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On the Stability of Boundary Layers of Incompressible Euler Equations

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In this paper we investigate the stability and instability of boundary layers of incompressible Euler equations. © 2000 Academic Press

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Dans cet article on étudie la stabilité et l'instabilité de couches limites des équations d'Euler. © 2000 Academic Press

Mots clés: mécanique des fluides; équations d'Euler; stabilité; couches limites.

1. INTRODUCTION

In this paper we study the stability of boundary layer type solutions $u = (u_1, u_2)$ for the $2D$ incompressible Euler equations

$$\partial_t u + (u.\nabla)\, u + \nabla p = 0, \tag{1}$$

$$\nabla . u = 0 \tag{2}$$

in the periodic half strip $(x, y) \in \Omega = \mathbb{T} \times \mathbb{R}_+$, with boundary condition

$$u_2 = 0 \quad \text{on} \quad y = 0 \tag{3}$$

and initial condition $u(0, .) = u_0$, where $u_0(x, y)$ is a given function. There are two main motivations for this study. First, the Prandtl boundary layers which appear in the inviscid limit of the Navier–Stokes equations have a size $\sqrt{\nu}$, where ν is the viscosity. Recent works [6, 14, 15] have shown that when the size of the layer is of order of the viscosity the layer is completely dominated by viscous effects: if the layer is small enough it is stable, else it can be *unstable* (and is unstable indeed in some cases [14]). If the size of Prandtl layer is much larger than ν, the viscosity is no longer able to

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stabilize the layer by itself and we have to find a stabilizing effect elsewhere, namely in the corresponding *inviscid* equations, in the Euler equations. Therefore we have first to investigate the stability of boundary layer type solutions for the Euler equations, which is moreover a classical approach in fluid mechanics; see [7, 20, 22]. The idea is the following: we know that the viscosity is not sufficient to control the Prandtl layer and that an eventual stabilization mechanism has to be found in Euler equations. Hence we first ignore the viscosity and study the stability of inviscid boundary layers, before trying to add viscous effects (which is a real difficulty: as noticed by Rayleigh, viscosity can have a destabilizing effect!). The methods of this paper will be applied to the Prandtl layer in [13] (with weaker results). The second motivation is the study of the semigeostrophic asymptotic, which arises in the study of frontogenesis in meteorology, following Hoskins [16], which is a limit more complicated than that studied here, but which contains it as a particular case.

So let us turn to the stability of a solution

$$u^\eta = \left(u_1^\eta\left(t, x, y, \frac{y}{\eta}\right), u_2^\eta\left(t, x, y, \frac{y}{\eta}\right)\right)$$

of (1), (2) on $\mathbb{T}_x \times \mathbb{R}_y^+$. It will be clear in the proof that the geometry of the domain is unimportant in the proof of the stability theorems, which can be extended to $\mathbb{R}_x \times \mathbb{R}_y^+$, to smooth exterior or interior domains. The dimension 2 is, however, crucial.

The study of the stability of profiles of the form $(0, u_2(t, x, y))$ has a long story and began with Rayleigh (1880), who proved that a necessary condition for instability is that u_2 should have an inflection point. It was then sharpened by Fjortoft, Tollmien, Lin (see the books of Drazin and Reid [7], Landau and Lifchitz [18], and Schlichting [22]). Mathematically the problem has been investigated by Arnold, who gives sufficient conditions for the *nonlinear* stability of *stationary* solutions of (1, 2). Using the functional

$$\int |\nabla\phi|^2 + \frac{\nabla\psi}{\nabla\Delta\psi}\,|\Delta\phi|^2,$$

where ϕ is the stream function of the perturbation and ψ the stream function of the stationary flow, he proves in particular that if this quadratic form is positive definite, or if

$$\int |\nabla\phi|^2 + \max\left(\frac{\nabla\psi}{\nabla\Delta\psi}\right)|\Delta\phi|^2$$

is negative definite, then the flow is stable.

Notice however that this criterion does not apply in our case since $\nabla\psi/\nabla\Delta\psi$ is neither positive nor negative. Moreover we would like to handle nonstationary solutions. The price to pay is that we get stability over times of order $O(1)$ and not *global* stability.

The paper is divided into two parts. In the first one we prove stability results using an energy method when basically there is no inflection point in the profile (the conditions are a little more technical). Namely let us assume

(H) there exist $\phi_\eta(t, x, y) > 0$ and $C > 0$, $\alpha > 0$ such that

$$\phi_\eta = \exp(-2\alpha y) \qquad \text{for} \quad y \geqslant 1, \tag{4}$$

$$\left|\frac{\partial_t \phi_\eta}{\phi_\eta}\right|_{L^\infty} + \left|\frac{u_1^\eta\, \partial_x \phi_\eta}{\phi_\eta}\right|_{L^\infty} + \left|\frac{u_2^\eta\, \partial_y \phi_\eta}{\phi_\eta}\right|_{L^\infty} \leqslant C, \tag{5}$$

$$|\partial_{xx}^2 u_2^\eta|_{L^\infty} + |\partial_{xy}^2 u_1^\eta|_{L^\infty} + |\partial_{xy}^2 u_2^\eta|_{L^\infty} \leqslant C\sqrt{\phi_\eta}, \tag{6}$$

$$\left|\frac{\partial_{yy}^2 u_1^\eta}{\phi_\eta} + 1\right| \leqslant \frac{C}{\sqrt{\phi_\eta}}, \tag{7}$$

and

$$C_1 \exp(-2\alpha y) + C_1 \frac{\mu}{\eta^2} \exp(-y/\eta) \leqslant \phi_\eta \leqslant C_2 \exp(-2\alpha y) + C_2 \frac{\mu}{\eta^2} \exp(-y/\eta). \tag{8}$$

Most of these assumptions are natural if u^η has a boundary layer and use the incompressibility condition in particular. For instance, as the layer is in the y direction, we enforce the x and t derivatives to be bounded: $|\partial_x u_1^\eta|_{L^\infty} \leqslant C$ and $|\partial_t \phi_\eta| + |\partial_x \phi_\eta| \leqslant C\,|\phi_\eta|$ are natural. By incompressibility condition this leads to $|\partial_y u_2^\eta|_{L^\infty} \leqslant C$ and $|u_2^\eta| \leqslant Cy$; hence, in the boundary layer, $|u_2^\eta\, \partial_y \phi_\eta| \leqslant Cy\, |\partial_y \phi_\eta| \leqslant C\,|\phi_\eta|$. The bound (5) is therefore natural, and similarly for (6). The main assumption is in fact (7), which essentially says $\phi_\eta = -\partial_{yy}^2 u_1^\eta$ in the boundary layer. Notice also that (6) is easily satisfied by $|\partial_{xx}^2 u_2^\eta|$ and $|\partial_{xy}^2 u_2^\eta|$ which are in fact bounded. Assumption (8) could be replaced by other decreasing properties. Notice that under assumption (7), there is no inflection point in the boundary layer. However, (7) is more strict since we enforce asymptotic behavior (8) and spatial regularity (which are classical requirements in boundary layer theory).

Let us first consider the linear equation on $v = (v_1, v_2)$,

$$\partial_t v + (u^\eta.\nabla)\, v + (v.\nabla)\, u^\eta + \nabla p = w, \tag{9}$$

$$\nabla . v = 0 \tag{10}$$

in $\mathbb{T}\times\mathbb{R}_+$, with boundary condition

$$v_2=0 \qquad \text{on} \quad y=0 \tag{11}$$

and initial data $v(0,.,.)=v_0$, where $v_0(x,y)$ and $w(t,x,y)$ are given functions.

Let

$$I(t)=\int \frac{|\operatorname{curl} v|^2}{\phi_\eta}\,dx\,dy.$$

THEOREM 1.1 (Linear Stability under Assumption (H)). *Let $u^\eta \in L^\infty([0,T], H^2(\mathbb{T}\times\mathbb{R}_+))$ be a sequence of functions such that* (H) *holds true. Then there exists a constant $C_0 \geqslant 0$ depending only on u^η such that if $w\in L^\infty(0,T; H^1(\mathbb{T}\times\mathbb{R}_+))$ and $v_0\in H^1(\mathbb{T}\times\mathbb{R}_+)$ then, for $0\leqslant t\leqslant T$,*

$$\partial_t I(t)\leqslant C_0 I(t)+\int \frac{|\operatorname{curl} w|^2}{\phi_\eta},$$

where v is the corresponding solution of (9, 10, 11).

Let us turn to the nonlinear equations

$$\partial_t v+(u^\eta.\nabla)\,v+(v.\nabla)\,u^\eta+(v.\nabla)\,v+\nabla p=w, \tag{12}$$

$$\nabla.v=0. \tag{13}$$

Let us assume moreover (H′) which consists of (14) and (15):

$$|\partial_x^\alpha\,\partial_y^\beta u^\eta|\leqslant\frac{C}{\eta^\beta}\exp(-y/\eta)+C\exp(-\alpha y) \qquad \text{for} \quad \alpha+\beta\leqslant\sigma, \tag{14}$$

$$|\sqrt{\phi_\eta}\,\partial_x u_2^\eta|_{L^\infty}+\left|\frac{\partial_y u_1^\eta}{\sqrt{\phi_\eta}}\right|_{L^\infty}\leqslant C. \tag{15}$$

Let

$$|||v|||_s^2=\sum_{|\alpha|+|\beta|\leqslant s}\eta^{8\alpha+8\beta}\int\frac{|\partial_x^\alpha\,\partial_y^\beta \operatorname{curl} v|^2}{\phi_\eta^{1+\beta}}.$$

THEOREM 1.2 (Nonlinear Stability under (H) and (H′)). *Let $s\geqslant 0$. Let σ be large enough. Under assumption* (H) *and* (H′) *there exists $C(t)\geqslant 0$ depending only on u^η such that*

$$\partial_t\,|||v|||_s^2\leqslant C(t)\,|||v|||_s^2+\frac{C(t)}{\eta^{16s}}\,|||v|||_s^3+|||w|||_s^2.$$

This estimate is not uniform in η but is sufficient to justify asymptotic expansions since the large factor η^{-16s} is compensated by the cube $|||v|||_s^3$ (see the proof of Section 5.4).

As an application of Theorems 1.1 and 1.2 we will justify asymptotic expansions of the boundary layer. Namely

THEOREM 1.3 (Asymptotic Expansion under (H)). *Let u_0^η be a given sequence of initial data, having for every N arbitrarily large an asymptotic expansion of the form*

$$u_0^\eta(x, y) = \sum_{j=0}^{N} \eta^j u_{j,\,\mathrm{int}}^0(x, y) + \sum_{j=0}^{N} \eta^j u_{j,\,\mathrm{b}}^0\left(x, \frac{y}{\eta}\right) + \eta^N \mathscr{R}_{N,\,0}^\eta, \tag{16}$$

where $u_{j,\,\mathrm{int}}^0$ and $u_{j,\,\mathrm{b}}^0$ are in H^s for every $s \geqslant 0$ and for every j, the functions $u_{j,\,\mathrm{int}}$ and $u_{j,\,\mathrm{b}}^0$ rapidly decreasing in their second variable, and where

$$\|\mathscr{R}_{N,\,0}^\eta\|_{H^s} \leqslant C_{N,\,s} \eta^{-s}.$$

Let $Y = y/\eta$. There exists a global solution $u_{0,\,\mathrm{int}}(t, x, y) \in L_{\mathrm{loc}}^\infty([0, +\infty[, H^s(\Omega))$ (for every s) of the Euler equations

$$\partial_t u_{0,\,\mathrm{int}} + (u_{0,\,\mathrm{int}} \,.\, \nabla)\, u_{0,\,\mathrm{int}} + \nabla p_{0,\,\mathrm{int}} = 0, \tag{17}$$

$$\nabla . u_{0,\,\mathrm{int}} = 0, \tag{18}$$

$$u_{0,\,\mathrm{int},\,2} = 0 \qquad \text{on } \partial\Omega \tag{19}$$

with initial data $u_{0,\,\mathrm{int}}^0$, where $u_{0,\,\mathrm{int},\,2}$ is the second componant of $u_{0,\,\mathrm{int}}$, and there exists a solution $u_{0,\,\mathrm{b}}(t, x, Y) \in L_{\mathrm{loc}}^\infty([0, T_*[, H^s(\Omega))$ for every s and some $T_* > 0$ of inviscid Prandtl type equations

$$\begin{aligned}
&\partial_t(u_{0,\,\mathrm{int},\,1}(t, x, 0) + u_{0,\,\mathrm{b},\,1}) + (u_{0,\,\mathrm{int},\,1}(t, x, 0)\\
&\quad + u_{0,\,\mathrm{b},\,1})\, \partial_x(u_{0,\,\mathrm{int},\,1}(t, x, 0) + u_{0,\,\mathrm{b},\,1})\\
&\quad + (Y\partial_y u_{0,\,\mathrm{int},\,2}(t, x, 0) + u_{0,\,\mathrm{b},\,2})\, \partial_Y u_{0,\,\mathrm{b},\,1} + \partial_x\, p_{0,\,\mathrm{int}}(t, x, 0) = 0,
\end{aligned} \tag{20}$$

$$\partial_x u_{0,\,\mathrm{b},\,1} + \partial_Y u_{0,\,\mathrm{b},\,2} = 0, \tag{21}$$

$$u_{0,\,\mathrm{b},\,2} = 0 \qquad \text{on } \partial\Omega \tag{22}$$

with initial data $u_{0,\,\mathrm{b}}^0$. Moreover for every T such that there exists $a > 0$ with

$$\sup_{0 \leqslant t \leqslant T,\, (x,\, Y) \in \Omega} |\exp(aY)\, \partial_{YY}^2 u_{0,\,\mathrm{b},\,1}(t, x, Y)| < +\infty, \tag{23}$$

$$\sup_{0 \leqslant t \leqslant T,\, (x,\, Y) \in \Omega} \exp(aY)\, \partial_{YY}^2 u_{0,\,\mathrm{b},\,1}(t, x, Y) < 0, \tag{24}$$

and

$$\sup_{0 \leqslant t \leqslant T,\, (x,\, Y) \in \Omega} \left(\frac{|\partial_t \partial^2_{YY} u_{0,\,\mathrm{b},\,1}|}{|\partial^2_{YY} u_{0,\,\mathrm{b},\,1}|} + \frac{|\partial_x \partial^2_{YY} u_{0,\,\mathrm{b},\,1}|}{|\partial^2_{YY} u_{0,\,\mathrm{b},\,1}|} + \frac{|Y \partial^2_{YYY} u_{0,\,\mathrm{b},\,1}|}{|\partial^2_{YY} u_{0,\,\mathrm{b},\,1}|} + \frac{|\partial_Y u_{0,\,\mathrm{b},\,1}|}{\sqrt{|\partial^2_{YY} u_{0,\,\mathrm{b},\,1}|}} \right) < +\infty, \tag{25}$$

there exist functions $u_{j,\,\mathrm{int}}(t, x, y)$ *and* $u_{j,\,\mathrm{b}}(t, x, y)$ *in* $L^\infty([0, T], H^s)$ *(for every* s*),* $u_{j,\,\mathrm{int}}$ *and* $u_{j,\,\mathrm{b}}$ *being moreover rapidly decreasing in their last variable, with*

$$u_{j,\,\mathrm{int}}(0, x, y) = u^0_{j,\,\mathrm{int}}, \qquad \text{and} \qquad u_{j,\,\mathrm{b}}(0, x, y) = u^0_{j,\,\mathrm{b}},$$

such that the solution u^η *of Euler equations with initial value* u^η_0 *satisfies*

$$u^\eta(t, x, y) = \sum_{j=0}^{N} \eta^j u_{j,\,\mathrm{int}}(t, x, y) + \sum_{j=0}^{N} \eta^j u_{j,\,\mathrm{b}}\left(t, x, \frac{y}{\eta}\right) + \eta^N \mathscr{R}^\eta_N(t, x, y) \tag{26}$$

on $[0, T]$*, with*

$$\|\mathscr{R}^\eta_N\|_{L^\infty([0,\,T],\,H^s)} \leqslant C_s \eta^{-s} \tag{27}$$

for $0 < \eta \leqslant 1$ *and for every* s *and every* N*.*

In order to simplify the presentation we assume that $u^0_{j,\,\mathrm{int}}$ and $u^0_{j,\,\mathrm{b}}$ are in $\bigcap_{s \geqslant 0} H^s(\Omega)$ (else we would lose some regularity at each step j). Therefore the boundary layer has completely regular behavior if initially there is no inflection point in it (more precisely if (H) holds true). This fact is physically well known. Similar theorems have been proved for the study of the inviscid limit of parabolic equations, in the noncharacteristic case in [14] and in the totally characteristic case in [12]. Notice that here the boundary layer is stable and survives over times of order $O(1)$, whereas there is no dissipation mechanism and no viscosity. The layer is therefore purely *inertial*. Totally characteristic hyperbolic systems, as noticed in the last section of [12], have similar behavior: in this case boundary layers can survive because of a particular algebraic property of the coefficients of the system. The stabilization effect is here much more complex. We refer to the beautiful mechanism suggested by Lin, as described for instance in [11], for physical insights. Condition (23) could be replaced by other decreasing properties. The main point is that $\partial^2_{YY} u_{0,\,\mathrm{b},\,1}$ never vanishes.

In the second part of the paper we prove an instability result for a particular profile having an inflection point. Namely we first recall

THEOREM 1.4 (Rayleigh, 1894, Revisited). *There exist a sequence of stationary solutions u^η of* (1, 2) *and a solution v^η of* (9, 10) *such that*

$$\|v^\eta\|_{L^2} = C_1 \exp\left(C_2 \frac{t}{\eta} \right) \tag{28}$$

for some positive constants C_1 and C_2.

We then specialize Theorem 1.4 to the following fully nonlinear instability result:

THEOREM 1.5 (Nonlinear Instability). *For every N and s arbitrarily large there exist two solutions u^η and v^η of* (1, 2), *u^η being moreover stationary and smooth and having boundary layer type behavior*

$$|\partial_x^\alpha \partial_y^\beta u^\eta| \leqslant \frac{C_{\alpha,\beta}}{\eta^\beta} \exp(-y/\eta)$$

for every α and $\beta \in \mathbb{N}$, such that

$$\|u^\eta(0,.) - v^\eta(0,.)\|_{H^s} \leqslant \eta^N$$

and

$$\|u^\eta(T_\eta,.) - v^\eta(T_\eta,.)\|_{L^\infty} \geqslant \sigma > 0$$

for some constant σ independent on η and for some sequence $T_\eta \to 0$ as $\eta \to 0$. As a consequence, Theorem 1.3 *is false without assumption* (H) *at* $t = 0$.

We also have

$$\|u^\eta(T_\eta) - v^\eta(T_\eta)\|_{L^2} \geqslant \sigma \eta^{1/2},$$

the $\eta^{1/2}$ factor coming from the small size of the boundary layer. The proof uses elementary tools and relies on upper bounds on the growth of solutions of inviscid Orr Sommerfeld equations, proved in Section 7.1 by O.D.E. arguments.

The situation is therefore highly chaotic when there are inflection points in the boundary layer profile: the smallest error on the initial data (measured in Sobolev spaces) leads to order one errors on the solution, even in very short times, of order $o(1)$. We conjecture that the same profile u^η would be stable over times of order $O(1)$ if we consider *analytic* perturbations instead of perturbations with Sobolev regularity, since it will be clear in the proof that the instability comes from eigenmodes which are highly oscillatory in the x direction, with period η, these eigenmodes being typically "killed" by any reasonable analytic type space (they are damped by a factor $\exp(-C/\eta)$). This remark is coherent and makes the link with the work [4] on the inviscid limit of Navier–Stokes equations in an analytic framework.

Notice that the stability results are valid only in two space dimensions, whereas Theorems 1.4 and 1.5 hold in fact in any space dimension.

After rescaling space and time by an η factor as explained in Section 6, Theorem 1.5 gives an example of stationary solution of 2D Euler equations, in $C^\infty(\mathbb{T} \times \mathbb{R}_+)$, which is linearly and nonlinearly unstable. Unstable stationary solutions have been recently constructed in the periodic case by Friedlander, Strauss and Vishik in [9] in the spirit of [10] (however, as stated they only proved instability in H^s with $s > 2$, whereas here we go to L^∞, a really physically relevant space). To the best of the author's knowledge, the present proof is the first rigorous proof of nonlinear instability in Euler equations, which leads to sup norm separation.

Notice also that a related problem (Oseen equations) has been studied by Temam and Wang [23].

2. PRELIMINARIES

2.1. *Vorticity Equation*

Let ζ be the two-dimensional vorticity. Nonlinear 2D Euler equations (1), (2) are equivalent to

$$\partial_t \zeta + (u . \nabla) \zeta = 0 \tag{29}$$

with $\operatorname{curl} u = \zeta$ and $\operatorname{div} u = 0$. The linearized version of (29) around (u_l, θ_l) is

$$\partial_t \Theta + (u_l . \nabla) \Theta + (v . \nabla) \Theta_l = \operatorname{curl} w, \tag{30}$$

where

$$\operatorname{curl} v = \Theta, \qquad \operatorname{div} v = 0, \tag{31}$$

$$\operatorname{curl} u_l = \Theta_l, \qquad \operatorname{div} u_l = 0, \tag{32}$$

which is equivalent to (9, 10).

2.2. *Green Functions*

Let us compute in this section the Green function $G_k(z_0, z)$ of

$$\partial^2_{zz} f - k^2 f = \delta_{z_0} \tag{33}$$

for $k \neq 0$, with boundary conditions

$$f = 0 \quad \text{for} \quad z = 0 \qquad \text{and} \qquad f \to 0 \quad \text{as} \quad z \to +\infty. \tag{34}$$

We get

$$G_k(z_0, z) = \frac{1}{2\,|k|} \exp(-|k|\, z_0) \left(\exp(|k|\, z) - \exp(-|k| z) \right) \qquad \text{for} \quad z \leqslant z_0$$

and

$$G_k(z_0, z) = \frac{1}{2\,|k|} \exp(-|k|\, z) \left(\exp(|k|\, z_0) - \exp(-|k|\, z_0) \right) \qquad \text{for} \quad z \geqslant z_0.$$

Notice that

$$|G_k(z_0, z)| \leqslant \frac{1}{2\,|k|}$$

for every z_0 and z, that G_k is symmetric,

$$G_k(z, z') = G_k(z', z), \tag{〉35}$$

for every z and z', and that G_k is increasing on $[0, z_0]$ and decreasing on $[z_0, +\infty[$. For $z \leqslant z_0$,

$$\exp(|k|\, z) - \exp(-|k|\, z) = \exp(|k|\, z)(1 - \exp(-2\,|k|\, z))$$

is bounded by $2\,|k|\, z \exp(|k|\, z)$; hence

$$|G_k(z_0, z)| \leqslant z \exp(|k|\, (z - z_0)) \leqslant z.$$

Similarly, for $z \geqslant z_0$,

$$|G_k(z_0, z)| \leqslant z_0;$$

hence in both cases

$$|G_k(z_0, z)| \leqslant \inf\left(z_0, z, \frac{1}{2\,|k|} \right). \tag{36}$$

2.3. *Weighted Inequalities*

For $0 < \eta < 1$, let

$$\phi_\eta = 1 + \frac{1}{\eta^2} \exp\left(-\frac{y}{\eta} \right). \tag{37}$$

Lemma 2.1. *Let $\psi_\eta(x, y)$ be a family of weights such that for $0 \leqslant y \leqslant 1$ and for $0 < \eta < 1$,*

$$C_1 \phi_\eta \leqslant \psi_\eta \leqslant C_2 \phi_\eta, \tag{38}$$

where C_1 and C_2 are two positive constants (independent of η). Let $\Psi \in H^2(\Omega)$ be supported in $0 \leqslant y \leqslant 1$, and let

$$(v_1, v_2) = \nabla^\perp \Psi, \qquad \Delta\Psi = \Theta. \tag{39}$$

Then

$$\|v_1\|^2_{L^2(\mathbb{T} \times [0,1])} + \|v_2\|^2_{L^2(\mathbb{T} \times [0,1])} \leqslant C \int_\Omega \frac{\Theta^2}{\psi_\eta}\, dx\, dy, \tag{40}$$

the constant C being independent of η.

Proof. Let us take the Fourier transforms of Θ and Ψ in the x variable only. We have

$$\Theta(t, x, y) = \sum_{k=-\infty}^{+\infty} \Theta_k(t, y) \exp(ikx)$$

and

$$\Psi(t, x, y) = \sum_{k=-\infty}^{+\infty} \Psi_k(t, y) \exp(ikx).$$

Using (39) we get

$$(\partial^2_{yy} - k^2)\, \Psi_k(t, y) = \Theta_k(t, y),$$

which can be solved using the Green function $G_k(y, y')$. Namely, for $k \neq 0$,

$$\Psi_k(t, y) = \int_0^{+\infty} dy'\, G_k(y', y)\, \Theta_k(t, y') = \int_0^1 dy'\, G_k(y', y)\, \Theta_k(t, y'). \tag{41}$$

Hence for $k \neq 0$,

$$\int_0^1 |\Psi_k(t, y)|^2\, dy \leqslant \left(\int_0^{+\infty} \frac{|\Theta_k|^2}{\phi_\eta} \right) \left(\int_0^1 dy \int_0^y \phi_\eta(y')\, G_k^2(y', y)\, dy' \right.$$

$$\left. + \int_0^1 dy \int_y^1 \phi_\eta(y')\, G_k^2(y', y)\, dy' \right).$$

But

$$\frac{\mu}{4k^2\eta^2}\int_0^{+\infty}\int_0^y e^{-y'/\eta}e^{-2ky}(e^{2ky'}-2+e^{-2ky'})\,dy\,dy' = \frac{\mu\eta k}{(1+2\eta k)(1+4\eta k)\,k^2},$$

$$\frac{\mu}{4k^2\eta^2}\int_0^{+\infty}\int_y^{+\infty} e^{-y'/\eta}e^{-2ky'}(e^{2ky}-2+e^{-2ky})\,dy\,dy' = \frac{2\mu\eta^2k^2}{(1+2k\eta)^2\,(1+4k\eta)\,k^2},$$

$$\frac{1}{4k^2}\int_0^1\int_0^y e^{-2ky}(e^{2ky'}-2+e^{-2ky'})\,dy\,dy'$$
$$= \frac{4k+8ke^{-2k}+4e^{-2k}+e^{-4k}}{32k^4} - \frac{5}{32k^4},$$

these three quantities being bounded by C/k^2, and similarly

$$\frac{1}{4k^2}\int_0^1\int_y^{+\infty} e^{-2ky'}(e^{2ky}-2+e^{-2ky})\,dy\,dy' \leqslant \frac{C}{k^2}.$$

Therefore

$$\int_0^1 |\Psi_k(t,\,y)|^2\,dy \leqslant \frac{C}{k^2}\int_0^{+\infty} \frac{|\Theta_k|^2}{\phi_\eta}\,dy.$$

But

$$v_2(t,\,x,\,y) = -\sum_{k=-\infty}^{+\infty} ik\Psi_k(t,\,y)\exp(ikx);$$

therefore

$$\|v_2\|_{L^2}^2 = \sum_{k=-\infty}^{+\infty} k^2\,\|\Psi_k\|_{L^2}^2 \leqslant C\sum_{k=-\infty}^{+\infty}\int_0^{+\infty}\frac{|\Theta_k|^2}{\phi_\eta}\,dy$$
$$\leqslant C\int \frac{|\Theta|^2}{\phi_\eta}\,dx\,dy \leqslant C'\int \frac{|\Theta|^2}{\psi_\eta}\,dx\,dy,$$

which proves half of the lemma (the case $k=0$ being straightforward).

Let us now bound v_1. For this we need to bound $\partial_y\Psi_k$. For $k\neq 0$,

$$\partial_y\Psi_k(t,\,y) = \int_0^{+\infty} dy'\,\partial_y G_k(y',\,y)\,\Theta_k(t,\,y'), \tag{42}$$

hence

$$\int_0^1 |\partial_y \Psi_k|^2 \, dy \leqslant \left(\int_0^{+\infty} \frac{|\Theta_k|^2}{\phi_\eta} \, dy \right) \int_0^1 dy \int_0^1 dy' \, |\partial_y G_k(y', y)|^2 \, \phi_\eta(y').$$

But

$$\frac{\mu}{\eta^2} \int_0^{+\infty} dy \int_0^y dy' \, \frac{e^{-2ky}}{4} \, (e^{2ky'} - 2 + e^{-2ky'}) \, e^{-y'/\eta} = \frac{\mu k \eta}{8k^2\eta^2 + 6k\eta + 1},$$

$$\frac{\mu}{\eta^2} \int_0^{+\infty} dy \int_y^{+\infty} dy' \, \frac{e^{-2ky'}}{4} \, (e^{2ky} + 2 + e^{-2ky}) \, e^{-y'/\eta} = \frac{\mu(2k^2\eta^2 + 4k\eta + 1)}{(1 + 2k\eta)^2 \, (1 + 4k\eta)},$$

$$\int_0^1 dy \int_0^y dy' \, e^{-2ky}(e^{2ky'} - 2 + e^{-2ky'}) = \frac{4k + 8ke^{-2k} + 4e^{-2k} + 1}{8k^2} - \frac{5}{8k^2},$$

these three quantities being bounded by some constant C independent on k, and

$$\int_0^1 dy \int_y^1 dy' \, e^{-2ky'}(e^{2ky} + 2 + e^{-2ky}) \leqslant C;$$

therefore

$$\int_0^{+\infty} |\partial_y \Psi_k(t, y)|^2 \leqslant C \left(\int \frac{|\Theta_k|^2}{\phi_\eta} \, dy \right) \leqslant C' \left(\int \frac{|\Theta_k|^2}{\psi_\eta} \, dy \right).$$

It remains to handle the case $k = 0$. As $\partial_y \Psi$ is supported in $y \in [0, 1]$,

$$\partial_y \Psi_0(y) = -\int_y^1 \partial^2_{yy} \Psi_0,$$

$$|\partial_y \Psi_0(y)|^2 \leqslant C \int_0^1 dy \, \frac{|\partial^2_{yy} \Psi_0|^2}{\phi_\eta} \int_y^1 \left(1 + \frac{\mu}{\eta^2} \exp(-y'/\eta) \right) dy'$$

$$\leqslant C \int_0^1 \frac{|\partial^2_{yy} \Psi_0|^2}{\phi_\eta} \left(2 + \frac{\mu}{\eta} \exp(-y/\eta) \right) dy.$$

Hence

$$\int_0^1 |\partial_y \Psi_0(y)|^2 \leqslant C \int_0^1 \frac{|\partial^2_{yy} \Psi_0|^2}{\phi_\eta} \, dy,$$

which ends the proof of the lemma. ∎

To handle large y we have to put a weight on v_1 and v_2. Let us define, for $0<\eta<1$ and $\alpha>0$,

$$\phi_{\alpha,\eta}(y)=\exp(-2\alpha y)+\frac{1}{\eta^2}\exp\left(-\frac{y}{\eta}\right). \tag{43}$$

LEMMA 2.2. *Let $\alpha>0$ and let ψ_η be a family of weights such that*

$$C_1\phi_{\alpha,\eta}\leqslant\psi_\eta\leqslant C_2\phi_{\alpha,\eta} \tag{44}$$

for some positive constants C_1 and C_2. Then the solution (v_1, v_2) of

$$(v_1, v_2)=\nabla^\perp\Psi, \qquad \Delta\Psi=\Theta, \tag{45}$$

where $\Psi\in H^2(\Omega)$, satisfies

$$\|v_1\|^2_{L^2(\mathbb{T}\times\mathbb{R}_+)}+\|v_2\|^2_{L^2(\mathbb{T}\times\mathbb{R}_+)}\leqslant C_\alpha\int\frac{\Theta^2}{\psi_\eta} \tag{46}$$

where C is independent of η.

Proof. Let χ be a smooth decreasing positive function with support in $0\leqslant y\leqslant 1$ and which equals 1 for $y\leqslant 1/2$. Splitting Ψ into $\chi\Psi$ and $(1-\chi)\Psi$ and using Lemma 2.1, we are led to prove (46) for Ψ and Θ with support in $y\geqslant 1/2$. Repeating the proof of the previous section we have to bound, for $k\neq 0$,

$$\int_0^{+\infty}dy\int_0^y dy' e^{-2\alpha y'}G_k^2(y', y)\,dy'\,dy$$

$$=\int_0^{+\infty}dy\int_0^y dy'\,e^{-2\alpha y'-2ky}(e^{ky'}-e^{-ky'})^2$$

$$=\frac{k}{2\alpha(2k^2+3\alpha k+\alpha^2)}\leqslant C,$$

$$\int_0^{+\infty}dy\int_y^{+\infty}dy'\,e^{-2\alpha y'}G_k^2(y', y)\,dy'\,dy$$

$$=\int_0^{+\infty}dy\int_y^{+\infty}dy'\,e^{-2\alpha y'}e^{-2ky'}(e^{ky}-e^{-ky})^2$$

$$=\frac{k^2}{2\alpha(2k^2+3\alpha k+\alpha^2)(\alpha+k)}\leqslant C,$$

and

$$\int_0^{+\infty} dy \int_0^y dy'\, e^{-2\alpha y'}\, |\partial_y G_k(y', y)|^2\, dy' dy$$

$$= \int_0^{+\infty} dy \int_0^y dy'\, e^{-2\alpha y' - 2ky} (e^{ky'} - e^{-ky'})^2 \leqslant C,$$

$$\int_0^{+\infty} dy \int_y^{+\infty} dy'\, e^{-2\alpha y'}\, |\partial_y G_k(y', y)|^2\, dy'\, dy$$

$$= \int_0^{+\infty} \int_y^{+\infty} e^{-2\alpha y'} e^{-2ky'} (e^{ky} + e^{-ky})^2$$

$$= \frac{1}{2} \frac{k^2 + 4\alpha k + 2\alpha^2}{(\alpha + k)(2k^2 + 3\alpha k + \alpha^2)\,\alpha} \leqslant C.$$

For $k = 0$,

$$\partial_y \Psi_0(y) = \int_y^{+\infty} \frac{\partial^2_{yy} \Psi_0}{\exp(-\alpha y')} \exp(-\alpha y')\, dy';$$

hence for $y \geqslant 1/2$,

$$|\partial_y \Psi_0(y)|^2 \leqslant C \exp(-\alpha y) \int_{1/2}^{+\infty} \frac{|\partial^2_{yy} \Psi_0|^2}{\phi_{\alpha, \eta}}$$

and

$$\int_0^{+\infty} |\partial_y \Psi_0|^2 \leqslant C \int \frac{\Theta_0^2}{\phi_{\alpha.\eta}},$$

which ends the proof of the lemma. ∎

3. LINEAR STABILITY: PROOF OF THEOREM 1.1

Let us prove Theorem 1.1 and let us drop the η index of ϕ_η for convenience. Let $\Theta = \operatorname{curl} v$ and let

$$I(t) = \int \frac{\Theta^2(t, x, y)}{\phi(t, x, y)}\, dx\, dy,$$

where we drop the index η. We have

$$\partial_t I = 2 \int \frac{\Theta \partial_t \Theta}{\phi} \, dx \, dy - \int \frac{\Theta^2}{\phi^2} \partial_t \phi \, dx \, dy.$$

Using (H) we get

$$\left| \int \frac{\Theta^2}{\phi^2} \partial_t \phi \, dx \, dy \right| \leqslant C I(t).$$

But, using (30),

$$\int \frac{\Theta \partial_t \Theta}{\phi} = -\int \frac{u_1 \Theta \, \partial_x \Theta}{\phi} - \int \frac{u_2 \Theta \, \partial_y \Theta}{\phi} - \int \frac{v_1 \, \partial_x \Theta_l}{\phi} \Theta - \int \frac{v_2 \, \partial_y \Theta_l}{\phi} \Theta + \int \frac{\Theta \operatorname{curl} w}{\phi}.$$

First

$$-\int \frac{u_1 \Theta \, \partial_x \Theta}{\phi} \, dx \, dy = -\int \frac{u_1}{\phi} \partial_x \frac{\Theta^2}{2} \, dx \, dy$$
$$= \int \frac{\partial_x u_1}{\phi} \frac{\Theta^2}{2} \, dx \, dy - \int u_1 \frac{\partial_x \phi}{\phi^2} \frac{\Theta^2}{2} \, dx \, dy$$

and using (H),

$$\left| \int u_1 \frac{\partial_x \phi}{\phi^2} \frac{\Theta^2}{2} \, dx \, dy \right| \leqslant C \int \frac{\Theta^2}{\phi} \, dx \, dy.$$

Next

$$-\int \frac{u_2 \Theta \, \partial_y \Theta}{\phi} \, dx \, dy = -\int \frac{u_2}{\phi} \partial_y \frac{\Theta^2}{2} \, dx \, dy$$
$$= \int \frac{\partial_y u_2}{\phi} \frac{\Theta^2}{2} \, dx \, dy - \int u_2 \frac{\partial_y \phi}{\phi^2} \frac{\Theta^2}{2} \, dx \, dy,$$

and using (H)

$$\left| \int u_2 \frac{\partial_y \phi}{\phi^2} \frac{\Theta^2}{2} \, dx \, dy \right| \leqslant C \int \frac{\Theta^2}{\phi}.$$

Moreover, by the incompressibility condition,

$$\int \frac{\partial_x u_1}{\phi} \frac{\Theta^2}{2} \, dx \, dy + \int \frac{\partial_y u_2}{\phi} \frac{\Theta^2}{2} \, dx \, dy = 0.$$

Now

$$\int \frac{v_1\,\partial_x \Theta_l}{\phi}\,\Theta\,dx\,dy = \int \frac{v_1\,\partial^2_{xy} u_1}{\phi}\,\Theta\,dx\,dy - \int \frac{v_1\,\partial^2_{xx} u_2}{\phi}\,\Theta\,dx\,dy.$$

Notice that $|\partial^2_{xx} u_2| \leqslant C\sqrt{\phi_\eta}$; therefore

$$\left|\int v_1 \frac{\partial^2_{xx} u_2}{\phi}\,\Theta\,dx\,dy\right| \leqslant \|v_1\|_{L^2}\left(\int \frac{\Theta^2}{\phi}\,dx\,dy\right)^{1/2} \leqslant C\int \frac{\Theta^2}{\phi}\,dx\,dy,$$

where we use Lemma 2.2. Moreover,

$$\left|\frac{\partial^2_{xy} u_1}{\phi}\right| \leqslant \frac{C}{\sqrt{\phi}};$$

hence

$$\left|\int \frac{v_1\,\partial^2_{xy} u_1}{\phi}\,\Theta\,dx\,dy\right| \leqslant C\,\|v_1\|_{L^2}\left(\int \frac{\Theta^2}{\phi}\,dx\,dy\right)^{1/2} \leqslant C\int \frac{\Theta^2}{\phi}\,dx\,dy,$$

where we use Lemma 2.2. Next

$$\int \frac{v_2\,\partial_y \Theta_l}{\phi}\,\Theta\,dx\,dy = \int \frac{v_2\,\partial^2_{yy} u_1}{\phi}\,\Theta\,dx\,dy - \int \frac{v_2\,\partial^2_{xy} u_2}{\phi}\,\Theta\,dx\,dy.$$

Let us introduce Ψ, the stream function such that $(v_1, v_2) = \nabla^\perp \Psi$ and $\Delta\Psi = \Theta$. We recall that, by (H),

$$\left|\frac{\partial^2_{yy} u_1}{\phi} + 1\right| \leqslant \frac{C}{\sqrt{\phi}}.$$

We have

$$\int v_2 \Theta\,dx\,dy = -\int \partial_x \Psi\,\Delta\Psi\,dx\,dy = \int \partial_x \nabla\Psi\,\nabla\Psi\,dx\,dy = 0$$

since $\partial_x \Psi = 0$ when $y = 0$. Therefore it remains to bound

$$\int \frac{|v_2|\,|\Theta|}{\sqrt{\phi}}\,dx\,dy \leqslant \left(\int |v_2|^2\,dx\,dy\right)^{1/2}\left(\int \frac{|\Theta|^2}{\phi}\,dx\,dy\right)^{1/2} \leqslant C\int \frac{\Theta^2}{\phi}\,dx\,dy,$$

using Lemma 2.2. Notice that by (H)

$$\left|\frac{\partial^2_{xy} u_2}{\phi}\right| \leqslant \frac{C}{\sqrt{\phi}};$$

hence

$$\left|\int \frac{v_2\,\partial^2_{xy}u_2}{\phi}\,\Theta\,dx\,dy\right| \leqslant C\int \frac{\Theta^2}{\phi}\,dx\,dy,$$

where we use Lemma 2.2. Next

$$\left|\int \frac{\Theta \operatorname{curl} w}{\phi}\,dx\,dy\right| \leqslant \left(\int \frac{\Theta^2}{\phi}\,dx\,dy\right)^{1/2}\left(\int \frac{|\operatorname{curl} w|^2}{\phi}\,dx\,dy\right)^{1/2}.$$

Summing up all these estimates, we get

$$\partial_t I(t) \leqslant CI(t) + \int \frac{|\operatorname{curl} w|^2}{\phi}\,dx\,dy,$$

which ends the proof of Theorem 1.1. ∎

4. NONLINEAR STABILITY: PROOF OF THEOREM 1.2

4.1. *First Order Derivatives*

Let us turn to the control of first order derivatives of (9, 10). Let

$$I_1(t) = \int \frac{|\partial_x \Theta|^2}{\phi} + \int \frac{|\partial_y \Theta|^2}{\phi^2}.$$

LEMMA 4.1. *Under the assumptions of Theorem* 1.1, *there exists C such that*

$$\partial_t I_1(t) \leqslant CI_1(t) + (C\eta^{-8} + C)\, I(t) + \int \frac{|\partial_x \operatorname{curl} w|^2}{\phi} + \int \frac{|\partial_y \operatorname{curl} w|^2}{\phi^2}. \tag{47}$$

Proof. We have

$$\partial_t(\partial_x\Theta) + u_1\,\partial_x(\partial_x\Theta) + u_2\,\partial_y(\partial_x\Theta) + (\partial_x v_1)\,\partial_x\Theta_l + (\partial_x v_2)\,\partial_y\Theta_l$$
$$+ (\partial_x u_1)\,\partial_x\Theta + (\partial_x u_2)\,\partial_y\Theta + v_1\,\partial^2_{xx}\Theta_l + v_2\,\partial^2_{xy}\Theta_l = \partial_x \operatorname{curl} w, \tag{48}$$

$$\partial_t(\partial_y\Theta) + u_1\,\partial_x(\partial_y\Theta) + u_2\,\partial_y(\partial_y\Theta) + (\partial_y v_1)\,\partial_x\Theta_l + (\partial_y v_2)\,\partial_y\Theta_l$$
$$+ (\partial_y u_1)\,\partial_x\Theta + (\partial_y u_2)\,\partial_y\Theta + v_1\,\partial^2_{xy}\Theta_l + v_2\,\partial^2_{yy}\Theta_l = \partial_y \operatorname{curl} w. \tag{49}$$

Many terms of (48) and (49) can be seen as source terms. Namely, let

$$S_1 = (\partial_x v_1)\,\partial_x\Theta_l + (\partial_x v_2)\,\partial_y\Theta_l + v_1\,\partial^2_{xx}\Theta_l + v_2\,\partial^2_{xy}\Theta_l - \partial_x \operatorname{curl} w$$

and

$$S_2 = (\partial_y v_1)\, \partial_x \Theta_l + (\partial_y v_2)\, \partial_y \Theta_l + v_1\, \partial^2_{xy} \Theta_l + v_2 \partial^2_{yy} \Theta_l - \partial_y \operatorname{curl} w.$$

Let us first bound S_1 and S_2. As div $(v_1, v_2) = 0$ we have

$$\|\nabla v_1\|^2_{L^2} + \|\nabla v_2\|^2_{L^2} \leqslant C\, \|\operatorname{curl}(v_1, v_2)\|^2_{L^2} \leqslant C\eta^{-2} \int \frac{|\Theta|^2}{\phi};$$

hence, using (H′), $\phi \geqslant 1$, $\phi \leqslant 1 + \eta^{-2}$ and Lemma 2.2, estimating each term of S_1 and S_2,

$$\int \frac{S_1^2}{\phi} + \int \frac{S_2^2}{\phi^2} \leqslant (C\eta^{-8} + C) \int \frac{|\Theta|^2}{\phi} + \int \frac{|\partial_x \operatorname{curl} w|^2}{\phi} + \int \frac{|\partial_y \operatorname{curl} w|^2}{\phi^2}.$$

For instance,

$$\int \frac{|v_2\, \partial^2_{yy} \Theta_l|^2}{\phi^2} \leqslant \eta^{-6} \int |v_2|^2 \leqslant \eta^{-6} \int \frac{|\Theta|^2}{\phi},$$

since $\phi \geqslant 1$ and $|\partial^2_{yy} \Theta_l| \leqslant C\eta^{-3}$, and using Lemma 46. Now

$$\int \frac{u_1\, \partial_x(\partial_x \Theta)\, \partial_x \Theta}{\phi} + \int \frac{u_2\, \partial_y(\partial_x \Theta)\, \partial_x \Theta}{\phi}$$

$$= \int \frac{u_1 (\partial_x \Theta)^2}{2\phi} \frac{\partial_x \phi}{\phi} + \int \frac{u_2 (\partial_x \Theta)^2}{2\phi} \frac{\partial_y \phi}{\phi} - \int \frac{\partial_x u_1\, |\partial_x \Theta|^2}{2\phi} - \int \frac{\partial_y u_2\, |\partial_x \Theta|^2}{2\phi},$$

which, using (H), is bounded by

$$C \int \frac{|\partial_x \Theta|^2}{\phi},$$

and similarly for the terms involving $\partial_y \Theta$.

Moreover,

$$\left| \int \frac{\partial_x u_1 (\partial_x \Theta)^2}{\phi} + \int \frac{\partial_x u_2\, \partial_x \Theta\, \partial_y \Theta}{\phi} \right| \leqslant C I_1(t),$$

using

$$|\partial_x u_1| + \sqrt{\phi}\, |\partial_x u_2| \leqslant C,$$

and similarly

$$\left| \int \frac{\partial_y u_1\, \partial_x \Theta\, \partial_y \Theta}{\phi^2} + \int \frac{\partial_y u_2 (\partial_y \Theta)^2}{\phi^2} \right| \leqslant C I_1(t),$$

using

$$\left|\frac{\partial_y u_1}{\sqrt{\phi}}\right| + |\partial_y u_2| \leqslant C.$$

Moreover

$$\left|\int \frac{S_1\, \partial_x \Theta}{\phi}\right| \leqslant C \int \frac{|\partial_x \Theta|^2}{\phi} + \int \frac{S_1^2}{\phi},$$

which ends the proof of the lemma. ∎

4.2. *Higher Order Derivatives*

Let

$$I_n(t) = \sum_{\alpha+\beta=n} \int \frac{|\partial_x^\alpha\, \partial_y^\beta \Theta|^2}{\phi^{\beta+1}}.$$

As in the previous section, we have

Lemma 4.2. *Let* $n \geqslant 1$. *Under the assumptions of Theorem* 1.1 *there exists* C *such that*

$$\partial_t I_n(t) \leqslant \sum_{i=0}^{n} (C\eta^{-8(n-i)} + C)\, I_i(t) + \sum_{\alpha+\beta=n} \int \frac{|\partial_x^\alpha\, \partial_y^\beta \operatorname{curl} w|^2}{\phi^{1+\beta}}. \tag{50}$$

As a corollary,

Lemma 4.3. *The solution* v *of linearized Euler equations* (9), (10) *satisfies, for* s *large enough,*

$$\partial_t\, |||v|||_s \leqslant C(t)\, |||v|||_s + |||w|||_s,$$

with $C(t)$ *independent of* η.

4.3. *Nonlinear Stability Result*

In addition to the terms already bounded in Lemmas 4.1 and 4.2 we have to bound

$$\int \frac{\partial_x^\alpha\, \partial_y^\beta \Theta\;\; \partial_x^\alpha\, \partial_y^\beta (v_1\, \partial_x \Theta)}{\phi^{1+\beta}} + \int \frac{\partial_x^\alpha\, \partial_y^\beta \Theta\; \partial_x^\alpha\, \partial_y^\beta (v_2\, \partial_y \Theta)}{\phi^{1+\beta}}, \tag{51}$$

which is a sum of terms of the form

$$
\begin{aligned}
J_{\alpha', \beta'} &= \int \frac{\partial_x^\alpha \partial_y^\beta \Theta \, \partial_x^{\alpha'} \partial_y^{\beta'} v_1 \, \partial_x^{\alpha-\alpha'+1} \partial_y^{\beta-\beta'} \Theta}{\phi^{1+\beta}} \\
&\quad + \int \frac{\partial_x^\alpha \partial_y^\beta \Theta \, \partial_x^{\alpha'} \partial_y^{\beta'} v_2 \, \partial_x^{\alpha-\alpha'} \partial_y^{\beta-\beta'+1} \Theta}{\phi^{1+\beta}} \\
&= J_{\alpha', \beta', 1} + J_{\alpha', \beta', 2}
\end{aligned}
$$

for $0 \leqslant \alpha' \leqslant \alpha$ and $0 \leqslant \beta' \leqslant \beta$.

For $\alpha' = \beta' = 0$,

$$
\begin{aligned}
J_{0,0} &= \int \frac{v_1}{\phi^{1+\beta}} \partial_x \frac{(\partial_x^\alpha \partial_y^\beta \Theta)^2}{2} + \int \frac{v_2}{\phi^{1+\beta}} \partial_y \frac{(\partial_x^\alpha \partial_y^\beta \Theta)^2}{2} \\
&= (1+\beta) \int \frac{v_1}{\phi^{1+\beta}} \frac{\partial_x \phi}{\phi} \frac{(\partial_x^\alpha \partial_y^\beta \Theta)^2}{2} + (1+\beta) \int \frac{v_2}{\phi^{1+\beta}} \frac{\partial_y \phi}{\phi} \frac{(\partial_x^\alpha \partial_y^\beta \Theta)^2}{2},
\end{aligned}
$$

since (v_1, v_2) is divergence free

$$
\leqslant \frac{C}{\eta^3} \left(\|v_1\|_{L^\infty(\mathbb{T} \times [0,1])} + \|v_2\|_{L^\infty(\mathbb{T} \times [0,1])} \right) \int \frac{(\partial_x^\alpha \partial_y^\beta \Theta)^2}{\phi^{1+\beta}},
$$

where we used $\partial_y \phi \leqslant C\eta^{-3}$.

For $\alpha' + \beta' \geqslant 1$ and for s large enough, either $\alpha' + \beta'$ or $\alpha + \beta + 2 - \alpha' - \beta'$ is less than $s - 3$. If $\alpha' + \beta' \leqslant s - 3$ we use

$$
\|\partial_x^{\alpha'} \partial_y^{\beta'} v_1\|_{L^\infty} + \|\partial_x^{\alpha'} \partial_y^{\beta'} v_2\|_{L^\infty} \leqslant \|v\|_{H^s} \leqslant \frac{C}{\eta^{8s}} |||v|||_s
$$

to get

$$
|J_{\alpha', \beta', 1}| \leqslant \frac{C}{\eta^{8s}} |||v|||_s \int \frac{|\partial_x^\alpha \partial_y^\beta \Theta| \, |\partial_x^{\alpha-\alpha'+1} \partial_y^{\beta-\beta'} \Theta|}{\phi^{1+\beta/2+(\beta-\beta')/2}} \leqslant \frac{C}{\eta^{16s}} |||v|||_s^3.
$$

For $\alpha + \beta + 2 - \alpha' - \beta' \leqslant s - 3$ we use

$$
\|\partial_x^{\alpha-\alpha'+1} \partial_y^{\beta-\beta'} \Theta\|_{L^\infty} \leqslant \frac{C}{\eta^{8s}} |||v|||_s
$$

to get a similar bound on $|J_{\alpha', \beta', 2}|$ and $|J_{\alpha', \beta'}|$, which ends the proof of Theorem 1.2. ∎

5. ASYMPTOTIC EXPANSION: THEOREM 1.3

5.1. *Inviscid Prandtl Equations*

Classical scalings in boundary layers lead to the study of

$$\partial_t u_1 + u_1\, \partial_x u_1 + u_2\, \partial_y u_1 = f(t, x), \tag{52}$$

$$\partial_x u_1 + \partial_y u_2 = 0 \tag{53}$$

$$u_2 = 0 \qquad \text{for} \quad y = 0 \tag{54}$$

in $\Omega = \mathbb{T} \times \mathbb{R}_+$, with initial data $(u_1(0, .), u_2(0, .)) = (u_1^0, u_2^0)$, where u_1^0 and u_2^0 are given, and where f is a given function which depends only on the x variable. This system can be seen as inviscid Prandtl equations.

It turns out that it is more convenient to work in Hölder spaces $C^s(\Omega)$ (with s an integer) than in Sobolev spaces H^s to study this system. Let

$$C^s(\Omega) = \{\phi,\ \|\phi\|_s = \sup_{\alpha+\beta \leqslant s}\ \sup_{(x, y) \in \Omega} |\partial_x^\alpha\, \partial_y^\beta \phi(x, y)| < +\infty\}.$$

Proposition 5.1. *Let u_1^0 and u_2^0 be given $C^s(\Omega)$ functions with s large enough, satisfying* (53), (54), *and let $f \in L^\infty([0, T_*], C^s(\mathbb{T}))$ (with $T_* > 0$). There exist $0 < T \leqslant T_*$ and solutions u_1 and u_2 in $L^\infty([0, T], C^{s-1}(\Omega))$ of* (52)–(54) *with initial data u_1^0 and u_2^0.*

Proof. System (52), (53) has a very deep structure (see [3] for other formulations, and in particular kinetic formulations). Let us extend u_1 and u_2 for $x \in \mathbb{T}$ to $x \in \mathbb{R}$ by periodicity, let us introduce the characteristics $X(t, x, y)$ and $Y(t, x, y)$ defined by

$$\partial_t X(t, x, y) = u_1(t, X(t, x, y), Y(t, x, y)),$$

$$\partial_t Y(t, x, y) = u_2(t, X(t, x, y), Y(t, x, y)),$$

with $X(0, x, y) = x$ and $Y(0, x, y) = y$, and let

$$\tilde{u}_1(t, x, y) = u_1(t, X(t, x, y), Y(t, x, y)).$$

Then (52)–(54) can be rewritten

$$\partial_t \tilde{u}_1 = f(t, X), \tag{55}$$

$$\partial_t X = \tilde{u}_1, \tag{56}$$

$$\partial_x X\, \partial_y Y - \partial_y X\, \partial_x Y = 1, \tag{57}$$

$$Y(t, x, 0) = 0, \tag{58}$$

Eq. (57) being the incompressibility condition (53). Equations (55) and (56) are straightforward to solve and we get

$$\tilde{u}_1 \in L^\infty([0, T_*], C^s(\Omega)), \qquad X - x \in L^\infty([0, T_*], C^s(\Omega)).$$

By definition of the initial conditions on X and Y, (57) can be solved in small time and we get $Y \in L^\infty([0, T], C^{s-1}(\Omega))$ for some $T \leqslant T_*$. Notice the loss of one derivative. Going back to the genuine variables x, y gives the proposition. ∎

Remarks. This proposition holds in any space dimension. Notice the loss of one derivative in the estimate. This is a crucial point, probably the main difficulty of (viscous) Prandtl equations. The proof fails when f depends on y! Notice that if $f = 0$ the proof is completely "geometric": X and u_1 are given explicitly, and Y can easily be deduced from the incompressibility condition. In general there is no global smooth solution since (57) can only be solved in small time. It is easy to construct explicit examples of solutions which blow up at a particular time. We refer to [8] for evidence of blow up for (viscous) Prandtl equations.

5.2. *Linearized Inviscid Prandtl Equations*

Let us turn to the study of

$$\partial_t v_1 + u_1 \partial_x v_1 + u_2 \partial_y v_1 + v_1 \partial_x u_1 + v_2 \partial_y u_1 = 0, \tag{59}$$

$$\partial_x v_1 + \partial_y v_2 = 0, \tag{60}$$

$$v_2 = 0 \qquad \text{for} \quad y = 0, \tag{61}$$

with $(v_1(0, .), v_2(0, .)) = (v_1^0, v_2^0)$, v_1^0 and v_2^0 being given, and where (u_1, u_2) is a solution of (52), (53) with some force f.

PROPOSITION 5.2. *Let v_1^0 and v_2^0 be given C^s functions with s large enough, and let u_1, u_2 be a solution of* (52)–(54) *on* $[0, T]$ *with initial data in $C^s(\Omega)$ and force term $f \in L^\infty([0, T], C^s(\Omega))$. Then there exist solutions v_1 and v_2 of* (59)–(61) *in $L^\infty([0, T], C^{s-2}(\Omega))$ with initial data v_1^0 and v_2^0.*

Proof. Let $\delta > 0$ and let u_1' and u_2' be the solution of (52), (53) with the same force f and with initial data

$$u_1'(0, x, y) = u_1(0, x, y) + \delta v_1^0, \qquad u_2'(0, x, y) = u_2(0, x, y) + \delta v_2^0.$$

Let X' and Y' be the characteristics associated to u_1' and u_2'. We have

$$\partial_t \tilde{u}_1 = f(t, X), \qquad \partial_t \tilde{u}_1' = f(t, X'),$$

$$\partial_t X = \tilde{u}_1, \qquad \partial_t X' = \tilde{u}_1'.$$

Therefore

$$\partial_t(|\tilde{u}_1-\tilde{u}_1'|+|X-X'|)\leqslant(1+|f|_{Lip})(|\tilde{u}_1-\tilde{u}_1'|+|X-X'|),$$

which leads to

$$|\tilde{u}_1-\tilde{u}_1'|_{L^\infty}+|X-X'|_{L^\infty}\leqslant C\delta\exp((1+|f|_{\mathrm{Lip}})\,t)$$

($|f|_{\mathrm{Lip}}$ denoting $\sup_{x\in\mathbb{T},\,t\in[0,T]}|\partial_x f(t,x)|$) and to

$$\partial_t(|\tilde{u}_1-\tilde{u}_1'|+|X-X'|)\leqslant C\delta\exp(Ct)$$

for $0\leqslant t\leqslant T$. Similarly,

$$\|\tilde{u}_1-\tilde{u}_1'\|_{C^s}+\|X-X'\|_{C^s}\leqslant C\delta\exp(Ct),$$
$$\|\partial_t(\tilde{u}_1-\tilde{u}_1')\|_{C^s}+\|\partial_t(X-X')\|_{C^s}\leqslant C\delta\exp(Ct).$$

We deduce in particular that for δ small enough, we can solve for Y' for $0\leqslant t\leqslant T$ and

$$\|Y-Y'\|_{C^{s-1}}\leqslant\delta C\exp(Ct).$$

Going back to u_1, u_2, and u_1',u_2' we get

$$\|u_1-u_1'\|_{L^\infty([0,T],\,C^{s-1})}+\|u_2-u_2'\|_{L^\infty([0,T],\,C^{s-1})}\leqslant C\delta$$

and

$$\|\partial_t(u_1-u_1')\|_{L^\infty([0,T],\,C^{s-1})}+\|\partial_t(u_2-u_2')\|_{L^\infty([0,T],\,C^{s-1})}\leqslant C\delta.$$

Let now

$$v_1^\delta=\frac{u_1'-u_1}{\delta},\qquad v_2^\delta=\frac{u_2'-u_2}{\delta}.$$

We have $v_1^\delta(0)=v_1^0$ and $v_2^\delta(0)=v_2^0$, and using a standard compactness argument (Aubin Lemma), v_1^δ and v_2^δ converge as δ goes to 0 to functions v_1 and v_2 in $L^\infty([0,T],C^{s-2})$. Passing to the limit in the equations on v_1^δ and v_2^δ,

$$\partial_t v_1^\delta+u_1\,\partial_x v_1^\delta+v_1^\delta\,\partial_x u_1+u_2\,\partial_y v_1^\delta+v_2^\delta\,\partial_y u_1+\delta v_1^\delta\,\partial_x v_1^\delta+\delta v_2^\delta\,\partial_y v_1^\delta=0,$$
$$\partial_x v_1^\delta+\partial_y v_2^\delta=0,$$

gives (59), (60). ∎

Remarks. The proof is in fact more elaborate than the proof of existence for *nonlinear* inviscid Prandtl equations. This proof opens many interesting questions: First is it crucial that u_1 and u_2 are solutions of (52), (53)? (Notice that f must be independent of y, therefore we cannot define f by (52) even if (u_1, u_2) is divergence-free.) Moreover there is again a loss of regularity in the solution, at $t=0$: is it possible to get this result with a "classical" energy method?

5.3. *Construction of an Approximate Solution*

Notice that the pressure p also has an asymptotic expansion, namely

$$p=\sum_{j=0}^{N} \eta^j p_{j,\,\mathrm{int}}(t, x, y)+\sum_{j=0}^{N} \eta^j p_{j,\,\mathrm{b}}\left(t, x, \frac{y}{\eta}\right). \tag{62}$$

As usual in boundary layer theory we will get $p_{0,\,\mathrm{b}}=0$.

Putting the Ansatz (26) in incompressible Euler equations we get that $u_{0,\,\mathrm{int}}$ satisfies Euler equations

$$\partial_t u_{0,\,\mathrm{int}}+(u_{0,\,\mathrm{int}}\,.\nabla)\, u_{0,\,\mathrm{int}}+\nabla p_{0,\,\mathrm{int}}=0, \tag{63}$$

$$\nabla . u_{0,\,\mathrm{int}}=0 \tag{64}$$

with boundary conditions

$$u_{0,\,\mathrm{int},\,2}=0 \tag{65}$$

where $u_{0,\,\mathrm{int},\,2}$ denotes the second component of $u_{0,\,\mathrm{int}}$. By standard results there exists a solution $u_{0,\,\mathrm{int}}$ in $L^\infty_{\mathrm{loc}}([0, +\infty[, H^s(\Omega))$ to (63)–(65) (for every s), and therefore in $L^\infty_{\mathrm{loc}}([0, +\infty[, C^s(\Omega))$ (for every s).

Let

$$u_{0,\,\mathrm{b}}(t, x, Y)=\begin{pmatrix} u_{0,\,\mathrm{b},\,1}(t, x, Y) \\ \eta u_{0,\,\mathrm{b},\,2}(t, x, Y)\end{pmatrix},$$

where Y is the fast variable $Y=y/\eta$. Notice the η factor in front of $u_{0,\,\mathrm{b},\,2}$, which comes from the incompressibility condition. This term could be rejected in $u_{1,\,\mathrm{b}}$, but the construction would then be more awkward. Let us derive the equation on $u_{0,\,\mathrm{b}}$. Putting (26) into the Euler equations we get, up to terms of order η,

$$\begin{aligned}\partial_t(u_{0,\,\mathrm{int},\,1}+u_{0,\,\mathrm{b},\,1})&+(u_{0,\,\mathrm{int},\,1}+u_{0,\,\mathrm{b},\,1})\,\partial_x(u_{0,\,\mathrm{int},\,1}+u_{0,\,\mathrm{b},\,1})\\ &+(u_{0,\,\mathrm{int},\,2}+\eta u_{0,\,\mathrm{b},\,2})\,\partial_y(u_{0,\,\mathrm{int},\,1}+u_{0,\,\mathrm{b},\,1})+\partial_x p_{0,\,\mathrm{int}}+\partial_x p_{0,\,\mathrm{b}}=0\end{aligned} \tag{66}$$

and

$$\partial_t(u_{0,\,\mathrm{int},\,2}+\eta u_{0,\,\mathrm{b},\,2})+(u_{0,\,\mathrm{int},\,1}+u_{0,\,\mathrm{b},\,1})\,\partial_x(u_{0,\,\mathrm{int},\,2}+\eta u_{0,\,\mathrm{b},\,2})$$
$$+(u_{0,\,\mathrm{int},\,2}+\eta u_{0,\,\mathrm{b},\,2})\,\partial_y(u_{0,\,\mathrm{int},\,2}+\eta u_{0,\,\mathrm{b},\,2})+\partial_y p_{0,\,\mathrm{int}}+\partial_y p_{0,\,\mathrm{b}}=0. \tag{67}$$

Making the change of variables $Y=y/\eta$ we get from (67)

$$p_{0,\,\mathrm{b}}=0$$

(as usual the pressure does not change in the boundary layer at first order). Moreover $(u_{0,\,\mathrm{int},\,2}+\eta u_{0,\,\mathrm{b},\,2})\,\partial_y u_{0,\,\mathrm{int},\,1}$ is of order η for Y bounded, and can therefore be forgotten. On the other side

$$(u_{0,\,\mathrm{int},\,2}+\eta u_{0,\,\mathrm{b},\,2})\,\partial_y u_{0,\,\mathrm{b},\,1}=\left(\frac{u_{0,\,\mathrm{int},\,2}}{\eta}+u_{0,\,\mathrm{b},\,2}\right)\partial_Y u_{0,\,\mathrm{b},\,1}$$

and

$$\frac{u_{0,\,\mathrm{int},\,2}}{\eta}=Y\,\partial_y u_{0,\,\mathrm{int},\,2}(t,x,0)+O(\eta);$$

therefore up to terms of order η, (66) can be rewritten

$$\partial_t(u_{0,\,\mathrm{int},\,1}(t,x,0)+u_{0,\,\mathrm{b},\,1})$$
$$+(u_{0,\,\mathrm{int},\,1}(t,x,0)+u_{0,\,\mathrm{b},\,1})\,\partial_x(u_{0,\,\mathrm{int},\,1}(t,x,0)+u_{0,\,\mathrm{b},\,1})$$
$$+(Y\,\partial_y u_{0,\,\mathrm{int},\,2}(t,x,0)+u_{0,\,\mathrm{b},\,2})\,\partial_Y u_{0,\,\mathrm{b},\,1}+\partial_x p_{0,\,\mathrm{int}}(t,x,0)=0. \tag{68}$$

Setting

$$\tilde u_{0,\,1}(t,x,Y)=u_{0,\,\mathrm{int},\,1}(t,x,0)+u_{0,\,\mathrm{b},\,1}(t,x,Y)$$

and

$$\tilde u_{0,\,2}(t,x,Y)=Y\,\partial_y u_{0,\,\mathrm{int},\,2}(t,x,0)+u_{0,\,\mathrm{b},\,2}(t,x,Y)$$

we get

$$\partial_t\tilde u_{0,\,1}+\tilde u_{0,\,1}\,\partial_x\tilde u_{0,\,1}+\tilde u_{0,\,2}\,\partial_Y\tilde u_{0,\,1}+\partial_x p_{0,\,\mathrm{int}}(t,x,0)=0 \tag{69}$$
$$\partial_x\tilde u_{0,\,1}+\partial_Y\tilde u_{0,\,2}=0 \tag{70}$$
$$\tilde u_{0,\,2}=0 \qquad \text{at} \quad Y=0. \tag{71}$$

Using Proposition 5.1 there exist solutions $(\tilde u_{0,\,1},\tilde u_{0,\,2})$ of (69)–(71) on a time interval $[0,T]$ for some $T>0$. We then recover $u_{0,\,\mathrm{b},\,1}$ and $u_{0,\,\mathrm{b},\,2}$ which are in $L^\infty([0,T],C^s)$ for every s. It is easy to prove that if $u_{0,\,\mathrm{b}}$ is initially rapidly decreasing in Y it remains so on $[0,T]$.

Let us turn to first order terms. First $u_{1,\,\mathrm{int}}$ satisfies

$$\partial_t u_{1,\,\mathrm{int}} + (u_{0,\,\mathrm{int}}.\nabla)\, u_{1,\,\mathrm{int}} + (u_{1,\,\mathrm{int}}.\nabla)\, u_{0,\,\mathrm{int}} + \nabla p_{1,\,\mathrm{int}} = 0, \tag{72}$$

$$\nabla . u_{1,\,\mathrm{int}} = 0, \tag{73}$$

$$u_{1,\,\mathrm{int},\,2}(t, x, 0) = 0. \tag{74}$$

By classical arguments there exists a solution $u_{1,\,\mathrm{int}}$ in $L^\infty([0, T], H^s)$ for every s. Higher order terms can be handled as previously. We will not detail them.

5.4. *End of the Proof of Theorem* 1.3

The proof of Theorem 1.3 starting from an approximate solution of high order N and using Theorem 1.2 follows closely (see [12, 14, 6]). Let us detail it now. Let N be arbitrarily large. By constructing high order terms we get an approximate solution v^{app} which satisfies (1) on $[0, T]$ up to $\eta^N R_N^\eta$, where $\| R_N^\eta \|_{H^s} \leqslant C\eta^{1-s}$ (for every s), hence $|||R_N^\eta|||_s \leqslant C$, and which satisfies (2). Namely

$$\partial_t v^{\mathrm{app}} + (v^{\mathrm{app}}.\nabla)\, v^{\mathrm{app}} + \nabla p^{\mathrm{app}} = \eta^N R_N^\eta, \tag{75}$$

$$\nabla . v^{\mathrm{app}} = 0. \tag{76}$$

The second step is to check that v^{app} satisfies (H) and (H$'$). Let

$$\phi_\eta(t, x, y) = \left(1 - \frac{1}{\eta^2}\, \partial_{YY}^2 u_{0,\,\mathrm{b},\,1}\left(t, x, \frac{y}{\eta}\right)\right) \chi(y) + \exp(-y)(1 - \chi(y)), \tag{77}$$

where χ is a smooth function with support in $[-1, 1]$ which equals 1 on $[-1/2, 1/2]$. Notice that (4) is satisfied. Using condition (25) together we the special form of the approximate solution, we check (5) and (6) (with u^η replaced by v^{app}). Using (23), (7) holds true, and using the special structure of v^{app}, (H$'$) is satisfied.

Let then $w = u^\eta - v^{\mathrm{app}}$, which satisfies

$$\partial_t w + (v^{\mathrm{app}}.\nabla)\, w + (w.\nabla)\, v^{\mathrm{app}} + (w.\nabla)\, w + \nabla p = -\eta^N R_N^\eta, \tag{78}$$

$$\nabla . w = 0. \tag{79}$$

Applying Theorem 1.2 to (75), (76) we get

$$\partial_t\, |||w|||_s^2 \leqslant C\, |||w|||_s^2 + \frac{C}{\eta^{16s}}\, |||w|||_s^3 + \eta^{2N}\, |||R_N^\eta|||_s^2. \tag{80}$$

But at $t=0$, $|||w|||_s \leqslant C\eta^N$. Let us restrict to N sufficiently large with respect to s so that $N > 32s+2$. Let T^η be the largest time t_0 for which $|||w(t)|||_s \leqslant \eta^{N/2}$ on $0 \leqslant t \leqslant t_0$. Let us prove that $T^\eta = T$ for η small enough.

If $T^\eta < T$ then on $[0, T^\eta]$, estimate (80) can be rewritten

$$\partial_t \, |||w|||_s^2 \leqslant (C+1)\, |||w|||_s^2 + C\eta^{2N-2};$$

hence, using the Gronwall Lemma, $|||w(T^\eta)|||_s^2 < \eta^{N/2}$ for η small enough, which leads to a contradiction. Hence, on $[0, T]$,

$$|||u^\eta - v^{\mathrm{app}}\,|||_s \leqslant \eta^{N/2}.$$

As N and s can be arbitrarily large, this ends the proof of the theorem.

6. LINEAR INSTABILITY: PROOF OF THEOREM 1.4

The construction is classical (see for instance [11, 18]) and can be traced back to Lord Rayleigh [21]. We will, however, recall it since it is the first step of the next section. The main point is that Euler equations are invariant under time and space changes of variables. Namely, if $u(t, x, y)$ is a solution of the Euler equations, $u(t', x', y')$ is also a solution for $t' = t/\eta$, $x' = x/\eta'$, $y' = y/\eta$. From now on we will work in the new variables (t', x', y') (in all the forthcoming sections). Instead of studying the stability of a profile $(\tilde{v}(y/\eta), 0)$ in times of order η, we study the stability of a profile $(\tilde{v}_l(y'), 0)$ in times of order one. To abbreviate the notation we drop the primes on t', x', y'. Let $\tilde{v}_l$ be a given profile and let

$$u^\eta = \begin{pmatrix} \tilde{v}_l(y) \\ 0 \end{pmatrix}.$$

Notice that as we have rescaled the variables, the profile is now fixed and independent of η.

We are led to look for an exponentially increasing eigenmode of

$$\partial_t v + \tilde{v}_l\, \partial_x v + v_2\, \partial_y \begin{pmatrix} \tilde{v}_l \\ 0 \end{pmatrix} + \nabla p = 0, \tag{81}$$

$$\nabla . v = 0. \tag{82}$$

Following the classical construction [11], we look for v of the form

$$v = \begin{pmatrix} \Psi'(y) \exp ik(x-ct) \\ -ik\Psi(y) \exp ik(x-ct) \end{pmatrix}, \tag{83}$$

where $\Psi(y) \exp ik(x-ct)$ plays the role of a current function. Taking the curl of (81), (82) we obtain the inviscid Orr–Sommerfeld equation

$$(\tilde{v}_l - c)(\partial^2_{yy} - k^2)\,\Psi - \Psi\,\partial^2_{yy}\tilde{v}_l = 0 \tag{84}$$

with boundary conditions

$$\Psi = 0 \quad \text{on } y = 0, \qquad \Psi \to 0 \quad \text{as } y \to +\infty. \tag{85}$$

As noticed in [18], the resolution of (84), (85) is straightforward when $\tilde{v}_l$ is piecewise linear. So let us consider $\tilde{v}_l$ defined by

$$\tilde{v}_l(y) = \begin{cases} \alpha y & \text{for} \quad y \leqslant 1, \\ \beta y + (\alpha - \beta) & \text{for} \quad 1 \leqslant y \leqslant 1 + \gamma, \\ \alpha + \beta\gamma & \text{for} \quad y \geqslant 1 + \gamma. \end{cases} \tag{86}$$

As $\partial^2_{yy}\tilde{v}_l$ is a sum of two Dirac masses, one in 1 and another in $1+\gamma$,

$$(\partial^2_{yy} - k^2)\,\Psi = \sigma_1\delta_1 + \sigma_2\delta_{1+\gamma}.$$

Hence

$$(\tilde{v}_l(1) - c)\,\sigma_1 = \Psi(1)(\beta - \alpha) = \sigma_1(\beta - \alpha)\,G(1, 1) + \sigma_2(\beta - \alpha)\,G(1+\gamma, 1)$$

and

$$(\tilde{v}_l(1+\gamma) - c)\,\sigma_2 = -\Psi(2)\,\beta = -\sigma_1\,\beta G(1, 1+\gamma) - \sigma_2\,\beta G(1+\gamma, 1+\gamma).$$

Therefore c is an eigenvalue of the matrix

$$\mathscr{M} = \begin{pmatrix} \tilde{v}_l(1) - (\beta - \alpha)\,G(1, 1) & -(\beta - \alpha)\,G(1+\gamma, 1) \\ \beta G(1, 1+\gamma) & \tilde{v}_l(1+\gamma) + \beta G(1+\gamma, 1+\gamma) \end{pmatrix}. \tag{87}$$

This matrix is completely explicit and depends on α, β, γ, and k. It has two eigenvalues, $c_+(k)$ and $c_-(k)$, depending on α, β, and γ. The main property is that c_+ and c_- are real except for particular values of k if $\beta > \alpha$ (which corresponds to the "inflection point" on $\tilde{v}_l$). A typical picture of $\Im c_+$ is given by Fig. 1. When the imaginary part of c_+ is strictly positive, the corresponding eigenvector of $\mathscr{M}$ leads to an exponentially increasing eigenmode v, which ends the proof of the proposition. ∎

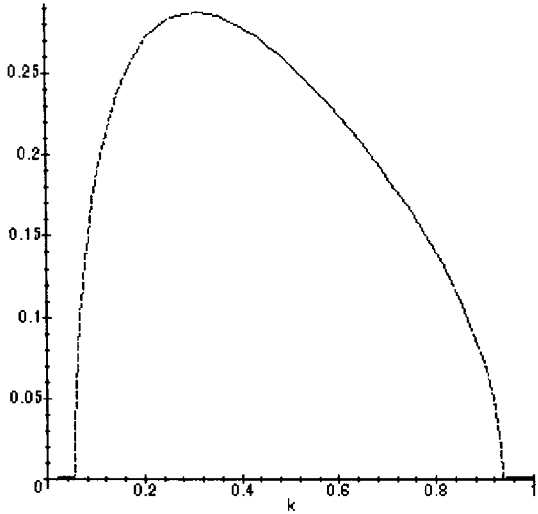

FIG. 1. $|\Im c_+|$ for $\alpha = 1/2$ and $\beta = 1$.

7. NONLINEAR INSTABILITY: THEOREM 1.5

7.1. *Growth of Solutions of the Inviscid Orr–Sommerfeld Equation*

Let us begin by majorations on solutions of the inviscid Orr–Sommerfeld equations.

PROPOSITION 7.1. *Let $K_0 > 0$ and let $\bar{\delta} > 0$. Let $\delta < 1$ be small enough (depending on K_0 and $\bar{\delta}$). Let $\tilde{u}_{\text{stat}}$ be a smooth increasing function with*

$$\tilde{u}_{\text{stat}}(y) = \begin{cases} \alpha y & \textit{for} \quad y \leqslant 1 - 2\delta \\ \beta y + (\alpha - \beta) & \textit{for} \quad 1 + 2\delta \leqslant y \leqslant 1 + \gamma - 2\delta \\ \alpha + \beta\gamma & \textit{for} \quad 1 + \gamma + 2\delta \leqslant y \end{cases}$$

such that $\partial^2_{yy} \tilde{u}_{\text{stat}}$ has a constant sign on $[1 - 2\delta, 1 + 2\delta]$ and $[1 + \gamma - 2\delta, 1 + \gamma + 2\delta]$ (with $\gamma > 4\delta$) and such that

$$|\partial^n_y \tilde{u}_{\text{stat}}| \leqslant C_n \delta^{-n}$$

for every $n \geqslant 0$. Let l be large enough. Let $|k| \leqslant K_0$, $k \neq 0$. Let ω be the solution of

$$(\partial_t + ik\tilde{u}_{\text{stat}})\,\omega - ik\Psi\,\partial^2_{yy}\tilde{u}_{\text{stat}} = \bar{\omega} \tag{88}$$

where

$$(\partial_{yy}^2 - k^2)\,\Psi = \omega \tag{89}$$

with boundary conditions

$$\Psi = 0 \quad on \;\; y = 0, \qquad \Psi \to 0 \quad as \;\; y \to +\infty, \tag{90}$$

with source term $\bar{\omega}$ *satisfying*

$$|\partial_y^n \bar{\omega}| \leqslant C_n \exp(-D_0 y) \exp(D_1 t) \tag{91}$$

for $0 \leqslant n \leqslant l$ *and initial data* ω^0 *such that*

$$|\partial_y^n \omega^0| \leqslant C_n \exp(-D_0 y) \tag{92}$$

for $0 \leqslant n \leqslant l$ *(with* $D_0 > 0$ *and* $D_1 > 0$*). Then* ω *satisfies*

$$|\partial_y^n \omega(t, y)| \leqslant C_{k, n, D_0} \exp(D_1 t) \exp(-\inf(D_0, k)\, y) \tag{93}$$

for $0 \leqslant n \leqslant l/2$, *provided* $D_1 > |\Im c_+(k)|\, k + \bar{\delta}$, *with* C_{k, n, D_0} *locally bounded in* k, n, *and* D_0.

We split the proof in several parts. Let $k \neq 0$, and let $I_1 = [1 - 2\delta, 1 + 2\delta]$ and $I_2 = [1 + \gamma - 2\delta, 1 + \gamma + 2\delta]$. We will strongly use the fact that $\partial_{yy}^2 \tilde{u}_{\mathrm{stat}}$ vanishes outside $I_1 \cup I_2$ which are small intervals in order to split Rayleigh's equation in two parts. The first one (on ω_1) deals with "Orr mechanism" and the essential spectrum by explicitly solving a simple linear equation. The second part (on ω_2) deals with vorticity creation by a perturbation analysis to go back to a two-by-two matrix.

Let us define the norm $\|\phi\|_{n, D_0, D_1}$ for a function $\phi(t, y)$ by

$$\|\phi\|_{n, D_0, D_1} = \sup_{t,\, y,\, \alpha \leqslant n} \left(|\partial_y^\alpha \phi(t, y)| \exp(D_0 y) \exp(-D_1 t) \right) \tag{94}$$

and $\|\phi\|_{n, D_0}$ for a function $\phi(y)$ by

$$\|\phi\|_{n, D_0} = \sup_{y,\, \alpha \leqslant n} \left(|\partial_y^\alpha \phi(y)| \exp(D_0 y) \right). \tag{95}$$

7.2. *First Step: Study of* ω_1

Let ω_1 be the solution of

$$(\partial_t + ik\tilde{u}_{\mathrm{stat}})\,\omega_1 = \bar{\omega} \tag{96}$$

with initial data $\omega_1(0, y) = \omega^0(y)$. Let Ψ_1 be defined by $\omega_1 = (\partial_{yy}^2 - k^2)\,\Psi_1$ with $\Psi_1 = 0$ for $y = 0$ and $\Psi_1 \to 0$ as $y \to +\infty$.

LEMMA 7.2. *There exist constants C_n such that*

$$\|\omega_1\|_{n, D_0, D_1} \leqslant \frac{C_n(1+k^n)}{\delta^n} (\|\omega^0\|_{n, D_0} + \|\bar{\omega}\|_{n, D_0, D_1}),$$

where the constants C_n are independent of t, δ, D_0, and k, but depend on D_1. Moreover, for $|y| \leqslant 2+2\gamma$,

$$|\Psi_1(y)| \leqslant C_0(\|\omega^0\|_{0, D_0} + \|\bar{\omega}\|_{0, D_0, D_1}) \exp(D_1 t),$$

C_0 depending on D_0 and D_1.

Proof of Lemma 7.2. The solution of (96) is explicit and we have

$$\begin{aligned}\omega_1(t, y) &= \exp\left(-ik \int_0^t \tilde{u}_{\text{stat}}(\tau, y)\, d\tau\right) \omega_1(0, y)\\ &\quad + \int_0^t \bar{\omega}(\tau, y) \exp\left(-ik \int_\tau^t \tilde{u}_{\text{stat}}(\tau', y)\, d\tau'\right) d\tau\\ &= \omega_{1,1}(t, y) + \omega_{1,2}(t, y).\end{aligned}$$

But

$$\begin{aligned}|\partial_y^n \omega_{1,1}| &\leqslant C_n \left(\frac{1+kt}{\delta}\right)^n \exp(-D_0 y)\, \|\omega^0\|_{n, D_0}\\ &\leqslant \frac{C_n(1+k^n)}{\delta^n} \exp(D_1 t) \exp(-D_0 y)\, \|\omega^0\|_{n, D_0}\end{aligned}$$

and

$$\begin{aligned}|\partial_y^n \omega_{1,2}| &\leqslant C_n \int_0^t \exp(D_1 \tau) \left(\frac{1+kt-k\tau}{\delta}\right)^n \exp(-D_0 y)\, \|\bar{\omega}\|_{n, D_0, D_1}\, d\tau\\ &\leqslant C_n(1+k^n)\, e^{D_1 t} \int_0^t e^{-D_1 \tau'} \left(\frac{1+\tau'}{\delta}\right)^n e^{-D_0 y}\, \|\bar{\omega}\|_{n, D_0, D_1}\, d\tau'\\ &\leqslant \frac{C_n(1+k^n)}{\delta^n} \exp(-D_0 y) \exp(D_1 t)\, \|\bar{\omega}\|_{n, D_0, D_1},\end{aligned}$$

since $D_1 > 0$. Therefore

$$\|\omega_1\|_{n, D_0, D_1} \leqslant \frac{C_n(1+k^n)}{\delta^n} (\|\omega^0\|_{n, D_0} + \|\bar{\omega}\|_{n, D_0, D_1}).$$

Using now

$$\Psi_1(t, y) = \int_0^{+\infty} G_k(y', y)\, \omega_1(t, y')\, dy'$$

together with $|G_k(z', z)| \leqslant \inf(z', z)$, we end the proof of Lemma 7.2. ▮

7.3. *Second Step: Study of* ω_2

7.3.1. *Vorticity Balance.* Let us now turn to $\omega_2 = \omega - \omega_1$, which satisfies

$$(\partial_t + ik\tilde{u}_{\text{stat}})\, \omega_2 - ik\Psi_2\, \partial_{yy}^2 \tilde{u}_{\text{stat}} = ik\Psi_1\, \partial_{yy}^2 \tilde{u}_{\text{stat}}, \tag{97}$$

$$\omega_2(0, y) = 0, \tag{98}$$

$$\Psi_2(t, 0) = 0, \qquad \Psi_2 \to 0 \quad \text{as } y \to +\infty, \tag{99}$$

where

$$\omega_2 = (\partial_{yy}^2 - k^2)\, \Psi_2.$$

The ideas of the majoration of ω_2 are the following: there are two areas where vorticity ω_2 can be created, namely I_1 and I_2 (where the flow $\tilde{u}_{\text{stat}}$ is not linear). Outside $I_1 \cup I_2$, the vorticity ω_2 vanishes and Ψ_2 has a completely known behavior. Moreover as $I_1 \cup I_2$ is small, we can use perturbation techniques to go back to the case "$\delta = 0$" (where $\partial_y \Psi_2$ has jumps at $y = 1$ and $y = 1 + \gamma$) which can be explicitly solved analytically.

More precisely, as $\partial_{yy}^2 \tilde{u}_{\text{stat}}$ vanishes outside $I_1 \cup I_2$,

$$\omega_2(t, y) = 0 \qquad \text{if} \quad y \notin I_1 \cup I_2.$$

Let

$$\sigma_1(t) = \int_{I_1} \omega_2(t, y)\, dy \qquad \text{and} \qquad \sigma_2(t) = \int_{I_2} \omega_2(t, y)\, dy.$$

Integrating (97) over I_1 and I_2 gives

$$\begin{aligned} &\tilde{v}_l(1)\, \sigma_1 - ik^{-1}\, \partial_t \sigma_1 - \sigma_1(\beta - \alpha)\, G_k(1, 1) - \sigma_2(\beta - \alpha)\, G_k(1 + \gamma, 1) = \phi_1 \\ &\tilde{v}_l(1 + \gamma)\, \sigma_2 - ik^{-1} \partial_t \sigma_2 + \sigma_1\, \beta G_k(1, 1 + \gamma) + \sigma_2\, \beta G_k(1 + \gamma, 1 + \gamma) = \phi_2, \end{aligned} \tag{100}$$

where ϕ_1 and ϕ_2 will be considered as perturbative terms,

$$\phi_1 = \phi_3 + \phi_4, \qquad \text{and} \qquad \phi_2 = \phi_5 + \phi_6,$$

ϕ_3 and ϕ_6 being the errors made by approximating the effects of ω_2 restricted to I_1 and I_2,

$$\phi_3 = -\int_{I_1} (\tilde{u}_{\text{stat}}(y) - \tilde{v}_l(1))\, \omega_2(t, y)\, dy$$
$$+ \int_{I_1} \partial^2_{yy} \tilde{u}_{\text{stat}} (\Psi_2(t, y) - \sigma_1 G_k(1, 1) - \sigma_2 G_k(1+\gamma, 1))\, dy,$$

keeping in mind that $\int_{I_1} \partial^2_{yy} \tilde{u}_{\text{stat}}\, dy = \beta - \alpha$,

$$\phi_5 = -\int_{I_2} (\tilde{u}_{\text{stat}}(y) - \tilde{v}_l(1+\gamma))\, \omega_2(t, y)\, dy$$
$$+ \int_{I_2} \partial^2_{yy} \tilde{u}_{\text{stat}} (\Psi_2(t, y) - \sigma_1 G_k(1, 1+\gamma) - \sigma_2 G_k(1+\gamma, 1+\gamma))\, dy,$$

and ϕ_4 and ϕ_6 being the terms induced by Ψ_1,

$$\phi_4 = \int_{I_1} \Psi_1\, \partial^2_{yy} \tilde{u}_{\text{stat}}\, dy, \qquad \phi_6 = \int_{I_2} \Psi_1\, \partial^2_{yy} \tilde{u}_{\text{stat}}\, dy.$$

We have, using Lemma 7.2,

$$|\phi_4| + |\phi_6| \leqslant C C_0 \exp(D_1 t)(\|\omega^0\|_{0, D_0} + \|\bar{\omega}\|_{0, D_0, D_1}). \tag{101}$$

7.3.2. *Error in the coupling I_1/I_2*

LEMMA 7.3. *There exists $\bar{C}$ independent of δ, D_0, D_1, and k, such that*

$$|\phi_3| + |\phi_5| \leqslant \bar{C}\delta \left(\int_{I_1} |\omega_2|\, dy + \int_{I_2} |\omega_2|\, dy \right) \tag{102}$$

and

$$\int_{I_1} |\omega_2(t, y)|\, dy + \int_{I_2} |\omega_2(t, y)|\, dy$$
$$\leqslant \bar{C}\, |k| \int_0^t \Bigg(C_0(\|\omega^0\|_{0, D_0} + \|\bar{\omega}\|_{0, D_0, D_1}) \exp(D_1 \tau)$$
$$+ |\sigma_1|(\tau) + |\sigma_2|(\tau) + \delta \int_{I_1} |\omega_2|(\tau, y')\, dy' + \delta \int_{I_2} |\omega_2|(\tau, y')\, dy' \Bigg)\, d\tau. \tag{103}$$

Proof. We immediately have

$$\left|\int_{I_1} (\tilde{u}_{\text{stat}}(y) - \tilde{u}_{\text{stat}}(1))\, \omega_2(t, y)\, dy\right| \leqslant \tilde{C}\delta \int_{I_1} |\omega_2|,$$

$\tilde{C}$ being independent of δ and similarly of I_2. Now

$$\Psi_2(t, y) = \int_{I_1 \cup I_2} dy'\, \omega_2(t, y')\, G_k(y', y). \tag{104}$$

If $y \in I_1$,

$$\begin{aligned}\int_{I_1} dy'\, \omega_2(t, y')\, G_k(y', y) = & \int_{I_1} dy'\, \omega_2(t, y')\, G_k(1, 1) \\ & + \int_{I_1} dy'\, \omega_2(t, y')(G_k(y', y) - G_k(1, 1)).\end{aligned}$$

Using then

$$|G_k(y', y) - G_k(1, 1)| \leqslant \tilde{C}\delta$$

with $\tilde{C}$ independent of δ and k we get

$$\left|\int_{I_1} dy'\, \omega_2(t, y')\, G_k(y', y) - G_k(1, 1)\, \sigma_1\right| \leqslant \tilde{C}\delta \int_{I_1} |\omega_2|,$$

hence

$$|\Psi_2(t, y) - G_k(1, 1)\, \sigma_1 - G_k(1 + \gamma, 1)\, \sigma_2| \leqslant \tilde{C}\delta \left(\int_{I_1} |\omega_2| + \int_{I_2} |\omega_2|\right), \tag{105}$$

which leads to (102).

Moreover, (105) gives

$$\sup_{y \in I_1 \cup I_2} |\Psi_2| \leqslant \tilde{C}(|\sigma_1| + |\sigma_2|) + \tilde{C}\delta \left(\int_{I_1} |\omega_2| + \int_{I_2} |\omega_2|\right),$$

where $\tilde{C}$ is independent of δ and k; therefore using (97) we get on $I_1 \cup I_2$

$$|(\partial_t + ik\tilde{u}_{\text{stat}})\, \omega_2| \leqslant \tilde{C}\, |k|\, |\partial^2_{yy} \tilde{u}_{\text{stat}}| \sup_{I_1 \cup I_2} (|\Psi_1| + |\Psi_2|).$$

As $\omega_2(0, y) = 0$,

$$|\omega_2(t, y)| \leqslant \bar{C}\, |k| \int_0^t d\tau\, |\partial_{yy}^2 \tilde{u}_{\mathrm{stat}}| \sup_{I_1 \cup I_2} (|\Psi_1| + |\Psi_2|). \tag{106}$$

The bound (103) is then straightforward. ▮

7.4. *Conclusion: L^∞ Bounds*

Equation (100) can be rewritten

$$\partial_t \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = -ik\mathcal{M} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} + ik \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{107}$$

with

$$\sigma_1(0) = \sigma_2(0) = 0,$$

where $\mathcal{M}$ is given by (87). Let A and B be two large constants, to be fixed later. Let T_ε be the largest time, such that for $0 \leqslant t \leqslant T_\varepsilon$,

$$\int_{I_1} |\omega_2| + \int_{I_2} |\omega_2| \leqslant A \exp(D_1 t) \tag{108}$$

and

$$|\sigma_1| + |\sigma_2| \leqslant B \exp(D_1 t). \tag{109}$$

At $t = 0$, $\omega_2 = 0$ therefore T_ε is strictly positive. We want to prove that for a suitable choice of A and B, $T_\varepsilon = +\infty$, which would imply that (108) and (109) are global in time. Let us assume by contradiction that $T_\varepsilon < +\infty$.

Using (101) and (102) we get

$$|\phi_1| + |\phi_2| \leqslant \bar{C}_0 \exp(D_1 t) + \bar{C} A \delta \exp(D_1 t),$$

where

$$\bar{C}_0 = C C_0 (\|\omega^0\|_{n, D_0} + \|\bar{\omega}\|_{n, D_0, D_1}),$$

therefore on $[0, T_\varepsilon]$, using (107) and $D_1 > |\Im c_+|\, k$,

$$|\sigma_1| + |\sigma_2| \leqslant (\bar{C}_0 + \bar{C} A \delta)(\exp(k\, |\Im c_+|\, t) + \exp(D_1 t))$$

(up to a change of the constant C in $\bar{C}_0$, depending on k) which is strictly less than $B \exp(D_1 t)$ provided

$$\bar{C}_0 \leqslant \frac{B}{4}, \qquad \bar{C} A \delta \leqslant \frac{B}{4}. \tag{110}$$

Moreover, using (103),

$$\int_{I_1} |\omega_2| + \int_{I_2} |\omega_2| \leqslant \tilde{C}\, |k| \exp(D_1 t) + \tilde{C}\, |k|\, B \exp(D_1 t) + \tilde{C}\, |k|\, \delta A \exp(D_1 t)$$

for some constant $\tilde{C}$ independent of δ and t, which is bounded by $A \exp(D_1 t)$ if δ is small enough ($\tilde{C}\, |k|\, \delta < 1/4$) and if

$$|k|\, \tilde{C} \leqslant \frac{A}{4}, \qquad |k|\, \tilde{C} B \leqslant \frac{A}{4}. \tag{111}$$

Take $B = 8\bar{C}_0$,

$$A = 4\, |k| \sup(\tilde{C}B, \tilde{C}) = 4\, |k|\, \tilde{C} \sup(8\bar{C}_0, 1),$$

and δ such that

$$16\, |k|\, \bar{C}\tilde{C} \sup(8\bar{C}_0, 1)\, \delta \leqslant 8\bar{C}_0;$$

then (110) and (111) hold true. Notice that δ only depends on $\bar{C}_0$, $\tilde{C}$, and k. For such a choice of A and B, at $t = T_\varepsilon$,

$$\int_{I_1} |\omega_2| + \int_{I_2} |\omega_2| < A \exp(D_1 t)$$

and

$$|\sigma_1| + |\sigma_2| < 8\bar{C}_0 \exp(D_1 t) < B \exp(D_1 t),$$

which is in contradiction with the definition of T_ε. Hence $T_\varepsilon = +\infty$: (108) and (109) are therefore global in time. Using (104), this leads to

$$|\Psi_2| \leqslant \bar{C}_0 \exp(D_1 t) \exp(-ky),$$

and using (106), this leads to

$$\|\omega_2\| \leqslant \frac{\hat{C}}{\delta} (\|\omega^0\|_{n, D_0} + \|\bar{\omega}\|_{n, D_0, D_1}) \exp(D_1 t) \exp(-\inf(D_0, k)\ y) \tag{112}$$

and therefore

$$\|\omega\| \leqslant \frac{\hat{C}}{\delta} (\|\omega^0\|_{n, D_0} + \|\bar{\omega}\|_{n, D_0, D_1}) \exp(D_1 t) \exp(-\inf(D_0, k)\ y). \tag{113}$$

Repeating these manipulations on derivatives of (97) ends the proof of the Proposition. ∎

7.5. *Instability near a Smooth Profile*

In Section 6 we have fulfilled the construction of an exponentially increasing eigenmode of linearized Euler equations near a profile u^η which was only continuous and piecewise smooth. In this section we want to make a similar construction for a smooth profile u^η, a regularized version of the linear by parts profile. Let us fix $\alpha < 1$ and $\beta = 1$. Let us fix k so that

$$|\Im c_+(k)| \, k > 0 \tag{114}$$

(where c_+ was defined in Section 6) and so that

$$|\Im c_+(jk)| \, jk = 0 \tag{115}$$

for $j = 2, 3, ...$. Let us take for instance $\alpha = 1/2$, $\beta = 1$, and $k = 0.6$ (see Fig. 1). We focus on the instability with wave number k and work on functions which are x periodic, with period $2\pi/k$. Therefore let $\mathbb{T}_k = \mathbb{R}/((2\pi/k)\,\mathbb{Z})$ be the periodic torus, of period $2\pi/k$. From now on we work in $\Omega = \mathbb{T}_k \times \mathbb{R}_+$.

Now let Ξ be a smooth positive function, with support in $[-1, 1]$ and $\int \Xi = 1$. Let

$$\tilde{v}_\mu = \tilde{v}_l * \frac{1}{\mu}\, \Xi\left(\frac{\cdot}{\mu}\right),$$

for $0 < \mu < \inf(1/2, \gamma/2)$, where $*$ is the convolution operator, and let $\tilde{v}_0 = \tilde{v}_l$. Notice that $\tilde{v}_\mu = \tilde{v}_l$ for $y \in [0, 1-2\mu] \cup [1+2\mu, 1+\gamma-2\mu] \cup [1+\gamma+2\mu, +\infty[$ and satisfies assumptions of Proposition 7.1.

Proposition 7.4. *Let $\nu > 0$ be small. For μ small enough there exists a solution $v_0(t, y)$ of the linearized Euler equations around $(\tilde{v}_\mu(t, y), 0)$ of the form*

$$v_0 = \begin{pmatrix} \Psi_0'(y) \exp ik(x - c_0 t) \\ -ik\Psi_0(y) \exp ik(x - c_0 t) \end{pmatrix} \tag{116}$$

with

$$|c_0 k - c_+(k)\, k| \leqslant \nu.$$

Proof. We have to prove that for μ small enough there exist c_0 near $c_+(k)$ and a solution v_0 of (81) with $\tilde{v}_l$ replaced by $\tilde{v}_\mu$. We will consider this problem as a shooting problem and apply an implicit function theorem. Let us associate to $0 \leqslant \mu \leqslant 1$ and c_0 the solution ϕ_{μ, c_0} of

$$\frac{d^2}{dz^2}\phi_{\mu, c_0} = k^2\phi_{\mu, c_0} + \phi_{\mu, c_0}\frac{\partial^2_{zz}\tilde{v}_\mu}{\tilde{v}_\mu - c_0},$$

$$\phi_{\mu, c_0}(0) = 0,$$

$$\frac{d}{dz}\phi_{\mu, c_0}(0) = 1.$$

We also impose $\phi_{\mu, c_0} \to 0$ as $z \to +\infty$ (else, as $\partial^2_{zz}\tilde{v}_l$ vanishes for z large enough, ϕ_{μ, c_0} has exponential growth at $+\infty$, which is not relevant physically). Notice that $\phi_{\mu, c_0} \to 0$ as $z \to +\infty$ if and only if

$$\frac{d}{dz}\phi_{\mu, c_0}(2) = -k\phi_{\mu, c_0}(2).$$

Therefore let

$$F(\mu, c_0) = \frac{d}{dz}\phi_{\mu, c_0}(2) + k\phi_{\mu, c_0}(2).$$

There exists a solution v_0 of (81) (with $\tilde{v}_l$ replaced by $\tilde{v}_\mu$ in (81)) with parameters μ and c_0 if and only if $F(\mu, c_0) = 0$. By definition of c_+,

$$F(0, c_+) = 0.$$

The function F is smooth in both variables and

$$\frac{dF}{dc}(0, c_+) \neq 0,$$

since c_+ is a simple root of $F(0, c) = 0$; therefore the application of the implicit function theorem ends the proof. ▮

7.6. *Construction of an Approximate Solution*

Let us turn to the proof of Theorem 1.5. The first step is to construct an approximate solution. Let ν be small enough such that $|2k\Im c_0| > |k\Im c_+(k)|$, and let $u^\eta = (\tilde{v}_\mu, 0)$. Notice that u^η is a stationary solution of (1, 2). We will build v^η starting from u^η and using several times (9, 10) and the estimates on this system given in Section 7.1 in order to construct a very precise approximate solution to (1, 2). The idea is to start from the unstable mode v_0 described in the last section and to add corrective terms in order to get an approximate solution up to times of order $\log \eta^{-1}$.

PROPOSITION 7.5. *For every $N>0$ and every $M>0$, there exist N functions $v^{(1)}, ..., v^{(N)}$ (with $v^{(1)}=\Re v_0$ given in the preceding paragraph), of the form*

$$v^{(j)}=\sum_{\alpha=1}^{N_\alpha} v^{j,\alpha} \tag{117}$$

for some integers N_α, with

$$v^{j,\alpha}=\Re\begin{pmatrix} \Psi'_{j,\alpha}(t,y)\exp ik^{j,\alpha}x \\ -ik^{j,\alpha}\Psi_{j,\alpha}(t,y)\exp ik^{j,\alpha}x \end{pmatrix} \tag{118}$$

for some functions $\Psi_{j,\alpha}$, satisfying for all $1\leqslant j\leqslant N$, for all α and for all $0\leqslant n\leqslant N$,

$$|\partial_y^n\Psi_{j,\alpha}(t,y)|\leqslant D_n^{(j,\alpha)}\exp(jk\,|\Im c_0|\,t-k_0\,y) \tag{119}$$

(with $k_0<k$) such that

$$v^{\mathrm{app}}=u^\eta+\sum_{j=1}^{N}\eta^{Mj}v^{(j)} \tag{120}$$

is an approximate solution of Euler equations in the sense

$$\partial_t v^{\mathrm{app}}+(v^{\mathrm{app}}.\nabla)\,v^{\mathrm{app}}+\nabla p=\eta^{MN}\mathscr{R}^\eta, \tag{121}$$

$$\operatorname{div} v^{\mathrm{app}}=0, \tag{122}$$

$$v_2^{\mathrm{app}}=0 \qquad \text{at} \quad y=0, \tag{123}$$

where

$$\|\mathscr{R}^\eta\|_{L^2}\leqslant C\exp(Nk\,|\Im c_0|\,t) \tag{124}$$

uniformly for $0<\eta\leqslant 1$.

Proof. To get the equations on $v^{(j)}$ we replace v^{app} by its expression in (121) and equal terms of order η^{Mj}. Therefore we study

$$\partial_t v^{(j)}+(u^\eta.\nabla)\,v^{(j)}+(v^{(j)}.\nabla)\,u^\eta+\nabla p=\mathscr{R}^{\eta,j}, \tag{125}$$

$$\operatorname{div} v^{(j)}=0, \tag{126}$$

$$v_2^{(j)}=0 \qquad \text{at} \quad z=0, \tag{127}$$

where $\mathscr{R}^{\eta, j}$ is given by the $v^{(j')}$ for $j' < j$ and is a sum of terms of the form $(v^{(j_1)}.\nabla)\, v^{(j_2)}$ for $j_1 < j$, $j_2 < j$, and $j = j_1 + j_2$. But the L^2 projection on divergence free vector fields of $(v^{(j_1)}.\nabla)\, v^{(j_2)}$ is a sum of terms of the form

$$\mathscr{R}_{j_1, j_2} = \Re \begin{pmatrix} \Psi'(t, y) \exp i(k^{j_1, \alpha_1} + k^{j_2, \alpha_2})\, x \\ -i(k^{j_1, \alpha_1} + k^{j_2, \alpha_2})\, \Psi(t, y) \exp i(k^{j_1, \alpha_1} + k^{j_2, \alpha_2})\, x \end{pmatrix}$$

for some function Ψ, with $\Psi(t, 0) = 0$,

$$|\partial_y^n \Psi(t, y)| \leqslant CC_n \exp(jk\, |\Im c_0|\, t) \exp(-k_0 y),$$

and $|k^{j_1, \alpha_1} + k^{j_2, \alpha_2}| \leqslant jk$ if we assume (119) for $j_1 < j$ and $j_2 < j$. But $|jk \Im c_0| \geqslant |2k \Im c_0| > |k \Im c_+(k)|$ (provided ν is small enough), hence using (115), $|jk \Im c_0| > |k^{j_1, \alpha_1} + k^{j_2, \alpha_2}|\, |\Im c_+(k^{j_1, \alpha_1} + k^{j_2, \alpha_2})|$. Proposition 7.1 then gives that the solution v^{j_1, j_2} of (125)–(127) is of the form

$$\begin{pmatrix} \Psi'_{j_1, j_2}(t, y) \exp i(k^{j_1, \alpha_1} + k^{j_2, \alpha_2})\, x \\ -ik \Psi_{j_1, j_2}(t, y) \exp i(k^{j_1, \alpha_1} + k^{j_2, \alpha_2})\, x \end{pmatrix}$$

with

$$|\partial_y^n \Psi_{j_1, j_2}(t, z)| \leqslant CC_n \exp(jk\, |\Im c_0|\, t) \exp(-k_0 y).$$

We will not detail the proof further. ▮

7.7. *Proof of Instability*

Let N be such that $Nk\, |\Im c_0| > 3$. We have

$$|\nabla v^{\mathrm{app}}|_{L^\infty(x, y)} \leqslant 1 + \sum_{j=1}^{N} C_j \eta^{Mj} \exp(jk\, |\Im c_0|\, t).$$

Let

$$T_0^\eta = \frac{M}{k\, |\Im c_0|} \ln \frac{1}{\eta}.$$

We have at time $t = T_0^\eta - \tau$

$$|\nabla v^{\mathrm{app}}|_{L^\infty(x, y)} \leqslant 1 + \sum_{j=1}^{N} C_j \exp(-jk\, |\Im c_0|\, \tau) \leqslant 2$$

for $\tau \geqslant \tau_0$ with τ_0 large enough, but independent on η.

Let v^η the solution of Euler equations with initial data $v^{\mathrm{app}}(0)$. Let $w^\eta = v^\eta - v^{\mathrm{app}}$ which satisfies

$$\partial_t w^\eta + (v^{\mathrm{app}}.\nabla)\, w^\eta + (w^\eta.\nabla)\, v^{\mathrm{app}} + (w^\eta.\nabla)\, w^\eta + \nabla p = -\eta^{MN} \mathscr{R}^\eta.$$

Multiplying by w^η and integrating, using $\operatorname{div} v^{\mathrm{app}} = \operatorname{div} w^\eta = 0$ and $|\nabla v^{\mathrm{app}}| \leqslant 2$ for $t \leqslant T_0^\eta - \tau_0$, we get

$$\tfrac{1}{2}\,\partial_t \int |w^\eta|^2 \leqslant 3 \int |w^\eta|^2 + C\eta^{2MN} \exp(2Nk\, |\Im c_0|\, t).$$

For N large enough, we then get

$$\int |w^\eta|^2 \leqslant C\eta^{2MN} \exp(2Nk\, |\Im c_0|\, t).$$

In particular, at $t = T_0^\eta - \tau$ with $\tau \geqslant \tau_0$,

$$\int |w^\eta|^2 \leqslant C \exp(-2\tau Nk\, |\Im c_0|). \tag{128}$$

But there exists $Y > 0$ such that

$$\|v_0\|_{L^2(\mathbb{T}_k \times [0,\, Y])} \geqslant \|v_0\|_{L^2(\mathbb{T}_k \times \mathbb{R}_+)}.$$

Thus

$$\|v^{\mathrm{app}} - u^\eta\|_{L^2(\mathbb{T}_k \times [0,\, Y])} \geqslant \frac{C_0}{2}\, \eta^M \exp(k\, |\Im c_0|\, t) - \sum_{j=2}^{N} C_j \eta^{Mj} \exp(jk\, |\Im c_0|\, t)$$

for some non-negative constant C_0; therefore at $t = T_0^\eta - \tau$,

$$\begin{aligned} \|v^{\mathrm{app}} - u^\eta\|_{L^2(\mathbb{T}_k \times [0,\, Y])} &\geqslant \frac{C_0}{2} \exp(-k\, |\Im c_0|\, \tau) - \sum_{j=2}^{N} C_j \exp(-jk\, |\Im c_0|\, \tau) \\ &\geqslant \frac{C_0}{4} \exp(-k\, |\Im c_0|\, \tau) \end{aligned} \tag{129}$$

for $\tau \geqslant \tau_1$ with $\tau_1 \geqslant \tau_0$ independent of η. Combining (128) and (129) gives at $t = T_0^\eta - \tau$

$$\begin{aligned} \|u^\eta - v^\eta\|_{L^2(\mathbb{T}_k \times [0,\, Y])} &\geqslant \frac{C_0}{4} \exp(-k\, |\Im c_0|\, \tau) - C \exp(-2\tau Nk\, |\Im c_0|) \\ &\geqslant \frac{C_0}{8} \exp(-k\, |\Im c_0|\, \tau) \end{aligned}$$

for $\tau \geqslant \tau_2$ with $\tau_2 \geqslant \tau_1$ independent of η. We therefore get at $t = T_0^\eta - \tau_2$

$$\|u^\eta - v^\eta\|_{L^\infty} \geqslant \sigma$$

with σ independent on η, which ends the proof of Theorem 1.5 after time and space are scaled back ($t \to \eta t$, $x \to \eta x$ and $y \to \eta y$). ∎

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REFERENCES

1. V. I. Arnold, "Méthodes mathématiques de la mécanique classique," Mir.
2. V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, *Ann. Institut Fourier* **16** (1996), 319–361.
3. Y. Brenier, Reformulations of some inviscid hydrostatic models, manuscript, 1997.
4. R. E. Caflisch and M. Sanmartino, Zero viscosity limit for analytic solutions of the Navier–Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, preprint.
5. R. E. Caflisch and M. Sanmartino, Zero viscosity limit for analytic solutions of the Navier–Stokes equation on a half-space. II. Construction of the Navier–Stokes solution, preprint.
6. B. Desjardins and E. Grenier, On the homogeneous model of wind driven ocean circulation, preprint, 1997.
7. P. G. Drazin and W. H. Reid, "Hydrodynamic Stability," Cambridge Monographs on Mechanics and Applied Mathematics, 1981.
8. W. E. B. Engquist, Blow up of solutions to the unsteady Prandtl equations, preprint, 1996.
9. S. Friedlander, W. Strauss, and M. Vishik, Nonlinear instability in an ideal fluid, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14**, No. 2 (1997), 187–209.
10. Y. Guo and W. Strauss, Nonlinear instability of double-humped equilibria, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **12**, No. 3 (1995), 339–352.
11. H. P. Greenspan, "The Theory of Rotating Fluids," Cambridge Monographs on Mechanics and Applied Mathematics, 1969.
12. E. Grenier, Boundary layers of characteristic nonlinear parabolic equations, *J. Math. Pures Appl.*, to appear.
13. E. Grenier, On the instability of boundary layers of incompressible Navier Stokes equations, in preparation.
14. E. Grenier and O. Guès, On the inviscid limit of noncharacteristic nonlinear parabolic systems, *J. Differential Equations*, to appear.
15. E. Grenier and N. Masmoudi, Ekman layers of rotating fluids, the case of well prepared initial data, *Comm. Partial Differential Equations* **22**, No. 5–6 (1997), 953–975.
16. B. J. Hoskins, The geostrophic momentum approximation and the semi-geostrophic equations, *J. Atmospheric Sci.* **32** (1975), 233–242.
17. T. Kato, Remarks on zero viscosity limit for nonstationary Navier Stokes flows with boundary, *in* "Seminar on Nonlinear Partial Differential Equations (Berkeley, CA, 1983)," pp. 85–98, Math. Sci. Res. Inst. Publ., Vol. 2, Springer-Verlag, New York/Berlin, 1984.

18. L. Landau and E. Lifchitz, “Mécanique des fluides,” Mir, 1959.
19. J. Marsden and T. Raţiu, Nonlinear stability in fluids and plasmas, *in* “Seminar on New Results in Nonlinear Partial Differential Equations (Bonn, 1984),” pp. 101–134, Aspects of Math., Vol. E10, Vieweg, Brauschweig, 1987.
20. C. Marchioro and M. Pulvirenti, “Mathematical Theory of Incompressible Nonviscous Fluids,” Applied Mathematical Sciences, Vol. 96, Springer-Verlag, 1994.
21. Lord Rayleigh, On the stability, or instability, of certain fluid motions, *Proc. London Math. Soc.* **11** (1880), 57–70.
22. H. Schlichting, “Boundary Layer Theory,” McGraw–Hill, New York, 1968.
23. Temam Wang, Asymptotic analysis of Oseen type equations in a channel at small viscosity, *Indiana Univ. Math. J.* **45** (1996).

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