2,B_0}(\omega). \end{aligned} \quad (3.17)$$

**Proof.** By (3.6) and (3.14),

$$\begin{aligned} & \frac{d}{dt} \left( \|\nabla v\|^2 + \|v\|^2 \right) \\ & \leq (2\epsilon z(\theta_{\tau-t}\omega) - \lambda) \left( \|\nabla v\|^2 + \|v\|^2 \right) + (1 + 2c_2) \|\nabla v\|^2 \\ & \quad + (c_4 + c_5) e^{-2\epsilon z(\theta_{\tau-t}\omega)}. \end{aligned} \quad (3.18)$$

Applying Gronwall inequality to (3.18) over  $[\tau - t, \tau]$ , we have

$$\begin{aligned} & \|\nabla v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))\|^2 + \|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))\|^2 \\ & \leq e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} \left( \|\nabla v_{\tau-t}(\theta_{-\tau}\omega)\|^2 + \|v_{\tau-t}(\theta_{-\tau}\omega)\|^2 \right) \\ & \quad + \frac{1+2c_2}{2} \left( e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} \|v_{\tau-t}(\theta_{-\tau}\omega)\|^2 + c_4 K_0(\omega) \right) \\ & \quad + (c_4 + c_5) \int_{\tau-t}^{\tau} e^{\int_s^{\tau} (2\epsilon z(\theta_l\omega) - \lambda) dl - 2\epsilon z(\theta_{s-\tau}\omega)} ds \\ & \leq \left( 1 + \frac{1+2c_2}{2} \right) e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} \left( M_0^2(\theta_{-t}\omega) + \tilde{M}_0^2(\theta_{-t}\omega) \right) + c_6 K_0(\omega). \end{aligned}$$

Thus, (3.17) holds.  $\square$

**Lemma 3.4.** Let

$$B_1(\omega) = \{u \in H^1(\mathbb{R}^3) : \|u\|_1 \leq M_1(\omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega} \in H^1(\mathbb{R}^3), \quad \forall \omega \in \Omega,$$

then for any  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$  and  $t \geq 0$ ,

$$\begin{cases} \Phi(t, \tau - t, \theta_{-t}\omega) B_0(\theta_{-t}\omega) \subseteq_{t \geq T_{B_0}(\omega)} B_0(\omega), \\ \Phi(t, \tau - t, \theta_{-t}\omega) B_0(\theta_{-t}\omega) \subseteq_{t \geq T_{B_0}(\omega) + T_{1, B_0}(\omega) + T_{2, B_0}(\omega)} B_1(\omega), \end{cases} \quad \text{in } L^2(\mathbb{R}^3). \quad (3.19)$$

**Proof.** It is a direct consequence of Lemmas 3.1–3.3.  $\square$

### 3.3. Estimation on tail of solutions

For every  $\tau \in \mathbb{R}$  and  $\omega \in \Omega$ , set

$$\begin{aligned} B_2(\tau, \omega) &= \cup_{s \geq T_{B_0}(\omega) + T_{1, B_0}(\omega) + T_{2, B_0}(\omega)} \Phi(s, \tau - s, \theta_{-s}\omega) B_0(\theta_{-s}\omega) \\ &= \cup_{s \geq T_{B_0}(\omega) + T_{1, B_0}(\omega) + T_{2, B_0}(\omega)} \cup_{v_{\tau-s}(\theta_{-\tau}\omega) \in B_0(\theta_{-s}\omega)} v(\tau, \tau - s, \theta_{-\tau}\omega, v_{\tau-s}(\theta_{-\tau}\omega)) \\ &\subseteq B_0(\omega) \cap B_1(\omega) \subset H^1(\mathbb{R}^3) \subset L^2(\mathbb{R}^3). \end{aligned}$$

Taking an increasing smooth function  $\xi \in C^1(\mathbb{R}_+, [0, 1])$  satisfying

$$\begin{cases} \xi(s) = 0, \forall s \in [0, 1]; \\ 0 \leq \xi(s) \leq 1, \forall s \in [1, 2]; \\ \xi(s) = 1, \forall s \in [2, \infty); \\ |\xi'(s)| \leq \tilde{C}, \forall s \in \mathbb{R}_+ \quad \text{for some positive constant } \tilde{C}. \end{cases} \quad (3.20)$$

First, we remark that all the numbers  $c_i$  ( $i \in \mathbb{N}$ ) below are positive constants independent of  $(R, \omega, \tau, t)$ . For the tail of solutions of (3.1), we have the following Lemma.

**Lemma 3.5.** *For every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $R > 1$  and  $t \geq 0$ , let  $v(r) = \varphi(r, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))$  ( $r \geq \tau - t$ ) be a solution of (3.1) with  $v_{\tau-t}(\theta_{-\tau}\omega) \in B_2(\tau - t, \theta_{-\tau}\omega)$ . Then the following statements hold.*

(i)

$$\int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |v(\tau)|^2 dx \leq e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_1^2(\theta_{-t}\omega) + \frac{K_1(\omega)}{R} + \gamma_R K_0(\omega), \quad (3.21)$$

where

$$\begin{aligned} K_1(\omega) &= 2\sqrt{2}\tilde{C} \int_{-\infty}^0 e^{\int_s^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_1^2(\theta_s\omega) ds, \\ \gamma_R &= 2 \int_{|x| \geq R} \beta_1(x) dx + \frac{2}{\lambda} \sup_{s \in \mathbb{R}} \int_{|x| \geq R} g^2(x, s) dx; \end{aligned}$$

(ii) there exist a number  $c_7 > 0$  and a tempered random variable  $K_2(\omega) > 0$  such that

$$\int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |\nabla v(\tau)|^2 dx \leq c_7 e^{\int_{-\tau}^0 (2\epsilon z(\theta_l \omega) - \lambda) dl} M_1^2(\theta_{-\tau} \omega) + \frac{K_2(\omega)}{R} + (\gamma_R + \tilde{\gamma}_R) K_0(\omega), \quad (3.22)$$

where

$$\tilde{\gamma}_R = \frac{1}{2} \sup_{s \in \mathbb{R}} \int_{|x| \geq R} g^2(x, s) dx + \int_{|x| \geq R} \beta_3^2(x) dx;$$

(iii) for any  $v > 0$ , there exists  $T_v(\omega) > 0$  such that

$$\begin{aligned} & \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) (|v(\tau, \tau - t, \theta_{-\tau} \omega, v_{\tau-t}(\theta_{-\tau} \omega))|^2 + |\nabla v(\tau, \tau - t, \theta_{-\tau} \omega, v_{\tau-t}(\theta_{-\tau} \omega))|^2) dx \\ & \leq v + \frac{K_1(\omega) + K_2(\omega)}{R} + (2\gamma_R + \tilde{\gamma}_R) K_0(\omega), \quad \forall t \geq T_v(\omega). \end{aligned} \quad (3.23)$$

**Proof.** By the continuity and cocycle property of  $\Phi(t, \tau, \omega)$  and (3.19), it follows that for any  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $t \geq 0$ ,

$$\begin{cases} v(r) \in B_2(r, \theta_{r-\tau} \omega) \subseteq B_0(\theta_{r-\tau} \omega) \cap B_1(\theta_{r-\tau} \omega) \subset H^1(\mathbb{R}^3), \\ \|v(r)\|^2 + \|\nabla v(r)\|^2 \leq M_1^2(\theta_{r-\tau} \omega), \end{cases} \quad \forall r \geq \tau - t. \quad (3.24)$$

(i) Taking the inner product of (3.1) with  $\xi(\frac{|x|^2}{R^2})v$  in  $L^2(\mathbb{R}^3)$ , we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |v|^2 dx + (2\lambda - 2\epsilon z(\theta_{r-\tau} \omega)) \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |v|^2 dx \\ & = 2 \int_{\mathbb{R}^3} (\Delta v) \xi\left(\frac{|x|^2}{R^2}\right) v dx + 2e^{-\epsilon z(\theta_{r-\tau} \omega)} \int_{\mathbb{R}^3} f(x, e^{\epsilon z(\theta_{r-\tau} \omega)}) \xi\left(\frac{|x|^2}{R^2}\right) v dx \\ & \quad + 2e^{-\epsilon z(\theta_{r-\tau} \omega)} \int_{\mathbb{R}^3} g(x, r) \cdot \xi\left(\frac{|x|^2}{R^2}\right) v dx. \end{aligned} \quad (3.25)$$

For the terms of (3.25), we have

$$\begin{aligned} & \int_{\mathbb{R}^3} (\Delta v) \xi\left(\frac{|x|^2}{R^2}\right) v dx = - \int_{\mathbb{R}^3} |\nabla v|^2 \xi\left(\frac{|x|^2}{R^2}\right) dx - \int_{R \leq |x| \leq \sqrt{2}R} v \xi'\left(\frac{|x|^2}{R^2}\right) \frac{2x}{R^2} (\nabla v) dx \\ & \leq - \int_{\mathbb{R}^3} |\nabla v|^2 \xi\left(\frac{|x|^2}{R^2}\right) dx + \frac{2\sqrt{2}\tilde{C}}{R} \int_{\mathbb{R}^3} |v| \cdot |\nabla v| dx \\ & \leq - \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |\nabla v|^2 dx + \frac{\sqrt{2}\tilde{C}}{R} M_1^2(\theta_{r-\tau} \omega), \end{aligned} \quad (3.26)$$

$$e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) \xi\left(\frac{|x|^2}{R^2}\right) v dx \leq e^{-2\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) \beta_1(x) dx, \quad (3.27)$$

$$e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} g \xi\left(\frac{|x|^2}{R^2}\right) v dx \leq \frac{\lambda}{4} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |v|^2 dx + \frac{e^{-2\epsilon z(\theta_{r-\tau}\omega)}}{\lambda} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) g^2(x, r) dx. \quad (3.28)$$

Then, from (3.25)–(3.28), it follows that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |v|^2 dx + 2 \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |\nabla v|^2 dx \\ & \leq (2\epsilon z(\theta_{r-\tau}\omega) - \lambda) \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |v|^2 dx + \frac{2\sqrt{2}\tilde{C}}{R} M_1^2(\theta_{r-\tau}\omega) \\ & \quad + 2e^{-2\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) \left( \beta_1(x) + \frac{1}{\lambda} g^2(x, r) \right) dx. \end{aligned} \quad (3.29)$$

By applying Gronwall inequality to (3.29) on  $[\tau - t, \tau]$ ,

$$\begin{aligned} & \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))|^2 dx \\ & + 2 \int_{\tau-t}^{\tau} e^{\int_s^{\tau} (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |\nabla v(s)|^2 dx ds \\ & \leq e^{\int_{-\tau}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} \|v_{\tau-t}(\theta_{-\tau}\omega)\|^2 + \frac{2\sqrt{2}\tilde{C}}{R} \int_{\tau-t}^{\tau} e^{\int_s^{\tau} (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} M_1^2(\theta_{s-\tau}\omega) ds \\ & \quad + \int_{\tau-t}^{\tau} e^{\int_s^{\tau} (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl - 2\epsilon z(\theta_{s-\tau}\omega)} 2 \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) \left( \beta_1(x) + \frac{1}{\lambda} g^2(x, s) \right) dx ds \\ & \leq e^{\int_{-\tau}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_1^2(\theta_{-\tau}\omega) + \frac{2\sqrt{2}\tilde{C}}{R} \int_{-\infty}^0 e^{\int_s^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_1^2(\theta_s\omega) ds \\ & \quad + 2 \left( \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) \beta_1(x) dx + \frac{1}{\lambda} \sup_{s \in \mathbb{R}} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) g^2(x, s) dx \right) K_0(\omega) \\ & \leq e^{\int_{-\tau}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_1^2(\theta_{-\tau}\omega) + \frac{K_1(\omega)}{R} + \gamma_R K_0(\omega). \end{aligned} \quad (3.30)$$

(ii) Taking the inner product of (3.1) with  $-\xi(\frac{|x|^2}{R^2})\Delta v$  in  $L^2(\mathbb{R}^3)$ , we obtain

$$\begin{aligned} & - \int_{\mathbb{R}^3} \frac{dv}{dt} \cdot \xi\left(\frac{|x|^2}{R^2}\right) \Delta v dx + \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) |\Delta v|^2 dx \\ & = - \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) \Delta v [\epsilon z(\theta_{r-\tau}\omega)v - \lambda v + e^{-\epsilon z(\theta_{r-\tau}\omega)} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)}v) + g(x, r)] dx. \end{aligned} \quad (3.31)$$

By (3.2) and (1.2),

$$\begin{aligned} \|v(r)\|_{L^6(\mathbb{R}^3)} &= \left( \int_{\mathbb{R}^3} v^6(r) dx \right)^{\frac{1}{6}} \leq C_0 (\|v(r)\|^2 + \|\nabla v(r)\|^2)^{\frac{1}{2}} \leq C_0 M_1(\theta_{r-\tau}\omega), \\ \|e^{-\epsilon z(\theta_{r-\tau}\omega)} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)}v(r))\|^2 &\leq e^{-2\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \left( c_1 |e^{\epsilon z(\theta_{r-\tau}\omega)}v(r)|^3 + \beta_2(x) \right)^2 dx \\ &\leq 2c_1^2 e^{4\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} v^6(r) dx + 2e^{-2\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \beta_2^2(x) dx \\ &\leq 2c_1^2 C_0^6 e^{4\epsilon z(\theta_{r-\tau}\omega)} M_1^6(\theta_{r-\tau}\omega) + 2\|\beta_2\|^2 e^{-2\epsilon z(\theta_{r-\tau}\omega)}, \end{aligned}$$

and by (3.1),

$$\begin{aligned} \left\| \frac{dv}{dt} \right\|^2 &\leq 4 \left( \|\Delta v\|^2 + [\epsilon z(\theta_{r-\tau}\omega) - \lambda]^2 \|v\|^2 + \|e^{-\epsilon z(\theta_{r-\tau}\omega)} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)}v(r))\|^2 \right) \\ &\quad + 4e^{-2\epsilon z(\theta_{r-\tau}\omega)} \|g\|^2 \\ &\leq 4\|\Delta v\|^2 + c_8[1 + z^2(\theta_{r-\tau}\omega)]M_1^2(\theta_{r-\tau}\omega) + 8c_1^2 C_0^6 e^{4\epsilon z(\theta_{r-\tau}\omega)} M_1^6(\theta_{r-\tau}\omega) \\ &\quad + 4e^{-2\epsilon z(\theta_{r-\tau}\omega)} (2\|\beta_2\|^2 + \|g\|^2). \end{aligned}$$

Thus,

$$\begin{aligned} & - \int_{\mathbb{R}^3} \frac{dv(t)}{dt} \cdot \xi\left(\frac{|x|^2}{R^2}\right) \Delta v dx \\ &= - \int_{\mathbb{R}^3} \Delta v \cdot \left( \xi\left(\frac{|x|^2}{R^2}\right) \frac{dv}{dt} \right) dx = \int_{\mathbb{R}^3} \nabla v \cdot \nabla \left( \xi\left(\frac{|x|^2}{R^2}\right) \frac{dv}{dt} \right) dx \\ &= \int_{\mathbb{R}^3} \nabla v \cdot \left( \frac{2x}{R^2} \xi'\left(\frac{|x|^2}{R^2}\right) \frac{dv}{dt} + \xi\left(\frac{|x|^2}{R^2}\right) \nabla \left( \frac{dv}{dt} \right) \right) dx \end{aligned}$$

$$\begin{aligned}
&\geq - \int_{R \leq |x| \leq \sqrt{2}R} \frac{2|x|}{R^2} \left| \xi' \left( \frac{|x|^2}{R^2} \right) \right| \cdot |\nabla v| \cdot \left| \frac{dv}{dt} \right| dx + \frac{1}{2} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) \frac{d}{dt} |\nabla v|^2 dx \\
&\geq - \frac{\sqrt{2}\tilde{C}}{R} \left( \|\nabla v\|^2 + \left\| \frac{dv}{dt} \right\|^2 \right) + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v|^2 dx \\
&\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v|^2 dx - \frac{c_9}{R} \|\Delta v\|^2 - \frac{c_{10}}{R} e^{-2\epsilon z(\theta_{r-\tau}\omega)} \\
&\quad - \frac{c_{10}}{R} \left( M_1^2(\theta_{r-\tau}\omega) + z^2(\theta_{r-\tau}\omega) M_1^2(\theta_{r-\tau}\omega) + e^{4\epsilon z(\theta_{r-\tau}\omega)} M_1^6(\theta_{r-\tau}\omega) \right). \tag{3.32}
\end{aligned}$$

$$\begin{aligned}
&- (-\lambda + \epsilon z(\theta_{r-\tau}\omega)) \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) v \Delta v dx \\
&\leq (\epsilon z(\theta_{r-\tau}\omega) - \lambda) \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v|^2 dx + \frac{\sqrt{2}\tilde{C}}{R} (\lambda + \epsilon |z(\theta_{r-\tau}\omega)|) M_1^2(\theta_{r-\tau}\omega), \tag{3.33}
\end{aligned}$$

$$\begin{aligned}
&- e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) \xi \left( \frac{|x|^2}{R^2} \right) \Delta v dx \\
&= e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \nabla v \cdot \nabla \left( f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) \xi \left( \frac{|x|^2}{R^2} \right) \right) dx \\
&= e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) \nabla v \cdot \frac{\partial f}{\partial x}(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) dx + \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) \frac{\partial f}{\partial u}(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) |\nabla v|^2 dx \\
&\quad + e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \nabla v \cdot f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v) \xi' \left( \frac{|x|^2}{R^2} \right) \frac{2x}{R^2} dx \\
&\leq e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v| \cdot |\beta_3(x)| dx + c_2 \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v|^2 dx \\
&\quad + \frac{2\sqrt{2}\tilde{C}}{R} \int_{\mathbb{R}^3} |\nabla v| \cdot |e^{-\epsilon z(\theta_{r-\tau}\omega)} f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v)| dx \\
&\leq \frac{1}{2} e^{-2\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) \beta_3^2(x) dx + c_{11} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v|^2 dx \\
&\quad + \frac{c_{12}}{R} \left( M_1^2(\theta_{r-\tau}\omega) + e^{4\epsilon z(\theta_{r-\tau}\omega)} M_1^6(\theta_{r-\tau}\omega) + e^{-2\epsilon z(\theta_{r-\tau}\omega)} \right), \tag{3.34}
\end{aligned}$$

$$\begin{aligned}
&- e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) g \Delta v dx \leq \frac{1}{4} e^{-2\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) g^2(x, t) dx + \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\Delta v|^2 dx. \tag{3.35}
\end{aligned}$$



Then, from (3.31)–(3.35), it follows that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v|^2 dx \\ & \leq (2\epsilon z(\theta_{r-\tau}\omega) - \lambda) \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v|^2 dx + c_{13} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v|^2 dx \\ & \quad + \frac{c_{14}}{R} \left( M_1^2(\theta_{r-\tau}\omega) + z^2(\theta_t\omega) M_1^2(\theta_{r-\tau}\omega) + e^{4\epsilon z(\theta_t\omega)} M_1^6(\theta_{r-\tau}\omega) \right) \\ & \quad + \frac{c_{15}}{R} \|\Delta v\|^2 + \left( \frac{c_{16}}{R} + \tilde{\gamma}_R \right) e^{-2\epsilon z(\theta_{r-\tau}\omega)}. \end{aligned} \quad (3.36)$$

By applying Gronwall inequality to (3.14) and (3.36) on  $[\tau - t, \tau]$ , respectively, and by  $R > 1$ , (3.24), (3.30),

$$\begin{aligned} & \|\nabla v(\tau)\|^2 + \int_{\tau-t}^{\tau} e^{\int_s^{\tau} (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} \|\Delta v(s)\|^2 ds \\ & \leq e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_1^2(\theta_{-t}\omega) + (1 + 2c_2) \int_{-\infty}^0 e^{\int_s^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_1^2(\theta_s\omega) ds + c_5 K_0(\omega) \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))|^2 dx \\ & \leq e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} \|v_{\tau-t}(\theta_{-\tau}\omega)\|^2 \\ & \quad + c_{13} \int_{\tau-t}^{\tau} e^{\int_s^{\tau} (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |\nabla v(s)|^2 ds \\ & \quad + \frac{c_{14}}{R} \int_{-t}^0 e^{\int_s^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} \left( M_1^2(\theta_s\omega) + z^2(\theta_s\omega) M_1^2(\theta_s\omega) + e^{4\epsilon z(\theta_s\omega)} M_1^6(\theta_s\omega) \right) ds \\ & \quad + \frac{2c_{15}}{R} \int_{\tau-t}^{\tau} e^{\int_s^{\tau} (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} \|\Delta v(s)\|^2 ds \\ & \quad + \left( \frac{c_{16}}{R} + \tilde{\gamma}_R \right) \int_{\tau-t}^{\tau} e^{-2\epsilon z(\theta_{s-\tau}\omega) + \int_s^{\tau} (2\epsilon z(\theta_{l-\tau}\omega) - \lambda) dl} ds \\ & \leq c_7 e^{\int_{-t}^0 (2\epsilon z(\theta_l\omega) - \lambda) dl} M_1^2(\theta_{-t}\omega) + \frac{c_{17}}{R} (K_0(\omega) + K_1(\omega) + K_3(\omega)) \\ & \quad + (\gamma_R + \tilde{\gamma}_R) K_0(\omega), \end{aligned}$$

where

$$K_3(\omega) = \int_{-\infty}^0 e^{\int_s^0 (2\epsilon z(\theta_l \omega) - \lambda) dl} \left( M_1^2(\theta_s \omega) + z^2(\theta_s \omega) M_1^2(\theta_s \omega) + e^{4\epsilon z(\theta_s \omega)} M_1^6(\theta_s \omega) \right) ds.$$

Then (3.22) holds, where

$$K_2(\omega) = c_{17} (K_0(\omega) + K_1(\omega) + K_3(\omega)).$$

(iii) By (3.21) and (3.22), we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) \left( |v(\tau, \tau - t, \theta_{-\tau} \omega, v_{\tau-t}(\theta_{-\tau} \omega))|^2 + |\nabla v(\tau, \tau - t, \theta_{-\tau} \omega, v_{\tau-t}(\theta_{-\tau} \omega))|^2 \right) dx \\ & \leq (1 + c_7) e^{\int_{-t}^0 (2\epsilon z(\theta_l \omega) - \lambda) dl} M_1^2(\theta_{-t} \omega) + \frac{K_1(\omega) + K_2(\omega)}{R} + (2\gamma_R + \tilde{\gamma}_R) K_0(\omega). \end{aligned} \quad (3.37)$$

By  $(1 + c_7) e^{\int_{-t}^0 (2\epsilon z(\theta_l \omega) - \lambda) dl} M_1^2(\theta_{-t} \omega) \xrightarrow{t \rightarrow \infty} 0$ , for any  $\nu > 0$ , taking

$$T_\nu(\omega) = \min \left\{ t : (1 + c_7) e^{\int_{-t}^0 (2\epsilon z(\theta_l \omega) - \lambda) dl} M_1^2(\theta_{-t} \omega) \leq \nu \right\} < \infty,$$

it follows from (3.37) that (3.23) holds. This complete the proof.  $\square$

Similar to [34,35], we have the existence of a random attractor for  $\Phi$ .

**Theorem 3.1.** Suppose conditions (A1)–(A2) hold, then the cocycle  $\Phi$  possesses a random attractor  $\mathcal{A} = \{\mathcal{A}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega} \in \mathcal{D}(L^2(\mathbb{R}^3))$  defined by

$$\mathcal{A}(\tau, \omega) = \bigcap_{r \geq 0} \overline{\bigcup_{t \geq r} \Phi(t, \tau - t, \theta_{-t} \omega, B_0(\theta_{-t} \omega))}$$

with the properties that for every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,

- (i)  $\mathcal{A}(\tau, \omega) \subseteq B_0(\omega) \cap B_1(\omega) \subset H^1(\mathbb{R}^3)$ ;
- (ii)  $\mathcal{A}(\tau, \omega)$  is compact in  $L^2(\mathbb{R}^3)$  and measurable in  $\omega$ ;
- (iii)  $\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(t + \tau, \theta_t \omega)$ ,  $\forall t \geq 0$ ;
- (iv) for every  $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(L^2(\mathbb{R}^3))$ ,  $\lim_{t \rightarrow +\infty} d_h(\Phi(t, \tau - t, \theta_{-t} \omega) \times B(\tau - t, \theta_{-t} \omega), \mathcal{A}(\tau, \omega)) = 0$ .

### 3.4. Existence of random exponential attractor

For every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ , set a tempered bounded random subset  $\{\chi(\tau, \omega)\}$  of  $L^2(\mathbb{R}^3)$  as

$$\begin{aligned} \chi(\tau, \omega) &= \bigcup_{s \geq T_{B_0}(\omega) + T_{1, B_0}(\omega) + T_{2, B_0}(\omega) + T_\nu(\omega)} \Phi(s, \tau - s, \theta_{-s} \omega) B_0(\theta_{-s} \omega) \\ &= \bigcup_{s \geq T_{B_0}(\omega) + T_{1, B_0}(\omega) + T_{2, B_0}(\omega) + T_\nu(\omega)} \bigcup_{v_{\tau-s}(\theta_{-\tau} \omega) \in B_0(\theta_{-s} \omega)} v(\tau, \tau - s, \theta_{-\tau} \omega, v_{\tau-s}(\theta_{-\tau} \omega)). \end{aligned}$$

Then it is easy to check from [Lemmas 3.1–3.4](#) that the family of tempered bounded random subsets  $\{\chi(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$  satisfies (H1) in [Theorem 2.1](#), exactly, it holds that

**(h-11)** for every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $\chi(\tau, \omega) \subseteq B_0(\omega) \cap B_1(\omega) \subset H^1(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$ , thus, the diameter of  $\chi(\tau, \omega)$  in  $L^2(\mathbb{R}^3)$  is bounded by  $2M_0(\omega)$ , where  $2M_0(\theta_t \omega)$  is continuous in  $t$  for all  $t \in \mathbb{R}$ ;

**(h-12)**  $\chi(\tau, \omega)$  is positively invariant with respect to  $\{\theta_t\}_{t \in \mathbb{R}}$  in the sense that for any  $t \geq 0$  and  $r \geq \tau - t$ ,

$$\begin{aligned} & \Phi(r, \tau - t, \theta_{-t}\omega) \chi(\tau - t, \theta_{-t}\omega) \\ &= \bigcup_{s \geq T_{B_0}(\omega) + T_{1, B_0}(\omega) + T_{2, B_0}(\omega) + T_v(\omega)} \Phi(r, \tau - t - s, \theta_{-s}\omega) B_0(\theta_{-s}\omega) \\ &\subseteq \bigcup_{s \geq T_{B_0}(\omega) + T_{1, B_0}(\omega) + T_{2, B_0}(\omega) + T_v(\omega)} \Phi(r, r - s, \theta_{-s}\omega) B_0(\theta_{-s}\omega) \\ &= \chi(r, \omega) \subseteq B_0(\omega) \cap B_1(\omega) \subset L^2(\mathbb{R}^3); \end{aligned}$$

**(h-13)**  $\chi(\tau, \omega)$  is pullback absorbing in  $\mathcal{D}(L^2(\mathbb{R}^3))$  that for any set  $B \in \mathcal{D}(L^2(\mathbb{R}^3))$ , there exists  $T_B = T_B(\tau, \omega) \geq 0$  such that  $\Phi(t, \tau - t, \theta_{-t}\omega) B(\tau - t, \theta_{-t}\omega) \subseteq \chi(\tau, \omega)$  for  $t \geq T_B$ .

Moreover, by [Lemma 3.5](#) (iii), it follows that

**(h-14)** for any  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $R > 1$  and  $v \in \chi(\tau, \omega)$ , it holds that

$$\int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right) (v^2(x) + |\nabla v(x)|^2) dx \leq v + \frac{K_1(\omega) + K_2(\omega)}{R} + (2\gamma_R + \tilde{\gamma}_R) K_0(\omega). \quad (3.38)$$

In the following, we show that  $\{\chi(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$  satisfies (H2)–(H4) in [Theorem 2.1](#) which implying the existence of a random exponential attractor for  $\Phi$ .

### 3.4.1. Lipschitz property of $\Phi$

For every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $t \geq 0$  and  $v_{j, \tau-t}(\theta_{-t}\omega) \in \chi(\tau - t, \theta_{-t}\omega)$ ,  $j = 1, 2$ , let

$$v_j(r) = v_j(r, \tau - t, \theta_{-t}\omega, v_{j, \tau-t}(\theta_{-t}\omega)), \quad y(r) = v_1(r) - v_2(r), \quad r \geq \tau - t, \quad j = 1, 2,$$

then

$$\begin{cases} \frac{dy}{dt} = \Delta y - \lambda y + \epsilon z(\theta_{r-\tau}\omega) y + e^{-\epsilon z(\theta_{r-\tau}\omega)} [f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_1) - f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_2)], \\ y(\tau - t, \tau - t, \theta_{-t}\omega, y_{\tau-t}(\theta_{-t}\omega)) = v_{1, \tau-t}(\theta_{-t}\omega) - v_{2, \tau-t}(\theta_{-t}\omega), \quad r \geq \tau - t, \tau \in \mathbb{R}. \end{cases} \quad (3.39)$$

By **(h-12)**, it holds that

$$\begin{cases} v_1(r), \quad v_2(r) \in \chi(r, \theta_{r-\tau}\omega) \subseteq B_1(\theta_{r-\tau}\omega) \subset H^1(\mathbb{R}^3), \\ \|v_1(r)\|_1 \leq M_1(\theta_{r-\tau}\omega), \quad \|v_2(r)\|_1 \leq M_1(\theta_{r-\tau}\omega), \end{cases} \quad \forall r \geq \tau - t. \quad (3.40)$$

Now we show the Lipschitz property of  $\Phi$  on  $\chi(\tau, \omega)$ .

**Lemma 3.6.** For every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $t \geq 0$  and  $v_{j,\tau-t}(\theta_{-t}\omega) \in \chi(\tau-t, \theta_{-t}\omega)$ ,  $j = 1, 2$ , it holds that for  $r \geq \tau-t$ ,

$$\|y(r)\|^2 + 2 \int_{\tau-t}^r e^{\int_s^r 2[\epsilon z(\theta_{r-\tau}\omega) + c_3]dl} \|\nabla y(s)\|^2 ds \leq e^{\int_{\tau-t}^r 2[\epsilon z(\theta_{r-\tau}\omega) + c_2]dl} \|y_{\tau-t}(\theta_{-\tau}\omega)\|^2. \quad (3.41)$$

Particularly,

$$\|y(\tau, \tau-t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega))\| \leq e^{\int_{-t}^0 \epsilon |z(\theta_s\omega)| ds + c_2 t} \|y_{\tau-t}(\theta_{-\tau}\omega)\|. \quad (3.42)$$

**Proof.** By (1.2), we have

$$e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} [f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_1(r)) - f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_2(r))] y(r) dx \leq c_2 \|y(r)\|^2.$$

Taking the inner product  $(\cdot, \cdot)$  of (3.39) with  $y(r)$ , we find that

$$\frac{d}{dt} \|y(r)\|^2 + 2 \|\nabla y(r)\|^2 \leq 2[\epsilon z(\theta_{r-\tau}\omega) + c_2] \|y(r)\|^2, \quad \forall r \geq \tau-t.$$

Thus, (3.41) and (3.42) hold. This completes the proof.  $\square$

### 3.4.2. Decomposition of solutions

To prove that  $\{\chi(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$  satisfies (2.2), we need to decompose the solutions of (3.39) into three parts, one part is finite-dimensional and other two parts are “decay” in mean sense.

Given  $0 < R < +\infty$ , denote the ball by  $\mathbb{U}_R = \{x \in \mathbb{R}^3 : |x| < R\}$ . Consider the following eigenvalue problem on  $\mathbb{U}_{2R} = \{x \in \mathbb{R}^3 : |x| < 2R\}$ :

$$-\Delta \tilde{u}(x) = \mu \tilde{u}(x), \quad \tilde{u}(x)|_{x \in \partial \mathbb{U}_{2R}} = 0, \quad x \in \mathbb{U}_{2R},$$

which has a family of solutions (eigenfunctions)  $\{\tilde{e}_{m,R}\}_{m \in \mathbb{N}}$  with eigenvalues  $\{\mu_{m,R}\}_{m \in \mathbb{N}}$  such that

$$0 < \mu_{1,R} \leq \mu_{2,R} \leq \cdots \leq \mu_{m,R} \leq \cdots, \quad \mu_{m,R} \rightarrow +\infty \quad \text{as } m \rightarrow +\infty \quad (3.43)$$

and for  $m \in \mathbb{N}$ ,  $-\Delta \tilde{e}_{m,R} = \mu_{m,R} \tilde{e}_{m,R}$ ,  $\tilde{e}_{m,R} \in H^2(\mathbb{U}_{2R}) \cap H_0^1(\mathbb{U}_{2R})$ . Moreover,  $\{\tilde{e}_{m,R}\}_{m \in \mathbb{N}}$  form an orthonormal basis of  $L^2(\mathbb{U}_{2R})$ . For given  $m \in \mathbb{N}$ , let

$$L_m^2(\mathbb{U}_{2R}) = \text{span}\{\tilde{e}_{1,R}, \tilde{e}_{2,R}, \dots, \tilde{e}_{m,R}\}, \quad L_m^2(\mathbb{U}_{2R})^\perp = \text{span}\{\tilde{e}_{m+1,R}, \tilde{e}_{m+2,R}, \dots\}$$

and

$$\tilde{P}_{m,R} : L^2(\mathbb{U}_{2R}) \rightarrow L_m^2(\mathbb{U}_{2R}), \quad \tilde{Q}_{m,R} = I_{L^2(\mathbb{U}_{2R})} - \tilde{P}_{m,R} : L^2(\mathbb{U}_{2R}) \rightarrow L_m^2(\mathbb{U}_{2R})^\perp,$$

where  $\tilde{P}_{m,R}$  is a  $m$ -dimensional orthonormal projector, and  $\mu_{m+1,R} \|\tilde{Q}_{m,R} v\|^2 \leq \|\nabla v\|^2$  for  $v \in L_m^2(\mathbb{U}_{2R})^\perp$ .

Write

$$e_{m,R}(x) = \begin{cases} \tilde{e}_{m,R}(x), & |x| < 2R, \\ 0, & |x| \geq 2R, \end{cases} \quad m \in \mathbb{N}.$$

Then  $\{e_{m,R}\}_{m \in \mathbb{N}}$  form a family of orthonormal functions of  $L^2(\mathbb{R}^3)$ . For given  $m \in \mathbb{N}$ , let

$$L^2_{m,R}(\mathbb{R}^3) = \text{span}\{e_{1,R}, e_{2,R}, \dots, e_{m,R}\}, \quad L^2_{m,R}(\mathbb{R}^3)^\perp = \text{span}\{e_{m+1,R}, e_{m+2,R}, \dots\},$$

$$P_{m,R} : L^2(\mathbb{R}^3) \rightarrow L^2_{m,R}(\mathbb{R}^3), \quad Q_{m,R} : L^2(\mathbb{R}^3) \rightarrow L^2_{m,R}(\mathbb{R}^3)^\perp,$$

then  $P_{m,R}$  is a  $m$ -dimensional projector form  $L^2(\mathbb{R}^3)$  into  $L^2_{m,R}(\mathbb{R}^3)$  and

$$\mu_{m+1,R} \|Q_{m,R} v\|^2 \leq \|\nabla v\|^2 \leq \|v\|_1^2, \quad \forall v \in L^2_{m,R}(\mathbb{R}^3)^\perp. \quad (3.44)$$

Let  $y(r) = y(r, \tau - t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega))$  ( $r \geq \tau - t$ ) be a solution of (3.39). Set

$$y_{1,m,R}(r) = P_{m,R}y(r) = \begin{cases} \tilde{P}_{m,R}y(r), & |x| < 2R, \\ 0, & |x| \geq 2R, \end{cases}$$

$$y_{2,m,R}(r) = Q_{m,R}y(r) = \begin{cases} \tilde{Q}_{m,R}y(r), & |x| < 2R, \\ 0, & |x| \geq 2R, \end{cases}$$

$$y_{3,m,R}(r) = (I - P_{m,R} - Q_{m,R})y(r) = \begin{cases} y(r), & |x| \geq 2R, \\ 0, & |x| < 2R. \end{cases}$$

Then the solution  $y(r)$  of (3.39) can be written as

$$y(r) = y_{1,m,R}(r) + y_{2,m,R}(r) + y_{3,m,R}(r), \quad r \geq \tau - t, \quad (3.45)$$

where

$$y_{1,m,R}(r) = P_{m,R}y(r) \in L^2_{m,R}(\mathbb{R}^3), \quad y_{2,m,R}(r) = Q_{m,R}y(r) \in L^2_{m,R}(\mathbb{R}^3)^\perp,$$

$$(I - P_{m,R})y(r) = y_{2,m,R}(r) + y_{3,m,R}(r),$$

$$(y_{1,m,R}(r), y_{2,m,R}(r)) = (y_{1,m,R}(r), y_{3,m,R}(r)) = (y_{2,m,R}(r), y_{3,m,R}(r)) = 0.$$

Next we estimate the components  $y_{2,m,R}(r)$  and  $y_{3,m,R}(r)$  in (3.45).

**Lemma 3.7.** For every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $R > 0$ ,  $t \geq 0$  and  $m \in \mathbb{N}$ , there exists a random variable  $C_1(\omega) \geq 0$  such that for any  $v_{j,\tau-t}(\omega) \in \chi(\tau - t, \theta_{-t}\omega)$ ,  $j = 1, 2$ , it holds that

$$\|y_{2,m,R}(\tau, \tau - t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega))\|$$

$$= \|Q_{m,R}\Phi(t, \tau - t, \theta_{-\tau}\omega)v_{1,\tau-t}(\theta_{-\tau}\omega) - Q_{m,R}\Phi(t, \tau - t, \theta_{-\tau}\omega)v_{2,\tau-t}(\theta_{-\tau}\omega)\|$$

$$\leq \left( e^{\int_{-t}^0 (\epsilon z(\theta_s\omega) - \frac{\lambda}{2}) ds} + \frac{1}{\sqrt{4\lambda + \mu_{m+1,R}}} e^{\int_{-t}^0 C_1(\theta_s\omega) ds} \right) \|v_{1,\tau-t}(\theta_{-\tau}\omega) - v_{2,\tau-t}(\theta_{-\tau}\omega)\|.$$

**Proof.** Taking the inner product of (3.39) with  $y_{2,m,R} = Q_{m,R}y$  in  $L^2(\mathbb{R}^3)$ , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|y_{2,m,R}\|^2 + \|\nabla y_{2,m,R}\|^2 \\ &= [\epsilon z(\theta_{r-\tau}\omega) - \lambda] \|y_{2,m,R}\|^2 \\ & \quad + e^{-\epsilon z(\theta_{r-\tau}\omega)} (f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_1(r)) - f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_2(r)), y_{2,m,R}). \end{aligned} \quad (3.46)$$

By Hölder inequality, (1.2), (3.40) and (3.2), we have

$$\begin{aligned} & \|e^{-\epsilon z(\theta_{r-\tau}\omega)} [f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_1(r)) - f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_2(r))]\|_{L^{\frac{6}{5}}(\mathbb{U}_{2R})}^2 \\ &= e^{-2\epsilon z(\theta_{r-\tau}\omega)} \left( \int_{\mathbb{U}_{2R}} |f'_u(x, e^{\epsilon z(\theta_{r-\tau}\omega)}(v_2(r) + \vartheta(v_1(r) - v_2(r))))|^{\frac{6}{5}} |v_1(r) - v_2(r)|^{\frac{6}{5}} dx \right)^{\frac{5}{3}} \\ &\leq e^{-2\epsilon z(\theta_{r-\tau}\omega)} \left( \int_{\mathbb{U}_{2R}} [c_3 e^{2\epsilon z(\theta_{r-\tau}\omega)} (|v_1(r)|^2 + |v_2(r)|^2) + \beta_4(x)]^3 dx \right)^{\frac{2}{3}} \|y_{2,m,R}(r)\|_{L^2(\mathbb{U}_{2R})}^2 \\ &\leq c_{18} e^{-2\epsilon z(\theta_{r-\tau}\omega)} \left( \int_{\mathbb{U}_{2R}} e^{6\epsilon z(\theta_{r-\tau}\omega)} (|v_1(r)|^6 + |v_2(r)|^6) dx + \int_{\mathbb{U}_{2R}} \beta_4^3(x) dx \right)^{\frac{2}{3}} \|y_{2,m,R}(r)\|^2 \\ &\leq c_{20} e^{-2\epsilon z(\theta_{r-\tau}\omega)} \left( e^{4\epsilon z(\theta_{r-\tau}\omega)} (\|\nabla v_1(r)\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla v_2(r)\|_{L^2(\mathbb{R}^3)}^4) + 1 \right) \|y_{2,m,R}(r)\|^2 \\ &\leq c_{21} e^{-2\epsilon z(\theta_{r-\tau}\omega)} \left( e^{4\epsilon z(\theta_{r-\tau}\omega)} M_1^4(\theta_{r-\tau}\omega) + 1 \right) \cdot \|y(r)\|^2 \end{aligned}$$

and

$$\begin{aligned} & (e^{-\epsilon z(\theta_{r-\tau}\omega)} [f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_2(r)) - f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_1(r))], y_{2,m,R}) \\ &= e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{U}_{2R}} [f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_2(r)) - f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_1(r))] y_{2,m,R} dx \\ &\leq \|e^{-\epsilon z(\theta_{r-\tau}\omega)} [f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_2(r)) - f(x, e^{\epsilon z(\theta_{r-\tau}\omega)} v_1(r))]\|_{L^{\frac{6}{5}}(\mathbb{U}_{2R})} \cdot \|\nabla y_{2,m,R}\|_{L^2(\mathbb{U}_R)} \\ &\leq \frac{1}{2} c_{21} \left( e^{2\epsilon z(\theta_{r-\tau}\omega)} M_1^4(\theta_{r-\tau}\omega) + e^{-2\epsilon z(\theta_{r-\tau}\omega)} \right) \|y(r)\|^2 + \frac{1}{2} \|\nabla y_{2,m,R}\|^2, \end{aligned} \quad (3.47)$$

where  $\vartheta \in (0, 1)$  and  $c_{21}$  is a positive constant depending on  $c_3$ ,  $C_0$  and  $\|\beta_4\|_{L^3(\mathbb{R}^3)}$ . Putting (3.44), (3.47) into (3.46), we have that for  $r \geq \tau - t$ ,

$$\begin{aligned} & \frac{d}{dt} \|y_{2,m,R}\|^2 \\ & \leq (2\epsilon z(\theta_{r-\tau}\omega) - \lambda - \mu_{m+1,R}) \|y_{2,m,R}\|^2 \end{aligned}$$

$$\begin{aligned}
 & + c_{21} \left( e^{2\epsilon z(\theta_r \omega)} M_1^4(\theta_{r-\tau} \omega) + e^{-2\epsilon z(\theta_{r-\tau} \omega)} \right) \|y(r)\|^2 \\
 & \leq (2\epsilon z(\theta_{r-\tau} \omega) - \lambda - \mu_{m+1,R}) \|y_{2,m,R}\|^2 \\
 & + c_{21} \left( e^{2\epsilon z(\theta_{r-\tau} \omega)} M_1^4(\theta_{r-\tau} \omega) + e^{-2\epsilon z(\theta_{r-\tau} \omega)} \right) e^{\int_{\tau-t}^r 2[\epsilon z(\theta_{r-\tau} \omega) + c_2] ds} \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2.
 \end{aligned} \tag{3.48}$$

By applying the Gronwall inequality to (3.48) on  $[\tau - t, \tau]$  and (3.28), we have that

$$\begin{aligned}
 & \|y_{2,m,R}(\tau, \tau - t, \theta_{-\tau} \omega, y_{\tau-t}(\theta_{-\tau} \omega))\|^2 \\
 & \leq \|y_{2,m,R}(\tau - t)\|^2 e^{\int_{-\tau}^0 (2\epsilon z(\theta_s \omega) - \lambda - \mu_{m+1,R}) ds} \\
 & + \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 \int_{\tau-t}^{\tau} c_{21} \left( e^{2\epsilon z(\theta_{r-\tau} \omega)} M_1^4(\theta_{r-\tau} \omega) + e^{-2\epsilon z(\theta_{r-\tau} \omega)} \right) \\
 & \quad \times e^{2 \int_{\tau-t}^r (\epsilon z(\theta_{s-\tau} \omega) + c_2) ds} e^{\int_r^{\tau} (2\epsilon z(\theta_{s-\tau} \omega) - \lambda - \mu_{m+1,R}) ds} dr \\
 & \leq \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 e^{\int_{-t}^0 (2\epsilon z(\theta_s \omega) - \lambda - \mu_{m+1,R}) ds} \\
 & + \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 e^{2 \int_{-t}^0 (2\epsilon |z(\theta_s \omega)| + c_2) ds} \\
 & \quad \times \int_{-t}^0 c_{21} \left( e^{2\epsilon z(\theta_r \omega)} M_1^4(\theta_r \omega) + e^{-2\epsilon z(\theta_r \omega)} \right) e^{(\lambda + \mu_{m+1,R})r} dr.
 \end{aligned} \tag{3.49}$$

Since  $\sqrt{x} \leq e^x$  for all  $x \geq 0$ , it follows that

$$\begin{aligned}
 & \int_{-t}^0 c_{21} \left( e^{2\epsilon z(\theta_r \omega)} M_1^4(\theta_r \omega) + e^{-2\epsilon z(\theta_r \omega)} \right) \cdot e^{(\lambda + \mu_{m+1,R})r} dr \\
 & \leq \left( \int_{-t}^0 c_{21}^2 \left( e^{2\epsilon z(\theta_r \omega)} M_1^4(\theta_r \omega) + e^{-2\epsilon z(\theta_r \omega)} \right)^2 dr \right)^{\frac{1}{2}} \left( \int_{-t}^0 e^{2(\lambda + \mu_{m+1,R})r} dr \right)^{\frac{1}{2}} \\
 & \leq \frac{1}{\sqrt{\lambda + \mu_{m+1,R}}} e^{2 \int_{-t}^0 c_{21}^2 (e^{4\epsilon z(\theta_r \omega)} M_1^8(\theta_r \omega) + e^{-4\epsilon z(\theta_r \omega)}) dr}
 \end{aligned}$$

and by (3.49),

$$\begin{aligned}
 & \|y_{2,m,R}(\tau, \tau - t, \theta_{-\tau} \omega, y_{\tau-t}(\theta_{-\tau} \omega))\| \\
 & \leq \left( e^{\int_{-t}^0 (\epsilon z(\theta_s \omega) - \frac{\lambda}{2}) ds} + \frac{1}{\sqrt{\lambda + \mu_{m+1,R}}} e^{\int_{-t}^0 C_1(\theta_s \omega) ds} \right) \|y_{\tau-t}(\theta_{-\tau} \omega)\|,
 \end{aligned}$$

where

$$C_1(\omega) = c_2 + 2\epsilon|z(\omega)| + c_{21}^2 e^{4\epsilon z(\omega)} M_1^8(\omega) + c_{21}^2 e^{-4\epsilon z(\omega)}.$$

The proof is completed.  $\square$

**Lemma 3.8.** *For every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $R > 2$ ,  $t \geq 0$  and  $m \in \mathbb{N}$ , there exists a random variable  $C_2(\omega) \geq 0$  such that for any  $v_{j,\tau-t}(\omega) \in \chi(\tau-t, \theta_{-t}\omega)$ ,  $j = 1, 2$ , it holds that*

$$\begin{aligned} & \|y_{3,m,R}(\tau, \tau-t, \theta_{-t}\omega, y_{\tau-t}(\theta_{-t}\omega))\| \\ &= \left( \int_{|x| \geq R} |\Phi(t, \tau-t, \theta_{-t}\omega)v_{1,\tau-t}(\theta_{-t}\omega) - \Phi(t, \tau-t, \theta_{-t}\omega)v_{2,\tau-t}(\theta_{-t}\omega)|^2 dx \right)^{\frac{1}{2}} \\ &\leq e^{\int_{-t}^0 (\epsilon z(\theta_s\omega) - \frac{\lambda}{2}) ds} \|v_{1,\tau-t}(\theta_{-t}\omega) - v_{2,\tau-t}(\theta_{-t}\omega)\| \\ &\quad + \sqrt{\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{4\lambda}}\right) I_{v,R} e^{\int_{-t}^0 C_2(\theta_s\omega) ds} \|v_{1,\tau-t}(\theta_{-t}\omega) - v_{2,\tau-t}(\theta_{-t}\omega)\|}, \end{aligned} \quad (3.50)$$

where

$$I_{v,R} = v + \frac{1}{R^2} + \frac{1}{R} + \gamma_{\frac{R}{2}} + \tilde{\gamma}_{\frac{R}{2}} + \beta_{4,R}, \quad \beta_{4,R} = \left( \int_{|x| \geq R} \beta_4^3(x) dx \right)^{\frac{1}{3}}.$$

**Proof.** For any  $v \in H^1(\mathbb{R}^3)$ , by (3.2) and (3.20), we have

$$\begin{aligned} & \|v\|_{L^6(\mathbb{R}^3 \setminus \mathbb{U}_{2R})}^2 \\ &= \left( \int_{|x| \geq 2R} v^6 dx \right)^{\frac{2}{6}} \leq \left( \int_{\mathbb{R}^3} \left( \xi \left( \frac{|x|^2}{R^2} \right) v \right)^6 dx \right)^{\frac{2}{6}} \\ &\leq C_0^2 \left( \int_{\mathbb{R}^3} \left( \nabla \left( \xi \left( \frac{|x|^2}{R^2} \right) v \right) \right)^2 dx + \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) v^2 dx \right) \\ &\leq C_0^2 \left( \int_{\mathbb{R}^3} \left( \xi' \left( \frac{|x|^2}{R^2} \right) \frac{2x}{R^2} v + \xi \left( \frac{|x|^2}{R^2} \right) \nabla v \right)^2 dx + \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) v^2 dx \right) \\ &\leq 2C_0^2 \left( \int_{\mathbb{R}^3} \left( \xi' \left( \frac{|x|^2}{R^2} \right) \frac{2x}{R^2} v \right)^2 dx + \int_{\mathbb{R}^3} \xi^2 \left( \frac{|x|^2}{R^2} \right) (\nabla v)^2 dx + \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) v^2 dx \right) \end{aligned}$$



$$\leq \frac{16C_0^2\tilde{C}^2}{R^2}||v||^2 + 2C_0^2 \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right)[(\nabla v)^2 + v^2]dx. \quad (3.51)$$

It follows from (3.38) and (3.51) that for  $i = 1, 2$  and  $R \geq 2$ ,

$$\begin{aligned} ||v_i(r)||_{L^6(\mathbb{R}^3 \setminus \mathbb{U}_R)}^2 &\leq \frac{64C_0^2\tilde{C}^2}{R^2}M_1^2(\theta_{r-\tau}\omega) \\ &\quad + 2C_0^2\left(v + \frac{2K_1(\theta_{r-\tau}\omega) + 2K_2(\theta_{r-\tau}\omega)}{R} + \left(2\gamma_{\frac{R}{2}} + \tilde{\gamma}_{\frac{R}{2}}\right)K_0(\theta_{r-\tau}\omega)\right) \\ &\leq 2C_0^2v + K_4(\theta_{r-\tau}\omega)\left(\frac{1}{R^2} + \frac{1}{R} + \gamma_{\frac{R}{2}} + \tilde{\gamma}_{\frac{R}{2}}\right), \end{aligned}$$

where

$$K_4(\omega) = 64C_0^2\tilde{C}^2M_1^2(\omega) + 2C_0^2(2K_0(\omega) + K_1(\omega) + K_2(\omega)).$$

Taking the inner product of (3.39) with  $\xi(\frac{|x|^2}{R^2})y$  in  $L^2(\mathbb{R}^3)$ , where  $R \geq 2$ , we obtain

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right)|y|^2dx + [2\lambda - 2\epsilon z(\theta_{r-\tau}\omega)] \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right)|y|^2dx - 2 \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right)y \Delta y dx \\ &= 2e^{-\epsilon z(\theta_{r-\tau}\omega)} \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right)[f(x, e^{\epsilon z(\theta_{r-\tau}\omega)}v_1(r)) - f(x, e^{\epsilon z(\theta_{r-\tau}\omega)}v_2(r))]y dx. \end{aligned} \quad (3.52)$$

For the terms of (3.52), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right)y \Delta y dx \leq \frac{2\sqrt{2}\tilde{C}}{R}||y|| \cdot ||\nabla y||, \quad (3.53) \\ &\int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right)[f(x, e^{\epsilon z(\theta_{r-\tau}\omega)}v_1(r)) - f(x, e^{\epsilon z(\theta_{r-\tau}\omega)}v_2(r))]y dx \\ &= \int_{\mathbb{R}^3} \xi\left(\frac{|x|^2}{R^2}\right)\frac{\partial f}{\partial u}(x, e^{\epsilon z(\theta_{r-\tau}\omega)}(v_2(r) + \vartheta(v_1(r) - v_2(r))))y^2 dx \\ &= \left( \int_{\mathbb{R}^3} [\xi\left(\frac{|x|^2}{R^2}\right)\frac{\partial f}{\partial u}(x, e^{\epsilon z(\theta_{r-\tau}\omega)}(v_2(r) + \vartheta(v_1(r) - v_2(r))))]^3 dx \right)^{\frac{1}{3}} \\ &\quad \times \left( \int_{\mathbb{R}^3} y^6 dx \right)^{\frac{1}{6}} \left( \int_{\mathbb{R}^3} y^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq C_0 \left( \int_{\mathbb{R}^3} \left[ \xi \left( \frac{|x|^2}{R^2} \right) 2c_3 e^{2\epsilon z(\theta_{r-\tau}\omega)} (v_1^2(r) + v_2^2(r)) + \xi \left( \frac{|x|^2}{R^2} \right) \beta_4(x) \right]^3 dx \right)^{\frac{1}{3}} \|y\|_1 \cdot \|y\| \\
&\leq 2c_3 C_0 e^{2\epsilon z(\theta_{r-\tau}\omega)} \left( \int_{\mathbb{R}^3} \xi^3 \left( \frac{|x|^2}{R^2} \right) (v_1^2(r) + v_2^2(r))^3 dx \right)^{\frac{1}{3}} \|y\|_1 \cdot \|y\| \\
&\quad + C_0 \left( \int_{\mathbb{R}^3} \xi^3 \left( \frac{|x|^2}{R^2} \right) \beta_4^3(x) dx \right)^{\frac{1}{3}} \|y\|_1 \cdot \|y\| \\
&\leq 2c_3 C_0 e^{2\epsilon z(\theta_{r-\tau}\omega)} \left( \left( \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) v_1^6(r) dx \right)^{\frac{1}{3}} + \left( \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) v_2^6(r) dx \right)^{\frac{1}{3}} \right) \|y\|_1 \cdot \|y\| \\
&\quad + C_0 \left( \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) \beta_4^3(x) dx \right)^{\frac{1}{3}} \|y\|_1 \cdot \|y\| \\
&\leq \left[ 2c_3 C_0 e^{2\epsilon z(\theta_{r-\tau}\omega)} \left( \|v_1(r)\|_{L^6(\mathbb{R}^3 \setminus \mathbb{U}_R)}^2 + \|v_2(r)\|_{L^6(\mathbb{R}^3 \setminus \mathbb{U}_R)}^2 \right) + C_0 \beta_{4,R} \right] \|y\|_1 \cdot \|y\| \\
&\leq 2c_3 C_0 e^{2\epsilon z(\theta_{r-\tau}\omega)} \left( 2C_0^2 v + K_4(\theta_{r-\tau}\omega) \left( \frac{1}{R^2} + \frac{1}{R} + \gamma_{\frac{R}{2}} + \tilde{\gamma}_{\frac{R}{2}} \right) \right) (\|y\|^2 + \|y\| \cdot \|\nabla y\|) \\
&\quad + C_0 \beta_{4,R} (\|y\|^2 + \|y\| \cdot \|\nabla y\|) \tag{3.54}
\end{aligned}$$

and

$$\begin{aligned}
&\|y(r)\|^2 + \|y(r)\| \cdot \|\nabla y(r)\| \\
&\leq e^{\int_{\tau-t}^r 2[\epsilon z(\theta_{r-\tau}\omega) + c_2] dl} \|y_{\tau-t}(\theta_{-\tau}\omega)\|^2 + \|\nabla y(r)\| \cdot e^{\int_{\tau-t}^r [\epsilon z(\theta_{r-\tau}\omega) + c_2] dl} \|y_{\tau-t}(\theta_{-\tau}\omega)\|. \tag{3.55}
\end{aligned}$$

Putting (3.53)–(3.55) into (3.52), we have that for  $r \geq \tau - t$ ,

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |y|^2 dx + [\lambda - 2\epsilon z(\theta_{r-\tau}\omega)] \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |y|^2 dx \\
&\leq I_{v,R} K_5(\theta_{r-\tau}\omega) e^{\int_{\tau-t}^r 2[\epsilon z(\theta_{r-\tau}\omega) + c_2] dl} \|y_{\tau-t}(\theta_{-\tau}\omega)\|^2 \\
&\quad + I_{v,R} K_5(\theta_{r-\tau}\omega) \|\nabla y\| e^{\int_{\tau-t}^r [\epsilon z(\theta_{r-\tau}\omega) + c_2] dl} \|y_{\tau-t}(\theta_{-\tau}\omega)\|, \tag{3.56}
\end{aligned}$$

where

$$K_5(\omega) = c_{22} e^{|\epsilon z(\omega)|} \left( 1 + M_1^2(\omega) + K_0(\omega) + K_1(\omega) + K_2(\omega) \right) \quad (\text{independent of } R)$$

and  $c_{22} > 0$  is a constant depending on  $c_3$ ,  $C_0$  and  $\tilde{C}$ . By applying the Gronwall inequality to (3.56) on  $[\tau - t, \tau]$  ( $t \geq 0$ ), we have

$$\begin{aligned}
 & \int_{\mathbb{R}^3} \xi \left( \frac{|x|^2}{R^2} \right) |y(\tau)|^2 dx \\
 & \leq e^{\int_{-\tau}^0 (2\epsilon z(\theta_s \omega) - \lambda) ds} \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 \\
 & \quad + I_{v,R} \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 \int_{\tau-t}^{\tau} K_5(\theta_{r-\tau} \omega) e^{\int_{\tau-t}^r 2(\epsilon z(\theta_{s-\tau} \omega) + c_2) ds} e^{\int_r^{\tau} (2\epsilon z(\theta_{s-\tau} \omega) - \lambda) ds} dr \\
 & \quad + I_{v,R} \|y_{\tau-t}(\theta_{-\tau} \omega)\| \int_{\tau-t}^{\tau} \|\nabla y(r)\| \cdot K_5(\theta_{r-\tau} \omega) e^{\int_{\tau-t}^r (\epsilon z(\theta_{s-\tau} \omega) + c_2) ds} e^{\int_r^{\tau} (2\epsilon z(\theta_{s-\tau} \omega) - \lambda) ds} dr \\
 & \leq e^{\int_{-\tau}^0 (2\epsilon z(\theta_s \omega) - \lambda) ds} \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 + I_{v,R} \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 e^{\int_{-\tau}^0 4(\epsilon |z(\theta_s \omega)| + c_2) ds} \int_{-\tau}^0 K_5(\theta_r \omega) e^{\lambda r} dr \\
 & \quad + I_{v,R} \|y_{\tau-t}(\theta_{-\tau} \omega)\| e^{\int_{-\tau}^0 2(\epsilon |z(\theta_s \omega)| + c_2) ds} \left( \int_{\tau-t}^{\tau} e^{\int_r^{\tau} 2(\epsilon z(\theta_{s-\tau} \omega) + c_2) ds} \|\nabla y(r)\|^2 dr \right)^{\frac{1}{2}} \\
 & \quad \times \left( \int_{-\tau}^0 K_5^2(\theta_r \omega) e^{2\lambda r} dr \right)^{\frac{1}{2}} \\
 & \leq \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 \left( e^{\int_{-\tau}^0 (2\epsilon z(\theta_s \omega) - \lambda) ds} + \frac{1}{\sqrt{2\lambda}} I_{v,R} e^{\int_{-\tau}^0 [4(\epsilon |z(\theta_s \omega)| + c_2) + K_5^2(\theta_s \omega)] ds} \right) \\
 & \quad + \frac{1}{\sqrt{2\lambda} \sqrt[4]{\lambda}} I_{v,R} \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 e^{\int_{-\tau}^0 [4(\epsilon |z(\theta_s \omega)| + c_2) + \frac{1}{2} K_5^4(\theta_s \omega)] ds} \\
 & \leq e^{\int_{-\tau}^0 (2\epsilon z(\theta_s \omega) - \lambda) ds} \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2 \\
 & \quad + \left( \frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt[4]{\lambda}} \right) I_{v,R} e^{\int_{-\tau}^0 [4(\epsilon |z(\theta_s \omega)| + c_2) + 1 + K_5^4(\theta_s \omega)] ds} \|y_{\tau-t}(\theta_{-\tau} \omega)\|^2. \tag{3.57}
 \end{aligned}$$

Thus, (3.50) holds, where

$$C_2(\omega) = 1 + 2c_2 + 2\epsilon |z(\omega)| + K_5^4(\omega).$$

The proof is completed.  $\square$

**Lemma 3.9.** For every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $t \geq 0$  and  $R \geq 2$ , there exist a random variable  $C_3(\omega) \geq 0$  and a  $m$ -dimensional orthonormal projector  $P_{m,R}: L^2(\mathbb{R}^3) \rightarrow L_{m,R}^2(\mathbb{R}^3)$  ( $m$  is independent of  $R$ ) such that for any  $v_{j,\tau-t}(\omega) \in \chi(\tau - t, \theta_{-t} \omega)$ ,  $j = 1, 2$ , it holds that

$$\begin{aligned} & |\Phi(t, \tau - t, \theta_{-\tau}\omega)v_{1, \tau-t}(\theta_{-\tau}\omega) - \Phi(t, \tau - t, \theta_{-\tau}\omega)v_{2, \tau-t}(\theta_{-\tau}\omega)| \\ & \leq e^{\int_{-\tau}^0 C_3(\theta_s\omega)ds} \|v_{1, \tau-t}(\theta_{-\tau}\omega) - v_{2, \tau-t}(\theta_{-\tau}\omega)\| \end{aligned} \quad (3.58)$$

and

$$\begin{aligned} & \|(I - P_{m,R})\Phi(t, \tau - t, \theta_{-\tau}\omega)v_{1, \tau-t}(\theta_{-\tau}\omega) - (I - P_{m,R})\Phi(t, \tau - t, \theta_{-\tau}\omega)v_{2, \tau-t}(\theta_{-\tau}\omega)\| \\ & \leq \|y_{2,m,R}(\tau, \tau - t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega))\| + \|y_{3,m,R}(\tau, \tau - t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega))\| \\ & \leq \left( 2e^{\int_{-\tau}^0 (\epsilon|z(\omega)| - \frac{\lambda}{2})ds} + \tilde{\delta}_{v,R,m} e^{\int_{-\tau}^0 C_3(\theta_s\omega)ds} \right) \|v_{1, \tau-t}(\theta_{-\tau}\omega) - v_{2, \tau-t}(\theta_{-\tau}\omega)\|, \end{aligned} \quad (3.59)$$

where

$$\begin{aligned} \tilde{\delta}_{v,R,m} &= \sqrt{\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt[4]{\lambda}}\right) I_{v,R} + \frac{1}{\sqrt[4]{\lambda} + \mu_{m+1,R}}}, \\ C_3(\omega) &= c_{23} + 2\epsilon|z(\omega)| + c_{25}e^{4|\epsilon z(\omega)|} + c_{24}e^{4\epsilon|z(\omega)|}M_1^8(\omega) \\ &\quad + c_{26}e^{4|\epsilon z(\omega)|} \left( K_0^4(\omega) + K_1^4(\omega) + K_2^4(\omega) \right). \end{aligned} \quad (3.60)$$

**Proof.** It is a direct consequence of [Lemmas 3.7–3.8](#).  $\square$

To show the boundedness of the expectations  $\mathbf{E}[\epsilon|z(\omega)| - \frac{\lambda}{2}]$ ,  $\mathbf{E}[(\epsilon|z(\omega)| - \frac{\lambda}{2})^2]$  and  $\mathbf{E}[C_3^2(\omega)]$ , we have the aid of the following result in [\[24,25,44\]](#).

**Lemma 3.10.** *The Ornstein–Uhlenbeck process  $z(\theta_t\omega)$  satisfies*

$$\mathbf{E}\left[e^{\gamma \int_{\tau}^{\tau+t} |z(\theta_s\omega)|ds}\right] \leq e^{\gamma t} \quad \text{for } 0 \leq \gamma^2 \leq 1, \quad \tau \in \mathbb{R}, \quad t \geq 0, \quad (3.61)$$

$$\mathbf{E}[|z(\theta_t\omega)|^p] = \frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}}, \quad \forall p > 0, \quad t \in \mathbb{R}, \quad (3.62)$$

where  $\Gamma$  is the Gamma function.

**Lemma 3.11.** *If the coefficient  $\epsilon$  of the random term in [\(3.1\)](#) is small enough such that*

$$\epsilon < \min \left\{ \frac{\lambda\sqrt{\pi}}{4}, \frac{\lambda}{768}, \frac{1}{768} \right\}, \quad (3.63)$$

then

$$\begin{cases} -\infty < \mathbf{E}[\epsilon|z(\omega)| - \frac{\lambda}{2}] < 0, \\ 0 \leq \mathbf{E}[(\epsilon|z(\omega)| - \frac{\lambda}{2})^2] < \infty, \\ 0 \leq \mathbf{E}[C_3^2(\omega)] < \infty. \end{cases} \quad (3.64)$$

**Proof.** First, by the property of Gamma function  $\Gamma$ ,

$$\begin{cases} \Gamma\left(\frac{1+2k}{2}\right) = \frac{(2k-1)(2k-3)\cdots 3\cdot 1}{2^k} \sqrt{\pi}, \\ \Gamma\left(\frac{1+(2k+1)}{2}\right) = k!, \end{cases} \quad \forall k \in \mathbb{N}. \quad (3.65)$$

By (3.62), (3.65) and  $\epsilon \leq \frac{\lambda\sqrt{\pi}}{4}$ ,

$$\mathbf{E}[|z(\omega)| - \frac{\lambda}{2}] = \frac{\epsilon}{\sqrt{\pi}} - \frac{\lambda}{2} < 0. \quad (3.66)$$

From (3.63), we have

$$\mathbf{E}\left(\left|\epsilon|z(\omega)| - \frac{\lambda}{2}\right|^2\right) \leq \frac{\epsilon^2}{2} + \frac{\lambda^2}{4} < \infty.$$

From (3.60), we have

$$C_3^2(\omega) \leq c_{27}[1 + z^2(\omega) + e^{16\epsilon|z(\omega)|} + K_0^{16}(\omega) + K_5^{16}(\omega) + K_6^{16}(\omega)], \quad (3.67)$$

where

$$\begin{aligned} K_5(\omega) &= \int_{-\infty}^0 e^{2\epsilon \int_s^0 z(\theta_l \omega) dl + \lambda s} M_1^2(\theta_s \omega) ds, \\ K_6(\omega) &= \int_{-\infty}^0 e^{\int_s^0 (2\epsilon z(\theta_l \omega) - \lambda) dl} \left( z^2(\theta_s \omega) M_1^2(\theta_s \omega) + e^{4\epsilon z(\theta_s \omega)} M_1^6(\theta_s \omega) \right) ds. \end{aligned}$$

Now let us estimate the expectation of each term in (3.67). Since

$$\begin{aligned} e^{|\epsilon z(\omega)|} &\leq 1 + \frac{\epsilon^2}{2!} |z(\omega)|^2 + \cdots + \frac{\epsilon^{2k}}{(2k)!} |z(\omega)|^{2k} + \cdots \\ &\quad + |\epsilon z(\omega)| + \frac{|\epsilon|^3}{3!} |z(\omega)|^3 + \cdots + \frac{|\epsilon|^{2k+1}}{(2k+1)!} |z(\omega)|^{2k+1} + \cdots. \end{aligned}$$

It follows from (3.62) and (3.65), that

$$\begin{aligned} \mathbf{E}[e^{|\epsilon z(\omega)|}] &\leq 1 + \frac{\epsilon^2}{2^2} + \frac{1}{2!} \left(\frac{\epsilon^2}{2^2}\right)^2 + \cdots + \frac{1}{k!} \left(\frac{\epsilon^2}{2^2}\right)^k + \cdots \\ &\quad + \frac{1}{\sqrt{\pi}} \left( |\epsilon| + \frac{|\epsilon|^3}{3!} + \frac{|\epsilon|^5}{3 \cdot 4 \cdot 5} + \cdots + \frac{|\epsilon|^{2k+1}}{(k+1)(k+2)\cdots(2k+1)} + \cdots \right) \end{aligned}$$

$$\begin{aligned}
&\leq e^{\frac{\epsilon^2}{2}} + \frac{|\epsilon|}{\sqrt{\pi}} \left( 1 + \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{3!} + \cdots + \frac{\epsilon^{2k}}{k!} + \cdots \right) \\
&\leq \left( 1 + \frac{|\epsilon|}{\sqrt{\pi}} \right) e^{\epsilon^2}.
\end{aligned} \tag{3.68}$$

Then

$$\mathbf{E}[z^2(\omega) + e^{16|\epsilon z(\omega)|}] \leq \frac{1}{2} + \left( 1 + \frac{16|\epsilon|}{\sqrt{\pi}} \right) e^{256\epsilon^2} \doteq \bar{K}_1. \tag{3.69}$$

For  $k \in \mathbb{N}$  and  $\epsilon < \min \left\{ \frac{\lambda}{8k}, \frac{1}{8k} \right\}$ ,

$$\begin{aligned}
\mathbf{E}[K_0^{2k}(\omega)] &\leq \mathbf{E} \left( \int_{-\infty}^0 e^{\lambda s + 2\epsilon \int_s^0 z(\theta_l \omega) dl - 2\epsilon z(\theta_s \omega)} ds \right)^{2k} \\
&\leq \mathbf{E} \left[ \left( \int_{-\infty}^0 e^{\lambda s} ds \right)^{\frac{2k-1}{2k}} \left( \int_{-\infty}^0 e^{\lambda s + 4k\epsilon \int_s^0 z(\theta_l \omega) dl + 4k\epsilon |z(\theta_s \omega)|} ds \right) \right] \\
&\leq \frac{2}{\sqrt[2k]{\lambda^{2k-1}}} \left( \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[e^{8k\epsilon \int_s^0 |z(\theta_l \omega)| dl}] ds + \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[e^{8k\epsilon |z(\theta_s \omega)|}] ds \right) \\
&\leq \frac{2}{\sqrt[2k]{\lambda^{2k-1}}} \left[ \frac{1}{\lambda - 8k\epsilon} + \frac{1}{\lambda} \left( 1 + \frac{8k|\epsilon|}{\sqrt{\pi}} \right) e^{64k^2\epsilon^2} \right] \doteq \hat{K}_{0,k},
\end{aligned}$$

$$\mathbf{E}[M_1^{4k}(\theta_s \omega)] \leq \tilde{C}_{2k} \mathbf{E}[1 + K_0^{2k}(\omega)] \leq \tilde{C}_{2k} (1 + \hat{K}_{0,k}) \doteq \hat{K}_{1,k}.$$

It follows that for  $\epsilon < \min \left\{ \frac{\lambda}{768}, \frac{1}{768} \right\}$ ,

$$\mathbf{E}[K_0^{16}(\omega)] \leq \frac{2}{\sqrt[16]{\lambda^{15}}} \left[ \frac{1}{\lambda - 64\epsilon} + \frac{1}{\lambda} \left( 1 + \frac{64|\epsilon|}{\sqrt{\pi}} \right) e^{64^2\epsilon^2} \right] \doteq \bar{K}_2, \tag{3.70}$$

$$\begin{aligned}
\mathbf{E}[K_5^{16}(\omega)] &\leq \mathbf{E} \left( \int_{-\infty}^0 e^{2\epsilon \int_s^0 z(\theta_l \omega) dl + \lambda s} M_1^2(\theta_s \omega) ds \right)^{16} \\
&\leq \frac{1}{\sqrt[16]{\lambda^{15}}} \mathbf{E} \left( \int_{-\infty}^0 e^{\lambda s + 32\epsilon \int_s^0 |z(\theta_l \omega)| dl} M_1^{32}(\theta_s \omega) ds \right) \\
&\leq \frac{2}{\sqrt[16]{\lambda^{15}}} \left( \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[e^{64\epsilon \int_s^0 z(\theta_l \omega) dl}] ds + \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[M_1^{64}(\theta_s \omega)] ds \right)
\end{aligned}$$

$$\leq \frac{2}{\sqrt[16]{\lambda^{15}}} \left( \frac{1}{\lambda - 64\epsilon} + \frac{\hat{K}_{1,16}}{\lambda} \right) \doteq \bar{K}_3, \quad (3.71)$$

$$\begin{aligned} K_6^{16}(\omega) &= \left( \int_{-\infty}^0 e^{2\epsilon \int_s^0 z(\theta_l \omega) dl + \lambda s} \left( z^2(\theta_s \omega) M_1^2(\theta_s \omega) + e^{4\epsilon z(\theta_s \omega)} M_1^6(\theta_s \omega) \right) ds \right)^{16} \\ &\leq \frac{2^{32}}{\sqrt[16]{\lambda^{15}}} \int_{-\infty}^0 e^{32\epsilon \int_s^0 |z(\theta_l \omega)| dl + \lambda s} \left( z^{32}(\theta_s \omega) M_1^{32}(\theta_s \omega) + e^{64\epsilon z(\theta_s \omega)} M_1^{96}(\theta_s \omega) \right) ds, \end{aligned}$$

$$\begin{aligned} &\mathbf{E} \int_{-\infty}^0 e^{32\epsilon \int_s^0 |z(\theta_l \omega)| dl + \lambda s} z^{32}(\theta_s \omega) M_1^{32}(\theta_s \omega) ds \\ &\leq 2\mathbf{E} \int_{-\infty}^0 e^{32\epsilon \int_s^0 |z(\theta_l \omega)| dl + \lambda s} [z^{64}(\theta_s \omega) + M_1^{64}(\theta_s \omega)] ds \\ &\leq 2 \left( 2 \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[e^{64\epsilon \int_s^0 z(\theta_l \omega) dl}] ds + \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[z^{128}(\theta_s \omega)] ds + \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[M_1^{128}(\theta_s \omega)] ds \right) \\ &\leq 2 \left[ \frac{2}{\lambda - 64\epsilon} + \frac{\Gamma(\frac{1+128}{2})}{\sqrt{\pi}} + \frac{\hat{K}_{1,32}}{\lambda} \right] \doteq \hat{K}_{4-1}, \\ &\mathbf{E} \int_{-\infty}^0 e^{32\epsilon \int_s^0 |z(\theta_l \omega)| dl + \lambda s} e^{64\epsilon z(\theta_s \omega)} M_1^{96}(\theta_s \omega) ds \\ &\leq 2\mathbf{E} \int_{-\infty}^0 e^{32\epsilon \int_s^0 |z(\theta_l \omega)| dl + \lambda s} [e^{128\epsilon z(\theta_s \omega)} + M_1^{192}(\theta_s \omega)] ds \\ &\leq 2 \left( 2 \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[e^{64\epsilon \int_s^0 z(\theta_l \omega) dl}] ds + \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[e^{256\epsilon z(\theta_s \omega)}] ds + \int_{-\infty}^0 e^{\lambda s} \mathbf{E}[M_1^{384}(\theta_s \omega)] ds \right) \\ &\leq 2 \left[ \frac{2}{\lambda - 64\epsilon} + \frac{1}{\lambda} \left( 1 + \frac{256|\epsilon|}{\sqrt{\pi}} \right) e^{256\epsilon^2} + \frac{\hat{K}_{1,96}}{\lambda} \right] \doteq \hat{K}_{4-2}, \end{aligned}$$

Thus,

$$\mathbf{E}[K_6^{16}(\omega)] \leq \frac{2^{32}}{\sqrt[16]{\lambda^{15}}} \left( \frac{1}{\lambda - 64\epsilon} + \frac{\hat{K}_{1,16}}{\lambda} + \hat{K}_{4-1} + \hat{K}_{4-2} \right) \doteq \bar{K}_4. \quad (3.72)$$

By (3.67)–(3.72), we have

$$\mathbf{E}[C_3^2(\omega)] \leq c_{27} (1 + \bar{K}_1 + \bar{K}_2 + \bar{K}_3 + \bar{K}_4) < \infty.$$

The proof is completed.  $\square$

**Lemma 3.12.** For any  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ , it holds that

$$\begin{cases} \lim_{r \searrow 0} \sup_{v \in \chi(\tau, \omega)} \|\Phi(t, \tau, \omega)v - v\| = 0, \\ \lim_{r \searrow 0} \sup_{v \in \chi(\tau-t, \theta_{-t}\omega)} \|\Phi(0, \tau-t, \theta_{-t}\omega)v - v\| = 0. \end{cases}$$

**Proof.** For every  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$  and  $v_\tau(\omega) \in \chi(\tau, \omega)$ , let

$$\Phi(r, \tau, \omega)v_\tau(\omega) = v(\tau + r, \tau, \theta_{-\tau}\omega, v_\tau(\theta_{-\tau}\omega)) = v(\tau + r) \quad (r \geq 0),$$

then by (h-12), it follows that

$$v(\tau + r) \in B_1(\theta_r\omega) \subset H^1(\mathbb{R}^3), \quad \|v(\tau + r)\|_1 \leq M_1(\theta_r\omega), \quad \forall r \geq 0 \quad (3.73)$$

and

$$\|e^{-\epsilon z(\theta_r\omega)} f(x, e^{\epsilon z(\theta_r\omega)} v(r))\|^2 \leq 2c_1^2 C_0^6 e^{4\epsilon z(\theta_r\omega)} M_1^6(\theta_r\omega) + 2\|\beta_2\|^2 e^{-2\epsilon z(\theta_r\omega)}.$$

From (3.14) and (3.73), we have that for any  $\tau \in \mathbb{R}$ ,  $\omega \in \Omega$ ,  $r \geq 0$ ,

$$\frac{d}{dt} \|\nabla v(\tau + r)\|^2 + \|\Delta v(\tau + r)\|^2 \leq (1 + 2c_2 + 2\epsilon z(\theta_r\omega) - \lambda) M_1^2(\theta_r\omega) + c_5 e^{-2\epsilon z(\theta_r\omega)}. \quad (3.74)$$

Taking  $t \in [0, 1]$ , integrating (3.74) and (3.1) on  $[\tau, \tau + t]$ , we have

$$\begin{aligned} \int_{\tau}^{\tau+t} \|\Delta v(s)\|^2 ds &\leq M_1^2(\omega) + \int_0^1 \left( (1 + 2c_2 + 2\epsilon z(\theta_r\omega) - \lambda) M_1^2(\theta_r\omega) + c_5 e^{-2\epsilon z(\theta_r\omega)} \right) dr \\ &\doteq M_2^2(\omega) \end{aligned} \quad (3.75)$$

and

$$\begin{aligned} &\Phi(t, \tau, \omega)v_\tau(\omega) - v_\tau(\omega) \\ &= v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau(\theta_{-\tau}\omega)) - v_\tau(\theta_{-\tau}\omega) \\ &= \int_{\tau}^{\tau+t} [\Delta v(s) + (\lambda + \epsilon z(\theta_{s-\tau}\omega))v(s)] ds \\ &\quad + \int_{\tau}^{\tau+t} [e^{-\epsilon z(\theta_{s-\tau}\omega)} f(x, e^{\epsilon z(\theta_{s-\tau}\omega)} v(s)) + g(x, s)] ds. \end{aligned} \quad (3.76)$$

It follows from (3.75) and (3.76) that on  $[\tau, \tau + t] \subseteq [\tau, \tau + 1]$ ,



$$\begin{aligned} & \|\Phi(t, \tau, \omega)v_\tau(\omega) - v_\tau(\omega)\| \\ & \leq M_2(\omega)\sqrt{t} + \int_0^t (\lambda + \epsilon z(\theta_s\omega))M_1(\theta_s\omega)ds \\ & \quad + \int_0^t \left(2c_1^2C_0^6e^{4\epsilon z(\theta_s\omega)}M_1^6(\theta_s\omega) + 2\|\beta_2\|^2e^{-2\epsilon z(\theta_s\omega)} + \|g(x, s)\|\right)ds \\ & \xrightarrow{t \searrow 0} 0. \end{aligned}$$

Similarly, it holds that  $\lim_{t \searrow 0} \sup_{v \in \chi(\tau-t, \theta_{-t}\omega)} \|\Phi(0, \tau-t, \theta_{-t}\omega)v - v\| = 0$ . The proof is completed.  $\square$

As a consequence of (h-11)–(h-13), Lemmas 3.9, 3.11–3.12 and Theorem 2.1, we have our main result in this section.

**Theorem 3.2.** Suppose (A1)–(A2) and (3.63) hold. Then  $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$  possesses a random exponential attractor  $\{\mathcal{K}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$  with properties: for any  $\tau \in \mathbb{R}$  and  $\omega \in \Omega$ ,

- (i)  $\mathcal{A}(\tau, \omega) \subseteq \mathcal{K}(\tau, \omega) \subseteq \overline{\chi(\tau, \omega)}$  and  $\mathcal{K}(\tau, \omega)$  is a compact set of  $L^2(\mathbb{R}^3)$  and measurable in  $\omega$ ;
- (ii)  $\Phi(t, \tau, \omega)\mathcal{K}(\tau, \omega) \subseteq \mathcal{K}(t + \tau, \theta_t\omega)$  for all  $t \geq 0$ ;
- (iii) there exists a positive number  $k_0 \in \mathbb{N}$  such that  $\dim_f \mathcal{A}(\tau, \omega) \leq \dim_f \mathcal{K}(\tau, \omega) \leq k_0 < \infty$ ;
- (iv) for any set  $B \in \mathcal{D}(L^2(\mathbb{R}^3))$ , there exist a random variable  $\tilde{T}_\omega \geq 0$  and a tempered random variable  $\check{b}_{\omega, B} > 0$  such that

$$d_h(\Phi(t, \tau, \omega)B(\tau, \omega), \mathcal{K}(t + \tau, \theta_t\omega)) \leq \check{b}_{\omega, B}e^{-\frac{\lambda \ln \frac{4}{3}}{4\lambda + 32 \ln \frac{32}{3}}t}, \quad t \geq T_B + \tilde{T}_\omega;$$

- (v) for any  $\tau \in \mathbb{R}, \omega \in \Omega$ ,  $\lim_{t \rightarrow 0} d_h(\mathcal{K}(\tau + t, \theta_t\omega), \mathcal{K}(\tau, \omega)) = 0$ .

**Proof.** From (3.63) and (3.66),

$$-\frac{\lambda}{2} \leq \mathbf{E}[\epsilon|z(\omega)| - \frac{\lambda}{2}] \leq -\frac{\lambda}{4} < 0.$$

Take  $t = t_0$  in (3.59) and (3.57) as

$$0 < \frac{\lambda + 4 \ln \frac{32}{3}}{\lambda} \leq t_0 = 1 + \frac{2 \ln \frac{3}{32}}{\mathbf{E}[\epsilon|z(\omega)| - \frac{\lambda}{2}]} \leq \frac{\lambda + 8 \ln \frac{32}{3}}{\lambda} < +\infty.$$

Then

$$-\frac{\lambda}{4\lambda + 16 \ln \frac{32}{3}} \leq -\frac{1}{4t_0} \leq -\frac{\lambda}{4\lambda + 32 \ln \frac{32}{3}} < 0.$$

From (3.64),

$$0 < 2\mathbf{E}\left[\left(\epsilon|z(\omega)| - \frac{\lambda}{2}\right)^2\right] + \mathbf{E}[C_3^2(\omega)] < \infty$$

and

$$0 < 2e^{-\frac{2t_0^2}{\ln \frac{3}{2}}\left(2\mathbf{E}\left[\left(\epsilon|z(\omega)| - \frac{\lambda}{2}\right)^2\right] + \mathbf{E}[C_3^2(\omega)]\right)} < +\infty.$$

Let

$$\tilde{\gamma} = \min \left\{ \frac{1}{8}, 2e^{-\frac{2t_0^2}{\ln \frac{3}{2}}\left(2\mathbf{E}\left[\left(\epsilon|z(\omega)| - \frac{\lambda}{2}\right)^2\right] + \mathbf{E}[C_3^2(\omega)]\right)} \right\} \in (0, +\infty)$$

be a bounded fixed positive number. Comparing (2.2) and (3.59), we see that

$$0 < \delta_2 = \tilde{\delta}_{v,R,m} = 2\sqrt{\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt[4]{\lambda}}\right)I_{v,R}} + \frac{2}{\sqrt[4]{\lambda} + \mu_{m+1,R}}, \quad (3.77)$$

and

$$\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt[4]{\lambda}}\right)I_{v,R} = \left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt[4]{\lambda}}\right)(v + \tilde{I}_R),$$

where

$$\tilde{I}_R = \frac{1}{R^2} + \frac{1}{R} + \gamma_{\frac{R}{2}} + \tilde{\gamma}_{\frac{R}{2}} + \beta_{4,R}. \quad (3.78)$$

Take  $v = v_0$  in (3.77) small enough such that  $\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt[4]{\lambda}}\right)v_0 \leq \frac{\tilde{\gamma}^2}{32}$ . By (A2) and (3.78),  $\tilde{I}_R \xrightarrow{R \rightarrow +\infty} 0$ , then there exists a large number  $\tilde{R} > 2$  such that  $\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt[4]{\lambda}}\right)\tilde{I}_{\tilde{R}} \leq \frac{\tilde{\gamma}^2}{32}$ . Thus,

$$2\sqrt{\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt[4]{\lambda}}\right)I_{v_0,\tilde{R}}} \leq \tilde{\gamma}.$$

For this fixed  $\tilde{R}$ , by (3.43),  $\mu_{m,\tilde{R}} \xrightarrow{m \rightarrow +\infty} +\infty$ , so, there exists a large finite integer  $m_0 = m_0(\tilde{R}) \in \mathbb{N}$  such that  $\frac{2}{\sqrt[4]{\lambda} + \mu_{m_0+1,\tilde{R}}} \leq \frac{\tilde{\gamma}}{2}$ . Hence,

$$0 < \tilde{\delta}_{v_0,\tilde{R},m_0} \leq \tilde{\gamma}.$$

Thus Theorem 2.1 implies the statements in Theorem 3.2 with

$$k_0 = \left\lceil \frac{2m_0 \ln \left( \frac{2\sqrt{m_0}}{\delta_{v_0, \tilde{R}, m_0}} + 1 \right)}{\ln \frac{4}{3}} \right\rceil + 1,$$

where  $[\cdot]$  denotes the integer part of number. The proof is completed.  $\square$

**Remark 3.1.** By the same method, we can prove the existence of a random exponential attractor for the following initial problem of stochastic non-autonomous reaction–diffusion equation with additive white noise in  $\mathbb{R}^3$ :

$$\begin{cases} du + (\lambda u - \Delta u)dt = (f(x, u) + g(x, t))dt + h(x)dW(t), & t > \tau \\ u(x, \tau) = u_\tau(x), & x \in \mathbb{R}^3, \tau \in \mathbb{R}, \end{cases} \quad (3.79)$$

when  $h \in H^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$  and  $g, f$  satisfy **(A1)**–**(A2)**, where the random term in (3.79) is understood in the Itô sense. But we should notice that for the system (1.1), we require that the coefficient  $\epsilon$  of the random term is small (see (3.63)), but for the system (3.79), we don't need such a restricted condition, this is because that the multiplicative noise depends on the state variable  $u$  but the additive noise term is independent of  $u$ .

**Remark 3.2.** The similar results in this paper are also true for the problems (1.1) and (3.79) in other dimensional whole spaced  $\mathbb{R}^p$  ( $p \in \mathbb{N}$ ) when the nonlinearity  $f(u, x)$  satisfies some conditions like **(A2)**.

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