



Solvability of singular integro-differential equations via Riemann–Hilbert problem [☆]

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Abstract

The article is to study singular integro-differential equations involving convolutional operators and Cauchy integral operators via Riemann–Hilbert problem. To do this, we adopt a new approach through Fourier transform on L^2 subspace which is Hölder-continuous with a certain decay at infinity. The Fourier transform converts the equations into a Riemann–Hilbert problem with Hölder-continuous coefficients and with nodal points, which allows us to construct the general solutions.

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1. Introduction

The theory of singular integro-differential equations has been widely studied. We refer to [13] for the pioneer work of the topic and [20,16] for the fundamental theory. In this area there arises many approaches, such as the numerical approach [10], Lie group approach [6], and the Fredholm operator approach [22]. Recently, much attention is focused on equations of convolution kernels with or without continuous coefficients; see ([1,21,3,14,15,19,27]).

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The purpose of this article is to study the singular integro-differential equations with convolution kernels of the form:

$$\begin{aligned} \sum_{j=0}^n \left\{ a_j \omega^{(j)}(t) + \frac{b_j}{\pi i} \int_{\mathbb{R}} \frac{\omega^{(j)}(s)}{s-t} ds + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} k_{j,1}(t-s) \omega^{(j)}(s) ds \right. \\ \left. + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} h_{j,1}(t-s) \mathcal{T} \omega^{(j)}(s) ds + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^-} k_{j,2}(t-s) \omega^{(j)}(s) ds \right. \\ \left. + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^-} h_{j,2}(t-s) \mathcal{T} \omega^{(j)}(s) ds \right\} = d(t), \quad \forall t \in \mathbb{R}. \end{aligned}$$

Here ω is the unknown function and $\omega^{(j)}$ is its derivative of order j , \mathcal{T} is the Cauchy integral transform.

Such kind of equations appear frequently in practical problems, such as fluid dynamics, shell theory, underwater acoustics, elasticity theory, and quantum mechanics ([4,2,12]).

We shall adopt a new approach in this article. We shall use Fourier transforms to the subspace of $L^2(\mathbb{R})$ in which functions are Hölder-continuous with a certain decay at infinity. This allows us to transform the problems of solving equations into boundary value problems for analytic function with discontinuous coefficients. By means of the classical Riemann boundary value problem, and of the principle of analytic continuation, we can prove the existence of the solution under certain conditions.

This work is organized as follows. In section 2 we introduce the function classes $\{0\}$ and $\{\{0\}\}$ and study the properties of the Fourier transforms and Cauchy transforms acting on them. In section 3 we adopt the Fourier transform approach to convert the equation into a Riemann–Hilbert problem with Hölder-continuous coefficients and with nodal points. In section 4 we establish the Plemelj formula for Fourier transform. This plays a crucial role in converting the equation into the boundary value problems. Finally, in section 5 we use the Cayley transform to identify the real line as a circle so that the Plemelj formula can be applied to study the Riemann–Hilbert problem. This allows us to show that the equation can be solved under certain conditions.

2. Fourier transform and Cauchy transform in Hölder spaces

Some notation and useful lemmas are given in this section. See [7,8,22,26].

Definition 2.1. Let \widehat{H} stand for the Hölder space in the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, in the sense that there exist positive constants A , M , and $\mu \in (0, 1]$ such that

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq A |x_1 - x_2|^\mu, & \forall x_1, x_2 \in [-M, -M]; \\ |f(x_1) - f(x_2)| &\leq A \left| \frac{1}{x_1} - \frac{1}{x_2} \right|^\mu, & \forall x_1, x_2 \in \mathbb{R} \setminus [-M, -M]. \end{aligned}$$

Definition 2.2. Let H stand for the Hölder space in the circle

$$\Gamma = \{z \in \mathbb{C} : |z + i/2| = 1/2\}.$$

Namely, there exist positive constants A and $\mu \in (0, 1]$ such that

$$|f(x_1) - f(x_2)| \leq A |x_1 - x_2|^\mu, \quad \forall x_1, x_2 \in \Gamma.$$

We remark that the two function class \widehat{H}, H are close related by the Cayley transform. The Cayley transform

$$\mathcal{C} : \overline{\mathbb{R}} \longrightarrow \Gamma,$$

is an involution, assigning each $x \in \overline{\mathbb{R}}$ to

$$\mathcal{C}x = -\frac{ix}{x+i} \in \Gamma.$$

Set

$$\mathcal{C}^* f = f \circ \mathcal{C}^{-1}.$$

It induces the bijection

$$\mathcal{C}^* : \widehat{H} \longrightarrow H$$

as well as the bijection

$$\mathcal{C}^* : L^2(\mathbb{R}) \longrightarrow L^2(\Gamma, d\mu),$$

where $d\mu$ denotes the measure

$$d\mu(\xi) = |\xi + i|^{-2} d\xi.$$

Therefore, it gives rise to a bijection

$$\mathcal{C}^* : \widehat{H} \cap L^2(\mathbb{R}) \longrightarrow H \cap L^2(\Gamma, d\mu).$$

The space $\{\{0\}\}$ denote the space of functions in $L^2(\mathbb{R})$ which are Hölder continuous on the real axis as well as at $\pm\infty$. $\{0\}$ denotes the space of inverse Fourier transforms of functions in $\{\{0\}\}$. See [5].

Definition 2.3. Let \widehat{H} be the Hölder spaces in $\overline{\mathbb{R}}$. We consider its square-integrable subspace

$$\{\{0\}\} := \widehat{H} \cap L^2(\mathbb{R}).$$

Under the Fourier inverse transform in $L^2(\mathbb{R})$, it correspondences to the space

$$\{0\} := \mathcal{F}^{-1}(\{\{0\}\}).$$

Under the Cayley transform, it correspondences to the space

$$\{\{0\}\}_\Gamma := H \cap L^2(\mathbb{R}, d\mu).$$

Here, the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is defined via

$$\mathcal{F}f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{ixt} dt, \quad \forall f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}).$$

By the theorem of bounded linear transform [24, P.9], we know that the Fourier transform can be extended to be a bounded bijection in $L^2(\mathbb{R})$. We shall formally denote this extension by

$$\mathcal{F}f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{ixt} dt, \quad \forall f \in L^2(\mathbb{R}).$$

We remark that all the three spaces from Lemma 2.3 are Hilbert spaces and the Cayley transform \mathcal{C} induced a bijection

$$\mathcal{C}^* : \{\{0\}\} \longrightarrow \{\{0\}\}_\Gamma.$$

A remarkable property of the Hölder space $\{\{0\}\}$ in $\overline{\mathbb{R}}$ is that it admits an involution operator via the operator of Cauchy principal value integral

$$\mathcal{T}f(t) = \text{P.V.} \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - t} d\tau, \quad \forall t \in \mathbb{R}.$$

In fact, we shall take a weaker Cauchy principal value integral in the sense that

$$\mathcal{T}f(t) = \lim_{\substack{R \rightarrow +\infty \\ \epsilon \rightarrow +0}} \frac{1}{\pi i} \int_{[-R, t-\epsilon] \cup [t+\epsilon, R]} \frac{f(\tau)}{\tau - t} d\tau, \quad \forall t \in \mathbb{R}.$$

As have been shown in Theorem 1.5.2 of [18] that \mathcal{T} is a self-map of the function class \widehat{H} under this modified concept of Cauchy principal value integral. From now on we shall omit the symbol P.V. and write the above integral simply as

$$\mathcal{T}f(t) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - t} d\tau, \quad \forall t \in \mathbb{R}.$$

Up to a constant, this is exactly the Hilbert transform. By the Riesz theorem, we knew that \mathcal{T} is an involution in the space $L^2(\mathbb{R})$.

Lemma 2.1. *Let \mathcal{T} be the operator of Cauchy principal value integral. Then \mathcal{T} is an involution self-map of $\{\{0\}\}$.*

Proof. The result follows directly from the facts that \mathcal{T} is an involution self-map in $L^2(\mathbb{R})$ as well as a self-map in $\{\{0\}\}$. \square

We now consider the behavior of the derivative of a function under the Fourier transform in the space $\{0\}$. For any function $\omega \in \{0\}$, we shall denote

$$\omega_{\pm}(t) = \frac{1}{2}(\operatorname{sgn} t \pm 1)\omega(t),$$

where $\operatorname{sgn} t$ denotes the classical sign function.

Lemma 2.2. *Under the assumption that all the derivative of ω up to the n -th order are all in the space $\{0\}$, i.e., $\omega^{(j)} \in \{0\}$ for any $j = 0, 1, 2, \dots, n$, we have*

$$\mathcal{F}[(\omega^{(j)})_{\pm}](x) = (-ix)^j \mathcal{F}[\omega_{\pm}](x) - \frac{1}{\sqrt{2\pi}} \sum_{m=0}^j (-ix)^m \omega^{(j-m)}(0).$$

Proof. Notice that

$$\mathcal{F}[\omega_{\pm}](\infty) = 0.$$

By induction on j , the result follows from the integration by parts. \square

Lemma 2.3. *Let $\omega^{(j)} \in \{0\}$ for every $j = 0, 1, 2, \dots, n$. Then*

$$\mathcal{F}[\omega^{(j)}](t) = (-ix)^j \mathcal{F}[\omega_{\pm}](x).$$

Proof. By Lemma 2.2 we have

$$\begin{aligned} \mathcal{F}[\omega^{(j)}](x) &= \mathcal{F}[\omega_+^{(j)}](x) - \mathcal{F}[\omega_-^{(j)}](x) \\ &= (-ix)^j \mathcal{F}[\omega_+](x) - (-ix)^j \mathcal{F}[\omega_-](x) \\ &= (-ix)^j \mathcal{F}[\omega](x), \end{aligned}$$

as desired. \square

Lemma 2.4. *Let $\omega \in \{0\}$ with $\mathcal{F}[\omega](0) = 0$. Then $\mathcal{T}\omega(t) \in \{0\}$ and*

$$\mathcal{F}[\mathcal{T}\omega](x) = -\operatorname{sgn} x \mathcal{F}[\omega](x).$$

Proof. By assumption, we have $\mathcal{F}[\omega] \in \{\{0\}\}$. Since $\mathcal{F}[\omega](\infty) = \mathcal{F}[\omega](0) = 0$, it follows that

$$\mathcal{F}[\omega](x) \operatorname{sgn} x \in \{\{0\}\}.$$

In this case we already know from [3] that

$$\mathcal{F}[\mathcal{T}\omega](x) = -\operatorname{sgn} x \mathcal{F}[\omega](x).$$

Combining the above results together, we have $\mathcal{F}[\mathcal{T}\omega(t)] \in \{\{0\}\}$ so that $\mathcal{T}\omega(t) \in \{0\}$. \square

Lemma 2.5. Assume ω and all of its derivatives up to order n are all in the space $\{0\}$ and assume $\mathcal{F}\omega(0) = 0$. We then have

$$\mathcal{F}[\mathcal{T}\omega^{(j)}](x) = -(-ix)^j \operatorname{sgn} x \mathcal{F}[\omega](x).$$

Proof. The result follows directly from Lemmas 2.3 and 2.4. \square

3. Plemelj formula of Fourier transform in L^2

We refer to [25, 11] for the theory of Hardy analytic spaces in the upper half plane.

For any $\omega \in L^2(\mathbb{R})$, we consider its Fourier transform $\mathcal{F}(\omega) \in L^2(\mathbb{R})$. The latter function can be regarded as a Hardy analytic function Ω in the upper half open plane \mathbb{C}^+ . Namely, $\Omega \in H^2(\mathbb{C}^+)$, the Hardy space in \mathbb{C}^+ , and its non-tangential limits at the boundary \mathbb{R} recovers $\mathcal{F}(\omega)$.

Now we assume $\omega \in C_0^\infty(\mathbb{R})$. The Paly–Wiener–Schwartz theorem [9, Theorem 7.3.1] shows that $\mathcal{L}\omega$, the Laplace transform of ω , is an entire analytic function in \mathbb{C} of exponential growth. In the upper half space, the Laplace transform is defined as

$$\mathcal{L}\omega(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} \omega(t) e^{izt} dt, \quad \forall z \in \mathbb{C}^+.$$

Its non-tangential boundary value in \mathbb{R} is given by

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} \omega(t) e^{ixt} dt, \quad \forall x \in \mathbb{R},$$

which is the Fourier transform of $f\chi_{\mathbb{R}}$. Here $\chi_{\mathbb{R}}$ denotes the characteristic function of \mathbb{R} . Therefore, the Laplace transform gives rise to a densely defined bounded linear operator

$$\mathcal{L} : L^2(\mathbb{R}) \longrightarrow H^2(\mathbb{C}^+).$$

Again by the theorem of bounded linear transform [24, P.9], the Laplace transform can be boundedly extended to the whole $L^2(\mathbb{R})$. We shall make a convention to denote formally the Laplace transform in the upper half space by

$$\mathcal{L}\omega(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} \omega(t) e^{izt} dt, \quad \forall \omega \in L^2(\mathbb{R}), \quad \forall z \in \mathbb{C}^+.$$

It is well-known that if $\omega \in L^2(\mathbb{R})$, then $\mathcal{F}(\omega) \rightarrow 0$ as $|x| \rightarrow \infty$ by the Riemann–Lebesgue lemma.

If $\Omega \in H^2(\mathbb{C}^+)$ and $\Omega(\infty) := \lim_{z \rightarrow \infty} \Omega(z) = 0$, then the residue theorem shows that

$$\Omega(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\Omega(t)}{t - z} dt.$$

We claim that

$$\Omega(z) = \mathcal{L}\omega(z), \quad \forall z \in \mathbb{C}^+,$$

whenever $\Omega(z)$ is the Hardy analytic function in $H^2(\mathbb{C}^+)$ corresponding to $\mathcal{F}\omega \in L^2(\mathbb{R})$ for any given $\omega \in L^2(\mathbb{R})$. Since both sides are functions from $H^2(\mathbb{C}^+)$, their non-tangential boundary value are all in $L^2(\mathbb{R})$ and equal by checking their inverse Fourier transform.

As a result, we let $\Omega \in H^2(\mathbb{C}^+)$ be the function such that its non-tangential boundary value is exactly $\mathcal{F}(\omega)$ for any given $\omega \in L^2(\mathbb{R})$, then the function

$$\Omega(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mathcal{F}(\omega)(t)}{t - z} dt \in H^2(\mathbb{C}^+).$$

Moreover, its boundary value is given by

$$\Omega^+(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \omega_+(t) e^{ixt} dt, \quad \forall x \in \mathbb{R}.$$

Similarly, one can consider the lower half plane to deduce that

$$\Omega(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mathcal{F}(\omega)(t)}{t - z} dt \in H^2(\mathbb{C}^-).$$

Also its boundary value is given by

$$\Omega^-(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \omega_-(t) e^{ixt} dt, \quad \forall x \in \mathbb{R}.$$

Consequently, if $\omega \in L^2(\mathbb{R})$, then

$$\Omega(z) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mathcal{F}(\omega)(t)}{t - z} dt \in H^2(\mathbb{C}^+ \sqcup \mathbb{C}^-)$$

such that their boundary values Ω^\pm from upper and lower half plane satisfies the equation

$$\Omega^+(x) - \Omega^-(x) = F(x), \quad \forall x \in \mathbb{R}.$$

Here

$$\Omega^\pm = \mathcal{F}(\omega_\pm), \quad F = \mathcal{F}(\omega)$$

for any given $\omega \in L^2(\mathbb{R})$. This can be regarded as the Plemelj formula of Fourier transform.

If $\omega \in \{0\}$ with $\mathcal{F}[\omega](0) = 0$, then $\omega_+ \in \{0\}$ and $\omega_- \in \{0\}$. In this case we already know that $\Omega(z) \in H^2(\mathbb{C}^+ \sqcup \mathbb{C}^-)$. Furthermore, $\Omega(z)$ is Hölder continuous in $\mathbb{C}^+ \sqcup \mathbb{C}^-$ as shown in [18].

Conversely, if both $\mathcal{F}(\omega_+)(0) = 0$ and $\mathcal{F}(\omega_-)(0) = 0$, then $\mathcal{F}(\omega)(0) = 0$. And in this case if both $\omega_+ \in \{0\}$ and $\omega_- \in \{0\}$, then $\omega \in \{0\}$.

4. Solvability of singular integro-differential equation

4.1. Singular integro-differential equation

We shall now study the following singular integro-different equation involving convolutional operators and Cauchy integral operators:

$$\begin{aligned} \sum_{j=0}^n \left\{ a_j \omega^{(j)}(t) + \frac{b_j}{\pi i} \int_{\mathbb{R}} \frac{\omega^{(j)}(s)}{s-t} ds + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} k_{j,1}(t-s) \omega^{(j)}(s) ds \right. \\ \left. + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} h_{j,1}(t-s) \mathcal{T} \omega^{(j)}(s) ds + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^-} k_{j,2}(t-s) \omega^{(j)}(s) ds \right. \\ \left. + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^-} h_{j,2}(t-s) \mathcal{T} \omega^{(j)}(s) ds \right\} = d(t), \quad \forall t \in \mathbb{R} \end{aligned} \quad (1)$$

Some explanations are needed for this equation.

In Eq. (1), all the integral kernels $k_{j,p}(t)$, $h_{j,p}(t)$, the function $d(t)$, and the constants a_j, b_j are given for any $j = 0, 1, 2, \dots, n$ and $p = 1, 2$. We take $\omega(t)$ to be the unknown function. Recall that $\omega^{(j)}(t)$ denotes the j -th derivative of $\omega(t)$ and \mathcal{T} is the operator of Cauchy principal value integral.

We shall make some restrictions on the given data:

Assumption A. We assume

$$\sum_{j=1}^n |b_j| \neq 0, \quad h_{j,p}(t), k_{j,p}(t), d(t) \in \{0\}$$

for all $j = 0, 1, 2, \dots, n$ and $p = 1, 2$.

Under Assumption A, we are intended to find such a solution ω that $\omega^{(j)} \in \{0\}$ for every $j = 0, 1, 2, \dots, n$ as well as $\mathcal{F}\omega(0) = 0$. If such a solution exists, we call that the singular integro-differential equation (1) admits a solution in the class $\{0\}$.

We can rewrite Eq. (1) as

$$\begin{aligned} \sum_{j=0}^n \left\{ a_j \omega^{(j)}(t) + \frac{b_j}{\pi i} \int_{\mathbb{R}} \frac{\omega^{(j)}(s)}{s-t} ds + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k_{j,1}(t-s) \omega_+^{(j)}(s) ds \right. \\ \left. + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h_{j,1}(t-s) \mathcal{T} \omega_+^{(j)}(s) ds - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k_{j,2}(t-s) \omega_-^{(j)}(s) ds \right. \\ \left. - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h_{j,2}(t-s) \mathcal{T} \omega_-^{(j)}(s) ds \right\} = d(t), \quad \forall t \in \mathbb{R}. \end{aligned} \quad (2)$$

4.2. Fourier transform and Riemann–Hilbert problem

Now we take the Fourier transforms and apply Lemmas 2.2 and 2.4. It turns out that Eq. (2) is equivalent to the following equation:

$$\Omega^+(x) = S(x)\Omega^-(x) + W(x), \quad x \in \mathbb{R}. \quad (3)$$

This is exactly a Riemann–Hilbert problem on an infinite straight line.

Here we denote

$$\begin{aligned} \Omega^\pm &= \mathcal{F}[\omega_\pm], \\ S(x) &= \frac{\sum_{j=0}^n (-i)^j (a_j - b_j \operatorname{sgn} x + K_{j,2}(x) - H_{j,2}(x) \operatorname{sgn} x) x^j}{\sum_{j=0}^n (-i)^j (a_j - b_j \operatorname{sgn} x + K_{j,1}(x) - H_{j,1}(x) \operatorname{sgn} x) x^j}, \\ W(x) &= \frac{D(x) + W_1(x)}{\sum_{j=0}^n (-i)^j (a_j - b_j \operatorname{sgn} x + K_{j,1}(x) - H_{j,1}(x) \operatorname{sgn} x) x^j}, \end{aligned}$$

where

$$W_1(x) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^n \left(\sum_{m=0}^{j-1} (-ix)^m \right) (K_{j,1}(x) - K_{j,2}(x) - (H_{j,1}(x) - H_{j,2}(x)) \operatorname{sgn} x),$$

and

$$D(x) = \mathcal{F}[d](x), \quad K_{j,p}(x) = \mathcal{F}[k_{j,p}](x), \quad H_{j,p}(x) = \mathcal{F}[h_{j,p}](x)$$

for any $j = 0, 1, 2, \dots, n$ and $p = 1, 2$.

Notice that by Assumption A we have $h_{j,p}(t), k_{j,p}(t), d(t) \in \{0\}$ so that all of their Fourier transforms $K_{j,p}$, $H_{j,p}$, and $D(x)$ belong to the class $\{\{0\}\}$. This means $K_{j,p}$, $H_{j,p}$, and $D(x)$ are Hölder continuous in \mathbb{R} .

Notice that in the definition of the functions $W(x)$ and $S(x)$, they share the same denominator. We denote it by $S_1(x)$, i.e.,

$$S_1(x) := \sum_{j=0}^n (-i)^j (a_j - b_j \operatorname{sgn} x + K_{j,1}(x) - H_{j,1}(x) \operatorname{sgn} x) x^j.$$

It is evident that $S_1(x)$ is continuous in \mathbb{R} except the origin. To make the Riemann–Hilbert problem (3) meaningful, we need to make an assumption:

Assumption B. We assume $S_1(x)$ has only a finite number of zeros in \mathbb{R} .

We remark that the Riemann boundary value problem (3) can be directly solved by the methods in [17] under certain conditions through Fredholm integral equations (see also Muskhelishvili [22]). But in this paper we shall apply Fourier theory to solve (3), which may enable us to deal with other equations.

4.3. Cayley transform and Riemann Hilbert problem in a circle

We shall now consider the Cayley transform

$$z = -\frac{i\xi}{\xi + i}, \quad \forall \xi \in \mathbb{C}.$$

This transform maps the real axis \mathbb{R} onto a circle

$$\Gamma := \partial B(-\frac{i}{2}, \frac{1}{2})$$

which surrounds an interior region Σ^+ and an exterior region Σ^- , and maps the upper half-plane \mathbb{C}^+ and lower half-plane \mathbb{C}^- onto the conformal Σ^+ , Σ^- , respectively. Here we denote by $B(a, s)$ the open ball centered at a with radius s .

Now we restrict the Cayley transform to the boundary. The Cayley transform thus becomes

$$x = -\frac{i\tau}{\tau + i}, \quad \forall \tau \in \Gamma, \quad x \in \mathbb{R}.$$

This Cayley transform induce a corresponding transform on functions in the real axis \mathbb{R} into the functions in the circle Γ via

$$F^\pm(\tau) = \Omega^\pm(x), \quad \forall \tau \in \Gamma.$$

This also allows us to translate the Riemann–Hilbert problem (3) on \mathbb{R} into that on the circle Γ .

Under this Cayley transform, the Riemann–Hilbert problem (3) thus can be written equivalently as

$$F^+(\tau) = P(\tau)F^-(\tau) + Q(\tau), \quad \forall \tau \in \Gamma. \quad (4)$$

Here

$$Q(\tau) = W(x), \quad P(\tau) = S(x).$$

By calculation, $Q(\tau)$ and $P(\tau)$ can be expressed by the given data explicitly as

$$Q(\tau) = \frac{C(\tau) + \frac{1}{\sqrt{2\pi}} \sum_{j=1}^n [\sum_{m=0}^{j-1} (\frac{-\tau}{\tau+i})^m] [E_{j,1}(\tau) - E_{j,2}(\tau) - (C_{j,1}(\tau) - C_{j,2}(\tau))\delta(\tau)]}{\sum_{j=0}^n (-\frac{\tau}{\tau+i})^j [a_j - b_j\delta(\tau) + E_{j,1}(\tau) - C_{j,1}(\tau)\delta(\tau)]},$$

$$P(\tau) = \frac{\sum_{j=0}^n (-1)^j [a_j - b_j\delta(\tau) + E_{j,2}(\tau) - C_{j,2}(\tau)\delta(\tau)] (\frac{\tau}{\tau+i})^j}{\sum_{j=0}^n (-1)^j [a_j - b_j\delta(\tau) + E_{j,1}(\tau) - C_{j,1}(\tau)\delta(\tau)] (\frac{\tau}{\tau+i})^j}.$$

Here, we denote

$$E_{j,p}(\tau) = K_{j,p}(x), \quad C_{j,p}(\tau) = H_{j,p}(x), \quad C(\tau) = D(x),$$

and denote

$$\delta(\tau) = \begin{cases} 1, & \tau \in \Gamma_1, \\ -1, & \tau \in \Gamma_2 \cup \{0, -i\}, \end{cases}$$

with Γ_1 and Γ_2 being the left and the right half open circles of Γ , respectively.

In this paper we only try to solve (4) in the irregular case. This means that $P(\tau)$ is permitted to have some zero-points and pole-points on Γ .

By the definition of $P(\tau)$, we can rewrite it as

$$P(\tau) = \frac{P_2(\tau)}{P_1(\tau)},$$

where

$$P_1(\tau) = \sum_{j=0}^n (-1)^j [a_j - b_j \delta(\tau) + E_{j,1}(\tau) - C_{j,1}(\tau) \delta(\tau)] \left(\frac{\tau}{\tau+i}\right)^j$$

$$P_2(\tau) = \sum_{j=0}^n (-1)^j [a_j - b_j \delta(\tau) + E_{j,2}(\tau) - C_{j,2}(\tau) \delta(\tau)] \left(\frac{\tau}{\tau+i}\right)^j$$

We now come to make an assumption on $P_1(\tau)$ and $P_2(\tau)$ that each of them has only finite zeros and pole-points on Γ .

Assumption C. We assume that both $P_1(\tau)$ and $P_2(\tau)$ have only finite zeros on Γ .

4.4. Riemann–Hilbert problem in a circle

Under the preceding assumptions, we try to solve the Riemann–Hilbert problem (4). In this case, (4) can be rewritten as

$$F^+(\tau) = \frac{\Pi_2(\tau)}{\Pi_1(\tau)} P_0(\tau) F^-(\tau) + Q(\tau), \quad \tau \in \Gamma. \quad (5)$$

Here, we assume $P_1(\tau)$ have zero points u_1, u_2, \dots, u_s with orders $\alpha_1, \alpha_2, \dots, \alpha_s$ respectively on Γ ; and $P_2(\tau)$ has some zero points v_1, v_2, \dots, v_l with the orders $\beta_1, \beta_2, \dots, \beta_l$ respectively on Γ . We further assume that they share the common and the same order zero points w_1, w_2, \dots, w_q with the orders r_1, r_2, \dots, r_q respectively on Γ .

We can thus rewrite $P(\tau)$ as

$$P(\tau) = \frac{P_2(\tau)}{P_1(\tau)} = \frac{\Pi_2(\tau)}{\Pi_1(\tau)} P_0(\tau),$$

where

$$\Pi_1(\tau) = \prod_{j=1}^s (\tau - u_j)^{\alpha_j}, \quad \Pi_2(\tau) = \prod_{j=1}^l (\tau - v_j)^{\beta_j}.$$

The point here is that $P_0(\tau)$ is non-vanishing and has no singular-points on the whole Γ except points 0 and $-i$.

Notice that the following statements are equivalent

- (i) $\omega^{(n)}(t) \in \{0\}$,
- (ii) $\Omega(x) \in \{\{0\}\}$,
- (iii) $F(\tau) \in \{\{0\}\}_\Gamma$.

We gather the above statements in a result.

Theorem 4.1. *Suppose Assumptions A–C holds true. If the Riemann–Hilbert problem (4) admits a solution in $\{\{0\}\}_\Gamma$ with $F^\pm(0) = 0$, then the singular integro-differential equation (1) admits a solution in the class $\{0\}$.*

5. Riemann–Hilbert problem

Theorem 4.1 reduces the original problem to finding a solution $F(\tau)$ of Riemann–Hilbert problem (4) in $\{\{0\}\}_\Gamma$.

Due to Hölder continuous assumption, we shall investigate a solution $F(\tau)$ of (4) with the property that it is continuous and bounded near all w_j ($1 \leq j \leq q$). We thus need some conditions of solvability.

Denote by $Q_2(\tau)$ the numerator in the definition of the function of $Q(\tau)$, i.e.,

$$Q_2(\tau) = C(\tau) + \frac{1}{\sqrt{2\pi}} \sum_{j=1}^n \left(\sum_{m=0}^{j-1} (-1)^m \left(\frac{\tau}{\tau+i} \right)^m \right) \times \\ (E_{j,1}(\tau) - E_{j,2}(\tau) - C_{j,1}(\tau)\delta(\tau) + C_{j,2}(\tau)\delta(\tau)) \quad (6)$$

Assumption D. We assume

- (i) The derivatives $C(\tau)$, $E_{j,p}(\tau)$, $C_{j,p}(\tau)$ ($p = 1, 2$) exist up to order $r_j - 1$ ($1 \leq j \leq q$) in a neighborhood of w_j , and all these derivatives satisfy the Hölder conditions.
- (ii) All the derivatives of $Q_2(\tau)$ vanishes at points w_j up to order $r_j - 1$ for any $j = 1, \dots, q$.

5.1. Nodal points

We come to consider the nodal points of the Riemann boundary value problem in (5). The nodal points of (5) depend on the values of $a_j \pm b_j$ ($j = 0, 1, \dots, n$) and given by

$$\tau = \begin{cases} 0, -i & \text{if all } a_j - b_j = 0, a_j + b_j \neq 0; \\ 0, & \text{if all } a_j - b_j \neq 0, a_j + b_j \neq 0; \\ -i & \text{if all } a_j - b_j \neq 0, a_j + b_j = 0; \\ \text{none} & \text{if all } a_j - b_j = 0, a_j + b_j = 0. \end{cases}$$

In this article we shall restrict our attention to the case that all $a_j - b_j = 0$, $a_j + b_j \neq 0$, since the approach remain valid in the remain cases.

Assumption E. We assume

$$a_j - b_j = 0, \quad a_j + b_j \neq 0, \quad \text{all } j.$$

Denote the nodes of the problem (5) by

$$C_1 = -i, \quad C_2 = 0.$$

Following the method used in [15,16], we find that near C_k ($k = 1, 2$) the function $P_0(\tau)$ in (5) can be written as

$$P_0(\tau) = (\tau - C_k)^{\alpha_k} P_k(\tau),$$

where

$$\alpha_k = \begin{cases} \beta_k^{(1)}, & \tau \in \Gamma_1; \\ \beta_k^{(2)}, & \tau \in \Gamma_2, \end{cases} \quad P_k(\tau) = \begin{cases} P_k^{(1)}(\tau), & \tau \in \Gamma_1; \\ P_k^{(2)}(\tau), & \tau \in \Gamma_2 \end{cases}$$

with $\beta_k^{(j)}$ ($k, j = 1, 2$) being real numbers and $P_k(\tau) \neq 0$ ($k = 1, 2$) on Γ .

5.2. Plemelj formula

Since $P_k(\tau) \neq 0$ ($k = 1, 2$), we can take a single-valued continuous branch of $\log P_0(\tau)$ on Γ . Now we introduce the following piece-wise holomorphic function:

$$\Upsilon(\xi) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\log P_0(\tau)}{\tau - \xi} d\tau, \quad \xi \notin \Gamma. \quad (7)$$

Note that $\Upsilon(\xi)$ is analytic in $\Sigma^+ \cup \Sigma^-$, and $\Upsilon(\infty) = 0$. In general, $\Upsilon(\xi)$ has a singularity of logarithmic type at $\xi = 0, -i$.

By applying the Plemelj formula for $\Upsilon(\xi)$ in (7), near C_k ($k = 1, 2$) we have

$$\Upsilon^{\pm}(\xi) = \Psi_k^{\pm}(\xi) \log(\xi - C_k) + \Delta_k^{\pm}(\xi). \quad (8)$$

Here we denote, for any $\xi \in S_k^{\pm} \setminus \{C_k\}$,

$$\Psi_k^{\pm}(\xi) = \frac{(\beta_k^{(1)} - \beta_k^{(2)})\theta_k(\xi)}{2\pi} + \frac{\arg P_k^{(2)}(C_k - 0) - \arg P_k^{(1)}(C_k + 0)}{2\pi} + \frac{\beta_k^{(2)} \pm \beta_k^{(1)}}{2},$$

$$\theta_k(\xi) = \arg(\xi - C_k),$$

where

$$S_k^+ = \Sigma^+ \cap G_k, \quad S_k^- = \Sigma^- \cap G_k$$

for some sufficiently small neighborhood G_k of C_k ($k = 1, 2$). And $\Delta_k^+(\xi)$ and $\Delta_k^-(\xi)$ are the functions with their real parts being bounded in S_k^+ and S_k^- .

Suppose that the tangential direction of Γ at C_k is the same as the forward direction of Γ . It is readily seen that, when z (near C_k) moves along the positive direction around C_k , according to the values of $\Psi_k^\pm(C_k \pm 0)$, we have the following two possible cases.

For any $k = 1, 2$, we set

$$A_k = \min\{\Psi_k^+(C_k + 0), \Psi_k^+(C_k - 0), \Psi_k^-(C_k + 0), \Psi_k^-(C_k - 0)\},$$

where the definitions of $\Psi_k^\pm(C_k \pm 0)$ are the same as that of $P(-i \pm 0)$. We denote

$$\lambda_k = -[A_k]$$

and define the index of (5) to be the integer

$$\mu = -(\lambda_1 + \lambda_2).$$

When A_k is an integer, we call C_k a special node of (5), otherwise, an ordinary node.

We consider a canonical function:

$$X(\xi) = (\xi - C_1)^{\lambda_1} (\xi - C_2)^{\lambda_2} e^{\Upsilon(\xi)}, \quad \xi \in \Sigma^\pm. \quad (9)$$

Taking the boundary values for $X(\xi)$ in (3.9), and using the principle of analytic continuation, we obtain

$$X^\pm(z) = (z - C_k)^{\Psi_k^\pm(z)} M(z) \prod_{j=1}^2 (z - C_j)^{\lambda_j}, \quad k = 1, 2, \quad (10)$$

where $M(z)$ and $M^{-1}(z)$ are bounded functions in a neighborhood of C_k .

In view of (10), we have

$$X^+(\tau) = P_0(\tau) X^-(\tau). \quad (11)$$

5.3. Homogeneous equation

We now consider the corresponding homogeneous problem of (5), which takes the form

$$F^+(\tau) = \frac{\Pi_2(\tau)}{\Pi_1(\tau)} P_0(\tau) F^-(\tau). \quad (12)$$

Let N_1 and N_2 be the degrees of the polynomials $\Pi_1(\tau)$ and $\Pi_2(\tau)$, respectively. Recall μ is the index of (5).

By the generalized Liouville theorem [14], (12) has a general solution of the form

$$\tilde{F}(\xi) = \begin{cases} X(\xi) \Pi_2(\xi) p_{\mu-N_1}(\xi), & \xi \in \Sigma^+; \\ X(\xi) \Pi_1(\xi) p_{\mu-N_1}(\xi), & \xi \in \Sigma^-. \end{cases} \quad (13)$$

Here $p_{\mu-N_1}(\xi)$ can be taken to be an arbitrary polynomial with degree $\mu - N_1$ whenever $\mu - N_1 > 0$, otherwise $p_{\mu-N_1}(\xi) \equiv 0$. In the latter case, (5) has only zero solution.

5.4. Solution via Cauchy principal value integral

With the help of the solution of the homogeneous equations above, we proceed to solve (5). To this end, we define a Cauchy principal value integral

$$f(\xi) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Pi_1(\tau)Q(\tau)}{X^+(\tau)(\tau - \xi)} d\tau. \quad (14)$$

To find a solution with $\Omega(x) \in \{0\}$, it is reasonable to assume that $f(\xi)$ is bounded and has no singularity at μ_j, v_k ($j = 1, 2, \dots, s; k = 1, 2, \dots, l$).

We set $\rho = N_1 + N_2 - 1$ and construct a Hermite interpolation polynomial $H_\rho(\xi)$, which is monic and take μ_j ($j = 1, 2, \dots, s$) and v_k ($k = 1, 2, \dots, l$) as exactly zero-points with the orders α_j, β_k , respectively.

We follow the discussion in [28,23]. In the same fashion, we can construct a special solution of (5) via

$$G(\xi) = \begin{cases} \frac{X(\xi)}{\Pi_1(\xi)}[f(\xi) - H_\rho(\xi)], & \xi \in \Sigma^+; \\ \frac{X(\xi)}{\Pi_2(\xi)}[f(\xi) - H_\rho(\xi)], & \xi \in \Sigma^-. \end{cases} \quad (15)$$

According to the theory of linear algebra, the general solutions of (5) are given by

$$F(\xi) = \begin{cases} G(\xi) + X(\xi)\Pi_2(\xi)p_{\mu-N_1}(\xi), & \xi \in \Sigma^+, \\ G(\xi) + X(\xi)\Pi_1(\xi)p_{\mu-N_1}(\xi), & \xi \in \Sigma^-. \end{cases} \quad (16)$$

In order to find a solution $F(\xi) \in H$, we are forced to assume $F(\xi)$ has no singularity at $\tau = \mu_j, \tau = v_k$ ($j = 1, 2, \dots, s; k = 1, 2, \dots, l$).

In this latter case, by carefully checking the boundary of $F(\xi)$ in (16), we finally conclude that (3.6) has the general solutions of the form

$$F(\xi) = \begin{cases} \frac{1}{2}Q(\xi) + \frac{X^+(\xi)}{\Pi_1(\xi)}[f(\xi) - H_\rho(\xi)] + X^+(\xi)\Pi_2(\xi)p_{\mu-N_1}(\xi), & \xi \in \Sigma^+; \\ -\frac{\Pi_1(\xi)Q(\xi)}{2P_0(\xi)\Pi_2(\xi)} + \frac{X^-(\xi)}{\Pi_2(\xi)}[f(\xi) - H_\rho(\xi)] + X^-(\xi)\Pi_1(\xi)p_{\mu-N_1}(\xi), & \xi \in \Sigma^-, \end{cases} \quad (17)$$

where $X^\pm(\xi)$ are expressed as in (10), and $p_{\mu-N_1}(\xi)$ is an arbitrary polynomial with degree $\max\{\mu - N_1, 0\}$. Moreover,

- (1) If $\mu - N_1 \geq 0$, then the solutions given by (17) which contains $\mu - N_1$ arbitrary constants.
- (2) If $\mu - N_1 < 0$, (17) has a unique solution with $p_{\mu-N_1}(\xi) \equiv 0$.

Now we make a final assumption.

Assumption F. We assume that there exists a function defined by the right side of (17) with its restriction to the boundary Γ in the function class $\{0\}_\Gamma$ with $F^\pm(0) = 0$.

Now we can state our main result.

Theorem 5.1. *Under Assumptions A–F, the Riemann–Hilbert problem (4) admits solutions.*

This result demonstrates that if a normal singular integro-differential equation is solvable, one may disturb the equation in a small scale along certain smooth functions to yield a non-normal singular integro-differential equations which is also solvable.

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