A note on the 2D generalized Zakharov–Kuznetsov equation: Local, global, and scattering results

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ABSTRACT

We consider the generalized two-dimensional Zakharov–Kuznetsov equation $u_t + \partial_x \Delta u + \partial_x(u^{k+1}) = 0$, where $k \geqslant 3$ is an integer number. For $k \geqslant 8$ we prove local well-posedness in the L^2-based Sobolev spaces $H^s(\mathbb{R}^2)$, where s is greater than the critical scaling index $s_k = 1 - 2/k$. For $k \geqslant 3$ we also establish a sharp criteria to obtain global $H^1(\mathbb{R}^2)$ solutions. A nonlinear scattering result in $H^1(\mathbb{R}^2)$ is also established assuming the initial data is small and belongs to a suitable Lebesgue space.

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1. Introduction

This note sheds new light on the local and global well-posedness of the initial-value problem (IVP) associated with the generalized Zakharov–Kuznetsov (gZK) equation in two-space dimensions:

$$\begin{cases} u_t + \partial_x \Delta u + \partial_x\big(u^{k+1}\big) = 0, & (x, y) \in \mathbb{R}^2, \ t > 0, \\ u(x, y, 0) = u_0(x, y), \end{cases} \tag{1.1}$$

where u is a real-valued function, $\Delta = \partial_x^2 + \partial_y^2$ stands for the Laplacian operator, and $k \geqslant 1$ is an integer number. Here we will concern with the L^2-supercritical case, i.e. $k \geqslant 3$ in (1.1).

In the case $k = 1$, the equation in (1.1) has a physical meaning and it was formally deduced by Zakharov and Kuznetsov [18] as an asymptotic model to describe the propagation of nonlinear

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ion-acoustic waves in a magnetized plasma. The gZK equation may also be seen as a natural, two-dimensional extension of the well-known generalized Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + \partial_x(u^{k+1}) = 0, \quad x \in \mathbb{R}, \ t > 0.$$

Our main purpose here lies in establishing local and global (in time) well-posedness results. These issues have already been studied in Faminskii [4], Biagioni and Linares [1], and Linares and Pastor [9,10]. In [4], Faminskii considered the case $k = 1$ and showed local and global well-posedness in $H^m(\mathbb{R}^2)$, $m \geqslant 1$ integer. In [1], Biagioni and Linares dealt with the case $k = 2$ and proved local well-posedness for data in $H^1(\mathbb{R}^2)$. By considering the cases $k = 1$ and $k = 2$ Linares and Pastor [9] improved the local results in [1,4] by showing that both IVP's are locally well-posed in $H^s(\mathbb{R}^2)$, $s > 3/4$. Moreover the authors also show that if $u_0 \in H^1(\mathbb{R}^2)$ and satisfies $\|u_0\|_{L^2} < \|Q\|_{L^2}$, where Q is the unique positive radial solution (hereafter refereed to as the ground state solution) of the elliptic equation

$$-\Delta Q + Q - Q^3 = 0, \tag{1.2}$$

then (for $k = 2$) global well-posedness holds in $H^1(\mathbb{R}^2)$. The case $k \geqslant 3$ was studied in [10] where the authors established local well-posedness in $H^s(\mathbb{R}^2)$, $s > 3/4$, if $3 \leqslant k \leqslant 7$ and in $H^s(\mathbb{R}^2)$, $s > s_k^* := 1 - 3/(2k-4)$, if $k \geqslant 8$. A global result for small initial data in $H^1(\mathbb{R}^2)$ was also proved.

The best local well-posedness results known are summarized in the following theorem.

Theorem 1.1. *(See [9,10].) The following statements hold.*

(i) *Assume $1 \leqslant k \leqslant 7$. Then for any $u_0 \in H^s(\mathbb{R}^2)$, $s > 3/4$, there exist $T = T(\|u_0\|_{H^s}) > 0$, a space $X_T \subset C([0, T]; H^s(\mathbb{R}^2))$ and a unique solution $u \in X_T$ of the IVP (1.1) defined in $[0, T]$. Moreover, continuous dependence upon the data holds.*

(ii) *Assume $k \geqslant 8$. Then for any $u_0 \in H^s(\mathbb{R}^2)$, $s > s_k^* := 1 - 3/(2k-4)$, there exist $T = T(\|u_0\|_{H^s}) > 0$, a space $X_T \subset C([0, T]; H^s(\mathbb{R}^2))$ and a unique solution $u \in X_T$ of the IVP (1.1) defined in $[0, T]$. Moreover, continuous dependence upon the data holds.*

Concerning other questions on the gZK equation we refer the reader to [2,3,11,13,14], and references therein.

To motivate the results to follow, let us perform a scaling argument: if u solves (1.1), with initial data u_0, then

$$u_\lambda(x, y, t) = \lambda^{2/k} u(\lambda x, \lambda y, \lambda^3 t)$$

also solves (1.1), with initial data $u_\lambda(x, y, 0) = \lambda^{2/k} u_0(\lambda x, \lambda y)$, for any $\lambda > 0$. Hence,

$$\|u_\lambda(\cdot, \cdot, 0)\|_{\dot{H}^s} = \lambda^{2/k+s-1} \|u_0\|_{\dot{H}^s}, \tag{1.3}$$

where $\dot{H}^s = \dot{H}^s(\mathbb{R}^2)$ denotes the homogeneous Sobolev space of order s. As a consequence of (1.3), the scale-invariant Sobolev space for the gZK equation is $H^{s_k}(\mathbb{R}^2)$, where $s_k = 1 - 2/k$. Therefore, one expects that the Sobolev spaces $H^s(\mathbb{R}^2)$ for studying the well-posedness of (1.1) are those with indices $s > s_k$.

It should be noted that $s_k < 3/4$ if $1 \leqslant k \leqslant 7$, $s_k = s_k^* = 3/4$ if $k = 8$, and $s_k^* > s_k$ if $k > 8$. Thus, in view of Theorem 1.1, except in the case $k = 8$, a gap for the local well-posedness is left between the index conjectured by the scaling argument and that one known in the current literature. One of our goals here is to fulfill this gap by reaching the critical index $s_k = 1 - 2/k$ (up to the endpoint) in the case $k > 8$. More precisely, we prove the following.

Theorem 1.2. *Let $k > 8$ and $s_k = 1 - 2/k$. For any $u_0 \in H^s(\mathbb{R}^2)$, $s > s_k$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution of the IVP (1.1), defined in the interval $[0, T]$, such that*

$$u \in C\big([0, T]; H^s(\mathbb{R}^2)\big), \tag{1.4}$$

$$\|u_x\|_{L_x^\infty L_{yT}^2} + \big\|D_x^s u_x\big\|_{L_x^\infty L_{yT}^2} + \big\|D_y^s u_x\big\|_{L_x^\infty L_{yT}^2} < \infty, \tag{1.5}$$

$$\|u\|_{L_T^{3k/2+} L_{xy}^\infty} + \|u_x\|_{L_T^{3k/(k+2)} L_{xy}^\infty} < \infty, \tag{1.6}$$

and

$$\|u\|_{L_x^{k/2} L_{yT}^\infty} < \infty. \tag{1.7}$$

Moreover, for any $T' \in (0, T)$ there exists a neighborhood U of u_0 in $H^s(\mathbb{R}^2)$ such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$ from U into the class defined by (1.4)–(1.7) is smooth.

The technique to show Theorem 1.2 will be the one developed by Kenig, Ponce, and Vega [8], which combines smoothing effects, Strichartz-type estimates, and a maximal function estimate together with the Banach contraction principle. One of the obstacles which prevent us in proving a similar result for $k \leqslant 7$ is that we have a maximal function estimate that holds in $H^s(\mathbb{R}^2)$ only for $s > 3/4$ (see Lemma 2.1).

After proving Theorem 1.2 we turn our attention to the issue of global well-posedness. As we already mentioned, such question has already been addressed in [4,1,9,10]. In particular, in [9] it was proved that if $k = 2$ and $\|u_0\|_{L^2} < \|Q\|_{L^2}$ (where Q is the ground state solution) then the solution is global in $H^1(\mathbb{R}^2)$ (for global results below $H^1(\mathbb{R}^2)$, see [10]). Also, in [10] it was shown if $k \geqslant 3$ and $\|u_0\|_{H^1}$ is small enough then global well-posedness holds in $H^1(\mathbb{R}^2)$. The proof of this last result is quite standard and relies on conservation laws and the Gagliardo–Nirenberg inequality,

$$\int u^{k+2} \, dx \, dy \leqslant c \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^k, \tag{1.8}$$

to get an a priori estimate. Indeed, first recall that the flow of the gZK is conserved by the quantities:

$$Mass \equiv M\big(u(t)\big) = \int u^2(t) \, dx \, dy \tag{1.9}$$

and

$$Energy \equiv E\big(u(t)\big) = \frac{1}{2} \int \big|\nabla u(t)\big|^2 \, dx \, dy - \frac{1}{k+2} \int u^{k+2}(t) \, dx \, dy, \tag{1.10}$$

where the symbol ∇ stands for the gradient in the space variables.

Combining (1.9), (1.10) and (1.8), we obtain that

$$\big\|u(t)\big\|_{H^1}^2 = M\big(u(t)\big) + 2E\big(u(t)\big) + \frac{2}{k+2} \int u^{k+2}(t) \, dx \, dy$$

$$\leqslant M(u_0) + 2E(u_0) + c \|u_0\|_{L^2}^2 \big\|\nabla u(t)\big\|_{L^2}^k. \tag{1.11}$$

Denote $X(t) = \|u(t)\|_{H^1}^2$. Since $k \geqslant 3$, we then have

$$X(t) \leqslant C\big(\|u_0\|_{H^1}\big) + c \|u_0\|_{L^2}^2 X(t)^{1 + \frac{k-2}{2}}.$$

Thus, if $\|u_0\|_{H^1}$ is small enough, a standard argument leads to $\|u(t)\|_{H^1} \leqslant C(\|u_0\|_{H^1})$ for $t \in [0, T]$. Therefore, we can apply the local theory to extend the solution globally.

Unfortunately, the above argument does not precise how small the initial data should be. Here, we study this question and obtain the following result.

Theorem 1.3. *Let $k \geqslant 2$ and $s_k = 1 - 2/k$. Assume $u_0 \in H^1(\mathbb{R}^2)$ and suppose that*

$$E(u_0)^{s_k} M(u_0)^{1-s_k} < E(Q)^{s_k} M(Q)^{1-s_k}, \quad E(u_0) \geqslant 0. \tag{1.12}$$

If

$$\|\nabla u_0\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} < \|\nabla Q\|_{L^2}^{s_k} \|Q\|_{L^2}^{1-s_k}, \tag{1.13}$$

then for any t as long as the solution exists,

$$\left\|\nabla u(t)\right\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} = \left\|\nabla u(t)\right\|_{L^2}^{s_k} \left\|u(t)\right\|_{L^2}^{1-s_k} < \|\nabla Q\|_{L^2}^{s_k} \|Q\|_{L^2}^{1-s_k}, \tag{1.14}$$

where Q is the unique positive radial solution of

$$\Delta Q - Q + Q^{k+1} = 0.$$

This in turn implies that $H^1(\mathbb{R}^2)$ solutions exist globally in time.

To prove Theorem 1.3, we follow closely our arguments in [5] where we have proved a similar result for the L^2-supercritical generalized KdV equation. We point out that these results are inspired by those ones obtained by Kenig and Merle [7] and Holmer and Roudenko [6].

Remark 1.4. In the limit case $k = 2$ (the modified ZK equation), conditions (1.12) and (1.13) reduce to the same one and it writes as

$$\|u_0\|_{L^2} < \|Q\|_{L^2}.$$

Such a condition was already used in [9] and [10] to show the existence of global solutions, respectively, in $H^1(\mathbb{R}^2)$ and $H^s(\mathbb{R}^2)$, $s > 53/63$.

Once Theorem 1.3 is established, we go on studying the asymptotic behavior of such global solutions as $t \to \pm\infty$. We prove that under a smallness condition the solution scatters to a solution of the linear problem. Precisely,

Theorem 1.5. *Let $k \geqslant 3$ and $p' = \frac{2(k+1)}{2k+1}$. Assume that $u_0 \in H^1(\mathbb{R}^2) \cap L^{p'}(\mathbb{R}^2)$ satisfies*

$$\|u_0\|_{L^{p'}} + \|u_0\|_{H^1} < \delta \tag{1.15}$$

for some δ small enough. Let $u(t)$ be the global solution of (1.1) given in Theorem 1.3. Then, there exist $f_\pm \in H^1(\mathbb{R}^2)$ such that

$$\left\|u(t) - U(t)f_\pm\right\|_{H^1} \to 0, \tag{1.16}$$

as $t \to \pm\infty$.

Note that the smallness condition (1.15) promptly implies the existence of global solutions in $H^1(\mathbb{R}^2)$. The proof of Theorem 1.5 is quite standard and it follows closely the arguments in [12,15,17].

Remark 1.6. Theorem 1.5 provides scattering whenever the initial data is small in $H^1(\mathbb{R}^2)$ and in $L^{p'}(\mathbb{R}^2)$. We do not know if the smallness condition in Theorem 1.3 is sharp in the sense that any global solution given by Theorem 1.3 scatters or not.

The paper is organized as follows. In Section 2 we introduce some notation and recall the useful linear estimates to our arguments. The local and global results, in Theorems 1.2 and 1.3, are proved in Sections 3 and 4, respectively. The concluding section, Section 5, is devoted to show Theorem 1.5.

2. Notation and preliminaries

Let us start this section by introducing the basic notation used throughout this note. We use c to denote various constants that may vary line by line. Given any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leqslant cb$. We use $a+$ and $a-$ to denote $a + \varepsilon$ and $a - \varepsilon$, respectively, for arbitrarily small $\varepsilon > 0$.

For $\alpha \in \mathbb{C}$, the operators D_x^α and D_y^α are defined via Fourier transform by $\widehat{D_x^\alpha f}(\xi, \eta) = |\xi|^\alpha \hat{f}(\xi, \eta)$ and $\widehat{D_y^\alpha f}(\xi, \eta) = |\eta|^\alpha \hat{f}(\xi, \eta)$, respectively. We use $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^s}$ to denote the norms in $L^p(\mathbb{R}^2)$ and $H^s(\mathbb{R}^2)$, respectively. If necessary, we use subscript to inform which variable we are concerned with. The mixed space–time norm is defined as (for $1 \leqslant p, q, r < \infty$)

$$\|f\|_{L_x^p L_y^q L_T^r} = \left(\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \left(\int_0^T |f(x, y, t)|^r dt \right)^{q/r} dy \right)^{p/q} dx \right)^{1/p},$$

with obvious modifications if either $p = \infty$, $q = \infty$ or $r = \infty$. Norms with interchanged subscript are similarly defined. If the subscript L_t^r appears in some norm, that means one is integrating the variable t on the whole $\mathbb{R}$.

Next we introduce the main tools to prove the local well-posedness. Consider the linear IVP

$$\begin{cases} u_t + \partial_x \Delta u = 0, & (x, y) \in \mathbb{R}^2, \ t \in \mathbb{R}, \\ u(x, y, 0) = u_0(x, y). \end{cases} \tag{2.17}$$

The solution of (2.17) is given by the unitary group $\{U(t)\}_{t=-\infty}^{\infty}$ such that

$$u(t) = U(t)u_0(x, y) = \int_{\mathbb{R}^2} e^{i(t(\xi^3 + \xi\eta^2) + x\xi + y\eta)} \hat{u}_0(\xi, \eta) \, d\xi \, d\eta. \tag{2.18}$$

The smoothing effect of Kato type, the Strichartz estimate, and the maximal function estimate for solution (2.18) are presented next.

Lemma 2.1. *The following statements hold.*

(i) (*Smoothing effect*) *If $u_0 \in L^2(\mathbb{R}^2)$ then*

$$\left\| \partial_x U(t) u_0 \right\|_{L_x^\infty L_{yT}^2} \lesssim \|u_0\|_{L_{xy}^2}. \tag{2.19}$$

(ii) (*Maximal function*) *For any $s > 3/4$ and $0 < T \leqslant 1$, we have*

$$\left\| U(t) f \right\|_{L_x^4 L_{yT}^\infty} \lesssim \|f\|_{H_{xy}^s}. \tag{2.20}$$

(iii) (*Strichartz-type estimate*) Let $0 \leqslant \varepsilon < 1/2$ and $0 \leqslant \theta \leqslant 1$. Then,

$$\left\| D_x^{\theta\varepsilon/2} U(t) f \right\|_{L_t^q L_{xy}^p} \lesssim \| f \|_{L_{xy}^2}, \tag{2.21}$$

where $p = \frac{2}{1-\theta}$ and $\frac{2}{q} = \frac{\theta(2+\varepsilon)}{3}$.

Proof. The proof of (i) is given in [4, Theorem 2.2] while proofs of (ii) and (iii) are given, respectively, in [9, Proposition 2.4] and [10, Corollary 2.7]. □

With Lemma 2.1 at hand, we prove the following.

Proposition 2.2. *Let* $s_k = 1 - 2/k$ *and* $0 < T \leqslant 1$. *Then, for any* $k > 8$,

  (i) $\| U(t) f \|_{L_x^{k/2} L_{yT}^\infty} \lesssim \| f \|_{H_{xy}^{s_k+}}$,
  (ii) $\| U(t) f \|_{L_T^{3k/2+} L_{xy}^\infty} \lesssim \| f \|_{H_{xy}^{s_k+}}$,
 (iii) $\| \partial_x U(t) f \|_{L_T^{3k/(k+2)} L_{xy}^\infty} \lesssim \| D_x^{s_k} f \|_{L_{xy}^2}$.

Proof. Inequality (i) follows interpolating the Sobolev embedding

$$\left\| U(t) f \right\|_{L_{Txy}^\infty} \lesssim \| f \|_{H^{1+}} \tag{2.22}$$

with the maximal function estimate $\| U(t) f \|_{L_x^4 L_{yT}^\infty} \leqslant \| f \|_{H^{s_k+}}$ (see (2.20)). To prove (ii) we first take $\varepsilon = 0$ and $\theta = 1$ in (2.21) to get $\| U(t) f \|_{L_T^3 L_{xy}^\infty} \lesssim \| f \|_{L^2}$. Thus (ii) follows interpolating such inequality with (2.22). Estimate (iii) is a particular case of (2.21) just taking $\theta = 1$ and $\varepsilon = 4/k$. □

Finally, we also recall the Chain rule and Leibniz rule for fractional derivatives.

Lemma 2.3 (*Chain rule*). *Let* $1 < p < \infty$, $r > 1$, *and* $h \in L_{\text{loc}}^{rp}(\mathbb{R})$. *Then*

$$\left\| D_z^\alpha F(f) h \right\|_{L_z^p(\mathbb{R})} \lesssim \left\| F'(f) \right\|_{L_z^\infty(\mathbb{R})} \left\| D_z^\alpha(f) M(h^{rp})^{1/rp} \right\|_{L_z^p(\mathbb{R})},$$

where M *denotes the Hardy–Littlewood maximal function.*

Proof. See Kenig, Ponce, and Vega [8, Theorem A.7]. □

Lemma 2.4 (*Leibniz rule*). *Let* $0 < \alpha < 1$ *and* $1 < p < \infty$. *Then,*

$$\left\| D_z^\alpha(fg) - f D_z^\alpha g - g D_z^\alpha f \right\|_{L_z^p(\mathbb{R})} \lesssim \| g \|_{L_z^\infty(\mathbb{R})} \left\| D_z^\alpha f \right\|_{L_z^p(\mathbb{R})}.$$

Proof. See Kenig, Ponce, and Vega [8, Theorem A.12]. □

3. Local well-posedness: proof of Theorem 1.2

As usual, we consider the integral operator

$$\Psi(u)(t) = \Psi_{u_0}(u)(t) := U(t) u_0 + \int\limits_0^t U(t - t') \partial_x(u^{k+1})(t') \, dt', \tag{3.23}$$

and define the metric spaces

$$\mathcal{Y}_T = \big\{ u \in C\big([0,T]; H^s(\mathbb{R}^2)\big); \ \vertiii{u} < \infty \big\}$$

and

$$\mathcal{Y}_T^a = \big\{ u \in \mathcal{Y}_T; \ \vertiii{u} \leqslant a \big\},$$

with

$$\vertiii{u} := \|u\|_{L_T^\infty H_{xy}^s} + \|u\|_{L_T^{3k/2+} L_{xy}^\infty} + \|u\|_{L_x^{k/2} L_{yT}^\infty} + \|u_x\|_{L_T^{3k/(k+2)} L_{xy}^\infty}$$

$$+ \|u_x\|_{L_x^\infty L_{yT}^2} + \big\|D_x^s u_x\big\|_{L_x^\infty L_{yT}^2} + \big\|D_y^s u_x\big\|_{L_x^\infty L_{yT}^2},$$

where $a, T > 0$ will be chosen later. We assume that $s_k < s < 1$ and $T \leqslant 1$.

First we estimate the H^s-norm of $\Psi(u)$. Let $u \in \mathcal{Y}_T^a$. By using Minkowski's inequality, group properties and then Hölder's inequality, we have

$$\big\|\Psi(u)(t)\big\|_{L_{xy}^2} \lesssim \|u_0\|_{H^s} + \int_0^T \big\|u^{3k/4}\big\|_{L_{xy}^\infty} \big\|u^{k/4} u_x\big\|_{L_{xy}^2} \, dt'$$

$$\lesssim \|u_0\|_{H^s} + \big\|u^{3k/4}\big\|_{L_T^2 L_{xy}^\infty} \big\|u^{k/4} u_x\big\|_{L_{xyT}^2}$$

$$\lesssim \|u_0\|_{H^s} + T^\gamma \|u\|_{L_T^{3k/2+} L_{xy}^\infty}^{} \|u\|_{L_x^{k/2} L_{yT}^\infty}^{k/4} \|u_x\|_{L_x^\infty L_{yT}^2}, \tag{3.24}$$

where $\gamma > 0$ is an arbitrarily small number.

On the other hand, using group properties and Minkowski's inequality, we have

$$\big\|D_x^s \Psi(u)(t)\big\|_{L_{xy}^2} \lesssim \big\|D_x^s u_0\big\|_{L_{xy}^2} + \int_0^T \big\|D_x^s\big(u^k u_x\big)\big\|_{L_{xy}^2} \, dt' = c\|u_0\|_{H^s} + A_0. \tag{3.25}$$

Applying Leibniz rule for fractional derivatives (see Lemma 2.4) and Hölder's inequality, we get

$$A_0 \lesssim \int_0^T \big\|u^{3k/4}\big\|_{L_{xy}^\infty} \big\|D_x^s\big(u^{k/4} u_x\big)\big\|_{L_{xy}^2} \, dt' + \int_0^T \big\|D_x^s\big(u^{3k/4}\big) u^{k/4} u_x\big\|_{L_{xy}^2} \, dt' = A_1 + A_2. \tag{3.26}$$

Moreover,

$$A_1 \lesssim \int_0^T \big\|u^{3k/4}\big\|_{L_{xy}^\infty} \|u_x\|_{L_{xy}^\infty} \big\|D_x^s u^{k/4}\big\|_{L_{xy}^2} \, dt' + \int_0^T \big\|u^{3k/4}\big\|_{L_{xy}^\infty} \big\|u^{k/4} D_x^s u_x\big\|_{L_{xy}^2} \, dt'$$

$$= A_{11} + A_{12}. \tag{3.27}$$

First we consider the term A_{11}. Thus, applying Hölder's inequality, Lemma 2.3 (with $h = 1$) and Hölder's inequality again, we have

$$A_{11} \lesssim \|u\|_{L_T^{3k/2}L_{xy}^\infty}^{3k/4} \left\| \|u_x\|_{L_{xy}^\infty} \|D_x^s u^{k/4}\|_{L_{xy}^2} \right\|_{L_T^2} \lesssim \|u\|_{L_T^{3k/2}L_{xy}^\infty}^{3k/4} \left\| \|u_x\|_{L_{xy}^\infty} \|u^{k/4-1}\|_{L_{xy}^\infty} \|D_x^s u\|_{L_{xy}^2} \right\|_{L_T^2}$$

$$\lesssim \|u\|_{L_T^{3k/2}L_{xy}^\infty}^{3k/4} \|u_x\|_{L_T^{3k/(k+2)}L_{xy}^\infty} \|u\|_{L_T^{3k/2}L_{xy}^\infty}^{(k-4)/4} \|u\|_{L_T^\infty H_{xy}^s}$$

$$\lesssim T^\gamma \|u\|_{L_T^{3k/2+}L_{xy}^\infty}^{k-1} \|u_x\|_{L_T^{3k/(k+2)}L_{xy}^\infty} \|u\|_{L_T^\infty H_{xy}^s}. \tag{3.28}$$

To bound A_{12} we just apply Hölder's inequality twice to obtain

$$A_{12} \lesssim \|u\|_{L_T^{3k/2}L_{xy}^\infty}^{3k/4} \|D_x^s u_x\|_{L_x^\infty L_{yT}^2} \|u\|_{L_x^{k/2}L_{yT}^\infty}^{k/4} \lesssim T^\gamma \|u\|_{L_T^{3k/2+}L_{xy}^\infty}^{3k/4} \|D_x^s u_x\|_{L_x^\infty L_{yT}^2} \|u\|_{L_x^{k/2}L_{yT}^\infty}^{k/4}. \tag{3.29}$$

Next we consider the term A_2. Lemma 2.3 (with $h = 1$) and Hölder's inequality yield

$$A_2 \lesssim \int_0^T \|D_x^s (u^{3k/4})\|_{L_{xy}^2} \|u^{k/4} u_x\|_{L_{xy}^\infty} \, dt' \lesssim \int_0^T \|u^{3k/4-1}\|_{L_{xy}^\infty} \|D_x^s u\|_{L_{xy}^2} \|u\|_{L_{xy}^\infty}^{k/4} \|u_x\|_{L_{xy}^\infty} \, dt'$$

$$\lesssim \|u\|_{L_T^{3k/2}L_{xy}^\infty}^{k-1} \|u_x\|_{L_T^{3k/(k+2)}L_{xy}^\infty} \|u\|_{L_T^\infty H_{xy}^s} \lesssim T^\gamma \|u\|_{L_T^{3k/2+}L_{xy}^\infty}^{k-1} \|u_x\|_{L_T^{3k/(k+2)}L_{xy}^\infty} \|u\|_{L_T^\infty H_{xy}^s}. \tag{3.30}$$

A similar analysis can be carried out to estimate the norm $\|D_y^s \Psi(u)(t)\|_{L_{xy}^2}$. Therefore, from (3.24)–(3.30), we deduce

$$\|\Psi(u)\|_{L_T^\infty H^s} \leqslant c\|u_0\|_{H^s} + cT^\gamma \|u\|^{k+1}. \tag{3.31}$$

The remaining norms are estimated similarly. Indeed, by combining the linear estimates (i)–(iii) in Proposition 2.2, Lemma 2.1(i), and group properties it is easy to see that all the problem reduces to the estimation of A_0. Therefore, we infer

$$\|\Psi(u)\| \leqslant c\|u_0\|_{H^s} + cT^\gamma \|u\|^{k+1}.$$

Choose $a = 2c\|u_0\|_{H^s}$, and $T > 0$ such that

$$ca^k T^\gamma \leqslant \frac{1}{4}.$$

Then, we see that $\Psi : \mathcal{Y}_T^a \mapsto \mathcal{Y}_T^a$ is well defined. Moreover, similar arguments show that Ψ is a contraction. To finish the proof we use standard arguments, thus, we omit the details. This completes the proof of Theorem 1.2.

4. Global well-posedness: proof of Theorem 1.3

We first note that from the discussion in (1.11) the smallness condition on $\|u_0\|_{H^1}$ should be closely related to the constant appearing in the Gagliardo–Nirenberg inequality (1.8). Thus, let us recall the classical result obtained by Weinstein [16], regarding the best constant for the Gagliardo–Nirenberg inequality.

Theorem 4.1. *Let $k > 0$, then the Gagliardo–Nirenberg inequality*

$$\|u\|_{L^{k+2}}^{k+2} \leqslant K_{\text{opt}}^{k+2} \|\nabla u\|_{L^2}^k \|u\|_{L^2}^2 \tag{4.32}$$

holds, and the sharp constant $K_{\mathrm{opt}} > 0$ is explicitly given by

$$K_{\mathrm{opt}}^{k+2} = \frac{k+2}{2\|\psi\|_{L^2}^k}, \tag{4.33}$$

where ψ is the unique non-negative, radially-symmetric, decreasing solution of the equation

$$\frac{k}{2}\Delta\psi - \psi + \psi^{k+1} = 0. \tag{4.34}$$

Proof. See [16, Corollary 2.1]. □

Remark 4.2. If ψ is the solution of (4.34), then by uniqueness

$$Q(x, y) = \psi\left(\sqrt{\frac{k}{2}}(x, y)\right)$$

is the solution of

$$\Delta Q - Q + Q^{k+1} = 0. \tag{4.35}$$

Moreover,

$$\|Q\|_{L^2}^2 = \frac{2}{k}\|\psi\|_{L^2}^2.$$

In view of Remark 4.2 and (4.33), we deduce that

$$K_{\mathrm{opt}}^{k+2} = \frac{2^{\frac{k-2}{2}}(k+2)}{k^{\frac{k}{2}}\|Q\|_{L^2}^k}. \tag{4.36}$$

Now, by multiplying (4.35) by Q, integrating on $\mathbb{R}^2$, and applying integration by parts, we obtain

$$\int_{\mathbb{R}^2} Q^{k+2}\,dx\,dy = \|Q\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2.$$

On the other hand, by multiplying (4.35) by $(x, y) \cdot \nabla Q$, integrating on $\mathbb{R}^2$, and applying integration by parts, we promptly obtain the identity

$$\int_{\mathbb{R}^2} Q^{k+2}\,dx\,dy = \frac{k+2}{2}\|Q\|_{L^2}^2.$$

Combining the last two relations, we have

$$\frac{k}{2}\|Q\|_{L^2}^2 = \|\nabla Q\|_{L^2}^2. \tag{4.37}$$

With these tools at hand, we are able to prove Theorem 1.3.

Proof of Theorem 1.3. We proceed as follows: write the $\dot{H}^1$-norm of $u(t)$ using the quantities $M(u(t))$ and $E(u(t))$. Then we use the sharp Gagliardo–Nirenberg inequality (4.32), with the sharp constant in (4.36), to yield

$$\left\|\nabla u(t)\right\|_{L^2}^2 = 2E(u_0) + \frac{2}{k+2}\int_{\mathbb{R}^2} u^{k+2}(t)\,dx\,dy \leqslant 2E(u_0) + \frac{2}{k+2}K_{\text{opt}}^{k+2}\|u_0\|_{L^2}^2\left\|\nabla u(t)\right\|_{L^2}^k$$

$$= 2E(u_0) + \left(\frac{2}{k}\right)^{\frac{k}{2}}\frac{1}{\|Q\|_{L^2}^k}\|u_0\|_{L^2}^2\left\|\nabla u(t)\right\|_{L^2}^k. \tag{4.38}$$

Let $X(t) = \|\nabla u(t)\|_{L^2}^2$, $A = 2E(u_0)$, and $B = (\frac{2}{k})^{\frac{k}{2}}\frac{\|u_0\|_{L^2}^2}{\|Q\|_{L^2}^k}$, then we can write (4.38) as

$$X(t) - BX(t)^{k/2} \leqslant A, \quad \text{for } t \in (0, T), \tag{4.39}$$

where T is given by Theorem 1.1 (or Theorem 1.2 if $k > 8$).

Now let $f(x) = x - Bx^{k/2}$, for $x \geqslant 0$. The function f has a local maximum at $x_0 = (\frac{2}{kB})^{2/(k-2)}$ with maximum value $f(x_0) = \frac{k-2}{k}(\frac{2}{kB})^{2/(k-2)}$. If we require that

$$2E(u_0) < f(x_0) \quad \text{and} \quad X(0) < x_0, \tag{4.40}$$

the continuity of $X(t)$ implies that $X(t) < x_0$ for any t as long as the solution exists.

Using relations (4.37), we have

$$E(Q) = \frac{k-2}{4}\|Q\|_{L^2}^2.$$

Therefore, a simple calculation shows that conditions (4.40) are exactly the inequalities (1.12) and (1.13). Moreover the inequality $X(t) < x_0$ reduces to (1.14). The proof of Theorem 1.3 is thus completed. □

5. Scattering: proof of Theorem 1.5

We start by recalling the following decay result for solutions $u(t) = U(t)f$, of the linear problem (2.17).

Proposition 5.1. *Let $0 \leqslant \varepsilon < 1/2$ and $0 \leqslant \theta \leqslant 1$. Then,*

$$\left\|D_x^{\theta\varepsilon}U(t)f\right\|_{L_{xy}^p} \leqslant C|t|^{-\theta\frac{(2+\varepsilon)}{3}}\|f\|_{L_{xy}^{p'}},$$

where $p = \frac{2}{1-\theta}$ and $p' = \frac{2}{1+\theta}$. In particular,

$$\left\|U(t)f\right\|_{L_{xy}^p} \leqslant C|t|^{-\frac{2\theta}{3}}\|f\|_{L_{xy}^{p'}}.$$

Proof. See Linares and Pastor [9, Lemma 2.3]. □

As a consequence, we have

Corollary 5.2. *Let p and p′ be as in Proposition* 5.1. *If* $0 \leqslant \theta < 1$ *and* $f \in L^{p'}(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, *then*

$$\big\| U(t)f \big\|_{L^p_{xy}} \leqslant C \big(1 + |t|\big)^{-\frac{2\theta}{3}} \big(\|f\|_{L^{p'}} + \|f\|_{H^1} \big).$$

Proof. The proof follows immediately from Proposition 5.1 and the embedding of $H^1(\mathbb{R}^2)$ in $L^p(\mathbb{R}^2)$, $2 \leqslant p < \infty$. $\quad\square$

Theorem 5.3 *(Decay). Let* $p = 2(k+1)$, $p' = \frac{2(k+1)}{2k+1}$, *and* $\theta = \frac{k}{k+1}$. *Assume* $u_0 \in L^{p'}(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ *satisfies*

$$\|u_0\|_{L^{p'}} + \|u_0\|_{H^1} < \delta.$$

Then, the solution u(t) given in Theorem 1.3 *satisfies*

$$\big(1 + |t|\big)^{\frac{2\theta}{3}} \big\| u(t) \big\|_{L^p} \leqslant C$$

for all $t \in \mathbb{R}$ *and some constant* $C > 0$.

Proof. From the integral formulation of (1.1), we have

$$u(t) = U(t)u_0 - \int_0^t U\big(t - t'\big) \partial_x\big(u^{k+1}\big)\big(t'\big)\, dt'.$$

Without loss of generality assume $t > 0$. Thus, from Proposition 5.1 and Corollary 5.2, we have

$$\big\| u(t) \big\|_{L^p} \leqslant \big\| U(t)u_0 \big\|_{L^p} + \int_0^t \big\| U\big(t - t'\big) \partial_x\big(u^{k+1}\big)\big(t'\big) \big\|_{L^p}\, dt'$$

$$\leqslant C(1 + t)^{-\frac{2\theta}{3}} \big(\|u_0\|_{L^{p'}} + \|u_0\|_{H^1} \big) + C \int_0^t \big(t - t'\big)^{-\frac{2\theta}{3}} \big\| \partial_x\big(u^{k+1}\big)\big(t'\big) \big\|_{L^{p'}}\, dt'$$

$$\leqslant C(1 + t)^{-\frac{2\theta}{3}} \delta + C \int_0^t \big(t - t'\big)^{-\frac{2\theta}{3}} \big\| u^k \partial_x u\big(t'\big) \big\|_{L^{p'}}\, dt'$$

$$\leqslant C(1 + t)^{-\frac{2\theta}{3}} \delta + C \int_0^t \big(t - t'\big)^{-\frac{2\theta}{3}} \big\| u^k \big\|_{L^{\frac{2(k+1)}{k}}} \| \partial_x u \|_{L^2}\, dt'$$

$$\leqslant C(1 + t)^{-\frac{2\theta}{3}} \delta + C \|u\|_{L^\infty_T H^1} \int_0^t \big(t - t'\big)^{-\frac{2\theta}{3}} \big\| u\big(t'\big) \big\|^k_{L^p}\, dt'.$$

Let

$$M(T) = \sup_{t \in [0,T]} (1 + t)^{\frac{2\theta}{3}} \big\| u(t) \big\|_{L^p}.$$

Then, we can write

$$M(T) \leqslant C\delta + C\delta(1+t)^{\frac{2\theta}{3}} M(T)^k \int\limits_0^t (t-t')^{-\frac{2\theta}{3}} (1+t')^{\frac{2\theta k}{3}} dt'. \tag{5.41}$$

Since $k \geqslant 3$, we then obtain

$$M(T) \leqslant C\delta + C\delta M(T)^k.$$

Hence, if $\delta \ll 1$, we deduce from a continuity argument that $M(T) \leqslant C$. This completes the proof. □

Remark 5.4. From (5.41) we see that it suffices to take $k > \frac{3+\sqrt{33}}{4} \simeq 2.186$. Note that the case $k = 2$ (L^2-critical) is not cover by our result and it is a very interesting open problem.

In the proof of Theorem 1.5, we only consider the case as $t \to -\infty$, since that as $t \to +\infty$ is similarly treated. Define

$$f_- = u_0 - \int\limits_{-\infty}^0 U(-t')\partial_x(u^{k+1}) dt'.$$

Then,

$$u(t) - U(t)f_- = \int\limits_{-\infty}^t U(t-t')\partial_x(u^{k+1}) dt'.$$

Lemma 5.5. *It also holds that,* $\|U(-t)u(t) - f_-\|_{L^{2(k+1)}} \to 0$, *as* $t \to -\infty$.

Proof. Indeed, from Proposition 5.1, we have

$$\|U(-t)u(t) - f_-\|_{L^{2(k+1)}} \leqslant C \int\limits_{-\infty}^t |t'|^{-\frac{2k}{3(k+1)}} \|u^k \partial_x u\|_{L^{\frac{2(k+1)}{2k+1}}} dt'.$$

From Hölder's inequality and Theorem 1.3, we then deduce

$$\|U(-t)u(t) - f_-\|_{L^{2(k+1)}} \leqslant C \int\limits_{-\infty}^t |t'|^{-\frac{2k}{3(k+1)}} \|u^k\|_{L^{\frac{2(k+1)}{k}}} \|\partial_x u\|_{L^2} dt'$$

$$\leqslant C\|u\|_{L_t^\infty H^1} C \int\limits_{-\infty}^t |t'|^{-\frac{2k}{3(k+1)}} \|u\|_{L^{2(k+1)}}^k dt'$$

$$\leqslant C\left(\int\limits_{-\infty}^t |t'|^{\frac{4k}{3}}\right)^{\frac{1}{2(k+1)}} \left(\int\limits_{-\infty}^t \|u\|_{L^{2(k+1)}}^{\frac{2k(k+1)}{2k+1}} dt'\right)^{\frac{2k+1}{2(k+1)}}.$$

Since $k \geqslant 3$ these last two integrals tend to zero as $t \to -\infty$. □

Lemma 5.6. *Let*

$$G(u) = \frac{1}{k+2} \int_{\mathbb{R}^2} u^{k+2} \, dx.$$

Then, $G(u(t)) \to 0$, as $t \to -\infty$.

Proof. From Hölder's inequality and Theorem 5.3, we have

$$\left| G\big(u(t)\big) \right| \leqslant C \int_{\mathbb{R}^2} \left| u(t) \right|^{k+1} \left| u(t) \right| dx \leqslant C \left\| u(t) \right\|_{L^2} \left(\int_{\mathbb{R}^2} \left| u(t) \right|^{2(k+1)} dx \right)^{\frac{1}{2}}$$

$$\leqslant C \left\| u(t) \right\|_{L^{2(k+1)}}^{k+1} \leqslant C \left(1 + |t| \right)^{-\frac{2k}{3}}. \qquad \square$$

Proof of Theorem 1.5. Since $U(t)$ is a unitary group, from Theorem 1.3, we obtain

$$\left\| U(-t)u(t) \right\|_{H^1} = \left\| u(t) \right\|_{H^1} \leqslant C \| u_0 \|_{H^1}.$$

Thus $U(-t)u(t) \rightharpoonup f_-$ in $H^1(\mathbb{R}^2)$, as $t \to -\infty$. Moreover,

$$\| f_- \|_{H^1} \leqslant \liminf_{t \to -\infty} \left\| U(-t)u(t) \right\|_{H^1} = \liminf_{t \to -\infty} \left\| u(t) \right\|_{H^1} = \liminf_{t \to -\infty} \big(\left\| u(t) \right\|_{H^1} - 2G\big(u(t)\big) \big) \leqslant \| f_- \|_{H^1}.$$

Hence, the weak limit is strong and we have

$$\left\| u(t) - U(t)f_- \right\|_{H^1} = \left\| U(-t)u(t) - f_- \right\|_{H^1} \to 0,$$

as $t \to -\infty$. This completes the proof of Theorem 1.5. $\square$

Acknowledgments

References

[1] H.A. Biagioni, F. Linares, Well-posedness results for the modified Zakharov–Kuznetsov equation, in: Nonlinear Equations: Methods, Models and Applications, in: Progr. Nonlinear Differential Equations Appl., vol. 54, Birkhäuser, Basel, 2003, pp. 181–189.

[2] E. Bustamante, P. Isaza, J. Mejia, On the support of solutions to the Zakharov–Kuznetsov equation, J. Differential Equations 251 (2011) 2728–2736.

[3] A. de Bouard, Stability and instability of some nonlinear dispersive solitary waves in higher dimension, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996) 89–112.

[4] A.V. Faminskii, The Cauchy problem for the Zakharov–Kuznetsov equation, Differ. Equ. 31 (1995) 1002–1012.

[5] L.G. Farah, F. Linares, A. Pastor, The supercritical generalized KdV equation: Global well-posedness in the energy space and below, Math. Res. Lett. 18 (2011) 357–377.

[6] J. Holmer, S. Roudenko, A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation, Comm. Math. Phys. 282 (2008) 435–467.

[7] C.E. Kenig, F. Merle, Global well-posedness, scattering, and blow-up for the energy-critical focusing nonlinear Schrödinger equation in the radial case, Invent. Math. 166 (2006) 645–675.

[8] C.E. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (1993) 527–620.

[9] F. Linares, A. Pastor, Well-posedness for the two-dimensional modified Zakharov–Kuznetsov equation, SIAM J. Math. Anal. 41 (2009) 1323–1339.

[10] F. Linares, A. Pastor, Local and global well-posedness for the 2D generalized Zakharov–Kuznetsov equation, J. Funct. Anal. 260 (2011) 1060–1085.
[11] F. Linares, A. Pastor, J.-C. Saut, Well-posedness for the ZK equation in a cylinder and on the background of a KdV soliton, Comm. Partial Differential Equations 35 (2010) 1674–1689.
[12] F. Linares, M. Scialom, Asymptotic behavior of solutions of a generalized Boussinesq type equation, Nonlinear Anal. 25 (1995) 1147–1158.
[13] M. Panthee, A note on the unique continuation property for Zakharov–Kuznetsov equation, Nonlinear Anal. 59 (2004) 425–438.
[14] M. Panthee, M. Scialom, Asymptotic behavior for a class of solutions to the critical modified Zakharov–Kuznetsov equation, Stud. Appl. Math. 124 (2010) 229–245.
[15] G. Ponce, L. Vega, Nonlinear small data scattering for the generalized Korteweg–de Vries equation, J. Funct. Anal. 90 (1990) 445–457.
[16] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 87 (1983) 567–576.
[17] W. Strauss, Nonlinear scattering theory at low energy, J. Funct. Anal. 41 (1981) 110–133.
[18] V.E. Zakharov, E.A. Kuznetsov, On three-dimensional solitons, Sov. Phys. JETP 39 (1974) 285–286.