



Asymptotic behavior of degenerate logistic equations [☆]

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Abstract

We analyze the asymptotic behavior of positive solutions of parabolic equations with a class of degenerate logistic nonlinearities of the type $\lambda u - n(x)u^\rho$. An important characteristic of this work is that the region where the logistic term $n(\cdot)$ vanishes, that is $K_0 = \{x : n(x) = 0\}$, may be non-smooth. We analyze conditions on λ , ρ , $n(\cdot)$ and K_0 guaranteeing that the solution starting at a positive initial condition remains bounded or blows up as time goes to infinity. The asymptotic behavior may not be the same in different parts of K_0 .

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1. Introduction

Let us consider the following reaction diffusion problem of logistic type

$$\begin{cases} u_t - \Delta u = \lambda u - n(x)u^\rho & \text{in } \Omega, t > 0 \\ u = 0 & \text{on } \partial\Omega, t > 0 \\ u(0) = u_0 \geq 0 \end{cases} \quad (1.1)$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, where $n(x) \geq 0$ in Ω is a bounded function not identically zero, $\rho > 1$, and $\lambda \in \mathbb{R}$. Observe that (1.1) is well posed for $0 \leq u_0 \in L^1(\Omega)$ and the solution, which will be denoted $u(t; u_0)$, becomes classical and positive for $t > 0$, see Section 2.

As for the asymptotic behavior of nonnegative solutions, note that (1.1) degenerates to a linear equation on the set in which $n(x)$ vanishes, denoted by K_0 . This set plays a crucial role in the dynamical properties and the asymptotic behavior of solutions of (1.1). In fact, one expects the solution to grow in this set, at least for large values of λ . On the other hand, in the complementary set of K_0 the nonlinear reaction term acts, and one expects the solutions to remain bounded in this set. Therefore, to determine the long term behavior of solutions, there should be a balance between the size of K_0 and the strength of the nonlinear term (measured in terms of ρ and the way that $n(x)$ vanishes near K_0) which we try to unveil here.

This type of question was studied before but always under the assumption that the function $n(x)$ is somehow smooth and the set $\{x \in \Omega : n(x) > 0\}$ is an open subset of Ω with regular boundary. Without pretending being exhaustive, we would like to mention some relevant contributions in this direction. For instance, Ouyang [25] studied the elliptic problem associated to (1.1). Then Fraile et al. [12] considered the elliptic but also the parabolic problem for the first time in the literature. Later Du and Huang [10] improved some of the technical conditions of the previous paper and proved results related to those of [16]. Since then, many papers have been devoted to this subject, many of them related to understanding the behavior of the solutions of some singular elliptic boundary value problem associated to (1.1); see e.g. [13–15]. A rather complete survey on this subject is [22]. All these papers contain references to other related results. Finally, see [26] for similar results in unbounded domains.

As a difference with respect to the above mentioned contributions, in the present paper we make no regularity assumption at all on $n(x)$ nor on K_0 , other than $n(x)$ being bounded and $K_0 \subset \Omega$ compact.

One particular and relevant example that may help us to grasp the scope of our results is the following. In dimension $N \geq 2$, consider the set K_0 is decomposed as $K_0 = K_1 \cup K_2$, where $K_1 = \bar{B}$ is the closure of a bounded, smooth, connected domain B (think for instance B is a ball) and K_2 is a segment (see Fig. 1 and ignore the dotted box appearing in the figure for a moment). If we denote by $\lambda_1(D)$ the first eigenvalue of the Laplacian operator with Dirichlet boundary conditions in a generic domain $D \subset \mathbb{R}^N$, we get the following picture. For $\lambda < \lambda_1(B)$ all solutions of (1.1) converge to a unique smooth globally asymptotically stable equilibria of (1.1). This equilibria is positive in Ω for $\lambda_1(\Omega) < \lambda < \lambda_1(B)$ and identically zero otherwise. On the other hand for $\lambda \geq \lambda_1(B)$ then all solutions of (1.1) become unbounded in a neighborhood of K_0 and bounded in the rest of Ω .

If K_0 is disconnected, that is $K_1 \cap K_2 = \emptyset$ then for all $\lambda \in \mathbb{R}$ all solutions remain bounded in K_2 . On the other hand, if $K_1 \cap K_2$ is a point, as depicted in Fig. 1 and we assume that

$$n(x) \geq \text{dist}(x, K_0)^\gamma \quad \text{for some } \gamma > 0$$

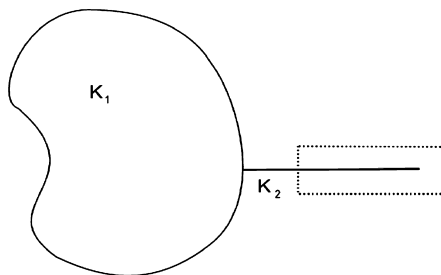


Fig. 1. Fat and slim set.

(which is an assumption on the rate at which $n(x)$ vanishes near K_0), and

$$\gamma + 2 < (\rho - 1)(N - 1) \quad (1.2)$$

then for $\lambda \geq \lambda_1(B)$ all solutions of (1.1) remain unbounded in B and bounded in K_2 . This type of balance conditions shows the great sensitivity of the asymptotic behavior of solutions with respect to the power of the nonlinearity and the rate at which $n(x)$ vanishes.

Turning back to our general setting, since we are interested here in the case in which $n(x)$ nor the region it vanishes are smooth we will assume that $0 \leq n \in L^\infty(\Omega)$ and there exists a compact set (which may be even empty)

$$K_0 \subset \Omega$$

such that $n(x) = 0$ for all $x \in K_0$ and for any compact set $C \subset \Omega \setminus K_0$ there exists $\alpha = \alpha(C) > 0$ such that

$$n(x) \geq \alpha \quad \text{for all } x \in C.$$

Note that this is equivalent to assume that there exists a positive, continuous function $n_0 \in C(\overline{\Omega} \setminus K_0)$, strictly positive near the boundary of Ω , such that

$$n(x) \geq n_0(x) > 0 \quad \text{for all } x \in \overline{\Omega} \setminus K_0. \quad (1.3)$$

By extending by zero n_0 to K_0 we get

$$n(x) \geq n_0(x) \geq 0 \quad \text{in } \overline{\Omega}$$

and then we will assume that either

$$n_0 \in C(\overline{\Omega}), \quad \text{or} \quad \inf\{n_0(x), x \in \overline{\Omega} \setminus K_0\} > 0 \quad (1.4)$$

and then, in the latter case, n_0 jumps across K_0 . In any case

$$K_0 = \{x \in \Omega : n(x) = 0\}. \quad (1.5)$$

Also observe that given a compact set $K_0 \subset \Omega$ and given any continuous real function $n^*(s) > 0$ with $n^*(0) = 0$, the function $n(x) = n^*(\text{dist}(x, K_0))$ is continuous with prescribed vanishing set K_0 . However, the “rate” at which $n(x)$ vanishes at K_0 is determined by $n^*(s)$.

To analyze the behavior of solutions of (1.1), we introduce in Section 3, by using approximation by smooth open sets, a positive quantity associated to the set K_0 that we denote $\lambda_0(K_0)$, which can be even ∞ , that will play a crucial role in the analysis. When K_0 is smooth, as in the references above, then $\lambda_0(K_0)$ coincides with the first eigenvalue of the Dirichlet Laplacian in the interior of K_0 which is a smooth open set. That $\lambda_0(K_0)$ is finite or infinite allows us to distinguish what we denote “fat” from “slim” sets, respectively. We also show that K_0 can be decomposed in different pairwise disjoint components, generically denoted K , each of them carrying its own $\lambda_0(K)$. By component we mean a compact subset K of K_0 such that $K_0 \setminus K$ is also compact. Although the decomposition is not necessarily finite, such components can be classified according to their fatness measured in the size of $\lambda_0(K)$. The smaller $\lambda_0(K)$, the fatter is the component. Slim components are those for which $\lambda_0(K) = \infty$.

Then in Section 4 we analyze the equilibria (i.e. stationary solutions) of (1.1). We show that, if we denote by $\lambda_1(\Omega) > 0$ the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions, then equilibria only exist for

$$\lambda_1(\Omega) < \lambda < \lambda_0(K_0)$$

and in such a case all solutions of (1.1) converge to a unique smooth positive equilibria of (1.1), φ_λ , which is therefore globally asymptotically stable. We also show that φ_λ is increasing in λ and becomes unbounded in $L^1(\Omega)$ as $\lambda \rightarrow \lambda_0(K_0)$; see Theorem 4.2. Of course, if K_0 is slim, then $\lambda_0(K_0) = \infty$ and this case concludes the analysis.

On the other hand, if K_0 is fat, when $\lambda \geq \lambda_0(K_0)$, (1.1) has no equilibria and all solutions become unbounded in Ω , as $t \rightarrow \infty$. Therefore the natural questions are where and how solutions become unbounded. We then prove in Section 5 that all solutions of (1.1) become unbounded in the same points and at the same rate, and that they remain bounded outside K_0 .

Using the decomposition of K_0 in components obtained in Section 3, we prove that the solutions of (1.1) remain bounded, as $t \rightarrow \infty$ and for all $\lambda \in \mathbb{R}$, in a neighborhood of each slim component. On the other hand, as λ increases the solutions of (1.1) become unbounded on a neighborhood of each fat component K as soon as $\lambda \geq \lambda_0(K)$ while if $\lambda < \lambda_0(K)$ they remain bounded for all times in a neighborhood of K . See Theorem 5.7.

Thus we obtain qualitatively the same picture than in the case of a smooth set K_0 , see [16], but without any regularity assumptions on this set.

This result leads us to consider the behavior of solutions near a fat component K of K_0 and still the goal is to determine in which points of K the solutions of (1.1) become unbounded or remain bounded when $\lambda \geq \lambda_0(K)$. From Lemma 5.4 we know that in case K is smooth (i.e. the closure of a smooth open set) all solutions become unbounded in the interior of K . Thus we consider the case in which K is composed of two compact sets glued together, $K = K_1 \cup K_2$ and $K_1 \cap K_2 \neq \emptyset$. Moreover we assume K_1 is fatter than K_2 that is $\lambda_0(K) = \lambda_0(K_1) < \lambda_0(K_2)$. A particular, but very relevant case of this situation appears when we consider that K_1 is the closure of a smooth open set and K_2 is a piece of a segment attached to K_1 , see Fig. 1 (ignore the dotted box appearing in the figure for a moment). As $\lambda \geq \lambda_0(K) = \lambda_0(K_1)$ we expect the solutions of (1.1) to become unbounded in K_1 but it is unclear whether or not they are also unbounded in K_2 , specially if $\lambda < \lambda_0(K_2)$. That is, the problem is to determine whether or not u is able to diffuse from K_1 to become unbounded in K_2 .

At this respect we give a sufficient condition guaranteeing that this does not happen, and therefore that all solutions of (1.1) remain bounded in K_2 while they are unbounded in K , thus in K_1 . This condition is stated in a general context in Theorem 5.15, and three quantities intervene in the characterization: i) the way K_1 and K_2 are glued together, measured basically in terms of the fractal dimension of their intersection, ii) the rate at which $n(x)$ vanishes near the intersection and iii) the strength of the nonlinearity measured by ρ .

Here we recover condition (1.2) in the example above; see Remark 5.17. Note that if instead of a segment, K_2 were a piece of a smooth manifold of dimension $d < N$, then (1.2) would read $\gamma + 2 < (\rho - 1)(N - d)$.

One of the main ingredients for this result is some universal upper bounds for solutions of (1.1), valid for all times, outside the set K_0 , see Lemma 5.9. This pointwise upper bound diverges to ∞ as $x \rightarrow K_0$ and agrees with the upper bounds known in the case K_0 is a smooth set, see [14,15,22]. When considering these pointwise bounds restricted to a box like the one depicted in Fig. 1 with a dotted stroke, we obtain a function which controls the behavior of the solution at the faces of this box. This function has suitable $(N - 1)$ -dimensional integrability properties, which are obtained from Lemma 3.13, of independent interest. Actually, it will be uniformly bounded in the three faces not touching K_2 and if condition (1.2) holds then it will be integrable in the face transversal to K_2 . Once we control the behavior of the solutions at the boundary of the box, we use suitable comparison results and parabolic regularity to obtain bounds on the solution in the inside of the box and therefore in the part of K_2 inside the box. If we are able to do this argument for boxes arbitrary near K_1 , then we will show that the function has to be bounded in K_2 .

Some partial previous results have been presented in [2,3].

2. Previous results and statement of the problem

In this section we collect some results on the solutions of (1.1) that will be used in this paper. First recall that we only assume $n(x) \geq 0$ in Ω and it is a bounded function not identically zero.

Observe that, as we consider only nonnegative initial data and $n(x) \geq 0$ in Ω , then, by comparison with the linear equation (throwing away the nonlinear term), any solution of (1.1) is below a solution of the linear equation and therefore, it is globally defined. In fact we have that (1.1) is well posed for $0 \leq u_0 \in L^1(\Omega)$ and the solution, which will be denoted $u(t; u_0)$, satisfies

$$u(\cdot; u_0) \in C([0, \infty), L^1(\Omega)) \cap C((0, \infty), W^{1,q}(\Omega) \cap W_0^{1,q}(\Omega)) \quad (2.1)$$

for any $1 < q < \infty$. In particular $u(\cdot; u_0)$ becomes classical for $t > 0$, see e.g. [29]. Also, when needed, we can assume the initial data is C^1 smooth in $\overline{\Omega}$ and, thanks to the strong maximum principle, strictly positive in Ω . Otherwise we can replace u_0 by $u(\varepsilon; u_0)$ for any $\varepsilon > 0$.

Some solutions that play an important role below are the *equilibria* or *stationary solutions* of (1.1), i.e., nonnegative solutions of the associated elliptic problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi - n(x)\varphi^\rho, & \text{in } \Omega \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

By this we mean a nonnegative $\varphi \in L^1(\Omega)$ such that

$$u(t; \varphi) = \varphi \quad \text{for all } t \geq 0.$$

In particular, from (2.1) we have that $\varphi \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ for all $1 < q < \infty$ and solves (2.2) in a strong sense. Conversely, if $\varphi \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^\rho(\Omega)$ for some $1 < q < \infty$, and solves (2.2) in a strong sense, then obviously φ is a stationary solution of (1.1).

To describe the asymptotic behavior of the solutions, denote $f(x, u) = \lambda u - n(x)u^\rho$. Then, some of the results in [28], which apply for more general type of nonlinear terms, can be summarized as:

Proposition 2.1. Denote $f(x, u) = \lambda u - n(x)u^\rho$ and assume there exist $C(x)$, $D(x)$ such that

$$f(x, u) \leq C(x)u + D(x) \quad (2.3)$$

for all $u \geq 0$ and $x \in \Omega$ and such that

- i) $C \in L^p(\Omega)$ with $p > N/2$,
- ii) the semigroup generated by $\Delta + C$ has exponential decay and
- iii) $D \in L^r(\Omega)$ for some $r > \frac{N}{2} \left(1 - \frac{1}{\rho}\right)$.

Then (1.1) has a unique nonnegative equilibria φ which is globally asymptotically stable for nonnegative nontrivial solutions of (1.1). That is, for every $u_0 \geq 0$ and nonzero in Ω , the solution of (1.1) satisfies

$$\lim_{t \rightarrow \infty} u(t, x; u_0) = \varphi(x)$$

uniformly in $x \in \overline{\Omega}$.

Moreover if $\lambda \leq \lambda_1(\Omega)$ then (2.3) is always satisfied and $\varphi = 0$, while if $\lambda > \lambda_1(\Omega)$ and (2.3) holds, then $\varphi(x) > 0$ in Ω .

Note that the case $\lambda < \lambda_1(\Omega)$ in Proposition 2.1 follows easily by comparison with the linear equation, that is, disregarding the nonlinear term in (1.1).

Now note that (1.1) degenerates to a linear equation on the set in which $n(x)$ vanishes, see (1.5). Hence this set plays a crucial role in whether or not (2.3) holds. The following results, taken from [28], distinguish the case in which $n > 0$ in Ω or vanishes slowly in a small region, from the case in which it vanishes very fast or in a large set. Observe that this result was stated in [28] for $n(x)$ continuous, but indeed that condition was not used; only boundedness of $n(x)$ is required.

Proposition 2.2.

- i) Suppose that either $n(x) \geq \gamma > 0$ in $\overline{\Omega}$ or $1/n \in L^s(\Omega)$, $s > N/2\rho$. Then for any $\lambda \in \mathbb{R}$ the hypotheses of Proposition 2.1 are satisfied.
- ii) Let $K_0 = \{x \in \Omega : n(x) = 0\}$ as in (1.5) and ω_δ be a neighborhood of K_0 such that $n(x) \geq \delta > 0$ for all $x \in \Omega \setminus \overline{\omega_\delta}$. Denote by $\lambda_1(\omega_\delta)$ the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in ω_δ .

Then, if

$$\lambda < \lambda_1(\omega_\delta)$$

the hypotheses of [Proposition 2.1](#) are satisfied.

Hence, in both cases [Proposition 2.1](#) applies.

The second part of the Proposition above suggests to proceed in the following way: let

$$K_0 = \{x \in \Omega : n(x) = 0\}$$

and consider $\delta \rightarrow 0$ and ω_δ a decreasing family of neighborhoods of K_0 with $\omega_\delta \subset \Omega$ and $\bigcap_{\delta>0} \omega_\delta = K_0$. Then, the family $\lambda_1(\omega_\delta)$ of the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in ω_δ is increasing as $\delta \rightarrow 0$ and we can define the monotonic limit

$$\lambda_0(K_0) = \lim_{\delta \rightarrow 0} \lambda_1(\omega_\delta) \quad (2.4)$$

which can be infinity. Note that always $\lambda_1(\Omega) < \lambda_0(K_0)$. Hence we get the following:

Corollary 2.3. *With the notations above, assume n satisfies [\(1.3\)](#). Then, for any*

$$\lambda < \lambda_0(K_0) \leq \infty$$

[Proposition 2.1](#) applies.

In particular [\(1.1\)](#) has a unique nonnegative equilibria φ_λ which is globally asymptotically stable for nonnegative nontrivial solutions of [\(1.1\)](#), as in [Proposition 2.1](#). Also, $\varphi_\lambda = 0$ for $\lambda \leq \lambda_1(\Omega)$ and is strictly positive in Ω for $\lambda > \lambda_1(\Omega)$.

Proof. Just note that for any Ω_δ of the decreasing family of neighborhoods of K_0 above, we can apply [Proposition 2.2](#), part ii). \square

Note that, in particular, if $\lambda_0(K_0) = \infty$ then the situation is the same as in part i) of [Proposition 2.2](#).

Therefore, our goal in the rest of the paper is to consider the case in which $\lambda_0(K_0) < \infty$ and to analyze two different questions. On one hand, we want to understand the behavior of φ_λ as λ increases to $\lambda_0(K_0)$. On the other hand, we want to describe the behavior of solutions of [\(1.1\)](#) when $\lambda > \lambda_0(K_0)$.

3. Fat and slim sets

Motivated by the definition [\(2.4\)](#), in this section we analyze some geometric-measure properties of compact sets in \mathbb{R}^N .

Consider $K \subset \mathbb{R}^N$ an arbitrary compact set and consider a *strictly nested* family of smooth bounded open sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$, that is $\bar{\Omega}_\varepsilon \subset \Omega_{\varepsilon'}$ if $\varepsilon < \varepsilon'$, such that

$$K = \bigcap_{\varepsilon>0} \Omega_\varepsilon.$$

Observe that we also have that $K = \cap_{\varepsilon>0} \bar{\Omega}_\varepsilon$ and the sequence $\{\bar{\Omega}_\varepsilon\}_{\varepsilon>0}$ is a decreasing sequence of compact sets.

Then for each $\varepsilon > 0$ consider

$$\lambda_1(\Omega_\varepsilon) = \inf_{u \in H_0^1(\Omega_\varepsilon)} \frac{\int_{\Omega_\varepsilon} |\nabla u|^2}{\int_{\Omega_\varepsilon} u^2} > 0, \quad (3.1)$$

the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω_ε . Then $\lambda_1(\Omega_\varepsilon)$ is increasing as $\varepsilon \rightarrow 0$ and we can define the monotonic limit

$$\lambda_0(K) = \lim_{\varepsilon \rightarrow 0} \lambda_1(\Omega_\varepsilon) \quad (3.2)$$

which may be infinity. Then we have the following:

Definition 3.1. A compact set $K \subset \mathbb{R}^N$ is **fat** if the sequence $\lambda_1(\Omega_\varepsilon)$ is bounded in ε , or equivalently if

$$0 < \lambda_0(K) < \infty.$$

Otherwise, $\lambda_0(K) = \infty$ and K is said **slim**.

Remark 3.2. Observe that from the properties of the Lebesgue measure

$$|\Omega_\varepsilon \setminus K| \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

In the proposition below and in the rest of the paper we will use the notion of “components” of a compact set. To clarify the meaning we define:

Definition 3.3. A component of a compact set K_0 is a compact set $K \subset K_0$ such that $K_0 \setminus K$ is also compact.

To avoid confusion let us clarify that a component of a compact set may not be connected, so the term component is not equivalent to “connected component”.

We have the following result.

Proposition 3.4.

- i) For any compact set $K \subset \mathbb{R}^N$, the definition of $\lambda_0(K)$ is independent of the decreasing family Ω_ε , which can be assumed to be C^∞ -smooth.
- ii) If $K_1 \subset K_2$ are compact sets, then $\lambda_0(K_2) \leq \lambda_0(K_1)$.
- iii) If $K = \overline{\Omega_0}$ where Ω_0 is a smooth bounded open set, then

$$\lambda_0(K) = \lambda_1(\Omega_0),$$

that is, the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω_0 .

iv) If $K = K_1 \cup K_2$ are compact sets, and K_1 and K_2 are separated in \mathbb{R}^N , that is $K_1 \cap K_2 = \emptyset$, then

$$\lambda_0(K) = \min\{\lambda_0(K_1), \lambda_0(K_2)\}.$$

v) *Decomposition of a compact set in components.*

For any compact set $K \subset \mathbb{R}^N$ there exists $m \in \mathbb{N}$ and K_1, \dots, K_m compact sets in \mathbb{R}^N such that

$$K = K_1 \cup \dots \cup K_m$$

and $K_i \cap K_j = \emptyset$ if $i \neq j$, i.e. K_i and K_j are separated in \mathbb{R}^N .

In particular,

$$\lambda_0(K) = \min\{\lambda_0(K_1), \dots, \lambda_0(K_m)\}.$$

Proof. i) Assume we have two strictly decreasing families of open sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ and $\{\Omega'_\varepsilon\}_{\varepsilon>0}$, and we denote by $\lambda_0(K) = \lim_{\varepsilon \rightarrow 0} \lambda_1(\Omega_\varepsilon)$, $\lambda'_0(K) = \lim_{\varepsilon \rightarrow 0} \lambda_1(\Omega'_\varepsilon)$. Since

$$K = \bigcap_{\varepsilon>0} \bar{\Omega}_\varepsilon = \bigcap_{\varepsilon>0} \bar{\Omega}'_\varepsilon$$

which are two decreasing sequences of compact sets, we have that for each $\eta > 0$ fixed there will exist $\varepsilon_0 > 0$, depending on η , such that $\bar{\Omega}'_\varepsilon \subset \Omega_\eta$, for each $\varepsilon < \varepsilon_0$, which implies that $\lambda_1(\Omega_\eta) \leq \lambda_1(\Omega'_\varepsilon)$ for each $\varepsilon < \varepsilon_0$. Passing to the limit as $\varepsilon \rightarrow 0$, we get $\lambda_1(\Omega_\eta) \leq \lambda'_0(K)$, for each $\eta > 0$. Passing to the limit as $\eta \rightarrow 0$, we get $\lambda_0(K) \leq \lambda'_0(K)$. With a completely symmetric argument, we get $\lambda'_0(K) \leq \lambda_0(K)$.

ii) If Ω_ε^2 is a decreasing family for K_2 and Ω_ε^1 is a decreasing family for K_1 , then we can construct the decreasing family for K_1 given by $\Omega'_\varepsilon = \Omega_\varepsilon^2 \cap \Omega_\varepsilon^1$, which satisfies $\Omega'_\varepsilon \subset \Omega_\varepsilon^2$. The monotonicity of the first eigenvalue of the Dirichlet Laplacian gives $\lambda_1(\Omega_\varepsilon^2) \leq \lambda_1(\Omega'_\varepsilon)$ and taking $\varepsilon \rightarrow 0$ we conclude.

iii) If Ω_0 is a bounded smooth domain and Ω_ε is a decreasing sequence of smooth domains satisfying $\bar{\Omega}_0 \subset \Omega_\varepsilon$, then from the stability properties of the Dirichlet eigenvalues for smooth domains, see [9,4], we get that $\lambda_1(\Omega_0) = \lim_{\varepsilon \rightarrow 0} \lambda_1(\Omega_\varepsilon)$.

iv) Since K_1 and K_2 are separated, we can construct a decreasing family for K , Ω_ε , of the form $\Omega_\varepsilon = \Omega_\varepsilon^1 \cup \Omega_\varepsilon^2$, where Ω_ε^i , $i = 1, 2$ is a decreasing family for K_i and $\Omega_\varepsilon^1 \cap \Omega_\varepsilon^2 = \emptyset$. Hence

$$\lambda_1(\Omega_\varepsilon) = \min\{\lambda_1(\Omega_\varepsilon^1), \lambda_1(\Omega_\varepsilon^2)\}$$

and taking $\varepsilon \rightarrow 0$ we conclude.

v) Let $\Omega_\varepsilon = \{x, \text{dist}(x, K) < \varepsilon\}$ for sufficiently small $\varepsilon > 0$. Then Ω_ε is a countable union of connected components. Since K is compact, there is a finite covering of K of such components, C_1, \dots, C_m . Then we take $K_i = C_i \cap K$. The rest is as in part iv). \square

Remark 3.5.

- i) Note that in part iii) of Proposition 3.4 the smoothness of Ω_0 is not a technical assumption. There exist a bounded domain Ω_0 such that for each sequence of *strictly* decreasing bounded smooth domains Ω_ε we have $\lambda_1(\Omega_0) < \lim_{\varepsilon \rightarrow 0} \lambda_1(\Omega_\varepsilon)$, see [18].

- ii) From part iv) of [Proposition 3.4](#) we get at once that the union of fat sets is fat, the union of slim sets is slim and the union of a fat set and a slim set is a fat set.
- iii) Note that in part v) of [Proposition 3.4](#) neither m nor the components of a compact set are uniquely determined. This is clearly seen if K is a converging sequence and its limit point. In fact if for $\varepsilon > 0$ we denote by $m(\varepsilon)$ the number of connected components of Ω_ε , then $m(\varepsilon)$ increases as $\varepsilon \rightarrow 0$.

Hence, we get the following result on the decomposition of a compact set.

Corollary 3.6. *For any compact set $K \subset \mathbb{R}^N$, there exist a (non-necessarily unique) decomposition on pairwise separated components*

$$K = K_1 \cup \dots \cup K_n \cup K_{n+1} \cup \dots \cup K_m \quad (3.3)$$

such that

$$K_1, \dots, K_n \text{ are fat}$$

in decreasing fatness, that is,

$$\lambda_0(K_1) \leq \dots \leq \lambda_0(K_n),$$

and

$$K_{n+1}, \dots, K_m \text{ are slim.}$$

Then $\lambda_0(K) = \min\{\lambda_0(K_1), \dots, \lambda_0(K_n)\} = \lambda_0(K_1)$.

Therefore, K can be split in fat and slim parts

$$K^{fat} := K_1 \cup \dots \cup K_n \quad \text{and} \quad K^{slim} := K_{n+1} \cup \dots \cup K_m$$

and $\lambda_0(K) = \lambda_0(K^{fat})$.

Now we turn our attention to fat sets.

Proposition 3.7.

- i) Any compact superset of a fat set is fat. Any compact subset of a slim set is slim.
- ii) If K contains a ball, then it is fat.
- iii) If the Lebesgue measure of K is $|K| = 0$ then K is slim.
- iv) For $N > 1$, there exist fat sets of empty interior and with arbitrary positive measure.

Proof. i) This is obvious from part ii) of [Proposition 3.4](#).

ii) If $K \supset B(x_0, R)$ and Ω_ε is a decreasing family for K , then for any ε small, $\lambda_1(\Omega_\varepsilon) \leq \lambda_1(B(x_0, R))$ and then $\lambda_0(K)$ is finite.

iii) If $|K| = 0$ and Ω_ε is a decreasing family for K then $|\Omega_\varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then, the Rayleigh–Faber–Krahn inequality, see for instance [\[17\]](#), implies that

$$\lambda_1(\Omega_\varepsilon) \geq \lambda_1(B_{R_\varepsilon})$$

where B_{R_ε} is a ball with the same measure as Ω_ε , that is, $|\Omega_\varepsilon| = C(N)R_\varepsilon^N \rightarrow 0$, as $\varepsilon \rightarrow 0$. Thus, $\lambda_1(B_{R_\varepsilon}) = R_\varepsilon^{-2}\lambda_1(B_1) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

iv) The basic technique to construct a fat set with empty interior is by removing an infinite number of balls of different radius from a bounded smooth domain. The construction follows closely the one provided, for different purposes, in [1, Proposition 6.1] and it is related to the well-known example by Cioranescu and Murat, [8]. See also [6].

Let us consider the ball $\Omega_0 = B(0, R) \subset \mathbb{R}^N$ and let us denote by Q the set of all points in Ω_0 with all coordinate rational numbers and denote by $\{q_n\}_{n=1}^\infty$ an ordering of Q . Obviously Q is dense in Ω .

In the construction we will use the fact that for any bounded open set U and $q \in U$, we have $\lambda_1(U \setminus \bar{B}(q, \varepsilon)) \rightarrow \lambda_1(U)$ as $\varepsilon \rightarrow 0$, see [9].

Let us fix a small positive number η . We start the construction defining $p_1 = q_1 \in \Omega_0$. Since $\lambda_1(\Omega_0 \setminus \bar{B}(p_1, \varepsilon)) \rightarrow \lambda_1(\Omega_0)$ as $\varepsilon \rightarrow 0$ we can choose $\varepsilon_1 > 0$ such that $B(p_1, 2\varepsilon_1) \subset \Omega_0$, $|B(p_1, \varepsilon_1)| \leq \eta/2$ and such that if $\Omega_1 = \Omega_0 \setminus \bar{B}(p_1, \varepsilon_1)$, then $\lambda_1(\Omega_1) \leq \lambda_1(\Omega_0) + \frac{1}{2}$. Observe that $\lambda_1(\Omega_0) \leq \lambda_1(\Omega_1)$ and Ω_1 is a domain with C^∞ -smooth boundary.

Let us denote now $p_2 = q_{n_2}$ with the property that $q_2, q_3, \dots, q_{n_2-1} \in \bar{B}(p_1, \varepsilon_1)$ that is, p_2 is the first q_j after q_1 such that $q_j \notin B(p_1, \varepsilon_1)$. Again, since $\lambda_1(\Omega_1 \setminus \bar{B}(p_2, \varepsilon)) \rightarrow \lambda_1(\Omega_1)$ as $\varepsilon \rightarrow 0$, we may choose $0 < \varepsilon_2$ small enough so that $|B(p_2, \varepsilon_2)| \leq \eta/2^2$ and $B(p_2, 2\varepsilon_2) \subset \Omega_1$ and if we denote by $\Omega_2 = \Omega_1 \setminus \bar{B}(p_2, \varepsilon_2)$, then $\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1) + \frac{1}{2^2}$. Observe that $\lambda_1(\Omega_1) \leq \lambda_1(\Omega_2)$ and Ω_2 is a domain with C^∞ -smooth boundary.

We may proceed now by induction to construct a sequence of points $\{p_m\}_m \in Q$, a sequence of numbers $\{\varepsilon_m\}_m$ with $|B(p_j, \varepsilon_j)| \leq \eta/2^j$ and a nested sequence of domains $\Omega_m = \Omega_{m-1} \setminus \bar{B}(p_m, \varepsilon_m) = \Omega_0 \setminus \bigcup_{i=1}^m \bar{B}(p_i, \varepsilon_i)$, which are C^∞ domains such that $\lambda_1(\Omega_{m-1}) \leq \lambda_1(\Omega_m) \leq \lambda_1(\Omega_0) + \sum_{i=1}^m \frac{1}{2^i} \leq \lambda_1(\Omega_0) + 1$. Notice that all balls $B(p_i, \varepsilon_i)$ are pairwise disjoint and

$$\forall \varepsilon > 0, \forall m \in \mathbb{N}, \forall k \geq m \text{ we have } B(p_m, \varepsilon) \cap \Omega_k^c \neq \emptyset. \quad (3.4)$$

Define

$$K_0 = \bigcap_{i=0}^\infty \bar{\Omega}_i = \bar{\Omega}_0 \setminus \bigcup_{i=1}^\infty B(p_i, \varepsilon_i).$$

Then, K_0 is a compact set. It is nonempty since the Lebesgue measure of

$$\left| \bigcup_{i=1}^\infty B(p_i, \varepsilon_i) \right| \leq \sum_{i=1}^\infty \eta/2^i = \eta$$

and therefore $|K_0| \geq |\Omega_0| - \eta > 0$, for η small enough.

But the interior of K_0 is empty. In other case, there will exist a point $\xi \in K_0$ and $R > 0$ small such that $B(\xi, R) \subset K_0$ and therefore, by construction, $B(\xi, R) \cap \bigcup_{i=1}^\infty B(p_i, \varepsilon_i) = \emptyset$. But, by density, there will exist $q_k \in B(\xi, R) \subset K_0$ and therefore $R' > 0$ such that $B(q_k, R') \subset B(\xi, R)$ and therefore $B(q_k, R') \cap \bigcup_{i=1}^\infty B(p_i, \varepsilon_i) = \emptyset$. This is in contradiction with (3.4).

Finally, if U is an open set containing $K_0 = \bigcap_{i=1}^\infty \bar{\Omega}_i$ we have that there exists i_0 such that $\bar{\Omega}_{i_0} \subset U$. This implies that $\lambda_1(U) \leq \lambda_1(\Omega_{i_0}) \leq \lambda_1(\Omega_0) + 1$. Since this estimate is obtained for all open $U \supset K_0$, we have $\lambda_0(K_0) \leq \lambda_1(\Omega_0) + 1$. Hence K_0 is fat. \square

Now we obtain further properties of fat sets by analyzing the eigenfunctions of the approximating family Ω_ε .

For a given compact set $K \subset \mathbb{R}^N$, we define the following subspace of $H^1(\mathbb{R}^N)$,

$$H_0^1(K) := \{\xi \in H^1(\mathbb{R}^N), \quad \xi(x) = 0 \quad \text{a.e. } x \in \mathbb{R}^N \setminus K\}. \quad (3.5)$$

Observe that we always have $0 \in H_0^1(K)$ but it is possible that this is the only function in $H_0^1(K)$, that is $H_0^1(K) = \{0\}$. This is the case, for example, when K has zero measure.

With this definition, we can prove:

Proposition 3.8. *A compact set $K \subset \mathbb{R}^N$ is fat if and only if the space $H_0^1(K)$ is different from $\{0\}$.*

Moreover, in this case,

$$\lambda_0(K) = \min \left\{ \frac{\int_{\mathbb{R}^N} |\nabla \xi|^2}{\int_{\mathbb{R}^N} |\xi|^2}, \quad \xi \in H_0^1(K), \quad \xi \neq 0 \right\}. \quad (3.6)$$

Also, the minimum is attained at least in a nonnegative function. Moreover, if ϕ is a minimizer, so is $|\phi|$.

Finally if ϕ is a minimizer, then ϕ satisfies

$$\int_{\mathbb{R}^N} \nabla \phi \nabla \xi = \lambda_0(K) \int_{\mathbb{R}^N} \phi \xi, \quad \forall \xi \in H_0^1(K). \quad (3.7)$$

Remark 3.9. Observe in particular that a compact set $K \subset \mathbb{R}^N$ is slim if and only if $H_0^1(K) = \{0\}$.

Remark 3.10. Notice that (3.6) is formally the Raleigh quotient for the Laplacian in the compact set K , which may have empty interior, see Proposition 3.7. Related to this, the Laplacian in non-smooth sets, even fractals, are studied for example in [33,7].

Proof. If there exists $0 \neq \phi \in H_0^1(K)$, we can assume $\int_{\mathbb{R}^N} |\phi|^2 = 1$ and then considering a decreasing family of domains Ω_ε for K we have that $\phi \in H_0^1(\Omega_\varepsilon)$ for all $\varepsilon > 0$. Therefore,

$$\lambda_1(\Omega_\varepsilon) \leq \int_{\Omega_\varepsilon} |\nabla \phi|^2 = \int_K |\nabla \phi|^2,$$

which implies the boundedness of the sequence $\lambda_1(\Omega_\varepsilon)$ as $\varepsilon \rightarrow 0$ and therefore K is fat. Moreover, we get

$$\lambda_0(K) \leq \int_K |\nabla \phi|^2, \quad \forall \phi \in H_0^1(K) \text{ with } \|\phi\|_{L^2(\mathbb{R}^N)}^2 = 1 \quad (3.8)$$

Conversely, assume now K is fat. Consider a family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of strictly decreasing bounded domains, so that $K = \bigcap_{\varepsilon>0} \bar{\Omega}_\varepsilon \subset \Omega_1$ and denote by ϕ_ε a normalized first eigenfunction of $-\Delta$ with Dirichlet boundary conditions in Ω_ε . We extend ϕ_ε to Ω_1 by zero.

Clearly $\lambda_1(\Omega_\varepsilon) = \int_{\Omega} |\nabla \phi_\varepsilon|^2 \leq \lambda_0(K)$. Then, taking a subsequence if necessary, we can assume that ϕ_ε converges weakly in $H_0^1(\Omega_1)$ and strongly in $L^2(\Omega_1)$ to $0 \leq \phi \in H_0^1(\Omega_1)$ satisfying $\|\phi\|_{L^2(\Omega_1)} = 1$ and $\phi(x) = 0$ a.e. $x \in \mathbb{R}^N \setminus K$. Also, by weak lower semicontinuity

$$\int_K |\nabla \phi|^2 = \int_{\Omega_1} |\nabla \phi|^2 \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_1} |\nabla \phi_\varepsilon|^2 = \lim_{\varepsilon \rightarrow 0} \lambda_1(\Omega_\varepsilon) = \lambda_0(K).$$

On the other hand, from (3.8) we get $\lambda_0(K) \leq \int_K |\nabla \phi|^2$. Therefore we have equal sign and moreover ϕ_ε converges strongly in $H_0^1(\Omega)$ to ϕ .

Now if ϕ is a minimizer, note that if $\xi \in H_0^1(K)$, $t \in \mathbb{R}$, and $\varphi = \phi + t\xi \in H_0^1(K)$, then from (3.6) we have $\lambda_0(K) \int_{\mathbb{R}^N} |\varphi|^2 \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2$ which translates into

$$\left(1 + 2t \int_{\mathbb{R}^N} \phi \xi + t^2 \int_{\mathbb{R}^N} |\xi|^2 \right) \lambda_0(K) \leq \lambda_0(K) + 2t \int_{\mathbb{R}^N} \nabla \phi \nabla \xi + t^2 \int_{\mathbb{R}^N} |\nabla \xi|^2.$$

Hence, for all $t \in \mathbb{R}$

$$0 \leq t^2 \left(\int_{\mathbb{R}^N} |\nabla \xi|^2 - \lambda_0(K) \int_{\mathbb{R}^N} |\xi|^2 \right) + 2t \left(\int_{\mathbb{R}^N} \nabla \phi \nabla \xi - \lambda_0(K) \int_{\mathbb{R}^N} \phi \xi \right)$$

which implies (3.7). The rest of the proof follows now easily. \square

Next we prove the following results. Note that all the results below say that we can shave off some parts of a compact set K without changing the value of $\lambda_0(K)$.

Proposition 3.11.

- i) Assume $\tilde{K} \subset K$ such that $|\tilde{K}| = 0$ and $K \setminus \tilde{K}$ is compact. Then $H_0^1(K \setminus \tilde{K}) = H_0^1(K)$ and therefore,

$$\lambda_0(K \setminus \tilde{K}) = \lambda_0(K).$$

- ii) Assume $\tilde{K} \subset K$ is such that there exists $\phi \neq 0$, a minimizer of (3.6), such that $\phi(x) = 0$ for a.e. $x \in \tilde{K}$ and $K \setminus \tilde{K}$ is compact. Then

$$\lambda_0(K \setminus \tilde{K}) = \lambda_0(K).$$

- iii) Assume $K = K_1 \cup K_2$ are compact sets with $K_1 \cap K_2 \neq \emptyset$ and denote $\tilde{K}_2 = \overline{K_2 \setminus K_1}$. Hence $K = K_1 \cup \tilde{K}_2$.

Then, if K is fat and \tilde{K}_2 is slim, we have $H_0^1(K_1) = H_0^1(K)$ and therefore,

$$\lambda_0(K_1) = \lambda_0(K).$$

Proof. i) Since $K \setminus \tilde{K} \subset K$, from the definition of $H_0^1(K)$, see (3.5), we have that $H_0^1(K \setminus \tilde{K}) \subset H_0^1(K)$. Moreover, if $\psi \in H_0^1(K)$ we have $\psi \in H^1(\mathbb{R}^N)$ and $\psi = 0$ a.e. $x \in \mathbb{R}^N \setminus K$. But since $|\tilde{K}| = 0$, then $\psi = 0$ a.e. $x \in (\mathbb{R}^N \setminus K) \cup \tilde{K} = \mathbb{R}^N \setminus (K \setminus \tilde{K})$. This implies that $\psi \in H_0^1(K \setminus \tilde{K})$ and therefore $H_0^1(K \setminus \tilde{K}) = H_0^1(K)$. The characterization of $\lambda_0(K)$ given by (3.6) implies now that $\lambda_0(K \setminus \tilde{K}) = \lambda_0(K)$.

ii) Now Proposition 3.8 applied to $K \setminus \tilde{K}$ with the function ϕ gives

$$\lambda_0(K \setminus \tilde{K}) \leq \int_{K \setminus \tilde{K}} |\nabla \phi|^2 \leq \int_K |\nabla \phi|^2 = \lambda_0(K).$$

The reverse inequality follows from part ii) in Proposition 3.4.

iii) We may assume $|\tilde{K}_2| > 0$ since otherwise part i) concludes. Also, observe that since $K_1 \subset K$, we have $H_0^1(K_1) \subset H_0^1(K)$. For the reverse inclusion consider a fixed function $\psi \in H_0^1(K)$. We will show that $\psi \equiv 0$ a.e. $x \in \tilde{K}_2 \setminus K_1$, which will imply that $\psi \in H_0^1(K_1)$.

Then, if we consider a decreasing family of bounded sets Ω_δ associated to K , we have $\psi \in H_0^1(\Omega_\delta)$. Also, we denote by

$$\Omega_\delta^2 = \{x \in \Omega_\delta, \text{dist}(x, \tilde{K}_2) < \delta\},$$

which is a decreasing family for \tilde{K}_2 .

Let $\xi_\delta \in H_0^1(\Omega_\delta^2)$ be the unique weak solution of

$$-\Delta \xi_\delta = \psi \quad \text{in } \Omega_\delta^2. \quad (3.9)$$

Then

$$\lambda_1(\Omega_\delta^2) \int_{\Omega_\delta^2} |\xi_\delta|^2 \leq \int_{\Omega_\delta^2} |\nabla \xi_\delta|^2 \leq \|\xi_\delta\|_{L^2(\Omega_\delta^2)} \|\psi\|_{L^2(\mathbb{R}^N)} \leq C \|\xi_\delta\|_{L^2(\Omega_\delta^2)}$$

Since \tilde{K}_2 is slim we have $\lambda_1(\Omega_\delta^2) \rightarrow \infty$. From the inequality above we get that $\int_{\Omega_\delta^2} |\xi_\delta|^2 \rightarrow 0$ and also $\int_{\Omega_\delta^2} |\nabla \xi_\delta|^2 \rightarrow 0$, as $\delta \rightarrow 0$.

Let us consider now a test function $\chi \in C_0^\infty(\mathbb{R}^N \setminus K_1)$, so that the function $\chi \psi \in H_0^1(\Omega_\delta^2)$ can be used as a test function in (3.9). Hence,

$$\begin{aligned} \left| \int_{\Omega_\delta^2} |\psi|^2 \chi \right| &= \left| \int_{\Omega_\delta^2} \nabla \xi_\delta \nabla (\chi \psi) \right| \\ &\leq \left(\int_{\Omega_\delta^2} |\nabla \xi_\delta|^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} |\nabla (\chi \psi)|^2 \right)^{1/2} \rightarrow 0, \quad \text{as } \delta \rightarrow 0. \end{aligned}$$

This implies that

$$\int_{\mathbb{R}^N} |\psi|^2 \chi = 0, \quad \text{for any } \chi \in C_0^\infty(\mathbb{R}^N \setminus K_1).$$

Hence, $\psi \equiv 0$ a.e. $x \in \mathbb{R}^N \setminus K_1$ and therefore $\psi \in H_0^1(K_1)$. This shows that $H_0^1(K_1) = H_0^1(K)$, which in particular implies that $\lambda_0(K_1) = \lambda_0(K)$. \square

Remark 3.12. In ii), we do not get $H_0^1(K) = H_0^1(K \setminus \tilde{K})$. As a matter of fact if for instance $p_0, p_1 \in \mathbb{R}^n$ with $|p_1 - p_0| = \tau > 0$ and if K is the union of the two disjoint closed balls $\bar{B}(p_0, \tau/2)$ and $\bar{B}(p_1, \tau/4)$, and we choose $\tilde{K} = \bar{B}(p_1, \tau/4)$, we easily get that $\lambda_0(K) = \lambda_0(K \setminus \tilde{K})$ but $H_0^1(K \setminus \tilde{K})$ is a proper subset of $H_0^1(K)$. Actually, any function in $H_0^1(K)$ which is not identically 0 in $B(p_1, \tau/4)$ lies in $H_0^1(K)$ but not in $H_0^1(K \setminus \tilde{K})$.

We conclude this section with the following. If $K \subset \mathbb{R}^N$ has zero Lebesgue measure, $|K| = 0$, then any decreasing family satisfies $|\Omega_\delta| \rightarrow 0$ as $\delta \rightarrow 0$. But the rate at which it goes to zero reflects some geometric-measure properties of K . In fact according to [11], Proposition 3.2, Chapter 3, if K has fractal dimension

$$0 \leq d = d_F(K) < N$$

and if we take

$$\Omega_\delta = \{x : \text{dist}(x, K) < \delta\}$$

then $|\Omega_\delta| \leq C\delta^{N-d} \rightarrow 0$, as $\delta \rightarrow 0$ for some constant $C > 0$.

Note that d above is denoted fractal dimension in [32, Chapter 5, Section 3, page 279], but box-counting dimension, Kolmogorov entropy, entropy or information dimension and even Minkowski dimension in [11, Chapter 3]. Also, recall that if K is locally a C^1 smooth m -dimensional manifold in \mathbb{R}^N , then $d_F(K) = m$, see e.g. [30, page 48].

The following result will be used further below and gives a criterion to check whether a function that is infinity on a compact set of measure zero is integrable. As shown below, this criterion depends on the fractal dimension of the set and the way the function diverges on the compact set.

Lemma 3.13. Assume $K \subset \mathbb{R}^N$ is a compact set with zero Lebesgue measure and fractal dimension $d_F(K) \leq d \leq N$ and consider a function defined on a bounded neighborhood Ω of K of the form

$$f(x) = f^*(\text{dist}(x, K)),$$

where $f^* : (0, \infty) \rightarrow (0, \infty)$ is continuous, strictly decreasing with $f^*(s) \xrightarrow{s \rightarrow 0^+} +\infty$.

Then $f \in L^r(\Omega)$ with $r \geq 1$ provided

$$\int_1^\infty \left[(f^*)^{-1}(y) \right]^{N-d} y^{r-1} dy < \infty.$$

In particular, if

$$f^*(s) = s^{-\alpha} \quad \text{for } 0 < s \leq s_0,$$

then the condition above reads $r\alpha < N - d$.

Proof. Note that since $f \geq 0$

$$\int_{\Omega} |f|^r(x) dx = \int_0^{\infty} |A_s| ds = \int_0^1 |A_s| ds + \int_1^{\infty} |A_s| ds$$

where $A_s = \{x \in \Omega, f^r(x) \geq s\}$. The first integral is bounded since $A_s \subset \Omega$ and Ω is a bounded set. For the second integral, note that $|f|^r(x) \geq s$ iff $\text{dist}(x, K) \leq (f^*)^{-1}(s^{\frac{1}{r}})$. Therefore $|A_s| = |\Omega_{\delta}|$ where

$$\Omega_{\delta} = \{x \in \Omega, \text{dist}(x, K) \leq \delta\} \quad \text{and } \delta = (f^*)^{-1}(s^{\frac{1}{r}}).$$

From the assumption on the fractal dimension of K we get $|\Omega_{\delta}| \leq C\delta^{N-d}$. Therefore,

$$\int_1^{\infty} |A_s| ds \leq \int_1^{\infty} \left[(f^*)^{-1} \left(s^{\frac{1}{r}} \right) \right]^{N-d} ds.$$

The change of variables $y = s^{\frac{1}{r}}$ for $s \geq 1$ gives the result.

If, $f^*(s) = s^{-\alpha}$ for $s \approx 0$ then $(f^*)^{-1}(y) = y^{-\alpha}$ for $y \approx \infty$ and the result follows. \square

4. Behavior of the positive equilibria

Let us consider the *equilibria* or *stationary solutions* of (1.1), i.e., nonnegative solutions of the associated elliptic problem

$$\begin{cases} -\Delta \varphi_{\lambda} = \lambda \varphi_{\lambda} - n(x) \varphi_{\lambda}^{\rho}, & \text{in } \Omega \\ \varphi_{\lambda} = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

see (2.2), where n satisfies (1.3).

Then we have the following.

Lemma 4.1. Assume that $\tilde{\varphi} \in L^1(\Omega)$ is a positive stationary solution of (1.1) for $\lambda = \tilde{\lambda}$. Then, for each λ in a neighborhood of $\tilde{\lambda}$ there exists a unique positive stationary solution of (1.1), φ_{λ} which is moreover a smooth function of λ .

Proof. Define the set $W_+^{2,q} = \{u \in W^{2,q}(\Omega) : u > 0, \text{ in } \Omega\}$ and consider $W_+^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ endowed with the topology of $W^{2,q}(\Omega)$. If $q > N$, we have that $W_+^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \hookrightarrow C^{1,\theta}(\Omega)$ for some $0 < \theta < 1$. Moreover, from Hopf lemma, we know that if $\tilde{\varphi}$ is a positive solution of (4.1) then $\tilde{\varphi}$ lies in the interior of $W_+^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$.

We consider the map

$$F : (\lambda, u) \rightarrow -\Delta u - \lambda u + n(x)u^\rho$$

from $\mathbb{R} \times W_+^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \rightarrow L^q(\Omega)$ for some $q > N$.

Then F is a continuously differentiable map, and we apply the implicit function theorem for $(\lambda, u) = (\tilde{\lambda}, \tilde{\varphi})$. By hypothesis $\tilde{\varphi}$ is a nonnegative stationary solution of (1.1), then $F(\tilde{\lambda}, \tilde{\varphi}) = 0$, and

$$\tilde{\lambda} = \mu_1(-\Delta + n(x)\tilde{\varphi}^{\rho-1}), \quad (4.2)$$

that is, $\tilde{\lambda}$ is the first eigenvalue of the operator $-\Delta + n(x)\tilde{\varphi}^{\rho-1}$ in Ω , with Dirichlet boundary conditions.

Moreover, the derivative with respect to u at $(\lambda, u) = (\tilde{\lambda}, \tilde{\varphi})$ is

$$D_u F(\tilde{\lambda}, \tilde{\varphi}) = -\Delta - \tilde{\lambda} + \rho n(x)\tilde{\varphi}^{\rho-1}.$$

Therefore, due to $\rho > 1$ and the monotonicity of the first eigenvalue

$$\mu_1(-\Delta - \tilde{\lambda} + \rho n(x)\tilde{\varphi}^{\rho-1}) > \mu_1(-\Delta - \tilde{\lambda} + n(x)\tilde{\varphi}^{\rho-1}) = 0, \quad (4.3)$$

or in other words, the derivative $D_u F(\tilde{\lambda}, \tilde{\varphi})$ is an isomorphism.

So, there exists a neighborhood of $\tilde{\lambda}$, such that for each λ in this neighborhood there is a unique solution $\varphi_\lambda \in W_+^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ of (1.1) lying in a neighborhood of $\tilde{\varphi}$. Therefore $\varphi_\lambda > 0$. Moreover, the map $\lambda \rightarrow \varphi_\lambda$ is continuously differentiable with $\varphi_{\tilde{\lambda}} = \tilde{\varphi}$. \square

The following result describes the existence and behavior of equilibria of (1.1). In particular it gives the converse of Corollary 2.3.

Theorem 4.2. Assume n satisfies (1.3).

i) Problem (1.1) has a positive stationary $L^1(\Omega)$ solution if and only if

$$\lambda_1(\Omega) < \lambda < \lambda_0(K_0).$$

ii) For $\lambda_1(\Omega) < \lambda < \lambda_0(K_0)$ the positive equilibria φ_λ is a smooth and increasing function of λ . Even more, as $\lambda \rightarrow \lambda_0(K_0)$, we have

$$\|\varphi_\lambda\|_{L^1(\Omega)} \rightarrow \infty. \quad (4.4)$$

iii) For each $\lambda_1(\Omega) < \lambda < \lambda_0(K_0)$, φ_λ is globally asymptotically stable for nonnegative nontrivial solutions of (1.1).

Proof. i) The “if” part comes from Corollary 2.3.

For the “only if” part, assume problem (1.1) has a positive stationary $L^1(\Omega)$ solution, for some value of λ . Then

$$\lambda = \mu_1(-\Delta + n(x)\varphi^{\rho-1}, \Omega), \quad (4.5)$$

that is, λ is the first eigenvalue of the operator $-\Delta + n(x)\varphi^{\rho-1}$ in Ω , with Dirichlet boundary conditions. Monotonicity with respect to the potential implies that $\lambda > \mu_1(-\Delta, \Omega) =: \lambda_1(\Omega)$. Equality (4.5) and the monotonicity with respect to the domain of this eigenvalue gives

$$\lambda < \mu_1(-\Delta + n(x)\varphi^{\rho-1}, \Omega_\delta).$$

Also note that, as $\delta \rightarrow 0$

$$\mu_1(-\Delta + n(x)\varphi^{\rho-1}, \Omega_\delta) \geq \mu_1(-\Delta, \Omega_\delta) = \lambda_1(\Omega_\delta) \rightarrow \lambda_0(K_0).$$

On the other hand, since n and φ are bounded, we have

$$\mu_1(-\Delta + n(x)\varphi^{\rho-1}, \Omega_\delta) \leq \mu_1(-\Delta + C\mathcal{X}_{\Omega_\delta \setminus K_0}, \Omega_\delta)$$

for some $C > 0$, where \mathcal{X} denotes the characteristic function. Take now the first normalized eigenfunction of $\lambda_1(\Omega_\delta)$ as a test function to get

$$\mu_1(-\Delta + C\mathcal{X}_{\Omega_\delta \setminus K_0}, \Omega_\delta) \leq \int_{\Omega_\delta} |\nabla \phi_\delta|^2 + C \int_{\Omega_\delta \setminus K_0} |\phi_\delta|^2.$$

Then note that the first term in the right hand side above converges as $\delta \rightarrow 0$ to $\lambda_0(K_0)$ while for the second we can assume, taking subsequences if necessary as in the proof of Proposition 3.8, that ϕ_δ converges to ϕ , weakly in $H_0^1(B)$ and strongly in $L^2(B)$, where B is a large ball such that $\Omega_\delta \subset B$. Then we have as $\delta \rightarrow 0$

$$\int_{\Omega_\delta \setminus K_0} |\phi_\delta|^2 = \int_{B \setminus K_0} |\phi_\delta|^2 \rightarrow \int_{B \setminus K_0} |\phi|^2 = 0$$

since $\phi(x) = 0$ a.e. $x \in \mathbb{R}^N \setminus K_0$. Hence the result is proved and $\lambda \leq \lambda_0(K_0)$.

However if $\lambda = \lambda_0(K_0)$ then by Lemma 4.1 there would be also stationary solutions of (1.1) for some $\tilde{\lambda} > \lambda_0(K_0)$, which contradicts the argument above.

ii) Note that for $\lambda_1(\Omega) < \lambda < \lambda_0(K_0)$, Corollary 2.3 gives the uniqueness of equilibria and from Lemma 4.1 the positive equilibria φ_λ is a smooth function of λ . Therefore we can compute the derivative $\frac{d\varphi_\lambda}{d\lambda}$ at $\lambda = \tilde{\lambda}$, and it is a continuous function of λ . Let $v := \frac{d\varphi_\lambda}{d\lambda}$, taking derivatives with respect to λ in (4.1) we obtain

$$\begin{cases} -\Delta v - \lambda v + \rho n(x)\varphi^{\rho-1}v = \varphi \geq 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.6)$$

Due to (4.2)–(4.3) and to the maximum principle, we deduce from (4.6) that

$$v := \frac{d\varphi_\lambda}{d\lambda} > 0,$$

and φ_λ is an increasing function of λ .

Therefore, there exists the monotonic limit

$$\varphi^*(x) = \lim_{\lambda \rightarrow \lambda_0(K_0)} \varphi_\lambda(x) \quad x \in \Omega. \quad (4.7)$$

We next prove (4.4). Assume on the contrary that $\varphi^* \in L^1(\Omega)$. Let us define

$$\begin{cases} -\Delta w^* = \lambda_0(K_0)\varphi^* & \text{in } \Omega \\ w^* = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.8)$$

then, by elliptic regularity, $w^* \in W_0^{1,p}(\Omega)$, for $1 \leq p < \frac{N}{N-1}$, see [23,31,5]. From Sobolev embeddings, $w^* \in L^q(\Omega)$, for $1 \leq q < \frac{N}{N-2}$. Moreover, $w^* \geq \varphi_\lambda$, for any $\lambda \leq \lambda_0(K_0)$, indeed

$$-\Delta(w^* - \varphi_\lambda) = \lambda_0(K_0)\varphi^* - \lambda\varphi_\lambda + n(x)(\varphi_\lambda)^\rho \geq 0, \quad \text{in } \Omega; \quad w^* - \varphi_\lambda = 0 \quad \text{on } \partial\Omega.$$

Therefore $\varphi_\lambda \leq w^*$ in Ω , and by Fatou's Lemma $\varphi^* \in L^q(\Omega)$ for $1 \leq q < \frac{N}{N-2}$.

Now we use a bootstrap argument combined with a comparison argument of the following type: elliptic regularity for w^* implies that $w^* \in W^{2,q}(\Omega)$; from Sobolev's embedding Theorems we obtain that $w^* \in L^{q^{**}}(\Omega)$ where $\frac{1}{q^{**}} = \frac{1}{q} - \frac{2}{N}$ if $q < N/2$, for any $q^{**} \in (1, \infty)$ if $q = N/2$, and $q^{**} = +\infty$ if $q > N/2$; now, the fact that $w^* \geq \varphi_\lambda$, and Fatou's Lemma gives that $\varphi^* \in L^{q^{**}}(\Omega)$. Bootstrapping, we can conclude that $\varphi^* \in L^\infty(\Omega)$.

Now again elliptic regularity implies that $\varphi_\lambda \in W^{2,p}(\Omega)$, for any $1 < p < \infty$, moreover $\|\varphi_\lambda\|_{W^{2,p}(\Omega)}$ is uniformly bounded. Therefore since $W^{2,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ is compact for $p > N/2$, we obtain that $\varphi_\lambda \rightarrow \varphi^*$ in $C(\bar{\Omega})$. Consequently φ^* is a strong solution of

$$\begin{cases} -\Delta\varphi^* = \lambda_0(K_0)\varphi^* - n(x)(\varphi^*)^\rho & \text{in } \Omega \\ \varphi^* = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.9)$$

But this contradicts part i).

iii) This follows from Corollary 2.3. \square

Remark 4.3. Theorem 4.2 was already known in the particular case when $n(x)$ is a smooth function, $K_0 = \overline{\Omega_0}$ where Ω_0 is an open set with regular boundary, see [25,13,12].

5. Behavior of the solutions of the evolution problem for $\lambda > \lambda_0(K_0)$

Observe that for $\lambda > \lambda_1(\Omega)$ the trivial solution of (1.1) becomes linearly unstable. Therefore if $u_0 \geq 0$ and not identically zero and $\lambda \geq \lambda_0(K_0)$ then the solution of (1.1) $u(t; u_0)$ is globally defined but cannot be bounded in Ω . Indeed if it were using compactness of the orbit we would obtain the existence of a bounded positive stationary solution of (1.1), which is absurd by Theorem 4.2 i). Therefore, for $\lambda \geq \lambda_0(K_0)$ we have, for any nonzero $u_0 \geq 0$

$$\limsup_{t \rightarrow \infty} \|u(t; u_0)\|_{L^1(\Omega)} = \infty.$$

Therefore the natural questions are where and how solutions become unbounded.

Remark 5.1. In case K_0 is a smooth compact set, in the sense that $K_0 = \overline{\Omega_0}$ where Ω_0 is a smooth open set, this problem has been studied in [14,12,20] and further developments in [16, 21,22], where the concept of metasolution is introduced as the limiting profile of the population as $t \rightarrow \infty$.

We first show that all solutions of (1.1) become unbounded at the same points and at the same rate. For this we use monotonicity properties of solutions. In fact we have the following.

Lemma 5.2. Assume $\lambda \in \mathbb{R}$, $u_0 \geq 0$ and non-zero and $\alpha > 0$. Then:

i) If $\alpha > 1$

$$\alpha u(t, x; u_0) \geq u(t, x; \alpha u_0).$$

ii) If $\alpha < 1$

$$\alpha u(t, x; u_0) \leq u(t, x; \alpha u_0).$$

Proof. Note that $v(t) = \alpha u(t; u_0)$ satisfies $v(0) = \alpha u_0$

$$v_t - \Delta v = \lambda v - \frac{n(x)}{\alpha^{\rho-1}} v^{\rho}.$$

Hence, if $\alpha > 1$, v is a supersolution of (1.1) with initial data αu_0 , and a subsolution if $\alpha < 1$. \square

In particular we get:

Corollary 5.3. For fixed $\lambda \geq \lambda_0(K_0)$, all solutions of (1.1) become unbounded at the same rate and at the same points, independently of the initial conditions.

Proof. Fix an initial condition $u_0 \in C^1(\overline{\Omega})$ with $u_0 > 0$ in Ω , $u_0 = 0$ on $\partial\Omega$ and with strictly negative normal derivative at the boundary of Ω . Then if $v_0 \geq 0$ is another initial condition, not identically zero, we show first that there exist $\varepsilon > 0$, small, and numbers $0 < \alpha < 1 < \beta$ such that

$$\alpha u_0 \leq u(\varepsilon; v_0) \leq \beta u_0, \quad \text{a.e. in } \Omega. \quad (5.1)$$

Indeed, for a given $\varepsilon > 0$, the function $u(\varepsilon; v_0)$ is smooth in Ω and from the strong maximum principle we get $u(\varepsilon; v_0) > 0$ in Ω . Moreover, since $u_t - \Delta u - \lambda u = -n(x)u^{\rho} \leq 0$, applying the parabolic Hopf maximum principle [27, Theorem 7, Chapter 3], we get $\frac{\partial}{\partial n} u(\varepsilon; v_0) < 0$ on $\partial\Omega$. From this, choosing $0 < \alpha < 1$ small enough and $\beta > 1$ large enough we get (5.1).

Observing now that $u(t, x, u(\varepsilon; v_0)) = u(t + \varepsilon, x, v_0)$ for all $t \geq 0$ and applying Lemma 5.2, we get

$$\alpha u(t, x; u_0) \leq u(t, x; \alpha u_0) \leq u(t + \varepsilon, x; v_0) \leq u(t, x; \beta u_0) \leq \beta u(t, x; u_0)$$

and we get the result. \square

Now, we have the following Lemma.

Lemma 5.4. Assume $K_0 \supset \Omega_0$ where Ω_0 is an open set. Then if $\lambda > \lambda_1(\Omega_0) \geq \lambda_0(K_0)$

$$\lim_{t \rightarrow \infty} u(x, t) = \infty, \quad \text{for all } x \in \Omega_0.$$

Proof. Let $z(x, t)$ be the solution of

$$\begin{cases} z_t - \Delta z = \lambda z & \text{in } \Omega_0 \\ z = 0 & \text{on } \partial\Omega_0 \\ z(0) = z_0 \geq 0 & \text{in } \Omega_0 \end{cases}$$

with $z_0 \leq u_0$. Then, by comparison and due to $n(x) \geq 0$ in K_0 , $z(x, t) \leq u(x, t)$ for $x \in \Omega_0$. Since $\lambda > \lambda_1(\Omega_0)$ then $z(x, t)$ grows exponentially in Ω_0 . \square

In particular if K_0 is regular, that is, the closure of the smooth open set Ω_0 then $\lambda_1(\Omega_0) = \lambda_0(K_0)$ and Lemma 5.4 shows that all solutions become unbounded in Ω_0 .

For the nonsmooth case, we proceed as follows. To get point-wise upper bounds on the solutions we will use the following, see [14], see also [19,24].

Lemma 5.5. Assume $\rho > 1$ and $\lambda, \beta > 0$ and consider a ball in \mathbb{R}^N of radius $a > 0$ and the following singular Dirichlet problem

$$\begin{cases} -\Delta z = \lambda z - \beta z^\rho & \text{in } B(0, a) \\ z = \infty & \text{on } \partial B(0, a). \end{cases}$$

Then

- i) There exists a unique positive radial solution, $z_a(x)$.
- ii) The solution satisfies

$$\left(\frac{\lambda}{\beta}\right)^{\frac{1}{\rho-1}} \leq z_a(0) = \inf_{B(0,a)} z_a(x) \leq \left(\frac{\lambda(\rho+1)}{2\beta} + \frac{B}{\beta a^2}\right)^{\frac{1}{\rho-1}}$$

for some constant $B > 0$.

Then we prove that all solutions of (1.1) remain bounded out of K_0 .

Proposition 5.6. Assume n satisfies (1.3).

Let $x_0 \in \Omega \setminus K_0$ and let $u_0 \geq 0$ be a bounded initial data for (1.1). Then for any given $\lambda \geq \lambda_0(K_0)$ there exists $b > 0$ and $M > 0$ such that

$$0 \leq u(t, x; u_0) \leq M, \quad x \in B(x_0, b) \subset \Omega \setminus K_0, \quad t \geq 0.$$

Proof. Let $x_0 \in \Omega \setminus K_0$ and let $a > 0$ be such that $B(x_0, a) \subset \Omega \setminus K_0$. Then by (1.3), denote $\beta = \inf\{n(x), x \in B(x_0, a)\} > 0$ and consider $z(x)$ the translation to $B(x_0, a)$ of the function in Lemma 5.5.

Given u_0 , for a sufficiently small we have that $u_0(x) \leq z(x_0) \leq z(x)$ for $x \in B(x_0, a)$. Hence $z(x)$ is a supersolution for $u(x, t)$ and then

$$u(x, t) \leq z(x), \quad x \in B(x_0, a), \quad t \geq 0.$$

Now in $B(x_0, a/2)$, $z(x)$ remains bounded and we conclude with $b = a/2$. \square

Now we prove that solutions of (1.1) become unbounded first on the fatter parts of K_0 . In fact the following result shows that, for all λ , all solutions of (1.1) remain bounded in the slim components of K_0 , while on the fat components of K_0 , as λ increases all solutions of (1.1) start to grow up in the fatter components of K_0 , progressively. Recall that by Definition 3.3 a component of K_0 is a compact set $K \subset K_0$ such that $K_0 \setminus K$ is also compact.

Theorem 5.7. *Let K be a component of K_0 . Then:*

- i) *Assume K is thin. Then for any $\lambda \in \mathbb{R}$ all solutions of (1.1) are bounded on K .*
- ii) *Assume K is fat and*

$$\lambda < \lambda_0(K)$$

then all solutions of (1.1) are bounded on K .

- iii) *Assume K is fat and*

$$\lambda_0(K) \leq \lambda$$

then, all solutions of (1.1) are unbounded in a neighborhood of K .

Proof. Given the component K , take a decreasing family Ω_δ for K . For δ small enough, from Proposition 5.6 we have that $u(x, t)$ remains bounded on $\partial\Omega_\delta$ by M . Then the solution of

$$\begin{cases} z_t - \Delta z = \lambda z & \text{in } \Omega_\delta \\ z = M & \text{on } \partial\Omega_\delta \\ z(0) = u_0 \geq 0 & \text{in } \Omega_\delta \end{cases}$$

is a supersolution for $u(x, t)$ on Ω_δ and then $u(x, t) \leq z(x, t)$.

If $\lambda < \lambda_0(K)$ then for δ small enough we have $\lambda < \lambda_1(\Omega_\delta)$ and then z is bounded on Ω_δ . This proves i) and ii).

To prove iii) assume now that K is fat and assume a solution of (1.1), $u(x, t)$, remains bounded in a neighborhood of K for all time.

Take a decreasing family Ω_δ for K and then

$$0 \leq n(x)u^{\rho-1} \leq C\mathcal{X}_{\Omega_\delta \setminus K}.$$

Then the solution of

$$\begin{cases} z_t - \Delta z + C\mathcal{X}_{\Omega_\delta \setminus K}z = \lambda z & \text{in } \Omega_\delta \\ z = 0 & \text{on } \partial\Omega_\delta \\ z(0) = z_0 \geq 0 \end{cases}$$

with $z_0 \leq u_0$ in Ω_δ is a subsolution for $u(x, t)$ on Ω_δ and then $u(x, t) \geq z(x, t)$.

If $\lambda > \lambda_0(K)$ then we choose $\delta > 0$ small such that $\lambda > \mu_1(-\Delta + C\mathcal{X}_{\Omega_\delta \setminus K}, \Omega_\delta) \rightarrow \lambda_0(K)$ and then z is unbounded on Ω_δ . This is in contradiction with the assumption of u being bounded.

It just remains to consider the case $\lambda = \lambda_0(K)$. In this case, from Lemma 5.2 it is enough to prove that some solution is unbounded in a neighborhood of K . For this consider a strictly positive, smooth initial data u_0 such that $-\Delta u_0 \leq \lambda u_0 - n(x)u_0^\rho$ in Ω . Such an initial data can be taken as a small multiple of the positive first eigenfunction of the Laplacian in Ω , i.e. $u_0 = \varepsilon \phi_1$, since

$$-\Delta u_0 = \varepsilon \lambda_1(\Omega) \phi_1 \leq \lambda u_0 - n(x)u_0^\rho = \varepsilon \lambda_0(K) \phi_1 - n(x)\varepsilon^\rho \phi_1^\rho$$

i.e. $n(x)\varepsilon^{\rho-1}\phi_1^{\rho-1} \leq \lambda_0(K) - \lambda_1(\Omega)$ which holds for sufficiently small $\varepsilon > 0$.

With this choice, we have that the solution $u(t, x; u_0)$ of (1.1) is monotone increasing in time; see e.g. Lemma 2.9 in [28].

Assume then $u(t, x; u_0)$ remains bounded in a neighborhood O of K . From (1.1) we have

$$\begin{cases} u_t - \Delta u = f(t, x) & \text{in } O, t > 0 \\ u = g(t, x) & \text{on } \partial O, t > 0 \end{cases}$$

with $f = \lambda u - n(x)u^\rho$ and g bounded in space and time. Then parabolic regularity implies uniform estimates on strong Sobolev norms in compact sets of O . Combining this with monotonicity in time, we have that $u(t, x; u_0) \rightarrow z(x)$ as $t \rightarrow \infty$ and z is smooth and strictly positive in a neighborhood of K . Take then in (1.1) a test function $\xi \in H_0^1(K)$, as defined in (3.5), and since n vanishes in K we get

$$\frac{d}{dt} \int_K u(t) \xi + \int_K \nabla u(t) \nabla \xi = \lambda_0(K) \int_K u(t) \xi.$$

Thus passing to the limit as $t \rightarrow \infty$ we get

$$\int_K \nabla z \nabla \xi = \lambda_0(K) \int_K z \xi.$$

Define now $\tilde{u} \in H_0^1(K)$ such that for any $\xi \in H_0^1(K)$ we have

$$\int_K \nabla \tilde{u} \nabla \xi = \lambda_0(K) \int_K z \xi. \quad (5.2)$$

We claim that $z - \tilde{u}$ is strictly positive in K . Once this is proved, take ϕ a nonnegative minimizer of (3.6) for which (3.7) holds. Hence, we have

$$\int_K \nabla \phi \nabla \tilde{u} = \lambda_0(K) \int_K \phi \tilde{u}$$

and using (5.2) with the test function $\xi = \phi$, we have

$$\int_K \nabla \tilde{u} \nabla \phi = \lambda_0(K) \int_K z \phi.$$

Subtracting these last two expressions, we get

$$\lambda_0(K) \int_K (z - \tilde{u}) \phi = 0$$

which is a contradiction with the claim that $z - \tilde{u}$ is a positive function.

To prove this claim take a decreasing family $\Omega_{\delta_n} \subset O$ for K and define $\tilde{u}_n \in H_0^1(\Omega_{\delta_n})$ such that

$$\begin{cases} -\Delta(z - \tilde{u}_n) = 0 & \text{in } \Omega_{\delta_n} \\ z - \tilde{u}_n = z > 0 & \text{on } \partial\Omega_{\delta_n}. \end{cases}$$

Then in Ω_{δ_n} we have $z - \tilde{u}_n \geq \min_{\partial\Omega_{\delta_n}} z \geq \min_{\overline{O}} z > 0$ which does not depend on n . The claim will be proved as soon as we show that $\tilde{u} = \lim_{n \rightarrow \infty} \tilde{u}_n$ pointwise in K . For this notice that for $f = -\Delta z$ we have, for any $\eta \in H_0^1(\Omega_{\delta_n})$

$$\int_{\Omega_{\delta_n}} \nabla \tilde{u}_n \nabla \eta = \int_{\Omega_{\delta_n}} f \eta = \int_{\Omega_{\delta_n}} \nabla z \nabla \eta.$$

Then, taking $\eta = \tilde{u}_n$, using the boundedness of f in $L^2(\Omega_{\delta_n})$ and that $\lambda_1(\Omega_{\delta_n}) \rightarrow \lambda_0(K)$, we get that $\{\tilde{u}_n\}_n$ (extended by zero) is bounded in $H_0^1(B)$, where $B \supset K$ is a suitable bounded set. We can assume then that \tilde{u}_n converges strongly in $L^2(B)$, weakly in $H_0^1(B)$ and a.e. B to a function w and $w = 0$ in $B \setminus K$. Taking now $\eta = \xi \in H_0^1(K)$ in the weak formulation above, we have

$$\int_K \nabla z \nabla \xi = \int_K f \xi = \int_K \nabla \tilde{u}_n \nabla \xi \rightarrow \int_K \nabla w \nabla \xi$$

and therefore $w = \tilde{u}$. Hence, the proof is complete. \square

In particular, we have the following. As in Corollary 3.6 we can assume

$$K_0 = K_1 \cup \dots \cup K_n \cup K_{n+1} \cup \dots \cup K_m$$

a decomposition in pairwise separated components, and such that K_{n+1}, \dots, K_m are slim and K_1, \dots, K_n are fat, labeled in decreasing fatness, that is,

$$\lambda_0(K_1) \leq \dots \leq \lambda_0(K_n).$$

Then $\lambda_0(K_0) = \min\{\lambda_0(K_1), \dots, \lambda_0(K_n)\} = \lambda_0(K_1)$.

Corollary 5.8. *With the notations above we have:*

- i) *For any $\lambda \in \mathbb{R}$ all solutions of (1.1) are bounded on $K_{n+1} \cup \dots \cup K_m$.*
- ii) *If for some $j = 1, \dots, n-1$*

$$\lambda_0(K_j) \leq \lambda < \lambda_0(K_{j+1})$$

then all solutions of (1.1) are bounded on $K_{j+1} \cup \dots \cup K_n$.

Moreover, all solutions of (1.1) are unbounded in a neighborhood of each K_1, \dots, K_j .

- iii) *If $\lambda \geq \lambda_0(K_n)$ then all solutions of (1.1) are unbounded in a neighborhood of each K_1, \dots, K_n .*

Now we want to discuss the behavior of solutions in fat parts of K_0 that cannot be separated in components, but still have fatter and slimmer parts glued together.

First using Lemma 5.5 we prove the following universal bounds for solutions of (1.1). For this we will assume that n satisfies (1.3) and (1.4).

Lemma 5.9. *Assume n satisfies (1.3) and (1.4).*

Then for any solution of (1.1) there exists a constant $A = A(u_0, \lambda)$ such that

$$0 \leq u(t, x; u_0) \leq h(x) = \left(\frac{A}{d_0^2(x) \inf_{B(x, \frac{1}{2}d_0(x))} n_0} \right)^{\frac{1}{\rho-1}}$$

with $d_0(x) = \text{dist}(x, K_0)$.

Proof. Let $x_0 \in \Omega \setminus K_0$, hence, $B_0 = B(x_0, \frac{1}{2}d_0(x_0)) \subset \Omega \setminus K_0$. Since u_0 is bounded in $\overline{\Omega}$ by $M > 0$, from (1.4), if n_0 is continuous in $\overline{\Omega}$ we can assume that for all x_0 close enough to K_0 , $\beta = \beta(x_0) = \inf\{n_0(x), x \in B_0\} > 0$ satisfies $\beta(x_0) \leq \frac{\lambda}{M^{\rho-1}}$. On the other hand, if $\inf\{n_0(x), x \in \overline{\Omega \setminus K_0}\} > 0$, we take $\beta = \beta(x_0) = C \inf\{n_0(x), x \in \overline{\Omega \setminus K_0}\}$ with $0 < C < 1$ such that $\beta \leq \frac{\lambda}{M^{\rho-1}}$ for all $x_0 \in \Omega \setminus K_0$.

Consider $z(x)$, the translation to B_0 of the function in Lemma 5.5. Then, using Lemma 5.5 we have, for all $x \in B_0$

$$u_0(x) \leq M \leq \left(\frac{\lambda}{\beta(x_0)} \right)^{\frac{1}{\rho-1}} \leq z(x_0) \leq z(x).$$

Hence $z(x)$ is a supersolution for $u(x, t)$ in B_0 and then

$$u(x, t) \leq z(x), \quad x \in B_0, \quad t \geq 0.$$

In particular, for $x = x_0$ we get, from Lemma 5.5 that for all $t \geq 0$,

$$u(x_0, t) \leq z(x_0) \leq \left(\frac{\lambda(\rho + 1)}{2\beta(x_0)} + \frac{B}{\beta(x_0)d_0(x_0)^2} \right)^{\frac{1}{\rho-1}}$$

for some constant $B > 0$. Since x_0 is close enough to K_0 we can assume

$$u(x_0, t) \leq z(x_0) \leq \left(\frac{A}{\beta(x_0)d_0(x_0)^2} \right)^{\frac{1}{\rho-1}}$$

for all $t \geq 0$ and some $A > 0$.

From [Proposition 5.6](#), far from K_0 , $u(x, t)$ remains bounded and for $x_0 \in K_0$ the result is obvious. \square

Now we obtain estimates on the solutions of [\(1.1\)](#) on suitable $N - 1$ dimensional transversal sections to K . For this we define

Definition 5.10. Let S be a bounded closed regular piece of a hyperplane in \mathbb{R}^N . That is, $S = \overline{S_0}$ with S_0 a bounded open set in the hyperplane.

We say S is **transversal** to the compact set K , if $K \not\subset S$, $K_S = K \cap S \neq \emptyset$ and there exists $c_1, c_2 > 0$ such that

$$c_1 \text{dist}_S(x, K_S) \leq \text{dist}(x, K) \leq c_2 \text{dist}_S(x, K_S), \quad \forall x \in S$$

where dist_S denotes the $(N - 1)$ -dimensional distance on the hyperplane containing S .

Then, we have:

Proposition 5.11. Assume S is transversal to K as in [Definition 5.10](#).

Also, assume $n(x)$ satisfies [\(1.3\)](#), [\(1.4\)](#) and there exists a real nonnegative continuous function n^* such that $n^*(s) > 0$ if $s > 0$, $n^*(0) \geq 0$ and for $x \in S$, close enough to K , and for $d_0(x) = \text{dist}(x, K_0)$, we have

$$\inf_{B(x, \frac{1}{2}d_0(x))} n_0 \geq n^*(d_0(x)), \quad x \in S. \quad (5.3)$$

Furthermore we assume $j(s) = s^2 n^*(s)$ is increasing in $s \geq 0$.

Finally assume that the $N - 1$ fractal dimension of $K_S = K \cap S$, that is, the fractal dimension of K_S as a subset of \mathbb{R}^{N-1} , is $0 \leq d^* < N - 1$.

If $r \geq 1$ is such that

$$\int_0^1 \frac{[j^{-1}(t)]^{N-1-d^*}}{t^{1+\frac{r}{\rho-1}}} dt < \infty$$

then for any solution of [\(1.1\)](#) there exists $h \in L^r(S)$ such that

$$0 \leq u(x, t) \leq h(x), \quad \text{for all } t \geq 0 \text{ and } x \in S.$$

In particular, if $n^*(s) = Cs^\gamma$ for $s \approx 0$, with $\gamma \geq 0$, then the above condition is satisfied, provided γ , d^* and ρ satisfy

$$(\gamma + 2)r < (\rho - 1)(N - 1 - d^*).$$

Specifically, whenever

$$\gamma + 2 < (\rho - 1)(N - 1 - d^*),$$

there exists an $r > 1$ satisfying the above condition.

Remark 5.12. Note that in (5.3), $B\left(x, \frac{1}{2}d_0(x)\right)$ is an N -dimensional ball and that this condition gives information in the way $n(x)$ behaves as $x \in S$ approaches K .

Proof. Observe that the function h in Lemma 5.9 satisfies for $x \in S$

$$h(x) \leq h^*(\text{dist}(x, K)), \quad h^*(s) = \left(\frac{C}{s^2 n^*(s)}\right)^{\frac{1}{\rho-1}}, \quad h^*(0) = \infty.$$

The assumption on the fractal dimension of K_S as a subset of \mathbb{R}^{N-1} implies that the $N - 1$ dimensional neighborhoods of K_S ,

$$\omega_\delta = \{x \in S, \quad \text{dist}_S(x, K_S) \leq \delta\}$$

have a $N - 1$ dimensional measure satisfying $|\omega_\delta|_{N-1} \leq C\delta^{N-1-d^*} \rightarrow 0$ as $\delta \rightarrow 0$.

Then Lemma 3.13 gives that $h \in L^r(S)$ provided

$$\int_1^\infty \left[j^{-1} \left(\frac{1}{y^{\rho-1}} \right) \right]^{N-1-d^*} y^{r-1} dy < \infty$$

and the change of variables $t = \frac{1}{y^{\rho-1}}$ gives the result.

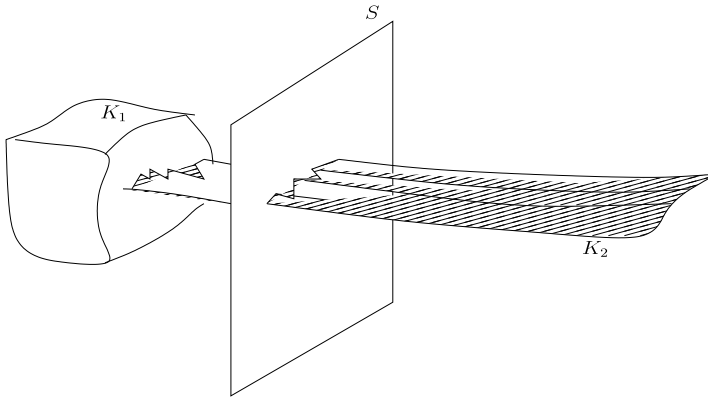
If, in particular, $n^*(s) = Cs^\gamma$ for $s \approx 0$ with $\gamma \geq 0$, the above reads

$$1 \leq r < \frac{(\rho - 1)(N - 1 - d^*)}{\gamma + 2}$$

which is possible, for some $\gamma \geq 0$, $\rho > 1$, $d^* \geq 0$, $r > 1$ provided

$$2 \leq \gamma + 2 < (\rho - 1)(N - 1 - d^*) \leq (\rho - 1)(N - 1).$$

This concludes the proof. \square



Remark 5.13. Note that both the “size” of the section K_S (in terms on its fractal dimension) and the way $n(x)$ vanishes near K_S , intervene in the result above.

Remark 5.14. The result also holds without assuming that S is a piece of a hyperplane. It will be enough to require that $S = F(S^*)$ where F is a C^1 -diffeomorphism in \mathbb{R}^{N-1} , where S^* is a regular piece of a hyperplane and both F and its inverse F^{-1} are globally Lipschitz maps. We will also need the existence of c_1 and c_2 such that

$$c_1 \text{dist}_S(x, K_S) \leq \text{dist}(x, K) \leq c_2 \text{dist}_S(x, K_S), \quad \forall x \in S$$

Note that now dist_S denotes the distance on S , which we define as the distance related by the map F , that is $\text{dist}_S(x, y) = \text{dist}(F^{-1}(x), F^{-1}(y))$.

Also, we require that $K_S = F(K^*)$ where $K^* \subset \mathbb{R}^{N-1}$ has fractal dimension $0 \leq d^* < N - 1$.

Now we prove one of the main results in this paper. For this, assume K is one of the fat components of K_0 , as in (3.3) and it is such that $K = K_1 \cup K_2$ where $K_1 \cap K_2 \neq \emptyset$. Assume also K is fatter than K_2 , that is, $\lambda_0(K) < \lambda_0(K_2)$.

Then, we prove that solutions of (1.1) become unbounded first on K_1 than in K_2 .

Theorem 5.15. Assume K is one of the fat components of K_0 , as in (3.3) and it is such that $K = K_1 \cup K_2$ where $K_1 \cap K_2 \neq \emptyset$. Assume also K is fatter than K_2 , that is, $\lambda_0(K) < \lambda_0(K_2)$.

Assume B is an “isolation box” for K_2 , that is, an open bounded set B such that $\overline{B} \supset K_2$, $K_1 \cap B = \emptyset$ and $K_1 \cap \overline{B} = K_1 \cap K_2$. Moreover, assume one of the “faces” of its boundary, say $S \subset \partial B$, is transversal to K as in Definition 5.10 and that we also have $\partial B \cap K = \partial B \cap K_2 \subset S$. Finally assume the conditions on $n(x)$ and the fractal dimension of $K_S = K \cap S$ are as in Proposition 5.11.

Then if $\lambda_0(K) \leq \lambda < \lambda_0(K_2)$, any solution of (1.1) remains bounded in K_2 although it is unbounded in a neighborhood of K , hence of K_1 .

Proof. We claim that, without loss of generality, we may assume that $\lambda < \lambda_1(B)$, where as usual $\lambda_1(U)$ is the first eigenvalue of the Laplace operator in the open set U with Dirichlet boundary conditions. For this, notice that by the definition of $\lambda_0(K_2)$, see (2.4), we have an open bounded set Ω_δ with $K_2 \subset \Omega_\delta \subset \Omega$ such that $\lambda < \lambda_1(\Omega_\delta) < \lambda_0(K_2)$. Observe now that if B is an isolating box, then $B \cap \Omega_\delta$ is also an isolating box and $\lambda < \lambda_1(\Omega_\delta) \leq \lambda_1(B \cap \Omega_\delta)$, which proves the claim.

Since $\partial B \cap K \subset S$, from Proposition 5.6 on $\partial B \setminus S$ we have L^∞ bounds on any given solution of (1.1). Also, Proposition 5.11 gives $u(x, t) \leq h(x)$ for $x \in S, t \geq 0$ and $h \in L^r(S)$, for $r > 1$ and we extend h to the rest of ∂B by a suitable constant. We denote by $\tilde{h} \in L^r(\partial B)$, this extension.

Thus, the solution of

$$\begin{cases} U_t - \Delta U = \lambda U & \text{in } B \\ U = \tilde{h}(x) \geq 0 & \text{on } \partial B \\ U(0) = u_0 \geq 0 & \text{in } B \end{cases}$$

which becomes a supersolution of $u(x, t)$ in B .

Now, since $\lambda_0(K) \leq \lambda < \lambda_1(B) < \lambda_0(K_2)$, Lemma 5.16 below gives L^∞ bounds for $U(x, t)$ uniformly for all times $t \geq 1$, on compact subsets of B . Hence, $u(x, t)$ remains bounded on K_2 while from Theorem 5.7 we know that it does become unbounded in a neighborhood of K , hence of K_1 . \square

Now we prove the result used above.

Lemma 5.16. *Consider the problem*

$$\begin{cases} U_t - \Delta U = \lambda U & \text{in } B \\ U = h(x) \geq 0 & \text{on } \partial B \\ U(0) = u_0 \geq 0 & \text{in } B \end{cases}$$

in a bounded open set $B \subset \mathbb{R}^N$, with $\lambda < \lambda_1(B)$ and $h \in L^r(\partial B)$ for some $1 < r < \infty$.

Then, for any compact set $K \subset B$ there exists $M = M(K)$ such that

$$0 \leq U(t, x) \leq M(K) \quad \text{for all } x \in K, t \geq 1.$$

Proof. Since $\lambda < \lambda_1(B)$, consider the solution of

$$\begin{cases} -\Delta H = \lambda H & \text{in } B \\ H = h(x) & \text{on } \partial B \end{cases} \quad (5.4)$$

where $h \in L^r(\partial B)$. Consider also the solution of the auxiliar elliptic problem

$$\begin{cases} -\Delta \varphi = \lambda \varphi + g & \text{in } B \\ \varphi = 0 & \text{on } \partial B \end{cases} \quad (5.5)$$

for some given function g in B . Multiplying (5.4) by φ and (5.5) by H and integrating by parts we get

$$\int_B g H = - \int_{\partial B} \frac{\partial \varphi}{\partial \vec{n}} h.$$

Therefore the mapping $h \mapsto L(h) = H$ is the adjoint of the mapping $g \mapsto M(g) = -\frac{\partial \varphi}{\partial \vec{n}}$.

If we take $g \in X = L^q(B)$, then elliptic regularity and traces give $M(g) \in Y = L^p(\partial B)$ for $p = \infty$, if $q > N$, or $1 \leq p \leq \frac{q(N-1)}{N-q}$, if $1 < q < N$. Then $L \in \mathcal{L}(Y^*, X^*)$. Denoting $1 < r = \frac{q(N-1)}{(q-1)N} < \infty$, we get $L \in \mathcal{L}(L^r(\partial B), L^{\frac{rN}{N-1}}(B))$. Even more, if $g \geq 0$ is smooth, then $M(g) > 0$ and then $h \geq 0$ implies $H \geq 0$.

Thus, the solution of (5.4) satisfies $0 \leq H \in L^{\frac{rN}{N-1}}(B)$ and truncating with a smooth function ϕ of compact support in B and which is equal to one in a subset ω of B , writing the equation for $v = \phi H$ and using elliptic regularity we get $H \in W^{1, \frac{rN}{N-1}}(\omega)$, i.e. $H \in W_{loc}^{1, \frac{rN}{N-1}}(B)$. Truncating once again and using this information we get $H \in W_{loc}^{2, \frac{rN}{N-1}}(B)$. Now Sobolev embedding gives $H \in L_{loc}^s(B)$ for some $s > \frac{rN}{N-1}$ and iterating this argument we finally get $H \in L_{loc}^\infty(B)$.

Now, clearly $U(t) = W(t) + H$, where H is the weak solution of (5.4) and $W(t)$ solves

$$\begin{cases} W_t - \Delta W = \lambda W & \text{in } B \\ W = 0 & \text{on } \partial B \\ W(0) = u_0 - H & \text{in } B. \end{cases}$$

Hence, using that $\lambda < \lambda_1(B)$ and from standard $L^1 - L^\infty$ parabolic estimates with Dirichlet boundary conditions we get that

$$\|W(t)\|_{L^\infty(B)} \leq C\|u_0 - H\|_{L^1(B)} \leq \tilde{C}, \quad \forall t \geq 1.$$

This and the L^∞ bounds in compact sets of B for H give the result. \square

Remark 5.17. In $N \geq 2$ dimensions consider $K_0 = K_1 \cup K_2$, where $K_1 = \overline{B}$, the closure of a ball, K_2 is a segment, and $K_1 \cap K_2$ is a point. In that case $d^* = 0$.

Assume for instance, that

$$n(x) \geq \left(\text{dist}(x, K_0)\right)^\gamma \quad \text{for some } \gamma > 0.$$

Then the above Theorem 5.15 reads: If

$$\gamma + 2 < (\rho - 1)(N - 1)$$

and if $\lambda > \lambda_0(K_0) = \lambda_1(B)$, then any solution of (1.1) remains unbounded in B and bounded in K_2 .

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