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Hyperbolic systems of balance laws via vanishing viscosity

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Abstract

Global weak solutions of a strictly hyperbolic system of balance laws in one-space dimension are constructed by the vanishing viscosity method of Bianchini and Bressan. For global existence, a suitable dissipativeness assumption has to be made on the production term g. Under this hypothesis, the viscous approximations u^ε, that are globally defined solutions to $u^\varepsilon_t + A(u^\varepsilon)\,u^\varepsilon_x + g(u^\varepsilon) = \varepsilon u^\varepsilon_{xx}$, satisfy uniform BV bounds exponentially decaying in time. Furthermore, they are stable in L^1 with respect to the initial data. Finally, as $\varepsilon \to 0$, u^ε converges in L^1_{loc} to the admissible weak solution u of the system of balance laws $u_t + (f(u))_x + g(u) = 0$ when $A = Df$.

Keywords: Hyperbolic systems; Balance laws; Vanishing viscosity; Dissipative

1. Introduction

The object of this work is to establish the existence of a global solution to the Cauchy problem for hyperbolic systems of balance laws

$$u_t + f(u)_x + g(u) = 0, \tag{1.1}$$

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$$u(0,x) = u_0(x), \tag{1.2}$$

by the method of vanishing viscosity. Here $x \in \mathbb{R}$, $u(t,x) \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$. We assume that the system is strictly hyperbolic, i.e. $A(u) = Df(u)$ has n real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{1.3}$$

and thereby n linearly independent right eigenvectors $r_i(u)$, $i = 1, \ldots, n$.

Over the years, four different techniques have been developed for constructing weak solutions, namely the random choice method of Glimm, the front tracking method, the vanishing viscosity method and the functional analytic method of compensated compactness. Expositions of the current state of the theory together with relevant bibliography may be found in the books [7,9,16,17].

For systems of balance laws, the existence of local in time BV solutions was first established by Dafermos and Hsiao [10], by the random choice method of Glimm [11]. Because of the presence of the production term $g(u)$, small oscillations in the solution may amplify in time, hence in general one does not have long-term stability in BV. Global existence was established in [10] under a suitable dissipativeness assumption on g. (See also [14,1]). Here, we study such problems by the vanishing viscosity method. Namely, we construct solutions to (1.1) as the $\varepsilon \downarrow 0$ limit of a family $\{u^\varepsilon\}$ of functions that satisfy the parabolic system

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x + g(u^\varepsilon) = \varepsilon u^\varepsilon_{xx}. \tag{1.4}$$

The vanishing viscosity method has been studied extensively. The scalar conservation law was treated by Oleinik [15] in one-space dimension and by Kruzkov [12] in several space dimensions. The case of systems of conservation laws, which had been open for a long time, has been recently treated in a fundamental paper by Bianchini and Bressan [6] (see also [2–5]). Here, we extend the analysis of Bianchini and Bressan to systems of balance laws.

One should not expect global existence unless the source $g(u)$ is dissipative. Consider a constant equilibrium solution u^*. In particular, $g(u^*) = 0$. If we linearize the hyperbolic system (1.1) about u^* and then decompose the solution u along the right eigenvectors of $A(u^*)$, the resulting linear system is

$$v_{i,t} + \lambda_i(u^*)v_{i,x} + \sum_{j=1}^{n} B_{ij}(u^*)v_j = 0, \tag{1.5}$$

where B_{ij} are the entries of the $n \times n$ matrix

$$B(u) = [r_1(u), \ldots, r_n(u)]^{-1} Dg(u)[r_1(u), \ldots, r_n(u)]. \tag{1.6}$$

The natural condition that renders the above linear system stable in L^1 is that the matrix $B(u^*)$ is *strictly column diagonally dominant*, i.e.

$$B_{ii}(u^*) - \sum_{j \neq i} |B_{ji}(u^*)| \geqslant \mu > 0, \quad i = 1, \ldots, n. \tag{1.7}$$

Under the above hypothesis, we prove global existence of solutions for system (1.1). The principal result is the following:

Theorem 1.1. *Consider the Cauchy problem*

$$u_t^\varepsilon + A(u^\varepsilon)\, u_x^\varepsilon + g(u^\varepsilon) = \varepsilon u_{xx}^\varepsilon, \tag{1.8}$$

$$u^\varepsilon(0, x) = u_0(x). \tag{1.9}$$

Assume that the matrices $A(u)$ have real distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$ and thereby n linearly independent eigenvectors $r_1(u), r_2(u), \ldots, r_n(u)$. Under the assumption that the matrix $B(u^*)$ defined by (1.6) *is strictly diagonally dominant, there exists a constant $\delta_0 > 0$ such that if $u_0 - u^* \in L^1$ and*

$$TV\{u_0\} < \delta_0, \tag{1.10}$$

then for each $\varepsilon > 0$ the Cauchy problem (1.8)–(1.9) *has a unique solution u^ε, defined for all $t \geqslant 0$. Moreover,*

$$TV\{u^\varepsilon(t, \cdot)\} \leqslant C\, e^{-\mu t} TV\{u_0\}, \tag{1.11}$$

where μ is a positive constant that depends on $B(u^*)$. Furthermore, if v^ε is another solution of (1.8) *with initial data v_0, then*

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1} \leqslant L\, e^{-\mu t} \|u_0 - v_0\|_{L^1}. \tag{1.12}$$

Finally, as $\varepsilon \to 0$, u^ε converges in L^1_{loc} to a function u, which is the admissible weak solution u of (1.1)–(1.2), *when the system is in conservation form, $A = Df$.*

The exponential decay is induced by the dissipative source term. It should be noted that the dissipativeness assumption (1.7) depends on the choice of right eigenvectors of $A(u)$. It is possible to state this condition in an invariant fashion (see [1]). In the proof of the above proposition, the heart of the matter is to obtain BV a priori bounds on solutions of (1.4), independent of ε; in particular, to show that $TV\{u^\varepsilon(t, \cdot)\}$ remains bounded, and indeed decays exponentially fast to zero, as $t \to \infty$. Our treatment follows closely the fundamental paper of Bianchini and Bressan [6] for conservation laws. To begin with, by rescaling the coordinates, $t \sim t/\varepsilon$, $x \sim x/\varepsilon$, system (1.8)–(1.9) reduces to

$$\begin{aligned} &u_t + A(u)u_x - u_{xx} + \varepsilon g(u) = 0, \\ &u(0,x) = u_0^\varepsilon(x) = u_0(\varepsilon x). \end{aligned} \tag{1.13}$$

The total variation of the initial data u_0^ε does not change with ε, while the L^1 norm does. Our goal is to establish a bound

$$TV\{u(t,\cdot)\} \leqslant Ce^{-\varepsilon\mu t}TV\{u_0\} \tag{1.14}$$

for all times $t \geqslant 0$, with C depending solely on the total variation of u_0 and not on $\|u_0^\varepsilon\|_{L^1}$.

In order to establish the stability estimate (1.12) in Theorem 1.1, we shall also work with the linearized evolution equation which governs an infinitesimal perturbation z of u:

$$z_t + A(u)z_x + \varepsilon Dg(u)z - z_{xx} + (z \bullet A(u))u_x = 0, \tag{1.15}$$

and establish a bound of the form

$$\|z(t)\|_{L^1} \leqslant L\, e^{-\varepsilon\mu t}\|z(0)\|_{L^1}. \tag{1.16}$$

The proof of Theorem 1.1 is carried out in two steps: In Section 2, we prove (1.14) and (1.16) over a time interval of length $\mathcal{O}(\delta_0^{-2})$ by using standard parabolic estimates. We also obtain exponential decay in L^1 of higher-order derivatives of u_x and z. In Sections 3–7, we extend the validity of these estimates up to $t = \infty$, by using the hyperbolic structure of the system. The reason of this two-step approach is that the parabolic estimates apply even when the derivatives of the initial data are large, but are only valid on a finite time interval, whose length depends on $TV\{u_0\}$; whereas, the hyperbolic estimates are valid for all times, but require initial values with small derivatives. Thus the parabolic estimates are employed in order to show that the size of derivatives attenuates in finite time and eventually enters and remains in the regime where the hyperbolic estimates apply. Finally, in Section 8 by rescaling the coordinates backwards, we complete the proof of Theorem 1.1.

2. Parabolic estimates

In this section, we establish bounds (1.14) and (1.16) over an initial time interval by means of parabolic estimates. We rewrite system (1.15) as

$$\begin{aligned} z_t + A^*z_x + \varepsilon Dg(u^*)z - z_{xx} &= (A^* - A(u))z_x - (z \bullet A(u))u_x, \\ &\quad + \varepsilon(Dg(u^*) - Dg(u))z, \end{aligned} \tag{2.1}$$

where $A^* \doteq A(u^*)$ and D is the gradient operator with respect to the state vector u.

Lemma 2.1. *Under hypothesis* (1.7), *for all* $t > 0$, *the Green's kernel* G *of the parabolic system*

$$w_t + A^* w_x - w_{xx} + \varepsilon Dg(u^*)w = 0 \tag{2.2}$$

satisfies the bounds

$$\|G(t)\|_{L^1} \leqslant \kappa e^{-\varepsilon\mu t}, \qquad \|G_x(t)\|_{L^1} \leqslant \frac{\kappa}{\sqrt{t}} e^{-\varepsilon\mu t}, \qquad \|G_{xx}(t)\|_{L^1} \leqslant \frac{\kappa}{t} e^{-\varepsilon\mu t}, \tag{2.3}$$

for some appropriate constant κ.

Proof. It suffices to establish the estimates for the corresponding Green's kernel of the system

$$w_t + \Lambda w_x - w_{xx} + \varepsilon B(u^*)w = 0 \tag{2.4}$$

obtained by diagonalizing the principal part of (2.2). The Green's function of (2.4) is given by $R(u^*)^{-1} G$, where $R(u^*)$ is the matrix of the right eigenvectors of $A(u^*)$ and $\Lambda = \mathrm{diag}(\lambda_i^*)$. Without loss of generality, we denote the Green's kernel of (2.4) by G, i.e.

$$\begin{aligned} &G_t + \Lambda G_x - G_{xx} + \varepsilon B\, G = 0, \\ &G(0, x) = \delta(x), \end{aligned} \tag{2.5}$$

where $B \doteq B(u^*)$ is strictly diagonally dominant. Perform the Fourier transform to (2.5) with respect to x. By using $\hat{G}$ to denote the Fourier transform of G and ξ to denote the Fourier variable, we arrive at

$$\begin{aligned} &\hat{G}_t = -\xi^2 \hat{G} - (i\xi\Lambda + \varepsilon B)\hat{G}, \\ &\hat{G}(0, \xi) = I. \end{aligned} \tag{2.6}$$

Thus

$$\hat{G}(t, \xi) = e^{-\xi^2 t} e^{-(i\xi\Lambda + \varepsilon B)t} \hat{G}(0, \xi), \tag{2.7}$$

which implies

$$G(t, x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} * \left[e^{-(i\xi\Lambda + \varepsilon B)t} \hat{G}(0, \xi) \right]^{\vee}. \tag{2.8}$$

Let G_2 be the Green's function to the system

$$w_t + \Lambda w_x + \varepsilon B w = 0, \tag{2.9}$$

then

$$G_2(t,x) = \left[e^{-(i\xi\Lambda+\varepsilon B)t} \hat{G}(0,\xi) \right]^{\vee}. \tag{2.10}$$

In view of (1.7), a straightforward calculation establishes the exponential decay of $G_2(t)$ in L^1

$$\|G_2(t)\|_{L^1} \leqslant c_1 e^{-\varepsilon\mu t}. \tag{2.11}$$

By employing the properties of convolution and (2.8), we deduce

$$\left\| \frac{\partial^\alpha}{\partial x^\alpha} G(t) \right\|_{L^1} \leqslant \left\| \frac{\partial^\alpha}{\partial x^\alpha} \left[\frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \right] \right\|_{L^1} \|G_2(t)\|_{L^1}. \tag{2.12}$$

Estimates (2.3) follow easily for $\alpha = 0, 1, 2$, if one recalls the corresponding bounds for the heat kernel. □

For future use, we define the following constants:

$$\hat{t} \doteq \left(\frac{1}{400\kappa\,\alpha\,\delta_0} \right)^2, \quad \varepsilon_0 \doteq \kappa^2 \alpha\,\delta_0, \quad \alpha \doteq \sup_u (\|DA\| + \|D^2 A\| + \|D^2 g\|), \tag{2.13}$$

where κ is the constant in (2.3). By means of parabolic arguments, we establish the following results:

Proposition 2.2. *Let u, z be solutions of systems* (1.13), (1.15) *respectively, satisfying the bounds*

$$\|u_x(t)\|_{L^1} \leqslant \delta_0 e^{-\varepsilon\mu t}, \quad \|z(t)\|_{L^1} \leqslant \delta_0 e^{-\varepsilon\mu t}, \tag{2.14}$$

for some constant $0 < \delta_0 < 1$ and for all $t \in [0, \hat{t}]$ and $\varepsilon \in [0, \varepsilon_0]$. Then for $t \in [0, \hat{t}]$ the following estimates hold:

$$\|u_{xx}(t)\|_{L^1}, \ \|z_x(t)\|_{L^1} \leqslant \frac{2\,\kappa\,\delta_0}{t^{1/2}} e^{-\varepsilon\mu t}, \tag{2.15}$$

$$\|u_{xxx}(t)\|_{L^1}, \ \|z_{xx}(t)\|_{L^1} \leqslant \frac{5\,\kappa^2\,\delta_0}{t} e^{-\varepsilon\mu t}, \tag{2.16}$$

$$\|u_{xxx}(t)\|_{L^\infty}, \ \|z_{xx}(t)\|_{L^\infty} \leqslant \frac{16\,\kappa^3\,\delta_0}{t^{3/2}} e^{-\varepsilon\mu t}, \tag{2.17}$$

where κ and $\mu > 0$ are the constants in (2.3) *and* (1.7).

Proof. The infinitesimal perturbation z of u is a solution to (2.1), so we can express z in terms of the Green's function G as follows

$$\begin{aligned} z(t) &= G(t) * z(0) + \int_0^t G(t-s) * [(A^* - A(u))z_x(s) - (z \bullet A(u))u_x(s)]\,ds \\ &\quad + \varepsilon \int_0^t G(t-s) * (Dg(u^*) - Dg(u))z(s)\,ds, \end{aligned} \tag{2.18}$$

where $*$ denotes convolution and $\beta \bullet \gamma$ is the derivative of γ in the direction of β, i.e. $\nabla_\beta \gamma$. Differentiating with respect to x yields

$$\begin{aligned} z_x(t) &= G_x(t) * z(0) + \int_0^t G_x(t-s) * [(A^* - A(u))z_x(s) - (z \bullet A(u))u_x(s)]\,ds \\ &\quad + \varepsilon \int_0^t G_x(t-s) * (Dg(u^*) - Dg(u))z(s)\,ds. \end{aligned} \tag{2.19}$$

Assume that $\tau < \hat{t}$ is the first time at which (2.15) holds as an equality, then we estimate $\|z_x(\tau)\|_{L^1}$ via (2.19). By (2.14) and Lemma 2.1, we get

$$\begin{aligned} \|z_x(\tau)\|_{L^1} &\leqslant \|G_x(\tau)\|_{L^1}\,\|z(0)\|_{L^1} + \int_0^\tau \|G_x(\tau-s)\|_{L^1} \Big[\|(A^* - A(u))z_x(s)\|_{L^1} \\ &\quad + \|(z \bullet A(u))u_x(s)\|_{L^1} + \varepsilon\|(Dg(u^*) - Dg(u))z(s)\|_{L^1}\Big] ds \\ &\leqslant \frac{\kappa}{\sqrt{\tau}}\, e^{-\varepsilon\mu\tau}\delta_0 + \int_0^\tau \frac{\kappa e^{-\varepsilon\mu(\tau-s)}}{\sqrt{\tau-s}} \Big[\|DA\|_{L^\infty}\|u_x(s)\|_{L^1}\|z_x(s)\|_{L^1} \\ &\quad + \|z(s)\|_{L^\infty}\|DA\|_{L^\infty}\|u_x(s)\|_{L^1} + \varepsilon\|D^2 g\|_{L^\infty}\|u_x(s)\|_{L^1}\|z(s)\|_{L^1}\Big] ds \\ &< \left[\frac{\kappa\delta_0}{\sqrt{\tau}} + 16\kappa^2\alpha\delta_0^2 + \varepsilon\kappa\,\alpha\delta_0^2 2\sqrt{\tau}\right] e^{-\varepsilon\mu\tau}. \end{aligned} \tag{2.20}$$

For values of ε in $[0, \varepsilon_0]$, the above estimate simplifies to

$$\|z_x(\tau)\|_{L^1} < \left[\frac{\kappa\delta_0}{\sqrt{\tau}} + \frac{1}{25}\,\frac{\kappa\delta_0}{\sqrt{\tau}} + \frac{1}{8\cdot 10^4}\,\frac{\kappa\delta_0}{\sqrt{\tau}}\right] e^{-\varepsilon\mu\tau} < \frac{2\kappa\delta_0}{\sqrt{\tau}} e^{-\varepsilon\mu\tau}, \tag{2.21}$$

which contradicts the choice of τ. Hence estimate (2.15) holds over the interval $[0, \hat{t}]$. Moreover, a solution of (2.1) is $z = u_x$, hence the same bound holds for $\|u_x(t)\|_{L^1}$.

To prove the remaining estimates, we argue in the same manner. We first express $z_{xx}(t)$ in terms of the Green's kernel G,

$$z_{xx}(t) = G_x\left(\frac{t}{2}\right) * z_x\left(\frac{t}{2}\right) + \varepsilon \int_{\frac{t}{2}}^{t} G_x(t-s) * \Big\{(Dg(u) - Dg(u^*))z(s)\Big\}_x ds$$
$$+ \int_{\frac{t}{2}}^{t} G_x(t-s) * \Big\{(A^* - A(u))z_x(s) - (z \bullet A(u))u_x(s)\Big\}_x ds \tag{2.22}$$

and then estimate $\|z_{xx}(t)\|_{L^1}$ and $\|z_{xx}(t)\|_{L^\infty}$ via the above equation. □

Proposition 2.3. *Let* $u = u(t, x)$, $z = z(t, x)$ *be solutions of* (1.13), (1.15), *respectively, such that*

$$TV\{u(0,\cdot)\} \leqslant \frac{\delta_0}{4\kappa}, \qquad \|z(0)\|_{L^1} \leqslant \frac{\delta_0}{4\kappa}. \tag{2.23}$$

Then u, z *are well-defined on the whole interval* $[0, \hat{t}]$ *satisfying the bounds*

$$\|u_x(t)\|_{L^1} \leqslant \frac{\delta_0}{2} e^{-\varepsilon\mu t}, \qquad \|z(t)\|_{L^1} \leqslant \frac{\delta_0}{2} e^{-\varepsilon\mu t}, \qquad t \in [0, \hat{t}]. \tag{2.24}$$

Proof. We follow the same idea as before. We write the solution z in terms of the Green's kernel G:

$$z(t,x) = G(t) * z(0) + \int_0^t G(t-s) * \big[(A^* - A(u))z_x(s) - (z \bullet A(u))u_x(s)\big] ds$$
$$+ \varepsilon \int_0^t G(t-s) * \big(Dg(u^*) - Dg(u)\big) z(s)\, ds. \tag{2.25}$$

Assume that $\tau < \hat{t}$ is the first time at which (2.24) holds as an equality, then

$$\|z(\tau)\|_{L^1} \leqslant \kappa e^{-\varepsilon\mu\tau} \|z(0)\|_{L^1} + \int_0^\tau \kappa e^{-\varepsilon\mu(\tau-s)} \Big[\|u_x(s)\|_{L^1} \|DA\|_{L^\infty} \|z_x(s)\|_{L^1}$$
$$+ \|z(s)\|_{L^\infty} \|DA\|_{L^\infty} \|u_x(s)\|_{L^1}$$
$$+ \varepsilon \|u_x(s)\|_{L^1} \|D^2 g\|_{L^\infty} \|z(s)\|_{L^1} \Big] ds. \tag{2.26}$$

If one sets $z = u_x$, then the above estimate proves bound (2.24.1) by a contradiction argument. Having this, we derive the bound (2.24.2) on z as follows: by (2.15), (2.23) and (2.24.1), the above inequality reduces to

$$\begin{aligned}\|z(\tau)\|_{L^1} &\leqslant \frac{\delta_0}{4}e^{-\varepsilon\mu\tau} + \int_0^\tau e^{-\varepsilon\mu\tau}2\left(\frac{\delta_0}{2}\|DA\|_{L^\infty}\frac{2\kappa\delta_0}{2\sqrt{s}}e^{-\varepsilon\mu s}\right) ds \\ &\quad + \varepsilon\int_0^\tau e^{-\varepsilon\mu\tau}\frac{\delta_0^2}{4}\|D^2g\|_{L^\infty}e^{-\varepsilon\mu s}\, ds \\ &\leqslant \frac{\delta_0}{4}e^{-\varepsilon\mu\tau} + 2\kappa\delta_0^2\|DA\|_{L^\infty}e^{-\varepsilon\mu\tau}\sqrt{\tau} + \varepsilon\frac{\delta_0^2}{4}\|D^2g\|_{L^\infty}\tau e^{-\varepsilon\mu\tau}. \qquad (2.27)\end{aligned}$$

For any ε, $0 < \varepsilon \leqslant \varepsilon_0$, we get $\|z(\tau)\|_{L^1} < \frac{\delta_0}{2}e^{-\varepsilon\mu\tau}$, which yields a contradiction by the choice of τ. This completes the proof. □

In particular, the assumption on $\|z(0)\|_{L^1}$ in the previous proposition can be relaxed because z satisfies a linear homogeneous equation.

The estimates in the following corollary show that as long as the total variation of $u(t)$ and $\|z(t)\|_{L^1}$ satisfy the desired bounds, all higher order derivatives of u_x and z are small, with exponentially decaying L^1 norms. This will enable us to use tools of hyperbolic type to extend estimates (1.14) and (1.16) to $[\hat{t}, \infty)$.

Corollary 2.4. *If the bounds* (2.14) *hold on a larger interval* $[0, T]$, *then for all* $t \in [\hat{t}, T]$,

$$\|u_{xx}(t)\|_{L^1},\ \|u_x(t)\|_{L^\infty},\ \|z_x(t)\|_{L^1} = \mathcal{O}(1)\delta_0^2 e^{-\varepsilon\mu t}, \qquad (2.28)$$

$$\|u_{xxx}(t)\|_{L^1},\ \|u_{xx}(t)\|_{L^\infty},\ \|z_{xx}(t)\|_{L^1} = \mathcal{O}(1)\delta_0^3 e^{-\varepsilon\mu t}, \qquad (2.29)$$

$$\|u_{xxx}(t)\|_{L^\infty},\ \|z_{xx}(t)\|_{L^\infty} = \mathcal{O}(1)\delta_0^4 e^{-\varepsilon\mu t}. \qquad (2.30)$$

Proof. Apply the proof of Proposition 2.2 on the time interval $[t - \hat{t}, t]$ and use $\hat{t} \approx \delta_0^{-2}$. □

3. Outline of the proof of BV estimates

In the previous section, we established the desired bounds on the time interval $[0, \hat{t}]$. The next task is to extend the estimates to $[\hat{t}, \infty)$. This will require laborious analysis. For the benefit of the reader we outline here a road map for the Sections 4–6.

The aim is to establish estimate (1.14) on the total variation of solutions of

$$u_t + A(u)u_x - u_{xx} + \varepsilon g(u) = 0 \qquad (3.1)$$

for small $TV\{u_0\}$ over the time interval $[\hat{t}, \infty)$. In Section 4, we quote the construction of viscous traveling waves selected by a center manifold technique, first presented in [6]. We decompose the gradient

$$u_x = \sum_{i=1}^{n} v_i \tilde{r}_i \tag{3.2}$$

into a sum of gradients of such waves. Using the same basis, the derivative u_t is written in the form

$$u_t + \varepsilon g(u) = \sum_{i=1}^{n} (w_i - \lambda_i^* v_i) \tilde{r}_i. \tag{3.3}$$

In Section 5, we establish validity of decomposition (3.2)–(3.3). In Section 6, we study the evolution of the component vector (v, w) and show that it satisfies a $2n \times 2n$ coupled-system of viscous balance laws with source terms of the form:

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \right]_x - \begin{pmatrix} v \\ w \end{pmatrix}_{xx} + \varepsilon \begin{pmatrix} B^\sharp & H \\ K & B^\flat \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \tag{3.4}$$

where Λ is a diagonal matrix. Under the dissipativeness hypothesis of Theorem 1.1, we show that the coupling matrix $\begin{pmatrix} B^\sharp & H \\ K & B^\flat \end{pmatrix}$ is strictly diagonally dominant.

In order to control the L^1 norms of $v(t, \cdot)$ and $w(t, \cdot)$ for $t > \hat{t}$, we have to show that $e^{\varepsilon \mu t}|\Phi|$ and $e^{\varepsilon \mu t}|\Psi|$ are integrable over the half-plane $\{t > \hat{t}, x \in \mathbb{R}\}$. We analyze the form of the various source terms $\Phi = (\phi_i)$ and $\Psi = (\psi_i)$ that may be regarded as the result of interactions between viscous waves. We employ a transversal interaction functional that controls interaction between waves of different families (in Section 6.1), as well as various "swept area" and "curve length" functionals that control the interaction between waves of the same family (in Section 6.2). We also use energy functionals on regions where the diffusion is dominant (in Section 6.3). Passing from the conservation law ($g \equiv 0$) to the balance law (1.1), system (3.4) becomes coupled because of the presence of the production term $g(u)$. An essential step in addressing this difficulty is to establish additional estimates for handling the coupling. For that purpose, supplementary Lyapunov functionals are constructed to those already devised in [6]. By employing these functionals, one finally estimates the integral

$$\int_{\hat{t}}^{\infty} \int e^{\varepsilon \mu t} \sum_i (|\phi_i| + |\psi_i|) \, dx \, dt = \mathcal{O}(1) \delta_0^2, \tag{3.5}$$

and this yields the desired a priori bound on $\|u_x(t, \cdot)\|_{L^1}$ in (1.14). Because of the diagonal dominance assumption, we succeed in establishing exponential decay of the total variation of $u(t, \cdot)$.

Following the above outline, we finally establish the following key result:

Lemma 3.1. *Let u be a solution of* (1.13) *such that for $t \in [\hat{t}, T]$, u satisfies*

$$\|u_x(t)\| \leqslant \delta_0 e^{-\varepsilon\mu t}. \tag{3.6}$$

Suppose the source terms in (3.4) *satisfy*

$$\int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \sum_{i=1}^{n} \big(|\phi_i(t,x)| + |\psi_i(t,x)|\big)\, dx\, dt \leqslant \delta_0 \tag{3.7}$$

for $t \in [\hat{t}, T]$, then we have the estimate

$$\int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \sum_{i=1}^{n} \big(|\phi_i(t,x)| + |\psi_i(t,x)|\big)\, dx\, dt = \mathcal{O}(1)\delta_0^2. \tag{3.8}$$

Assuming that the above lemma holds, we conclude the proof of the a priori BV bound (1.14) as follows: Consider any initial data $u_0 : \mathbb{R} \rightarrowtail \mathbb{R}^n$, with

$$TV\{u_0\} \leqslant \frac{\delta_0}{8\sqrt{n}\kappa}, \qquad \lim_{x\to-\infty} u_0(x) = u^* \in K, \tag{3.9}$$

where κ is given by (2.3). By Proposition 2.3, the solution to (1.13) exists on an initial time interval $[0, \hat{t}]$, satisfying the bound

$$\|u_x(\hat{t})\|_{L^1} \leqslant \frac{\delta_0}{4\sqrt{n}}\, e^{-\varepsilon\mu\hat{t}}. \tag{3.10}$$

According to Proposition 2.2, this solution can be prolonged in time as long as its total variation remains small. Define the time

$$T \doteq \sup\left\{\tau\,;\ \sum_i \int_{\hat{t}}^{\tau} \int e^{\varepsilon\mu t} \big(|\phi_i(t,x)| + |\psi_i(t,x)|\big)\, dx\, dt \leqslant \frac{\delta_0}{2}\right\}. \tag{3.11}$$

Suppose $T < \infty$, then for all $t \in [\hat{t}, T]$, decomposition (3.2) holds and one has

$$\begin{aligned}
\|u_x(t)\|_{L^1} &\leqslant \sum_{i=1}^{n} \|v_i(t)\|_{L^1} \\
&\leqslant \sum_{i=1}^{n} e^{-\varepsilon\mu t}\left(e^{\varepsilon\mu\hat{t}}\|v_i(\hat{t})\|_{L^1} + \int_{\hat{t}}^{t}\int e^{\varepsilon\mu s}|\phi_i(s,x)|\, dx\, ds\right) \\
&\leqslant e^{-\varepsilon\mu(t-\hat{t})} 2\sqrt{n}\|u_x(\hat{t})\|_{L^1} + e^{-\varepsilon\mu t}\frac{\delta_0}{2} \leqslant \delta_0 e^{-\varepsilon\mu t}.
\end{aligned} \tag{3.12}$$

Assuming Lemma 3.1 is valid, by (3.11) and (3.12) we obtain

$$\sum_i \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \left(|\phi_i(t,x)| + |\psi_i(t,x)|\right) dx\, dt = \mathcal{O}(1)\delta_0^2 < \frac{\delta_0}{2}, \tag{3.13}$$

for sufficiently small δ_0. This yields a contradiction with the choice of T in (3.11). Thus, the solution u is globally defined and the total variation remains uniformly bounded and, indeed, decays exponentially fast in time, $TV\{u(t)\} < \delta_0 e^{-\varepsilon\mu t}$ for all $t \in [\hat{t}, \infty)$.

For convenience, we use throughout the following terminology:

Definition 3.2. We call a scalar function $\xi = \xi(t,x)$ controllable if (3.7) implies

$$\int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \xi(t,x)\, dx\, dt = \mathcal{O}(1)\delta_0^2. \tag{3.14}$$

Thus Lemma 3.1 amounts to showing that $|\phi_i(t,x)|$ and $|\psi_i(t,x)|$ are controllable for all $i = 1, \ldots, n$.

4. Construction of viscous traveling waves

In this section, we consider the viscous traveling waves first constructed by Bianchini and Bressan [6,5] by a center manifold argument [8]. We quote a summary of their results and the main estimates. For each i, we consider the viscous traveling i-waves $U(x - \sigma_i t)$ that are solutions to the system of conservation laws with no source ($g \equiv 0$), having speed $\sigma_i \approx \lambda_i(u^*)$ and corresponding to trajectories that lie on the center manifold $\mathcal{C}_i \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$

$$\mathcal{C}_i = \left\{ (u, v, \sigma_i) : \, v = v_i \tilde{r}_i, \; |u - u^*| < \delta, \; |v_i| < \delta, \; |\sigma_i - \lambda_i^*| < \delta \right\} \tag{4.1}$$

of the flow described in Section 4 in [6]. The unit vector $\tilde{r}_i = \tilde{r}_i(u, v_i, \sigma_i)$ is defined in a small neighborhood of $(u^*, 0, \lambda_i^*)$ and tends to $r_i^* \doteq r_i(u^*)$ as $(u, v_i, \sigma_i) \to (u^*, 0, \lambda_i^*)$. We use the same notation as in [6]:

$$\tilde{r}_{i,u} \doteq \frac{\partial \tilde{r}_i}{\partial u}, \qquad \tilde{r}_{i,v} \doteq \frac{\partial \tilde{r}_i}{\partial v_i} \qquad \tilde{r}_{i,\sigma} \doteq \frac{\partial \tilde{r}_i}{\partial \sigma_i}. \tag{4.2}$$

By the same convention, we denote higher-order derivatives of $\tilde{r}_i$, by $\tilde{r}_{i,uu}$, $\tilde{r}_{i,uv}$, $\tilde{r}_{i,u\sigma}$ etc.

The following identity plays a key role in controlling the component source terms ϕ_i and ψ_i:

$$(A(u) - \tilde{\lambda}_i I)\tilde{r}_i = v_i(\tilde{r}_{i,u}\tilde{r}_i + (\tilde{\lambda}_i - \sigma_i)\tilde{r}_{i,v}), \tag{4.3}$$

where $\tilde{\lambda}_i$ is the "generalized eigenvalue" defined as $\tilde{\lambda}_i \doteq \langle \tilde{r}_i, A(u)\tilde{r}_i \rangle$. This identity is fundamental since it corresponds to $(A(u) - \lambda_i I)r_i = 0$. By continuity, (4.3) implies

$$\tilde{r}_i(u, 0, \sigma_i) = r_i(u), \tag{4.4}$$

and $\tilde{\lambda}_i \to \lambda_i(u)$ as $v_i \to 0$. Furthermore, using (4.4), we deduce the following important estimates:

$$\tilde{r}_i(u, v_i, \sigma_i) - r_i(u) = \mathcal{O}(1) \cdot v_i, \quad \tilde{r}_{i,\sigma} = \mathcal{O}(1) \cdot v_i, \tag{4.5}$$

$$\tilde{r}_{i,u\sigma} = \mathcal{O}(1) \cdot v_i \quad \tilde{r}_{i,\sigma\sigma} = \mathcal{O}(1) \cdot v_i, \tag{4.6}$$

$$|\tilde{\lambda}_i(u, v_i, \sigma_i) - \lambda_i(u)| = \mathcal{O}(1) \cdot v_i, \quad \tilde{\lambda}_{i,\sigma} = \mathcal{O}(1) \cdot v_i. \tag{4.7}$$

Without loss of generality, we can assume that the eigenvectors $r_i(u^*)$ form an orthonormal basis. Having this, we conclude

$$\begin{aligned} \langle \tilde{r}_i(u, v_i, \sigma_i), \tilde{r}_j(u, v_j, \sigma_j) \rangle &= \delta_{ij} + \mathcal{O}(1)\delta_0, \\ \langle \tilde{r}_i, A(u)\tilde{r}_j \rangle &= \mathcal{O}(1)\delta_0 \qquad \text{for } i \neq j. \end{aligned} \tag{4.8}$$

These estimates will be used in the forthcoming sections in order to control the component source terms ϕ_i, ψ_i.

5. Decomposition of the derivatives

Here, we decompose the gradient u_x and the time derivative u_t pointwise along a suitable basis using the construction of $\tilde{r}_i$ defining $\mathcal{C}_i$ as presented in the previous section. Let u be a smooth solution of the system of viscous balance laws (1.13). For each point x, given (u, u_x, u_{xx}) we determine u_t via $u_t = u_{xx} - A(u)u_x + \varepsilon g(u)$. We seek $(v, w) \in \mathbb{R}^{2n}$ such that

$$\begin{aligned} u_x &= \sum_{i=1}^{n} v_i \tilde{r}_i(u, v_i, \sigma_i), \\ u_t + \varepsilon g(u) &= \sum_{i=1}^{n} (w_i - \lambda_i^* v_i)\tilde{r}_i(u, v_i, \sigma_i), \end{aligned} \tag{5.1}$$

while the speed is given by

$$\sigma_i = \lambda_i^* - \theta\left(\frac{w_i}{v_i}\right). \tag{5.2}$$

The cutoff function θ satisfies

$$\theta(s) = \begin{cases} s \text{ if } |s| \leqslant \delta_1, \\ 0 \text{ if } |s| \geqslant 3\delta_1, \end{cases} \qquad |\theta'(s)| \leqslant 1, \quad |\theta''(s)| \leqslant 4/\delta_1, \tag{5.3}$$

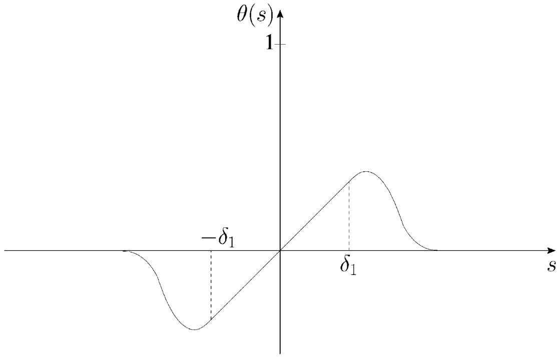

Fig. 1. Cutoff function $\theta(s)$.

as shown in Fig. 1. It is necessary to insert the cutoff function θ in order to guarantee that the speed σ_i remains close to λ_i^* and therefore $\tilde{r}_i(u, v_i, \sigma_i)$ is well defined. From now on, we use the abbreviation $\theta_i \doteq \theta(\frac{w_i}{v_i})$. The presence of $\varepsilon g(u)$ in the decomposition of u_t is crucial, as it corrects the speed of the viscous traveling waves when passing from the system of conservation laws to the system of balance laws. Decomposition (5.1) corresponds to viscous traveling waves U_i such that

$$U_i(x) = u(x), \quad {U_i}'(x) = v_i \tilde{r}_i, \quad {U_i}'' = (A(u) - \sigma_i) U_i.$$

From the first equation in (5.1) it follows, $u_x(x) = \sum_i U_i'(x)$. In addition, if for all $i = 1, \ldots, n$, the cutoff function θ is the identity map, i.e. $\sigma_i = \lambda_i^* - w_i/v_i$, then there is also a good fit of the second derivative u_{xx}:

$$\begin{aligned} u_{xx}(x) &= u_t + \varepsilon g(u) + A(u)u_x = \sum_i \left(w_i - \lambda_i^* v_i\right) \tilde{r}_i + A(u) \sum_i v_i \tilde{r}_i \\ &= \sum_i (A(u) - \sigma_i I) v_i \tilde{r}_i = \sum_i U_i''(x). \end{aligned} \tag{5.4}$$

The following lemma establishes the decomposition.

Lemma 5.1. *For $|u - u^*|$, $|u_x|$ and $|u_{xx}|$ sufficiently small, there is a unique solution (v, w) to system* (5.1). *Moreover, the map $(u, u_x, u_{xx}) \mapsto (v, w)$ is smooth outside the manifolds $\mathcal{N}_i = \{(v, w);\ v_i = w_i = 0\}$, $i = 1, \ldots, n$; more precisely it is $\mathcal{C}^{1,1}$, i.e. continuously differentiable with Lipschitz continuous derivatives on a neighborhood of the point $(u^*, 0, 0)$.*

Proof. The uniqueness is immediate. To verify existence, we define the map:

$$\begin{aligned}\mathcal{G}: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n &\mapsto \mathbb{R}^{2n}\\ (u, v, w) &\mapsto \sum_{i=1}^{n} \mathcal{G}_i(u, v_i, w_i),\end{aligned} \tag{5.5}$$

$$\mathcal{G}_i(u, v_i, w_i) = \begin{pmatrix} v_i \tilde{r}_i\left(u, v_i, \lambda_i^* - \theta\left(\frac{w_i}{v_i}\right)\right) \\ \left(w_i - \lambda_i^* v_i\right)\tilde{r}_i\left(u, v_i, \lambda_i^* - \theta\left(\frac{w_i}{v_i}\right)\right) \end{pmatrix}, \tag{5.6}$$

and apply the implicit function theorem. We easily observe that the mapping $\mathcal{G}$ is well defined and the Jacobian matrices are

$$\begin{aligned}\frac{\partial \mathcal{G}_i}{\partial(v_i, w_i)} &= \begin{pmatrix} \tilde{r}_i & 0 \\ -\lambda_i^* \tilde{r}_i & \tilde{r}_i \end{pmatrix}\\ &+ \begin{pmatrix} v_i \tilde{r}_{i,v} + \theta_i' \frac{w_i}{v_i} \tilde{r}_{i,\sigma} & -\theta_i' \tilde{r}_{i,\sigma} \\ w_i \tilde{r}_{i,v} - \lambda_i^* v_i \tilde{r}_{i,v} + \frac{w_i^2}{v_i^2} \theta_i' \tilde{r}_{i,\sigma} - \lambda_i^* \theta_i' \frac{w_i}{v_i} \tilde{r}_{i,\sigma} & \lambda_i^* \theta_i' \tilde{r}_{i,\sigma} - \frac{w_i}{v_i} \theta_i' \tilde{r}_{i,\sigma} \end{pmatrix}\\ &\doteq \mathcal{M}_{i,0} + \mathcal{M}_{i,1}.\end{aligned} \tag{5.7}$$

For (v, w) in a small neighborhood of $(0, 0)$, $\mathcal{M}_0(u; \cdot)$ is invertible and uniformly bounded and $\mathcal{M}_1(u; \cdot) \to 0$ as $(v, w) \to 0$. Therefore the map $(v, w) \mapsto \mathcal{G}(u; v, w)$ is $\mathcal{C}^1$ and invertible in a small neighborhood of $(0, 0)$. Thus given (u, u_x, u_{xx}), there exists unique (v, w) such that

$$\mathcal{G}(u, v, w) = (u_x, u_{xx} - A(u)u_x) = (u_x, u_t + \varepsilon g(u)) \tag{5.8}$$

and (5.1) holds. Following the details in Lemma 5.2 in [6], one can show that $\mathcal{G}$ is $\mathcal{C}^{1,1}$ on a neighborhood of $(u^*, 0, 0)$ with uniformly bounded second derivatives outside $\mathcal{N}_i$. This completes the proof. □

The following lemma states the bounds on v and w that correspond to the bounds on the u_x, u_{xx} and u_{xxx} derived in Section 2.

Lemma 5.2. *Assume that bounds* (2.14) *hold on* $[0, T]$. *Then, the components* v_i, w_i *in* (5.1) *satisfy the estimates*

$$\|v_i(t)\|_{L^1},\ \|w_i(t)\|_{L^1} = \mathcal{O}(1)\delta_0 e^{-\varepsilon\mu t}, \tag{5.9}$$

$$\|v_i(t)\|_{L^\infty},\ \|w_i(t)\|_{L^\infty},\ \|v_{i,x}(t)\|_{L^1},\ \|w_{i,x}(t)\|_{L^1} = \mathcal{O}(1)\delta_0^2 e^{-\varepsilon\mu t}, \tag{5.10}$$

$$\|v_{i,x}(t)\|_{L^\infty},\ \|w_{i,x}(t)\|_{L^\infty} = \mathcal{O}(1)\delta_0^3 e^{-\varepsilon\mu t}, \tag{5.11}$$

for all $t \in [\hat{t}, T]$.

Proof. As shown in the previous lemma, the map

$$\mathcal{G}(u;\cdot,\cdot):(v,w)\mapsto(u_x,u_{xx}-A(u)u_x)=(u_x,u_t+\varepsilon g(u)) \tag{5.12}$$

is locally invertible and continuously differentiable with Lipshitz continuous derivatives. To prove the estimates follow closely the proof of Lemma 5.4 in [6] using the map $\mathcal{G}$ in (5.12). □

6. The evolution of the components

This section investigates the evolution of the components v_i and w_i in the decomposition (5.1).

Lemma 6.1. *The unique solution* (v,w) *of decomposition* (5.1) *satisfies a* $2n\times 2n$ *system of balance laws with source of the form*:

$$\begin{pmatrix} v\\ w\end{pmatrix}_t+\left[\begin{pmatrix}\Lambda & 0\\ 0 & \Lambda\end{pmatrix}\begin{pmatrix} v\\ w\end{pmatrix}\right]_x-\begin{pmatrix} v\\ w\end{pmatrix}_{xx}+\varepsilon\left[\begin{pmatrix}B^\sharp & H\\ K & B^\flat\end{pmatrix}\begin{pmatrix} v\\ w\end{pmatrix}\right]=\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}, \tag{6.1}$$

where Λ *is the* $n\times n$ *diagonal matrix with entries* $\{\tilde\lambda_i\}$ *and the matrix* $\begin{pmatrix}B^\sharp & H\\ K & B^\flat\end{pmatrix}$ *is diagonally dominant. More precisely*,

$$|B^\sharp_{ij}-B^\flat_{ij}|=\mathcal{O}(|u-u^*|,|v|,|w|),\qquad |H_{ij}|,\ |K_{ij}|=\mathcal{O}(|u-u^*|,|v|,|w|), \tag{6.2}$$

and if $(u,v,w)=(u^*,0,0)$,

$$B^\sharp=B^\flat=B(u^*)=[r_1(u^*),\dots,r_n(u^*)]^{-1}Dg(u^*)[r_1(u^*),\dots,r_n(u^*)]. \tag{6.3}$$

Furthermore, *the component source terms in* (6.1) *satisfy the bounds*

$$\begin{aligned}\phi_i,\psi_i\ =\ &O(1)\cdot\sum_{j\neq i}\big(|v_jv_i|+|v_{j,x}v_i|+|v_jw_i|+|v_jw_{i,x}|+|w_jw_i|+|v_{j,x}w_i|\\ &+|v_{j,x}v_{i,x}|+|v_{j,x}w_{i,x}|+|w_jw_{i,x}|+|w_{j,x}w_{i,x}|\big)\longrightarrow \textit{Transversal}\\ &+O(1)\cdot\sum_j|v_{j,x}w_j-v_jw_{j,x}|\longrightarrow \textit{Change in speed},\ \textit{Linear}\\ &+O(1)\cdot\sum_j\left|v_j\left(\frac{w_j}{v_j}\right)_x\right|^2\cdot\chi_{\{|w_j/v_j|<3\delta_1\}}\longrightarrow \textit{Change in speed},\ \textit{Quadratic}\\ &+O(1)\cdot\sum_j\big(|v_{j,x}|+|w_{j,x}|\big)\cdot|w_j-\theta_jv_j|.\longrightarrow \textit{Cutoff}\end{aligned} \tag{6.4}$$

Proof. Explicit computations are given in Appendix A. □

In view of the above lemma, the component source terms Φ, Ψ belong to the four categories given in (6.4). See page 250 in [6] for a short explanation on why these terms arise.

In the following subsections, we prove that all terms that appear in the component source of (6.1) are controllable. We employ an interaction potential to control the transversal terms in Section 6.1. The terms that belong to the change in speed category whether linear or quadratic, are handled by means of the area and length functionals of suitable planar curves introduced in Section 6.2. Last, in Section 6.3, by employing energy methods, it is shown that the cutoff terms are monitored by terms in the preceding categories and are therefore controllable.

6.1. Transversal terms

To begin with, we establish an a priori bound on the contribution of the transversal terms. Having a solution $u(t,x)$ of parabolic system (1.13) and assuming

$$\|u_x(t)\|_{L^1} \leqslant \delta_0 e^{-\varepsilon\mu t}, \quad t \in [0,T], \tag{6.5}$$

by the results in Section 5, decomposition (5.1) holds and therefore, the components v_i and w_i exist according to Lemma 5.1 and satisfy the bounds in Lemma 5.2. Assuming

$$\sum_i \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t}\big(|\phi_i(t,x)| + |\psi_i(t,x)|\big)\,dx\,dt \leqslant \delta_0, \tag{6.6}$$

we must prove the following estimate

$$\begin{aligned}\int_{\hat{t}}^{T} \int \sum_{j\neq k} e^{\varepsilon\mu t}\big(&|v_j v_k| + |v_{j,x} v_k| + |v_j w_k| + |v_j w_{k,x}| + |w_j w_k| + |w_j w_{k,x}| \\ &+ |v_{j,x} v_{k,x}| + |v_{j,x} w_k| + |v_{j,x} w_{k,x}| + |w_{j,x} w_{k,x}|\big)\,dx\,dt = O(1)\delta_0^2. \end{aligned}\tag{6.7}$$

As a consequence of strict hyperbolicity there exists a constant $c > 0$ such that

$$\inf_{t,x} \tilde{\lambda}_j - \sup_{t,x} \tilde{\lambda}_i \geqslant c > 0, \quad \forall i < j. \tag{6.8}$$

The proof of the following lemma follows closely that one of Lemma 7.1 in [6]. However, it worths stating it in order to indicate the role of the ε-term in (6.1) and the treatment of the dissipation.

Lemma 6.2. *Let* (v, w) *be solution of* (6.1) *defined for* $t \in [0, T]$. *Then for* $i \neq j$,

$$\int_0^T \int e^{\varepsilon\mu t} \left(|v_i\, v_j| + |v_i\, w_j| + |w_i\, w_j|\right) dx\, dt = \frac{\mathcal{O}(1)}{c}\, \delta_0^2. \tag{6.9}$$

Proof. We introduce the interaction potential functional

$$\mathcal{Q}(z, \tilde{z}) = \iint \mathcal{K}(x-y) z(x) \tilde{z}(y)\, dx\, dy, \tag{6.10}$$

where the kernel $\mathcal{K}$ is given by

$$\mathcal{K}(s) = \begin{cases} \dfrac{1}{c} & s \geqslant 0, \\ \dfrac{1}{c} e^{cs/2} & s < 0 \end{cases} \tag{6.11}$$

and apply it on the pairs (v_i, v_j), (v_i, w_j), (w_i, w_j) for $i < j$. Suppose first that system (6.1) is homogeneous. Then for $i < j$, the derivative of $\mathcal{Q}(v_i, v_j)$ with respect to time is

$$\begin{aligned} \frac{d}{dt}\mathcal{Q}(v_i(t), v_j(t)) = \iint \mathcal{K}(x-y) \Bigg[& sgn v_i(x) \Bigg(v_{i,xx}(x) - (\tilde{\lambda}_i v_i(x))_x \\ & - \varepsilon \sum_k B^\sharp_{ik}(x) v_k(x) + H_{ik}(x) w_k(x) \Bigg) |v_j(y)| \\ & + |v_i(x)| sgn v_j(y) \Bigg(v_{j,xx}(y) - (\tilde{\lambda}_j\, v_j(y))_x \\ & - \varepsilon \sum_k \left(B^\sharp_{jk}(y) v_k(y) + H_{jk}(y) w_k(y) \right) \Bigg) \Bigg] dy\, dx. \end{aligned}$$

Integrating by parts and using the fact that $c\mathcal{K}'(s) - 2\mathcal{K}''(s)$ is the delta function yield

$$\begin{aligned} \frac{d}{dt}\mathcal{Q}(v_i(t), v_j(t)) \leqslant & - \int |v_i(x)|\, |v_j(x)|\, dx - 2\varepsilon\mu \mathcal{Q}(v_i(t), v_j(t)) \\ & + \varepsilon \iint \mathcal{K}(x-y) \Bigg[\Bigg(\sum_{k \neq i} |B^\sharp_{ik} v_k(x)| + \sum_k |H_{ik} w_k(x)| \Bigg) |v_j(y)| \\ & + |v_i(x)| \Bigg(\sum_{k \neq i} |B^\sharp_{jk} v_k(y)| + \sum_k |H_{jk} w_k(y)| \Bigg) \Bigg] dx\, dy. \end{aligned}$$

Let $M \doteq sup\{\|B^\sharp\|_\infty, \|H\|_\infty, \|B^\flat\|_\infty, \|K\|_\infty\}$. Multiplying by $e^{\varepsilon\mu t}$, integrating over $[\hat{t}, T]$ and using (5.9) yields

$$\begin{aligned}
&\int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} |v_i(x) v_j(x)|\, dx\, dt \\
&\quad \leqslant \frac{\varepsilon}{c} M \sum_k \int_{\hat{t}}^{T} e^{\varepsilon\mu t} \Big[(\|v_k(t)\|_{L^1} + \|w_k(t)\|_{L^1}) \|v_j\|_{L^1} \\
&\qquad + \|v_i\|_{L^1} (\|v_k(t)|_{L^1} + \|w_k(t)\|_{L^1}) \Big] dt + e^{\varepsilon\mu\hat{t}} \mathcal{Q}(v_i(\hat{t}), v_j(\hat{t})) \\
&\quad \leqslant \frac{1}{c} \|v_i(\hat{t})\|_{L^1} \|v_j(\hat{t})\|_{L^1} + \frac{\mathcal{O}(1)\delta_0^2}{c\mu} M e^{-\varepsilon\mu\hat{t}}.
\end{aligned} \tag{6.12}$$

Similarly we prove the corresponding estimates for the transversal terms $|v_i\, w_j|$, $|w_i\, w_j|$ always for the homogeneous system (6.1). To prove bound (6.9) for the non-homogeneous system, we employ the fundamental solution that satisfies system (6.1) in the homogeneous case with initial data the delta function. Suppose that it has components Γ_i and $\tilde{\Gamma}_i$ corresponding to v_i and w_i. Therefore, in view of the above analysis, it follows

$$\int_{max\{s,s'\}}^{T} \int e^{\varepsilon\mu t} |\Gamma_i(t,x,s,y) \Gamma_j(t,x,s',y')|\, dx\, dt \leqslant \frac{1}{c} \left(1 + \mathcal{O}(1)\delta_0^2\right), \tag{6.13}$$

for $i<j$. The same result holds for all pairs $(\Gamma_i, \tilde{\Gamma}_j)$, $(\tilde{\Gamma}_i, \Gamma_j)$ and $(\tilde{\Gamma}_i, \tilde{\Gamma}_j)$. Now we can write the solution (v, w) of system (6.1) in the form

$$v_i(t,x) = \int \Gamma_i(t,x,0,y) v_i(0,y)\, dy + \int_0^t \int \Gamma_i(t,x,s,y) \phi_i(s,y)\, dy\, ds, \tag{6.14}$$

$$w_j(t,x) = \int \tilde{\Gamma}_j(t,x,0,y) w_j(0,y)\, dy + \int_0^t \int \tilde{\Gamma}_j(t,x,s,y) \psi_j(s,y)\, dy\, ds. \tag{6.15}$$

Using (6.13)–(6.15) we deduce

$$\begin{aligned}
&\int_0^T \int e^{\varepsilon\mu t} |v_i(t,x)\, v_j(t,x)|\, dx\, dt \\
&\quad = \frac{\mathcal{O}(1)}{c} \left(\int |v_i(0)| dx + \int_0^T \int |\phi_i|\, dx dt \right) \left(\int |v_j(0)| dx + \int_0^T \int |\phi_j|\, dx\, dt \right)
\end{aligned}$$

as well as the corresponding estimates for $|v_i w_j|$ and $|w_i w_j|$. The proof follows easily. □

The following lemma shows how to treat the transversal terms $|v_{i,x}v_j|$, $|v_{i,x}w_j|$, $|v_i w_{j,x}|$ and $|w_i w_{j,x}|$ for all $i \neq j$. Here, we choose to present only the proof for $|v_{i,x}v_j|$.

Lemma 6.3. *Let v_i and v_j be the i-th and j-th components of the solution v of* (6.1), *for $i \neq j$. Assume* (6.5) *and* (6.6) *hold, then for $i \neq j$*

$$\int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} |v_{i,x}(t,x)||v_j(t,x)|\, dx\, dt = O(1)\delta_0^2. \tag{6.16}$$

Proof. It suffices to show that the quantity

$$I(T) = \sup_{(\tau,\xi)\in[0,T]\times\mathbb{R}} \int_0^{T-\tau} \int e^{\varepsilon\mu t} |v_{i,x}(t,x) v_j(t+\tau, x+\xi)|\, dx\, dt \tag{6.17}$$

is of order δ_0^2, since $I(T)$ is an upper bound of the integral (6.16). By (6.5) and Lemma 5.2, for all $t \in [\hat{t}, T]$

$$\|v_i(t)\|_{L^1}, \|v_j(t)\|_{L^1} \leqslant \delta_0\, e^{-\varepsilon\mu t}, \qquad \|v_{i,x}(t)\|_{L^1}, \|v_j(t)\|_{L^\infty} \leqslant k\delta_0^2 e^{-\varepsilon\mu t}, \tag{6.18}$$

$$\|\tilde{\lambda}_{i,x}(t)\|_{L^\infty}, \|\tilde{\lambda}_{i,x}(t)\|_{L^1} \leqslant k\delta_0 e^{-\varepsilon\mu t}, \tag{6.19}$$

for some constant k. Without loss of generality, we can assume $\lim_{x\to-\infty} \tilde{\lambda}_i(t,x) = 0$. These imply that $I(T)$ is bounded, $I(T) \leqslant (k\delta_0^2)^2 T$. We perform the integration in (6.17) first over $[0,1]$ and then over $[1, T-\tau]$. On the time interval $[0,1]$,

$$\begin{aligned} \int_0^1 \int e^{\varepsilon\mu t} |v_{i,x}(t,x) v_j(t+\tau, x+\xi)|\, dx\, dt &\leqslant \int_0^1 e^{\varepsilon\mu t} \|v_{i,x}\|_{L^1} \|v_j\|_{L^\infty}\, dt \\ &\leqslant (k\delta_0^2)^2. \end{aligned} \tag{6.20}$$

For $t > 1$, we express $v_{i,x}$ in terms of the Green's kernel G of the heat equation,

$$\begin{aligned} G(t,x) &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}; \\ v_{i,x}(t,x) &= \int G_x(1,y) v_i(t-1, x-y)\, dy \\ &\quad + \int_0^1 \int G_x(s,y) \Big[\phi_i - (\tilde{\lambda}_i v_i)_x\Big](t-s, x-y)\, dy\, ds \\ &\quad - \varepsilon \int_0^1 \int G_x(s,y) \left[\sum_{k=1}^{n} (B_{ik}^{\sharp} v_k + H_{ik} w_k)(t-s, x-y)\right] dy\, ds. \end{aligned} \tag{6.21}$$

We now control the integration (6.17) over $[1, T-\tau]$,

$$
\begin{aligned}
&\int_1^{T-\tau}\int e^{\varepsilon\mu t}|v_{i,x}(t,x)v_j(t+\tau,x+\xi)|\,dx\,dt \\
&\quad\leqslant \int |G_x(1,y)|dy\cdot \sup_{s,y,\tau,\xi}\int_1^{T-\tau}\int e^{\varepsilon\mu t}|v_i(t-s,x-y)v_j(t+\tau,x+\xi)|\,dx\,dt \\
&\qquad + \|\tilde{\lambda}_{i,x}\|_{L^\infty}\int_0^1\int |G_x(s,y)|\,dyds \sup_{s,y,\tau,\xi}\int_1^{T-\tau}\int e^{\varepsilon\mu t}|v_i(t-s,x-y)v_j(t+\tau,x+\xi)|\,dx\,dt \\
&\qquad + \|\tilde{\lambda}_i\|_{L^\infty}\int_0^1\int |G_x(s,y)|\,dyds \sup_{s,y,\tau,\xi}\int_1^{T-\tau}\int e^{\varepsilon\mu t}|v_{i,x}(t-s,x-y)v_j(t+\tau,x+\xi)|\,dx\,dt \\
&\qquad + \sup_{t'}\|v_j(t')\|_{L^\infty}\int_0^1\int |G_x(s,y)|\,dy\,ds\int_0^T\int e^{\varepsilon\mu t}|\phi_i(t,x)|\,dx\,dt \\
&\qquad + \varepsilon\int_0^1\int |G_x(s,y)|\,dyds \sup_{s,y,\tau,\xi}\int_1^{T-\tau}\int e^{\varepsilon\mu t}\big|B^\sharp_{i1}v_1+\ldots B^\sharp_{in}v_n+H_{i1}w_1 \\
&\qquad + \ldots + H_{in}w_n|(t-s,x-y)|v_j(t+\tau,x+\xi)\big|\,dx\,dt.
\end{aligned}
\tag{6.22}
$$

The first two terms belong to the category of transversal terms that are already treated in Lemma 6.2. In fact, for fixed $t'>0$ and sufficiently small δ_0, we can prove

$$
\int_0^T\int e^{\varepsilon\mu t}|v_i(t,x)\,v_j(t+t',x)|\,dx\,dt \leqslant \frac{1+\mathcal{O}(1)\delta_0^2}{c}\,4\delta_0^2 < \frac{8\delta_0^2}{c} \tag{6.23}
$$

by following the same steps as in the proof of Lemma 6.2 and by employing as interaction potential functional

$$
\mathcal{Q}(v_i(t), v_j(t+t')) = \iint \mathcal{K}(x-y)v_i(t,x)v_j(t+t',y)\,dx\,dy. \tag{6.24}
$$

Using (6.23) as well as the bounds (6.18), (6.19), we obtain

$$\int_1^{T-\tau}\int e^{\varepsilon\mu t}|v_{i,x}(t,x)v_j(t+\tau,x+\xi)|\,dx\,dt$$

$$\leqslant \left(\frac{1}{\sqrt{\pi}}+\|\tilde{\lambda}_{i,x}\|_{L^\infty}\frac{2}{\sqrt{\pi}}\right)\cdot\frac{8\delta_0^2}{c}+\|\tilde{\lambda}_i\|_{L^\infty}\frac{2}{\sqrt{\pi}}I(T)+k\delta_0^2\frac{2}{\sqrt{\pi}}\delta_0$$

$$+\varepsilon\frac{2}{\sqrt{\pi}}M2n\,\delta_0^2\frac{e^{-\varepsilon\mu}}{\varepsilon\mu}. \tag{6.25}$$

Hence by virtue of (6.20) and (6.25) and for sufficiently small $\delta_0>0$, we deduce

$$I(T)\leqslant 2(k\delta_0^2)^2+\left(\frac{1}{\sqrt{\pi}}+k\delta_0\frac{2}{\sqrt{\pi}}\right)\frac{16\delta_0^2}{c}+k\delta_0^3\frac{4}{\sqrt{\pi}}+8n\frac{M\delta_0^2e^{-\varepsilon\mu}}{\sqrt{\pi}\mu}=\mathcal{O}(1)\delta_0^2 \tag{6.26}$$

and this yields the result. $\square$

Similarly we estimate the contribution of the remaining transversal terms $|v_{i,x}w_{j,x}|$, $|v_{i,x}\,w_{j,x}|$ and $|w_{i,x}w_{j,x}|$. Thus we have shown that all transversal terms are controllable, hence (6.7) is valid.

6.2. Swept area and length curve functionals

In this section, we study the interaction of viscous waves of the same family that arise from the rate of change of speed σ_i. For each i, we prove the following estimates:

$$\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\,|w_{i,x}v_i-w_iv_{i,x}|\,dx\,dt=\mathcal{O}(1)\delta_0^2, \tag{6.27}$$

$$\int_{\hat{t}}^{T}\int_{|w_i/v_i|<3\delta_1}e^{\varepsilon\mu t}\,|v_i|^2\left|\left(\frac{w_i}{v_i}\right)_x\right|^2dx\,dt=\mathcal{O}(1)\delta_0^3. \tag{6.28}$$

To accomplish this goal, we should first introduce the following planar curves: For $i=1,\dots,n$

$$\gamma_i(t,x)=\left(\int_{-\infty}^{x}v_i(t,y)\,dy\,,\ \int_{-\infty}^{x}w_i(t,y)\,dy\right), \tag{6.29}$$

$$\tilde{\gamma}_i(t,x) = \left(\int_{-\infty}^{x} w_i(t,y)\,dy\,,\ \int_{-\infty}^{x} v_i(t,y)\,dy \right). \tag{6.30}$$

These curves γ_i and $\tilde{\gamma}_i$ evolve in time according to the following equations:

$$\gamma_{i,t} + \tilde{\lambda}_i \gamma_{i,x} - \gamma_{i,xx} + \varepsilon \sum_{k=1}^{n} \left(\int_{-\infty}^{x} B^{\sharp}_{ik}\, v_k + H_{ik}\, w_k\, dy,\ \int_{-\infty}^{x} K_{ik}\, v_k + B^{\flat}_{ik}\, w_k\, dy \right)$$

$$= \left(\int_{-\infty}^{x} \phi_i(t,y)\,dy,\ \int_{-\infty}^{x} \psi_i(t,y)\,dy \right), \tag{6.31}$$

$$\tilde{\gamma}_{i,t} + \tilde{\lambda}_i \tilde{\gamma}_{i,x} - \tilde{\gamma}_{i,xx} + \varepsilon \sum_{k=1}^{n} \left(\int_{-\infty}^{x} K_{i,k}\, v_k + B^{\flat}_{i,k}\, w_k\, dy,\ \int_{-\infty}^{x} B^{\sharp}_{i,k}\, v_k + H_{i,k}\, w_k\, dy \right)$$

$$= \left(\int_{-\infty}^{x} \psi_i(t,y)\,dy,\ \int_{-\infty}^{x} \phi_i(t,y)\,dy \right). \tag{6.32}$$

With an ordered pair of curves (γ, ξ), we associate the *area functional* $\mathcal{A}$ to be

$$\mathcal{A}(\gamma, \xi) = \frac{1}{2} \iint_{x<y} |\gamma_x(x) \wedge \xi_x(y)|\, dx\, dy, \tag{6.33}$$

More precisely, we are interested in the functionals $\mathcal{A}(\gamma_i, \gamma_j)$ and $\mathcal{A}(\gamma_i, \tilde{\gamma}_j)$. Lemma 5.2 implies

$$0 \leqslant \mathcal{A}(\gamma_i, \gamma_j)\,,\ \mathcal{A}(\gamma_i, \tilde{\gamma}_j) = \mathcal{O}(1)\delta_0^3 e^{-\varepsilon\mu t}. \tag{6.34}$$

If γ_i is a closed curve, $\mathcal{A}(\gamma_i, \gamma_i)$ provides an upper bound for the sum of the areas of the regions enclosed by the curve counting the corresponding winding number. See also [4]. This is the only one functional considered in the case of conservation laws, i.e. when (6.1) is decoupled. In order to eliminate the effect of the coupling of system (6.1) on integral (6.27), it is necessary to introduce the functionals $\mathcal{A}$ applied on the pair (γ_i, γ_j) for $i \neq j$ and $(\gamma_i, \tilde{\gamma}_j)$. It will be shown that the area functionals are monotonically decreasing and in particular exponentially decaying, because of the presence of the dissipative source, which appears as the ε-term in system (6.31)–(6.32). The following lemma is a generalization of Lemma 8.1 in [6].

Lemma 6.4. *Let* (v, w) *be solution to* (6.1) *for* $t \in [0, T]$ *and assume that the maps* $x \mapsto v(t,x)$, $x \mapsto w(t,x)$ *and* $x \mapsto \tilde{\lambda}(t,x)$ *are* $\mathcal{C}^{1,1}$. *Then the area functionals* *satisfy*:

$$\sum_{i,j} \frac{d}{dt} e^{\varepsilon\mu t} \left[\mathcal{A}(\gamma_i(t), \gamma_j(t)) + \mathcal{A}(\gamma_i(t), \tilde{\gamma}_j(t))\right]$$

$$\leqslant - \sum_i \int e^{\varepsilon\mu t} |\gamma_{i,x}(x) \wedge \gamma_{i,xx}(x)|\, dx$$

$$+ \sum_{i,j} e^{\varepsilon\mu t} \Big(\|v_i(t)\|_{L^1} \|\psi_j(t)\|_{L^1} + \|w_i(t)\|_{L^1} \|\phi_j(t)\|_{L^1} + \|v_i(t)\|_{L^1} \|\phi_j(t)\|_{L^1}$$

$$+ \|w_i(t)\|_{L^1} \|\psi_j(t)\|_{L^1} \Big) + O(1)\varepsilon\, e^{-\varepsilon\mu t} \delta_0^2. \tag{6.35}$$

Proof. For $i = 1, \ldots, n$, at each x where $\gamma_{i,x} \neq 0$, define the vector $\eta_i(x)$ in $\mathbb{R}^2$ to be

$$\gamma_{i,x}(x) \wedge \overrightarrow{v} = |\gamma_{i,x}| \langle \eta_i(x), \overrightarrow{v} \rangle, \qquad \text{i.e.} \quad \eta_i(x) = \left(-\frac{w_i(x)}{|\gamma_{i,x}|}, \frac{v_i(x)}{|\gamma_{i,x}|} \right). \tag{6.36}$$

For fixed x, consider the projection of γ_j along the vector $\eta_i(x)$,

$$y \mapsto \chi^{\eta_i, \gamma_j}(y) = \langle \eta_i(x), \gamma_j(y) \rangle. \tag{6.37}$$

The time derivatives of the area functionals $\mathcal{A}(\gamma_i, \gamma_j)$ and $\mathcal{A}(\gamma_j, \gamma_i)$ are

$$\frac{d}{dt}\mathcal{A}(\gamma_i, \gamma_j) = \frac{1}{2} \iint_{x<y} sgn(\gamma_{i,x}(x) \wedge \gamma_{j,x}(y)) \Big\{ \gamma_{i,xt}(x) \wedge \gamma_{j,x}(y)$$

$$+ \gamma_{i,x}(x) \wedge \gamma_{j,xt}(y) \Big\}\, dx\, dy,$$

$$\frac{d}{dt}\mathcal{A}(\gamma_j, \gamma_i) = \frac{1}{2} \iint_{x<y} sgn(\gamma_{j,x}(x) \wedge \gamma_{i,x}(y)) \Big\{ \gamma_{j,xt}(x) \wedge \gamma_{i,x}(y)$$

$$+ \gamma_{j,x}(x) \wedge \gamma_{i,xt}(y) \Big\}\, dx\, dy.$$

By adding the above derivatives, one observes that the terms pair together and the space of integration becomes the whole space $\mathbb{R}^2$:

$$\frac{d}{dt}\big(\mathcal{A}(\gamma_i, \gamma_j) + \mathcal{A}(\gamma_j, \gamma_i)\big) = \frac{1}{2} \iint sgn(\gamma_{i,x}(x) \wedge \gamma_{j,x}(y)) [\gamma_{i,x}(x) \wedge \gamma_{j,xt}(y)]\, dx\, dy$$

$$+ \frac{1}{2} \iint sgn(\gamma_{j,x}(x) \wedge \gamma_{i,x}(y)) [\gamma_{j,x}(x) \wedge \gamma_{i,xt}(y)]\, dx\, dy.$$

From (6.36) and (6.37), it follows

$$\begin{aligned}
&\frac{d}{dt}\big(\mathcal{A}(\gamma_i,\gamma_j)+\mathcal{A}(\gamma_j,\gamma_i)\big)\\
&\quad=\frac{1}{2}\int|\gamma_{i,x}(x)|\left[\int sgn\langle\eta_i(x),\gamma_{j,x}(y)\rangle\langle\eta_i(x),\gamma_{j,xt}(y)\rangle\,dy\right]dx\\
&\qquad+\frac{1}{2}\int|\gamma_{j,x}(x)|\left[\int sgn\langle\eta_j(x),\gamma_{i,x}(y)\rangle\langle\eta_j(x),\gamma_{i,xt}(y)\rangle\,dy\right]dx\\
&\quad=\frac{1}{2}\int|\gamma_{i,x}(x)|\frac{d}{dt}\left(TV_y\chi^{\eta_i(x),\gamma_j}\right)+|\gamma_{j,x}(x)|\frac{d}{dt}\left(TV_y\chi^{\eta_j(x),\gamma_i}\right)dx. \qquad (6.38)
\end{aligned}$$

Having x fixed, investigate $\chi^{\eta_i(x),\gamma_j}$ as a function of y in order to compute the time derivative

$$\frac{d}{dt}\left(TV_y\chi^{\eta_i(x),\gamma_j}\right).$$

By an approximation argument, one can assume that the projection has the required regularity for almost every $(t,x)\in\mathbb{R}^2$ and a finite number of local extremum points attained at

$$y_{-p}^{\eta_i,\gamma_j},\cdots<y_{-1}^{\eta_i,\gamma_j}<y_0^{\eta_i,\gamma_j}<y_1^{\eta_i,\gamma_j}<\cdots<y_{p'}^{\eta_i,\gamma_j}, \qquad (6.39)$$

changing monotonicity across every point, as shown in Fig. 2. Denote $y_\alpha^{\eta_i,\gamma_j}$ by $y_\alpha^{i,j}$, for simplicity. Note $y_0^{i,i}=x$. For $i\neq j$, choose $y_0^{i,j}$ to be near x so that $sgn\langle\eta_i(x),\gamma_{j,xx}(x)\rangle=sgn\langle\eta_i(x),\gamma_{j,xx}(y_0^{i,j})\rangle$. In view of the above,

$$\begin{gathered}
sgn\langle\eta_i(x),\gamma_{j,x}(y)\rangle=-(-1)^\alpha sgn\langle\eta_i(x),\gamma_{j,xx}(y_0^{i,j})\rangle,\quad\text{for } y\in\left(y_{\alpha-1}^{i,j},y_\alpha^{i,j}\right),\\
\langle\eta_i(x),\gamma_{i,x}(x)\rangle=0,\quad\langle\eta_i(x),\gamma_{j,x}(y_\alpha^{i,j})\rangle=0, \qquad (6.40)\\
sgn\langle\eta_i(x),\gamma_{j,xx}(y_\alpha^{i,j})\rangle=(-1)^\alpha sgn\langle\eta_i(x),\gamma_{j,xx}(x)\rangle.
\end{gathered}$$

Hence, the time derivative becomes

$$\begin{aligned}
\frac{d}{dt}\left(TV_y\chi^{\eta_i,\gamma_j}\right)&=\sum_\alpha\int_{y_{\alpha-1}^{i,j}}^{y_\alpha^{i,j}}sgn\langle\eta_i(x),\gamma_{j,x}(y)\rangle\langle\eta_i(x),\gamma_{j,xt}(y)\rangle\,dy\\
&=\sum_\alpha-\int_{y_{\alpha-1}^{i,j}}^{y_\alpha^{i,j}}sgn\langle\eta_i(x),\gamma_{j,xx}(x)\rangle(-1)^\alpha\langle\eta_i(x),\gamma_{j,xt}(y)\rangle\,dy\\
&=-sgn\langle\eta_i(x),\gamma_{j,xx}(x)\rangle 2\sum_\alpha(-1)^\alpha\langle\eta_i(x),\gamma_{j,t}(y_\alpha^{i,j})\rangle. \qquad (6.41)
\end{aligned}$$

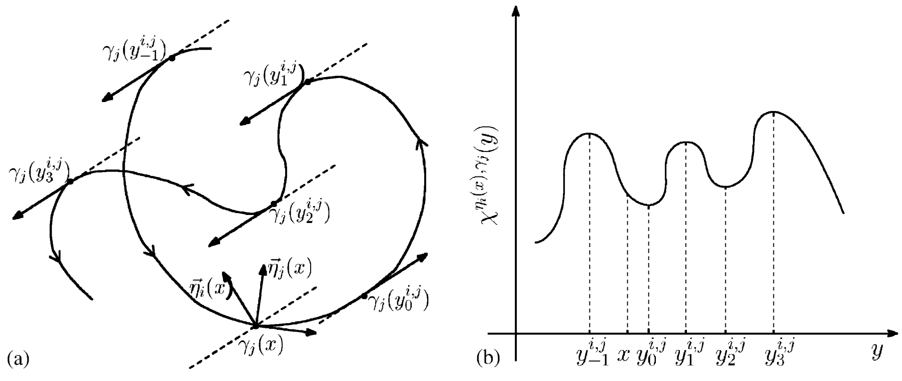

Fig. 2. (a) The curve $\gamma_j(y)$ and the vector $\eta_i(x)$ perpendicular to $\gamma_{j,x}(y_\alpha^{i,j})$. (b) Graph of $\chi^{\eta_i(x),\gamma_j}(y)$.

A similar computation yields

$$\frac{d}{dt}\left(TV_y\chi^{\eta_j,\gamma_i}\right) = -sgn\langle\eta_j(x),\gamma_{i,xx}(x)\rangle 2\sum_{\alpha'}(-1)^{\alpha'}\langle\eta_j(x),\gamma_{i,t}(y_{\alpha'}^{j,i})\rangle. \tag{6.42}$$

Substitute (6.41), (6.42) in (6.38) and apply identities (6.40), to get

$$\begin{aligned}
&\frac{d}{dt}\big(\mathcal{A}(\gamma_i,\gamma_j)+\mathcal{A}(\gamma_j,\gamma_i)\big)\\
&\quad= \frac{1}{2}\int|\gamma_{i,x}(x)|\frac{d}{dt}\left(TV_y\chi^{\eta_i(x),\gamma_j}\right)dx + \frac{1}{2}\int|\gamma_{j,x}(x)|\frac{d}{dt}\left(TV_y\chi^{\eta_j(x),\gamma_i}\right)dx\\
&\quad= -\int|\gamma_{i,x}(x)|sgn\langle\eta_i(x),\gamma_{j,xx}(x)\rangle\sum_\alpha(-1)^\alpha\langle\eta_i(x),\gamma_{j,xx}\left(y_\alpha^{i,j}\right)-\tilde{\lambda}_j\gamma_{j,x}\left(y_\alpha^{i,j}\right)\\
&\qquad-\varepsilon\sum_k\int_{-\infty}^{y_\alpha^{i,j}}\left(B_{jk}^{\sharp}v_k+H_{jk}w_k,K_{jk}v_k+B_{jk}^{\flat}w_k\right)dy+\int_{-\infty}^{y_\alpha^{i,j}}(\phi_j,\psi_j)\,dy\rangle\,dx\\
&\qquad-\int|\gamma_{j,x}(x)|sgn\langle\eta_j(x),\gamma_{i,xx}(x)\rangle\sum_{\alpha'}(-1)^{\alpha'}\langle\eta_j(x),\gamma_{i,xx}\left(y_{\alpha'}^{j,i}\right)-\tilde{\lambda}_i\gamma_{i,x}\left(y_{\alpha'}^{j,i}\right)\\
&\qquad-\varepsilon\sum_k\int_{-\infty}^{y_\alpha^{j,i}}\left(B_{ik}^{\sharp}v_k+H_{ik}w_k,K_{ik}v_k+B_{ik}^{\flat}w_k\right)dy+\int_{-\infty}^{y_\alpha^{j,i}}(\phi_i,\psi_i)\,dy\rangle\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leqslant -\int |\gamma_{i,x}(x) \wedge \gamma_{j,xx}\left(y_0^{i,j}\right)| \, dx + \|v_i(t)\|_{L^1} \|\psi_j(t)\|_{L^1} + \|w_i(t)\|_{L^1} \|\phi_j(t)\|_{L^1} \\
&\quad - \frac{\varepsilon}{2} \iint |\gamma_{i,x}(x)| sgn\langle \eta_i(x), \gamma_{j,x}(y)\rangle \langle \eta_i(x), m_j(y)\rangle \, dy \, dx \\
&\quad - \int |\gamma_{j,x}(x) \wedge \gamma_{i,xx}\left(y_0^{j,i}\right)| \, dx + \|v_j(t)\|_{L^1} \|\psi_i(t)\|_{L^1} + \|w_j(t)\|_{L^1} \|\phi_i(t)\|_{L^1} \\
&\quad - \frac{\varepsilon}{2} \iint |\gamma_{j,x}(x)| sgn\langle \eta_j(x), \gamma_{i,x}(y)\rangle \langle \eta_j(x), m_i(y)\rangle \, dy \, dx,
\end{aligned} \tag{6.43}
$$

where

$$
m_l(y) = \left(\sum_k B^{\sharp}_{lk} \, v_k(y) + H_{lk} \, w_k(y), \sum_k K_{lk} \, v_k(y) + B^{\flat}_{lk} \, w_k(y) \right).
$$

In a similar way, the area functional applied on the curves γ and $\tilde{\gamma}$ is treated:

$$
\begin{aligned}
&\frac{d}{dt}(\mathcal{A}(\gamma_i, \tilde{\gamma}_j) + \mathcal{A}(\gamma_j, \tilde{\gamma}_i)) \\
&\quad \leqslant \|v_i(t)\|_{L^1} \|\phi_j(t)\|_{L^1} + \|w_i(t)\|_{L^1} \|\psi_j(t)\|_{L^1} \\
&\qquad + \|v_j(t)\|_{L^1} \|\phi_i(t)\|_{L^1} + \|w_j(t)\|_{L^1} \|\psi_i(t)\|_{L^1} \\
&\qquad - \frac{\varepsilon}{2} \iint |\gamma_{i,x}(x)| sgn\langle \eta_i(x), \tilde{\gamma}_{j,x}(y)\rangle \langle \eta_i(x), k_j(y)\rangle \, dy \, dx \\
&\qquad - \frac{\varepsilon}{2} \iint |\gamma_{j,x}(x)| sgn\langle \eta_j(x), \tilde{\gamma}_{i,x}(y)\rangle \langle \eta_j(x), k_i(y)\rangle \, dy \, dx,
\end{aligned} \tag{6.44}
$$

and

$$
k_l(y) \doteq \left(\sum_k K_{lk} \, v_k(y) + B^{\flat}_{lk} \, w_k(y), \sum_k B^{\sharp}_{lk} \, v_k(y) + H_{lk} \, w_k(y) \right).
$$

From (6.43), (6.44) and (5.9), it follows

$$
\begin{aligned}
&\frac{d}{dt} \sum_{i,j} (\mathcal{A}(\gamma_i, \gamma_j) + \mathcal{A}(\gamma_i, \tilde{\gamma}_j)) \\
&\quad \leqslant \sum_i - \int |\gamma_{i,x}(x) \wedge \gamma_{i,xx}(x)| \, dx + \mathcal{O}(1) \delta_0 e^{-\varepsilon \mu t} \sum_i (\|\phi_i(t)\|_{L^1} + \|\psi_i(t)\|_{L^1})
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i,j} \frac{\varepsilon}{2} \Bigg\{ \iint -|\gamma_{i,x}(x)| sgn\langle \eta_i(x), \gamma_{j,x}(y)\rangle \Big\langle \eta_i(x), \Big(B^{\sharp}_{jj}(y)\, v_j(y),\, B^{\flat}_{jj}(y)\, w_j(y)\Big)\Big\rangle \\
&+ \sum_{k\neq j} \Big|\Big\langle (-w_i(x), v_i(x)), \Big(B^{\sharp}_{jk}(y)\, v_k(y),\, B^{\flat}_{jk}(y)\, w_k(y)\Big)\Big\rangle\Big| \\
&+ \sum_{k} \big|\langle (-w_i(x), v_i(x)), (H_{jk}(y)\, w_k(y),\, K_{jk}(y)\, v_k(y))\rangle\big| \; dx\, dy \\
&-|\gamma_{i,x}(x)| sgn\langle \eta_i(x), \tilde\gamma_{j,x}(y)\rangle \Big\langle \eta_i(x), \Big(B^{\flat}_{jj}(y)\, v_j(y),\, B^{\sharp}_{jj}(y)\, w_j(y)\Big)\Big\rangle \\
&+ \sum_{k\neq j} \Big|\langle (-w_i(x), v_i(x)), \Big(B^{\flat}_{jk}(y)\, w_k(y),\, B^{\sharp}_{jk}(y)\, v_k(y)\Big)\rangle\Big| \\
&+ \sum_{k} \big|\langle (-w_i(x), v_i(x)), (K_{jk}(y)\, v_k(y),\, H_{jk}(y)\, w_k(y))\rangle\big| \; dx\, dy \Bigg\}.
\end{aligned}
$$

Observe that the integrand $\gamma_{i,x}(x) \wedge \gamma_{i,xx}(x)$ is exactly the component source term $w_{i,x} v_i - w_i v_{i,x}$. Now, interchange j and k and use the approximate estimates (6.2) for the matrices $B^{\sharp}$, $B^{\flat}$, K and H. It is important that the $B^{\sharp}$ and $B^{\flat}$ are close to B. Thus,

$$
\begin{aligned}
&\frac{d}{dt} \sum_{i,j} (\mathcal{A}(\gamma_i, \gamma_j) + \mathcal{A}(\gamma_i, \tilde\gamma_j)) \\
&\quad \leqslant \sum_i - \int |\gamma_{i,x}(x) \wedge \gamma_{i,xx}(x)|\, dx + \mathcal{O}(1)\delta_0 \sum_i e^{-\varepsilon\mu t} (\|\phi_i(t)\|_{L^1} + \|\psi_i(t)\|_{L^1}) \\
&\qquad + \sum_{i,j=1,n} \Bigg\{ -\frac{\varepsilon}{2} \iint \Bigg\{ B^{\flat}_{jj} - \sum_{k\neq j} |B^{\flat}_{kj}| - \sum_k |H_{kj}| \Bigg\} |v_i(x) w_j(y) - w_i(x) v_j(y)|\, dy\, dx \\
&\qquad - \frac{\varepsilon}{2} \iint \Bigg\{ B^{\flat}_{jj} - \sum_{k\neq j} |B^{\flat}_{kj}| - \sum_k |H_{kj}| \Bigg\} |v_i(x) v_j(y) - w_i(x) w_j(y)|\, dy\, dx \\
&\qquad + \varepsilon \mathcal{O}(1) \big(\|v_j\|_{L^1} \|w_i\|_{L^1} + \|v_j\|_{L^1} \|v_i\|_{L^1} + \|w_j\|_{L^1} \|w_i\|_{L^1} \big) \Bigg\}.
\end{aligned}
\tag{6.45}
$$

Therefore, the diagonal dominance property and (5.9) imply

$$\begin{aligned}
&\frac{d}{dt}\sum_{i,j}(\mathcal{A}(\gamma_i,\gamma_j)+\mathcal{A}(\gamma_i,\tilde{\gamma}_j))\\
&\quad\leqslant \sum_i -\int |\gamma_{i,x}(x)\wedge\gamma_{i,xx}(x)|\,dx\\
&\qquad+\mathcal{O}(1)\delta_0 e^{-\varepsilon\mu t}\sum_i(\|\phi_i(t)\|_{L^1}+\|\psi_i(t)\|_{L^1})\\
&\qquad-\varepsilon\mu\sum_{i,j}\big(\mathcal{A}(\gamma_i,\gamma_j)+\mathcal{A}(\gamma_i,\tilde{\gamma}_j)\big)+\varepsilon\mathcal{O}(1)e^{-2\varepsilon\mu t}\delta_0^2,
\end{aligned}\tag{6.46}$$

and this yields the result. $\square$

Finally, to get estimate (6.27), multiply (6.35) by $e^{\varepsilon\mu t}$ and integrate over $[\hat{t}, T]$. Then (6.6), and (6.34) establish the result.

The *length functional* $\mathcal{L}$ applied to the curve γ is defined to be the length of this curve. If γ is a curve that moves along the curvature then the length is monotonically decreasing. See [4]. We consider the γ_i curves defined in (6.29), so the length functional takes the form

$$\mathcal{L}_i(t)=\mathcal{L}(\gamma_i(t))=\int\sqrt{v_i^2(t,x)+w_i^2(t,x)}\,dx \tag{6.47}$$

and satisfies

$$0\leqslant\mathcal{L}_i(t)\leqslant\|v_i(t)\|_{L^1}+\|w_i(t)\|_{L^1}=\mathcal{O}(1)\delta_0 e^{-\varepsilon\mu t}. \tag{6.48}$$

Since the evolution equation (6.31) of γ_i contains a dissipative term, $\mathcal{L}(\gamma_i)$ decays exponentially as γ_i evolves in time.

Lemma 6.5. *Let (v,w) be solution of (6.1) for $t\in[0,T]$. Assume that the maps $x\mapsto v(t,x)$, $x\mapsto w(t,x)$ and $x\mapsto\tilde{\lambda}(t,x)$ are $\mathcal{C}^{1,1}$ and that $\gamma_{i,x}(t,x)\neq 0$ for every x. Then*

$$\begin{aligned}
\frac{d}{dt}(\mathcal{L}_i(t)) \leqslant{} & -\frac{1}{(1+9\delta_1^2)^{3/2}}\int_{|\frac{w_i}{v_i}|\leqslant 3\delta_1}|v_i|\left|\left(\frac{w_i}{v_i}\right)_x\right|^2dx-\varepsilon\mu\mathcal{L}_i(t)\\
&+\|\phi_i(t)\|_{L^1}+\|\psi_i(t)\|_{L^1}+\mathcal{O}(1)\,\varepsilon M\delta_0 e^{-\varepsilon\mu t}.
\end{aligned}\tag{6.49}$$

Proof. To begin with, it is easy to verify that

$$|\gamma_{i,xx}|^2\,|\gamma_{i,x}|^2-\langle\gamma_{i,x},\gamma_{i,xx}\rangle^2=v_i^4\left|\left(\frac{w_i}{v_i}\right)_x\right|^2,\qquad \frac{|v_i|^3}{|\gamma_{i,x}|^3}=\frac{1}{\left(1+\left(\frac{w_i}{v_i}\right)^2\right)^{3/2}}.$$

Because of the assumption that $\gamma_{i,x} \neq 0$, one can integrate by parts and employ the above identities and (5.9)

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}_i(t) &= \int \frac{\langle \gamma_{i,x}, \gamma_{i,xt}\rangle}{\sqrt{\langle \gamma_{i,x}, \gamma_{i,x}\rangle}}\, dx \\
&= \int \frac{\langle \gamma_{i,x}, \gamma_{i,xx}\rangle}{|\gamma_{i,x}|} - \frac{\langle \gamma_{i,x}, (\tilde{\lambda}_i \gamma_{i,x})_x\rangle}{|\gamma_{i,x}|} + \frac{\langle \gamma_{i,x}, (\phi_i, \psi_i)\rangle}{|\gamma_{i,x}|}\, dx \\
&\quad - \int \frac{\left\langle \gamma_{i,x}, \varepsilon\left(\sum_k \left(B^{\sharp}_{ik}\, v_k + H_{ik}\, w_k\right), \sum_k \left(K_{ik}\, v_k + B^{\flat}_{ik}\, w_k\right)\right)\right\rangle}{|\gamma_{i,x}|}\, dx. \\
&\leqslant -\int \frac{|v_i| \left|\left(\dfrac{w_i}{v_i}\right)_x\right|^2}{\left(1 + \left(\dfrac{w_i}{v_i}\right)^2\right)^{3/2}}\, dx - \varepsilon \int \frac{B^{\sharp}_{ii} v_i^2 + B^{\flat}_{ii} w_i^2}{\sqrt{v_i^2 + w_i^2}}\, dx \\
&\quad + \|\phi_i(t)\|_{L^1} + \|\psi_i(t)\|_{L^1} \\
&\quad + \varepsilon \int \left(\sum_{k\neq i} |B^{\sharp}_{ik}||v_k| + \sum_k |H_{ik}||w_k| + \sum_k |K_{ik}||v_k| + \sum_{k\neq i} |B^{\flat}_{ik}||w_k| \right) dx \\
&\leqslant -\frac{1}{(1+9\delta_1^2)^{3/2}} \int_{|\frac{w_i}{v_i}| \leqslant 3\delta_1} |v_i| \left|\left(\frac{w_i}{v_i}\right)_x\right|^2 dx + \|\phi_i(t)\|_{L^1} + \|\psi_i(t)\|_{L^1} \\
&\quad - \varepsilon\mu\mathcal{L}_i(t) + \varepsilon\, 4nM\delta_0 e^{-\varepsilon\mu t}
\end{aligned}
$$

and this yields the result. □

Now, we establish bound (6.28) via (6.49). The L^∞ bound of $v_i(t)$ is given in (5.10), and hence

$$
\begin{aligned}
&\int_{|\frac{w_i}{v_i}| \leqslant 3\delta_1} e^{\varepsilon\mu t} |v_i|^2 \left|\left(\frac{w_i}{v_i}\right)_x\right|^2 dx \leqslant \delta_0^2 \int_{\left|\frac{w_i}{v_i}\right| \leqslant 3\delta_1} |v_i| \left|\left(\frac{w_i}{v_i}\right)_x\right|^2 dx \\
&\quad \leqslant \delta_0^2 (1+9\delta_1^2)^{3/2} \left(-\frac{d}{dt}\mathcal{L}_i(t) - \varepsilon\mu\mathcal{L}_i(t) + \|\phi_i(t)\|_{L^1} + \|\psi_i(t)\|_{L^1} + \varepsilon M\delta_0 e^{-\varepsilon\mu t} \right).
\end{aligned}
$$

Finally, we integrate over $[\hat{t}, T]$ and apply (6.48) and (6.6), to get (6.28).

6.3. Energy method

It now remains to show that the cutoff terms in (6.4) are controllable, i.e. for each $i = 1, \dots, n$,

$$\int_{\hat{t}}^{T} \int e^{\varepsilon \mu t} \big(|v_{i,x}| + |w_{i,x}|\big)|w_i - \theta_i v_i| \, dx \, dt = \mathcal{O}(1)\delta_0^2. \tag{6.50}$$

It is easy to see that there is no contribution from the component source terms of this category, when the cut-off function θ_i is not active, i.e. in the region $\left|\frac{w_i}{v_i}\right| < \delta_1$. The interaction of waves of the same family due to the wrong choice of speed is treated by employing energy functionals. We introduce, in advance, additional cutoff functions, η and ζ, that simplify the interaction. Consider a smooth even function $\eta : \mathbb{R} \to [0, 1]$ that satisfies

$$\eta(s) = \begin{cases} 0 \text{ if } |s| \leqslant \dfrac{3\delta_1}{5} \\ 1 \text{ if } |s| \geqslant \dfrac{4\delta_1}{5} \end{cases} \tag{6.51}$$

with bounded derivatives

$$|\eta'(s)| \leqslant \frac{21}{\delta_1}, \qquad |\eta''(s)| \leqslant \frac{101}{\delta_1^2}. \tag{6.52}$$

Similarly, we define ζ to be

$$\zeta(s) = \eta\left(|s| - \frac{\delta_1}{5}\right) \leqslant \eta(s), \tag{6.53}$$

as shown in Fig. 3. For shorthand, we will be using $\eta_i = \eta(\frac{w_i}{v_i})$ and $\zeta_i = \zeta(\frac{w_i}{v_i})$. In the following lemma, we establish some useful relations between v_i, w_i and $v_{i,x}$, which we apply afterwards to the cutoff terms. The aim is to reduce the integrand of (6.50) into controllable terms and other new terms and to treat the latter by means of energy methods.

Lemma 6.6. *If* $|w_i/v_i| \geqslant 3\delta_1/5$, *then*

$$|w_i| \leqslant 2|v_{i,x}| + \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} (|v_j| + |v_{j,x}| + |w_j|), \tag{6.54}$$

$$|v_i| \leqslant \frac{5}{2\delta_1}|v_{i,x}| + \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} (|v_j| + |v_{j,x}| + |w_j|). \tag{6.55}$$

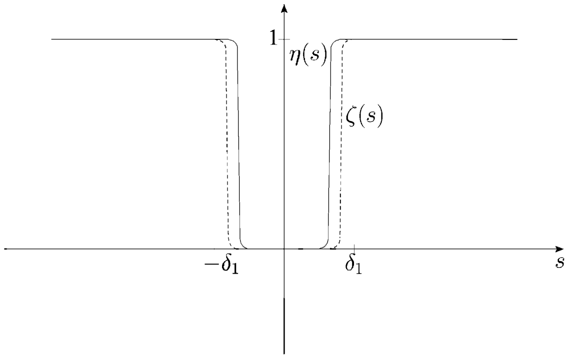

Fig. 3. The cutoff functions $\eta(s)$ and $\zeta(s)$.

If $|w_i/v_i| \leqslant \delta_1$, *then*

$$|v_{i,x}| \leqslant 2\delta_1 |v_i| + \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} (|v_j| + |v_{j,x}| + |w_j|). \tag{6.56}$$

Proof. As a result of the decomposition (5.1), the identity

$$u_t + \varepsilon g(u) + A(u) u_x = u_{xx} \tag{6.57}$$

can be written as

$$\sum_i (w_i - \lambda_i^* v_i)\tilde{r}_i + \sum_i v_i A(u)\tilde{r}_i$$
$$= \sum_i v_{i,x}\tilde{r}_i + \sum_i v_i \left[\sum_j v_j \tilde{r}_{i,u}\tilde{r}_j + v_{i,x}\tilde{r}_{i,v} - \theta_{i,x}\tilde{r}_{i,\sigma} \right].$$

Taking the inner product with $\tilde{r}_i$, yields

$$v_{i,x} = w_i + (\tilde{\lambda}_i - \lambda_i^*) v_i + \Theta_i, \tag{6.58}$$

where

$$\begin{aligned} \Theta_i &= \sum_{j \neq i} (w_j - \lambda_j^* v_j)\langle \tilde{r}_i, \tilde{r}_j \rangle + \sum_{j \neq i} \langle \tilde{r}_i, A(u))\tilde{r}_j \rangle v_j - \sum_{j \neq i} \sum_k \langle \tilde{r}_i, \tilde{r}_{j,u}\tilde{r}_k \rangle v_j v_k \\ &\quad - \sum_{j \neq i} \langle \tilde{r}_i, \tilde{r}_{j,v} \rangle v_j v_{j,x} + \sum_{j \neq i} \langle \tilde{r}_i, \tilde{r}_{j,\sigma} \rangle v_j \theta_{j,x} - \sum_{j \neq i} \langle \tilde{r}_i, \tilde{r}_j \rangle v_{j,x} \\ &= \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} (|v_j| + |v_{j,x}| + |w_j|). \end{aligned} \tag{6.59}$$

From this point, the proof follows easily if one adjusts the argument in the proof of Lemma 9.1 in [6] to this setting. □

Now, we reduce the integrand of (6.50) to an expression that is easier to deal with. As it is mentioned in the beginning of this section, the quantity $|w_i - \theta_i v_i|$ vanishes when $|w_i/v_i| \leqslant \delta_1$. Otherwise, $|w_i - \theta_i v_i| \leqslant |w_i|$. By means of the cutoff function ζ and the relation (6.54), one has

$$|w_i - \theta_i v_i| \leqslant |\zeta_i w_i| \leqslant \zeta_i \left(2|v_{i,x}| + \mathcal{O}(1)\delta_0 \sum_{j\neq i} (|v_j| + |v_{j,x}| + |w_j|) \right).$$

Applying Schwarz's inequality, the integrand in (6.50) becomes

$$\begin{aligned}
&(|v_{i,x}| + |w_{i,x}|)|w_i - \theta_i v_i| \\
&\quad \leqslant (|v_{i,x}| + |w_{i,x}|)\zeta_i \left(2|v_{i,x}| + \mathcal{O}(1)\delta_0 \sum_{j\neq i} |v_j| + |v_{j,x}| + |w_j| \right) \\
&\quad \leqslant 3\eta_i v_{i,x}^2 + \zeta_i w_{i,x}^2 + \mathcal{O}(1)\cdot\delta_0 \sum_{j\neq i} \big(|v_{i,x}| + |w_{i,x}|\big)\big(|v_j| + |v_{j,x}| + |w_j|\big),
\end{aligned} \tag{6.60}$$

because $\zeta_i \leqslant \eta_i$ and one may take $\mathcal{O}(1)\delta_0 < 1$. By virtue of the results in Section 6.1, it suffices to show

$$\int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \eta_i v_{i,x}^2 \, dx\, dt = \mathcal{O}(1)\cdot\delta_0^2, \qquad \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \zeta_i w_{i,x}^2 \, dx\, dt = \mathcal{O}(1)\cdot\delta_0^2, \tag{6.61}$$

for all $i = 1, \ldots, n$.

To begin with, we study the term $\eta_i v_{i,x}^2$. Upon multiplying the evolution equation of v_i given in (6.1) by $\eta_i v_i$ and integrating by parts, we get

$$\begin{aligned}
\int \eta_i v_{i,x}^2 \, dx = &-\int \left(\eta_i \frac{v_i^2}{2}\right)_t + \varepsilon \sum_k \eta_i v_i \left(B_{ik}^{\sharp} v_k + H_{ik} w_k \right) dx \\
&+ \int \left(\eta_{i,t} + \tilde{\lambda}_i \eta_{i,x} - \eta_{i,xx} \right) \frac{v_i^2}{2} \, dx \\
&- \int \eta_i \tilde{\lambda}_{i,x} \frac{v_i^2}{2} \, dx - 2\int \eta_{i,x} v_i v_{i,x} \, dx + \int \eta_i v_i \phi_i \, dx.
\end{aligned} \tag{6.62}$$

In what follows, we show that the integrals on the right-hand side of (6.62) can be bounded by controllable functions. We start by computing $\eta_{i,t}+\tilde{\lambda}_i\eta_{i,x}-\eta_{i,xx}$:

$$\begin{aligned}
&\eta_{i,t}+\tilde{\lambda}_i\eta_{i,x}-\eta_{i,xx}\\
&\quad=\eta_i'\left(\psi_i\frac{1}{v_i}-\phi_i\frac{w_i}{v_i^2}\right)+2\eta_i'\frac{v_{i,x}}{v_i}\left(\frac{w_i}{v_i}\right)_x-\eta_i''\left(\frac{w_i}{v_i}\right)_x^2\\
&\qquad-\varepsilon\eta_i'\sum_k\left(\left(K_{ik}v_k+B^\flat_{ik}w_k\right)\frac{1}{v_i}-\left(B^\sharp_{ik}v_k+H_{ik}w_k\right)\frac{w_i}{v_i^2}\right).
\end{aligned}\tag{6.63}$$

Applying (4.7) and (6.54) and by account of (6.52), we deduce that for sufficiently small δ_0, $|\tilde{\lambda}_i-\lambda_i^*|=\mathcal{O}(1)\delta_0<<\delta_1$, therefore the third integral on the right-hand side of (6.62) is expressed by

$$\begin{aligned}
\left|\int\tilde{\lambda}_{i,x}\eta_i\frac{v_i^2}{2}\,dx\right|&=\left|\int\left(\tilde{\lambda}_i-\lambda_i^*\right)_x\eta_i\frac{v_i^2}{2}\,dx\right|=\left|\int\left(\tilde{\lambda}_i-\lambda_i^*\right)\left(\eta_{i,x}\frac{v_i^2}{2}+\eta_i v_i v_{i,x}\right)dx\right|\\
&\leqslant\|\tilde{\lambda}_i-\lambda_i^*\|_{L^\infty}\cdot\int\Bigg[\frac{1}{2}|\eta_i'||w_{i,x}v_i-v_{i,x}w_i|+\frac{5}{2\delta_1}\eta_i v_{i,x}^2\\
&\qquad+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\left(|v_{i,x}v_j|+|v_{i,x}v_{j,x}|+|v_{i,x}w_j|\right)\Bigg]dx\\
&\leqslant\int|w_{i,x}v_i-v_{i,x}w_i|+\frac{1}{2}\eta_i v_{i,x}^2\,dx\\
&\qquad+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\Big(|v_{i,x}v_j|+|v_{i,x}v_{j,x}|+|v_{i,x}w_j|\Big).
\end{aligned}\tag{6.64}$$

By the definition of η_i, the derivative η_i' is nonzero on the region $\frac{3\delta_1}{5}<|w_i/v_i|<\delta_1$. This, together with Lemma 6.6 yields

$$\begin{aligned}
|\eta_{i,x}v_iv_{i,x}|&=\left|\eta_i'v_iv_{i,x}\left(\frac{w_i}{v_i}\right)_x\right|\leqslant 2\left|\delta_1\eta_i'v_i^2\left(\frac{w_i}{v_i}\right)_x\right|\\
&\quad+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\left[\left|\eta_i'v_iv_j\left(\frac{w_i}{v_i}\right)_x\right|+\left|\eta_i'v_iv_{j,x}\left(\frac{w_i}{v_i}\right)_x\right|\right.\\
&\quad\left.+\left|\eta_i'v_iw_j\left(\frac{w_i}{v_i}\right)_x\right|\right]\\
&\leqslant\mathcal{O}(1)|w_{i,x}v_i-w_iv_{i,x}|+\mathcal{O}(1)\cdot\delta_0|\eta_i'|\sum_{j\neq i}\left(|v_jw_{i,x}|+|v_jv_{i,x}|\left|\frac{w_i}{v_i}\right|\right.\\
&\quad\left.+|v_{j,x}w_{i,x}|+|v_{j,x}v_{i,x}|\left|\frac{w_i}{v_i}\right|+|w_jw_{i,x}|+|w_jv_{i,x}|\left|\frac{w_i}{v_i}\right|\right).
\end{aligned}\tag{6.65}$$

Combining (6.63)–(6.65) and (6.62), we arrive at the following estimate

$$\begin{aligned}
&\frac{1}{2}\int \eta_i v_{i,x}^2\,dx\\
&\quad\leqslant -\int \left(\eta_i \frac{v_i^2}{2}\right)_t + \varepsilon \sum_k \eta_i v_i \left(B_{ik}^{\sharp} v_k + H_{ik} w_k\right) dx\\
&\qquad + \frac{1}{2}\int |\eta_i'| \left(|v_i \psi_i| + |w_i \phi_i|\right) dx + \frac{1}{2}\int \left|\eta_i'' v_i^2 \left(\frac{w_i}{v_i}\right)_x^2\right| dx\\
&\qquad + \mathcal{O}(1)\Bigg[\int |w_{i,x} v_i - w_i v_{i,x}|\,dx\\
&\qquad + \delta_0 \int |\eta_i'| \sum_{j\neq i}\Bigg(|v_j w_{i,x}| + |v_j v_{i,x}|\left|\frac{w_i}{v_i}\right| + |v_{j,x} w_{i,x}|\\
&\qquad + |v_{j,x} v_{i,x}|\left|\frac{w_i}{v_i}\right| + |w_j w_{i,x}| + |w_j v_{i,x}|\left|\frac{w_i}{v_i}\right|\Bigg) dx\Bigg] + \int |\eta_i v_i \phi_i|\,dx\\
&\qquad + \mathcal{O}(1)\int |w_{i,x} v_i - v_{i,x} w_i|\,dx + \mathcal{O}(1)\delta_0 \sum_{j\neq i}\int |v_{i,x} v_j| + |v_{i,x} w_j| + |v_{i,x} v_{j,x}|\,dx\\
&\qquad - \frac{\varepsilon}{2}\int \eta_i' \sum_k \Big((K_{ik} v_k + B_{ik}^{\flat} w_k) v_i - (B_{ik}^{\sharp} v_k + H_{ik} w_k) w_i\Big)\,dx. \qquad (6.66)
\end{aligned}$$

Recalling the dissipativeness hypothesis and multiplying by $e^{\varepsilon\mu t}$, we obtain the desired integral

$$\begin{aligned}
&\frac{1}{2}\int e^{\varepsilon\mu t}\eta_i v_{i,x}^2\,dx\\
&\quad\leqslant -\int \left(e^{\varepsilon\mu t}\eta_i \frac{v_i^2}{2}\right)_t dx + \mathcal{O}(1)\varepsilon\delta_0^3 e^{-2\varepsilon\mu t} M\\
&\qquad + \frac{1}{2}\int e^{\varepsilon\mu t}|\eta_i'|(|v_i\psi_i| + |w_i\phi_i|)\,dx + \mathcal{O}(1)\int e^{\varepsilon\mu t}|w_{i,x} v_i - w_i v_{i,x}|\,dx\\
&\qquad + \mathcal{O}(1)\delta_0 \int e^{\varepsilon\mu t}|\eta_i'| \sum_{j\neq i}\Bigg(|v_j w_{i,x}| + |v_j v_{i,x}|\left|\frac{w_i}{v_i}\right| + |v_{j,x} w_{i,x}| + |v_{j,x} v_{i,x}|\left|\frac{w_i}{v_i}\right|\\
&\qquad + |w_j w_{i,x}| + |w_j v_{i,x}|\left|\frac{w_i}{v_i}\right|\Bigg) dx + \frac{1}{2}\int e^{\varepsilon\mu t}\left|\eta_i'' v_i^2 \left(\frac{w_i}{v_i}\right)_x^2\right| dx\\
&\qquad + \delta_0 \sum_{j\neq i}\int e^{\varepsilon\mu t}\left(|v_{i,x} v_j| + |v_{i,x} w_j| + |v_{i,x} v_{j,x}|\right) dx + \int e^{\varepsilon\mu t}|\eta_i v_i \phi_i|\,dx. \qquad (6.67)
\end{aligned}$$

Finally, to show that $\eta_i v_{i,x}^2$ is controllable, we integrate over $[\hat{t}, T]$. Assuming (6.6) holds and recalling (6.5), (6.9), (6.27) and (6.28), we conclude

$$\begin{aligned}
&\int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \eta_i v_{i,x}^2 \, dx\, dt \\
&\quad \leqslant \int e^{\varepsilon\mu\hat{t}} \eta_i v_i^2(\hat{t}) \, dx + \mathcal{O}(1)\, \varepsilon \delta_0^3 M \int_{\hat{t}}^{T} e^{-\varepsilon\mu t} \, dt \\
&\quad + \mathcal{O}(1) \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \left(|v_i \psi_i| + |w_i \phi_i| \right) dx + \mathcal{O}(1) \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} |w_{i,x} v_i - w_i v_{i,x}| \, dx \\
&\quad + \mathcal{O}(1)\delta_0 \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \sum_{j\neq i} \Big(|v_j w_{i,x}| + |v_j v_{i,x}| + |v_{j,x} w_{i,x}| + |v_{j,x} v_{i,x}| \\
&\quad + |w_j v_{i,x}| + |w_j w_{i,x}| \Big) \, dx\, dt \\
&\quad + \mathcal{O}(1) \int_{\hat{t}}^{T} \int_{|\frac{w_i}{v_i}| \leqslant \delta_1} e^{\varepsilon\mu t} \left| v_i^2 \left(\frac{w_i}{v_i} \right)_x^2 \right| dx\, dt + 2 \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} |v_i \phi_i| \, dx\, dt \\
&\quad + \delta_0 \sum_{j\neq i} \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \left(|v_{i,x} v_j| + |v_{i,x} w_j| + |v_{i,x} v_{j,x}| \right) dx\, dt = \mathcal{O}(1) \cdot \delta_0^2. \qquad (6.68)
\end{aligned}$$

Now, we proceed to derive the corresponding estimate for $\zeta_i w_{i,x}^2$. We start out again by multiplying the evolution equation of w_i given in (6.1) by $\zeta_i w_i$ and integrating by parts

$$\begin{aligned}
\int \zeta_i w_{i,x}^2 \, dx &= -\int \left(\zeta_i \frac{w_i^2}{2} \right)_t + \varepsilon \sum_k \zeta_i w_i \left(K_{ik} v_k + B_{ik}^{\flat} w_k \right) dx \\
&\quad + \int \left(\zeta_{i,t} + \tilde{\lambda}_i \zeta_{i,x} - \zeta_{i,xx} \right) \frac{w_i^2}{2} \, dx \\
&\quad - \int \zeta_i \tilde{\lambda}_{i,x} \frac{w_i^2}{2} \, dx - 2 \int \zeta_{i,x} w_i w_{i,x} \, dx + \int \zeta_i w_i \psi_i \, dx. \qquad (6.69)
\end{aligned}$$

We investigate the integrals on the right-hand side of (6.69) one by one. The expression $\zeta_{i,t} + \tilde{\lambda}_i \zeta_{i,x} - \zeta_{i,xx}$ is given by (6.63), replacing η_i by ζ_i. If $\zeta_i \neq 0$, then $|\frac{w_i}{v_i}| > \frac{4\delta_1}{5}$ and Lemma 6.6 implies

$$\begin{aligned}
|w_i w_{i,x}| &\leqslant 2|v_{i,x} w_{i,x}| + \mathcal{O}(1)\delta_0 \sum_{j\neq i} \left(|v_j| + |v_{j,x}| + |w_j| \right) |w_{i,x}| \\
&\leqslant v_{i,x}^2 + w_{i,x}^2 + \mathcal{O}(1)\delta_0 \sum_{j\neq i} \left(|v_j w_{i,x}| + |v_{j,x} w_{i,x}| + |w_j w_{i,x}| \right). \qquad (6.70)
\end{aligned}$$

By employing the above bound, we derive an estimate analogous to (6.64):

$$\left|\int \tilde{\lambda}_{i,x}\zeta_i \frac{w_i^2}{2}\,dx\right| \leqslant \mathcal{O}(1)\int |w_{i,x}v_i - v_{i,x}w_i|\,dx + \frac{1}{2}\int \eta_i v_{i,x}^2\,dx + \frac{1}{2}\int \zeta_i w_{i,x}^2$$

$$+\mathcal{O}(1)\cdot\delta_0^2\int \sum_{j\neq i}\big(|w_{i,x}v_j|+|w_{i,x}v_{j,x}|+|w_{i,x}w_j|\big)\,dx. \tag{6.71}$$

Furthermore, if $\frac{4\delta_1}{5} < \left|\frac{w_i}{v_i}\right| < \delta_1$, then

$$\left|v_i\left(\frac{w_i}{v_i}\right)_x\right|^2 = \frac{|w_{i,x}v_i - v_{i,x}w_i|^2}{v_i^2} \geqslant w_{i,x}^2 - 2|w_{i,x}v_{i,x}|\left|\frac{w_i}{v_i}\right| - \left|\frac{w_i}{v_i}\right|^2 v_{i,x}^2$$

$$\geqslant w_{i,x}^2 - \left(w_{i,x}^2 + v_{i,x}^2\right)\delta_1 - \delta_1^2 v_{i,x}^2 \geqslant \frac{1}{2}\left(w_{i,x}^2 - v_{i,x}^2\right). \tag{6.72}$$

Consequently, the upper bound of $w_{i,x}^2$ yields

$$|\zeta_{i,x}w_i w_{i,x}| \leqslant \delta_1\left|\zeta_i'\left(\frac{w_i}{v_i}\right)_x v_i w_{i,x}\right| \leqslant \frac{\delta_1}{2}|\zeta_i'|\left(3\left|v_i\left(\frac{w_i}{v_i}\right)_x\right|^2 + |v_{i,x}|^2\right)$$

$$= \mathcal{O}(1)\left|v_i\left(\frac{w_i}{v_i}\right)_x\right|^2 \chi_{\left\{|\frac{w_i}{v_i}|<\delta_1\right\}} + \mathcal{O}(1)\cdot\eta_i|v_{i,x}|^2. \tag{6.73}$$

We now substitute (6.73), (6.71) in (6.69) and also use (6.63) as well as (6.65). By virtue of the diagonal dominance property, yields

$$\begin{aligned}
&\frac{1}{2}\int e^{\varepsilon\mu t}\zeta_i w_{i,x}^2\,dx \\
&\quad\leqslant -\int\left(e^{\varepsilon\mu t}\zeta_i\frac{w_i^2}{2}\right)_t dx + \mathcal{O}(1)\int e^{\varepsilon\mu t}(|v_i\psi_i| + |w_i\phi_i|)\,dx \\
&\qquad+\mathcal{O}(1)\delta_0^3\varepsilon M e^{-\varepsilon\mu t} + \mathcal{O}(1)\int_{|\frac{w_i}{v_i}|<\delta_1} e^{\varepsilon\mu t}\left|v_i\left(\frac{w_i}{v_i}\right)_x\right|^2 dx \\
&\qquad+\mathcal{O}(1)\int e^{\varepsilon\mu t}\eta_i v_{i,x}^2\,dx + \int e^{\varepsilon\mu t}|\zeta_i w_i\psi_i|\,dx + \mathcal{O}(1)\int e^{\varepsilon\mu t}|w_{i,x}v_i - v_{i,x}w_i|\,dx \\
&\qquad+\mathcal{O}(1)\delta_0^2\int e^{\varepsilon\mu t}\sum_{j\neq i}\big(|w_{i,x}v_j| + |w_{i,x}v_{j,x}| + |w_{i,x}w_j|\big)\,dx \\
&\qquad+\mathcal{O}(1)\delta_0\sum_{j\neq i}\int e^{\varepsilon\mu t}|\zeta_i'|\Bigg(|v_j w_{i,x}| + |v_j v_{i,x}|\left|\frac{w_i}{v_i}\right| + |v_{j,x}w_{i,x}| + |v_{j,x}v_{i,x}|\left|\frac{w_i}{v_i}\right| \\
&\qquad+|w_j w_{i,x}| + |w_j v_{i,x}|\left|\frac{w_i}{v_i}\right|\Bigg).
\end{aligned} \tag{6.74}$$

In view of (6.9), (6.27) and (6.28), (6.6) implies

$$\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\zeta_i w_{i,x}^2\,dx\,dt = \mathcal{O}(1)\cdot\delta_0^2. \tag{6.75}$$

Therefore, $\eta_i v_{i,x}^2$ and $\zeta_i w_{i,x}^2$ are controllable, and thus (6.60) implies the validity of (6.50).

The four estimates (6.9), (6.27), (6.28) and (6.50) together with Lemma 6.1 prove Lemma 3.1. Thus, we have proven that the solution u is globally defined and satisfies a priori the BV bound

$$TV\{u(t)\} = \mathcal{O}(1)\,e^{-\varepsilon\mu t}\,TV\{u_0\}. \tag{6.76}$$

More precisely, under assumption (6.6) it is proven that the interaction of the waves is in fact quadratic with respect to $TV\{u_0\}$:

$$\int_{\hat{t}}^{\infty}\int e^{\varepsilon\mu t}\big(|\phi_i(t,x)| + |\psi_i(t,x)|\big)\,dx\,dt = \mathcal{O}(1)\cdot\delta_0^2. \tag{6.77}$$

Hence, all the estimates (6.27), (6.28) and (6.50) are of order δ_0^3.

7. Stability of approximate solutions

Let u be the solution of the viscous hyperbolic system of balance laws

$$u_t + A(u)u_x + \varepsilon g(u) - u_{xx} = 0, \tag{7.1}$$

$$u(0,x) = \bar{u}(x) \doteq u_0(\varepsilon x), \tag{7.2}$$

where g is dissipative. If the total variation of u_0 is sufficiently small, then the solution $u(t,x)$ is globally defined and satisfies the BV bound

$$TV_x\,u(t) \leqslant C\ e^{-\varepsilon\mu t}TV\,u_0, \tag{7.3}$$

for all $t\geqslant 0$, where C is a constant independent of ε. This section discusses the stability of solutions to (7.1)–(7.2). Consider the infinitesimal perturbation $z = z(t,x)$ of u and its evolution equation

$$z_t + (A(u)z)_x + \varepsilon Dg(u)z - z_{xx} = (u_x \bullet A(u))z - (z \bullet A(u))u_x. \tag{7.4}$$

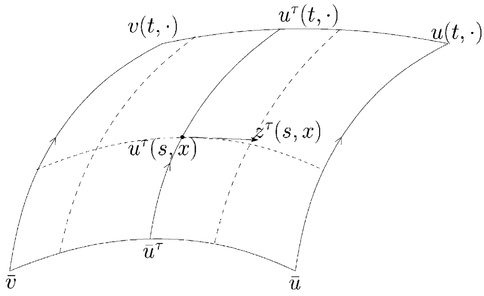

Fig. 4. The solution u^τ to (7.1)–(7.2), $\tau \in [0, 1]$.

Lemma 7.1. *Under the assumptions of Theorem* 1.1, *we have the following bound on the infinitesimal perturbation z of u*

$$\|z(t)\|_{L^1} \leqslant Le^{-\varepsilon\mu t}\|z(0)\|_{L^1}, \tag{7.5}$$

for some constant L.

The proof of the above lemma generalizes the proof of the BV bound of u and follows the same strategy. After decomposing z along a suitable basis obtained in a similar fashion as that for the gradient u_x and studying the evolution of the components of this 'generalized' decomposition, we proceed with the proof of this lemma.

Beforehand, assuming that the above lemma holds, we claim that (7.5) implies that the solutions to (7.1)–(7.2) are stable in L^1. Indeed, having initial data $\bar{u}$ and $\bar{v}$ with small total variation and

$$u^* = \lim_{x\to-\infty} \bar{u} = \lim_{x\to-\infty} \bar{v},$$

for each $\tau \in [0, 1]$ we construct a path of initial data to system (7.1)

$$\bar{u}^\tau(x) = \tau\bar{u}(x) + (1-\tau)\bar{v}(x). \tag{7.6}$$

Then the total variation of $\bar{u}^\tau$ is small and the solution $u^\tau(t,\cdot)$ to (7.1)–(7.6) is globally defined satisfying the BV a priori bound (7.3).

Consider the tangent vector

$$z^\tau(t,x) = \frac{du^\tau}{d\tau}(t,x) \tag{7.7}$$

to the path $\tau \mapsto u^\tau(t,x)$, as shown in Fig. 4. By direct computation, we obtain that z^τ is a solution to (7.4) with initial data

$$z^\tau(0,x) = \bar{u}(x) - \bar{v}(x). \tag{7.8}$$

Therefore, by Lemma 7.1, z^τ satisfies (7.5) for every $\tau \in [0,1]$. Thus, for every $t > 0$, we deduce

$$\begin{aligned}\|u(t) - v(t)\|_{L^1} &\leqslant \int_0^1 \left\| \frac{du^\tau}{d\tau}(t) \right\|_{L^1} d\tau \\ &\leqslant L\, e^{-\varepsilon\mu t} \int_0^1 \|\bar{u} - \bar{v}\|_{L^1}\, d\tau = L\, e^{-\varepsilon\mu t} \|\bar{u} - \bar{v}\|_{L^1}. \qquad (7.9)\end{aligned}$$

This proves that all solutions with small total variation are uniformly stable in L^1. In view of the above discussion, the bound in Lemma 7.1 is crucial to the stability estimate (7.9).

In Section 5, we discussed extensively the decomposition given in (5.1). Observe that the speeds $\sigma_i(t,x)$ of the viscous traveling waves selected in the decomposition might be discontinuous as a function of (t,x). However, for each i, we can modify the speeds to smooth functions and decomposition (5.1) still holds with the new choice of speeds. By this modification, it is not difficult to see that the components v_i and w_i still satisfy a system of form (6.1) and the component source terms are controled as before

$$\sum_{i=1}^{n} \int_{\hat{t}}^{\infty} \int e^{\varepsilon\mu t} \big(|\phi_i| + |\psi_i|\big)\, dx\, dt < \delta_0. \qquad (7.10)$$

From now on, we consider decomposition (5.1) along the gradients of viscous traveling waves whose speeds σ_i are smooth functions of (t,x).

We now return to Lemma 7.1. Having u as a reference solution of (7.1), we consider the infinitesimal perturbation z of u, which satisfies the perturbed equation (7.4). Observe that $z = u_x$ or u_t are solutions to (7.4). In order to prove the L^1 bound (7.5), we employ the same techniques applied to prove the BV bound (7.3) on u with some appropriate modifications. By Proposition 2.3, bound (7.5) holds over an initial time interval $[0,\hat{t}]$. For later times, we decompose z into a suitable basis $\hat{r}_i$ $i = 1\ldots,n$, and study the evolution of the components of this decomposition.

We denote the flux of z by

$$y = z_x - A(u)z. \qquad (7.11)$$

We decompose (z,y) pointwise along a suitable basis so that the decomposition must be compatible with (5.1), when $(z,y) = (u_x, u_t + \varepsilon g(u))$. Given u and v, we seek $(p,q) \in \mathbb{R}^{2n}$ so that

$$\begin{aligned} z &= \sum_{i=1}^{n} p_i \tilde{r}_i \left(u, v_i, \lambda_i^* - \theta\left(\frac{q_i}{p_i}\right)\right), \\ y &= \sum_{i=1}^{n} (q_i - \lambda_i^* p_i) \tilde{r}_i \left(u, v_i, \lambda_i^* - \theta\left(\frac{q_i}{p_i}\right)\right), \end{aligned} \qquad (7.12)$$

where θ is the cutoff function defined at (5.3). We use the following notation:

$$\hat{r}_i \doteq \tilde{r}_i\left(u, v_i, \lambda_i^* - \theta\left(\frac{q_i}{p_i}\right)\right), \tag{7.13}$$

$$\hat{\theta}_i \doteq \theta\left(\frac{q_i}{p_i}\right) \quad \text{and} \quad \hat{\lambda}_i \doteq \langle \hat{r}_i, A(u)\hat{r}_i\rangle. \tag{7.14}$$

In general, $\hat{r}_i \neq \tilde{r}_i$. Using the above notation, the fundamental identity (4.3) becomes

$$\left(A(u) - \hat{\lambda}_i I\right)\hat{r}_i = v_i\big(\hat{r}_{i,u}\hat{r}_i + \hat{r}_{i,v}(\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i)\big). \tag{7.15}$$

The following lemma corresponds to Lemma 5.1 and establishes the validity of the decomposition (7.12).

Lemma 7.2. *If $|u-u^*|$ and $|v|$ are sufficiently small, then given $(z, y) \in \mathbb{R}^{2n}$, system (7.12) has a unique solution (p, q). Moreover, the map $(z, y) \mapsto (p, q)$ is Lipschitz continuous and in particular, it is smooth outside the manifolds $\widehat{\mathcal{N}}_i = \left\{(p, q) \in \mathbb{R}^{2n} : p_i = q_i = 0\right\}$, $i = 1, \dots, n$.*

Proof. The proof is similar to that of Lemma 5.1. The uniqueness is immediate, because z and y are uniquely determined by p and q. We define the map

$$\begin{aligned} \widehat{\mathcal{G}} : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}^{2n} \\ (p, q) &\mapsto \sum_{i=1}^n \widehat{\mathcal{G}}_i(p_i, q_i), \end{aligned} \tag{7.16}$$

$$\widehat{\mathcal{G}}_i(p_i, q_i) = \begin{pmatrix} p_i\tilde{r}_i\left(u, v_i, \lambda_i^* - \theta\left(\frac{q_i}{p_i}\right)\right) \\ (q_i - \lambda_i^* p_i)\tilde{r}_i\left(u, v_i, \lambda_i^* - \theta\left(\frac{q_i}{p_i}\right)\right) \end{pmatrix}. \tag{7.17}$$

We prove that the Jacobian matrices of $\widehat{\mathcal{G}}_i$ are uniformly bounded and invertible for small v_i. Indeed,

$$\begin{aligned} \frac{\partial \widehat{\mathcal{G}}_i}{\partial(p_i, q_i)} &= \begin{pmatrix} \hat{r}_i & 0 \\ -\lambda_i^*\hat{r}_i & \hat{r}_i \end{pmatrix} + \begin{pmatrix} \hat{\theta}_i' \frac{q_i}{p_i}\hat{r}_{i,\sigma} & -\hat{\theta}_i'\hat{r}_{i,\sigma} \\ \frac{q_i^2}{p_i^2}\hat{\theta}_i'\hat{r}_{i,\sigma} - \lambda_i^*\hat{\theta}_i'\frac{q_i}{p_i}\hat{r}_{i,\sigma} & \lambda_i^*\hat{\theta}_i'\hat{r}_{i,\sigma} - \frac{q_i}{p_i}\hat{\theta}_i'\hat{r}_{i,\sigma} \end{pmatrix} \\ &\doteq \widehat{\mathcal{M}}_{i,0} + \widehat{\mathcal{M}}_{i,1}. \end{aligned} \tag{7.18}$$

By virtue of (4.5), $\hat{r}_{i,\sigma} = \mathcal{O}(1)v_i$, hence $\widehat{\mathcal{M}}_1 \to 0$ for $|v|$ small and therefore, the Jacobian matrix is invertible. By the implicit function theorem, for each (z, y) in a small neighborhood of the origin of $\mathbb{R}^{2n}$, there exist unique p and q so that $(z, y) = \widehat{\mathcal{G}}(p, q)$, i.e. (7.12) holds. Moreover, the map $\widehat{\mathcal{G}}$ is Lipschitz continuous on the whole space and in particular, it is smooth outside the manifolds $\widehat{\mathcal{N}}_i$, $i = 1, \dots, n$. This completes the proof. □

Our goal is to study the evolution of the components p_i and q_i of the decomposition (7.12) and in particular, to prove that they satisfy a parabolic system of a similar form to (6.1). However, the map $\widehat{\mathcal{G}}$ defined in the previous lemma is only Lipschitz continuous and hence the same holds for p and q as functions of t and x. This raises some difficulties because the derivatives of p and q may not even exist everywhere. We overcome these difficulties by approximating z and y by smooth functions. See more details in [6]. Therefore, without loss of generality, we shall assume that z and y are smooth functions.

The following lemma passes the information for the perturbation z and y stated in Corollary 2.4 to the components p_i and q_i.

Lemma 7.3. *Assume that bound* (7.5) *holds for* $t \in [0, T]$. *Then for all* $t \in [\hat{t}, T]$ *the perturbed components* p_i, q_i *satisfy the estimates*

$$\|p_i(t)\|_{L^1},\ \|q_i(t)\|_{L^1} = \mathcal{O}(1)\delta_0 e^{-\varepsilon\mu t}, \tag{7.19}$$

$$\|p_i(t)\|_{L^\infty},\ \|q_i(t)\|_{L^\infty},\ \|p_{i,x}(t)\|_{L^1},\ \|q_{i,x}(t)\|_{L^1} = \mathcal{O}(1)\delta_0^2 e^{-\varepsilon\mu t}, \tag{7.20}$$

$$\|p_{i,x}(t)\|_{L^\infty},\quad \|q_{i,x}(t)\|_{L^\infty} = \mathcal{O}(1)\delta_0^3 e^{-\varepsilon\mu t}. \tag{7.21}$$

Proof. As long as (7.5) holds, using standard parabolic theory, we can control the L^1 norms of the derivatives of z as stated in Corollary 2.4. Since the map $\widehat{\mathcal{G}} : (p, q) \mapsto (z, y)$ is uniformly Lipschitz continuous, to justify (7.19)–(7.21), one needs to repeat the argument in the proof of Lemma 5.2 for the perturbation z, using (7.5) and (2.28)–(2.30).

In the following lemma, we prove that (p, q) satisfy a $2n \times 2n$ system of viscous balance laws with source and we establish some bounds on the various terms that appear in this system.

Lemma 7.4. *For* $|u - u^*|$ *and* $|v|$ *small, decomposition* (7.12) *holds. The unique solution* (p, q) *of decomposition* (7.12) *satisfies a* $2n \times 2n$ *viscous hyperbolic system of balance laws with source of the form*

$$\begin{pmatrix} p \\ q \end{pmatrix}_t + \left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}\right]_x - \begin{pmatrix} p \\ q \end{pmatrix}_{xx} + \varepsilon\left[\begin{pmatrix} \widehat{B}^\sharp & \widehat{H} \\ \widehat{K} & \widehat{B}^\flat \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}\right] = \begin{pmatrix} \widehat{\Phi} \\ \widehat{\Psi} \end{pmatrix}, \tag{7.22}$$

where Λ *is the* $n \times n$ *diagonal matrix with entries* $\{\tilde{\lambda}_i\}$ *and the matrix* $\begin{pmatrix} \widehat{B}^\flat & \widehat{H} \\ \widehat{K} & \widehat{B}^\flat \end{pmatrix}$ *is diagonally dominant. More precisely*,

$$|\widehat{B}^\sharp_{ij} - \widehat{B}^\flat_{ij}| = \mathcal{O}(|u-u^*|, |v|), \qquad |\widehat{H}_{ij}|,\ |\widehat{K}_{ij}| = \mathcal{O}(|u-u^*|, |v|) \tag{7.23}$$

and when $(u, v) = (u^*, 0)$, $\widehat{B}^\sharp = \widehat{B}^\flat = B(u^*) \doteq R(u^*)^{-1} Dg(u^*) R(u^*)$. *Furthermore, the source terms in* (7.22) *satisfy the bounds*

$$\begin{aligned}
\widehat{\phi}_i, \widehat{\psi}_i \;=\; & \mathcal{O}(1) \cdot \sum_{j \neq i} \Big(|p_i v_j| + |q_i v_j| + |p_i v_{j,x}| + |q_i v_{j,x}| + |p_i w_j| + |q_i w_j| + |p_{i,x} w_j| \\
& + |q_{i,x} w_j| + |p_{i,x} v_j| + |q_{i,x} v_j| + |p_{i,x} v_{j,x}| + |q_{i,x} v_{j,x}| + |p_i q_{j,x}| + |p_i p_{j,x}| \\
& + |p_{i,x} q_{j,x}| + |p_{i,x} p_{j,x} + |q_i p_{j,x}| + |q_i q_{j,x}| \Big) \longrightarrow \textit{Transversal} \\
& + \mathcal{O}(1) \cdot \sum_j \Big(|p_{j,x} w_j - p_j w_{j,x}| + |q_{j,x} w_j - q_j w_{j,x}| + |q_{i,x} p_i - p_{i,x} q_i| \\
& + |p_{j,x} v_j - p_j v_{j,x}| + |q_{j,x} v_j - q_j v_{j,x}| \Big) \longrightarrow \textit{Change in speed, Linear} \\
& + \mathcal{O}(1) \cdot \sum_j \big(|v_j| + |p_j| \big) \, |p_j| \left| \left(\frac{q_j}{p_j} \right)_x \right|^2 \chi_{\left\{ |\frac{q_j}{p_j}| < 3\delta_1 \right\}} \rightarrow \textit{Change in speed, Quadr.} \\
& + \mathcal{O}(1) \cdot \sum_j \big(|p_{j,x}| + |q_{j,x}| + |p_j v_j| + |q_j v_j| \big) \cdot |w_j - \theta_j v_j| \longrightarrow \textit{Cutoff} \\
& + \mathcal{O}(1) \cdot \sum_j |p_j \phi_j| + |q_j \phi_j| \longrightarrow \textit{Source of gradient component } v \\
& + \mathcal{O}(1) \cdot \varepsilon \sum_{j,k} \Big(|p_j v_k| + |p_j w_k| + |q_j v_k| + |q_j w_k| \Big). \rightarrow \varepsilon - \textit{Order Term}
\end{aligned} \tag{7.24}$$

Proof. The proof is given in Appendix B.

In what follows we study the relation between p_i, q_i and $p_{i,x}$ as in Lemma 6.6. We first substitute (7.12) in the identity $y = z_x - A(u)z$, to get

$$\begin{aligned}
\sum_i \big(q_i - \lambda_i^* p_i \big) \hat{r}_i \;=\; & \sum_i p_{i,x} \hat{r}_i - \sum_i A(u) p_i \hat{r}_i + \sum_{i,j} p_i v_j \hat{r}_{i,u} \tilde{r}_j \\
& + \sum_i p_i v_{i,x} \hat{r}_{i,v} - \sum_i p_i \hat{\theta}_{i,x} \hat{r}_{i,\sigma}.
\end{aligned} \tag{7.25}$$

By taking the inner product with $\hat{r}_i$, then

$$q_i = p_{i,x} - \left(\hat{\lambda}_i - \lambda_i^*\right) p_i + \widehat{\Theta}_i, \tag{7.26}$$

with

$$\widehat{\Theta}_i = \mathcal{O}(1)\delta_0 \sum_{j\neq i} \left(|p_j| + |p_{j,x}| + |q_j|\right), \tag{7.27}$$

by estimates (4.5), (4.8), (5.10) and (5.11). Having (7.26), we establish the following relations:

Lemma 7.5. *If* $|q_i/p_i| \geqslant 3\delta_1/5$, *then*

$$|q_i| \leqslant 2|p_{i,x}| + \mathcal{O}(1) \cdot \delta_0 \sum_{j\neq i} \left(|p_j| + |p_{j,x}| + |q_j|\right), \tag{7.28}$$

$$|p_i| \leqslant \frac{5}{2\delta_1}|p_{i,x}| + \mathcal{O}(1) \cdot \delta_0 \sum_{j\neq i} \left(|p_j| + |p_{j,x}| + |q_j|\right). \tag{7.29}$$

If $|q_i/p_i| \leqslant \delta_1$, *then*

$$|p_{i,x}| \leqslant 2\delta_1|p_i| + \mathcal{O}(1) \cdot \delta_0 \sum_{j\neq i} (|p_j| + |p_{j,x}| + |q_j|). \tag{7.30}$$

Proof. The proof is analogous to that of Lemma 6.6 in Section 6.3. Use (7.26) and (7.27). □

We naturally extend Definition 3.2 of controllable terms to this setting, i.e. a scalar function $\widehat{\xi}(t,x)$ is called controllable if for fixed $T > \hat{t}$,

$$\sum_{i=1}^{n} \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} \left(|\widehat{\phi}_i(t,x)| + |\widehat{\psi}_i(t,x)|\right) dx dt < \delta_0 \tag{7.31}$$

implies

$$\int_{\hat{t}}^{T} \int e^{\varepsilon\mu t} |\widehat{\xi}(t,x)| \, dx \, dt = \mathcal{O}(1)\delta_0^2. \tag{7.32}$$

By employing the same tools with those implemented in Sections 6.1–6.3, we prove that all perturbed source terms (7.24) are controllable. In particular, if (7.31) holds, all transversal terms in (7.24) satisfy

$$\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\Big(|p_i v_j| + |q_i v_j| + |p_i v_{j,x}| + |q_i v_{j,x}| + |p_i w_j| + |q_i w_j| + |p_{i,x} v_j| + |q_{i,x} v_j| + |p_{i,x} v_{j,x}| + |q_{i,x} v_{j,x}| + |p_{i,x} w_j| + |q_{i,x} w_j| + |p_i q_{j,x}| + |p_i p_{j,x}| + |p_{i,x} q_{j,x}| + |p_{i,x} p_{j,x} + |q_i p_{j,x}| + |q_i q_{j,x}|\Big)\,dx\,dt = \mathcal{O}(1)\delta_0^2 \tag{7.33}$$

for $j \neq i$. This is an immediate consequence of Section 6.1. Moreover, for each $i = 1, \ldots, n$, we introduce the curves

$$\gamma_i^{(v,p)}(t,x) = \left(\int_{-\infty}^{x} v_i(t,y)\,dy,\ \int_{-\infty}^{x} p_i(t,y)\,dy\right) \tag{7.34}$$

and similarly $\gamma_i^{(v,q)}(t,x)$, $\gamma_i^{(w,p)}(t,x)$, $\gamma_i^{(w,q)}(t,x)$, $\gamma_i^{(p,q)}(t,x)$ etc. We study the evolution of these planar curves as presented in Section 6.2 and introduce the corresponding area and length functionals. For example,

$$\mathcal{A}(\gamma_i^{(v,p)}, \gamma_j^{(v,p)}) = \frac{1}{2}\iint_{x<y} |\gamma_{i,x}^{(v,p)}(x) \wedge \gamma_{j,x}^{(v,p)}(y)|\,dx\,dy \tag{7.35}$$

$$\mathcal{L}_i^{(v,p)}(t) = \mathcal{L}(\gamma_i^{(v,p)}(t)) = \int \sqrt{v_i^2(t,x) + p_i^2(t,x)}\,dx \tag{7.36}$$

for $i, j = 1, \ldots, n$. We obtain bounds on the time derivatives of these functionals similar to those in Lemmas 6.4 and 6.5. By integrating over $[\hat{t}, T]$,

$$\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\Big(|p_{j,x} v_j - p_j v_{j,x}| + |q_{j,x} v_j - q_j v_{j,x}| + |p_{j,x} w_j - p_j w_{j,x}| + |q_{j,x} w_j - q_j w_{j,x}| + |q_{i,x} p_i - q_i p_{i,x}|\Big)\,dx\,dt = \mathcal{O}(1)\cdot\delta_0^2, \tag{7.37}$$

and

$$\int_{\hat{t}}^{T}\int_{\left|\frac{q_i}{p_i}\right|<3\delta_1} e^{\varepsilon\mu t}\left|p_i\left(\frac{q_i}{p_i}\right)_x^2\right|\,dx\,dt = \mathcal{O}(1)\cdot\delta_0. \tag{7.38}$$

From (5.10) and (7.20), it follows

$$\int_{\hat{t}}^{T}\int_{\left|\frac{q_i}{p_i}\right|<3\delta_1} e^{\varepsilon\mu t}\,(|v_i| + |p_i|)\left|p_i\left(\frac{q_i}{p_i}\right)_x^2\right|\,dx\,dt = \mathcal{O}(1)\cdot\delta_0^3. \tag{7.39}$$

The terms in the last two categories in (7.24) are controllable. Indeed, by (7.10) and (7.20), we have

$$\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\left(|p_i\phi_i|+|q_i\phi_i|\right)dx\,dt=\mathcal{O}(1)\cdot\delta_0^3, \tag{7.40}$$

for all $i=1,\dots,n$, and by Lemmas 5.2 and 7.3, we also have

$$\varepsilon\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\left(|p_jv_k|+|p_jw_k|+|q_jv_k|\right)dx\,dt=\mathcal{O}(1)\cdot\delta_0^3, \tag{7.41}$$

for all $j,k=1,\dots,n$. It remains to show that the cutoff terms are controllable. By employing energy methods as in Section 6.3, we can prove the estimates

$$\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\widehat{\eta}_i p_{i,x}^2\,dx\,dt=\mathcal{O}(1)\cdot\delta_0^2,\qquad \int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\widehat{\eta}_i q_{i,x}^2\,dx\,dt=\mathcal{O}(1)\cdot\delta_0^2 \tag{7.42}$$

Having the above bounds, it is easy to derive that the cutoff terms are bounded by controllable functions. For explicit calculations, see pp. 287–289 in [6]. Here, we just state the result.

$$\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\left(|p_{i,x}|+|q_{i,x}|+|p_iv_i|+|q_iv_i|\right)|w_i-\theta_iv_i|\,dx\,dt=\mathcal{O}(1)\cdot\delta_0^2. \tag{7.43}$$

Thus we have shown that all perturbed source terms in the parabolic system (7.22) are controllable.

Lemma 7.6. *Let z be a solution of* (7.4) *such that for $t\in[\hat{t},T]$, z satisfies*

$$\|z(t)\|\leqslant\delta_0e^{-\varepsilon\mu t}. \tag{7.44}$$

Suppose the source terms satisfy

$$\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\sum_{i=1}^{n}\left(|\widehat{\phi}_i(t,x)|+|\widehat{\psi}_i(t,x)|\right)dx\,dt\leqslant\delta_0. \tag{7.45}$$

Then we have the estimate

$$\int_{\hat{t}}^{T}\int e^{\varepsilon\mu t}\sum_{i=1}^{n}\left(|\widehat{\phi}_i(t,x)|+|\widehat{\psi}_i(t,x)|\right)dx\,dt=\mathcal{O}(1)\delta_0^2. \tag{7.46}$$

Proof. In view of the discussion before this lemma, estimates (7.33), (7.37), (7.39), (7.40), (7.41), (7.43) yield the result. □

Here, we conclude the proof of the uniform L^1 bound (7.5) of the infinitesimal perturbation z:

Proof of Lemma 7.1. Having

$$\|z(0)\|_{L^1} \leqslant \frac{\delta_0}{8\sqrt{n}\kappa}, \tag{7.47}$$

where κ is given by (2.3), the solution $z(t,x)$ to (7.4) exists on an initial time interval $[0,\hat{t}]$. This is established in Proposition 2.3. Moreover, the perturbation z satisfies the bound,

$$\|z(t)\|_{L^1} \leqslant \frac{\delta_0}{4\sqrt{n}} e^{-\varepsilon\mu t}, \quad t \in [0,\hat{t}]. \tag{7.48}$$

According to Proposition 2.2, this solution can be prolonged in time as long as the L^1 bound remains small. To extend bound (7.48) to all times, we argue by contradiction. Choose T to be the time

$$T \doteq \sup\left\{\tau\,;\ \sum_i \int_{\hat{t}}^{\tau} \int e^{\varepsilon\mu t}\left(|\widehat{\phi}_i(t,x)| + |\widehat{\psi}_i(t,x)|\right) dx\,dt \leqslant \frac{\delta_0}{2}\right\}. \tag{7.49}$$

Suppose $T < \infty$, then by Lemma 7.2, decomposition (7.12) holds for all $t \in [\hat{t}, T]$ and

$$\begin{aligned}
\|z(t)\|_{L^1} &\leqslant \sum_{i=1}^{n} \|p_i(t)\|_{L^1} \leqslant \sum_{i=1}^{n} e^{-\varepsilon\mu t}\left(e^{\varepsilon\mu\hat{t}}\|p_i(\hat{t})\|_{L^1} + \int_{\hat{t}}^{t}\int e^{\varepsilon\mu s}|\widehat{\phi}_i(s,x)|\,dx\,ds\right) \\
&\leqslant e^{-\varepsilon\mu(t-\hat{t})} 2\sqrt{n}\|z(\hat{t})\|_{L^1} + e^{-\varepsilon\mu t}\frac{\delta_0}{2} \leqslant \delta_0 e^{-\varepsilon\mu t}.
\end{aligned} \tag{7.50}$$

By (7.49) and (7.50), for sufficiently small δ_0, Lemma 7.6 implies

$$\sum_i \int_{\hat{t}}^{T} \int e^{\varepsilon\mu t}\left(|\widehat{\phi}_i(t,x)| + |\widehat{\psi}_i(t,x)|\right) dx\,dt = \mathcal{O}(1)\delta_0^2 < \frac{\delta_0}{2}, \tag{7.51}$$

which contradicts the choice of T in (7.49). This completes the proof. □

In the view of the above discussion, one has the stability estimate

$$\|u(t) - v(t)\| \leqslant L\, e^{-\varepsilon\mu t}\|u(0) - v(0)\|_{L^1}, \quad t > 0 \tag{7.52}$$

for any two solutions to (7.1).

8. Convergence of approximate solutions.

Up to this point, we proved that the solution $u(t,x)$ to

$$u_t + A(u)u_x + \varepsilon g(u) = u_{xx} \tag{8.1}$$

$$u(0,x) = \bar{u}(x) = u_0(\varepsilon x), \tag{8.2}$$

is globally defined in time, its total variation remains small according to

$$TVu(t) = C\, e^{-\varepsilon\mu t}\, TV\bar{u} = C\, e^{-\varepsilon\mu t}\, TVu_0\,, \tag{8.3}$$

and it is uniformly stable, i.e.

$$\|u(t) - v(t)\|_{L^1} \leqslant L e^{-\varepsilon\mu t} \|\bar{u} - \bar{v}\|_{L^1}. \tag{8.4}$$

An immediate implementation of the stability estimate is a uniform L^1 bound

$$\|u(t) - u^*\|_{L^1} \leqslant L\, e^{-\varepsilon\mu t} \|\bar{u} - u^*\|_{L^1} = L\, e^{-\varepsilon\mu t} \frac{1}{\varepsilon} \|u_0 - u^*\|_{L^1}. \tag{8.5}$$

We now focus on the continuous dependence of u in time. By Proposition 2.2, Corollary 2.4 and (8.3),

$$\|u_x(t)\|_{L^1} \leqslant C\delta_0 e^{-\varepsilon\mu t} \quad \text{for all} \quad t > 0, \qquad \|u_{xx}(t)\|_{L^1} \leqslant \begin{cases} \dfrac{2\kappa\delta_0}{\sqrt{t}} e^{-\varepsilon\mu t} & \text{for } t < \hat{t}, \\ \dfrac{2\kappa\delta_0}{\sqrt{\hat{t}}} e^{-\varepsilon\mu t} & \text{for } t \geqslant \hat{t}. \end{cases}$$

Hence the identity $u_t = u_{xx} - A(u)u_x - \varepsilon g(u)$ implies

$$\begin{aligned} \|u_t(t)\|_{L^1} &\leqslant \|u_{xx}(t)\|_{L^1} + \|A(u)u_x\|_{L^1} + \varepsilon \|g(u)\|_{L^1} \\ &\leqslant L' \left(1 + \frac{1}{2\sqrt{t}}\right) e^{-\varepsilon\mu t}, \end{aligned} \tag{8.6}$$

for an appropriate constant L'. For fixed times $t > s \geqslant 0$, by integrating the above estimate over $[s,t]$, we get

$$\begin{aligned} \|u(t) - u(s)\|_{L^1} &\leqslant \int_s^t \|u_t(\tau)\|_{L^1}\, d\tau \leqslant L' \int_s^t e^{-\varepsilon\mu\tau} \left(1 + \frac{1}{2\sqrt{\tau}}\right) d\tau \\ &\leqslant L' e^{-\varepsilon\mu s} \big(|t - s| + |\sqrt{t} - \sqrt{s}|\big). \end{aligned} \tag{8.7}$$

Our results so far refer to the solution of the parabolic system (8.1)–(8.2). However, in view of the discussion in Section 1, by rescaling the coordinates, our analysis can be extended to the vanishing viscosity approximations $u^\varepsilon = u^\varepsilon(t,x)$. Consider the viscous hyperbolic system of balance laws (1.8)–(1.9) and recall that

$$u^\varepsilon(t,x) = u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \tag{8.8}$$

where u satisfies (8.1)–(8.2). Hence the parabolic system (1.8)–(1.9) has a unique solution u^ε globally defined and for $u_0 - u^* \in L^1$ with $TV\{u_0\} < \delta_0$, then

$$TV\{u^\varepsilon(t)\} = TV\{u(t/\varepsilon)\} \leqslant C\, e^{-\mu t} TV\{u_0\}. \tag{8.9}$$

If v^ε is another solution of (1.8) with initial data $v_0 \in L^1$, then by (8.4), we obtain the stability of solutions to (1.8):

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1} = \varepsilon\|u(t) - v(t)\|_{L^1} \leqslant L\, e^{-\mu t}\|u_0 - v_0\|_{L^1}. \tag{8.10}$$

Finally, the continuous dependence with respect to time for solutions of (1.8) is expressed by

$$\begin{aligned} \|u^\varepsilon(t) - u^\varepsilon(s)\|_{L^1} &= \varepsilon\|u(t/\varepsilon) - u(s/\varepsilon)\|_{L^1} \\ &\leqslant \varepsilon L'\left(\left|\frac{t}{\varepsilon} - \frac{s}{\varepsilon}\right| + \left|\sqrt{\frac{t}{\varepsilon}} - \sqrt{\frac{s}{\varepsilon}}\right|\right) e^{-\mu s} \\ &= L'\left(|t-s| + \sqrt{\varepsilon}|\sqrt{t} - \sqrt{s}|\right) e^{-\mu s} \end{aligned} \tag{8.11}$$

for $t > s \geqslant 0$. By Helly's Compactness Theorem, a convergent sequence $\{u^{\varepsilon_m}\}_m$ may be extracted with $\varepsilon_m \downarrow 0$ as $m \to \infty$, whose limit is denoted by u, i.e.

$$u^{\varepsilon_m}(t) \longrightarrow u(t) \qquad \text{in } L^1_{\text{loc}} \tag{8.12}$$

for all $t > 0$. The limit $u(t,\cdot)$ is a BV function which satisfies

$$TV u(t) \leqslant \liminf_{m\to\infty} TV\{u^{\varepsilon_m}(t)\} \leqslant C\, e^{-\mu t} TV\{u_0\}. \tag{8.13}$$

Moreover, by construction, u is the admissible weak solution to

$$u_t + f(u)_x + g(u) = 0, \tag{8.14}$$

$$u(0,x) = u_0(x), \tag{8.15}$$

when (1.8) is in conservative form, i.e. $A(u) = Df(u)$. This completes the proof of Theorem 1.1. □

Appendix A. The evolution of the components (v, w) in the decomposition

This appendix presents the explicit calculations that derive the system of viscous balance laws (6.1) of (v, w). It further studies the various terms that appear in the component source (ϕ, ψ) of system (6.1) and by laborious work, demonstrates that ϕ and ψ are bounded by terms in the four categories given in (6.4).

To begin with, upon differentiating (1.13) with respect to x and t respectively, one obtains the evolution equations of u_x and u_t

$$\begin{aligned} &u_{xt} + (A(u)u_x)_x + \varepsilon(g(u))_x - (u_x)_{xx} = 0, \\ &u_{tt} + (A(u)u_t)_x + \varepsilon(g(u))_t - (u_t)_{xx} = (u_x \bullet A(u))u_t - (u_t \bullet A(u))u_x. \end{aligned} \tag{A.1}$$

System (6.1) arises by rewriting (A.1) via decomposition (5.1)

$$\begin{aligned} u_x &= \sum_{i=1}^{n} v_i \tilde{r}_i(u, v_i, \sigma_i), \\ u_t + \varepsilon g(u) &= \sum_{i=1}^{n} (w_i - \lambda_i^* v_i)\tilde{r}_i(u, v_i, \sigma_i). \end{aligned} \tag{A.2}$$

Differentiating (5.1) with respect to x and using the identity (4.3) yields

$$\begin{aligned} u_{xx} - A(u)u_x &= \sum_i v_{i,x}\tilde{r}_i + \sum_i v_i \tilde{r}_{i,x} - \sum_i A(u) v_i \tilde{r}_i \\ &= \sum_i v_{i,x}\tilde{r}_i + \sum_i v_i [v_i \tilde{r}_{i,u}\tilde{r}_i - A(u)\tilde{r}_i] \\ &\quad + \sum_i v_i \left[v_{i,x}\tilde{r}_{i,v} - \theta_i' \left(\frac{w_i}{v_i}\right)_x \tilde{r}_{i,\sigma} \right] + \sum_{i \neq j} v_i v_j \tilde{r}_{i,u}\tilde{r}_j, \\ &= \sum_i v_{i,x}\tilde{r}_i + \sum_i v_i \left[-\tilde{\lambda}_i \tilde{r}_i + \left(-\tilde{\lambda}_i + \lambda_i^* - \theta_i \right) v_i \tilde{r}_{i,v} \right] \\ &\quad + \sum_i v_i \left[v_{i,x}\tilde{r}_{i,v} - \theta_i' \left(\frac{v_i w_{i,x} - v_{i,x} w_i}{v_i^2} \right) \tilde{r}_{i,\sigma} \right] + \sum_{i \neq j} v_i v_j \tilde{r}_{i,u}\tilde{r}_j \\ &= \sum_i \left(v_{i,x} - \tilde{\lambda}_i v_i \right) \left[\tilde{r}_i + v_i \tilde{r}_{i,v} + \theta_i' \left(\frac{w_i}{v_i} \right) \tilde{r}_{i,\sigma} \right] \\ &\quad - \sum_i \left(w_{i,x} - \tilde{\lambda}_i w_i \right) \theta_i' \tilde{r}_{i,\sigma} + \sum_i v_i^2 (\lambda_i^* - \theta_i) \tilde{r}_{i,v} \\ &\quad + \sum_{i \neq j} v_i v_j \tilde{r}_{i,u}\tilde{r}_j, \end{aligned} \tag{A.3}$$

and

$$
\begin{aligned}
&u_{tx} - A(u)u_t \\
&= \left(\sum_i \left(w_i - \lambda_i^* v_i\right)\tilde{r}_i\right)_x - A(u)\sum_i (w_i - \lambda_i^* v_i)\tilde{r}_i \\
&\quad -\varepsilon Dg(u)u_x + \varepsilon A(u)g(u) \\
&= \sum_i \left(w_{i,x} - \lambda_i^* v_{i,x}\right)\tilde{r}_i + \sum_i \left(w_i - \lambda_i^* v_i\right)\left[\tilde{r}_{i,u}u_x + \tilde{r}_{i,v}v_{i,x} - \tilde{r}_{i,\sigma}\theta_{i,x} - A(u)\tilde{r}_i\right] \\
&\quad -\varepsilon Dg(u)u_x + \varepsilon A(u)g(u) \\
&= \sum_i \left[(w_{i,x} - \lambda_i^* v_{i,x})\tilde{r}_i + (w_i - \lambda_i^* v_i)(v_i\tilde{r}_{i,u}\tilde{r}_i - A(u)\tilde{r}_i)\right] \\
&\quad + \sum_{i\neq j}(w_i - \lambda_i^* v_i)v_j\tilde{r}_{i,u}\tilde{r}_j + \sum_i (w_i - \lambda_i^* v_i)\left[v_{i,x}\tilde{r}_{i,v} - \theta_i' \frac{w_{i,x}v_i - v_{i,x}w_i}{v_i^2}\tilde{r}_{i,\sigma}\right] \\
&\quad -\varepsilon Dg(u)u_x + \varepsilon A(u)g(u) \\
&= \sum_i \left(w_{i,x} - \lambda_i^* v_{i,x}\right)\tilde{r}_i + \sum_i \left(w_i - \lambda_i^* v_i\right)\left(-\tilde{\lambda}_i\tilde{r}_i + \left(-\tilde{\lambda}_i + \lambda_i^* - \theta_i\right)v_i\tilde{r}_{i,v}\right) \\
&\quad + \sum_{i\neq j}\left(w_i - \lambda_i^* v_i\right)v_j\tilde{r}_{i,u}\tilde{r}_j + \sum_i \left(w_i - \lambda_i^* v_i\right)\left[v_{i,x}\tilde{r}_{i,v} - \theta_i' \frac{w_{i,x}v_i - v_{i,x}w_i}{v_i^2}\tilde{r}_{i,\sigma}\right] \\
&\quad -\varepsilon Dg(u)u_x + \varepsilon A(u)g(u) \\
&= \sum_i \left(w_{i,x} - \tilde{\lambda}_i w_i\right)\left[\tilde{r}_i - \theta_i' \frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right] + \sum_i \left(v_{i,x} - \tilde{\lambda}_i v_i\right)\left[w_i\tilde{r}_{i,v} + \theta_i' \frac{w_i^2}{v_i^2}\tilde{r}_{i,\sigma}\right] \\
&\quad + \sum_i w_i v_i \left(\lambda_i^* - \theta_i\right)\tilde{r}_{i,v} + \sum_{i\neq j} w_i v_j \tilde{r}_{i,u}\tilde{r}_j \\
&\quad - \sum_i \lambda_i^* \left\{ \left(v_{i,x} - \tilde{\lambda}_i v_i\right)\left[\tilde{r}_i + v_i\tilde{r}_{i,v} + \theta_i' \frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right] + \left(w_{i,x} - \tilde{\lambda}_i w_i\right)\left(-\theta_i'\tilde{r}_{i,\sigma}\right) \right. \\
&\quad \left. + v_i^2(\lambda_i^* - \theta_i)\tilde{r}_{i,v} + \sum_{i\neq j} v_i v_j \tilde{r}_{i,u}\tilde{r}_j \right\} - \varepsilon Dg(u)u_x + \varepsilon A(u)g(u).
\end{aligned}
\tag{A.4}
$$

In addition, by differentiating (5.1) with respect to t, one gets

$$\begin{aligned} u_{xt} &= \sum_i v_{i,t}\tilde{r}_i + \sum_i v_i \left(\tilde{r}_{i,u}u_t + \tilde{r}_{i,v}v_{i,t} - \tilde{r}_{i,\sigma}\theta_i' \frac{w_{i,t}v_i - v_{i,t}w_i}{v_i^2} \right) \\ &= \sum_i v_{i,t}\left[\tilde{r}_i + v_i\tilde{r}_{i,v} + \theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right] + \sum_{i,j} v_i\left(w_j - \lambda_j^* v_j\right)\tilde{r}_{i,u}\tilde{r}_j \\ &\quad + \sum_i w_{i,t}[-\theta_i'\tilde{r}_{i,\sigma}] - \varepsilon\sum_i v_i\tilde{r}_{i,u}g(u), \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} u_{tt} &= \sum_i \left(w_{i,t} - \lambda_i^* v_{i,t}\right)\tilde{r}_i + \sum_i \left(w_i - \lambda_i^* v_i\right)\tilde{r}_{i,t} - \varepsilon Dg(u)u_t \\ &= \sum_i \left(w_{i,t} - \lambda_i^* v_{i,t}\right)\tilde{r}_i + \sum_i \left(w_i - \lambda_i^* v_i\right)\left[\tilde{r}_{i,u}u_t + \tilde{r}_{i,v}v_{i,t} - \theta_i'\frac{w_{i,t}v_i - v_{i,t}w_i}{v_i^2}\tilde{r}_{i,\sigma}\right] \\ &\quad - \varepsilon Dg(u)u_t \\ &= \sum_i v_{i,t}\left[w_i\tilde{r}_{i,v} + \theta_i'\left(\frac{w_i}{v_i}\right)^2\tilde{r}_{i,\sigma}\right] + \sum_i w_{i,t}\left[\tilde{r}_i - \theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right] \\ &\quad - \sum_i \lambda_i^*\left\{v_{i,t}\left[\tilde{r}_i + v_i\tilde{r}_{i,v} + \theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right] - \theta_i' w_{i,t}\tilde{r}_{i,\sigma} + \sum_j v_i\left(w_j - \lambda_j^* v_j\right)\tilde{r}_{i,u}\tilde{r}_j\right\} \\ &\quad + \sum_{i,j} w_i\left(w_j - \lambda_j^* v_j\right)\tilde{r}_{i,u}\tilde{r}_j - \varepsilon\sum_i\left(w_i - \lambda_i^* v_i\right)\tilde{r}_{i,u}g(u) - \varepsilon Dg(u)u_t. \end{aligned} \tag{A.6}$$

Once more, taking the derivative of $u_{xx} - A(u)u_x$ and $u_{tx} - A(u)u_t$, which are given in (A.3) and (A.4), with respect to x, it follows that

$$\begin{aligned} &(u_x)_{xx} - (A(u)u_x)_x \\ &\quad = \sum_i \left(v_{i,xx} - \left(\tilde{\lambda}_i v_i\right)_x\right)\left[\tilde{r}_i + v_i\tilde{r}_{i,v} + \theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right] \\ &\qquad + \sum_i\left(w_{i,xx} - \left(\tilde{\lambda}_i w_i\right)_x\right)[-\theta_i'\tilde{r}_{i,\sigma}] + \sum_i\left(v_{i,x} - \tilde{\lambda}_i v_i\right)\left[\sum_j v_j\tilde{r}_{i,u}\tilde{r}_j + 2v_{i,x}\tilde{r}_{i,v}\right. \\ &\qquad + \left(-\theta_{i,x} + \left(\theta_i'\frac{w_i}{v_i}\right)_x\right)\tilde{r}_{i,\sigma} + \sum_j v_i v_j\tilde{r}_{i,vu}\tilde{r}_j + v_i v_{i,x}\tilde{r}_{i,vv} \end{aligned}$$

$$+\left(-v_i\theta_{i,x}+\theta_i'\frac{w_i}{v_i}v_{i,x}\right)\tilde{r}_{i,v\sigma}+\sum_j v_j\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma u}\tilde{r}_j-\theta_{i,x}\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma\sigma}\Bigg]$$

$$+\sum_i\left(w_{i,x}-\tilde{\lambda}_i w_i\right)\left[-\theta_{i,x}'\tilde{r}_{i,\sigma}-\sum_j\theta_i'v_j\tilde{r}_{i,\sigma u}\tilde{r}_j-\theta_i'v_{i,x}\tilde{r}_{i,\sigma v}+\theta_i'\theta_{i,x}\tilde{r}_{i,\sigma\sigma}\right]$$

$$+\sum_i v_i^2\left(\lambda_i^*-\theta_i\right)\left[\sum_j v_j\tilde{r}_{i,vu}\tilde{r}_j+v_{i,x}\tilde{r}_{i,vv}-\theta_{i,x}\tilde{r}_{i,v\sigma}\right]+\sum_i\left(v_i^2(\lambda_i^*-\theta_i)\right)_x\tilde{r}_{i,v}$$

$$+\sum_{i\neq j}v_iv_j\Bigg[\sum_k v_k\left(\tilde{r}_{i,uu}\left(\tilde{r}_j\otimes\tilde{r}_k\right)+\tilde{r}_{i,u}\tilde{r}_{j,u}\tilde{r}_k\right)+v_{i,x}\tilde{r}_{i,uv}\tilde{r}_j+v_{j,x}\tilde{r}_{i,u}\tilde{r}_{j,v}$$

$$-\theta_{i,x}\tilde{r}_{i,u\sigma}\tilde{r}_j-\theta_{j,x}\tilde{r}_{i,u}\tilde{r}_{j,\sigma}\Bigg]+\sum_{i\neq j}\left(v_iv_j\right)_x\tilde{r}_{i,u}\tilde{r}_j, \tag{A.7}$$

and

$$(u_t)_{xx}-(A(u)u_t)_x$$

$$=\sum_i\left(v_{i,xx}-\left(\tilde{\lambda}_i v_i\right)_x\right)\left[w_i\tilde{r}_{i,v}+\theta_i'\frac{w_i^2}{v_i^2}\tilde{r}_{i,\sigma}\right]$$

$$+\sum_i\left(w_{i,xx}-\left(\tilde{\lambda}_i w_i\right)_x\right)\left[\tilde{r}_i-\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right]$$

$$+\sum_i\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\Bigg[w_{i,x}\tilde{r}_{i,v}+\sum_j w_iv_j\tilde{r}_{i,vu}\tilde{r}_j$$

$$+w_iv_{i,x}\tilde{r}_{i,vv}+\left(-w_i\theta_{i,x}+\theta_i'\left(\frac{w_i}{v_i}\right)^2v_{i,x}\right)\tilde{r}_{i,v\sigma}$$

$$+\sum_j\theta_i'\left(\frac{w_i}{v_i}\right)^2v_j\tilde{r}_{i,\sigma u}\tilde{r}_j-\theta_i'\left(\frac{w_i}{v_i}\right)^2\theta_{i,x}\tilde{r}_{i,\sigma\sigma}+\left(\theta_i'\left(\frac{w_i}{v_i}\right)^2\right)_x\tilde{r}_{i,\sigma}\Bigg]$$

$$+\sum_i(w_{i,x}-\tilde{\lambda}_i w_i)\Bigg[\sum_j v_j\tilde{r}_{i,u}\tilde{r}_j+v_{i,x}\tilde{r}_{i,v}-\left(\theta_{i,x}+\left(\theta_i'\frac{w_i}{v_i}\right)_x\right)\tilde{r}_{i,\sigma}$$

$$-\sum_j v_j\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma u}\tilde{r}_j-v_{i,x}\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma v}+\theta_{i,x}\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma\sigma}\Bigg]$$

$$
\begin{aligned}
&+\sum_i v_i w_i\left(\lambda_i^*-\theta_i\right)\left[\sum_j v_j \tilde{r}_{i,vu}\tilde{r}_j+v_{i,x}\tilde{r}_{i,vv}-\theta_{i,x}\tilde{r}_{i,v\sigma}\right]\\
&+\sum_i\left(v_i w_i\left(\lambda_i^*-\theta_i\right)\right)_x\tilde{r}_{i,v}+\sum_{i\neq j}(w_i v_j)_x\tilde{r}_{i,u}\tilde{r}_j\\
&+\sum_{i\neq j} w_i v_j\left[\sum_k v_k\left(\tilde{r}_{i,uu}\left(\tilde{r}_j\otimes\tilde{r}_k\right)+\tilde{r}_{i,u}\tilde{r}_{j,u}\tilde{r}_k\right)+v_{i,x}\tilde{r}_{i,uv}\tilde{r}_j\right.\\
&\qquad\qquad\left.+v_{j,x}\tilde{r}_{i,u}\tilde{r}_{j,v}-\theta_{i,x}\tilde{r}_{i,u\sigma}\tilde{r}_j-\theta_{j,x}\tilde{r}_{i,u}\tilde{r}_{j,\sigma}\right]\\
&-\sum_i\lambda_i^*\left\{\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\left[\tilde{r}_i+v_i\tilde{r}_{i,v}+\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right]\right.\\
&\qquad\qquad\left.+\left(w_{i,x}-\tilde{\lambda}_i w_i\right)\left(-\theta_i'\tilde{r}_{i,\sigma}\right)+v_i^2\left(\lambda_i^*-\theta_i\right)\tilde{r}_{i,v}+\sum_{i\neq j}v_i v_j\tilde{r}_{i,u}\tilde{r}_j\right\}_x\\
&-\varepsilon(Dg(u)u_x)_x+\varepsilon(u_x\bullet A(u))g(u)+\varepsilon A(u)Dg(u)u_x. \qquad\qquad \text{(A.8)}
\end{aligned}
$$

Finally, combining the above and by account of (A.1), we end up with two vector equations that describe the evolution of v and w:

$$
\begin{aligned}
&\sum_i\left(v_{i,t}+\left(\tilde{\lambda}_i v_i\right)_x-v_{i,xx}\right)\left[\tilde{r}_i+v_i\tilde{r}_{i,v}+\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right]\\
&\quad+\sum_i\left(w_{i,t}+\left(\tilde{\lambda}_i w_i\right)_x-w_{i,xx}\right)\left[-\theta_i'\tilde{r}_{i,\sigma}\right]\\
&\quad+\varepsilon\sum_i v_i\left[Dg(u)\tilde{r}_i-g(u)\bullet\tilde{r}_i\right]=\sum_i\alpha_i, \qquad\qquad \text{(A.9)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_i\left(v_{i,t}+\left(\tilde{\lambda}_i v_i\right)_x-v_{i,xx}\right)\left[w_i\tilde{r}_{i,v}+\theta_i'\left(\frac{w_i}{v_i}\right)^2\tilde{r}_{i,\sigma}\right]\\
&\quad+\sum_i\left(w_{i,t}+\left(\tilde{\lambda}_i w_i\right)_x-w_{i,xx}\right)\left[\tilde{r}_i-\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right]\\
&\quad-\sum_i\lambda_i^*\left\{\left(v_{i,t}+\left(\tilde{\lambda}_i v_i\right)_x-v_{i,xx}\right)\left[\tilde{r}_i+v_i\tilde{r}_{i,v}+\theta_i'\frac{w_i}{v_i}\tilde{r}_{i,\sigma}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\left(w_{i,t}+\left(\tilde{\lambda}_i w_i\right)_x - w_{i,xx}\right)\left[\theta_i'\tilde{r}_{i,\sigma}\right]\Bigg\}\\
&+\varepsilon\sum_i (w_i-\lambda_i^* v_i)\left(Dg(u)\tilde{r}_i-\tilde{r}_{i,u}g(u)\right)\\
&+\varepsilon\sum_i v_i\Big(Dg(u)A(u)-A(u)Dg(u)-g(u)\bullet A(u)+v_i\tilde{r}_i\bullet Dg(u)\Big)\tilde{r}_i\\
&\quad=\sum_i \beta_i-\lambda_i^*\alpha_i,
\end{aligned}
\tag{A.10}
$$

with

$$
\begin{aligned}
\sum_i \alpha_i \;\doteq\; & \sum_i \tilde{r}_{i,u}\tilde{r}_i\left[v_i\left(v_{i,x}-\tilde{\lambda}_i v_i\right)-v_i\left(w_i-\lambda_i^* v_i\right)\right]\\
&+\sum_{i\neq j}\tilde{r}_{i,u}\tilde{r}_j\left[v_j\left(v_{i,x}-\tilde{\lambda}_i v_i\right)-v_i\left(w_j-\lambda_j^* v_j\right)+(v_i v_j)_x\right]\\
&+\sum_i \tilde{r}_{i,v}\left[2v_{i,x}\left(v_{i,x}-\tilde{\lambda}_i v_i\right)+\left(v_i^2\left(\lambda_i^*-\theta_i\right)\right)_x\right]\\
&+\sum_i \tilde{r}_{i,\sigma}\left[\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\left(-\theta_{i,x}+\left(\theta_i'\frac{w_i}{v_i}\right)_x\right)-\theta_{i,x}'\left(w_{i,x}-\tilde{\lambda}_i w_i\right)\right]\\
&+\sum_i \tilde{r}_{i,vu}\tilde{r}_i\left[v_i^3\left(\lambda_i^*-\theta_i\right)+v_i^2\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\right]\\
&+\sum_{i\neq j}\tilde{r}_{i,vu}\tilde{r}_j\left[v_i v_j\left(v_{i,x}-\tilde{\lambda}_i v_i\right)+v_i^2 v_j\left(\lambda_i^*-\theta_i\right)\right]\\
&+\sum_i \tilde{r}_{i,vv}\left[v_i v_{i,x}\left(v_{i,x}-\tilde{\lambda}_i v_i\right)+v_{i,x}v_i^2\left(\lambda_i^*-\theta_i\right)\right]\\
&+\sum_i \tilde{r}_{i,v\sigma}\Bigg[\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\left(-v_i\theta_{i,x}+\theta_i'\frac{w_i}{v_i}v_{i,x}\right)-\theta_i' v_{i,x}\left(w_{i,x}-\tilde{\lambda}_i w_i\right)\\
&\qquad\qquad -\;v_i^2\theta_{i,x}(\lambda_i^*-\theta_i)\Bigg]\\
&+\sum_i \tilde{r}_{i,\sigma u}\tilde{r}_i\left[\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\theta_i' w_i-\theta_i' v_i\left(w_{i,x}-\tilde{\lambda}_i w_i\right)\right]\\
&+\sum_{i\neq j}\tilde{r}_{i,\sigma u}\tilde{r}_j\left[v_j\theta_i'\frac{w_i}{v_i}\left(v_{i,x}-\tilde{\lambda}_i v_i\right)-\theta_i' v_j\left(w_{i,x}-\tilde{\lambda}_i w_i\right)\right]
\end{aligned}
$$

$$+\sum_i \tilde{r}_{i,\sigma\sigma}\left[-\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\theta_{i,x}\theta_i'\frac{w_i}{v_i}+\theta_i'\theta_{i,x}\left(w_{i,x}-\tilde{\lambda}_i w_i\right)\right]$$

$$+\sum_{i\neq j} v_i v_j\left[\sum_k v_k\left(\tilde{r}_{i,uu}(\tilde{r}_j\otimes\tilde{r}_k)+\tilde{r}_{i,u}\tilde{r}_{j,u}\tilde{r}_k\right)+v_{j,x}\tilde{r}_{i,u}\tilde{r}_{j,v}-\theta_{j,x}\tilde{r}_{i,u}\tilde{r}_{j,\sigma}\right.$$
$$\left.+v_{i,x}\tilde{r}_{i,uv}\tilde{r}_j-\theta_{i,x}\tilde{r}_{i,u\sigma}\tilde{r}_j\right], \tag{A.11}$$

and

$$\sum_i \beta_i-\lambda_i^*\alpha_i$$
$$\doteq \sum_i \tilde{r}_{i,u}\tilde{r}_i\left[\left(w_{i,x}-\tilde{\lambda}_i w_i\right)v_i-w_i\left(w_i-\lambda_i^* v_i\right)\right]$$
$$+\sum_{i\neq j}\tilde{r}_{i,u}\tilde{r}_j\left[\left(w_{i,x}-\tilde{\lambda}_i w_i\right)v_j-w_i\left(w_j-\lambda_j^* v_j\right)+(w_i v_j)_x\right]$$
$$+\sum_i \tilde{r}_{i,v}\left[\left(v_{i,x}-\tilde{\lambda}_i v_i\right)w_{i,x}+v_{i,x}\left(w_{i,x}-\tilde{\lambda}_i w_i\right)+\left(v_i w_i(\lambda_i^*-\theta_i)\right)_x\right]$$
$$+\sum_i \tilde{r}_{i,\sigma}\left[\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\left(\theta_i'\left(\frac{w_i}{v_i}\right)^2\right)_x-\left(w_{i,x}-\tilde{\lambda}_i w_i\right)\left(\theta_{i,x}+\left(\theta_i'\frac{w_i}{v_i}\right)_x\right)\right]$$
$$+\sum_i \tilde{r}_{i,vu}\tilde{r}_i\left[w_i v_i\left(v_{i,x}-\tilde{\lambda}_i v_i\right)+w_i v_i^2\left(\lambda_i^*-\theta_i\right)\right]$$
$$+\sum_{i\neq j}\tilde{r}_{i,vu}\tilde{r}_j\left[\left(v_{i,x}-\tilde{\lambda}_i v_i\right)w_i v_j+w_i v_i v_j\left(\lambda_i^*-\theta_i\right)\right]$$
$$+\sum_i \tilde{r}_{i,vv}\left[w_i v_{i,x}\left(v_{i,x}-\tilde{\lambda}_i v_i\right)+w_i v_i v_{i,x}\left(\lambda_i^*-\theta_i\right)\right]$$
$$+\sum_i \tilde{r}_{i,v\sigma}\left[\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\left(-w_i\theta_{i,x}+\theta_i'\left(\frac{w_i}{v_i}\right)^2 v_{i,x}\right)\right.$$
$$\left.-\left(w_{i,x}-\tilde{\lambda}_i w_i\right)v_{i,x}\theta_i'\frac{w_i}{v_i}-w_i v_i(\lambda_i^*-\theta_i)\theta_{i,x}\right]$$
$$+\sum_i \tilde{r}_{i,\sigma u}\tilde{r}_i\left[\left(v_{i,x}-\tilde{\lambda}_i v_i\right)\theta_i'\left(\frac{w_i}{v_i}\right)^2 v_i-\left(w_{i,x}-\tilde{\lambda}_i w_i\right)\theta_i' w_i\right]$$

$$
\begin{aligned}
&+\sum_{i\neq j}\tilde r_{i,\sigma u}\tilde r_j\left[\left(v_{i,x}-\tilde\lambda_i v_i\right)\theta_i'\left(\frac{w_i}{v_i}\right)^2 v_j-\left(w_{i,x}-\tilde\lambda_i w_i\right)\theta_i'\frac{w_i}{v_i}v_j\right]\\
&+\sum_{i}\tilde r_{i,\sigma\sigma}\left[-\left(v_{i,x}-\tilde\lambda_i v_i\right)\theta_i'\left(\frac{w_i}{v_i}\right)^2 \theta_{i,x}+\left(w_{i,x}-\tilde\lambda_i w_i\right)\theta_{i,x}\theta_i'\frac{w_i}{v_i}\right]\\
&+\sum_{i\neq j}w_i v_j\left[\sum_k v_k\left(\tilde r_{i,uu}(\tilde r_j\otimes\tilde r_k)+\tilde r_{i,u}\tilde r_{j,u}\tilde r_k\right)+v_{i,x}\tilde r_{i,uv}\tilde r_j+v_{j,x}\tilde r_{i,u}\tilde r_{j,v}\right.\\
&\qquad\qquad\left.-\theta_{i,x}\tilde r_{i,u\sigma}\tilde r_j-\theta_{j,x}\tilde r_{i,u}\tilde r_{j,\sigma}\right]\\
&+\sum_{i\neq j}(w_i-\lambda_i^* v_i)v_j\left[(\tilde r_j\bullet A(u))\tilde r_i-(\tilde r_i\bullet A(u))\tilde r_j\right]\\
&-\sum_i\lambda_i^*\left\{\left(v_{i,x}-\tilde\lambda_i v_i\right)\left[\tilde r_i+v_i\tilde r_{i,v}+\theta_i'\frac{w_i}{v_i}\tilde r_{i,\sigma}\right]_x+\left(w_{i,x}-\tilde\lambda_i w_i\right)\left[-\theta_i'\tilde r_{i,\sigma}\right]_x\right.\\
&\left.+\left[v_i^2(\lambda_i^*-\theta_i)\tilde r_{i,v}+\sum_{j\neq i}v_i v_j\tilde r_{i,u}\tilde r_j\right]_x-\sum_j v_i\left(w_j-\lambda_j^* v_j\right)\tilde r_{i,u}\tilde r_j\right\}\\
&-\varepsilon\sum_{j\neq i}v_i(\tilde r_i\bullet Dg(u))v_j\tilde r_j.
\end{aligned}\tag{A.12}
$$

System (A.9)–(A.10) can be interpreted as a $2n\times 2n$ system in vector form

$$
\left(\frac{\partial\mathcal G}{\partial(v,w)}\right)
\begin{pmatrix}
v_{1,t}+\left(\tilde\lambda_1 v_1\right)_x-v_{1,xx}\\
v_{2,t}+\left(\tilde\lambda_2 v_2\right)_x-v_{2,xx}\\
\vdots\\
v_{n,t}+\left(\tilde\lambda_n v_n\right)_x-v_{n,xx}\\
w_{1,t}+\left(\tilde\lambda_1 w_1\right)_x-w_{1,xx}\\
\vdots\\
w_{n,t}+\left(\tilde\lambda_n w_n\right)_x-w_{n,xx}
\end{pmatrix}
+\varepsilon\mathcal D
\begin{pmatrix}
v_1\\ v_2\\ \vdots\\ w_1\\ \vdots\\ w_n
\end{pmatrix}
=
\begin{pmatrix}
\sum_i\alpha_i\\
\sum_i\beta_i
\end{pmatrix},
\tag{A.13}
$$

with

$$
\mathcal D\doteq\begin{pmatrix}
[g(u),\tilde r_1] & \dots & [g(u),\tilde r_n] & 0 & \dots & 0\\
D_{21} & \dots & D_{2n} & [g(u),\tilde r_1] & \dots & [g(u),\tilde r_n]
\end{pmatrix}
\tag{A.14}
$$

$$
\begin{aligned}
[g(u), \tilde{r}_j] &\doteq Dg(u)\tilde{r}_j - \tilde{r}_{j,u} g(u) && \text{(A.15)}\\
D_{2j} &\doteq \left[Dg(u)A(u) - g(u) \bullet A(u) + v_j(\tilde{r}_j \bullet Dg(u)) - A(u)Dg(u)\right]\tilde{r}_j \\
&\quad -\lambda_j^* \left[Dg(u)\tilde{r}_j - \tilde{r}_{j,u} g(u)\right]. && \text{(A.16)}
\end{aligned}
$$

$j = 1, \ldots, n$, while the map $\mathcal{G}$ is defined in (5.5). In Lemma 5.1, it is shown that

$$
\frac{\partial \mathcal{G}}{\partial(v, w)} = \mathcal{M}_0 + \mathcal{M}_1 \to \mathcal{M}_0, \qquad \text{as } (v, w) \to (0, 0).
$$

It is easily verified that

$$
\left(\frac{\partial \mathcal{G}}{\partial(v, w)}\right)^{-1} \mathcal{D} \to \begin{pmatrix} R(u^*)^{-1} Dg(u^*) R(u^*) & 0 \\ 0 & R(u^*)^{-1} Dg(u^*) R(u^*) \end{pmatrix}
$$

as $(u, v, w) \to (u^*, 0, 0)$, where $R(u)$ is the matrix of right eigenvectors of $A(u)$. Since $\left(\frac{\partial \mathcal{G}}{\partial(v,w)}\right)^{-1}$ is uniformly bounded, system (A.13) takes form

$$
\begin{pmatrix} v_t + (\Lambda v)_x - v_{xx} \\ w_t + (\Lambda w)_x - w_{xx} \end{pmatrix} + \varepsilon \begin{pmatrix} B^\sharp & H \\ K & B^\flat \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \tag{A.17}
$$

where $\Lambda = diag\{\tilde{\lambda}_i\}$ and $\Phi = \mathcal{O}(1) \sum_i \alpha_i$ and $\Psi = \mathcal{O}(1) \sum_i \beta_i$. In addition, the matrix $\begin{pmatrix} B^\sharp & H \\ K & B^\flat \end{pmatrix}$ that induces the coupling in (A.17) is strictly diagonally dominant. More precisely, we have

$$
B^\sharp_{ij} - B^\flat_{ij} = \mathcal{O}(|u - u^*|, |v|, |w|), \qquad H_{ij},\ K_{ij} = \mathcal{O}(|u - u^*|, |v|, |w|), \tag{A.18}
$$

and if we regard these matrices as functions of (u, v, w), then

$$
B^\sharp(u^*, 0, 0) = B^\flat(u^*, 0, 0) = B(u^*) = R(u^*)^{-1} Dg(u^*) R(u^*).
$$

It remains to investigate the various component source terms that appear in the force of (A.17), given explicitly in (A.11) and (A.12). We observe that all terms in Φ and Ψ that involve a product of distinct components belong to the category of transversal terms. To treat the remaining component source terms, we should first note the following facts: We infer from (6.58) that

$$
\left(|v_i| + |w_i| + |v_{i,x}| + |w_{i,x}|\right) |v_{i,x} - \left(\tilde{\lambda}_i - \lambda_i^*\right) v_i - w_i| = \text{transversal terms}.
$$

In addition, if $|w_i| > \delta_1 |v_i|$, then by (6.54) one gets

$$v_i = \mathcal{O}(1) v_{i,x} + \mathcal{O}(1)\delta_0 \sum_{j \neq i} \left(|v_j| + |w_j| + |v_{j,x}|\right). \tag{A.19}$$

Moreover, from (4.5) and (4.6), it follows

$$\tilde{r}_{i,\sigma},\ \tilde{r}_{i,\sigma\sigma},\ \tilde{r}_{i,\sigma u} = \mathcal{O}(1) v_i. \tag{A.20}$$

Now we list all terms that appear in α_i and β_i and show, one by one, that are controllable by account of the above facts.
Coefficients of $\tilde{r}_{i,u}\tilde{r}_i$:

- $v_i\left(v_{i,x} - \tilde{\lambda}_i v_i\right) - v_i\left(w_i - \lambda_i^* v_i\right) = v_i\left[v_{i,x} - \left(\tilde{\lambda}_i - \lambda_i^*\right) v_i - w_i\right],$

- $$v_i\left(w_{i,x} - \tilde{\lambda}_i w_i\right) - w_i\left(w_i - \lambda_i^* v_i\right) = \left[v_i w_{i,x} - v_{i,x} w_i\right] + w_i\left[v_{i,x} - \left(\tilde{\lambda}_i - \lambda_i^*\right) v_i - w_i\right].$$

Coefficients of $\tilde{r}_{i,v}$:

- $$2v_{i,x}\left(v_{i,x} - \tilde{\lambda}_i v_i\right) + \left(v_i^2\left(\lambda_i^* - \theta_i\right)\right)_x = 2v_{i,x}\left[v_{i,x} - \left(\tilde{\lambda}_i - \lambda_i^*\right) v_i - w_i\right] + 2v_{i,x}[w_i - \theta_i v_i] + \theta_i'[v_{i,x} w_i - v_i w_{i,x}],$$

- $$\begin{aligned} & w_{i,x}\left(v_{i,x} - \tilde{\lambda}_i v_i\right) + v_{i,x}\left(w_{i,x} - \tilde{\lambda}_i w_i\right) + \left(w_i v_i\left(\lambda_i^* - \theta_i\right)\right)_x \\ &\quad = 2w_{i,x}[w_i - \theta_i v_i] + 2w_{i,x}\left[v_{i,x} - \left(\tilde{\lambda}_i - \lambda_i^*\right) v_i - w_i\right] \\ &\qquad + \left(\lambda_i^* - \theta_i - \tilde{\lambda}_i + \theta_i' \frac{w_i}{v_i}\right)[v_{i,x} w_i - w_{i,x} v_i]. \end{aligned}$$

Coefficients of $\tilde{r}_{i,\sigma}/v_i$:

- $$\begin{aligned} & v_i\left(v_{i,x} - \tilde{\lambda}_i v_i\right)\left(-\theta_{i,x} + \left(\theta_i' \frac{w_i}{v_i}\right)_x\right) - v_i\left(w_{i,x} - \tilde{\lambda}_i w_i\right)\theta_{i,x}' \\ &\quad = -(v_i w_{i,x} - v_{i,x} w_i)\theta_i'' \left(\frac{w_i}{v_i}\right)_x \\ &\quad = -\theta_i'' \left[v_i \left(\frac{w_i}{v_i}\right)_x\right]^2, \end{aligned}$$

- $$v_i\left(v_{i,x}-\tilde\lambda_i v_i\right)\left(\theta_i'\left(\frac{w_i}{v_i}\right)^2\right)_x - v_i\left(w_{i,x}-\tilde\lambda_i w_i\right)\left(\theta_{i,x}+\left(\theta_i'\frac{w_i}{v_i}\right)_x\right)$$
$$= \theta_i'' v_i\left(v_{i,x}-\tilde\lambda_i v_i\right)\left(\frac{w_i}{v_i}\right)^2\left(\frac{w_i}{v_i}\right)_x + \theta_i' v_i\left(v_{i,x}-\tilde\lambda_i v_i\right)2\left(\frac{w_i}{v_i}\right)\left(\frac{w_i}{v_i}\right)_x$$
$$-v_i\left(w_{i,x}-\tilde\lambda_i w_i\right)\theta_i'\left(\frac{w_i}{v_i}\right)_x - v_i\left(w_{i,x}-\tilde\lambda_i w_i\right)\theta_i''\left(\frac{w_i}{v_i}\right)\left(\frac{w_i}{v_i}\right)_x$$
$$-v_i\left(w_{i,x}-\tilde\lambda_i w_i\right)\theta_i'\left(\frac{w_i}{v_i}\right)_x$$
$$= -\left(\theta_i''\left(\frac{w_i}{v_i}\right)+2\theta_i'\right)\left[v_i\left(\frac{w_i}{v_i}\right)_x\right]^2.$$

Coefficients of $\tilde r_{i,vu}\tilde r_i$:

- $$v_i^2\left(v_{i,x}-\tilde\lambda_i v_i\right)+v_i^3(\lambda_i^*-\theta_i) = v_i^2\left[v_{i,x}-\left(\tilde\lambda_i-\lambda_i^*\right)v_i-w_i\right]+v_i^2[w_i-\theta_i v_i],$$

- $$v_i w_i\left(v_{i,x}-\tilde\lambda_i v_i\right)+v_i^2 w_i(\lambda_i^*-\theta_i) = v_i w_i\left[v_{i,x}-\left(\tilde\lambda_i-\lambda_i^*\right)v_i-w_i\right]$$
$$+v_i w_i[w_i-v_i\theta_i].$$

Coefficients of $\tilde r_{i,vv}$:

- $$v_i v_{i,x}\left(v_{i,x}-\tilde\lambda_i v_i\right)+v_{i,x}v_i^2(\lambda_i^*-\theta_i) = v_i v_{i,x}\left[v_{i,x}-\left(\tilde\lambda_i-\lambda_i^*\right)v_i-w_i\right]$$
$$+v_i v_{i,x}[w_i-\theta_i v_i],$$
- $$w_i v_{i,x}\left(v_{i,x}-\tilde\lambda_i v_i\right)+v_{i,x}v_i w_i(\lambda_i^*-\theta_i) = w_i v_{i,x}\left[v_{i,x}-\left(\tilde\lambda_i-\lambda_i^*\right)v_i-w_i\right]$$
$$+w_i v_{i,x}[w_i-\theta_i v_i].$$

Coefficients of $\tilde r_{i,v\sigma}$:

- $$\left(v_{i,x}-\tilde\lambda_i v_i\right)\left(-v_i\theta_{i,x}+\theta_i' v_{i,x}\frac{w_i}{v_i}\right)-\left(w_{i,x}-\tilde\lambda_i w_i\right)\theta_i' v_{i,x}-v_i^2(\lambda_i^*-\theta_i)\theta_{i,x}$$
$$= v_{i,x}\theta_i'\left(\frac{v_{i,x}w_i-v_i w_{i,x}}{v_i}\right)-\theta_{i,x}v_i\left(v_{i,x}-\left(\tilde\lambda_i-\lambda_i^*\right)v_i-\theta_i v_i\right)$$

$$= 2\theta_i'\left(v_{i,x}\frac{w_i}{v_i} - w_{i,x}\right)\left(\left[v_{i,x} - \left(\tilde{\lambda}_i - \lambda_i^*\right)v_i - w_i\right] + [w_i - \theta_i v_i]\right)$$
$$+\left(\tilde{\lambda}_i - \lambda_i^* + \theta_i\right)\theta_i'[v_{i,x}w_i - v_i w_{i,x}],$$

- $$\left(v_{i,x} - \tilde{\lambda}_i v_i\right)\left(-w_i\theta_{i,x} + \theta_i'\left(\frac{w_i}{v_i}\right)^2 v_{i,x}\right) - \left(w_{i,x} - \tilde{\lambda}_i w_i\right)\theta_i' v_{i,x}\frac{w_i}{v_i}$$
$$-v_i w_i(\lambda_i^* - \theta_i)\theta_{i,x}$$
$$= \theta_i'\left(\frac{w_i}{v_i}\frac{w_i}{v_i}v_{i,x}^2 - \tilde{\lambda}_i w_i\frac{w_i}{v_i}v_{i,x}\right) - \theta_i' w_i v_{i,x}\left(\frac{w_{i,x}}{v_i} - \frac{v_{i,x}w_i}{v_i^2}\right)$$
$$+\tilde{\lambda}_i\theta_i' w_i\left(w_{i,x} - v_{i,x}\frac{w_i}{v_i}\right) - \theta_i'\frac{w_i}{v_i}\left(v_{i,x}w_{i,x} - \tilde{\lambda}_i w_i v_{i,x}\right)$$
$$-\frac{w_i}{v_i}(\lambda_i^* - \theta_i)\theta_i'(w_{i,x}v_i - v_{i,x}w_i)$$
$$= 2\theta_i'\frac{w_i}{v_i}\left(\frac{w_i}{v_i}v_{i,x} - w_{i,x}\right)v_{i,x} - \theta_i'\frac{w_i}{v_i}\left(\frac{w_i}{v_i}v_{i,x} - w_{i,x}\right)\tilde{\lambda}_i v_i$$
$$+\theta_i'\frac{w_i}{v_i}\left(v_{i,x}\frac{w_i}{v_i} - w_{i,x}\right)\lambda_i^* v_i + \theta_i\theta_i'(w_{i,x}v_i - v_{i,x}w_i)\frac{w_i}{v_i}$$
$$= 2\theta_i'\frac{w_i}{v_i}\left(v_{i,x}\frac{w_i}{v_i} - w_{i,x}\right)\left([v_{i,x} - (\tilde{\lambda}_i - \lambda_i^*)v_i - w_i] + [w_i - \theta_i v_i]\right)$$
$$-\left(\tilde{\lambda}_i - \lambda_i^* + \theta_i\right)\theta_i'[w_{i,x}v_i - w_i v_{i,x}]\frac{w_i}{v_i}.$$

Coefficients of $\tilde{r}_{i,\sigma u}\tilde{r}_i/v_i$:

- $\left(v_{i,x} - \tilde{\lambda}_i v_i\right)w_i v_i\theta_i' - \left(w_{i,x} - \tilde{\lambda}_i w_i\right)v_i^2\theta_i' = \theta_i' v_i[v_{i,x}w_i - w_{i,x}v_i],$
- $\left(v_{i,x} - \tilde{\lambda}_i v_i\right)w_i^2\theta_i' - \left(w_{i,x} - \tilde{\lambda}_i w_i\right)w_i v_i\theta_i' = \theta_i' w_i[v_{i,x}w_i - w_{i,x}v_i].$

Coefficients of $\tilde{r}_{i,\sigma\sigma}/v_i$:

- $-\left(v_{i,x} - \tilde{\lambda}_i v_i\right)w_i\theta_i'\theta_{i,x} + \left(w_{i,x} - \tilde{\lambda}_i w_i\right)v_i\theta_i'\theta_{i,x} = \theta_i'^2\left[v_i\left(\frac{w_i}{v_i}\right)_x\right]^2,$
- $-\left(v_{i,x} - \tilde{\lambda}_i v_i\right)w_i\frac{w_i}{v_i}\theta_i'\theta_{i,x} + \left(w_{i,x} - \tilde{\lambda}_i w_i\right)w_i\theta_i'\theta_{i,x} = \theta_i'^2\frac{w_i}{v_i}\left[v_i\left(\frac{w_i}{v_i}\right)_x\right]^2.$

To sum up, observe that all component source terms are expressed in terms of transversal, change in speed linear and quadratic and cutoff terms as stated in (6.4). It is easy to see that the coefficients of the controllable terms are bounded using the results of Sections 4 and 5. This completes the proof of Lemma 6.1.

Appendix B. The evolution of perturbed components

In this appendix, we study the evolution of the perturbed components p and q of the decomposition (7.12) and prove Lemma 7.4. The procedure retraces the steps presented in Appendix **??**, however more terms appear in the perturbed source $\hat{\phi}$ and $\hat{\psi}$ than before. Also, explicit straightforward calculations establish the form of the perturbed source (7.24).

The evolution equations of the infinitesimal perturbation z and the flux $y = z_x - A(u)z$ are:

$$\begin{aligned}
&z_t + (A(u)z)_x + \varepsilon Dg(u)z - z_{xx} = (u_x \bullet A(u))z - (z \bullet A(u))u_x, \\
&y_t + (A(u)y)_x + \varepsilon Dg(u)y - y_{xx} \\
&\quad = [u_x \bullet A(u)z - z \bullet A(u)u_x]_x - A(u)[(u_x \bullet A(u))z - (z \bullet A(u))u_x] \\
&\qquad + (u_x \bullet A(u))y - (u_t \bullet A(u))z \\
&\qquad - \varepsilon(u_x \bullet Dg(u))z + \varepsilon[A(u)Dg(u)z - Dg(u)A(u)]z.
\end{aligned} \tag{B.1}$$

The goal is to rewrite the above system in terms of (p, q) by employing the decomposition (7.12);

$$\begin{aligned}
z &= \sum_{i=1}^{n} p_i \tilde{r}_i \left(u, v_i, \lambda_i^* - \theta\left(\frac{q_i}{p_i}\right)\right), \\
y &= \sum_{i=1}^{n} \left(q_i - \lambda_i^* p_i\right) \tilde{r}_i \left(u, v_i, \lambda_i^* - \theta\left(\frac{q_i}{p_i}\right)\right).
\end{aligned} \tag{B.2}$$

Here, we only show the calculations of two of the terms, z_t and y_t, and the rest can be obtained in a similar fashion.

$$\begin{aligned}
z_t &= \sum_i p_{i,t}\hat{r}_i + \sum_i p_i \left[\hat{r}_{i,u}u_t + v_{i,t}\hat{r}_{i,v} - \hat{\theta}_{i,t}\hat{r}_{i,\sigma}\right] \\
&= \sum_i p_{i,t}\hat{r}_i + \sum_i p_i \left[v_{i,t}\hat{r}_{i,v} - \hat{\theta}_i' \frac{q_{i,t}p_i - p_{i,t}q_i}{p_i^2}\hat{r}_{i,\sigma}\right] \\
&\quad + \sum_{i,j} p_i \left(w_j - \lambda_j^* v_j\right)\hat{r}_{i,u}\tilde{r}_j - \varepsilon \sum_i p_i \hat{r}_{i,u} g(u)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i p_{i,t}\left[\hat r_i + \hat\theta_i' \frac{q_i}{p_i}\hat r_{i,\sigma}\right] - \sum_i \hat\theta_i' q_{i,t}\hat r_{i,\sigma} + \sum_i p_i v_{i,t}\hat r_{i,v}\\
&\quad + \sum_{i,j} p_i\left(w_j - \lambda_j^* v_j\right)\hat r_{i,u}\tilde r_j - \varepsilon\sum_i p_i \hat r_{i,u} g(u),
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
y_t &= \sum_i \left(q_{i,t} - \lambda_i^* p_{i,t}\right)\hat r_i + \sum_i (q_i - \lambda_i^* p_i)[\hat r_{i,u} u_t + \hat r_{i,v} v_{i,t} - \hat r_{i,\sigma}\hat\theta_{i,t}]\\
&= \sum_i \left(q_{i,t} - \lambda_i^* p_{i,t}\right)\hat r_i + \sum_i (q_i - \lambda_i^* p_i)\left[v_{i,t}\hat r_{i,v} - \hat\theta_i' \frac{q_{i,t} p_i - p_{i,t} q_i}{p_i^2}\hat r_{i,\sigma}\right]\\
&\quad + \sum_{i,j}\left(q_i - \lambda_i^* p_i\right)(w_j - \lambda_j^* v_j)\hat r_{i,u}\tilde r_j - \varepsilon\sum_i (q_i - \lambda_i^* p_i)\hat r_{i,u} g(u)\\
&= \sum_i p_{i,t}\left[\hat\theta_i'\left(\frac{q_i}{p_i}\right)^2\hat r_{i,\sigma}\right] + \sum_i q_{i,t}\left[\hat r_i - \hat\theta_i'\frac{q_i}{p_i}\hat r_{i,\sigma}\right] + \sum_i q_i v_{i,t}\hat r_{i,v}\\
&\quad + \sum_{i,j} q_i (w_j - \lambda_j^* v_j)\hat r_{i,u}\tilde r_j - \varepsilon\sum_i q_i\hat r_{i,u} g(u)\\
&\quad - \sum_i \lambda_i^*\left\{ p_{i,t}\left[\hat r_i + \hat\theta_i'\frac{q_i}{p_i}\hat r_{i,\sigma}\right] - \hat\theta_i' q_{i,t}\hat r_{i,\sigma} + v_{i,t} p_i \hat r_{i,v}\right.\\
&\qquad\qquad \left. + \sum_j p_i (w_j - \lambda_j^* v_j)\hat r_{i,u}\tilde r_j - \varepsilon p_i\hat r_{i,u} g(u)\right\}.
\end{aligned}
\tag{B.4}
$$

In turn system (B.1) becomes a system of two vector equations of the perturbed components p and q:

$$
\begin{aligned}
&\sum_i \left(p_{i,t} + \left(\hat\lambda_i p_i\right)_x - p_{i,xx}\right)\left[\hat r_i + \hat\theta_i\frac{q_i}{p_i}\hat r_{i,\sigma}\right] + \left(q_{i,t} + \left(\hat\lambda_i q_i\right)_x - q_{i,xx}\right)\left[-\hat\theta_i'\hat r_{i,\sigma}\right]\\
&\quad + \varepsilon\sum_i p_i[Dg(u)\hat r_i - \hat r_{i,u} g(u)] = \sum_i \widehat{\alpha}_i,
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
&\sum_i \left(p_{i,t} + \left(\hat\lambda_i p_i\right)_x - p_{i,xx}\right)\left[\hat\theta_i'\left(\frac{q_i}{p_i}\right)^2\hat r_{i,\sigma}\right]\\
&\quad + \left(q_{i,t} + (\hat\lambda_i q_i)_x - q_{i,xx}\right)\left[\hat r_i - \hat\theta_i'\frac{q_i}{p_i}\hat r_{i,\sigma}\right]\\
&\quad - \sum_i \lambda_i^*\left\{\left(p_{i,t} + \left(\hat\lambda_i p_i\right)_x - p_{i,xx}\right)\left[\hat r_i + \hat\theta_i'\frac{q_i}{p_i}\hat r_{i,\sigma}\right] + \left(q_{i,t} + \left(\hat\lambda_i q_i\right)_x - q_{i,xx}\right)[-\hat\theta_i'\hat r_{i,\sigma}]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon\sum_i (q_i-\lambda_i^* p_i)[Dg(u)\hat{r}_i-\hat{r}_{i,u}g(u)]-\varepsilon\sum_i p_i(g(u)\bullet A(u))\hat{r}_i\\
&-\varepsilon\sum_i p_i[(A(u)Dg(u)-Dg(u)A(u))\hat{r}_i-v_i(\tilde{r}_i\bullet Dg(u))\hat{r}_i]\\
&=\sum_i \widehat{\beta}_i-\lambda_i^*\widehat{\delta}_i,
\end{aligned}\tag{B.6}
$$

where

$$
\begin{aligned}
\sum_i \widehat{\alpha}_i \doteq\ & \sum_i \hat{r}_{i,u}\tilde{r}_i\left[\left(p_{i,x}-\hat{\lambda}_i p_i\right)v_i-p_i\left(w_i-\lambda_i^* v_i\right)\right]\\
&+\sum_{i\neq j}\hat{r}_{i,u}\tilde{r}_j[(p_{i,x}-\hat{\lambda}_i p_i)v_j+(p_i v_j)_x-p_i(w_j-\lambda_j^* v_j)]\\
&+\sum_i \hat{r}_{i,v}\left[\left(p_{i,x}-\hat{\lambda}_i p_i\right)v_{i,x}+\left(p_i\left(v_{i,x}-\left(\hat{\lambda}_i-\lambda_i^*+\hat{\theta}_i\right)v_i\right)\right)_x-p_i v_{i,t}\right]\\
&+\sum_i \hat{r}_{i,\sigma}\left[(p_{i,x}-\hat{\lambda}_i p_i)\left(-\hat{\theta}_{i,x}+\left(\hat{\theta}_i'\frac{q_i}{p_i}\right)_x\right)-(q_{i,x}-\hat{\lambda}_i q_i)\hat{\theta}_i''\left(\frac{q_i}{p_i}\right)_x\right]\\
&+\sum_i \hat{r}_{i,vu}\tilde{r}_i[p_i v_i(v_{i,x}-(\hat{\lambda}_i-\lambda_i^*+\hat{\theta}_i)v_i)]\\
&+\hat{r}_{i,vv}[p_i v_{i,x}(v_{i,x}-(\hat{\lambda}_i-\lambda_i^*+\hat{\theta}_i)v_i)]\\
&+\sum_{i\neq j}\hat{r}_{i,vu}\tilde{r}_j[p_i v_j(v_{i,x}-(\hat{\lambda}_i-\lambda_i^*+\hat{\theta}_i)v_i)]\\
&+\sum_i \hat{r}_{i,v\sigma}\left[(p_{i,x}-\hat{\lambda}_i p_i)\hat{\theta}_i' v_{i,x}\frac{q_i}{p_i}-(q_{i,x}-\hat{\lambda}_i q_i)\hat{\theta}_i' v_{i,x}\right.\\
&\left.-p_i(v_{i,x}-(\hat{\lambda}_i-\lambda_i^*+\hat{\theta}_i)v_i)\hat{\theta}_{i,x}\right]\\
&+\hat{r}_{i,\sigma u}\tilde{r}_i\left[(p_{i,x}-\hat{\lambda}_i p_i)\hat{\theta}_i' v_i\frac{q_i}{p_i}-(q_{i,x}-\hat{\lambda}_i q_i)\hat{\theta}_i' v_i\right]\\
&+\sum_{i\neq j}\hat{r}_{i,\sigma u}\tilde{r}_j\left[(p_{i,x}-\hat{\lambda}_i p_i)\hat{\theta}_i' v_j\frac{q_i}{p_i}-(q_{i,x}-\hat{\lambda}_i q_i)\hat{\theta}_i' v_j\right]\\
&+\sum_i \hat{r}_{i,\sigma\sigma}\left[-(p_{i,x}-\hat{\lambda}_i p_i)\hat{\theta}_i'\hat{\theta}_{i,x}\frac{q_i}{p_i}+(q_{i,x}-\hat{\lambda}_i q_i)\hat{\theta}_i'\hat{\theta}_{i,x}\right]\\
&+\sum_i \left(p_i v_i\hat{r}_{i,u}(\tilde{r}_i-\hat{r}_i)\right)_x
\end{aligned}
$$

$$+\sum_{i\neq j} p_i v_j \left[\sum_k v_k \left(\hat{r}_{i,u}\tilde{r}_{j,u}\tilde{r}_k + \hat{r}_{i,uu}(\tilde{r}_j \otimes \tilde{r}_k)\right) + v_{j,x}\hat{r}_{i,u}\tilde{r}_{j,v} + v_{i,x}\hat{r}_{i,uv}\tilde{r}_j \right.$$

$$\left. - \theta_{j,x}\hat{r}_{i,u}\tilde{r}_{j,\sigma} - \hat{\theta}_{i,x}\hat{r}_{i,u\sigma}\tilde{r}_j\right] + \sum_{i,j} p_i v_j \left[(\tilde{r}_j \bullet A(u))\hat{r}_i - (\hat{r}_i \bullet A(u))\tilde{r}_j\right], \quad \text{(B.7)}$$

and

$$\sum_i \widehat{\beta}_i - \lambda_i^* \widehat{\delta}_i$$

$$\doteq \sum_i \hat{r}_{i,u}\tilde{r}_i \left[\left(q_{i,x} - \hat{\lambda}_i q_i\right) v_i - q_i \left(w_i - \lambda_i^* v_i\right)\right]$$

$$+ \sum_{i\neq j} \hat{r}_{i,u}\tilde{r}_j \left[\left(q_{i,x} - \hat{\lambda}_i q_i\right) v_j - q_i \left(w_j - \lambda_j^* v_j\right) + \left(q_i v_j\right)_x\right]$$

$$+ \sum_i \hat{r}_{i,v} \left[\left(q_{i,x} - \hat{\lambda}_i q_i\right) v_{i,x} + \left(q_i \left(v_{i,x} - \left(\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i\right) v_i\right)\right)_x - q_i v_{i,t}\right]$$

$$+ \sum_i \hat{r}_{i,\sigma} \left[(p_{i,x} - \hat{\lambda}_i p_i) \left(\hat{\theta}_i' \left(\frac{q_i}{p_i}\right)^2\right)_x - (q_{i,x} - \hat{\lambda}_i q_i) \left(\hat{\theta}_{i,x} + \left(\hat{\theta}_i' \frac{q_i}{p_i}\right)_x\right)\right]$$

$$+ \sum_i \hat{r}_{i,vu}\tilde{r}_i [(v_{i,x} - (\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i)v_i)q_i v_i]$$

$$+ \sum_i \hat{r}_{i,vv} [(v_{i,x} - (\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i)v_i)q_i v_{i,x}]$$

$$+ \sum_{i\neq j} \hat{r}_{i,vu}\tilde{r}_j [(v_{i,x} - (\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i)v_i)q_i v_j]$$

$$+ \sum_i \hat{r}_{i,v\sigma} \left[(p_{i,x} - \hat{\lambda}_i p_i)\hat{\theta}_i' v_{i,x} \left(\frac{q_i}{p_i}\right)^2 - (q_{i,x} - \hat{\lambda}_i q_i) v_{i,x} \hat{\theta}_i' \frac{q_i}{p_i} \right.$$

$$\left. - q_i (v_{i,x} - (\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i)v_i)\hat{\theta}_{i,x}\right]$$

$$+ \sum_i \hat{r}_{i,\sigma u}\tilde{r}_i \left[(p_{i,x} - \hat{\lambda}_i p_i)\hat{\theta}_i' \left(\frac{q_i}{p_i}\right)^2 v_i - (q_{i,x} - \hat{\lambda}_i q_i)\hat{\theta}_i \frac{q_i}{p_i} v_i\right]$$

$$+ \sum_{i\neq j} \hat{r}_{i,\sigma u}\tilde{r}_j \left[(p_{i,x} - \hat{\lambda}_i p_i)\hat{\theta}_i' \left(\frac{q_i}{p_i}\right)^2 v_j - (q_{i,x} - \hat{\lambda}_i q_i)\hat{\theta}_i \frac{q_i}{p_i} v_j\right]$$

$$
\begin{aligned}
&+\sum_i \hat r_{i,\sigma\sigma}\left[-(p_{i,x}-\hat\lambda_i p_i)\hat\theta_i'\left(\frac{q_i}{p_i}\right)^2\hat\theta_{i,x}+(q_{i,x}-\hat\lambda_i q_i)\hat\theta_{i,x}\hat\theta_i'\frac{q_i}{p_i}\right]\\
&+\sum_i (q_i v_i \hat r_{i,u}(\tilde r_i-\hat r_i))_x+\sum_{i,j}[v_i p_j((\tilde r_i\bullet A(u))\hat r_j-(\hat r_j\bullet A(u))\tilde r_i)]_x\\
&+\sum_{i\neq j} q_i v_j\left[\sum_k v_k(\hat r_{i,u}\tilde r_{j,u}\tilde r_k+\hat r_{i,uu}(\tilde r_j\otimes\tilde r_k))+v_{i,x}\hat r_{i,uv}\tilde r_j+v_{j,x}\hat r_{i,u}\tilde r_{j,v}\right.\\
&\qquad\left.-\hat\theta_{i,x}\hat r_{i,u\sigma}\tilde r_j-\theta_{j,x}\hat r_{i,u}\tilde r_{j,\sigma}\right]+\sum_{i,j}(v_i q_j-w_i p_j)(\tilde r_i\bullet A(u))\hat r_j\\
&+\sum_{i,j} A(u)v_i p_j[(\hat r_j\bullet A(u))\tilde r_i-(\tilde r_i\bullet A(u))\hat r_j]\\
&+\sum_{i\neq j}(\lambda_i^*-\lambda_j^*)v_i p_j(\tilde r_i\bullet A(u))\hat r_j-\varepsilon\sum_{i\neq j} p_i v_j(\tilde r_j\bullet Dg(u))\hat r_i\\
&-\sum_i \lambda_i^*\left\{(p_{i,x}-\hat\lambda_i p_i)\left[\hat r_i+\hat\theta_i'\frac{q_i}{p_i}\hat r_{i,\sigma}\right]_x-(q_{i,x}-\hat\lambda_i q_i)\left[\hat\theta_i'\hat r_{i,\sigma}\right]_x\right.\\
&\qquad+\left[p_i(v_{i,x}-(\hat\lambda_i-\lambda_i^*+\theta_i)v_i)\hat r_{i,v}\right]_x+\left[p_i v_i\hat r_{i,u}(\tilde r_i-\hat r_i)\right]_x\\
&\qquad\left.+\sum_{i\neq j}(p_i v_j\hat r_{i,u}\tilde r_j)_x-v_{i,t}p_i\hat r_{i,v}-\sum_j p_i(w_j-\lambda_j^* v_j)\hat r_{i,u}\tilde r_j\right\}, \qquad \text{(B.8)}
\end{aligned}
$$

where $\widehat{\delta}_i \doteq \widehat{\alpha}_i - \sum_j \big(p_i v_j(\tilde r_j\bullet A(u))\hat r_i-(\hat r_i\bullet A(u))\tilde r_j\big)$.

System (B.5)–(B.6) can be written in the form

$$
\left(\frac{\partial\widehat{\mathcal{G}}}{\partial(p,q)}\right)\begin{pmatrix} p_t+(\hat\Lambda p)_x-p_{xx}\\ q_t+(\hat\Lambda w)_x-q_{xx}\end{pmatrix}+\varepsilon\widehat{\mathcal{D}}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}\sum_i\widehat{\alpha}_i\\ \sum_i\widehat{\beta}_i-\lambda_i^*\widehat{\delta}_i\end{pmatrix}, \qquad \text{(B.9)}
$$

where $\widehat{\mathcal{D}}$ is given in (A.14), if one replaces $\tilde r$ by $\hat r$ as well as $\hat\Lambda = diag\{\hat\lambda_i\}$. The map $\widehat{\mathcal{G}}(p,q)$ is defined at (7.16), while

$$
\frac{\partial\widehat{\mathcal{G}}}{\partial(p,q)}=\widehat{\mathcal{M}}_0+\widehat{\mathcal{M}}_1\to\widehat{\mathcal{M}}_0 \quad \text{as } v\to 0. \qquad \text{(B.10)}
$$

Again, this implies

$$\left(\frac{\partial\widehat{\mathcal{G}}}{\partial(p,q)}\right)^{-1}\widehat{\mathcal{D}} \to \begin{pmatrix} R(u^*)^{-1}Dg(u^*)R(u^*) & 0 \\ 0 & R(u^*)^{-1}Dg(u^*)R(u^*) \end{pmatrix}$$

for (u, v) tending to $(u^*, 0)$. Because $\left(\frac{\partial\hat{\mathcal{G}}}{\partial(p,q)}\right)^{-1}$ is uniformly bounded, the system (B.9) takes the following form:

$$\begin{pmatrix} p_t + (\Lambda p)_x - p_{xx} \\ q_t + (\Lambda q)_x - q_{xx} \end{pmatrix} + \varepsilon \begin{pmatrix} \widehat{B}^\sharp & \widehat{H} \\ \widehat{K} & \widehat{B}^\flat \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \widehat{\Phi} \\ \widehat{\Psi} \end{pmatrix},$$

where the matrix $\begin{pmatrix} \widehat{B}^\sharp & \widehat{H} \\ \widehat{K} & \widehat{B}^\flat \end{pmatrix}$ is strictly diagonally dominant and

$$\widehat{\Phi} = \mathcal{O}(1)\sum_i \alpha_i + \Big((\Lambda - \hat{\Lambda})p\Big)_x, \quad \widehat{\Psi} = \mathcal{O}(1)\sum_i \beta_i + \Big((\Lambda - \hat{\Lambda})q\Big)_x. \tag{B.11}$$

In particular, we have

$$\begin{pmatrix} \widehat{B}^\sharp & \widehat{H} \\ \widehat{K} & \widehat{B}^\flat \end{pmatrix} \doteq \left(\frac{\partial\widehat{\mathcal{G}}}{\partial(p,q)}\right)^{-1}\widehat{\mathcal{D}},$$

$$\widehat{B}^\sharp_{ij} - \widehat{B}^\flat_{ij} = \mathcal{O}(|u-u^*|, |v|), \qquad \widehat{H}_{ij},\ \widehat{K}_{ij} = \mathcal{O}(|u-u^*|,\ |v|), \tag{B.12}$$

and when $(u, v) = (u^*, 0)$, we have $\widehat{B}^\sharp = \widehat{B}^\flat = B(u^*)$.

In what follows, we investigate the various perturbed source terms that appear in the source $\widehat{\Phi}$, $\widehat{\Psi}$ given in (B.11), i.e.

$$\widehat{\alpha}_i, \qquad \widehat{\beta}_i - \lambda_i^*\widehat{\delta}_i, \qquad \big((\tilde{\lambda}_i - \hat{\lambda}_i)p_i\big)_x, \qquad \big((\tilde{\lambda}_i - \hat{\lambda}_i)q_i\big)_x, \tag{B.13}$$

and show that these terms are monitored by the controllable functions given in (7.24).

First, we list some additional functions that are controllable. By means of (6.58)–(6.59), we get

$$\begin{aligned}
&\Big(|p_i| + |q_i| + |p_{i,x}| + |q_{i,x}|\Big)|v_{i,x} - (\tilde{\lambda}_i - \lambda_i^*)v_i - w_i| \\
&\quad = \Big(|p_i| + |q_i| + |p_{i,x}| + |q_{i,x}|\Big) \cdot |\Theta_i| \\
&\quad = \text{Transversal terms.}
\end{aligned}$$

Moreover, we have

$$p_i[w_{i,x}v_i - w_i v_{i,x}] = [p_i w_{i,x} - w_i p_{i,x}]v_i + [p_{i,x}v_i - p_i v_{i,x}]w_i,$$

$$q_i[w_{i,x}v_i - w_i v_{i,x}] = [q_i w_{i,x} - q_{i,x} w_i]v_i + [q_{i,x}v_i - q_i v_{i,x}]w_i.$$

Hence, $p_i[w_{i,x}v_i - w_i v_{i,x}]$ and $q_i[w_{i,x}v_i - w_i v_{i,x}]$ are controllable. In addition, the following bound holds

$$|\hat{\lambda}_i - \tilde{\lambda}_i| = \mathcal{O}(1)|\hat{r}_i - \tilde{r}_i| = \mathcal{O}(1)v_i|\hat{\theta}_i - \theta_i| = \mathcal{O}(1)\delta_0|\hat{\theta}_i - \theta_i|. \tag{B.14}$$

Since $|\theta'| \leqslant 1$,

$$|\hat{\theta}_i - \theta_i| \leqslant \left| \frac{q_i}{p_i} - \frac{w_i}{v_i} \right|, \tag{B.15}$$

and by (6.54) and (7.28) yields

$$\begin{aligned}
|p_i v_i||\hat{\theta}_i - \theta_i| &\leqslant |q_i v_i - w_i p_i| \\
&= \Bigg| \left[p_{i,x} - \left(\hat{\lambda}_i - \lambda_i^*\right) p_i + \mathcal{O}(1)\delta_0 \sum_{j\neq i} \left(|p_j| + |q_j| + |p_{j,x}|\right) \right] v_i \\
&\quad - \left[v_{i,x} - \left(\tilde{\lambda}_i - \lambda_i^*\right) v_i + \mathcal{O}(1)\delta_0 \sum_{j\neq i} \left(|v_j| + |w_j| + |v_{j,x}|\right) \right] p_i \Bigg| \\
&= \Bigg| (p_{i,x}v_i - v_{i,x}p_i) - (\hat{\lambda}_i - \tilde{\lambda}_i)v_i p_i \\
&\quad + \mathcal{O}(1)\delta_0 \sum_{j\neq i} \left(|v_i p_j| + |q_j v_i| + |p_{j,x}v_i| + |w_j p_i| + |v_{j,x}p_i|\right) \Bigg| \\
&\leqslant |p_{i,x}v_i - v_{i,x}p_i| + \mathcal{O}(1)\delta_0|\hat{\theta}_i - \theta_i||v_i p_i| \\
&\quad + \mathcal{O}(1)\delta_0 \sum_{j\neq i} \left(|v_i p_j| + |q_j v_i| + |p_{j,x}v_i| + |w_j p_i| + |v_{j,x}p_i|\right).
\end{aligned} \tag{B.16}$$

For small $\delta_0 > 0$, $\mathcal{O}(1)\delta_0 < \frac{1}{2}$, hence

$$\begin{aligned}
|p_i v_i(\hat{\theta}_i - \theta_i)| &\leqslant |q_i v_i - w_i p_i| \leqslant 2|p_{i,x}v_i - v_{i,x}p_i| \\
&\quad + \mathcal{O}(1)\delta_0 \sum_{j\neq i} (|v_i p_j| + |q_j v_i| + |p_{j,x}v_i| + |w_j p_i| + |v_{j,x}p_i|),
\end{aligned}$$

whence both $p_i v_i(\hat{\theta}_i - \theta_i)$ and $q_i v_i - w_i p_i$ are controllable.

We consider two regions in which $\hat{\theta}_i - \theta_i$ can be nonzero i.e. $\left|\frac{q_i}{p_i}\right| \leqslant 6\delta_1$ or $\left|\frac{q_i}{p_i}\right| \geqslant 6\delta_1$ and $\left|\frac{w_i}{v_i}\right| \leqslant 3\delta_1$. Let χ be the characteristic function, then

$$\begin{aligned}
|q_i v_i(\hat{\theta}_i - \theta_i))| &\leqslant \left|\frac{q_i}{p_i}\right| |p_i v_i(\hat{\theta}_i - \theta_i)| \chi_{\left\{\left|\frac{q_i}{p_i}\right| \leqslant 6\delta_1\right\}} + |q_i v_i| 2\delta_1 \cdot \chi_{\left\{\left|\frac{q_i}{p_i}\right| \geqslant 6\delta_1,\, \left|\frac{w_i}{v_i}\right| \leqslant 3\delta_1\right\}} \\
&\leqslant 6\delta_1 |p_i v_i(\hat{\theta}_i - \theta_i)| + 2\delta_1 \cdot 2|q_i v_i - w_i p_i|, \\
|p_i w_i(\hat{\theta}_i - \theta_i))| &\leqslant \left|\frac{w_i}{v_i}\right| |v_i p_i(\hat{\theta}_i - \theta_i)| \chi_{\left\{\left|\frac{w_i}{v_i}\right| \leqslant 6\delta_1\right\}} + |w_i p_i| 2\delta_1 \cdot \chi_{\left\{\left|\frac{w_i}{v_i}\right| \geqslant 6\delta_1,\, \left|\frac{q_i}{p_i}\right| \leqslant 3\delta_1\right\}} \\
&\leqslant 6\delta_1 \left| p_i v_i(\hat{\theta}_i - \theta_i)\right| + 4\delta_1 |q_i v_i - w_i p_i|.
\end{aligned}$$

Now, by virtue of (7.26) and (6.58),

$$\begin{aligned}
p_{i,x} v_i(\hat{\theta}_i - \theta_i) &= q_i v_i(\hat{\theta}_i - \theta_i) + (\hat{\lambda}_i - \lambda_i^*) p_i v_i(\hat{\theta}_i - \theta_i) \\
&\quad + \mathcal{O}(1)\delta_0 \sum_{j \neq i} \big(|p_j v_i| + |q_j v_i| + |p_{j,x} v_i|\big), \qquad \text{(B.17)} \\
p_i v_{i,x}(\hat{\theta}_i - \theta_i) &= p_i w_i(\hat{\theta}_i - \theta_i) + (\tilde{\lambda}_i - \lambda_i^*) p_i v_i(\hat{\theta}_i - \theta_i) \\
&\quad + \mathcal{O}(1)\delta_0 \sum_{j \neq i} \big(|v_j p_i| + |w_j p_i| + |v_{j,x} p_i|\big).
\end{aligned}$$

So $p_{i,x} v_i(\hat{\theta}_i - \theta_i)$ and $p_i v_{i,x}(\hat{\theta}_i - \theta_i)$ are controllable.

Here, consider the two regions $\left|\frac{w_i}{v_i}\right| < 3\delta_1$ and $\left|\frac{q_i}{p_i}\right| < 3\delta_1$ on which $\hat{\theta}_i - \theta_i \neq 0$. In general, these regions overlap. By (6.54), we deduce

$$\begin{aligned}
|q_i w_i(\hat{\theta}_i - \theta_i))| &\leqslant \left|\frac{w_i}{v_i}\right| |q_i v_i(\hat{\theta}_i - \theta_i)| \chi_{\left\{\left|\frac{w_i}{v_i}\right| < 3\delta_1\right\}} + |q_i w_i(\hat{\theta}_i - \theta_i)| \cdot \chi_{\left\{\left|\frac{q_i}{p_i}\right| < 3\delta_1\right\}} \\
&\leqslant 3\delta_1 |q_i v_i(\hat{\theta}_i - \theta_i)| \chi_{\left\{|\frac{w_i}{v_i}| < 3\delta_1\right\}} + |p_i w_i(\hat{\theta}_i - \theta_i)| \chi_{\left\{|\frac{q_i}{p_i}| < 3\delta_1\right\}} \\
&= 3\delta_1 |q_i v_i|\, |\hat{\theta}_i - \theta_i| + |p_i(v_{i,x} - (\tilde{\lambda}_i - \lambda_i^*) v_i)| |\hat{\theta}_i - \theta_i| \\
&\quad + \mathcal{O}(1)\delta_0 \sum_{j \neq i} \big(|p_i v_j| + |p_i w_j| + |p_i v_{j,x}|\big) 2\delta_1,
\end{aligned}$$

$$\begin{aligned}
q_{i,x} v_i(\hat{\theta}_i - \theta_i) &= (\hat{\theta}_i - \theta_i)[q_{i,x} v_i - v_{i,x} q_i] + q_i v_{i,x}(\hat{\theta}_i - \theta_i) \\
&= (\hat{\theta}_i - \theta_i)[q_{i,x} v_i - v_{i,x} q_i] + q_i w_i(\hat{\theta}_i - \theta_i) + q_i(\tilde{\lambda}_i - \lambda_i^*) v_i(\hat{\theta}_i - \theta_i) \\
&\quad + \mathcal{O}(1)\delta_0 \sum_{j \neq i} \big(|v_j q_i| + |w_j q_i| + |v_{j,x} q_i|\big). \qquad \text{(B.18)}
\end{aligned}$$

Next, we are going to show that the following terms:

$$p_i(\tilde{r}_i - \hat{r}_i)_x, \qquad q_i(\tilde{r}_i - \hat{r}_i)_x, \qquad p_i(\tilde{\lambda}_i - \hat{\lambda}_i)_x, \qquad q_i(\tilde{\lambda}_i - \hat{\lambda}_i)_x \tag{B.19}$$

are controllable.

$$\begin{aligned} p_i(\tilde{r}_i - \hat{r}_i)_x =\ & p_i v_i(\hat{\theta}_i - \theta_i)\left[\sum_j v_j \frac{(\tilde{r}_{i,u} - \hat{r}_{i,u})}{v_i(\hat{\theta}_i - \theta_i)}\tilde{r}_j + v_{i,x}\frac{\tilde{r}_{i,v} - \hat{r}_{i,v}}{v_i(\hat{\theta}_i - \theta_i)}\right] \\ & + \hat{\theta}_i'\left(\frac{q_i}{p_i}\right)_x \hat{r}_{i,\sigma} p_i - \theta_i'\left(\frac{w_i}{v_i}\right)_x p_i \tilde{r}_{i,\sigma} \\ =\ & p_i v_i(\hat{\theta}_i - \theta_i)\left[\sum_j v_j \frac{(\tilde{r}_{i,u} - \hat{r}_{i,u})}{v_i(\hat{\theta}_i - \theta_i)}\tilde{r}_j + v_{i,x}\frac{\tilde{r}_{i,v} - \hat{r}_{i,v}}{v_i(\hat{\theta}_i - \theta_i)}\right] \\ & + \hat{\theta}_i'\frac{q_i}{p_i}\frac{\hat{r}_{i,\sigma}}{v_i}[v_{i,x}p_i - p_{i,x}v_i] + \hat{\theta}_i'\frac{\hat{r}_{i,\sigma}}{v_i}[v_i q_{i,x} - q_i v_{i,x}] \\ & + \theta_i'\frac{w_i}{v_i}\frac{\tilde{r}_{i,\sigma}}{v_i}[v_{i,x}p_i - p_{i,x}v_i] + \theta_i'\frac{\tilde{r}_{i,\sigma}}{v_i}[w_i p_{i,x} - p_i w_{i,x}], \end{aligned} \tag{B.20}$$

$$\begin{aligned} q_i(\tilde{r}_i - \hat{r}_i)_x =\ & q_i v_i(\hat{\theta}_i - \theta_i)\left[\sum_j v_j \frac{(\tilde{r}_{i,u} - \hat{r}_{i,u})}{v_i(\hat{\theta}_i - \theta_i)}\tilde{r}_j + v_{i,x}\frac{\tilde{r}_{i,v} - \hat{r}_{i,v}}{v_i(\hat{\theta}_i - \theta_i)}\right] \\ & + \hat{\theta}_i'\left(\frac{q_i}{p_i}\right)_x \hat{r}_{i,\sigma} p_i - \theta_i'\left(\frac{w_i}{v_i}\right)_x p_i \tilde{r}_{i,\sigma} \\ =\ & p_i v_i(\hat{\theta}_i - \theta_i)\left[\sum_j v_j \frac{(\tilde{r}_{i,u} - \hat{r}_{i,u})}{v_i(\hat{\theta}_i - \theta_i)}\tilde{r}_j + v_{i,x}\frac{\tilde{r}_{i,v} - \hat{r}_{i,v}}{v_i(\hat{\theta}_i - \theta_i)}\right] \\ & + \hat{\theta}_i'\left(\frac{q_i}{p_i}\right)^2\frac{\hat{r}_{i,\sigma}}{v_i}[v_{i,x}p_i - p_{i,x}v_i] + \hat{\theta}_i'\frac{q_i}{p_i}\frac{\hat{r}_{i,\sigma}}{v_i}[v_i q_{i,x} - q_i v_{i,x}] \\ & + \theta_i'\frac{w_i}{v_i}\frac{\tilde{r}_{i,\sigma}}{v_i}[v_{i,x}q_i - q_{i,x}v_i] + \theta_i'\frac{\tilde{r}_{i,\sigma}}{v_i}[w_i q_{i,x} - q_i w_{i,x}]. \end{aligned} \tag{B.21}$$

Thus the first two terms in (B.19) are controllable. We can show that $p_i(\tilde{\lambda}_i - \hat{\lambda}_i)_x$ and $p_i(\tilde{\lambda}_i - \hat{\lambda}_i)_x$ are controllable if we repeat the above calculations where $\tilde{r}$ and $\hat{r}$ are replaced by $\tilde{\lambda}$, $\hat{\lambda}$. From (B.17) and (B.18), it follows that the terms $\Big((\tilde{\lambda}_i - \hat{\lambda}_i)p_i\Big)_x$ and $\Big((\tilde{\lambda}_i - \hat{\lambda}_i)q_i\Big)_x$ are controllable.

In the following calculations, we complete the investigation of the terms $\widehat{\alpha}_i$ and $\widehat{\beta}_i$. First, observe that all terms in (B.5)–(B.6) that involve product of two components of different families belong to the category of transversal terms. We show that all other terms in $\widehat{\alpha}_i$ and $\widehat{\beta}_i$ are controllable by treating one by one. Here we select to present only two of these terms, the coefficients of $\hat{r}_{i,v}$

- $$\left(p_{i,x} - \hat{\lambda}_i p_i\right) v_{i,x} + \left(p_i \left(v_{i,x} - \left(\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i\right) v_i\right)\right)_x - p_i v_{i,t}$$
$$\begin{aligned} &= 2\left[p_{i,x}\left(v_{i,x} - \left(\tilde{\lambda}_i - \lambda_i^*\right) v_i - w_i\right)\right] + 2\left[p_{i,x}\left(w_i - \theta_i v_i\right)\right] \\ &\quad + \left[p_i v_{i,x} - p_{i,x} v_i\right]\left(\lambda_i^* - \hat{\theta}_i - \hat{\lambda}_i - \hat{\theta}_i' \frac{q_i}{p_i}\right) + \hat{\theta}_i' \left[v_{i,x} q_i - q_{i,x} v_i\right] \\ &\quad -2\left[p_{i,x} v_i \left(\hat{\lambda}_i + \hat{\theta}_i - \tilde{\lambda}_i - \theta_i\right)\right] + \left[p_i \left(\left(\tilde{\lambda}_i - \hat{\lambda}_i\right) v_i\right)_x\right] \\ &\quad - p_i \phi_i + \varepsilon \sum_k \left(B_{ik}^{\sharp} p_i v_k + H_{ik} p_i w_k\right), \end{aligned}$$
- $$\left(q_{i,x} - \hat{\lambda}_i q_i\right) v_{i,x} + \left(q_i \left(v_{i,x} - \left(\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i\right) v_i\right)\right)_x - q_i v_{i,t}$$
$$\begin{aligned} &= 2\left[q_{i,x}\left(v_{i,x} - \left(\tilde{\lambda}_i - \lambda_i^*\right) v_i - w_i\right)\right] + 2[q_{i,x}(w_i - \theta_i v_i)] \\ &\quad + \left[q_i v_{i,x} - q_{i,x} v_i\right]\left(\lambda_i^* - \hat{\theta}_i - \hat{\lambda}_i + \hat{\theta}_i' \frac{q_i}{p_i}\right) \\ &\quad + \hat{\theta}_i' \left(\frac{q_i}{p_i}\right)^2 \left[p_{i,x} v_i - v_{i,x} p_i\right] \\ &\quad -2\left[q_{i,x} v_i \left(\hat{\lambda}_i + \hat{\theta}_i - \tilde{\lambda}_i - \theta_i\right)\right] + \left[q_i \left((\tilde{\lambda}_i - \hat{\lambda}_i) v_i\right)_x\right] \\ &\quad - q_i \phi_i + \varepsilon \sum_k \left(B_{ik}^{\sharp} q_i v_k + H_{ik} q_i w_k\right). \end{aligned}$$

In view of the above analysis, we conclude that all terms in the perturbed source $\widehat{\Phi}$, $\widehat{\Psi}$ are controllable. This completes the proof of Lemma 7.4.

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