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Blow-up prevention by nonlinear diffusion in a 2D Keller-Segel-Navier-Stokes system with rotational flux

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Abstract

This paper deals with a boundary-value problem for a coupled chemotaxis-Navier-Stokes system involving tensor-valued sensitivity with saturation

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(nS(x,n,c)\nabla c), & x\in\Omega, t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega, t>0,\\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u + n\nabla\phi, & x\in\Omega, t>0,\\ \nabla\cdot u = 0, & x\in\Omega, t>0, \end{cases} \tag{$KSNS$}$$

which describes chemotaxis-fluid interaction in cases when the evolution of the chemoattractant is essentially dominated by production through cells, where $\kappa\in\mathbb{R}$, $\phi\in W^{2,\infty}(\Omega)$ and S is a given function with values in $\mathbb{R}^{2\times 2}$ which fulfills

$$|S(x,n,c)| \le C_S$$

with some $C_S > 0$. If $m > 1$ and $\Omega\subset\mathbb{R}^2$ is a **bounded** domain with smooth boundary, then for all reasonably regular initial data, a corresponding initial-boundary value problem for $(KSNS)$ possesses a global (weak) solution which is bounded. Our main tool is consideration of the energy functional

$$\int_\Omega n^m + \int_\Omega |\nabla c|^2.$$

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1. Introduction

Chemotaxis, the biased movement of cells in response to chemical gradients, plays an important role in coordinating cell migration in many biological phenomena (see Hillen and Painter [7]). For example, the fruit fly Drosophila melanogaster navigates up gradients of attractive odours during food location, and male moths follow pheromone gradients released by the female during mate location. In 1970 Keller and Segel [11] proposed a mathematical model describing chemotactic aggregation of cellular slime molds

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\nabla c), & x \in \Omega,\ t > 0, \\ c_t = \Delta c - c + n & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, n and c denote the density of the cell population and the concentration of the attracting chemical substance, respectively. One of the most characteristic mathematical features of system (1.1) is the possibility of blow-up of solutions in a finite or infinite time. It is well known that solutions of system (1.1) blow up when $N = 2$ with large total mass of cells and $N \geq 3$ with arbitrarily small prescribed total mass of cells (see [1,8,14,24]). In order to describe the nonlinear dependence on the cell density in cell movement, the following variant has also been widely studied

$$\begin{cases} n_t = \Delta n^m - \nabla \cdot (n\nabla c), & x \in \Omega,\ t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $m > 0$. The main problem is whether the solutions of (1.2) are bounded or blow-up. In fact, all solutions are global and uniformly bounded if $m > 2 - \frac{2}{N}$ (see [18,23]); whereas if $m < 2 - \frac{2}{N}$, (1.2) has some solutions which blow up in a finite time (see [2,23]). Therefore,

$$m = 2 - \frac{2}{N}$$

is the critical blow-up exponent, which is related to the presence of a so-called volume-filling effect. But in the models (1.1) and (1.2), the authors did not take into account the relationship between cells and their environment. So the above models can be used to describe that the bacterial chemotaxis was occurred in the quiescent fluid. Yet suspensions of aerobic bacteria often develop flows from the interplay of chemotaxis and buoyancy. Tuval and his collaborators [20] described the above biological phenomena and proposed the mathematical model consisting of oxygen diffusion and consumption, chemotaxis, and fluid dynamics

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(nS(n)\nabla c), & x\in\Omega, t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega, t>0,\\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u + n\nabla\phi, & x\in\Omega, t>0,\\ \nabla\cdot u = 0, & x\in\Omega, t>0, \end{cases} \tag{1.3}$$

where n, c, u and P denote, respectively, the density of cells, chemical concentration, velocity field and pressure of the fluid. The coefficient κ is related to the strength of nonlinear fluid convection, ϕ stands for the potential of the gravitational field within which the cells are driven through buoyant forces, the function $S(n)$ measures the chemotactic sensitivity, and $f(c)$ represents the oxygen consumption rate. It is worth noticing that the results obtained so far indicate that in contrast to the standard Keller-Segel model, phenomena of finite-time blow-up, which represents maybe the most extreme facet of bacterial aggregation, cannot be determined whether they will occur or not for the above system (1.3) involving chemical signal consumption. Even for the Stokes-fluid ($\kappa = 0$) system, the simplified system of (1.3), there was no result in this respect. For the signal production case, when $\kappa = 0$, $S(n) = (n+1)^{-\alpha}$, Wang and Xiang [22] and Winkler [27] considered the system in 2D and 3D space respectively. They concluded that if $\alpha > \frac{N-2}{N}$, the global bounded classical solutions for all sufficiently regular initial data would be existed. Some modeling approaches suggested that an adequate description of bacterial motion near surfaces of their surrounding fluid should involve rotational components in the cross-diffusive flux (see [28,29]), so the natural generalizations of chemotaxis-fluid systems should model the evolution of the cell density, as the following form

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(nS(x,n,c)\nabla c), & x\in\Omega, t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega, t>0,\\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u + n\nabla\phi, & x\in\Omega, t>0,\\ \nabla\cdot u = 0, & x\in\Omega, t>0, \end{cases}$$

where the chemotactic sensitivity $S(x,n,c)$ is tensor-valued and

$$|S(x,n,c)| \le C_S(n+1)^{-\alpha}$$

with some constants $\alpha > 0$ and $C_S > 0$. Wang, Winkler and Xiang ([21]) and Ke and Zheng ([10]) considered the global existence of the solution for the case $N = 2$ and $N = 3$, respectively. They concluded that when $\alpha > \frac{N-2}{N}$, the global solutions would be existed. When we consider the porous medium type nonlinear diffusion, the system can be rewritten as

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(nS(x,n,c)\nabla c), & x\in\Omega, t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega, t>0,\\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u + n\nabla\phi, & x\in\Omega, t>0,\\ \nabla\cdot u = 0, & x\in\Omega, t>0, \end{cases} \tag{1.4}$$

and the Frobenius norm of S would satisfy

$$|S(x,n,c)| \le C_S$$

for some positive constant C_S. When $\kappa = 0$, Li et al. [12] and Zheng [31] considered the chemotaxis-Stokes system (1.4) for $N = 2$ and $N = 3$, respectively. They concluded that when $m > 2 - \frac{2}{N}$, the weak solutions of the simplified system (1.4) ($\kappa = 0$) are global existent and

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bounded. But till now, as far as we know, it is still not clearly that in the case that $\kappa \neq 0$, whether the solution of the chemotaxis-Navier-Stokes system (1.4) is bounded or not.

The emergence of degenerate diffusion, full Navier-Stokes fluid ($\kappa \neq 0$) and rotational flux (tensor-valued sensitivity S) makes the system (1.4) contain more complex cross-diffusion mechanism, which brings more mathematical difficulties to the problem. In fact, if $\kappa = 0$, by utilizing the L^1 estimate on n, one can invoke Lemma 2.4 in [22] and the Sobolev embedding theorem (Theorem 5.6.6 in [3]) to obtain the regularity of u in arbitrary L^p spaces (see Lemma 2.4 in [12]). Then one can also obtain L^p estimate on c, by using the variation-of-constants representation for c (see the proof of Lemma 2.6 in [22] and Lemma 2.6 in [12]). By using the estimates on c and u, one can finally derive the entropy-like estimate involving the functional $\int_\Omega n^p + \int_\Omega |\nabla c|^{2q}$ (see Lemma 2.9 in [12] or Lemma 2.10 in [22]). Once the crucial step has been accomplished, the main results could be easily obtained by using the standard Alikakos-Moser iteration. However, when $\kappa \neq 0$, we can not acquire the regularity of u in arbitrary L^p spaces directly. It makes us to find a new method. We develop some L^p-estimate techniques to raise the a priori estimates of solutions from $L^1(\Omega) \to L^{m-1}(\Omega) \to L^m(\Omega) \to L^p(\Omega)$ (for $\forall p > 1$), which seems a new method in the case of fluid-free system.

In this paper, we shall subsequently consider the chemotaxis-Navier-Stokes system (1.4) along with the initial data

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.5}$$

and under the boundary conditions

$$\big(\nabla n^m - nS(x,n,c)\nabla c\big) \cdot \nu = \nabla c \cdot \nu = 0, \quad u = 0, \qquad x \in \partial\Omega, t > 0 \tag{1.6}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, where we assume that the chemotactic sensitivity tensor $S(x,n,c)$ be satisfied

$$S \in C^2(\bar{\Omega} \times [0,\infty)^2; \mathbb{R}^{2\times 2}) \tag{1.7}$$

and

$$|S(x,n,c)| \le C_S \quad \text{for all } (x,n,c) \in \Omega \times [0,\infty)^2 \tag{1.8}$$

with some $C_S > 0$. Throughout this paper, we assume that

$$\phi \in W^{2,\infty}(\Omega) \tag{1.9}$$

and the initial data (n_0, c_0, u_0) fulfills

$$\begin{cases} n_0 \in C^\kappa(\bar{\Omega}) \text{ for certain } \kappa > 0 \text{ with } n_0 \ge 0 \text{ in } \Omega, \\ c_0 \in W^{2,\infty}(\Omega) \text{ with } c_0 \ge 0 \text{ in } \bar{\Omega}, \\ u_0 \in D(A), \end{cases} \tag{1.10}$$

where A denotes the Stokes operator with domain $D(A) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega)$, and $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) | \nabla \cdot \varphi = 0\}$ (see [17]).

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Within the above frameworks, our main result concerning global existence and boundedness of solutions to (1.4)-(1.6) is as follows.

Theorem 1.1. *Let $m > 1$, $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and assume (1.7)-(1.10) hold. Then the problem (1.4)-(1.6) admits a global-in-time weak solution (n, c, u, P), which is uniformly bounded in the sense that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \ \text{for all} \ \ t > 0 \tag{1.11}$$

with some positive constant C.

Remark 1.1. (i) If $u \equiv 0$, Theorem 1.1 coincides with Theorem 5.1 in [23], which is **optimal** according to the 2D fluid-free system.

(ii) Theorem 1.1 extends the results of Li, Wang and Xiang [12], in which the authors discussed the chemotaxis-Stokes system ($\kappa = 0$) in a 2D **convex** domain. As mentioned earlier, we not only extend the results to the chemotaxis-Navier-Stokes system ($\kappa \neq 0$), but also remove the convexity assumption of the domain. In [12], in order to get the regularity of ∇c, the authors added the assumption that the domain should be convex. In this paper, we apply the boundedness of $\|\nabla c\|_{L^2(\Omega)}$ (see Lemma 3.4) and the fractional Gagliardo–Nirenberg inequality (see Lemma 2.5 in [9]) to gain the regularity of ∇c in arbitrary L^p spaces and drop the hypothesis of convexity for Ω.

This paper is organized as follows. In Section 2, we do some preliminary works and propose a approximate problem. In Section 3, we use some iteration technique to establish the necessary a priori estimates. Finally, in Section 4, we obtain the global existence and boundedness of the solutions for the system (1.4)-(1.6) in a bounded domain.

2. Preliminaries

In order to construct the weak solutions by an approximation procedure, we construct the approximate problems as follows

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta(n_\varepsilon + \varepsilon)^m - \nabla \cdot (n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon), & x \in \Omega, \ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, & x \in \Omega, \ t > 0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \Delta u_\varepsilon - \kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega, \ t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega, \ t > 0, \\ \nabla n_\varepsilon \cdot \nu = \nabla c_\varepsilon \cdot \nu = 0, u_\varepsilon = 0, & x \in \partial\Omega, \ t > 0, \\ n_\varepsilon(x,0) = n_0(x), c_\varepsilon(x,0) = c_0(x), \ u_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where

$$S_\varepsilon(x,n,c) := \rho_\varepsilon(x)\chi_\varepsilon(u)S(x,n,c), \quad x \in \bar{\Omega}, \quad n \ge 0, \quad c \ge 0,$$

$$\rho_\varepsilon \in C_0^\infty(\Omega) \ \text{ such that } \ 0 \le \rho_\varepsilon \le 1 \ \text{ in } \ \Omega \ \text{ and } \ \rho_\varepsilon \nearrow 1 \ \text{ in } \ \Omega \ \text{ as } \ \varepsilon \searrow 0,$$

$$\chi_\varepsilon \in C_0^\infty([0,\infty)) \ \text{ such that } \ 0 \le \chi_\varepsilon \le 1 \ \text{ in } \ [0,\infty) \ \text{ and } \ \chi_\varepsilon \nearrow 1 \ \text{ in } \ [0,\infty) \ \text{ as } \ \varepsilon \searrow 0,$$

and

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$$Y_\varepsilon w := (1+\varepsilon A)^{-1} w \text{ for all } w \in L^2_\sigma(\Omega)$$

is a standard Yosida approximation.

By the well-established fixed-point arguments (see Lemma 2.1 in [26], [25] and Lemma 2.1 in [15]), we could show the local solvability of system (2.1).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and assume (1.7)-(1.10) hold. For any $\varepsilon \in (0,1)$, there exist $T_{max,\varepsilon} \in (0,\infty]$ and a classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ of system (2.1) in $\Omega \times [0, T_{max,\varepsilon})$. Here*

$$\begin{cases} n_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \\ c_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})) \cap \bigcap_{p>1} L^\infty([0, T_{max,\varepsilon}); W^{1,p}(\Omega)), \\ u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})) \cap \bigcap_{\gamma\in(0,1)} C^0([0, T_{max,\varepsilon}); D(A^\gamma)), \\ P_\varepsilon \in C^{1,0}(\bar{\Omega} \times (0, T_{max,\varepsilon})). \end{cases} \tag{2.2}$$

Moreover, n_ε and c_ε are nonnegative in $\Omega \times (0, T_{max,\varepsilon})$, and if $T_{max,\varepsilon} < +\infty$, then

$$\limsup_{t \nearrow T_{max,\varepsilon}} [\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}] = \infty$$

for all $p > 2$ and $\gamma \in (\frac{1}{2}, 1)$.

Lemma 2.2. *([19]) Let $T \in (0,\infty]$, $\sigma \in (0,T)$, $A > 0$ and $B > 0$, and suppose that $y : [0,T) \to [0,\infty)$ is absolutely continuous such that*

$$y'(t) + Ay(t) \le h(t) \ \textit{for a.e.} \ \ t \in (0,T)$$

with some nonnegative function $h \in L^1_{loc}([0,T))$ satisfying

$$\int_t^{t+\sigma} h(s)ds \le B \ \textit{ for all } \ t \in (0, T-\sigma).$$

Then

$$y(t) \le \max\{y_0 + B, \frac{B}{A\tau} + 2B\} \ \textit{ for all } \ t \in (0,T).$$

3. Some basic a priori estimates

In order to establish the global solvability of system (2.1), in this section, we plan to derive some estimates for the approximate system (2.1), which plays a significant role in obtaining the main result. Let us first state two basic estimates on n_ε and c_ε.

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Lemma 3.1. *([10]) The solution of (2.1) satisfies*

$$\int_\Omega n_\varepsilon = \int_\Omega n_0 \text{ for all } t \in (0, T_{max,\varepsilon}) \tag{3.1}$$

as well as

$$\int_\Omega c_\varepsilon \le \max\{\int_\Omega n_0, \int_\Omega c_0\} \text{ for all } t \in (0, T_{max,\varepsilon}).$$

According to Lemma 3.1, we can obtain the following energy-type equality.

Lemma 3.2. *Let $m > 1$. Then there exists $C > 0$ independent of ε such that the solution of (2.1) satisfies*

$$\int_\Omega n_\varepsilon + \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} + \int_\Omega c_\varepsilon^2 + \int_\Omega |u_\varepsilon|^2 \le C \quad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.2}$$

Moreover, for all $t \in (0, T_{max,\varepsilon} - \tau)$, it holds that one can find a constant $C > 0$ independent of ε such that

$$\int_t^{t+\tau} \int_\Omega \left[(n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + |\nabla c_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right] \le C, \tag{3.3}$$

where $\tau = \min\{1, \frac{1}{6} T_{max,\varepsilon}\}$.

In order to obtain the boundedness of n_ε, we need to give higher norm estimates on c_ε.

Lemma 3.3. *Let $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ be the solution of (2.2) and $\tau = \min\{1, \frac{1}{6} T_{max,\varepsilon}\}$. Then for any $q > 2$, there exists $C := C(q, K)$ independent of ε such that*

$$\|c_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \le C \text{ for all } t \in (0, T_{max,\varepsilon}). \tag{3.4}$$

Proof. Let $p > 3 + 4(m - 1)$. Multiplying the second equation in (2.1) by c_ε^{p-1}, using the fact $\nabla \cdot u_\varepsilon = 0$, and applying the Hölder inequality, we have

$$\begin{aligned}
&\frac{1}{p} \frac{d}{dt} \int_\Omega c_\varepsilon^p + (p-1) \int_\Omega c_\varepsilon^{p-2} |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^p \\
&= \int_\Omega c_\varepsilon^{p-1} n_\varepsilon \\
&\le \int_\Omega c_\varepsilon^{p-1} (n_\varepsilon + \varepsilon)
\end{aligned}$$

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$$\leq \|n_\varepsilon + \varepsilon\|_{L^{\frac{p-2(m-1)}{p-4(m-1)}}(\Omega)} \left(\int_\Omega c_\varepsilon^{\frac{(p-1)[p-2(m-1)]}{m-1}}\right)^{\frac{m-1}{p-2(m-1)}} \quad \text{for all } \ t \in (0, T_{max,\varepsilon}). \tag{3.5}$$

Now, due to the Gagliardo-Nirenberg inequality and (3.1), for some positive constants κ_0 and κ_1, we derive

$$\begin{aligned}
&\left(\int_\Omega c_\varepsilon^{\frac{(p-1)[p-2(m-1)]}{m-1}}\right)^{\frac{m-1}{p-2(m-1)}}\\
&= \|c_\varepsilon^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^{\frac{(p-1)[p-2(m-1)]}{p(m-1)}}(\Omega)}\\
&\leq \kappa_0(\|\nabla c_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{p[p-2(m-1)-1]}{[p-1][p-2(m-1)]}} \|c_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(m-1)}{(p-1)[p-2(m-1)]}} + \|c_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)})^{\frac{2(p-1)}{p}}\\
&\leq \kappa_1(\|\nabla c_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2[p-2(m-1)-1]}{p-2(m-1)}} + 1).
\end{aligned}$$

So that, in light of (3.5) and the Young inequality, we derive that for all $t \in (0, T_{max,\varepsilon})$,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega c_\varepsilon^p + (p-1)\int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^p\\
&\leq \kappa_1 \|n_\varepsilon + \varepsilon\|_{L^{\frac{p-2(m-1)}{p-4(m-1)}}(\Omega)} (\|\nabla c_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2[p-2(m-1)-1]}{p-2(m-1)}} + 1)\\
&\leq \frac{(p-1)}{2}\int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2 + C_1(p)\kappa_1^{p-2(m-1)}\|n_\varepsilon + \varepsilon\|^{p-2(m-1)}_{L^{\frac{p-2(m-1)}{p-4(m-1)}}(\Omega)} + \kappa_1\|n_\varepsilon + \varepsilon\|_{L^{\frac{p-2(m-1)}{p-4(m-1)}}(\Omega)},
\end{aligned}$$

where we have used the fact that $\frac{p-2(m-1)-1}{p-2(m-1)} + \frac{1}{p-2(m-1)} = 1$. In view of $p > 3 + 4(m-1)$, again, from the Young inequality, there exist positive constants C_3 and C_4 such that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega c_\varepsilon^p + \frac{(p-1)}{2}\int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^p\\
&\leq C_2\|n_\varepsilon + \varepsilon\|^{p-2(m-1)}_{L^{\frac{p-2(m-1)}{p-4(m-1)}}(\Omega)} + C_3 \text{ for all } t \in (0, T_{max,\varepsilon}).
\end{aligned} \tag{3.6}$$

In the following, we will estimate the integrals on the right-hand side of (3.6). In view of the Gagliardo-Nirenberg inequality, for some C_4, C_5 and $C_6 > 0$ which are independent of ε, we may derive from (3.3) that

$$\int_t^{t+\tau}\left(\|n_\varepsilon + \varepsilon\|^{p-2(m-1)}_{L^{\frac{p-2(m-1)}{p-4(m-1)}}(\Omega)} + C_3\right)ds$$

$$
\begin{aligned}
&= \int_t^{t+\tau} \left(\|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{p-2(m-1)}{[p-4(m-1)](m-1)}}(\Omega)}^{\frac{p-2(m-1)}{m-1}} + C_3 \right) ds \\
&\le C_4 \int_t^{t+\tau} \left(\|\nabla (n_\varepsilon+\varepsilon)^{m-1}\|_{L^2(\Omega)}^2 \|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{p}{m-1}} + \|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{p-2(m-1)}{m-1}} \right) + C_3 \\
&\le C_5 \int_t^{t+\tau} \left(\|\nabla (n_\varepsilon+\varepsilon)^{m-1}\|_{L^2(\Omega)}^2 \right) + C_3 \\
&\le C_6,
\end{aligned}
$$

where $\tau = \min\{1, \frac{1}{6}T_{max,\varepsilon}\}$. Therefore, (3.4) holds by applying Lemma 2.2 and the Hölder inequality. □

Based on Lemma 3.2 and Lemma 3.3, we can get a series of important estimates on n_ε and c_ε.

Lemma 3.4. *Let $m > 1$. Then the solution of (2.1) satisfies*

$$
\int_\Omega (n_\varepsilon+\varepsilon)^m + \int_\Omega |\nabla c_\varepsilon|^2 \le C \quad \textit{for all } t \in (0, T_{max,\varepsilon}) \textit{ and any } \varepsilon > 0 \tag{3.7}
$$

and

$$
\int_t^{t+\tau}\int_\Omega (n_\varepsilon+\varepsilon)^{2m} \le C \quad \textit{for all } t \in (0, T_{max,\varepsilon}-\tau) \textit{ and any } \varepsilon > 0, \tag{3.8}
$$

where $\tau = \min\{1, \frac{1}{6}T_{max,\varepsilon}\}$.

Proof. Multiplying the first equation of (2.1) by $(n_\varepsilon+\varepsilon)^{m-1}$, integrating the product in Ω, and noticing $\nabla \cdot u_\varepsilon = 0$, one obtains

$$
\begin{aligned}
&\frac{1}{m}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|_{L^m(\Omega)}^m + (m-1)\int_\Omega (n_\varepsilon+\varepsilon)^{2m-3}|\nabla n_\varepsilon|^2 \\
&= -\int_\Omega (n_\varepsilon+\varepsilon)^{m-1}\nabla\cdot(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon) \\
&= (m-1)\int_\Omega (n_\varepsilon+\varepsilon)^{m-2} n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\
&\le C_S(m-1)\int_\Omega (n_\varepsilon+\varepsilon)^{m-1}|\nabla n_\varepsilon||\nabla c_\varepsilon| \quad \text{for all } t \in (0, T_{max,\varepsilon})
\end{aligned}
$$

by using (1.8). Then, by using the Young inequality, we have

$$\begin{aligned}&\frac{1}{m}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^m_{L^m(\Omega)}+(m-1)\int_\Omega(n_\varepsilon+\varepsilon)^{2m-3}|\nabla n_\varepsilon|^2\\ \le&\frac{m-1}{2}\int_\Omega(n_\varepsilon+\varepsilon)^{2m-3}|\nabla n_\varepsilon|^2\\ &+\frac{(m-1)C_S^2}{2}\int_\Omega(n_\varepsilon+\varepsilon)|\nabla c_\varepsilon|^2\ \text{ for all }\ t\in(0,T_{max,\varepsilon}).\end{aligned}\tag{3.9}$$

On the other hand, in view of Lemma 3.2 and the Gagliardo–Nirenberg inequality, we infer that for some $\gamma_0>0$ and $\gamma_1>0$,

$$\begin{aligned}&\int_\Omega(n_\varepsilon+\varepsilon)^{2m}\\ =&\|(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^{\frac{4m}{2m-1}}_{L^{\frac{4m}{2m-1}}(\Omega)}\\ \le&\gamma_0(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^{\frac{2m-1}{2m}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^{\frac{1}{2m}}_{L^{\frac{2}{2m-1}}(\Omega)}+\|(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|_{L^{\frac{2}{2m-1}}(\Omega)})^{\frac{4m}{2m-1}}\\ \le&\gamma_1\|\nabla(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^2_{L^2(\Omega)}+\gamma_1.\end{aligned}$$

We then achieve, with the help of the above inequality, that

$$\begin{aligned}&m(m-1)\int_\Omega(n_\varepsilon+\varepsilon)^{2m-3}|\nabla n_\varepsilon|^2\\ =&\frac{4m(m-1)}{(2m-1)^2}\|\nabla(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^2_{L^2(\Omega)}\\ \ge&\frac{1}{\gamma_1}\frac{4m(m-1)}{(2m-1)^2}(\int_\Omega(n_\varepsilon+\varepsilon)^{2m}-1).\end{aligned}\tag{3.10}$$

Here, the Young inequality allows to be written as

$$\frac{(m-1)C_S^2}{2}\int_\Omega(n_\varepsilon+\varepsilon)|\nabla c_\varepsilon|^2\le\varepsilon_1\int_\Omega(n_\varepsilon+\varepsilon)^{2m}+C_1(\varepsilon_1)\int_\Omega|\nabla c_\varepsilon|^{\frac{4m}{2m-1}},$$

where

$$\varepsilon_1=\frac{1}{\gamma_1}\frac{m-1}{(2m-1)^2}\tag{3.11}$$

and

$$C_1(\varepsilon_1)=\frac{2m-1}{2m}(\varepsilon_1 2m)^{-\frac{1}{2m-1}}\left(\frac{(m-1)C_S^2}{2}\right)^{\frac{2m}{2m-1}}.$$

In light of (3.4), there exist positive constants $l_0>\frac{1}{m-1}$ and C_2, such that

$$\|c_\varepsilon(\cdot,t)\|_{L^{l_0}(\Omega)}\le C_2 \ \text{ for all } \ t\in(0,T_{max,\varepsilon}). \tag{3.12}$$

Next, with the help of the Gagliardo–Nirenberg inequality and (3.12), we derive that

$$\begin{aligned}
&C_1(\varepsilon_1)\int_\Omega |\nabla c_\varepsilon|^{\frac{4m}{2m-1}}\\
&\le C_3\|\Delta c_\varepsilon\|_{L^2(\Omega)}^{a\frac{4m}{2m-1}}\|c_\varepsilon\|_{L^{l_0}(\Omega)}^{(1-a)\frac{4m}{2m-1}}+C_3\|c_\varepsilon\|_{L^{l_0}(\Omega)}^{\frac{4m}{2m-1}}\\
&\le C_4\|\Delta c_\varepsilon\|_{L^2(\Omega)}^{a\frac{4m}{2m-1}}+C_4
\end{aligned}$$

with some positive constants C_3 and C_4, where

$$a=\frac{\frac{1}{2}+\frac{1}{l_0}-\frac{2m-1}{4m}}{\frac{1}{2}+\frac{1}{l_0}}\in(0,1).$$

This, together with the Young inequality and $a\frac{4m}{2m-1}<2$ (due to $l_0>\frac{1}{m-1}$), yields

$$C_1(\varepsilon_1)\int_\Omega |\nabla c_\varepsilon|^{\frac{4m}{2m-1}}\le\frac{1}{4}\|\Delta c_\varepsilon\|_{L^2(\Omega)}^2+C_5. \tag{3.13}$$

Taking $-\Delta c_\varepsilon$ as the test function for the second equation of (2.1), and using the Young inequality, it yields that for all $t\in(0,T_{max,\varepsilon})$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla c_\varepsilon\|_{L^2(\Omega)}^2+\int_\Omega|\Delta c_\varepsilon|^2+\int_\Omega|\nabla c_\varepsilon|^2\\
&=-\int_\Omega n_\varepsilon\Delta c_\varepsilon+\int_\Omega(u_\varepsilon\cdot\nabla c_\varepsilon)\Delta c_\varepsilon\\
&=-\int_\Omega n_\varepsilon\Delta c_\varepsilon-\int_\Omega\nabla c_\varepsilon\nabla(u_\varepsilon\cdot\nabla c_\varepsilon)\\
&=-\int_\Omega n_\varepsilon\Delta c_\varepsilon-\int_\Omega\nabla c_\varepsilon\nabla(\nabla u_\varepsilon\cdot\nabla c_\varepsilon),
\end{aligned} \tag{3.14}$$

where we have used the fact that

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$$\int_\Omega \nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot u_\varepsilon) = \frac{1}{2}\int_\Omega u_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2 = 0 \ \text{ for all } \ t \in (0, T_{max,\varepsilon}).$$

Meanwhile, we can further use the Gagliardo-Nirenberg inequality and the elliptic regularity ([5]) to conclude that for some $C_6 > 0$,

$$\|\nabla c_\varepsilon\|^2_{L^4(\Omega)} \le C_6 \|\Delta c_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\Omega)} \ \text{ for all } \ t \in (0, T_{max,\varepsilon}).$$

This, together with the Cauchy-Schwarz inequality and the Young inequality, yields

$$\begin{aligned}
& -\int_\Omega \nabla c_\varepsilon \nabla(\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \\
\le\ & \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^4(\Omega)} \\
\le\ & C_6 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\Delta c_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\Omega)} \\
\le\ & C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{4}\|\Delta c_\varepsilon\|^2_{L^2(\Omega)} \ \text{ for all } \ t \in (0, T_{max,\varepsilon}).
\end{aligned} \tag{3.15}$$

Applying the Cauchy-Schwarz inequality, one obtains

$$-\int_\Omega n_\varepsilon \Delta c_\varepsilon \le \ \frac{1}{4}\int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 \ \text{ for all } \ t \in (0, T_{max,\varepsilon}). \tag{3.16}$$

From (3.14) and (3.15), we thus infer that

$$\frac{d}{dt}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\Delta c_\varepsilon|^2 + 2\int_\Omega |\nabla c_\varepsilon|^2 \le \ 2\int_\Omega n_\varepsilon^2 + 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}. \tag{3.17}$$

Collecting (3.9), (3.13)–(3.17), we derive that for all $t \in (0, T_{max,\varepsilon})$,

$$\begin{aligned}
& \frac{d}{dt}(\|n_\varepsilon + \varepsilon\|^m_{L^m(\Omega)} + \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}) + m(m-1)\int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \\
& + \frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2 + 2\int_\Omega |\nabla c_\varepsilon|^2 \\
\le\ & m\varepsilon_1 \int_\Omega (n_\varepsilon + \varepsilon)^{2m} + 2\int_\Omega n_\varepsilon^2 + 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_7 \\
\le\ & m\varepsilon_1 \int_\Omega (n_\varepsilon + \varepsilon)^{2m} + 2\int_\Omega (n_\varepsilon + \varepsilon)^2 + 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_7.
\end{aligned}$$

Moreover, it follows from the Young inequality and $m > 1$, that

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$$\begin{aligned}
&\frac{d}{dt}(\|n_\varepsilon+\varepsilon\|^m_{L^m(\Omega)}+\|\nabla c_\varepsilon\|^2_{L^2(\Omega)})+m(m-1)\int_\Omega (n_\varepsilon+\varepsilon)^{2m-3}|\nabla n_\varepsilon|^2\\
&+\frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2+2\int_\Omega|\nabla c_\varepsilon|^2\\
\le\ & 2m\varepsilon_1\int_\Omega (n_\varepsilon+\varepsilon)^{2m}+2C_6^2\|\nabla u_\varepsilon\|^2_{L^2(\Omega)}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)}+C_8 \text{ for all } t\in(0,T_{max,\varepsilon}).
\end{aligned}\tag{3.18}$$

By substituting (3.10) into (3.18) and using (3.11), we find that

$$\begin{aligned}
&\frac{d}{dt}(\|n_\varepsilon+\varepsilon\|^m_{L^m(\Omega)}+\|\nabla c_\varepsilon\|^2_{L^2(\Omega)})+(\frac{1}{\gamma_1}\frac{4m(m-1)}{(2m-1)^2}-2m\varepsilon_1)\int_\Omega (n_\varepsilon+\varepsilon)^{2m}\\
&+\frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2+2\int_\Omega|\nabla c_\varepsilon|^2\\
=\ &\frac{d}{dt}(\|n_\varepsilon+\varepsilon\|^m_{L^m(\Omega)}+\|\nabla c_\varepsilon\|^2_{L^2(\Omega)})+\frac{1}{\gamma_1}\frac{2m(m-1)}{(2m-1)^2}\int_\Omega (n_\varepsilon+\varepsilon)^{2m}\\
&+\frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2+2\int_\Omega|\nabla c_\varepsilon|^2\\
\le\ & 2C_6^2\|\nabla u_\varepsilon\|^2_{L^2(\Omega)}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)}+C_9 \text{ for all } t\in(0,T_{max,\varepsilon}).
\end{aligned}$$

Therefore, we derive from the Young inequality that

$$\begin{aligned}
&\frac{d}{dt}(\|n_\varepsilon+\varepsilon\|^m_{L^m(\Omega)}+\|\nabla c_\varepsilon\|^2_{L^2(\Omega)})+2\int_\Omega n_\varepsilon^m+2\int_\Omega|\nabla c_\varepsilon|^2+\frac{1}{\gamma_1}\frac{m(m-1)}{(2m-1)^2}\int_\Omega (n_\varepsilon+\varepsilon)^{2m}\\
&\le 2C_6^2\|\nabla u_\varepsilon\|^2_{L^2(\Omega)}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)}+C_{10}\\
&\le 2C_6^2\|\nabla u_\varepsilon\|^2_{L^2(\Omega)}(\|\nabla c_\varepsilon\|^2_{L^2(\Omega)}+\|n_\varepsilon+\varepsilon\|^m_{L^m(\Omega)})+C_{10} \text{ for all } t\in(0,T_{max,\varepsilon}),
\end{aligned}\tag{3.19}$$

where we have used the fact that $2\int_\Omega n_\varepsilon^m\le\frac{1}{\gamma_1}\frac{m(m-1)}{(2m-1)^2}\int_\Omega(n_\varepsilon+\varepsilon)^{2m}+C_{10}$, $m>1$ and the Young inequality. Now, again, from the Gagliardo–Nirenberg inequality, (3.3), and Lemma 3.2, there exist constants $\gamma_3>0$ and $\gamma_4>0$, such that

$$\begin{aligned}
&\int_t^{t+\tau}\int_\Omega (n_\varepsilon+\varepsilon)^m\\
=&\int_t^{t+\tau}\|(n_\varepsilon+\varepsilon)^{m-1}\|^{\frac{m}{m-1}}_{L^{\frac{m}{m-1}}(\Omega)}\\
\le\ &\gamma_3(\int_t^{t+\tau}\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|^{\frac{m-1}{m}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{m-1}\|^{\frac{1}{m}}_{L^{\frac{1}{m-1}}(\Omega)}+\int_t^{t+\tau}\|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)})^{\frac{m}{m-1}}
\end{aligned}$$

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$$\leq \gamma_4 \int_t^{t+\tau} \|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|^2_{L^2(\Omega)} + \gamma_4 \text{ for all } t \in (0, T_{max,\varepsilon}-\tau), \tag{3.20}$$

where $\tau = \min\{1, \frac{1}{6}T_{max,\varepsilon}\}$. Therefore, by (3.20), we conclude that

$$\int_t^{t+\tau}\int_\Omega (n_\varepsilon+\varepsilon)^m \leq \gamma_5 \text{ for all } t \in (0, T_{max,\varepsilon}-\tau). \tag{3.21}$$

Thus, for $t \in (0, T_{max,\varepsilon})$, if we write

$$y(t) := \|n_\varepsilon(\cdot,t)+\varepsilon\|^m_{L^m(\Omega)} + \|\nabla c_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}$$

and

$$\rho(t) = 2C_6^2 \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2,$$

(3.19) implies that

$$y'(t) + h(t) \leq \rho(t)y(t) + C_{11} \text{ for all } t \in (0, T_{max,\varepsilon}), \tag{3.22}$$

where

$$h(t) = \frac{1}{\gamma_1}\frac{m(m-1)}{(2m-1)^2}\int_\Omega (n_\varepsilon+\varepsilon)^{2m}(\cdot,t) \geq 0.$$

Next, by using estimates (3.21) and (3.3), one obtains

$$\int_t^{t+\tau} \rho(s)ds \leq C_{12}$$

and

$$\int_t^{t+\tau} y(s)ds \leq C_{13},$$

for all $t \in (0, T_{max,\varepsilon}-\tau)$. For given $t \in (0, T_{max,\varepsilon})$, by using estimates (3.21) and (3.3) again, one can choose $t_0 \geq 0$ such that $t_0 \in [t-\tau, t)$ and

$$y(\cdot, t_0) \leq C_{14}.$$

This, together with (3.22) and the Gronwall lemma, yields

$$
\begin{aligned}
y(t) &\le y(t_0)e^{\int_{t_0}^t \rho(s)ds} + \int_{t_0}^{t} e^{\int_s^t \rho(\tau)d\tau} C_{11} ds \\
&\le C_{14}e^{C_{12}} + \int_{t_0}^{t} e^{C_{12}} C_{11} ds \\
&\le C_{14}e^{C_{12}} + e^{C_{12}} C_{11} \text{ for all } t \in (0, T_{max,\varepsilon}). \quad (3.23)
\end{aligned}
$$

Finally, combining (3.22) with (3.23), it yields (3.7) and (3.8). □

Lemma 3.5. *Let $m > 1$. There exists a positive constant C independent of ε, such that*

$$
\int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2 \le C \textit{ for all } t \in (0, T_{max,\varepsilon}). \quad (3.24)
$$

Proof. Firstly, applying the Helmholtz projection to both sides of the first equation in (2.1), then multiplying the result identified by Au_ε, integrating by parts, and using the Young inequality, we find that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |Au_\varepsilon|^2 \\
&= \int_\Omega Au_\varepsilon \mathcal{P}(-\kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon) + \int_\Omega \mathcal{P}(n_\varepsilon \nabla\phi)Au_\varepsilon \quad (3.25) \\
&\le \frac{1}{2}\int_\Omega |Au_\varepsilon|^2 + \kappa^2 \int_\Omega |(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)} \int_\Omega n_\varepsilon^2 \text{ for all } t \in (0, T_{max,\varepsilon}).
\end{aligned}
$$

Noticing that $\|Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} \le \|u_\varepsilon\|_{L^2(\Omega)}$, it follows from the Gagliardo-Nirenberg inequality and the Cauchy-Schwarz inequality that with some $C_1 > 0$ and $C_2 > 0$

$$
\begin{aligned}
&\kappa^2 \int_\Omega |(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon|^2 \\
&\le \kappa^2 \|Y_\varepsilon u_\varepsilon\|^2_{L^4(\Omega)} \|\nabla u_\varepsilon\|^2_{L^4(\Omega)} \\
&\le \kappa^2 C_1 [\|\nabla Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} \|Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)}][\|Au_\varepsilon\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)}] \\
&\le \kappa^2 C_1 C_2 \|\nabla Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} [\|Au_\varepsilon\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)}] \text{ for all } t \in (0, T_{max,\varepsilon}). \quad (3.26)
\end{aligned}
$$

Now, from the fact that $D(A^{\frac{1}{2}}) := W_0^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_\sigma(\Omega)$ and (3.2), it follows that

$$
\|\nabla Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} = \|A^{\frac{1}{2}} Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} = \|Y_\varepsilon A^{\frac{1}{2}} u_\varepsilon\|_{L^2(\Omega)} \le \|A^{\frac{1}{2}} u_\varepsilon\|_{L^2(\Omega)} \le \|\nabla u_\varepsilon\|_{L^2(\Omega)}. \quad (3.27)
$$

Due to Theorem 2.1.1 in [17], $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $D(A)$. This, together with the Young inequality and estimates (3.27) and (3.26), yields

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$$\begin{aligned}
&\kappa^2 \int_\Omega |(Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon|^2 \\
\leq\ & C_3 \|A u_\varepsilon\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \\
\leq\ & \frac{1}{4} \|A u_\varepsilon\|_{L^2(\Omega)} + \kappa^4 C_1^2 C_2^2 \|\nabla u_\varepsilon\|^4_{L^2(\Omega)} \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}),
\end{aligned}$$

which combining with (3.26) implies that

$$\frac{1}{2}\frac{d}{dt} \|A^{\frac{1}{2}} u_\varepsilon\|^2_{L^2(\Omega)} \leq \kappa^4 C_1^2 C_2^2 \|\nabla u_\varepsilon\|^4_{L^2(\Omega)} + \|\nabla \phi\|^2_{L^\infty(\Omega)} \int_\Omega n_\varepsilon^2 \text{ for all } t \in (0, T_{max,\varepsilon}).$$

By the fact that $\|A^{\frac{1}{2}} u_\varepsilon\|^2_{L^2(\Omega)} = \|\nabla u_\varepsilon\|^2_{L^2(\Omega)}$, we conclude that

$$z'(t) \leq \rho(t) z(t) + h(t) \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}), \tag{3.28}$$

where

$$z(t) := \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2,$$

as well as

$$\rho(t) = 2\kappa^4 C_1^2 C_2^2 \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2$$

and

$$h(t) = 2\|\nabla \phi\|^2_{L^\infty(\Omega)} \int_\Omega n_\varepsilon^2(\cdot, t).$$

However, (3.3) along with (3.8) warrants that for some positive constant α_0,

$$\int_t^{t+\tau} \int_\Omega |\nabla u_\varepsilon|^2 \leq \alpha_0 \quad \text{for all} \quad t \in (0, T_{max,\varepsilon} - \tau) \tag{3.29}$$

and

$$\int_t^{t+\tau} \int_\Omega n_\varepsilon^2 \leq \alpha_0 \quad \text{for all} \quad t \in (0, T_{max,\varepsilon} - \tau) \tag{3.30}$$

with $\tau = \min\{1, \frac{1}{6} T_{max,\varepsilon}\}$. Now, (3.29) and (3.30) ensure that for all $t \in (0, T_{max,\varepsilon} - \tau)$

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$$\int_t^{t+\tau} \rho(s)ds \le 2C_3^2\alpha_0$$

and

$$\int_t^{t+\tau} h(s)ds \le 4\|\nabla\phi\|^2_{L^\infty(\Omega)}\alpha_0.$$

For given $t \in (0, T_{max,\varepsilon})$, applying (3.29) again, we can choose $t_0 \ge 0$ such that $t_0 \in [t-\tau, t)$ and

$$\int_\Omega |\nabla u_\varepsilon(\cdot, t_0)|^2 \le C_4,$$

which combining with (3.28) implies that

$$\begin{aligned} z(t) \le\ & z(t_0)e^{\int_{t_0}^t \rho(s)ds} + \int_{t_0}^t e^{\int_s^t \rho(\tau)d\tau} h(s)ds \\ \le\ & C_4 e^{2C_3^2\alpha_0} + \int_{t_0}^t e^{2C_3^2\alpha_0} h(s)ds \\ \le\ & C_4 e^{2C_3^2\alpha_0} + e^{2C_3^2\alpha_0} 4\|\nabla\phi\|^2_{L^\infty(\Omega)}\alpha_0 \text{ for all } t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{3.31}$$

One can easily obtain the inequality (3.24) from (3.31). □

Lemma 3.6. *Let $m > 1$. Then there exists a positive constant C independent of ε such that the solution of (2.1) satisfies*

$$\|\nabla c_\varepsilon(\cdot, t)\|_{L^{2m}(\Omega)} \le C \ \textit{for all}\ t \in (0, T_{max,\varepsilon}). \tag{3.32}$$

Proof. Considering the fact that $\nabla c_\varepsilon \cdot \nabla\Delta c_\varepsilon = \frac{1}{2}\Delta|\nabla c_\varepsilon|^2 - |D^2 c_\varepsilon|^2$, using the second equation in (2.1) and several integrations by parts, we find that

$$\begin{aligned} &\frac{1}{2m}\frac{d}{dt}\|\nabla c_\varepsilon\|^{2m}_{L^{2m}(\Omega)} \\ =& \int_\Omega |\nabla c_\varepsilon|^{2m-2}\nabla c_\varepsilon \cdot \nabla(\Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon) \\ =& \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2m-2}\Delta|\nabla c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^{2m-2}|D^2 c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^{2m} \\ & - \int_\Omega n_\varepsilon \nabla\cdot(|\nabla c_\varepsilon|^{2m-2}\nabla c_\varepsilon) + \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\nabla\cdot(|\nabla c_\varepsilon|^{2m-2}\nabla c_\varepsilon) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{m-1}{2}\int_\Omega |\nabla c_\varepsilon|^{2m-4}\left|\nabla|\nabla c_\varepsilon|^2\right|^2 + \frac{1}{2}\int_{\partial\Omega}|\nabla c_\varepsilon|^{2m-2}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} - \int_\Omega |\nabla c_\varepsilon|^{2m} \\
&\quad -\int_\Omega |\nabla c_\varepsilon|^{2m-2}|D^2c_\varepsilon|^2 - \int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2m-2}\Delta c_\varepsilon - \int_\Omega n_\varepsilon\nabla c_\varepsilon\cdot\nabla(|\nabla c_\varepsilon|^{2m-2}) \\
&\quad +\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2m-2}\Delta c_\varepsilon + \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\nabla c_\varepsilon\cdot\nabla(|\nabla c_\varepsilon|^{2m-2}) \\
&= -\frac{2(m-1)}{m^2}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^m\right|^2 + \frac{1}{2}\int_{\partial\Omega}|\nabla c_\varepsilon|^{2m-2}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} - \int_\Omega |\nabla c_\varepsilon|^{2m-2}|D^2c_\varepsilon|^2 \\
&\quad -\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2m-2}\Delta c_\varepsilon - \int_\Omega n_\varepsilon\nabla c_\varepsilon\cdot\nabla(|\nabla c_\varepsilon|^{2m-2}) - \int_\Omega |\nabla c_\varepsilon|^{2m} \\
&\quad +\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2m-2}\Delta c_\varepsilon + \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\nabla c_\varepsilon\cdot\nabla(|\nabla c_\varepsilon|^{2m-2})
\end{aligned} \tag{3.33}
$$

for all $t\in(0,T_{max,\varepsilon})$. Since $|\Delta c_\varepsilon|\le\sqrt{2}|D^2c_\varepsilon|$, by utilizing the Young inequality, we can estimate

$$
\begin{aligned}
&\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2m-2}\Delta c_\varepsilon \\
\le\ & \sqrt{2}\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2m-2}|D^2c_\varepsilon| \\
\le\ & \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2m-2}|D^2c_\varepsilon|^2 + 2\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2m-2} \\
\le\ & \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2m-2}|D^2c_\varepsilon|^2 + 2\int_\Omega (n_\varepsilon+\varepsilon)^2|\nabla c_\varepsilon|^{2m-2}
\end{aligned} \tag{3.34}
$$

and

$$
\begin{aligned}
&\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2m-2}\Delta c_\varepsilon \\
\le\ & \sqrt{2}\int_\Omega |u_\varepsilon\cdot\nabla c_\varepsilon||\nabla c_\varepsilon|^{2m-2}|D^2c_\varepsilon| \\
\le\ & \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2m-2}|D^2c_\varepsilon|^2 + 2\int_\Omega |u_\varepsilon\cdot\nabla c_\varepsilon|^2|\nabla c_\varepsilon|^{2m-2} \\
\le\ & \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2m-2}|D^2c_\varepsilon|^2 + 2\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^{2m} \\
\le\ & \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2m-2}|D^2c_\varepsilon|^2 + 2\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^{2m}
\end{aligned} \tag{3.35}
$$

for all $t \in (0, T_{max,\varepsilon})$. Again, from the Young inequality, we have

$$\begin{aligned}
& -\int_\Omega n_\varepsilon \nabla c_\varepsilon \cdot \nabla(|\nabla c_\varepsilon|^{2m-2}) \\
= & -(m-1)\int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2(m-2)} \nabla c_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2 \\
\leq & \frac{m-1}{8}\int_\Omega |\nabla c_\varepsilon|^{2m-4} \left|\nabla |\nabla c_\varepsilon|^2\right|^2 + 2(m-1)\int_\Omega |n_\varepsilon|^2 |\nabla c_\varepsilon|^{2m-2} \\
\leq & \frac{(m-1)}{2m^2}\int_\Omega \left|\nabla |\nabla c_\varepsilon|^m\right|^2 + 2(m-1)\int_\Omega |n_\varepsilon|^2 |\nabla c_\varepsilon|^{2m-2}
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
& \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon) \nabla c_\varepsilon \cdot \nabla(|\nabla c_\varepsilon|^{2m-2}) \\
& = (m-1)\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon) |\nabla c_\varepsilon|^{2(m-2)} \nabla c_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2 \\
& \leq \frac{m-1}{8}\int_\Omega |\nabla c_\varepsilon|^{2m-4} \left|\nabla |\nabla c_\varepsilon|^2\right|^2 \\
& \quad + 2(m-1)\int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^2 |\nabla c_\varepsilon|^{2m-2} \\
& \leq \frac{(m-1)}{2m^2}\int_\Omega \left|\nabla |\nabla c_\varepsilon|^m\right|^2 + 2(m-1)\int_\Omega |u_\varepsilon|^2 |\nabla c_\varepsilon|^{2m}.
\end{aligned} \tag{3.37}$$

Observe that

$$\begin{aligned}
& \int_{\partial\Omega} \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} |\nabla c_\varepsilon|^{2m-2} \\
\leq & \; C_\Omega \int_{\partial\Omega} |\nabla c_\varepsilon|^{2m} \\
= & \; C_\Omega \| |\nabla c_\varepsilon|^m \|^2_{L^2(\partial\Omega)}.
\end{aligned} \tag{3.38}$$

Let us take $r \in (0, \frac{1}{2})$. Due to Proposition 4.22 (ii) in [6], we have that $W^{r+\frac{1}{2},2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ is compact, so that,

$$\| |\nabla c_\varepsilon|^m \|^2_{L^2(\partial\Omega)} \leq C_1 \| |\nabla c_\varepsilon|^m \|^2_{W^{r+\frac{1}{2},2}(\Omega)}. \tag{3.39}$$

Now, let us pick $a = \frac{2m+2r-1}{2m}$. By $r \in (0, \frac{1}{2})$ and $m > 1$, it implies that $r + \frac{1}{2} \le a < 1$. Therefore, from the fractional Gagliardo–Nirenberg inequality and Lemma 3.4, for some positive constants δ_0, δ_1 and C_1, we conclude

$$\begin{aligned} & \| |\nabla c_\varepsilon|^m \|^2_{W^{r+\frac{1}{2},2}(\Omega)} \\ \le\ & \delta_0 \|\nabla |\nabla c_\varepsilon|^m\|^a_{L^2(\Omega)} \| |\nabla c_\varepsilon|^m \|^{1-a}_{L^{\frac{2}{m}}(\Omega)} + \delta_1 \| |\nabla c_\varepsilon|^m \|_{L^{\frac{2}{m}}(\Omega)} \\ \le\ & C_1 \|\nabla |\nabla c_\varepsilon|^m\|^a_{L^2(\Omega)} + C_1. \end{aligned} \tag{3.40}$$

Combining (3.38)–(3.40), using the Young inequality and the fact that $a \in (0, 1)$, it yields

$$\begin{aligned} & \int_{\partial\Omega} \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} |\nabla c_\varepsilon|^{2m-2} \\ \le\ & C_2 \|\nabla |\nabla c_\varepsilon|^m\|^a_{L^2(\Omega)} + C_2 \\ \le\ & \frac{(m-1)}{2m^2} \int_\Omega \left|\nabla |\nabla c_\varepsilon|^m\right|^2 + C_3. \end{aligned} \tag{3.41}$$

Now, together with (3.33)–(3.37) and (3.41), we can derive that, for some positive constant C_4,

$$\begin{aligned} & \frac{1}{2m} \frac{d}{dt} \|\nabla c_\varepsilon\|^{2m}_{L^{2m}(\Omega)} + \frac{m-1}{2m^2} \int_\Omega \left|\nabla |\nabla c_\varepsilon|^m\right|^2 + \frac{1}{2} \int_\Omega |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^{2m} \\ & \le 2m \int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2m-2} + 2m \int_\Omega |u_\varepsilon|^2 |\nabla c_\varepsilon|^{2m} + C_4 \ \text{ for all } \ t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{3.42}$$

We proceed to estimate the first term on the right-hand side of (3.42). By using the Young inequality, we conclude that

$$\begin{aligned} & 2m \int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2m-2} \\ & \le 2m \int_\Omega (n_\varepsilon + \varepsilon)^2 |\nabla c_\varepsilon|^{2m-2} \\ & \le \frac{1}{2} \int_\Omega |\nabla c_\varepsilon|^{2m} + C_5 \int_\Omega (n_\varepsilon + \varepsilon)^{2m} \ \text{ for all } \ t \in (0, T_{max,\varepsilon}) \end{aligned} \tag{3.43}$$

and

$$2m \int_\Omega |u_\varepsilon|^2 |\nabla c_\varepsilon|^{2m} \le \int_\Omega |\nabla c_\varepsilon|^{2m+1} + C_6 \int_\Omega u_\varepsilon^{4m+2} \ \text{ for all } \ t \in (0, T_{max,\varepsilon}), \tag{3.44}$$

where $C_5 = \frac{m}{m-1} \left(\frac{1}{2}m\right)^{-\frac{1}{m-1}} (2m)^m$ and $C_6 = (2m)^{2m+1}$. On the other hand, due to (3.7), we derive from the Gagliardo–Nirenberg inequality that for some positive constants C_7 and C_8

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$$\begin{aligned}
&\int_\Omega |\nabla c_\varepsilon|^{2m+1}\\
= \ & \| |\nabla c_\varepsilon|^m \|_{L^{\frac{2m+1}{m}}(\Omega)}^{\frac{2m+1}{m}}\\
\le \ & C_7 (\| \nabla |\nabla c_\varepsilon|^m \|_{L^2(\Omega)}^{\frac{2m-1}{2m+1}} \| |\nabla c_\varepsilon|^m \|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2}{2m+1}} + \| |\nabla c_\varepsilon|^m \|_{L^{\frac{2}{m}}(\Omega)})^{\frac{2m+1}{m}}\\
\le \ & C_8 (\| \nabla |\nabla c_\varepsilon|^m \|_{L^2(\Omega)}^{\frac{2m-1}{m}} + 1),
\end{aligned}$$

which together with the Young inequality provides a constant C_9 such that

$$\int_\Omega |\nabla c_\varepsilon|^{2m+1} \le \frac{m-1}{2m^2} \int_\Omega \left|\nabla |\nabla c_\varepsilon|^m\right|^2 + C_9. \tag{3.45}$$

Inserting (3.45) into (3.44), we derive that

$$\begin{aligned}
&2m \int_\Omega |u_\varepsilon|^2 |\nabla c_\varepsilon|^{2m}\\
\le \ & \frac{m-1}{2m^2} \int_\Omega \left|\nabla |\nabla c_\varepsilon|^m\right|^2 + C_6 \int_\Omega u_\varepsilon^{4m+2} + C_9 \text{ for all } t \in (0, T_{max,\varepsilon}).
\end{aligned} \tag{3.46}$$

Substituting (3.43) and (3.46) into (3.42), we have

$$\begin{aligned}
&\frac{1}{2m}\frac{d}{dt} \|\nabla c_\varepsilon\|_{L^{2m}(\Omega)}^{2m} + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2m}\\
\le \ & C_5 \int_\Omega (n_\varepsilon + \varepsilon)^{2m} + C_6 \int_\Omega u_\varepsilon^{4m+2} + C_{10} \text{ for all } t \in (0, T_{max,\varepsilon}).
\end{aligned}$$

Next, since $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for any $p > 1$, the boundedness of $\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ (see Lemma 3.5) implies that there exists a positive constant C_{11} such that

$$\|u_\varepsilon(\cdot,t)\|_{L^{4m+2}(\Omega)} \le C_{11} \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}),$$

which together with (3.8) yields to (3.32) by using Lemma 2.2. This completes the proof of Lemma 3.6. □

Lemma 3.7. *Let $m > 1$. Then for all $p > 1$, there exists a positive constant C independent of ε, such that the solution of (2.1) from Lemma 2.1 satisfies*

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \quad \textit{for all} \quad t \in (0, T_{max,\varepsilon}). \tag{3.47}$$

Proof. Let $p > \max\{1, m-1\}$. Taking $(n_\varepsilon + \varepsilon)^{p-1}$ as the test function for the first equation of (2.1), combining with the second equation, and using (1.8), the Young inequality and the fact $\nabla \cdot u_\varepsilon = 0$, we obtain, for all $t \in (0, T_{max,\varepsilon})$,

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$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}+m(p-1)\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\\
\le\ & (p-1)\int_\Omega (n_\varepsilon+\varepsilon)^{p-2}n_\varepsilon|\nabla n_\varepsilon||S_\varepsilon(x,n_\varepsilon,c_\varepsilon)||\nabla c_\varepsilon|\\
\le\ & (p-1)C_S\int_\Omega (n_\varepsilon+\varepsilon)^{p-1}|\nabla n_\varepsilon||\nabla c_\varepsilon|\\
\le\ & \frac{m(p-1)}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2+\frac{(p-1)C_S^2}{2m}\int_\Omega (n_\varepsilon+\varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}+\frac{m(p-1)}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\\
\le\ & \frac{(p-1)C_S^2}{2m}\int_\Omega (n_\varepsilon+\varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2
\end{aligned}
\tag{3.48}
$$

for all $t\in(0,T_{max,\varepsilon})$. In the following, we will estimate the right-hand side of (3.48). In fact, due to $m>1$, we conclude from (3.32) that

$$
\begin{aligned}
&\int_\Omega (n_\varepsilon+\varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2\\
\le\ & \left(\int_\Omega (n_\varepsilon+\varepsilon)^{\frac{m(p+1-m)}{m-1}}\right)^{\frac{m-1}{m}}\left(\int_\Omega |\nabla c_\varepsilon|^{2m}\right)^{\frac{1}{m}}\\
\le\ & C_1\left(\int_\Omega (n_\varepsilon+\varepsilon)^{\frac{m(p+1-m)}{m-1}}\right)^{\frac{m-1}{m}} \quad \text{for all } t\in(0,T_{max,\varepsilon})
\end{aligned}
$$

by using the Hölder inequality. Combining the above inequality with (3.2), considering the fact $m>1$ and $\frac{2(mp-m^2+1)}{m(p+m-1)}<2$, and using the Gagliardo–Nirenberg inequality and the Young inequality, we obtain

$$
\begin{aligned}
&C_1\left(\int_\Omega (n_\varepsilon+\varepsilon)^{\frac{m(p+1-m)}{m-1}}\right)^{\frac{m-1}{m}}\\
&=C_1\|(n_\varepsilon+\varepsilon)^{\frac{m(p+1-m)}{m-1}}\|^{\frac{2(p+1-m)}{m+p-1}}_{L^{\frac{2m(p+1-m)}{(m-1)(m+p-1)}}(\Omega)}\\
&\le C_2(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|^{\frac{mp-m^2+1}{m(p+1-m)}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|^{\frac{m-1}{m(p+1-m)}}_{L^{\frac{2}{p+m-1}}(\Omega)}
\end{aligned}
$$

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$$
\begin{aligned}
&+\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)})^{\frac{2(p+1-m)}{m+p-1}}\\
&\le C_3(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(mp-m^2+1)}{m(p+m-1)}}+1)\\
&\le \frac{m(p-1)}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2+C_4 \text{ for all } t\in(0,T_{max,\varepsilon}).
\end{aligned}\tag{3.49}
$$

Inserting (3.49) into (3.48), we have

$$
\frac{1}{p}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}+\frac{m(p-1)}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\le C_5.
$$

Therefore, (3.47) holds by using Lemma 2.2 and some basic calculation. This completes the proof of Lemma 3.7. □

Lemma 3.8. *Let $m>1$ and $\gamma\in(\frac{1}{2},1)$. Then one can find a positive constant C independent of ε, such that*

$$
\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C \text{ for all } t\in(0,T_{max,\varepsilon}),
$$
$$
\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le C \text{ for all } t\in(0,T_{max,\varepsilon})
$$

as well as

$$
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C \text{ for all } t\in(0,T_{max,\varepsilon})
$$

and

$$
\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\le C \text{ for all } t\in(0,T_{max,\varepsilon}).
$$

Proof. Firstly, applying the variation-of-constants formula to the projected version of the third equation in (2.1), we derive that

$$
u_\varepsilon(\cdot,t)=e^{-tA}u_0+\int_0^t e^{-(t-\tau)A}\mathcal{P}[n_\varepsilon(\cdot,t)\nabla\phi-\kappa(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon]d\tau \text{ for all } t\in(0,T_{max,\varepsilon}).
$$

Now, picking $h_\varepsilon=\mathcal{P}[n_\varepsilon(\cdot,t)\nabla\phi-\kappa(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon]$, then, in view of the standard smoothing properties of the Stokes semigroup, we derive that for all $t\in(0,T_{max,\varepsilon})$ and $\gamma\in(\frac{1}{2},1)$, there exist $C_1>0$ and $C_2>0$ such that

$$
\begin{aligned}
&\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\\
&\le \|A^\gamma e^{-tA}u_0\|_{L^2(\Omega)}+\int_0^t \|A^\gamma e^{-(t-\tau)A}h_\varepsilon(\cdot,\tau)d\tau\|_{L^2(\Omega)}d\tau
\end{aligned}
$$

$$\leq \|A^\gamma u_0\|_{L^2(\Omega)} + C_1 \int_0^t (t-\tau)^{-\gamma-\frac{2}{2}(\frac{1}{p_0}-\frac{1}{2})} e^{-\lambda(t-\tau)} \|h_\varepsilon(\cdot,\tau)\|_{L^{p_0}(\Omega)} d\tau$$

$$\leq C_2 + C_1 \int_0^t (t-\tau)^{-\gamma-\frac{2}{2}(\frac{1}{p_0}-\frac{1}{2})} e^{-\lambda(t-\tau)} \|h_\varepsilon(\cdot,\tau)\|_{L^{p_0}(\Omega)} d\tau \tag{3.50}$$

by using (1.10), where $p_0 \in (1, 2)$ satisfies that

$$p_0 > \frac{2}{3-2\gamma}. \tag{3.51}$$

In light of (3.47), for some positive constant C_3, it has

$$\|n_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)} \leq C_3 \text{ for all } t \in (0, T_{max,\varepsilon}).$$

Employing the Hölder inequality and the continuity of $\mathcal{P}$ in $L^p(\Omega;\mathbb{R}^2)$ (see [4]), there exist positive constants C_4, C_5, C_6 and C_7 such that

$$\begin{aligned}
&\|h_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)} \\
&\leq C_4\|(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)} + C_4\|n_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)} \\
&\leq C_5\|Y_\varepsilon u_\varepsilon\|_{L^{\frac{2p_0}{2-p_0}}(\Omega)} \|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + C_5 \\
&\leq C_6\|\nabla Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)}\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + C_5 \\
&\leq C_7 \quad \text{for all } t \in (0, T_{max,\varepsilon}),
\end{aligned} \tag{3.52}$$

where we have used the fact that $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2p_0}{2-p_0}}(\Omega)$ and the boundedness of $\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$. Collecting (3.50), (3.51) and (3.52), we conclude that

$$\begin{aligned}
&\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \\
\leq\ & C_8 \int_0^t (t-\tau)^{-\gamma-\frac{2}{2}(\frac{1}{p_0}-\frac{1}{2})} e^{-\lambda(t-\tau)} \|h_\varepsilon(\cdot,\tau)\|_{L^{p_0}(\Omega)} d\tau \\
\leq\ & C_9 \text{ for all } t \in (0, T_{max,\varepsilon}),
\end{aligned}$$

which together with the fact that $D(A^\gamma)$ is continuously embedded into $L^\infty(\Omega)$ by $\gamma > \frac{1}{2}$ yields

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{10} \text{ for all } t \in (0, T_{max,\varepsilon}). \tag{3.53}$$

In view of (3.53) and (3.32), we may use (1.10), the fact that $m > 1$, and the smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ to see that there exists $C_{11} > 0$ such that

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{11} \text{ for all } t \in (0, T_{max,\varepsilon}). \tag{3.54}$$

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Then, the boundedness of n_ε can be obtained by the well-known Moser-Alikakos iteration procedure (see e.g. Lemma A.1 in [18]). Indeed, by using (3.53) and (3.54), we see that the hypotheses of Lemma A.1 in [18] are valid provided that we take the parameter p in Lemma 3.7 appropriately large. Thus, we obtain

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_{12} \ \text{ for all } \quad t \in (0, T_{max,\varepsilon}).$$

The proof of Lemma 3.8 is completed. □

With all above regularization properties of each component n_ε, c_ε and u_ε at hand, we can show the existence of global bounded solutions to the regularized system (2.1).

Lemma 3.9. *Let $m > 1$ and $\gamma \in (\frac{1}{2}, 1)$. Let $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)_{\varepsilon\in(0,1)}$ be classical solutions of (2.1) constructed in Lemma 2.1 on $[0, T_{max,\varepsilon})$. Then the solution is global on $[0, \infty)$. Moreover, one can find $C > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \ \textit{ for all } \ t \in (0, \infty)$$

and

$$\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \ \textit{ for all } \ t \in (0, \infty)$$

as well as

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \ \textit{ for all } \ t \in (0, \infty).$$

In addition, we also have

$$\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \ \textit{ for all } \ t \in (0, \infty).$$

Then, with the help of Lemma 3.9, we can straightforwardly deduce the uniform Hölder properties of c_ε, ∇c_ε and u_ε by the standard parabolic regularity theory as the proof of Lemmas 3.18–3.19 in [25] (see also [30]).

Lemma 3.10. *Let $m > 1$. Then one can find $\mu \in (0, 1)$ such that for some $C > 0$*

$$\|c_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} \le C \ \textit{ for all } \ t \in (0, \infty)$$

as well as

$$\|u_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} \le C \ \textit{ for all } \ t \in (0, \infty),$$

and for any $\tau > 0$, there exists $C(\tau) > 0$ fulfilling

$$\|\nabla c_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} \le C(\tau) \ \textit{ for all } \ t \in (\tau, \infty).$$

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4. Proof the main result

In this section, we will give the proof of the main result. Based on the above lemmas, we will construct the weak solution as the limitation of classical solutions to the approximating systems (2.1). Applying the idea of [30] (see also [13] and [25]), we first state the definition of the solution as follows.

Definition 4.1. Let $T > 0$ and (n_0, c_0, u_0) fulfills (1.10). Then a triple of functions (n, c, u) is called a weak solution of (1.4)-(1.6) if the following conditions are satisfied

$$\begin{cases} n \in L^1_{loc}(\bar{\Omega} \times [0, T)), \\ c \in L^1_{loc}([0, T); W^{1,1}(\Omega)), \\ u \in L^1_{loc}([0, T); W^{1,1}(\Omega)), \end{cases}$$

where $n \geq 0$ and $c \geq 0$ in $\Omega \times (0, T)$ as well as $\nabla \cdot u = 0$ in the distributional sense in $\Omega \times (0, T)$, moreover,

$$\begin{gathered} n^m \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0, \infty)), \\ cu, \quad nu \quad \text{and} \quad n\nabla c \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^2) \end{gathered}$$

and

$$-\int_0^T \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot, 0) = \int_0^T \int_\Omega n^m \Delta\varphi + \int_0^T \int_\Omega n\nabla c \cdot \nabla\varphi + \int_0^T \int_\Omega nu \cdot \nabla\varphi$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ satisfying $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0, T)$, as well as

$$\begin{aligned} & -\int_0^T \int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot, 0) \\ = \quad & -\int_0^T \int_\Omega \nabla c \cdot \nabla\varphi - \int_0^T \int_\Omega c\varphi + \int_0^T \int_\Omega n\varphi + \int_0^T \int_\Omega cu \cdot \nabla\varphi \end{aligned}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ and

$$\begin{aligned} & -\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot, 0) \\ = \quad & \kappa \int_0^T \int_\Omega u \otimes u \cdot \nabla\varphi - \int_0^T \int_\Omega \nabla u \cdot \nabla\varphi - \int_0^T \int_\Omega n\nabla\phi \cdot \varphi \end{aligned}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T); \mathbb{R}^2)$ fulfilling $\nabla\varphi \equiv 0$ in $\Omega \times (0, T)$.

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If for each $T > 0$, (n, c, u): $\Omega \times (0, \infty) \longrightarrow \mathbb{R}^4$ is a weak solution of (1.4)-(1.6) in $\Omega \times (0, T)$, then we call (n, c, u) a global weak solution of (1.4)-(1.6).

In order to use the Aubin-Lions Lemma (see [16]), we will need the regularity of the time derivative of bounded solutions. Employing almost exactly the same arguments as that in the proof of Lemmas 3.22–3.23 in [25] (the minor necessary changes are left as an easy exercise to the reader), and taking advantage of Lemma 3.9, we conclude the following Lemma.

Lemma 4.1. *Let $m > 1$ and $\varsigma > \max\{m, 2(m-1)\}$. Then for all $\varepsilon \in (0, 1)$, there exists a positive constant C independent of ε such that*

$$\|\partial_t n_\varepsilon(\cdot, t)\|_{(W_0^{2,2}(\Omega))^*} \leq C \ \ \text{for all} \ \ t \in (0, \infty).$$

Moreover, let $\varsigma > \max\{m, 2(m-1)\}$. Then for all $T > 0$ and $\varepsilon \in (0, 1)$, one can find $C(T)$ independent of ε such that

$$\int_0^T \|\partial_t (n_\varepsilon + \varepsilon)^\varsigma(\cdot, t)\|_{(W_0^{2,2}(\Omega))^*} dt \leq C(T) \ \ \text{for all} \ \ t \in (0, T)$$

and

$$\int_0^T \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^\varsigma|^2 \leq C(T) \ \ \text{for all} \ \ t \in (0, T).$$

Finally, we can prove the main result.

Proof of Theorem 1.1. In conjunction with Lemma 3.9 and the Aubin-Lions compactness lemma (see Simon [16]), we thus infer the existence of a sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ along which

$$n_\varepsilon \rightharpoonup n \ \ \text{weakly star in} \ \ L^\infty(\Omega \times (0, \infty)), \tag{4.1}$$

$$n_\varepsilon \to n \ \ \text{in} \ \ C^0_{loc}([0, \infty); (W_0^{2,2}(\Omega))^*), \tag{4.2}$$

$$c_\varepsilon \to c \ \ \text{in} \ \ C^0_{loc}(\bar{\Omega} \times [0, \infty)), \tag{4.3}$$

$$\nabla c_\varepsilon \to \nabla c \ \ \text{in} \ \ C^0_{loc}(\bar{\Omega} \times (0, \infty)), \tag{4.4}$$

$$\nabla c_\varepsilon \rightharpoonup \nabla c \ \ \text{weakly star in} \ \ L^\infty(\Omega \times (0, \infty)) \tag{4.5}$$

as well as

$$u_\varepsilon \to u \ \ \text{in} \ \ C^0_{loc}(\bar{\Omega} \times [0, \infty)) \tag{4.6}$$

and

$$Du_\varepsilon \rightharpoonup Du \ \ \text{weakly star in} \ \ L^\infty(\Omega \times (0, \infty)) \tag{4.7}$$

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holds for some limit $(n, c, u) \in (L^\infty(\Omega \times (0, \infty)))^4$ with nonnegative n and c. On the other hand, Lemma 4.1 implies that for each $T > 0$, $(n_\varepsilon^\varsigma)_{\varepsilon \in (0,1)}$ is bounded in $L^2((0, T); W^{1,2}(\Omega))$. By using Aubin-Lions lemma again, one can obtain $n_\varepsilon^\varsigma \to z^\varsigma$ for some nonnegative measurable $z : \Omega \times (0, \Omega) \to \mathbb{R}$. Thus, by utilizing (4.1) and the Egorov theorem, we get $z = n$. Thereby

$$n_\varepsilon \to n \ \text{ a.e. in } \ \Omega \times (0, \infty) \tag{4.8}$$

holds.

Due to these convergence properties (see (4.1)–(4.8)), by applying the standard arguments, we may take $\varepsilon = \varepsilon_j \searrow 0$ in each term of the natural weak formulation of (2.1) separately. Then we can verify that (n, c, u) can be complemented by some pressure function P in such a way that (n, c, u, P) is a weak solution of (1.4)-(1.6). Finally, we can infer from the boundedness of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ and the Banach-Alaoglu theorem that (n, c, u) is bounded. □

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