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Well-posedness for a higher-order Benjamin–Ono equation

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ABSTRACT

In this paper we prove that the initial value problem associated to the following higher-order Benjamin–Ono equation

$$\partial_t v - b\mathcal{H}\partial_x^2 v + a\partial_x^3 v = cv\partial_x v - d\partial_x\big(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)\big),$$

where $x, t \in \mathbb{R}$, v is a real-valued function, $\mathcal{H}$ is the Hilbert transform, $a \in \mathbb{R}$, b, c and d are positive constants, is locally well-posed for initial data

$$v(0) = v_0 \in H^s(\mathbb{R}), \quad s \geqslant 2 \quad \text{or}$$

$$v_0 \in H^k(\mathbb{R}) \cap L^2\big(\mathbb{R}; x^2\,dx\big), \quad k \in \mathbb{Z}_+,\ k \geqslant 2.$$

1. Introduction

We study the initial value problem (IVP) associated to the following higher-order Benjamin–Ono equation

$$\begin{cases} \partial_t v - b\mathcal{H}\partial_x^2 v + a\partial_x^3 v = cv\partial_x v - d\partial_x\big(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)\big), \\ v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

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where $x, t \in \mathbb{R}$, v is a real-valued function, $a \in \mathbb{R}$, $a \neq 0$, b, c and d are positive constants, and $\mathcal{H}$ is the Hilbert transform, *i.e.*

$$\mathcal{H}f(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy. \tag{1.2}$$

The equation above corresponds to a second-order approximation of the unidirectional evolution of weakly nonlinear dispersive internal long waves at the interface of a two-layer system, one being infinitely deep. It was derived by Craig, Guyenne and Kalisch [4], using Hamiltonian perturbation theory.

In this work we are interested in establishing a well-posedness theory for the IVP (1.1) in usual Sobolev spaces. We first observe that the L^2-norm as well as the quantity

$$H(v) = \int_{\mathbb{R}} \left(a(\partial_x v)^2 + bv\mathcal{H}\partial_x v + \frac{c}{3}v^3 - dv^2\mathcal{H}\partial_x v \right) dx \tag{1.3}$$

is conserved by solutions of the equation in (1.1).

In contrast with the IVP associated to the Benjamin–Ono equation (see [1,3,8,9,11,16,20,22]) there are no well-posedness results for the IVP (1.1) available in the literature. In [18] it was shown that the map data-solution of the IVP (1.1) from $H^s(\mathbb{R})$ to $C([0,T]; H^s(\mathbb{R}))$, for any $s \in \mathbb{R}$, is not C^2. Thus one has that local well-posedness in $H^s(\mathbb{R})$ cannot be established by a fixed point argument using the integral equation. The techniques in [18] follow the ideas used in [17] to obtain a similar result for the Benjamin–Ono equation.

The purpose of this paper is to investigate local well-posedness for the IVP (1.1). In this direction our main results are next.

Theorem 1.1. *Let $k \in \mathbb{Z}_+$ be such that $k \geqslant 3$. Then for any*

$$v_0 \in Z^k = H^k(\mathbb{R}) \cap L^2\big(\mathbb{R}; x^2\,dx\big), \tag{1.4}$$

there exist $T = T(\|v_0\|_{Z^k}) > 0$ with $T(\alpha) \nearrow +\infty$ as $\alpha \to 0$, a space Z_T^k such that

$$Z_T^k \hookrightarrow C\big([0,T] : Z^k\big),$$

and a unique solution v of (1.1) in Z_T^k satisfying $v(\cdot, 0) = v_0$.

Moreover, for any $T' \in (0,T)$, there exists a neighborhood Ω^k of v_0 in Z^k such that the flow map data-solution is smooth from Ω^k into $Z_{T'}^k$.

Our argument of proof follows the idea of the gauge transformation introduced by Hayashi and Ozawa [7] in the one-dimensional nonlinear Schrödinger equation, and used in [14,15] in related context. However due to the nonsmooth character of the symbol modeling the dispersive relation and other features our problem does not fall in the scope of these works.

After projecting into the positive and negative frequencies and applying the gauge transforms to a system formally equivalent to the equation in (1.1) one performs a fixed point argument using the smoothing properties associated to the linear problem however one needs to include the L^2 norm with weight x^2 to control the gauge transform. One also should observe that one key tool in our analysis to deal with the second-order derivatives in the nonlinear terms was the use of a commutator estimate (see Lemma 2.5 below) recently proved in [5]. The result in Theorem 1.1 should be the best possible using a fixed point argument which agrees with the result in [18].

One should also mention that the solution obtained in Theorem 1.1 solves the IVP (1.1). Actually one can prove that under suitable data the system (3.46) (below) and the IVP (1.1) are equivalent.

We can still improve the result in Theorem 1.1 noticing that H^2 would be sufficient to obtain solutions via contraction mapping principle except to control $\|xv\|_{L^2}$. Thus we can use an energy method argument to show the following result.

Theorem 1.2. *For any v_0 satisfying*

$$v_0 \in Z^2 = H^2(\mathbb{R}) \cap L^2\big(\mathbb{R}; x^2\,dx\big), \tag{1.5}$$

there exist $T = T(\|v_0\|_{Z^2}) > 0$ with $T(\alpha) \nearrow +\infty$ as $\alpha \to 0$, and a unique solution v of (1.1) *satisfying*

$$v \in C\big([0,T] : Z^2\big), \tag{1.6}$$

$$\partial_x^l P_\pm v \in L^6\big([0,T] : L^\infty(\mathbb{R})\big), \quad l = 0, 1, 2, \tag{1.7}$$

$$\partial_x^3 P_\pm v \in L^2\big([0,T]; L^2_{\mathrm{loc}}(\mathbb{R})\big), \tag{1.8}$$

$$\partial_x(xv) \in L^2\big([0,T]; L^2_{\mathrm{loc}}(\mathbb{R})\big), \tag{1.9}$$

and

$$\sum_{j=-\infty}^{+\infty} \sup_{0 \leqslant t \leqslant T} \sup_{j/N \leqslant x \leqslant j/N+1} \big|\partial_x^l P_\pm v\big|^2 < \infty, \quad l = 0, 1, \tag{1.10}$$

where $N = N(T) \in \mathbb{Z}_+$. Moreover, the flow map data-solution: $v_0 \mapsto v$ is continuous from Z^2 in the class (1.6)–(1.10).

The next question is whether one can obtain local well-posedness for the IVP (1.1) without imposing the solution being in a weighted space. As we commented above the restriction comes up when the gauge transform is implemented. More precisely, the gauge transforms in our argument can be chosen either as

$$\Phi(x,t) = \exp\left(i \int_{-\infty}^{x} v(y,t)\,dy\right) \tag{1.11}$$

or

$$\Phi(x,t) = \exp\left(i \int_{0}^{x} v(y,t)\,dy\right). \tag{1.12}$$

In the first case since we only have $v \in L^2(\mathbb{R})$ we need to use the $\|x(\cdot)\|_{L^2}$-norm to control Φ. The second case looks better since one just requires $v(\cdot,t) \in L^2_{\mathrm{loc}}(\mathbb{R})$ to make sense of Φ. However in this case the application associated to the integral equation fails to be a contraction.

To overcome that obstruction we use a compactness argument. We still use the gauge transform and the same kind of estimates established in Theorem 1.1 to obtain a priori estimates for smooth solutions of the IVP (1.1) provided by the argument used in [21]. Here is essential to select the gauge transform as in (1.12), this will allow us to take the limit in H^2 without restriction on the data. The result is as follows.

Theorem 1.3. *Let $s \geqslant 2$. For any $v_0 \in H^s(\mathbb{R})$, there exist a positive time $T = T(\|v_0\|_{H^s})$ and a unique solution v of* (1.1) *satisfying*

$$v \in C\big([0,T]; H^s(\mathbb{R})\big), \tag{1.13}$$

$$D_x^r \partial_x^l P_\pm v \in L^6\big([0,T]; L^\infty(\mathbb{R})\big), \quad l = 0, 1, 2, \tag{1.14}$$

$$P_\pm v \in L^2\big([0,T]; H^{s+1}_{\mathrm{loc}}(\mathbb{R})\big) \tag{1.15}$$

and

$$\sum_{j=-\infty}^{+\infty} \sup_{0 \leqslant t \leqslant T} \sup_{j/N \leqslant x \leqslant j/N+1} \big|D_x^r \partial_x P_\pm v\big|^2 < \infty, \tag{1.16}$$

where $0 \leqslant r \leqslant s-2$, $N = N(T) \in \mathbb{Z}_+$.

Moreover, for any $T' < T$, there exists a neighborhood V of v_0 in $H^s(\mathbb{R})$ such that the flow map data-solution: $\tilde{v}_0 \mapsto \tilde{v}$ from V into the class defined by (1.13)–(1.16) *with T' instead of T is continuous.*

In the light of these results, the above comments on those in [18] and the conservation law (1.3) (which gives an a priori estimate of the H^1-norm of the local solutions) the question of the local well-posedness in H^1 (which implies global well-posedness) naturally arises.

Our arguments may be further refined to obtain the result for $s > 7/4$. In fact, it may be possible that a modification of the original equation with an appropriate gauge transform as in the work of Tao [22] for the BO equation can be used to lower the regularity required for the existence. However in this case one still needs to rely on the existence of solutions which as pointed out before is first established here.

This paper is organized as follows: In Section 2, we introduce some notation and derive several estimates useful in the proof of our main results. The proof of Theorems 1.1 and 1.2 will be given in Section 3. Finally in Section 4 we show Theorem 1.3.

2. Notations and preliminary estimates

The following notation will be used throughout this article: $(\cdot,\cdot)_{L^2}$ denotes the L^2-scalar product for real-valued functions, while $J^s = (1-\Delta)^{\frac{s}{2}}$ and $D^s = (-\Delta)^{\frac{s}{2}}$ will denote the Bessel and Riesz potential of order $-s$. Note that $D^1 = \mathcal{H}\partial_x$. We will use k to denote a positive constant; moreover, for any positive numbers a and b, the notation $a \lesssim b$ means that $a \leqslant kb$. And we denote $a \sim b$, when $a \lesssim b$ and $b \lesssim a$.

Since the linear differential operators $a\partial_x^3 \pm ib\partial_x^2$ will appear in our analysis, we shall begin by considering the associated problems

$$\begin{cases} \big(\partial_t + a\partial_x^3 \pm ib\partial_x^2\big) w = F, \\ w(x,0) = w_0(x), \end{cases} \tag{2.17}$$

whose solutions when $F \equiv 0$ are given by the unitary groups $\{W_\pm(t)\}_{t\in\mathbb{R}}$ in $H^s(\mathbb{R})$, where $W_\pm(t) = e^{-t(a\partial_x^3 \pm ib\partial_x^2)}$. We also define the unitary group associated to the linear part of (1.1), $V(t) = e^{-t(a\partial_x^3 - b\mathcal{H}\partial_x^2)}$. We shall reduce the estimates for (2.17) to the known ones for the linearized KdV equation

$$\begin{cases} \partial_t z + a\partial_x^3 z = \widetilde{F}, \\ z(x,0) = z_0(x). \end{cases} \tag{2.18}$$

We will treat for example the case of W_+. Multiply the equation in (2.17) by $e^{i\frac{b}{3a}x}$ and define $f(x,t) = e^{i\frac{b}{3a}x}w(x,t)$, so

$$\partial_t f + a\partial_x^3 f + \frac{b^2}{3a}\partial_x f - i\frac{2}{27}\frac{b^3}{a^2}f = e^{i\frac{b}{3a}x}F. \tag{2.19}$$

It is deduced by setting $h(x,t) = f(x + \frac{b^2}{3a}t, t)$ that

$$\partial_t h + a\partial_x^3 h - i\frac{2}{27}\frac{b^3}{a^2}h = e^{i\frac{b}{3a}(x+\frac{b^2}{3a}t)}F\left(x + \frac{b^2}{3a}t, t\right). \tag{2.20}$$

Finally define

$$z(x,t) = e^{-i\frac{2}{27}\frac{b^3}{a^2}t}h(x,t) = e^{i\frac{b^3}{27a^2}t}e^{i\frac{b}{3a}x}w\left(x + \frac{b^2}{3a}t, t\right). \tag{2.21}$$

Then z is a solution to (2.18) with

$$z_0(x) = e^{i\frac{b}{3a}x}w_0(x) \quad \text{and} \quad \widetilde{F}(x,t) = e^{i\frac{b^3}{27a^2}t}e^{i\frac{b}{3a}x}F\left(x + \frac{b^2}{3a}t, t\right), \tag{2.22}$$

or equivalently if $z(x,t)$ is a solution to (2.18), then

$$w(x,t) = e^{-i\frac{b^3}{27a^2}t}e^{-i\frac{b}{3a}x}z\left(x - \frac{b^2}{3a}t, t\right) \tag{2.23}$$

solves (2.17) with the appropriate modifications on F and w_0.

Lemma 2.1. *There exist three positive constants c_1, c_2 and c_3 such that*

$$\left\| W_\pm(t)w_0 \right\|_{L_t^6 L_x^\infty} \leqslant c_1 \|w_0\|_{L^2}, \tag{2.24}$$

$$\left\{ \sup_{j\in\mathbb{Z}} \int_0^T \int_{-\infty}^{+\infty} \left|\chi_{j/N}\partial_x W_\pm(t)w_0(x)\right|^2 dx\,dt \right\}^{1/2} \leqslant c_2(1+T)^{\frac{1}{2}}\|w_0\|_{L^2}, \tag{2.25}$$

and

$$\left\{ \sum_{j=-\infty}^{+\infty} \sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{R}} \left|\chi_{j/N}(x)W_\pm(t)w_0(x)\right|^2 \right\}^{1/2} \leqslant c_3(1+T)^2\|w_0\|_{H^1}, \tag{2.26}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $[0,1]$, $\operatorname{supp}\chi \subset (-1,3)$, $\chi_{j/N} = \chi(\cdot - j/N)$ and $N = N(T) \in \mathbb{Z}_+$, $N \sim (1+T)$.

Remark 2.2. Estimates (2.24)–(2.26) still hold with V instead of $W_\pm$.

Proof. Fix $F \equiv 0$ in (2.17)–(2.23) for the rest of the proof. It is known from [13] that

$$\|z\|_{L_t^6 L_x^\infty} \leqslant c\|z_0\|_{L^2},$$

which implies (2.24), since all the transformations used in (2.20)–(2.23) preserve the $L_t^6 L_x^\infty$ norm.

Now, for $R > 0$, let $\chi^R \in C_0^\infty(\mathbb{R})$ such that $0 \leqslant \chi^R \leqslant 1$, $\chi^R \equiv 1$ on $[0, R]$, $\operatorname{supp}\chi^R \subset (-R, 3R)$ and $\chi_j^R = \chi^R(\cdot - j)$. It was proved in [13] that

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \left|\partial_x z(x,t)\right|^2 dt = \|z_0\|_{L^2}^2 = \|w_0\|_{L^2}^2.$$

Then it follows that

$$\sup_{j\in\mathbb{Z}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\chi_j^R(x)\partial_x z(x,t)\right|^2 dx\,dt \leqslant 3R\|w_0\|_2^2. \tag{2.27}$$

On the other hand, fix $R \geqslant 1 + \frac{b^2 T}{3a}$ and $N = N(T) \in \mathbb{Z}_+$ such that

$$\#\left\{l \in \mathbb{Z} \;\middle|\; 0 \leqslant l \leqslant R + \frac{b^2 T}{3a}\right\} < 2R + 1 \leqslant N.$$

Then, it is deduced from (2.21), (2.27) and the L^2-norm conservation for (2.17) that

$$\begin{aligned}
&\sup_{j\in\mathbb{Z}} \int_0^T \int_{-\infty}^{\infty} \left|\chi_{j/N}^1(x)\partial_x w(x,t)\right|^2 dx\,dt \\
&\leqslant \sup_{j\in\mathbb{Z}} \int_0^T \int_{-\infty}^{\infty} \left|\chi_j^{2R}\left(x - \frac{b^2 t}{3a}\right)\partial_x w(x,t)\right|^2 dx\,dt \\
&\leqslant \sup_{j\in\mathbb{Z}} \iint \left|\chi_j^{2R}(x)\partial_x z(x,t)\right|^2 dx\,dt + \sup_{j\in\mathbb{Z}} \iint \left|\chi_j^{2R}(x)\frac{b}{3a} w\left(x + \frac{b^2 t}{3a}, t\right)\right|^2 dx\,dt \\
&\leqslant \left(6R + \frac{bT}{3a}\right)\|w_0\|_{L^2}^2,
\end{aligned}$$

which implies (2.25).

The following maximal function estimate was derived in [12]:

$$\left(\sum_{j=-\infty}^{\infty} \sup_{|t|\leqslant T} \sup_{j<x<j+R} \left|z(x,t)\right|^2\right)^{1/2} \leqslant cR(1+T)\|z_0\|_{H^s}, \tag{2.28}$$

for any $s > \frac{3}{4}$. Moreover, it follows from (2.21) that

$$\|z_0\|_{H^1}^2 \leqslant \left(1 + \left(\frac{b}{3a}\right)^2\right)\|w_0\|_2^2 + \|\partial_x w_0\|_2^2. \tag{2.29}$$

Therefore, it is concluded from (2.28) and (2.29) that

$$\sum_{j=-\infty}^{\infty} \sup_{|t|\leqslant T} \sup_{x\in\mathbb{R}} \left|\chi_{j/N}^{1}(x)w(x,t)\right|^2 \leqslant \sum_{j=-\infty}^{\infty} \sup_{|t|\leqslant T} \sup_{x\in\mathbb{R}} \left|\chi_j^R\left(x-\frac{b^2t}{3a}\right)w(x,t)\right|^2$$
$$\leqslant c\left(1+\left(\frac{b}{3a}\right)^2\right)R^2(1+T)^2\|w_0\|_{H^1}^2,$$

which yields (2.26). □

Remark 2.3. Note that, contrarily to c_2 and c_3, the constant c_1 does not depend on the parameters a and b.

A result to commute V with x, which is proved in [19], will be useful.

Lemma 2.4. *Let*

$$\Gamma(x,t) = x - 3at\partial_x^2 + 2bt\mathcal{H}\partial_x.$$

Then,

$$\Gamma(x,t)V(t)\phi = V(t)(x\phi), \tag{2.30}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

The following lemma, proved in [5], will also be needed to estimate commutators involving the Hilbert transform and derivatives.

Lemma 2.5.

(i) *Let L denote one of the following operators: P_+, P_-, or $\mathcal{H}$, where $\widehat{P_\pm f}(\xi) = \chi_{\mathbb{R}_\pm}\widehat{f}(\xi)$. Then for any $p \in (1,\infty)$ and any $l, m \in \mathbb{Z}^+ \cup \{0\}$ with $l+m \geqslant 1$, there exists $c = c(p;l;m) > 0$ such that*

$$\left\|\partial_x^l[L,a]\partial_x^m f\right\|_p \leqslant c\left\|\partial_x^{(l+m)}a\right\|_\infty \|f\|_p. \tag{2.31}$$

(ii) *Let $\alpha \in [0,1)$, $\beta \in (0,1)$ with $\alpha+\beta \in [0,1]$. Then for any $p, q \in (1,+\infty)$ and for any $\delta > \frac{1}{q}$, there exists $c = c(\alpha,\beta,p,q,\delta) > 0$ such that*

$$\left\|D^\alpha\left[D^\beta,a\right]D^{1-(\alpha+\beta)}f\right\|_{L_x^p} \leqslant c\left\|J^\delta\partial_x a\right\|_{L_x^q}\|f\|_{L_x^p}. \tag{2.32}$$

We also recall some identities involving $\mathcal{H}$, P_+ and P_-.

Lemma 2.6. *It holds that*

$$\mathcal{H} = -i(P_+ - P_-) \quad \text{and} \quad 1 = P_+ + P_-, \tag{2.33}$$

$$\|\mathcal{H}\phi\|_{H^s} = \|\phi\|_{H^s}, \quad \forall s \in \mathbb{R} \text{ and } \forall \phi \in H^s(\mathbb{R}), \tag{2.34}$$

and

$$x\mathcal{H}(\phi) = \mathcal{H}(x\phi), \quad \forall \phi \in \mathcal{S}(\mathbb{R}) \quad \textit{satisfying} \quad \int_{\mathbb{R}} \phi \, dx = 0. \tag{2.35}$$

3. Proof of Theorems 1.1 and 1.2

3.1. The gauge transformation

First we perform a gauge transformation on (1.1) in order to eliminate the higher-order derivative terms in the nonlinearity. Taking the derivative of (1.1) leads to

$$\begin{aligned}\left(\partial_t + a\partial_x^3 - b\mathcal{H}\partial_x^2\right)\partial_x v &= c\left(\partial_x v \partial_x v + v\partial_x^2 v\right) - d\left(\mathcal{H}\partial_x v \partial_x^2 v + 2\partial_x v \mathcal{H}\partial_x^2 v\right) \\ &\quad - 3d\mathcal{H}\left(\partial_x v \partial_x^2 v\right) - d\left(v\mathcal{H}\partial_x^3 v + \mathcal{H}\left(v\partial_x^3 v\right)\right).\end{aligned} \tag{3.36}$$

To get rid of the Hilbert transform we shall project the equation in (3.36) into the positive and negative frequencies, using the identities (2.33), so

$$\begin{aligned}\left(\partial_t + a\partial_x^3 - b\mathcal{H}\partial_x^2\right)\partial_x P_+ v &= cP_+\left(\partial_x v \partial_x v + v\partial_x^2 v\right) - dP_+\left(\mathcal{H}\partial_x v \partial_x^2 v + 2\partial_x v \mathcal{H}\partial_x^2 v\right) \\ &\quad + 3idP_+\left(\partial_x v \partial_x^2 v\right) - dP_+\left(v\mathcal{H}\partial_x^3 v\right) + idP_+\left(v\partial_x^3 v\right).\end{aligned} \tag{3.37}$$

Observe that the nonlinear terms on the right-hand side of (3.37) are of two types (i) and (ii). The terms in class (i) involve up to “second-order derivatives” in v, and when it does appear (the second-order derivatives) is multiplied by at worst order one (or order zero). So it suffices to consider the term $-2dP_+(\partial_x v\mathcal{H}\partial_x^2 v)$ as a representative of this class, which can be rewritten using (2.33) as

$$2idP_+\left(\partial_x v\left(\partial_x^2 P_+ v - \partial_x^2 P_- v\right)\right). \tag{3.38}$$

The terms in the second class (ii) involving “third-order derivatives” in v can be handled as follows

$$\begin{aligned}-dP_+\left(v\mathcal{H}\partial_x^3 v\right) + idP_+\left(v\partial_x^3 v\right) &= -dvP_+\mathcal{H}\partial_x^3 v - d[P_+, v]\partial_x\mathcal{H}\partial_x^2 v \\ &\quad + id\left\{vP_+\partial_x^3 v + [P_+, v]\partial_x\partial_x^2 v\right\} \\ &= 2idv\partial_x^3 P_+ v + 2id[P_+, v]\partial_x^3 P_+ v.\end{aligned} \tag{3.39}$$

Thus Eq. (3.36) is rewritten as

$$\begin{aligned}\left(\partial_t + a\partial_x^3 + ib\partial_x^2\right)\partial_x P_+ v &= P_+ Q_+(v, \partial_x P_+ v, \partial_x P_- v) \\ &\quad + 2idv\partial_x^3 P_+ v + 2id[P_+, v]\partial_x^3 P_+ v.\end{aligned} \tag{3.40}$$

Taking the complex conjugate of (3.40) and using the fact that $\overline{P_+ v} = P_- \bar{v}$, since v is real-valued, we get a similar equation for $P_- v$, *i.e.*,

$$\begin{aligned}\left(\partial_t + a\partial_x^3 - ib\partial_x^2\right)\partial_x P_- v &= P_- Q_-(v, \partial_x P_+ v, \partial_x P_- v) \\ &\quad - 2idv\partial_x^3 P_- v - 2id[P_-; v]\partial_x^3 P_- v.\end{aligned} \tag{3.41}$$

Here Q_+ and Q_- are quadratic polynomials, whose "worse" terms are of order 2 in v (or order 1 in $\partial_x P_\pm v$) multiplied by one of order 1 or 0 in v. Moreover, the last terms on the right-hand side of (3.40) and (3.41) can be considered as terms in the first class (i) because of the commutator estimate (2.31).

Next, we perform a gauge transformation on (3.40)–(3.41) in order to eliminate the higher-order derivative terms in the nonlinearity. Multiply the equation in (3.40) by $\Psi = \Psi(x,t)$ and use the identities

$$\Psi \partial_t \partial_x P_+ v = \partial_t(\Psi \partial_x P_+ v) - \partial_t \Psi \partial_x P_+ v, \tag{3.42a}$$

$$\begin{aligned} \Psi a \partial_x^3 \partial_x P_+ v = {} & a\partial_x^3(\Psi \partial_x P_+ v) - 3a\partial_x \Psi \partial_x^3 P_+ v \\ & - 3a\partial_x^2 \Psi \partial_x^2 P_+ v - a\partial_x^3 \Psi \partial_x P_+ v, \end{aligned} \tag{3.42b}$$

and

$$\Psi \partial_x^2 \partial_x P_+ v = ib\partial_x^2(\Psi \partial_x P_+ v) - 2ib\partial_x \Psi \partial_x^2 P_+ v - ib\partial_x^2 \Psi \partial_x P_+ v. \tag{3.42c}$$

We collect the "real" terms of order 3 in v

$$3a\partial_x \Psi \partial_x^3 P_+ v + 2idv\Psi \partial_x^3 P_+ v, \tag{3.43}$$

and want to choose Ψ to vanish this expression, so $-3a\partial_x \Psi + i2dv\Psi = 0$, *i.e.*

$$\Psi(x,t) = \exp\left(-i\frac{2d}{3a}\int_{-\infty}^{x} v(s,t)\,ds\right). \tag{3.44}$$

In the sequel, we will denote Ψ by Ψ_+ and its complex conjugate $\overline{\Psi}$ by Ψ_-. The term $\partial_t \Psi \partial_x P_+ v$ appearing in (3.42a) can be rewritten by making use of the equation in (1.1) as

$$-\partial_x P_+ v \Psi i \frac{2d}{3a}\left\{-a\partial_x^2 v + b\mathcal{H}\partial_x v + c\frac{v^2}{2} - d\big(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)\big)\right\}(x,t), \tag{3.45}$$

which falls in the class considered in (i).

Therefore, after defining the new variables $w_\pm = \Psi_\pm \partial_x P_\pm v$, we get the following dispersive system

$$\begin{cases} \big(\partial_t + a\partial_x^3 - b\mathcal{H}\partial_x^2\big)v = cv\partial_x v + id\partial_x\big(v(\Psi_- w_+ - \Psi_+ w_-)\big) \\ \qquad\qquad - d\mathcal{H}\partial_x\big(v(\Psi_- w_+ + \Psi_+ w_-)\big), \\ \big(\partial_t + a\partial_x^3 \pm ib\partial_x^2\big)w_\pm = \Psi_\pm P_\pm Q_\pm^\dagger(\Psi_\pm, v, w_\pm) + \Psi_\pm Q_\pm^\ddagger(\Psi_\pm, v, w_\pm) \\ \qquad\qquad + N_\pm(\Psi_\pm, v, w_\pm), \end{cases} \tag{3.46}$$

where $Q_\pm^\dagger$ and $Q_\pm^\ddagger$ are polynomials at least quadratic involving the first derivative of $v_\pm$ and $w_\pm$ multiplied by a term of order zero, for example

$$F_1 = \Psi_+ P_+(w_+ \partial_x w_+), \tag{3.47}$$

and $N_\pm$ are the last terms on the right-hand side of (3.40) and (3.41), which is to say

$$F_2 = [P_+, v]\partial_x^3 P_+ v = [P_+, v]\partial_x^2(\Psi_- w_+). \tag{3.48}$$

We will also choose

$$F_3 = \mathcal{H}\partial_x(v\Psi_- w_+) \tag{3.49}$$

as a representative term for the nonlinearity on the right-hand side of the first equation in (3.46).

Finally, the gauge transformed system (3.46) is solved by using a fixed point argument.

Proposition 3.1. *Let $k \in \mathbb{Z}_+$, $k \geqslant 2$. Then for any $(v_0, w_{\pm 0})$ such that*

$$(v_0, w_{\pm 0}) \in \big(H^k(\mathbb{R}) \cap L^2\big(\mathbb{R}; x^2\,dx\big)\big) \times H^k(\mathbb{R})^2 = X^k, \tag{3.50}$$

there exist $T = T(\|(v_0, w_{\pm 0})\|_{X^k}) > 0$ with $T(\alpha) \nearrow +\infty$ as $\alpha \to 0$, a space X_T^k such that $X_T^k \hookrightarrow C([0,T]; X^k)$, and a unique solution $(v, w_\pm)$ of (3.46) in X_T^k satisfying

$$\big(v(\cdot,0), w_\pm(\cdot,0)\big) = (v_0, w_{\pm 0}). \tag{3.51}$$

Moreover, for any $T' \in (0,T)$, there exists a neighborhood Ω^k of $(v_0, w_{\pm 0})$ in X^k such that the flow map data-solution is smooth from Ω^k into $X_{T'}^k$.

Proof. For sake of simplicity, we will only consider the case $k = 2$. The integral system associated to (3.46) can be written as

$$\begin{cases} v = \mathcal{F}(v, w_\pm) = V(t)v_0 + \displaystyle\int_0^t V\big(t-t'\big)\mathcal{N}(v, w_\pm)\big(t'\big)\,dt', \\ w_\pm = \mathcal{G}_\pm(v, w_\pm) = W_\pm(t)w_{\pm 0} + \displaystyle\int_0^t W_\pm\big(t-t'\big)\mathcal{M}_\pm(v, w_\pm)\big(t'\big)\,dt', \end{cases} \tag{3.52}$$

where $\mathcal{N}$ and $\mathcal{M}_\pm$ denote the nonlinearities on the right-hand side of (3.46). Let $T > 0$. Define the following semi-norms

$$\begin{aligned}
\lambda_1^T(f) &= \sup_{0\leqslant t\leqslant T} \big\|f(t)\big\|_{H^2}, \\
\lambda_2^T(f) &= \sum_{l=0}^{3} \big\|\partial_x^l f\big\|_{L_T^6 L_x^\infty}, \\
\lambda_3^T(f) &= \left\{\sup_j \int_0^T \int \big|\chi_{j/N}(x)\partial_x^3 f(x,t)\big|^2\,dx\,dt\right\}^{1/2}, \\
\lambda_4^T(f) &= \sum_{l=0}^{1} \left\{\sum_j \sup_{|t|\leqslant T} \sup_x \big|\chi_{j/N}(x)\partial_x^l f(x,t)\big|^2\right\}^{1/2}, \\
\lambda_5^T(f) &= \sup_{0\leqslant t\leqslant T} \|xf\|_{L_x^2}, \\
\lambda_6^T(f) &= \left\{\sup_j \int_0^T \int \big|\chi_{j/N}(x)\partial_x\big(xf(x,t)\big)\big|^2\,dx\,dt\right\}^{1/2}.
\end{aligned} \tag{3.53}$$

Since N depends on T, we fix $0 < T \leqslant 1$, so that the constants appearing on the estimates (2.25) and (2.26) are fixed. Then, we define the Banach space X_T^2 by

$$X_T^2 = \{(v, w_\pm) \in C([0,T]; X^2) \mid \|(v, w_\pm)\|_{X_T^2} < \infty\},$$

where

$$\|(v, w_\pm)\|_{X_T^2} = \sum_{j=1}^{6} \lambda_j^T(v) + \sum_{j=1}^{4} \lambda_j^T(w_\pm),$$

and X^2 is defined in (3.50). Note that if $(v, w_\pm) \in X_T^2$, it follows that $\Psi_\pm$ are well defined so that the gauge transform system (3.46) makes sense. Moreover, we have that $\|\Psi_\pm\|_{L^\infty_{x,T}} \leqslant 1$, since v is a real-valued function.

Using the integral equation, Minkowski's integral inequality, identity (2.30) and the linear estimates obtained in Lemma 2.1, it is deduced that

$$\begin{aligned}
&\sum_{j=1}^{6} \lambda_j(\mathcal{F}) + \sum_{j=1}^{4} \lambda_j(\mathcal{G}_+) + \sum_{j=1}^{4} \lambda_j(\mathcal{G}_-) \\
&\quad \lesssim \|(v_0, w_{\pm 0})\|_{X^2} + \int_0^T \big(\|\mathcal{N}(t)\|_{H^2} + \|x\mathcal{N}(t)\|_{L^2} + \|\mathcal{M}_\pm(t)\|_{H^2}\big)\, dt. \qquad (3.54)
\end{aligned}$$

Then it remains to estimate $\|\partial_x^2 F_j\|_{L^1_T L^2_x}$, $j = 1, 2, 3$, and $\|xF_3\|_{L^1_T L^2_x}$ where F_j are defined in (3.47)–(3.49), since they are the representative terms of the nonlinearities $\mathcal{N}$ and $\mathcal{M}_\pm$. We have, using Hölder's inequality,

$$\begin{aligned}
\|\partial_x^2 F_1\|_{L^1_T L^2_x} &\lesssim \|\partial_x^2 \Psi_+ P_+(w_+ \partial_x w_+)\|_{L^1_T L^2_x} + \|\partial_x \Psi_+ P_+(\partial_x w_+ \partial_x w_+)\|_{L^1_T L^2_x} \\
&\quad + \|\partial_x \Psi_+ P_+(w_+ \partial_x^2 w_+)\|_{L^1_T L^2_x} + \|\Psi_+ P_+(\partial_x w_+ \partial_x^2 w_+)\|_{L^1_T L^2_x} \\
&\quad + \|\Psi_+ P_+(w_+ \partial_x^3 w_+)\|_{L^1_T L^2_x} \\
&\lesssim \int_0^T \|w_+\|_{L^\infty_x}\big(1 + \|\partial_x v\|_{L^\infty_x} + \|v\|^2_{L^\infty_x}\big)\|\partial_x w_+\|_{L^2_x}\, dt \\
&\quad + \int_0^T \|\partial_x v\|_{L^\infty_x} \|\partial_x w_+\|_{L^\infty_x} \|\partial_x w_+\|_{L^2_x}\, dt \\
&\quad + \int_0^T \big(\|\partial_x v\|_{L^\infty_x} \|w_+\|_{L^\infty_x} + \|\partial_x w_+\|_{L^\infty_x}\big)\|\partial_x^2 w_+\|_{L^2_x}\, dt \\
&\quad + T^{\frac{1}{2}} \|\chi_{j/N} w_+ \partial_x^3 w_+\|_{l^2_j L^2_{x,T}} \\
&\lesssim T\big(1 + \lambda_1^T(v) + \lambda_1^T(v)^2\big)\lambda_1^T(w_+)^2 + T^{\frac{1}{2}} \lambda_3^T(w_+)\lambda_4^T(w_+). \qquad (3.55)
\end{aligned}$$

Next it is deduced from estimate (2.31) that

$$\begin{aligned}\left\|\partial_x^2 F_2\right\|_{L_T^1 L_x^2} &\leqslant \left\|\partial_x^2[P_+, v]\partial_x^2(\Psi_- w_+)\right\|_{L_T^1 L_x^2} \\ &\leqslant \int_0^T \left\|\partial_x^2 v\right\|_{L_x^\infty}\left\|\partial_x^2(\Psi_- w_+)\right\|_{L_x^2} dt \\ &\lesssim T^{\frac{5}{6}}\lambda_2^T(v)\big(1+\lambda_1^T(v)+\lambda_1^T(v)^2\big)\lambda_1^T(w_+). \end{aligned} \tag{3.56}$$

The estimate for $\|\partial_x^2 F_3\|_{L_T^1 L_x^2}$ follows similarly to the one for $\|\partial_x^2 F_1\|_{L_T^1 L_x^2}$. Hence, it remains to bound $\|xF_3\|_{L_T^1 L_x^2}$. In this direction, observe from (2.35) that

$$\begin{aligned} xF_3 &= \mathcal{H}\big(x\partial_x(v\Psi_- w_+)\big) \\ &= \mathcal{H}\big(\partial_x(xv)\Psi_- w_+\big) - \mathcal{H}(v\Psi_- w_+) + \mathcal{H}\big(xv\,\partial_x(\Psi_- w_+)\big). \end{aligned}$$

Thus, Hölder's inequality yields

$$\begin{aligned}\|xF_3\|_{L_T^1 L_x^3} \leqslant{}& T^{\frac{1}{2}}\left\|\chi_{j/N}\Psi_- w_+\partial_x(xv)\right\|_{l_j^2 L_{x,T}^2} \\ &+ \int_0^T \left(\|v\|_{L_x^\infty}\|w_+\|_{L_x^2} + \|xv\|_{L_x^2}\left\|\partial_x(\Psi_- w_+)\right\|_{L_x^\infty}\right) dt \\ \lesssim{}& T^{\frac{1}{2}}\lambda_6^T(v)\lambda_4^T(w_+) + T\lambda_1^T(v)\lambda_1^T(w_+) \\ &+ T\lambda_5^T(v)\big(1+\lambda_1^T(v)\big)\lambda_1^T(w_+). \end{aligned} \tag{3.57}$$

Therefore, it is deduced combining estimates (3.54)–(3.57) that there exist positive constants θ and k and a polynomial p with all terms at least quadratic such that

$$\left\|\big(\mathcal{F}(v, w_\pm), \mathcal{G}_\pm(v, w_\pm)\big)\right\|_{X_T^2} \leqslant k\left\|(v_0, w_{\pm 0})\right\|_{X^2} + kT^\theta p\big(\left\|(v, w_\pm)\right\|_{X_T^2}\big), \tag{3.58}$$

which concludes the proof of Proposition 3.1 by a fixed point argument (see [14] for example). □

Theorem 1.1 follows by applying Proposition 3.1 to the gauge transformed system (3.46) with initial data $(v_0, \Psi_{0,-}P_+\partial_x v_0, \Psi_{0,+}P_-\partial_x v_0)$.

3.2. A priori estimates in $H^2(\mathbb{R}) \cap L^2(\mathbb{R}; x^2\,dx)$

To prove Theorem 1.2, it would be sufficient to bound $\|xv\|_{L_T^\infty L_x^2}$ using energy estimates for the $H^3(\mathbb{R}) \cap L^2(|x|^2\,dx)$ solutions given by Theorem 1.1 in terms of "H^2-solution". To do so we will use the equation in (1.1) directly.

We multiply the equation in (1.1) by xv and integrate with respect to x. We have that

$$\int x\partial_t v xv = \frac{1}{2}\frac{d}{dt}\int (xv)^2. \tag{3.59}$$

Using (2.35) and integration by parts

$$\begin{aligned}\int x\partial_x^2\mathcal{H}vxv &= -2\int x\partial_x\mathcal{H}vv - \int x\partial_x\mathcal{H}vx\partial_x v\\ &= -2\int x\partial_x\mathcal{H}vv + 2\int x\mathcal{H}v\partial_x v + \int x\mathcal{H}vx\partial_x^2 v\\ &= -4\int x\partial_x\mathcal{H}vv + 2\int x\partial_x(\mathcal{H}vv) - \int v\mathcal{H}\big(x^2\partial_x^2 v\big)\\ &= -4\int x\partial_x\mathcal{H}vv - 2\int \mathcal{H}vv - \int vx^2\mathcal{H}\partial_x^2 v. \end{aligned}\tag{3.60}$$

Thus

$$\int x\partial_x^2\mathcal{H}vxv = -2\int x\partial_x\mathcal{H}vv \leqslant 2\|\partial_x v\|_{L^2}\|xv\|_{L^2}. \tag{3.61}$$

By integration by parts

$$\int x\partial_x^3 vxv = -2\int x\partial_x^2 vv - \int x\partial_x^2 vx\partial_x v. \tag{3.62}$$

Then

$$-2\int x\partial_x^2 vv \leqslant 2\big\|\partial_x^2 v\big\|_{L^2}\|xv\|_{L^2} \tag{3.63}$$

and

$$\begin{aligned}-\int x\partial_x^2 vx\partial_x v &= -\int x^2\partial_x\left(\frac{(\partial_x v)^2}{2}\right) = \int x\partial_x v\partial_x v = -\int \partial_x vv - \int xv\partial_x^2 v\\ &\leqslant \|xv\|_{L^2}\big\|\partial_x^2 v\big\|_{L^2}. \end{aligned}\tag{3.64}$$

Thus combining (3.62)–(3.64) we get

$$\left|\int x\partial_x^3 vxv\right| \leqslant 3\big\|\partial_x^2 v\big\|_{L^2}\|xv\|_{L^2}. \tag{3.65}$$

Using the Sobolev inequality

$$\int xv\partial_x vxv \leqslant \|\partial_x v\|_{L^\infty}\|xv\|_{L^2}^2 \leqslant c\|v\|_{H^2}\|xv\|_{L^2}^2. \tag{3.66}$$

Finally, using Leibniz's rule we have that

$$\begin{aligned}&\int x\partial_x(v\mathcal{H}\partial_x v)xv + \int x\partial_x\mathcal{H}(v\partial_x v)xv\\ &\quad= \int x\partial_x v\mathcal{H}\partial_x vxv + \int (xv)^2\mathcal{H}\partial_x^2 v + \int \mathcal{H}\big(x\partial_x v\partial_x v + xv\partial_x^2 v\big)xv. \end{aligned}\tag{3.67}$$

Using Hölder's inequality it follows that

$$\left|\int (xv)^2\mathcal{H}\partial_x^2 v + \int \mathcal{H}\big(xv\partial_x^2 v\big)xv\right| \leqslant c\big(\big\|\mathcal{H}\partial_x^2 v\big\|_{L^\infty} + \big\|\partial_x^2 v\big\|_{L^\infty}\big)\|xv\|_{L^2}^2. \tag{3.68}$$

On the other hand, integration by parts, Hilbert transform properties and the Cauchy–Schwarz inequality yield

$$\begin{aligned}\int x\partial_x v\mathcal{H}\partial_x v xv &= -\int \mathcal{H}\left(x^2\partial_x\frac{v^2}{2}\right)\partial_x v = \int \mathcal{H}\big(xv^2\big)\partial_x v - \frac{1}{2}\int (xv)^2\mathcal{H}\partial_x^2 v \\ &\leqslant \|v\|_{H^2}^2\|xv\|_{L^2} + \frac{1}{2}\big\|\mathcal{H}\partial_x^2 v\big\|_{L^\infty}\|xv\|_{L^2}^2,\end{aligned} \tag{3.69}$$

and

$$\begin{aligned}\int \mathcal{H}(x\partial_x v\partial_x v)xv &= \int v\partial_x v\mathcal{H}(xv) + \int xv\partial_x^2 v\mathcal{H}(xv) \\ &\quad + \int xv\partial_x v\mathcal{H}v + \int x^2\partial_x\left(\frac{v^2}{2}\right)\mathcal{H}\partial_x v \\ &= \int v\partial_x v\mathcal{H}(xv) + \int xv\partial_x^2 v\mathcal{H}(xv) + \int xv\partial_x v\mathcal{H}v \\ &\quad - \int xv^2\mathcal{H}\partial_x v - \frac{1}{2}\int (xv)^2\mathcal{H}\partial_x^2 v \\ &\leqslant \|v\|_{H^2}^2\|xv\|_{L^2} + \left(\big\|\partial_x^2 v\big\|_{L^\infty} + \frac{1}{2}\big\|\mathcal{H}\partial_x^2 v\big\|_{L^\infty}\right)\|xv\|_{L^2}^2.\end{aligned} \tag{3.70}$$

Getting together (3.59)–(3.61) and (3.65)–(3.70) we have that

$$\frac{1}{2}\frac{d}{dt}\|xv\|_{L^2}^2 \leqslant c\big(\|v\|_{H^2} + \|v\|_{H^2}^2\big)\|xv\|_{L^2} + c\big(\big\|\partial_x^2 v\big\|_{L^\infty} + \big\|\mathcal{H}\partial_x^2 v\big\|_{L^\infty}\big)\|xv\|_{L^2}^2. \tag{3.71}$$

Now our $H^3(\mathbb{R}) \cap L^2(|x|^2\,dx)$ solution can be extended in $H^2(\mathbb{R}) \cap L^2(|x|^2\,dx)$ in a time interval depending only on

$$\delta = \|v_0\|_{H^2} + \|xv_0\|_{L^2}.$$

Since we are using the fixed point in the norms in (3.53) except $\|xv(t)\|_{L^2}$, our solution for given v_0 as in (1.5) satisfies (1.6)–(1.10) and is unique in this class.

4. Proof of Theorem 1.3

4.1. Proof of the uniqueness

Let v^1 and v^2 be two solutions of (1.1) associated with the same initial data v_0, defined on a time interval $[0,T]$, and satisfying (1.13)–(1.16). We will denote by M_T^j, $j = 1,2$, the maximum of the quantities defined on (1.13)–(1.16) for v^j and by M_T the maximum of M_T^1 and M_T^2.

Introducing $u = v^1 - v^2$, one has that u satisfies the equation

$$\partial_t u - b\mathcal{H}\partial_x^2 u + a\partial_x^3 u = \frac{c}{2}\partial_x\big(\big(v^1 + v^2\big)u\big) - d\partial_x\big(u\mathcal{H}\partial_x v^1 + v^2\mathcal{H}\partial_x u\big)$$
$$- d\partial_x\big(\mathcal{H}\big(u\partial_x v^1\big) + \mathcal{H}\big(v^2\partial_x u\big)\big), \tag{4.72}$$

with initial data $u(\cdot, 0) = 0$. We deduce multiplying (4.72) by u, integrating in the space variable x and then integrating by parts that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 &= \frac{c}{4}\big(\partial_x\big(v^1 + v^2\big)u, u\big)_{L^2} + d\big(u\mathcal{H}\partial_x v^1, \partial_x u\big)_{L^2} + d\big(v^2\mathcal{H}\partial_x u, \partial_x u\big)_{L^2} \\ &\quad - d\big(u\partial_x v^1, \mathcal{H}\partial_x u\big)_{L^2} - d\big(v^2\partial_x u, \mathcal{H}\partial_x u\big)_{L^2} \\ &= \frac{c}{4}\big(\partial_x\big(v^1 + v^2\big)u, u\big)_{L^2} - \frac{d}{2}\big(u\mathcal{H}\partial_x^2 v^1, u\big)_{L^2} \\ &\quad + d\big(u\partial_x^2 v^1, \mathcal{H}u\big)_{L^2} + d\big(\partial_x u\partial_x v^1, \mathcal{H}u\big)_{L^2}.\end{aligned}$$

Therefore it follows from Hölder and Young's inequalities that, for all $\delta > 0$, there exists a positive constant k such that

$$\frac{d}{dt}\|u\|_{L^2}^2 \leqslant k\big(\big\|\partial_x\big(v^1 + v^2\big)\big\|_{L^\infty} + \big\|P_\pm\partial_x^2 v^1\big\|_{L^\infty} + \delta^{-1}\big)\|u\|_{L^2}^2 + \delta\int\big(\partial_x v^1\partial_x u\big)^2\,dx.$$

This leads, after integration in time between 0 and T, to

$$\big\|u(T)\big\|_{L^2}^2 \leqslant k\big(M_T + \delta^{-1}\big)\int_0^T\big\|u(t)\big\|_{L^2}^2\,dt + \delta f(T), \quad \forall\delta > 0, \tag{4.73}$$

where

$$f(T) = \int_0^T\int\big(\partial_x v^1\partial_x u\big)^2\,dx\,dt. \tag{4.74}$$

Following the ideas of Ponce for the fifth-order KdV equation [21], we derive a Kato type smoothing effect for u.

Lemma 4.1. *There exists a positive constant k such that for all* $\delta > 0$

$$\int_0^T\int\big(\partial_x u + D^1 u\big)^2\psi'\,dx\,dt$$
$$\leqslant k(1 + M_T)\big\|u(T)\big\|_{L^2}^2 + k\big(M_T + M_T^2 + \delta^{-1}\big)\int_0^T\big\|u(t)\big\|_{L^2}^2\,dt + \delta f(T), \tag{4.75}$$

where $f(T)$ *is defined in* (4.74) *and* $\psi \in C^\infty(\mathbb{R})$ *with* $\psi' \in C_0^\infty(\mathbb{R})$ *and* $\psi, \psi' \geqslant 0$.

Assuming Lemma 4.1 (which will be proved below), we shall prove the uniqueness part of Theorem 1.3. We use the splitting argument of Ginibre and Tsutsumi to get that

$$
\begin{aligned}
f(T) &\leqslant \sum_{j=-\infty}^{+\infty} \int_0^T \int \left(\partial_x v^1 \partial_x u\right)^2 \chi_{j/N}^3 \, dx \, dt \\
&\leqslant \left(\sum_{j=-\infty}^{+\infty} \sup_{\mathbb{R}\times[0,T]} \left(\partial_x v^1 \chi_{j/N}\right)^2 \right) \left(\sup_{j\in\mathbb{Z}} \int_0^T \int (\partial_x u)^2 \chi_{j/N} \, dx \, dt \right),
\end{aligned}
$$

where $\chi_{j/N}$ is defined as in Lemma 2.1. Thus, we deduce applying (4.75) with $\psi' = \chi_{j/N}$ and by the definition of M_T that for all $\delta > 0$,

$$
\begin{aligned}
f(T) \leqslant{}& k M_T (1 + M_T) \left\| u(T) \right\|_{L^2}^2 \\
&+ k M_T \left(M_T + M_T^2 + \delta^{-1} \right) \int_0^T \left\| u(t) \right\|_{L^2}^2 \, dt + M_T \delta f(T).
\end{aligned} \tag{4.76}
$$

Inserting (4.76) in (4.73) and fixing $\delta > 0$ such that

$$
M_T \delta + 2k M_T (1 + M_T) \delta \leqslant \frac{1}{2},
$$

we deduce that

$$
\left\| u(T) \right\|_{L^2}^2 \leqslant k \left(M_T + \delta^{-1} \right) \int_0^T \left\| u(t) \right\|_{L^2}^2 \, dt.
$$

Therefore we conclude from Gronwall's inequality and the fact that $u(\cdot, 0) = 0$ that $\|u(t)\|_{L^2} = 0$ for all $t \in [0, T]$, which yields the uniqueness result.

It remains to prove Lemma 4.1. For this purpose the following commutator estimate, which can be found in [6, pp. 249–252], will be useful.

Lemma 4.2. *For any $s \in \mathbb{R}$ and $\sigma > \frac{1}{2}$, there exists $c = c_{s,\sigma} > 0$ such that*

$$
\left\| \left[J^s, \phi \right] f \right\|_{L^2} \leqslant c \|\phi\|_{H^{s+2+\sigma}} \|f\|_{H^{s-1}}, \tag{4.77}
$$

for all $\phi \in H^{s+2+\sigma}(\mathbb{R})$, $f \in H^{s-1}(\mathbb{R})$.

Proof of Lemma 4.1. *Step* 1: *gain of* 1/2 *derivative*. Let ψ be as in Lemma 4.1. We apply $J^{-\frac{1}{2}}$ to Eq. (4.72), multiply by $J^{-\frac{1}{2}} u \psi$ and integrate in space to deduce that

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left(J^{-\frac{1}{2}} u, J^{-\frac{1}{2}} u \psi \right)_{L^2} ={}& -a \left(J^{-\frac{1}{2}} \partial_x^3 u, J^{-\frac{1}{2}} u \psi \right)_{L^2} + b \left(J^{-\frac{1}{2}} \mathcal{H} \partial_x^2 u, J^{-\frac{1}{2}} u \psi \right)_{L^2} \\
&+ \frac{c}{2} \left(J^{-\frac{1}{2}} \partial_x \left(\left(v^1 + v^2 \right) u \right), J^{-\frac{1}{2}} u \psi \right)_{L^2} \\
&- d \left(J^{-\frac{1}{2}} \partial_x \left(u \mathcal{H} \partial_x v^1 + \mathcal{H} \left(u \partial_x v^1 \right) \right), J^{-\frac{1}{2}} u \psi \right)_{L^2}
\end{aligned}
$$

$$\begin{aligned}&-d\big(J^{-\frac{1}{2}}\partial_x\big(v^2\mathcal{H}\partial_x u+\mathcal{H}\big(v^2\partial_x u\big)\big),\,J^{-\frac{1}{2}}u\psi\big)_{L^2}\\&:=I+II+III+IV+V,\end{aligned}\tag{4.78}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(J^{-\frac{1}{2}}\mathcal{H}u,\,J^{-\frac{1}{2}}\mathcal{H}u\psi\big)_{L^2}&=-a\big(J^{-\frac{1}{2}}\partial_x^3\mathcal{H}u,\,J^{-\frac{1}{2}}\mathcal{H}u\psi\big)_{L^2}-b\big(J^{-\frac{1}{2}}\partial_x^2u,\,J^{-\frac{1}{2}}\mathcal{H}u\psi\big)_{L^2}\\&\quad+\frac{c}{2}\big(J^{-\frac{1}{2}}\partial_x\mathcal{H}\big(\big(v^1+v^2\big)u\big),\,J^{-\frac{1}{2}}\mathcal{H}u\psi\big)_{L^2}\\&\quad-d\big(J^{-\frac{1}{2}}\partial_x\mathcal{H}\big(u\mathcal{H}\partial_x v^1+\mathcal{H}\big(u\partial_x v^1\big)\big),\,J^{-\frac{1}{2}}\mathcal{H}u\psi\big)_{L^2}\\&\quad-d\big(J^{-\frac{1}{2}}\partial_x\mathcal{H}\big(v^2\mathcal{H}\partial_x u+\mathcal{H}(v_2\partial_x u)\big),\,J^{-\frac{1}{2}}\mathcal{H}u\psi\big)_{L^2}\\&:=\widetilde{I}+\widetilde{II}+\widetilde{III}+\widetilde{IV}+\widetilde{V}.\end{aligned}\tag{4.79}$$

First, we observe integrating by parts and using the Cauchy–Schwarz inequality that

$$I=-\frac{3a}{2}\big(J^{-\frac{1}{2}}\partial_x u,\,J^{-\frac{1}{2}}\partial_x u\psi'\big)_{L^2}+\mathcal{O}\big(\|u\|_{L^2}^2\big),\tag{4.80}$$

and

$$\widetilde{I}=-\frac{3a}{2}\big(J^{-\frac{1}{2}}\partial_x\mathcal{H}u,\,J^{-\frac{1}{2}}\partial_x\mathcal{H}u\psi'\big)_{L^2}+\mathcal{O}\big(\|u\|_{L^2}^2\big),\tag{4.81}$$

where $\mathcal{O}(f)$ denotes big O of f. We also get integrating by parts that

$$II=-b\big(J^{-\frac{1}{2}}\mathcal{H}\partial_x u,\,J^{-\frac{1}{2}}\partial_x u\psi\big)_{L^2}-b\big(J^{-\frac{1}{2}}\mathcal{H}\partial_x u,\,J^{-\frac{1}{2}}u\psi'\big)_{L^2}$$

and

$$\widetilde{II}=b\big(J^{-\frac{1}{2}}\partial_x u,\,J^{-\frac{1}{2}}\mathcal{H}\partial_x u\psi\big)_{L^2}+b\big(J^{-\frac{1}{2}}\partial_x u,\,J^{-\frac{1}{2}}\mathcal{H}u\psi'\big)_{L^2}.$$

Then

$$\begin{aligned}II+\widetilde{II}&=2b\big(J^{-\frac{1}{2}}\partial_x u,\,J^{-\frac{1}{2}}\mathcal{H}u\psi'\big)_{L^2}+b\big(J^{-\frac{1}{2}}u,\,J^{-\frac{1}{2}}\mathcal{H}u\psi''\big)_{L^2}\\&=2b\big(J^{-1}\partial_x u,\,J^{-\frac{1}{2}}\mathcal{H}u\psi'\big)_{L^2}+2b\big(\big[J^{-\frac{1}{2}},\psi'\big]J^{-\frac{1}{2}}\partial_x u,\mathcal{H}u\big)_{L^2}+\mathcal{O}\big(\|u\|_{L^2}^2\big),\end{aligned}$$

so that we deduce from (4.77) that

$$II+\widetilde{II}\lesssim\|u\|_{L^2}^2.\tag{4.82}$$

Using similar arguments, we deduce that

$$III+IV+\widetilde{III}+\widetilde{IV}\lesssim\big(\big\|v^1\big\|_{H^2}+\big\|v^2\big\|_{H^2}+\big\|P_{\pm}\partial_x^2v^1\big\|_{L^\infty}\big)\|u\|_{L^2}^2\lesssim M_T\|u\|_{L^2}^2.\tag{4.83}$$

Next observe that

$$V = -d\big(J^{-\frac{1}{2}}\big(\partial_x v^2\mathcal{H}\partial_x u\big),\, J^{-\frac{1}{2}}u\psi\big)_{L^2} + d\big(J^{-\frac{1}{2}}\partial_x\mathcal{H}\big(\partial_x v^2 u\big),\, J^{-\frac{1}{2}}u\psi\big)_{L^2}$$
$$- d\big(J^{-\frac{1}{2}}\big(v^2\mathcal{H}\partial_x^2 u\big),\, J^{-\frac{1}{2}}u\psi\big)_{L^2} - d\big(J^{-\frac{1}{2}}\partial_x^2\mathcal{H}\big(v^2 u\big),\, J^{-\frac{1}{2}}u\psi\big)_{L^2}. \tag{4.84}$$

The first two terms can be handled using a similar argument as for IV. The remaining terms can be bounded applying the commutator estimates (2.31) and (4.77) by

$$\begin{aligned}
&-d\big(J^{-\frac{1}{2}}\big(v^2\mathcal{H}\partial_x^2 u\big),\, J^{-\frac{1}{2}}u\psi\big)_{L^2} + d\big(J^{-\frac{1}{2}}\big(v^2u\big), \mathcal{H}\big(\partial_x^2 J^{-\frac{1}{2}}u\psi\big)\big)_{L^2}\\
&\quad= -d\big(\big[J^{-\frac{1}{2}}, \psi\big]J^{-\frac{1}{2}}\big(v^2\mathcal{H}\partial_x^2 u\big), u\big)_{L^2} - d\big(J^{-1}\big(v^2\mathcal{H}\partial_x^2 u\big), u\psi\big)_{L^2}\\
&\qquad+ d\big(J^{-\frac{1}{2}}\big(v^2u\big), [\mathcal{H}, \psi]J^{-\frac{1}{2}}\partial_x^2 u\big)_{L^2} + d\big(J^{-\frac{1}{2}}\big(v^2u\big), \psi J^{-\frac{1}{2}}\mathcal{H}\partial_x^2 u\big)_{L^2}\\
&\quad= -d\big(\big[J^{-1}, v^2\big]\mathcal{H}\partial_x^2 u, u\psi\big)_{L^2} - d\big(v^2 J^{-1}\mathcal{H}\partial_x^2 u, u\psi\big)_{L^2} + \mathcal{O}\big(M_T\|u\|_{L^2}^2\big)\\
&\qquad+ d\big(v^2u, \big[J^{-\frac{1}{2}}, \psi\big]J^{-\frac{1}{2}}\mathcal{H}\partial_x^2 u\big)_{L^2} + d\big(v^2u, \psi J^{-1}\mathcal{H}\partial_x^2 u\big)_{L^2}.
\end{aligned} \tag{4.85}$$

Then, we get observing that the second and fourth terms on the right-hand side of (4.85) cancel and repeating a similar argument for $\widetilde{V}$, that

$$V + \widetilde{V} \lesssim M_T\|u\|_{L^2}^2. \tag{4.86}$$

Finally, we notice that

$$\begin{aligned}
\int \big(D^{\frac{1}{2}}u\big)^2\psi'\,dx &= \int \big(D^1 J^{-\frac{1}{2}}u + \big(D^{\frac{1}{2}} - D^1 J^{-\frac{1}{2}}\big)u\big)^2\psi'\,dx\\
&\lesssim \big(J^{-\frac{1}{2}}\mathcal{H}\partial_x u,\, J^{-\frac{1}{2}}\mathcal{H}\partial_x u\psi'\big)_{L^2} + \big\|\big(D^{\frac{1}{2}} - D^1 J^{-\frac{1}{2}}\big)u\big\|_{L^2}^2\\
&\lesssim \big(J^{-\frac{1}{2}}\mathcal{H}\partial_x u,\, J^{-\frac{1}{2}}\mathcal{H}\partial_x u\psi'\big)_{L^2} + \|u\|_{L^2}^2,
\end{aligned} \tag{4.87}$$

since the symbol of the operator $D^{\frac{1}{2}} - D^1 J^{-\frac{1}{2}}$, $m(\xi) = |\xi|^{\frac{1}{2}} - |\xi|(1+\xi^2)^{-\frac{1}{4}}$ belongs to $L^\infty(\mathbb{R})$. Therefore, we conclude integrating (4.78) and (4.79) between 0 and T and using estimates (4.80)–(4.87) that

$$\int_0^T \int \big(\big(D^{\frac{1}{2}}u\big)^2 + \big(D^{\frac{1}{2}}\mathcal{H}u\big)^2\big)\psi'\,dx\,dt \lesssim \big\|u(T)\big\|_{L^2}^2 + (1+M_T)\int_0^T \big\|u(t)\big\|_{L^2}^2\,dt. \tag{4.88}$$

Step 2: *gain of* 1 *derivative*. For this, multiply Eq. (4.72) by $u\psi$ and integrate in space to deduce that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(u, u\psi)_{L^2} &= -a\big(\partial_x^3 u, u\psi\big)_{L^2} + b\big(\mathcal{H}\partial_x^2 u, u\psi\big)_{L^2} + \frac{c}{2}\big(\partial_x\big(\big(v^1+v^2\big)u\big), u\psi\big)_{L^2}\\
&\quad- d\big(\partial_x\big(u\mathcal{H}\partial_x v^1 + \mathcal{H}\big(u\partial_x v^1\big)\big), u\psi\big)_{L^2}\\
&\quad- d\big(\partial_x\big(v^2\mathcal{H}\partial_x u + \mathcal{H}\big(v^2\partial_x u\big)\big), u\psi\big)_{L^2}\\
&:= \mathcal{I} + \mathcal{II} + \mathcal{III} + \mathcal{IV} + \mathcal{V},
\end{aligned} \tag{4.89}$$

and similarly

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\mathcal{H}u,\mathcal{H}u\psi)_{L^2} &= -a\big(\partial_x^3\mathcal{H}u,\mathcal{H}u\psi\big)_{L^2}-b\big(\partial_x^2u,\mathcal{H}u\psi\big)_{L^2}\\&\quad+\frac{c}{2}\big(\partial_x\mathcal{H}\big(\big(v^1+v^2\big)u\big),\mathcal{H}u\psi\big)_{L^2}\\&\quad-d\big(\partial_x\mathcal{H}\big(u\mathcal{H}\partial_xv^1+\mathcal{H}\big(u\partial_xv^1\big)\big),\mathcal{H}u\psi\big)_{L^2}\\&\quad-d\big(\mathcal{H}\partial_x\big(v^2\mathcal{H}\partial_xu+\mathcal{H}\big(v^2\partial_xu\big)\big),\mathcal{H}u\psi\big)_{L^2}\\&:=\widetilde{\mathcal{I}}+\widetilde{\mathcal{II}}+\widetilde{\mathcal{III}}+\widetilde{\mathcal{IV}}+\widetilde{\mathcal{V}}.\end{aligned}\tag{4.90}$$

It is deduced integrating by parts as in (4.80)–(4.81) that

$$\mathcal{I}+\widetilde{\mathcal{I}}=-\frac{3a}{2}\int\big(\partial_xu+D^1u\big)^2\psi'\,dx+\mathcal{O}\big(\|u\|_{L^2}^2\big),\tag{4.91}$$

and using the same trick as in (4.82), Plancherel's identity and estimate (2.32) that

$$\begin{aligned}\mathcal{II}+\widetilde{\mathcal{II}}&=-2b\big(D^1u,u\psi'\big)_{L^2}+\mathcal{O}\big(\|u\|_{L^2}^2\big)\\&=-2b\int\big(D^{\frac12}u\big)^2\psi'\,dx-2b\big(u,D^{\frac12}\big[D^{\frac12},\psi'\big]u\big)_{L^2}\\&=-2b\int\big(D^{\frac12}u\big)^2\psi'\,dx+\mathcal{O}\big(\|u\|_{L^2}^2\big).\end{aligned}\tag{4.92}$$

One can also easily see integrating by parts that

$$\mathcal{III}+\mathcal{IV}+\widetilde{\mathcal{III}}+\widetilde{\mathcal{IV}}=\mathcal{O}\big(M_T\|u\|_{L^2}^2\big)+k\big(\partial_xv^1\partial_xu,\mathcal{H}u\psi\big)_{L^2},\tag{4.93}$$

and for all $\delta>0$ one can use Young's inequality to estimate the integral in time between 0 and T of the last term on the right-hand side of (4.93) by

$$\int_0^T\big(\partial_xv^1\partial_xu,\mathcal{H}u\psi\big)_{L^2}\,dt\leqslant k\delta^{-1}\int_0^T\big\|u(t)\big\|_{L^2}^2\,dt+\delta f(T).\tag{4.94}$$

Furthermore, it follows integrating by parts and using the commutator estimate (2.31) that

$$\begin{aligned}\mathcal{V}&=d\big(v^2\mathcal{H}\partial_xu,\partial_xu\psi\big)_{L^2}+d\big(v^2\mathcal{H}\partial_xu,u\psi'\big)_{L^2}\\&\quad-d\big(v^2\partial_xu,\mathcal{H}(\partial_xu\psi)\big)_{L^2}+d\big(\mathcal{H}\big(v^2\partial_xu\big),u\psi'\big)_{L^2}\\&=d\big(v^2\mathcal{H}\partial_xu,\partial_xu\psi\big)_{L^2}+2d\big(v^2\mathcal{H}\partial_xu,u\psi'\big)_{L^2}-d\big(v^2\partial_xu,\psi\mathcal{H}\partial_xu\big)_{L^2}\\&\quad-d\big(v^2\partial_xu,[\mathcal{H},\psi]\partial_xu\big)_{L^2}+d\big(\big[\mathcal{H},v^2\big]\partial_xu,u\psi'\big)_{L^2}\\&=2d\big(v^2\mathcal{H}\partial_xu,u\psi'\big)_{L^2}+d\big(\partial_xv^2u,[\mathcal{H},\psi]\partial_xu\big)_{L^2}\\&\quad+d\big(v^2u,\partial_x[\mathcal{H},\psi]\partial_xu\big)_{L^2}+d\big(\big[\mathcal{H},v^2\big]\partial_xu,u\psi'\big)_{L^2}\\&=2d\big(v^2\mathcal{H}\partial_xu,u\psi'\big)_{L^2}+\mathcal{O}\big(M_T\|u\|_{L^2}^2\big).\end{aligned}\tag{4.95}$$

Note that $\widetilde{\mathcal{V}}$ could be treated similarly. To handle the integral in time between 0 and T of the first term on the right-hand side of (4.93) we can use (2.32) and (4.88)

$$\int_0^T \left(v^2\mathcal{H}\partial_x u, u\psi'\right)_{L^2} = \int_0^T\int v^2\left(D^{\frac{1}{2}}u\right)^2\psi'\,dx\,dt + \int_0^T\int uD^{\frac{1}{2}}\left[D^{\frac{1}{2}}, v^2\psi'\right]u\,dx\,dt$$

$$\lesssim M_T\left\|u(T)\right\|_{L^2}^2 + M_T(1+M_T)\int_0^T\left\|u(t)\right\|_{L^2}^2\,dt. \tag{4.96}$$

Integrating in time between 0 and T (4.89)–(4.90) and using estimates (4.91)–(4.96) together with (4.88) to treat the first term appearing on the right-hand side of (4.92) yield (4.75). □

Remark 4.3. Combining the argument used to derive the uniqueness result with parabolic regularization method implies the existence of smooth solutions for initial data in $H^\infty(\mathbb{R})$. Similar results were derived by Ponce for the fifth-order KdV equation in [21].

4.2. Proof of the existence

Let $v_0 \in H^s(\mathbb{R})$, with $s \geqslant 2$. From Remark 4.3, we know that for all $\epsilon > 0$, there exist $T_\epsilon > 0$, with $T_\epsilon \underset{\epsilon\to 0}{\longrightarrow} 0$, and a unique solution $v_\epsilon \in C([0, T_\epsilon]; H^\infty(\mathbb{R}))$ of

$$\begin{cases} \partial_t v - b\mathcal{H}\partial_x^2 v + a\partial_x^3 v = cv\partial_x v - d\partial_x\left(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)\right), \\ v(x,0) = v_{0,\epsilon}(x) = \rho_\epsilon * v_0(x), \end{cases} \tag{4.97}$$

where $\rho \in \mathcal{S}(\mathbb{R})$ with $\rho \geqslant 0$, $\int \rho\,dx = 1$, and $\int x^k\rho(x)\,dx = 0$, $k \in \mathbb{Z}_+$, $0 \leqslant k \leqslant [s]+1$. The following properties of the smoothing operators will be used in this section:

Lemma 4.4. *Let $s \geqslant 0$, $\phi \in H^s(\mathbb{R})$ and for any $\epsilon > 0$, $\phi_\epsilon = \rho_\epsilon * \phi$. Then,*

$$\|\phi_\epsilon\|_{H^{s+\alpha}} \lesssim \epsilon^{-\alpha}\|\phi\|_{H^s}, \quad \forall\alpha \geqslant 0, \tag{4.98}$$

and

$$\|\phi - \phi_\epsilon\|_{H^{s-\beta}} \underset{\epsilon\to 0}{=} o\left(\epsilon^\beta\right), \quad \forall\beta \in [0, s]. \tag{4.99}$$

The proof of Lemma 4.4 can be found in [2] or [10].

The first step is to derive an a priori H^2-estimate on the solution v_ϵ in a time interval $[0, T_0]$, where T_0 is a positive time independent of ϵ. In this direction, fix $\epsilon > 0$ and let us denote, for sake of simplicity, $v = v_\epsilon$. Then, arguing as in Section 3.1, it is deduced that $\tilde{w}_\pm = \Phi_\pm\partial_x P_\pm v$ are solutions to the dispersive equations

$$\left(\partial_t + a\partial_x^3 \pm ib\partial_x^2\right)\tilde{w}_\pm = \Phi_\pm P_\pm Q_\pm^\dagger(\Phi_\pm, v, \tilde{w}_\pm) + \Phi_\pm Q_\pm^\ddagger(\Phi_\pm, v, \tilde{w}_\pm)$$
$$+ L_\pm(\Phi_\pm, v, \tilde{w}_\pm) + N_\pm(\Phi_\pm, v, \tilde{w}_\pm), \tag{4.100}$$

where

$$\Phi_\pm(x,t) = \exp\left(\mp i\frac{2d}{3a}\int_0^x v(s,t)\,ds\right). \tag{4.101}$$

Notice that we choose the "primitive" starting at $x = 0$. This has the advantage that the integral is defined if $v(\cdot, t) \in L^2_{\mathrm{loc}}(\mathbb{R})$ (which will be a consequence of our estimates below). However we pay a price on it, because of the term $\partial_t \Phi \partial_x P_+ v$ appearing in (3.42a), which can be rewritten by making use of the equation in (1.1) as

$$-\partial_x P_+ v \Phi i \frac{2d}{3a}\left\{-a\partial_x^2 v + b\mathcal{H}\partial_x v + c\frac{v^2}{2} - d\big(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)\big)\right\}(x,t)$$

$$+ \partial_x P_+ v \Phi i \frac{2d}{3a}\left\{-a\partial_x^2 v + b\mathcal{H}\partial_x v + c\frac{v^2}{2} - d\big(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)\big)\right\}(0,t). \tag{4.102}$$

The first term in (4.102) is nonlinear of the class considered in (i), while the second is a linear one. Here, recall that $Q^{\dagger}_{\pm}$ and $Q^{\ddagger}_{\pm}$ are polynomials at least quadratic involving the first derivative of $v_{\pm}$ and $\tilde{w}_{\pm}$ multiplied by a term of order zero, for example

$$F_1 = \Phi_+ P_+(\tilde{w}_+ \partial_x \tilde{w}_+), \tag{4.103}$$

$L_{\pm}$ are the linear terms obtained in the second part of (4.102) and whose a representative term is

$$F_2 = \tilde{w}_+(x,t)\partial_x^2 v(0,t), \tag{4.104}$$

and $N_{\pm}$ are the last terms on the right-hand side of (3.40) and (3.41), which is to say

$$F_3 = [P_+, v]\partial_x^3 P_+ v = [P_+, v]\partial_x^2(\Phi_- \tilde{w}_+). \tag{4.105}$$

Proposition 4.5. *Let $s \in \mathbb{R}$ and $v_0 \in H^s(\mathbb{R})$. There exists $T_0 = T_0(\|v_0\|_{H^2}) > 0$ such that for all $\epsilon > 0$, the solution v_ϵ of* (4.97) *satisfies*

$$\sup_{[0,T_0]} \big\| v_\epsilon(t)\big\|_{H^s_x} + \sum_{j=1}^{2} \big\| D_x^r \partial_x^j P_{\pm} v_\epsilon \big\|_{L^6_{T_0} L^\infty_x} \leqslant M_1\big(\|v_0\|_{H^2}\big), \tag{4.106}$$

$$\sup_{j \in \mathbb{Z}} \int_0^{T_0} \int_{\mathbb{R}} \big|\chi_{j/N} D_x^r \partial_x^2 \tilde{w}_{\epsilon,\pm}\big|^2 \, dx\, dt \leqslant M_2\big(\|v_0\|_{H^2}\big), \tag{4.107}$$

and

$$\sum_{j \in \mathbb{Z}} \sup_{[0,T_0]\times\mathbb{R}} \big|\chi_{j/N} D_x^r \tilde{w}_{\epsilon,\pm}\big|^2 \leqslant M_3\big(\|v_0\|_{H^2}\big), \tag{4.108}$$

for all $0 \leqslant r \leqslant s-2$, where M_1, M_2 and M_3 are independent of ϵ, $\chi \in C_0^\infty(\mathbb{R})$, $0 \leqslant \chi \leqslant 1$, $\chi_j = \chi(\cdot - j)$, and $N = N(T) \in \mathbb{Z}_+$.

Proof. For sake of simplicity, we will only consider the case $s = 2$. Recall here the notation $\tilde{w}_{\epsilon,\pm} = \Phi_{\epsilon,\pm}\partial_x P_{\pm} v_\epsilon$ where $\Phi_{\epsilon,\pm}(x,t) = e^{-i\frac{2d}{3a}\int_0^x v_\epsilon(s,t)\,ds}$. For any $T > 0$, we define the following quantities:

$$\lambda_1(T) = \sup_{0\leqslant t\leqslant T} \|v_\epsilon(t)\|_{L^2_x} + \sup_{0\leqslant t\leqslant T} \|\tilde{w}_{\epsilon,\pm}(t)\|_{H^1_x},$$
$$\lambda_2(T) = \|\tilde{w}_{\epsilon,\pm}\|_{L^6_T L^\infty_x} + \|\partial_x \tilde{w}_{\epsilon,\pm}\|_{L^6_T L^\infty_x},$$
$$\lambda_3(T) = \left\{\sup_j \int\int \left|\chi_{j/N}(x)\partial_x^2 \tilde{w}_{\epsilon,\pm}(x,t)\right|^2 dx\,dt\right\}^{1/2},$$
$$\lambda_4(T) = \left\{\sum_j \sup_{|t|\leqslant T} \sup_x \left|\chi_{j/N}(x)\tilde{w}_{\epsilon,\pm}(x,t)\right|^2\right\}^{1/2}.$$

Since N depends on T, we fix $0 < T \leqslant T^*$ with $T^* = (1 + \|v_0\|_{H^2})^{-1}$, so that the constants appearing on the estimates (2.25) and (2.26) are fixed. Note that $\|\Phi_{\epsilon,\pm}\|_{L^\infty_{x,T}} \leqslant 1$, since v_ϵ is a real-valued function. Moreover, due to the L^2-norm conservation of (1.1), we have that

$$\sup_{0\leqslant t\leqslant T} \|v_\epsilon(t)\|_{L^2} = \|v_{0,\epsilon}\|_{L^2} \leqslant \|v_0\|_{L^2}. \tag{4.109}$$

To estimate the other part of the $\lambda_j(T)$, consider the integral system associated to (4.100)

$$\tilde{w}_{\epsilon,\pm} = \tilde{w}_\pm(t)\tilde{w}_{0,\epsilon,\pm} + \int_0^t \tilde{w}_\pm(t-t')\mathcal{N}_\pm(v_\epsilon, \tilde{w}_{\epsilon,\pm})(t')\,dt', \tag{4.110}$$

where $\mathcal{N}_\pm$ denote the nonlinearities on the right-hand side of (4.100). Using (4.109), (4.110), Minkowski's integral inequality and the linear estimates obtained in Lemma 2.1, it is deduced that

$$\begin{aligned}
&\max\left\{\lambda_1(T), \lambda_2(T), (1+T)^{-\frac{1}{2}}\lambda_3(T), (1+T)^{-2}\lambda_4(T)\right\} \\
&\quad\lesssim \|v_{0,\epsilon}\|_{L^2} + \|\tilde{w}_{0,\epsilon,\pm}\|_{H^1} + \int_0^T \|\mathcal{N}_\pm(t)\|_{H^1_x}\,dt.
\end{aligned} \tag{4.111}$$

Then it remains to estimate $\|F_j\|_{L^1_T H^1_x}$, $j = 1, 2, 3$, where F_j are defined in (4.103)–(4.105), since they are the representative terms of the nonlinearities $\mathcal{N}_\pm$. We have, using Hölder's inequality,

$$\begin{aligned}
\|F_1\|_{L^1_T H^1_x} &\leqslant \|\Phi_+ P_+(\tilde{w}_+\partial_x\tilde{w}_+)\|_{L^1_T L^2_x} + \|\partial_x\Phi_+ P_+(\tilde{w}_+\partial_x\tilde{w}_+)\|_{L^1_T L^2_x} \\
&\quad + \|\Phi_+ P_+((\partial_x\tilde{w}_+)^2)\|_{L^1_T L^2_x} + \|\Phi_+ P_+(\tilde{w}_+\partial_x^2\tilde{w}_+)\|_{L^1_T L^2_x} \\
&\lesssim \int_0^T \|\tilde{w}_+\|_{L^\infty_x}(1 + \|v\|_{L^\infty_x})\|\partial_x\tilde{w}_+\|_{L^2_x}\,dt \\
&\quad + \int_0^T \|\partial_x\tilde{w}_+\|_{L^\infty_x}\|\partial_x\tilde{w}_+\|_{L^2_x}\,dt + T^{\frac{1}{2}}\|\chi_{j/N}\tilde{w}_+\partial_x^2\tilde{w}_+\|_{l^2_j L^2_{x,T}} \\
&\lesssim T(1+\lambda_1(T))\lambda_1(T)^2 + T^{\frac{5}{6}}\lambda_1(T)\lambda_2(T) + T^{\frac{1}{2}}\lambda_3(T)\lambda_4(T).
\end{aligned} \tag{4.112}$$

Next it follows from Hölder's inequality that

$$\begin{aligned}
\|F_2\|_{L_T^1 H_x^1} &\leqslant \int_0^T \left\|\partial_x^2 v(t)\right\|_{L_x^\infty} \left\|\tilde{w}_+(t)\right\|_{H_x^1} dt \\
&\leqslant \int_0^T \left(\left\|\partial_x(\Phi_- \tilde{w}_+)\right\|_{L_x^\infty} + \left\|\partial_x(\Phi_+ \tilde{w}_-)\right\|_{L_x^\infty}\right) \left\|\tilde{w}_+(t)\right\|_{H_x^1} dt \\
&\lesssim T^{\frac{5}{6}} \big(1 + \lambda_1(T)\big) \lambda_2(T) \lambda_1(T). \qquad (4.113)
\end{aligned}$$

Finally, estimate (2.31) yields

$$\begin{aligned}
\|F_3\|_{L_T^1 H_x^1} &\leqslant \left\|[P_+, v]\partial_x^2(\Phi_- \tilde{w}_+)\right\|_{L_T^1 L_x^2} + \left\|\partial_x [P_+, v]\partial_x^2(\Phi_- \tilde{w}_+)\right\|_{L_T^1 L_x^2} \\
&\lesssim \int_0^T \left\|\partial_x^2 v(t)\right\|_{L_x^\infty} \left\|(\Phi_- \tilde{w}_+)(t)\right\|_{H_x^1} dt \\
&\lesssim T^{\frac{5}{6}} \lambda_2(T) \big(1 + \lambda_1(T)\big) \lambda_1(T). \qquad (4.114)
\end{aligned}$$

Therefore, gathering (4.111)–(4.114), it is deduced that there exist two polynomials p_1, p_2, with all terms of order at least 1, such that

$$\begin{aligned}
&\max\left\{\lambda_1(T), \lambda_2(T), (1+T)^{-\frac{1}{2}} \lambda_3(T), (1+T)^{-2} \lambda_4(T)\right\} \\
&\quad \lesssim \|v_0\|_{H^2} + T p_1\big(\lambda_1(T)\big) \lambda_1(T) + T^{\frac{5}{6}} p_2\big(\lambda_1(T)\big) \lambda_2(T) + T^{\frac{1}{2}} \lambda_3(T) \lambda_4(T). \qquad (4.115)
\end{aligned}$$

Notice that λ_j, $j = 1, \ldots, 4$ are continuous nondecreasing functions; hence, defining T_0 as the smallest T such that

$$\begin{aligned}
&\max\left\{T_0 p_1\big(\lambda_1(T_0)\big), T_0^{\frac{5}{6}} p_2\big(\lambda_1(T_0)\big), T_0^{\frac{5}{6}} p_2\big(\lambda_1(T_0)\big) \lambda_1(T_0)^{-1} \lambda_2(T_0), \right. \\
&\quad \left. T_0^{\frac{1}{2}} (1+T_0)^2 \lambda_3(T_0), T_0^{\frac{1}{2}} (1+T_0)^{\frac{1}{2}} \lambda_4(T_0)\right\} = \frac{1}{10k}, \qquad (4.116)
\end{aligned}$$

where k is the implicit positive constant appearing in (4.115), it follows from (4.115) that

$$\max\left\{\lambda_1(T_0), \lambda_2(T_0), (1+T_0)^{-\frac{1}{2}} \lambda_3(T_0), (1+T_0)^{-2} \lambda_4(T_0)\right\} \lesssim \|v_0\|_{H^2}. \qquad (4.117)$$

Furthermore, it is deduced from (4.116) and (4.117) that at time T_0 one of the following inequalities must hold:

$$\frac{1}{10} \leqslant k T_0 p_1\big(\|v_0\|_{H^2}\big), \qquad \frac{1}{10} \leqslant k T_0^{\frac{5}{6}} p_2\big(\|v_0\|_{H^2}\big),$$

or

$$\frac{1}{10} \leqslant k T_0^{\frac{1}{2}} (1+T_0)^{\frac{1}{2}} (1+T_0)^2 \|v_0\|_{H^2},$$

which implies that there exists a constant $M = M(\|v_0\|_{H^2})$ such that $T_0 \geqslant M$. $\square$

Proposition 4.5 implies that the sequence $(v_\epsilon)_{\epsilon>0}$ defined on the time interval $[0, T_0]$, independent of ϵ, is bounded in the norms defined in (4.106)–(4.108). Moreover, using a similar argument to the one employed in the proof of the uniqueness and estimate (4.99), one can prove that for $\epsilon > \epsilon' > 0$, the function $u = u_{\epsilon,\epsilon'} = v_\epsilon - v_{\epsilon'}$ satisfies

$$\sup_{[0,T_0]} \|u(t)\|_{L^2} \leqslant k\|v_{0,\epsilon} - v_{0,\epsilon'}\|_{L^2} \underset{\epsilon\to 0}{=} o(\epsilon^s). \tag{4.118}$$

Then the sequence $(v_\epsilon)_{\epsilon>0}$ converges in $C([0, T_0]; L^2(\mathbb{R}))$, as ϵ tends to zero, to a solution v of (1.1).

Furthermore, since $(v_\epsilon)_\epsilon$ is bounded in $C([0, T_0]; H^s(\mathbb{R}))$, we deduce by an interpolation argument that

$$v_\epsilon \underset{\epsilon\to 0}{\longrightarrow} v \quad \text{in } C\big([0, T_0]; H^r(\mathbb{R})\big), \text{ for any } r < s. \tag{4.119}$$

The next proposition fills the gap between (4.119) and the persistence property.

Proposition 4.6. *We have that*

$$v \in C\big([0, T_0]; H^s(\mathbb{R})\big). \tag{4.120}$$

Proof. It suffices to prove that (v_ϵ) tends to v in $C([0, T_0]; H^s(\mathbb{R}))$ as ϵ tends to zero. In this direction, fix $\epsilon > 0$ and define

$$\tilde{w}_{\epsilon,\pm} = \Phi_{\epsilon,\pm}\partial_x P_\pm v_\epsilon, \quad \text{where } \Phi_{\epsilon,\pm}(x,t) = e^{\mp i\frac{2d}{3a}\int_0^x v_\epsilon(y,t)\,dy}, \tag{4.121}$$

$$u = u_\epsilon = v_\epsilon - v, \quad \text{and} \quad \alpha_\pm = \alpha_{\epsilon,\pm} = \tilde{w}_{\epsilon,\pm} - \tilde{w}_\pm. \tag{4.122}$$

Arguing as before, we deduce that α_+ satisfies

$$\begin{aligned}
\big(\partial_t + a\partial_x^3 \pm ib\partial_x^2\big)\alpha_\pm = {} & \Phi_{\epsilon,\pm}P_\pm Q_\pm^\dagger(\Phi_{\epsilon,\pm}, v_\epsilon, \tilde{w}_{\epsilon,\pm}) + \Phi_{\epsilon,\pm}Q_\pm^\ddagger(\Phi_{\epsilon,\pm}, v_\epsilon, \tilde{w}_{\epsilon,\pm}) \\
& + \tilde{w}_{\epsilon,\pm}\partial_x^2 v_\epsilon(0,t) + 2id\Phi_{\epsilon,\pm}[P_\pm, v_\epsilon]\partial_x^2\tilde{w}_{\epsilon,\pm} \\
& - \Phi_\pm P_\pm Q_\pm^\dagger(\Phi_\pm, v, \tilde{w}_\pm) - \Phi_\pm Q_\pm^\ddagger(\Phi_\pm, v, \tilde{w}_\pm) \\
& - \tilde{w}_\pm\partial_x^2 v(0,t) - 2id\Phi[P_\pm, v]\partial_x^2\tilde{w}_\pm,
\end{aligned} \tag{4.123}$$

where $Q_\pm^\dagger$ and $Q_\pm^\ddagger$ are defined as in (3.39). Then we can rewrite (4.123) as

$$\begin{aligned}
\big(\partial_t + a\partial_x^3 \pm ib\partial_x^2\big)\alpha_\pm = {} & \Phi_{\epsilon,\pm}P_\pm\big\{Q_\pm^\dagger(\Phi_{\epsilon,\pm}, v_\epsilon, \tilde{w}_{\epsilon,\pm}) - Q_\pm^\dagger(\Phi_\pm, v, \tilde{w}_\pm)\big\} \\
& + (\Phi_{\epsilon,\pm} - \Phi_\pm)P_\pm Q_\pm^\dagger(\Phi_\pm, v, \tilde{w}_\pm) \\
& + \Phi_{\epsilon,\pm}\big\{Q_\pm^\ddagger(\Phi_{\epsilon,\pm}, v_\epsilon, \tilde{w}_{\epsilon,\pm}) - Q_\pm^\ddagger(\Phi_\pm, v, \tilde{w}_\pm)\big\} \\
& + (\Phi_{\epsilon,\pm} - \Phi_\pm)Q_\pm^\ddagger(\Phi_\pm, v, \tilde{w}_\pm) \\
& + \alpha_\pm\partial_x^2 v_\epsilon(0,t) + \tilde{w}_\pm\partial_x^2 u(0,t) \\
& + 2id\Phi_{\epsilon,\pm}[P_\pm, u]\partial_x^2\tilde{w}_{\epsilon,\pm} + 2id\Phi_{\epsilon,\pm}[P_\pm, v]\partial_x^2\alpha_\pm \\
& + 2id(\Phi_{\epsilon,\pm} - \Phi_\pm)[P_\pm, v]\partial_x^2\tilde{w}_\pm.
\end{aligned} \tag{4.124}$$

In the sequel, we will need the following assertion:

$$\left\|(\Phi_{\epsilon,\pm}-\Phi_{\pm})f\right\|_{L^2} \underset{\epsilon\to 0}{=} o(1), \quad \forall f\in L^2(\mathbb{R}). \tag{4.125}$$

Observe that we do not have that $\|(\Phi_{\epsilon,\pm}-\Phi_{\pm})\|_{L^\infty} \underset{\epsilon\to 0}{\longrightarrow} 0$. By the dominated convergence theorem, it suffices to prove that $|\Phi_{\epsilon,\pm}(x,t)-\Phi_{\pm}(x,t)|^2 \underset{\epsilon\to 0}{\longrightarrow} 0$, for almost every $x\in\mathbb{R}$. But, using the mean value and Cauchy–Schwarz inequalities and (4.118), we get that for every $x\in\mathbb{R}$,

$$\left|\Phi_{\epsilon,\pm}(x,t)-\Phi_{\pm}(x,t)\right|^2 \leqslant \frac{2d}{3a}\left|\int_0^x (v_\epsilon - v)(y,t)\,dy\right|^2 \leqslant |x|\left\|(v_\epsilon-v)(\cdot,t)\right\|_{L^2}^2 \underset{\epsilon\to 0}{\longrightarrow} 0,$$

which implies (4.125).

Next we define (still in the case $s=2$)

$$\tilde{\lambda}_1(T)=\tilde{\lambda}_1(T,\alpha_\pm)=\sup_{[0,T]}\left\|\alpha_\pm(t)\right\|_{H^1},$$

$$\tilde{\lambda}_2(T)=\tilde{\lambda}_2(T,\alpha_\pm)=\|\alpha_\pm\|_{L^6_T L^\infty_x}+\|\partial_x\alpha_\pm\|_{L^6_T L^\infty_x},$$

$$\tilde{\lambda}_3(T)=\tilde{\lambda}_3(T,\alpha_\pm)=\sup_{j\in\mathbb{Z}}\int_0^T\int_{\mathbb{R}}\left|\chi_{j/N}\partial_x^2\alpha_\pm\right|^2,$$

and

$$\tilde{\lambda}_4(T)=\tilde{\lambda}_4(T,\alpha_\pm)=\sum_{j\in\mathbb{Z}}\sup_{[0,T]\times\mathbb{R}}|\chi_{j/N}\alpha_\pm|^2.$$

We deduce arguing as in the proof of Proposition 4.5 with Eq. (4.124), and using (4.113), (4.114), (4.118) and (4.125) that

$$\max\left\{\tilde{\lambda}_1(T_0),\tilde{\lambda}_2(T_0),(1+T_0)^{-\frac{1}{2}}\tilde{\lambda}_3(T_0),(1+T_0)^{-2}\tilde{\lambda}_4(T_0)\right\} \underset{\epsilon\to 0}{=} o(1). \tag{4.126}$$

To conclude the proof of Proposition 4.6, we observe that

$$\begin{aligned}\partial_x(v_\epsilon - v) &= \partial_x P_+(v_\epsilon - v)+\partial_x P_-(v_\epsilon - v)\\ &= \Phi_{\epsilon,-}\alpha_+ + (\Phi_{\epsilon,-}-\Phi_-)\tilde{w}_+ + \Phi_{\epsilon,+}\alpha_- + (\Phi_{\epsilon,+}-\Phi_+)\tilde{w}_-,\end{aligned} \tag{4.127}$$

so that we deduce (4.120) gathering (4.118), (4.125)–(4.127). □

Finally, the proof of the continuity of the flow map data-solution follows by using the classical Bona–Smith argument (see [2,10] or [20]).

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