Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 7800–7820

Journal of Differential Equations

www.elsevier.com/locate/jde

Prevalence of stable periodic solutions for Duffing equations

Jifeng Chu [a,*,1], Feng Wang [a,b,2]

[a] Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China
[b] School of Mathematics and Physics, Changzhou University, Changzhou 213164, China

Received 22 August 2015; revised 14 January 2016

Available online 12 February 2016

Abstract

We study the prevalence of stable periodic solutions of the Duffing equations for external force which guarantees the existence of periodic solutions. Both the dissipative case and the conservative case are considered. The nonlinearity may be periodic or satisfies the condition of Landesman–Lazer type.

MSC: 34C25

Keywords: Prevalence; Stable periodic solutions; Duffing equations

1. Introduction

Let $X = \mathbb{C}(\mathbb{R}/T\mathbb{Z})$ be the space of continuous T-periodic functions. Then X is a separable Banach space with the norm $\|f\|_\infty = \max_{t\in\mathbb{R}} |f(t)|$. We consider the Duffing equation

$$\ddot{x} + c\dot{x} + g(x) = f(t), \tag{1.1}$$

* Corresponding author.
E-mail addresses: jifengchu@126.com (J. Chu), fengwang188@163.com (F. Wang).
[1] Jifeng Chu was supported by the National Natural Science Foundation of China (Grant Nos. 11171090, 11401166) and the Fundamental Research Funds for the Central Universities (Grant No. 2015B19214).
[2] Feng Wang was supported by Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No. 15KJB110001).

http://dx.doi.org/10.1016/j.jde.2016.02.003
0022-0396/

where $c > 0$, $g \in \mathbb{C}(\mathbb{R}, \mathbb{R})$ and $f \in X$. It is well-known that equation (1.1) may not admit T-periodic solutions if we just take f from the space X, even when some additional conditions are satisfied. For example, it was shown in [1,18,25] that the following equation

$$\ddot{x} + c\dot{x} + a \sin x = h(t)$$

may not have T-periodic solutions in some cases, where $c > 0$ and $h \in X_0$, where

$$X_0 = \{f \in X : \bar{f} = \frac{1}{T}\int_0^T f(t)dt = 0\}.$$

Therefore, the possible space to guarantee the existence of T-periodic solutions of (1.1) may be smaller than X. If we have no damped term in (1.1), it becomes the conservative equation,

$$\ddot{x} + g(x) = f(t). \tag{1.2}$$

When g is a periodic function, we know that the periodic solutions of (1.2) can always be found as critical points of the corresponding action functional. See [17]. Compared with (1.2), the periodic solutions of equation (1.1) cannot be found as critical points because it is dissipative.

In this paper, we introduce the following two subspaces

$$\tilde{X}_* = \{f \in X : \text{equation } (*) \text{ has at least one } T\text{-periodic solution}\},$$

and

$$S_* = \{f \in X : \text{equation } (*) \text{ has at least one stable } T\text{-periodic solution}\},$$

in which $(*)$ denotes (1.1) or (1.2). Obviously, $S_* \subset \tilde{X}_*$. Our aim is to prove that S_* is prevalent in $\tilde{X}_*$. A prevalent set can be seen as the analogue of a set of full measure in infinite dimension. For more information on prevalence, we refer to the paper by Ott and York [26].

In the last three decades, the stability of periodic solutions of (1.1) has attracted many researchers since Ortega initiated the study by using the relations between the stability and the Brouwer degree of Poincaré map. See, for example, [3–5,19,20]. For the conservative equation (1.2), looking for stable periodic solutions is a difficult topic because the study requires sophisticated techniques which use the information on nonlinear approximation. An important progress along this topic is ‘third order approximation’. Based on Moser’s twist theorem, Ortega [21] obtained the formula of the first twist coefficient of time periodic scalar Lagrangian equations. Later, such formula was rewritten by Zhang [31] and has been used to study the stable periodic solutions of different differential equations, such as swing, the pendulum and singular equations [6–9,15].

Very recently, in [22,23] Ortega proved the prevalent property of stable periodic solutions of the forced pendulum equations

$$\ddot{x} + \beta \sin x = f(t). \tag{1.3}$$

In particular, he proved that if

$$0 < \beta \le \left(\frac{\pi}{T}\right)^2, \tag{1.4}$$

equation (1.3) has at least one stable T-periodic solution for almost every forcing $f \in X_0$. In other words, the set

$$\{f \in X_0 : \text{(1.3) has a stable } T\text{-periodic solution}\}$$

is prevalent in X_0. Two examples were constructed in [24] to explain that the upper bound $(\pi/T)^2$ in (1.4) is optimal.

The motivation of our study is to obtain the prevalence of stable periodic solutions for (1.1) and (1.2). We emphasize that the extension of (1.3) to equations (1.1) and (1.2) has essential difficulties and necessities. First, we consider the dissipative case, which was not covered in [22–24]. Secondly, we can deal with a large class of functions g. For example, g may be periodic or satisfies the condition of Landesman–Lazer type. Thirdly, because we deal with general functions g instead of $\sin x$, we have to overcome some essential difficulties.

The rest part is organized as follows. In Section 2, we recall two prevalent results and the eigenvalue theory for Hill equations. In Section 3, we prove the prevalence of stable periodic solutions for equation (1.1) when g is periodic, or g satisfies the condition of Landesman–Lazer type. See [28] for recent surveys on results of the Landesman–Lazer type. Finally in Section 4, we prove the prevalence of stable periodic solutions for equation (1.2).

2. Preliminaries

2.1. Two prevalent results

Let $\mathbb{E}$ be a separable Banach space with the norm $\|\cdot\|$. We consider a map

$$H : \mathbb{R}^d \times \mathbb{E} \to \mathbb{R}^d, \qquad (\xi, e) \mapsto H(\xi, e)$$

and define

$$Z = \{(\xi, e) \in \mathbb{R}^d \times \mathbb{E} : H(\xi, e) = 0\}.$$

We say that 0 is a regular value of $H(\cdot, e)$ if $\det[\partial_1 H(\xi, e)] \ne 0$ for each $\xi \in \mathbb{R}^d$ such that $H(\xi, e) = 0$.

The following prevalent transversality theorem was proved in [22, Theorem 2]. See [11, Lemma 1] and [27, Lemma 3.2] for related results.

Lemma 2.1. *(See [22].) Assume that the following conditions hold:*

- (C_1) $H \in C^1(\mathbb{R}^n \times \mathbb{E}, \mathbb{R}^n)$;
- (C_2) *given $b > 0$, there exists $B > 0$ such that if $(\xi, e) \in Z$ and $\|e\| \le b$, then $|\xi| \le B$;*
- (C_3) *there exists a compact set $K \subset \mathbb{E}$ such that the linear operator $\partial_2 H(\xi, e) : \mathbb{E} \to \mathbb{R}^d$ is onto if $(\xi, e) \in Z$ and $e \notin K$.*

Then the set

$$\tilde{\mathbb{E}} = \{e \in \mathbb{E} : 0 \textit{ is a regular value of } H(\cdot, e)\}$$

is open and prevalent. Moreover, if

$$H(T(\xi), e) = H(\xi, e) \textit{ with } T(\xi_1, \xi_2, \dots, \xi_n) = (\xi_1 + T, \xi_2, \dots, \xi_n),$$

then condition (C_2) *can be replaced by*

$(\mathrm{C}_2)_{per}$ *given* $b > 0$*, there exists* $B > 0$ *such that if* $(\xi, e) \in Z$ *and* $\|e\| \le b$*, then* $|\hat{\xi}| \le B$*, where* $\hat{\xi} = (\xi_2, \dots, \xi_n)$*.*

Lemma 2.2. *(See [23].) Let G be an open and prevalent subset of* $\mathbb{E}$*. Assume that there exist a family* $\{U_\alpha\}_{\alpha \in A}$ *of open subsets of* $\mathbb{E}$ *and functions* $d_\alpha \in C^1(U_\alpha, \mathbb{R})$ *such that*

$$G \subset \bigcup_{\alpha \in A} U_\alpha,$$

$$d'_\alpha(e) \neq 0 \textit{ for each } e \in U_\alpha, \alpha \in A.$$

Let C be a Borel subset of $\mathbb{R}$ *with zero measure. Then the set*

$$\widetilde{G} = \bigcup_{\alpha \in A} d_\alpha^{-1}(\mathbb{R} \backslash C)$$

is prevalent in $\mathbb{E}$*.*

2.2. Some facts on Hill's equation

Given $a \in X$, let us consider Hill's equation

$$\ddot{x} + a(t)x = 0. \tag{2.1}$$

We refer the reader to the book [16] for details on Hill's equations. Denote by $\Psi(t) = \phi_1(t) + i\phi_2(t)$ the complex-valued solution of (2.1) with the initial data $\Psi(0) = 1$ and $\dot{\Psi}(0) = i$. The Floquet multipliers of (2.1) are the eigenvalues ρ_i, $i = 1, 2$ of the monodromy matrix

$$\Phi_T = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \dot{\phi}_1(T) & \dot{\phi}_2(T) \end{pmatrix}.$$

Since Φ_T is symplectic, we know that $\rho_1 \cdot \rho_2 = 1$. We can classify (2.1) into three cases, according to the Floquet multipliers, as either *hyperbolic* when $|\rho_{1,2}| \neq 1$, or *elliptic* when $|\rho_{1,2}| = 1$ but $\rho_{1,2} \neq \pm 1$, or *parabolic* when $\rho_{1,2} = \pm 1$, respectively.

The discriminant of (2.1) is defined as

$$\Delta := \mathrm{trace}(\Phi_T) = \rho_1 + \rho_2,$$

which can be thought as a functional

$$\Delta : X \to \mathbb{R}, \quad a \mapsto \Delta[a]. \tag{2.2}$$

It is well known that Δ is continuous. Moreover, it was shown in [23] that Δ is Gateaux differentiable. In fact, for a given function $\delta \in X$, we know from [23, Lemma 7] that

$$\Delta'[a]\delta = \int_0^T \chi(s,a)\delta(s)ds,$$

where

$$\chi(s,a) = -\phi_2(T)\phi_1^2(s) + (\phi_1(T) - \dot{\phi}_2(T))\phi_1(s)\phi_2(s) + \dot{\phi}_1(T)\phi_2^2(s). \tag{2.3}$$

The function $\chi(\cdot,a)$ is continuous and the derivative $\Delta'[a]$ can be interpreted as an element of the dual space of X.

Next we recall the theory of eigenvalues of (2.1). Consider the eigenvalue problems

$$\ddot{x} + (\lambda + a(t))x = 0 \tag{2.4}$$

subject to the periodic boundary condition

$$x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0, \tag{2.5}$$

or to the anti-periodic boundary condition

$$x(0) + x(T) = \dot{x}(0) + \dot{x}(T) = 0. \tag{2.6}$$

Denote by

$$\lambda_1^D(a) < \lambda_2^D(a) < \cdots < \lambda_n^D(a) < \cdots$$

all eigenvalues of (2.4) with the Dirichlet boundary condition

$$x(0) = x(T) = 0. \tag{2.7}$$

Theorem 2.3. *(See [16].) There exist two real-valued sequences $\{\underline{\lambda}_n(a) : n \in \mathbb{N}\}$ and $\{\overline{\lambda}_n(a) : n \in \mathbb{Z}^+\}$ such that*

- (P$_1$) $-\infty < \overline{\lambda}_0(a) < \underline{\lambda}_1(a) \le \overline{\lambda}_1(a) < \cdots < \underline{\lambda}_n(a) \le \overline{\lambda}_n(a) < \cdots$ *and* $\underline{\lambda}_n(a) \to +\infty$, $\overline{\lambda}_n(a) \to +\infty$ *as* $n \to \infty$*;*
- (P$_2$) λ *is an eigenvalue of (2.4)–(2.5) (or (2.4)–(2.6)) if and only if* $\lambda = \underline{\lambda}_n(a)$ *or* $\overline{\lambda}_n(a)$ *for some even (or odd) integer* n*;*
- (P$_3$) *for any* $n \in \mathbb{N}$, $\underline{\lambda}_n(a) = \min\{\lambda_n^D(a_{t_0}) : t_0 \in \mathbb{R}\}$, $\overline{\lambda}_n(a) = \max\{\lambda_n^D(a_{t_0}) : t_0 \in \mathbb{R}\}$, *where* $a_{t_0}(t) = a(t + t_0)$*;*
- (P$_4$) *if* $a_1(t) \ge a_2(t)$ *for all* t*, then for any* $n \in \mathbb{N}$, $\underline{\lambda}_n(a_1) \le \underline{\lambda}_n(a_2)$, $\overline{\lambda}_n(a_1) \le \overline{\lambda}_n(a_2)$, $\lambda_n^D(a_1) \le \lambda_n^D(a_2)$*; moreover, if* $\bar{a}_1 > \bar{a}_2$*, then all inequalities above are strict;*
- (P$_5$) *the eigenfunction of* $\overline{\lambda}_0(a)$ *do not vanish everywhere; for* $n \ge 1$*, the eigenfunctions of* $\underline{\lambda}_n(a)$ *or* $\overline{\lambda}_n(a)$ *have exactly* $n-1$ *zeros in the intervals of the form* $(t_0, t_0 + T)$.

Lemma 2.4. *Assume that* $\underline{\lambda}_1(a) > 0$. *Then the following conclusions are true.*

(1) *Equation* (2.1) *does not admit any negative Floquet multipliers.*

(2) *The possible* T*-periodic solution* x *of equation* (2.1) *is either trivial or different from zero for each* $t \in \mathbb{R}$.

Proof. (1) Suppose that there exists a nontrivial solution x of (2.1) with a negative Floquet multiplier, i.e. $x(t+T) = \rho x(t)$, $t \in \mathbb{R}$ for some $\rho < 0$. Then there exists $t_0 \in [0, T]$ such that $x(t_0) = x(t_0 + T) = 0$, and therefore x is a nontrivial solution of (2.1) with the Dirichlet boundary condition (2.7), which implies that x is an eigenfunction associated with zero eigenvalue $\lambda_k^D(a) = 0$ for some $k \geq 1$. Therefore $\underline{\lambda}_1(a) \leq \lambda_1^D(a) \leq \lambda_k^D(a) = 0$, which is a contradiction.

(2) Assume that (2.1) admits a nontrivial T-periodic solution x which vanishes at some t_0. Then it must vanishes also at $t_0 + T$. Now the proof is finished by the same reasoning as in (1). □

Let us consider the linear periodic equation

$$\ddot{x} + c\dot{x} + a(t)x = 0. \tag{2.8}$$

Lemma 2.5. *Assume that*

$$\underline{\lambda}_1(a) + \frac{c^2}{4} > 0.$$

Then the following conclusions are true.

(1) *Equation* (2.8) *does not admit any negative Floquet multipliers.*

(2) *The possible* T*-periodic solution* x *of equation* (2.8) *is either trivial or different from zero for each* $t \in \mathbb{R}$.

Proof. Let $x(t) = e^{-\frac{1}{2}ct}u(t)$. Then (2.8) becomes

$$\ddot{u} + \left[a(t) - \frac{c^2}{4}\right]u = 0.$$

Obviously x and u have the same zeros. Notice the following fact, which can be found in the proof of [29, Lemma 2.1],

$$\underline{\lambda}_1(a - \frac{c^2}{4}) = \underline{\lambda}_1(a) + \frac{c^2}{4}.$$

Now the results follow from Lemma 2.4. □

2.3. Index

Given $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Let $x(t; \xi)$ be the solution of (1.1) satisfying

$$x(0) = \xi_1, \quad \dot{x}(0) = \xi_2.$$

We assume that $x(t;\xi)$ is unique for each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. The Poincaré map is defined as the mapping

$$P : D_T \subset \mathbb{R}^2 \to \mathbb{R}^2, \qquad P(\xi) = (x(T;\xi), \dot{x}(T;\xi)),$$

where

$$D_T = \{\xi \in \mathbb{R}^2 : x(t;\xi) \text{ is defined in } [0,T]\}.$$

It is well known that D_T is open in $\mathbb{R}^2$ and P is a homeomorphism between D_T and $P(D_T)$. Moreover, the search of T-periodic solution of (1.1) is equivalent to looking for the fixed point of $\xi = P(\xi)$.

Let x be a T-periodic solution of (1.1) with $\xi_0 = (x(0), \dot{x}(0))$. The solution x is said to be isolated (periodic T) if ξ_0 is an isolated fixed point of P. In such case the index of x is defined by

$$\mathrm{ind}_T(x) = i[P, \xi_0],$$

where i is the notion of the local fixed point index in the plane. We refer to [12] for more information about the theory of the index.

3. Prevalence of stable periodic solutions of (1.1)

3.1. Connection between asymptotical stability and index

We say that T-periodic solution x of (1.1) is non-degenerate if $y \equiv 0$ is the unique T-periodic solution of

$$\ddot{y} + c\dot{y} + g'(x(t))y = 0. \tag{3.1}$$

Lemma 3.1. *(See [10, Theorem 2.1].) Assume that the characteristic exponents associated with (3.1) all have negative real parts. Then the corresponding T-periodic solution x of (1.1) is asymptotically stable.*

Lemma 3.2. *Assume that $c > 0$ and x is a non-degenerate T-periodic solution of (1.1) such that*

$$\underline{\lambda}_1(g'(x(t))) + \frac{c^2}{4} \geq 0. \tag{3.2}$$

Then x is asymptotically stable (resp. unstable) if and only if $\mathrm{ind}_T(x) = 1$ (resp. $\mathrm{ind}_T(x) = -1$).

Proof. Denote by ρ_1 and ρ_2 ($|\rho_1| \geq |\rho_2|$) the Floquet multipliers of (3.1). Since x is non-degenerate, 1 cannot be a Floquet multiplier. Because x is non-degenerate, we can assume that the inequality (3.2) is strict. By Lemma 2.5 the multipliers are either conjugate complex numbers or positive real numbers.

If ρ_1 and ρ_2 are a pair of conjugate numbers, by applying the Jacobi–Liouville formula, since $\rho_1\rho_2 = e^{-cT}$, we have that

$$|\rho_1| = |\rho_2| = e^{-\frac{cT}{2}} < 1,$$

and

$$\mathrm{Re}(\mu_1) = \mathrm{Re}(\mu_2) = \frac{1}{2}\mathrm{Re}(\mu_1 + \mu_2) = \frac{1}{2T}\ln(\rho_1\rho_2) = -\frac{c}{2} < 0.$$

Thus

$$\mathrm{ind}_T(x) = \mathrm{sign}\{\det(I_2 - M(T))\} = \mathrm{sign}\{|1 - \rho_1|^2)\} = 1$$

and x is asymptotically stable by Lemma 3.1.

If ρ_1 and ρ_2 are real and positive. Then $\mathrm{ind}_T(x) = 1$ if and only if both ρ_1 and ρ_2 are smaller than one because we have the fact $\rho_1\rho_2 < 1$. Thus the characteristic exponents are both negative and the statement about the asymptotical stability is proved also in this case.

Finally, x is unstable if and only if ρ_1 has absolute value greater than one, which can hold only if they are real. Thus we know that $0 < \rho_2 < 1 < \rho_1$ and therefore $\mathrm{ind}_T(x) = \mathrm{sign}\{(1 - \rho_1)(1 - \rho_2)\} = -1$. □

3.2. Periodic cases

Consider the differential equation (1.1) with $c > 0$ and $f \in X$. We assume that $g : \mathbb{R} \to \mathbb{R}$ is of class C^1 and satisfies

(σ_1) $g(x + 2\pi) = g(x)$ for each $x \in \mathbb{R}$,
(σ_2) g is not locally trivial, which means that for a fixed constant ξ and every open and non-empty interval $I \subset \mathbb{R}$ there exists some $x \in I$ such that $g(x) \neq \xi$.

Theorem 3.3. *Assume that* (σ_1) *and* (σ_2) *hold. Then the set*

$$\mathfrak{R}_{per} = \{f \in \tilde{X}_{(1.1)} : \textit{ every } T\textit{-periodic solution of } (1.1) \textit{ is non-degenerate}\}$$

is open and prevalent in $\tilde{X}_{(1.1)}$.

Proof. Given $\xi = (\xi_1, \xi_2)^* \in \mathbb{R}^2$ and $f \in \tilde{X}_{(1.1)}$, the solution of the initial value problem

$$\ddot{x} + c\dot{x} + g(x) = f(t), \ x(0) = \xi_1, \ \dot{x}(0) = \xi_2 \tag{3.3}$$

will be denoted by $x(t; \xi, f)$. Since g is bounded, this solution is globally defined. Let $\phi_1(t)$ and $\phi_2(t)$ be real-valued solutions of (3.1) satisfying

$$\phi_1(0) = 1, \ \dot{\phi}_1(0) = 0, \ \phi_2(0) = 0, \ \dot{\phi}_2(0) = 1.$$

Let

$$\Phi(t; \xi, f) =: \Phi(t) = \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) \end{pmatrix}$$

be the fundamental matrix solution of

$$\dot{Y} = A(t)Y, \ Y(0) = I_2, \tag{3.4}$$

with

$$A(t) = \begin{pmatrix} 0 & 1 \\ -g'(x(t;\xi,f)) & -c \end{pmatrix}.$$

The theorem on continuous dependence can be applied to the Cauchy problems (3.3) and (3.4). It implies that the map

$$(t;\xi,f) \in \mathbb{R} \times \mathbb{R}^2 \times X \to \Phi(t;\xi,f) \in \mathbb{R}^{2\times 2}$$

is continuous. In particular it is uniformly continuous on compact sets. This implies that if $\xi_n \to \xi$ and $\|f_n - f\|_\infty \to 0$, then

$$\Phi(t;\xi_n,f_n) \to \Phi(t;\xi,f)$$

uniformly in $t \in [0,T]$. It is easy to verify that $W(t) = \det \Phi(t)$ satisfies the following equation

$$W'(t) = -cW(t),$$

which implies that

$$W(t) = W(0)\exp\Big(-\int_0^t c\,ds\Big) = \exp(-ct). \tag{3.5}$$

The Poincaré matrix of (3.1) is

$$\Phi(T) = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \dot{\phi}_1(T) & \dot{\phi}_2(T) \end{pmatrix}.$$

Using (3.5), we have

$$W(T) = \det \Phi(T) = \exp(-cT).$$

Let us consider the map

$$H : \mathbb{R}^2 \times X \to \mathbb{R}^2, \ \ H(\xi,f) = (x(T;\xi,f) - \xi_1, \dot{x}(T;\xi,f) - \xi_2)^*.$$

Obviously, the zeros of $H(\cdot,f)$ are the initial conditions producing T-periodic solutions. This map is continuous and the theorem on differentiability with respect to initial conditions and parameters implies that it is Gâteaux differentiable with partial derivatives $\partial_1 H(\xi,f) \in \mathbb{R}^{2\times 2}$ and $\partial_2 H(\xi,f) \in L(X,\mathbb{R}^2)$ given by

$$\partial_1 H(\xi,f) = \Phi(T;\xi,f) - I_2, \ \partial_2 H(\xi,f)p = (y(T), \dot{y}(T))^*,$$

where p is an arbitrary function in X and $y(t)$ is the solution of

$$\ddot{y} + c\dot{y} + g'(x(t;\xi,f))y = p(t),\ \ y(0) = \dot{y}(0) = 0.$$

Using the formula of variation of constants, one may easily see that

$$y(t) = \int_0^T G(t,s;\xi,f)\frac{p(s)}{W(s)}ds, \tag{3.6}$$

where

$$G(t,s;\xi,f) = \phi_2(t)\phi_1(s) - \phi_2(s)\phi_1(t).$$

Similarly,

$$\dot{y}(t) = \int_0^T \frac{\partial G}{\partial t}(t,s;\xi,f)\frac{p(s)}{W(s)}ds. \tag{3.7}$$

The continuity of Φ and the formulas (3.6) and (3.7) can be employed to prove the continuity of the partial derivatives of H. In particular the continuity of

$$(\xi,f) \in \mathbb{R}^2 \times X \to \partial_2 H(\xi,f) \in L(X,\mathbb{R}^2)$$

is a consequence of the estimate

$$\|\partial_2 H(\xi,f) - \partial_2 H(\hat{\xi},\hat{f})\| \le$$
$$\frac{1}{W(T)}\int_0^T \{|G(T,s;\xi,f) - G(T,s;\hat{\xi},\hat{f})| + |\frac{\partial G}{\partial t}(T,s;\xi,f) - \frac{\partial G}{\partial t}(T,s;\hat{\xi},\hat{f})|\}ds.$$

The previous discussions show that H is Fréchet differentiable and (C_1) of Lemma 2.1 holds. The condition (σ_1) implies that

$$x(t;T(\xi),f) = x(t;\xi,f) + 2\pi.$$

We can deduce that H satisfies the periodicity condition $H(T(\xi),f) = H(\xi,f)$, where $T(\xi) = T(\xi_1,\xi_2) = (\xi_1 + 2\pi,\xi_2)$. Let x be a T-periodic solution of (1.1). Multiplying (1.1) by $\ddot{x}$ and integrating from 0 to T, we have that

$$\int_0^T \ddot{x}^2 dt + c\int_0^T \dot{x}\ddot{x}dt + \int_0^T g(x(t;\xi,f))\ddot{x}dt = \int_0^T f(t)\ddot{x}dt,$$

from which we can deduce that there exists a constant C (dependent only on T, f and g) such that $\|\ddot{x}\|_2 \le C$. By the Sobolev inequality,

$$\|\dot{x}\|_\infty \le \sqrt{\frac{T}{12}}C.$$

Then

$$|\xi_2| = |\dot{x}(0;\xi,f)| \le \|\dot{x}\|_\infty \le \sqrt{\frac{T}{12}}C. \tag{3.8}$$

Thus the condition $(C_2)_{per}$ holds with $B = \sqrt{\frac{T}{12}}C$.

To check [(C_3), Lemma 2.1] we define

$$K = \{k \in \mathbb{R} : \min_{x\in\mathbb{R}} g(x) \le k \le \max_{x\in\mathbb{R}} g(x)\}$$

and prove that $\partial_2 H(\xi, f) : X \to \mathbb{R}^2$ is onto if $(\xi, f) \in Z$ and $f \notin K$. After some computations with the formulas (3.6) and (3.7) we obtain

$$\partial_2 H(\xi, f)p = \Phi(T;\xi,f)J\int_0^T p(t)\Big(\frac{\phi_1(t)}{W(t)}, \frac{\phi_2(t)}{W(t)}\Big)^* dt$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since Φ and J are inverse, it is enough to prove that

$$L : p \in X \to \int_0^T p(t)\Big(\frac{\phi_1(t)}{W(t)}, \frac{\phi_2(t)}{W(t)}\Big)^* dt \in \mathbb{R}^2$$

is onto. In view of [22, Lemma 6], we need to prove that

$$\Big(\frac{\phi_1(t)}{W(t)}\Big)', \quad \Big(\frac{\phi_2(t)}{W(t)}\Big)'$$

are linearly independent. Actually we will prove that the Wronskian of these functions

$$D(t;\xi,f) = \begin{vmatrix} \Big(\frac{\phi_1(t)}{W(t)}\Big)' & \Big(\frac{\phi_2(t)}{W(t)}\Big)' \\ \Big(\frac{\phi_1(t)}{W(t)}\Big)'' & \Big(\frac{\phi_2(t)}{W(t)}\Big)'' \end{vmatrix}$$

is not identically zero. From (3.1) and (3.5), we have

$$D(t;\xi,f) = \frac{g'(x(t;\xi,f))}{W^2(t)}.$$

Assume by contradiction that $D(t;\xi,f)\equiv 0$. Then $g'(x(t;\xi,f))$ vanishes identically and so $g(x(t;\xi,f))$ is a constant k. From the equation (1.1),

$$f(t)=\ddot{x}+c\dot{x}+k.$$

The solution $x(t;\xi,f)$ is T-periodic and hence $k=\frac{1}{T}\int_0^T f(t)dt=\overline{f}$. In consequence, we obtain that

$$g(x(t;\xi,f))\equiv\overline{f},$$

and the assumption (σ_2) implies that $x(t;\xi,f)$ must be constant. Then

$$f(t)=k=\overline{f},$$

which has been excluded since $f\notin K$. The proof is completed using Lemma 2.1. □

Lemma 3.4. *Assume that $f\in\mathfrak{R}_{per}$. Then there exists only a finite number of T-periodic solutions of* (1.1) *satisfying $x(0)\in[0,2\pi]$.*

Proof. Given a T-periodic solution x. If $f\in\mathfrak{R}_{per}$, we know that $\det(I-P'(x))\neq 0$. The implicit function theorem can be applied to deduce that all T-periodic solutions of (1.1) are isolated. If we combine this fact with the bound in (3.8) we can conclude that the set of fixed points

$$\{(\xi_1,\xi_2)\in\mathbb{R}^2 : P(\xi)=\xi,\ \xi_1\in[0,2\pi]\}$$

is finite. □

Let $f\in\mathfrak{R}_{per}$ and a number $\sigma\in\mathbb{R}$ such that $x(0)\neq\sigma$ for any T-periodic solution. Let $x_1,\ldots,x_n$ be the family of T-periodic solutions satisfying

$$\sigma<x_i(0)<\sigma+2\pi,\quad i=1,\ldots,n.$$

Each of these solutions has an associated index that can be computed by the formula

$$\mathrm{ind}_T(x_i)=\mathrm{sgn}\{(1-\rho_1)(1-\rho_2)\},$$

where ρ_1 and ρ_2 are the Floquet multipliers associated to the variational equation (3.1) with $x=x_i$.

Lemma 3.5. *(See [20].) Let $x_1,\ldots,x_n$ be the family of T-periodic solutions of* (1.1). *Then*

$$\sum_{i=1}^{n}\mathrm{ind}_T(x_i)=0.$$

Theorem 3.6. *Assume that* (σ_1), (σ_2) *are satisfied and*

$$g'(x) < (\frac{\pi}{T})^2 + \frac{c^2}{4}, \qquad \textit{for all } x \in \mathbb{R}. \tag{3.9}$$

Then for each $f \in \mathfrak{R}_{per}$, *equation* (1.1) *has at least one asymptotically stable* T*-periodic solution and one unstable* T*-periodic solution.*

Proof. We remark that condition (3.9) means that

$$\underline{\lambda}_1(g'(x(t))) + \frac{c^2}{4} > 0$$

holds for every T-periodic solution x of (1.1). See [29] for the related discussions. For each $f \in \mathfrak{R}_{per}$, (1.1) has T-periodic solutions. Let $x_1, \dots, x_n$ be T-periodic solutions of (1.1). From Lemma 3.5, each index $\mathrm{ind}_T(x_i)$ can only take the values 1 or -1 and so at least one of the solutions, say x_1, must satisfy $\mathrm{ind}_T(x_1) = 1$. x_1 is asymptotically stable by Lemma 3.2. Similarly, there exists at least one T-periodic solution x_2 satisfying $\mathrm{ind}_T(x_2) = -1$, which is unstable. □

The following result follows directly from Theorem 3.6 and it is very related to [19, Theorem 3.1], where category was employed instead of prevalence.

Theorem 3.7. *Under the conditions of Theorem* 3.6, *the set* $S_{(1.1)}$ *is prevalent in* $\tilde{X}_{(1.1)}$.

3.3. Cases of Landesman–Lazer type

In this subsection, we always assume that $c > 0$ and $g \in C^1(\mathbb{R}, \mathbb{R})$. Moreover, we assume that

(σ_3) g is bounded and

$$\lim_{x \to -\infty} g(x) = g(-\infty) < \overline{f} < g(+\infty) = \lim_{x \to +\infty} g(x).$$

The condition (σ_3) is known as Landesman–Lazer type, which was introduced in [13]. The classical results by Lazer in [14] and Ward in [30] for equations (1.1) and (1.2) imply that the Landesman–Lazer condition (σ_3) is sufficient for the existence of a T-periodic solution, and therefore in this case

$$\tilde{X} = \{f \in X : g(-\infty) < \overline{f} < g(+\infty)\}.$$

Theorem 3.8. *Assume that* (σ_2) *and* (σ_3) *hold. Then the set*

$$\mathfrak{R}_{LL} = \{f \in \tilde{X} : \textit{ every } T\textit{-periodic solution of } (1.1) \textit{ is non-degenerate}\}$$

is open and prevalent in $\tilde{X}$.

Proof. We will show that the assumptions of Lemma 2.1 are satisfied. Following along the same lines of Theorem 3.3, we can obtain that assumptions (C_1) and (C_3) hold. Therefore, we only need to prove that (C_2) holds.

Let x be a T-periodic solution of (1.1). Multiplying (1.1) by $\dot{x}$ and integrating, we get

$$c\int_0^T \dot{x}^2(t)dt = \int_0^T \dot{x}(t)f(t)dt.$$

Using the Hölder inequality, we obtain that $\|\dot{x}\|_2 \le \frac{1}{c}\|f\|_2$. On the other hand, integrating (1.1) shows that there is a $t_0 \in [0, T]$ such that $g(x(t_0)) = \overline{f}$. Since (σ_2), $x(t_0)$ is bounded. Thus we have for all $t \in \mathbb{R}$,

$$|x(t)| = |x(t_0) + \int_{t_0}^t \dot{x}(s)ds| \le |x(t_0)| + \frac{1}{c}\|f\|_2.$$

The condition (C_2) holds with $B = |x(t_0)| + \frac{1}{c}\|f\|_2$. □

Lemma 3.9. *Assume that conditions (σ_2), (σ_3) and (3.9) hold. Then for each $f \in \mathfrak{R}_{LL}$, equation (1.1) has only finite T-periodic solutions $x_1, \dots, x_n$, which satisfy* $\sum_{i=1}^n \mathrm{ind}_T(x_i) = 1$.

Proof. The proof of the first part is the same as that of Lemma 3.4. Consider the following homotopy equation

$$\ddot{x} + c\dot{x} + g(x) = \tau f(t), \quad \tau \in [0, 1]. \tag{3.10}$$

Using the same way in the proof of Theorem 3.8, we can show that there exists a constant $l > 0$, independently of τ, such that all possible T-periodic solutions x_τ of (3.10) satisfies

$$\|x_\tau\| \le l.$$

Let P_τ denote the Poincaré associated to (3.10). Since (3.10) is an autonomous equation, by [2, Theorem 1] we know that the degree $\deg(I - P_0, B_l, 0)$ can be reduced to the Brouwer degree $\deg(g, B_l \cap \mathbb{R}, 0)$. Under the condition (σ_3), we know that

$$\deg(g, B_l \cap \mathbb{R}, 0) = 1.$$

By applying homotopy invariance property, we have that

$$\deg(I - P_1, B_l, 0) = \deg(I - P_0, B_l, 0) = 1.$$

Now the result follows from the additive property of degree theory. □

Theorem 3.10. *Assume that (σ_2), (σ_3) and (3.9) hold. Then for each $f \in \mathfrak{R}_{LL}$, equation (1.1) has at least one asymptotically stable solution.*

Proof. Let $x_1, \ldots, x_n$ be the family of T-periodic solutions of (1.1). From Lemma 3.9, each index $\mathrm{ind}_T(x_i)$ can only take the values 1 or -1 and so at least one of the solutions, say x_1, must satisfy $\mathrm{ind}_T(x_1) = 1$ and be asymptotically stable by Lemma 3.2. □

Theorem 3.11. *Under the conditions of Theorem 3.10, the set $S_{(1.1)} \cup (X \setminus \tilde{X})$ is prevalent in X.*

4. Prevalence of stable periodic solutions of (1.2)

In this section, we will study the prevalence of stable periodic solutions for the conservative equation (1.2). We assume that $f \in X_0$, g is bounded and satisfies (σ_2), (σ_3). We also assume that the following condition holds

(σ_4) $g \in \mathbb{C}^2(\mathbb{R}, \mathbb{R})$ and $g^2(x) + (g'(x))^2 > 0$ for any $x \in \mathbb{R}$. The zeros of $g''(x)$ are isolated and $g''(x) = 0$ if $g(x) = 0$.

For example, it is easy to verify that $g(x) = \arctan x$ satisfies (σ_2), (σ_3) and (σ_4). Under the above conditions, we will prove that the set $S_{(1.2)}$ is prevalent in the space X_0.

4.1. Discriminant of the variational equation

We say that a T-periodic solution x of (1.2) is non-degenerate if $y \equiv 0$ is the unique T-periodic solution of the variational equation

$$\ddot{y} + g'(x(t))y = 0. \tag{4.1}$$

Given $f \in X_0$ and a T-periodic solution x of equation (1.2), we define $D = D[x]$ as the discrimination of the variational equation (4.1). To be precise on the domain of the functional, we introduce the set

$$M = \{x \in C^2(\mathbb{R}/T\mathbb{Z}) : \int_0^T g(x(t))dt = 0\}.$$

The tangent space at $x \in M$ is

$$T_x(M) = \{y \in C^2(\mathbb{R}/T\mathbb{Z}) : \int_0^T g'(x(t))y(t)dt = 0\}.$$

The rigorous definition of the functional is

$$D : M \to \mathbb{R},\ D[x] = \Delta[g'(x)].$$

From the chain rule we deduce that D is C^1 and for each $y \in T_x(M)$,

$$D'[x]y = \Delta'[g'(x)](g''(x))y = \int_0^T \chi_x(t)(g''(x))y(t)dt.$$

Under the condition (σ_4), we know that the zeros of g are also isolated. Then let us take

$$M_* = M \backslash \{x_n : n \in \mathbb{Z}\},$$

where x_n satisfies $g(x_n) = 0$. Of course we also have that $g''(x_n) = 0$. The restriction of the functional will be denoted by $D_* : M_* \to \mathbb{R}$.

Lemma 4.1. *Assume that g satisfies (σ_4). Then all real numbers different from ± 2 are regular values of D_*.*

Proof. Assume that $x \in M_*$ is a critical point of D_*. Then $D'[x] = 0$ and

$$\int_0^T \chi_x(t)(g''(x))y(t)dt = 0$$

for each $y \in T_x(M)$. By the general theory of Hilbert spaces we know that

$$T_x(M)^\perp = V^\perp,$$

where V is the closure of $T_x(M)$ in $L^2(\mathbb{R}/T\mathbb{Z})$. The space V can also be described as the hyperplane orthogonal to the line spanned by $g'(x(t))$,

$$V = L^\perp,\ L = \{\lambda g'(x) : \lambda \in \mathbb{R}\}.$$

Hence the function x is a critical point of D if and only if

$$\chi_x g''(x) \in V^\perp = (L^\perp)^\perp = L.$$

This means that

$$\chi_x(t) g''(x(t)) = \lambda g'(x(t)), \tag{4.2}$$

for some $\lambda \in \mathbb{R}$. Since $x \in M$, there exists an instant τ such that $g(x(\tau)) = 0$. By condition (σ_4), we know that $g''(x(\tau)) = 0$ and therefore

$$\lambda g'(x(\tau)) = 0.$$

Using condition (σ_4) again, $g'(x(\tau)) \neq 0$ and we obtain $\lambda = 0$. Hence

$$\chi_x(t) g''(x(t)) = 0, \quad \text{for every } t.$$

Since the zeros of g'' are isolated, there exists an interval I such that

$$g''(x(t)) \neq 0, \quad \text{for every } t \in I.$$

Thus $\chi_x(t) = 0$ for each $t \in I$. It follows from (2.3) that

$$\dot{\phi}_1(T) = \phi_2(T) = \phi_1(T) - \dot{\phi}_2(T) = 0,$$

which means that the monodromy matrix $\Phi_T = \pm\mathrm{Id}$ and therefore $D[x] = \pm 2$. □

Consider the operator

$$F : M \to X_0,\ \ F[x] = \ddot{x} + g(x),$$

then equation (1.2) with $f \in X_0$ is equivalent to the operator equation

$$F[x] = f.$$

The map F is smooth with derivative

$$F'[x] : T_x(M) \to X_0,\ \ \ F'[x]y = \ddot{y} + g'(x)y.$$

Lemma 4.2. *Given $x \in M$, the derivation $F'[x]$ is an isomorphism if and only if x is a non-degenerate T-periodic solution of (1.2).*

Proof. Suppose that $F'[x]$ is an isomorphism and let y be a T-periodic solution of the linearized equation (4.1). Integrating over a period, we obtain $\int_0^T g'(x(t))y(t)dt = 0$ and so $y \in T_x(M)$. From the linearized equation, $F'[x]y = 0$ and this implies $y = 0$ because Ker $F'[x] = \{0\}$.

Conversely, assume that x is non-degenerate, then the kernel of $F'[x]$ is trivial. To prove that $F'[x]$ is onto, let p be a given function in X_0. By Fredholm's alternative we know that the non-homogeneous equation

$$\ddot{y} + g'(x(t))y = p(t)$$

has a unique T-periodic solution. By integrating the above equality over one period, we have

$$\int_0^T g'(x(t))y(t)dt = 0.$$

Then $y \in T_x(M)$ and $F'[x]y = p$. Thus $F'[x]$ is onto. □

Lemma 4.3. *(See [23].) Let $C = \varphi(L \cup \mathbb{Q})$, where $\varphi(x) = 2\cos(2\pi x)$ and L denotes the set of Liouville numbers. Assume that g is real analytic, x is a T-periodic solution of (1.2) and the discriminant of the linearized equation (4.1) satisfies $|\Delta| < 2$ and $\Delta \notin C$. Then x is stable.*

4.2. Connection between ellipticity and index

Lemma 4.4. *Assume that x is a non-degenerate T-periodic solution of (1.2) such that*

$$\underline{\lambda}_1(g'(x(t))) > 0. \tag{4.3}$$

Then x is elliptic (resp. hyperbolic) if and only if $\mathrm{ind}_T(x) = 1$ (resp. $\mathrm{ind}_T(x) = -1$).

Proof. Denote by ρ_1 and $\rho_2(|\rho_1| \geq |\rho_2|)$ the Floquet multipliers of (4.1). Since x is non-degenerate 1 cannot be a Floquet multiplier. By Lemma 2.4 the multipliers are either conjugate complex numbers or positive real numbers.

If ρ_1 and ρ_2 are a pair of conjugate numbers, then x is elliptic if and only if $\mathrm{ind}_T(x) = \mathrm{sign}\{|1-\rho_1|^2)\} = 1$. If ρ_1 and ρ_2 are positive real numbers, then $0 < \rho_2 < 1 < \rho_1$, and therefore x is hyperbolic if and only if $\mathrm{ind}_T(x) = \mathrm{sign}\{(1-\rho_1)(1-\rho_2)\} = -1$. $\square$

Theorem 4.5. *Assume that* (σ_2), (σ_3) *and* (σ_4) *hold. Suppose further that* g *is real analytic and* $g'(x) < (\frac{\pi}{T})^2$ *for all* $x \in \mathbb{R}$. *Then there exists an open and prevalent set* $\widetilde{G} \subset X_0$ *such that for each* $f \in \widetilde{G}$, *equation* (1.2) *has at least one stable solution.*

Proof. The proof is divided into four steps.
Step 1. *Non-degeneracy*. First we note that if $g'(x) < (\frac{\pi}{T})^2$ for every $x \in \mathbb{R}$, then $\underline{\lambda}_1(g'(x(t))) > 0$. Let the set

$$U = \{f \in X_0 : \text{ every } T\text{-periodic solution of (1.2) is non-degenerate}\}.$$

Now we prove that U is open and prevalent in X_0.

Given $\xi = (\xi_1, \xi_2)^* \in \mathbb{R}^2$ and $f \in X_0$, $x(t; \xi, f)$ is a solution of the initial value problem

$$\ddot{x} + g(x) = f(t), \; x(0) = \xi_1, \; \dot{x}(0) = \xi_2.$$

Let

$$\Phi(t; \xi, f) =: \Phi(t) = \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) \end{pmatrix}$$

be the fundamental matrix solution of

$$\dot{Y} = \begin{pmatrix} 0 & 1 \\ -g'(x(t; \xi, f)) & 0 \end{pmatrix} Y, \; Y(0) = I_2,$$

where $\phi_1(t)$ and $\phi_2(t)$ be real-valued solutions of (4.1) satisfying

$$\phi_1(0) = 1, \; \dot{\phi}_1(0) = 0, \; \phi_2(0) = 0, \; \dot{\phi}_2(0) = 1.$$

Then Φ is continuous by the theorem on continuous dependence.

Consider

$$H : \mathbb{R}^2 \times X \to \mathbb{R}^2, \quad H(\xi, f) = (x(T; \xi, f) - \xi_1, \dot{x}(T; \xi, f) - \xi_2)^*,$$

which is Gâteaux differentiable with partial derivatives $\partial_1 H(\xi, f) \in \mathbb{R}^{2\times 2}$ and $\partial_2 H(\xi, f) \in L(X, \mathbb{R}^2)$ defined by

$$\partial_1 H(\xi, f) = \Phi(T; \xi, f) - I_2, \; \partial_2 H(\xi, f)p = (y(T), \dot{y}(T))^*,$$

where p is an arbitrary function in X_0 and y is the solution of

$$\ddot{y} + g'(x(t;\xi,f))y = p(t),\ \ y(0) = \dot{y}(0) = 0.$$

The formula of variation of constants implies that

$$y(t) = \int_0^T G(t,s;\xi,f)p(s)ds, \tag{4.4}$$

where

$$G(t,s;\xi,f) = \phi_2(t)\phi_1(s) - \phi_2(s)\phi_1(t).$$

Similarly,

$$\dot{y}(t) = \int_0^T \frac{\partial G}{\partial t}(t,s;\xi,f)p(s)ds. \tag{4.5}$$

Hence, by the continuity of Φ and the formulas (4.4) and (4.5) we get the estimate

$$\|\partial_2 H(\xi,f) - \partial_2 H(\hat{\xi},\hat{f})\| \leq$$
$$\int_0^T \{|G(T,s;\xi,f) - G(T,s;\hat{\xi},\hat{f})| + |\frac{\partial G}{\partial t}(T,s;\xi,f) - \frac{\partial G}{\partial t}(T,s;\hat{\xi},\hat{f})|\}ds.$$

Now we prove the continuity of the partial derivatives of H and H is Fréchet differentiable and (C_1) of Lemma 2.1 holds.

Given $(\xi,f) \in Z$ we know that $x(t;\xi,f)$ is a T-periodic solution of (1.2). The mean value theorem and the equation lead to the estimate

$$|\dot{x}(t)| = |\int_\tau^t \ddot{x}(s)ds| \leq |\ddot{x}(\eta)||t-\tau| \leq \frac{3}{2}T(\|g\|_\infty + \|f\|_\infty),\ \ \eta \in [\tau,t]. \tag{4.6}$$

Integrating (1.2) from 0 to T, we have that $\int_0^T g(x(t))dt = \int_0^T f(t)dt = 0$. Since (σ_2), there exists $t_2 \in [0,T]$ such that $x(t_2)$ is bounded and $g(x(t_2)) = 0$. Thus for all $t \in \mathbb{R}$

$$|x(t)| = |x(t_2) + \int_{t_2}^t \dot{x}(s)ds| \leq |x(t_2)| + T\|\dot{x}\|_\infty.$$

This implies that $|\xi|$ is bounded by (4.6). Hence $[(C_2)$, Lemma 2.1$]$ holds.

Following the same lines as in the proof of Theorem 3.3, we can show that the condition (C_3) of Lemma 2.1 is satisfied if we use [23, Lemma 6] with $K = \{0\}$. The proof of non-degeneracy is completed by Lemma 2.1.

Step 2. *Ellipticity*. By the same method in Lemma 3.4, we can verify that there is only a finite number of T-periodic solution of (1.2) if $f \in U$ and denote $x_1,\ldots,x_n$ are the family of T-periodic solutions of (1.2). An important global property of these index is that

$$\sum_{i=1}^{n} \mathrm{ind}_T(x_i) = 1.$$

The proof is the same as that of Lemma 3.9. Thus each index $\mathrm{ind}_T(x_i)$ can only take the values 1 or -1 and so at least one of the solutions, say x_1, must satisfy $\mathrm{ind}_T(x_1) = 1$ and be elliptic by Lemma 4.4.

Step 3. *Prevalence*. Let $V = U\backslash\{0\}$. Given $f \in V$, we can select $x \in M$ such that $F[x] = f$ and the linearized equation at $x(t)$ is elliptic. Then $x \in M_* = M\backslash\{x_n : n \in \mathbb{Z}\}$ since $f \neq F[x_n]$.

From Lemma 4.2, we know $F'[x]$ is an isomorphism because the solution x is non-degenerate. Applying the inverse function theorem, we can find open sets $M_f \subset M_*$, $V_f \subset V$ with $x \in V_f$, $f \in M_f$, and such that the restriction $F : M_f \to V_f$ is a diffeomorphism. After restricting the size of V_f we can assume that the functional

$$d_f : V_f \to \mathbb{R},\ d_f(\widetilde{f}) = D[F^{-1}(\widetilde{f})]$$

is smooth and take values in the interval $(-2, 2)$. From Lemma 4.1 we deduce that $d'_f(\widetilde{f}) \neq 0$ for each $\widetilde{f} \in V_f$.

Define $\mathbb{E} = X_0$, $G = V$. Hence Lemma 2.2 shows that

$$\widetilde{G} = \bigcup_{f \in V} d_f^{-1}(\mathbb{R}\backslash C)$$

is prevalent in X_0.

Step 4. *Stability*. For each $\widetilde{f} \in \widetilde{G}$, we claim that the equation

$$\ddot{x} + g(x) = \widetilde{f}(t)$$

has a stable T-periodic solution. By the definition of $\widetilde{G}$ there exists a T-periodic solution x such that the discriminant of the linearized equation is given by $\Delta = d_f(\widetilde{f})$, for some $f \in V$ with $\widetilde{f} \in V_f$. Moreover, $|\Delta| < 2$ and $\Delta \notin C$. By using now Lemma 4.3, x is stable. So we have completed the proof of Theorem 4.5. $\square$

The following result follows directly from Theorem 4.5.

Corollary 4.6. *Under the conditions of Theorem 4.5, the set $S_{(1.2)}$ is prevalent in X_0.*

Acknowledgments

We would like to thank Professor Rafael Ortega for his careful reading of the manuscript and valuable suggestions, especially bringing our attention to the References [18,19,25] and helping us to introduce the condition (σ_4). We would like to show our thanks to the anonymous referee for his/her valuable suggestions and comments.

References

[1] J.M. Alonso, Nonexistence of periodic solutions for a damped pendulum equation, Differential Integral Equations 10 (1997) 1141–1148.

[2] A. Capietto, J. Mawhin, F. Zanolin, Continuation theorems for periodic perturbations of autonomous systems, Trans. Amer. Math. Soc. 329 (1992) 41–72.
[3] H. Chen, Y. Li, Rate of decay of stable periodic solutions of Duffing equations, J. Differential Equations 236 (2007) 493–503.
[4] H. Chen, Y. Li, Existence, uniqueness, and stability of periodic solutions of an equation of Duffing type, Discrete Contin. Dyn. Syst. 18 (2007) 793–807.
[5] H. Chen, Y. Li, Bifurcation and stability of periodic solutions of Duffing equations, Nonlinearity 21 (2008) 2485–2503.
[6] J. Chu, J. Lei, M. Zhang, The stability of the equilibrium of a nonlinear planar system and application to the relativistic oscillator, J. Differential Equations 247 (2009) 530–542.
[7] J. Chu, J. Lei, M. Zhang, Lyapunov stability for conservative systems with lower degrees of freedom, Discrete Contin. Dyn. Syst. Ser. B 16 (2011) 423–443.
[8] J. Chu, P.J. Torres, F. Wang, Radial stability of periodic solutions of the Gylden–Meshcherskii-type problem, Discrete Contin. Dyn. Syst. 35 (2015) 1921–1932.
[9] J. Chu, M. Zhang, Rotation number and Lyapunov stability of elliptic periodic solutions, Discrete Contin. Dyn. Syst. 21 (2008) 1071–1094.
[10] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, New York, 1955.
[11] V.Y. Kaloshin, Prevalence in spaces of finitely smooth mappings, Funct. Anal. Appl. 31 (1997) 95–99.
[12] M. Krasnoselskii, P.P. Zabreiko, Geometrical Methods of Nonlinear Analysis, Springer-Verlag, Berlin, 1984.
[13] E.M. Landesman, A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970) 609–623.
[14] A.C. Lazer, On Schauder's fixed point theorem and forced second-order nonlinear oscillations, J. Math. Anal. Appl. 21 (1968) 421–425.
[15] J. Lei, X. Li, P. Yan, M. Zhang, Twist character of the least amplitude periodic solution of the forced pendulum, SIAM J. Math. Anal. 35 (2003) 844–867.
[16] W. Magnus, S. Winkler, Hill's Equations, Wiley, New York, 1966.
[17] J. Mawhin, Periodic oscillations of forced pendulum-like equations, in: Ordinary and Partial Differential Equations, in: Lecture Notes in Math., vol. 964, Springer, Berlin–New York, 1982, pp. 458–476.
[18] R. Ortega, A counterexample for the damped pendulum equation, Acad. Roy. Belg. Bull. Cl. Sci. (6) LXXIII (1987) 405–409.
[19] R. Ortega, Stability and index of periodic solutions of an equation of Duffing equation, Boll. Unione Mat. Ital. 3-B (1989) 533–546.
[20] R. Ortega, Some applications of the topological degree to stability theory, in: Topological Methods in Differential Equations and Inclusions, Montreal, PQ, 1994, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 472, Kluwer Acad. Publ., Dordrecht, 1995, pp. 377–409.
[21] R. Ortega, Periodic solution of a Newtonian equation: stability by the third approximation, J. Differential Equations 128 (1996) 491–518.
[22] R. Ortega, Prevalence of non-degenerate periodic solutions in the forced pendulum equation, Adv. Nonlinear Stud. 13 (2013) 219–229.
[23] R. Ortega, Stable periodic solutions in the forced pendulum equation, Regul. Chaotic Dyn. 18 (2013) 585–599.
[24] R. Ortega, A forced pendulum equation without stable periodic solutions of a fixed period, Port. Math. 71 (2014) 193–216.
[25] R. Ortega, E. Serra, M. Tarallo, Non-continuation of the periodic oscillations of a forced pendulum in the presence of friction, Proc. Amer. Math. Soc. 128 (2000) 2659–2665.
[26] W. Ott, J.A. Yorke, Prevalence, Bull. Amer. Math. Soc. 42 (2005) 263–290.
[27] C. Shannon, A prevalent transversality theorem for Lipschitz functions, Proc. Amer. Math. Soc. 134 (2006) 2755–2765.
[28] N. Soave, G. Verzini, Bounded solutions for a forced bounded oscillator without friction, J. Differential Equations 256 (2014) 2526–2558.
[29] P.J. Torres, Existence and stability of periodic solutions of a Duffing equation by using a new maximum principle, Mediterr. J. Math. 1 (2004) 479–486.
[30] J. Ward, Periodic solutions for a class of ordinary differential equations, Proc. Amer. Math. Soc. 78 (1980) 350–352.
[31] M. Zhang, The best bound on the rotations in the stability of periodic solutions of a Newtonian equation, J. Lond. Math. Soc. 67 (2003) 137–148.