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Regularity results for nonlinear parabolic obstacle problems with subquadratic growth

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Abstract

In [26] it was shown that the spatial gradient of the solution u to the parabolic obstacle problem with superquadratic growth is local Hölder continuous provided the obstacle is regular enough. In this paper, we extend this regularity result to the subquadratic case. This means we establish the local Hölder continuity of the spatial gradient of the solution u to the parabolic obstacle problem with subquadratic growth. More precisely, we prove that

$$Du \in C^{0;\alpha,\frac{\alpha}{2}}_{\mathrm{loc}} \quad \text{for some } \alpha \in (0,1),$$

provided the coefficients and the obstacle are regular enough. Moreover, we use the local Hölder continuity to prove the local Lipschitz continuity of the solution u, i.e.

$$u \in C^{0;1,\frac{1}{2}}_{\mathrm{loc}}.$$

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1. Introduction

The aim of this paper is to establish the local Hölder continuity of the spatial gradient of the solution u to the parabolic obstacle problem in the subquadratic case, i.e. the growth exponent satisfies $\frac{2n}{n+2} < p < 2$ with $n \geq 2$. Here, we will extend the result shown in [26] for the case $p \geq 2$ and thereby, we will complete the theory of the local Hölder continuity of the spatial gradient of the solution u to the parabolic obstacle problem. This result we will also use to prove that u is local Lipschitz continuous.

In general, obstacle problems are interesting objects in the theory of partial differential equations and the calculus of variations. The obstacle problem is a classic motivating example in the mathematical study of variational inequalities and free boundary problems. The theory of obstacle problems is also motivated by numerous applications, e.g. in mathematical physics, in mechanics, in control theory or in mathematical biology. We refer to [1,33] for an overview of the classical theory and applications. Up to now, the theory for elliptic problems is well understood, as well as the theory for elliptic obstacle problems. However, the case of non-linear parabolic problems with general obstacle functions remained open for a long time. First results were achieved by Bögelein, Duzaar and Mingione [5] and then, by Scheven [41]. Here, we want to highlight that in [5] the authors established the first existence result to parabolic problems with irregular obstacles, which are not necessarily non-increasing in time. They consider general obstacles with the only additional assumption that the time derivative of the obstacle lies in $L^{p'}$. This is required since their method relies on a time mollification argument, combined with a maximum construction in order to recover the obstacle condition, where the pointwise maximum construction is not compatible with distributional time derivatives. Moreover, they established the Calderón–Zygmund theory for a large class of parabolic obstacle problems, i.e. they proved that the (spatial) gradient of solutions is as integrable as that of the assigned obstacles. Then, in [41] Scheven introduced a new concept of solution to parabolic obstacle problems of p-Laplacian type with highly irregular obstacles, the so-called *localizable solutions*, see Definition 1. The main feature of localizable solutions is that they solve the obstacle problem locally, which is a priori not clear by the formulation of the problem, cf. the remarks preceding Definition 1. This new concept allows to consider more general settings, i.e. it is no more necessary to assume that the time derivative of the obstacle function lies in $L^{p'}$. It suffices to consider obstacles with distributional time derivatives. Moreover, we want to emphasize that the concept of localizable solutions allows to prove more general regularity results. Scheven also proved Calderón–Zygmund estimates for parabolic obstacle problems. The main difference between the result of Scheven and the result of Bögelein, Duzaar and Mingione is that in [5] they need an additional assumption on the boundary data, which seems to be unnatural for proving regularity in the interior. The reason for the additional assumption on the boundary data arises from the fact that the formulation of the obstacle problem is not of local nature. Bögelein, Duzaar and Mingione used a complex approximation argument to approximate the solutions by more regular ones and since the given solution was not known to be localizable, this approximation procedure had to be implemented on the whole domain. This problem could be avoid by the concept of localizable solutions.

These existence results enable also many regularity results for parabolic problems with irregular obstacle, see e.g. [2,8,6,26]. In the context of higher integrability of solutions, we have to mention a further result, which is given by Bögelein and Scheven in [6]. They proved the self-improving property of integrability for parabolic obstacle problems without any monotonicity assumption in time on the obstacle function. Thus, the higher integrability of the spatial gradient of solutions is by now well known, see also [7–9]. Moreover, we want to highlight that the theory of localizable solutions also permits to establish the existence theory to parabolic obstacle problems with nonstandard $p(x,t)$-growth in [24]. Furthermore, the concept of localizable solutions permits to prove the regularity of these solutions, i.e. the Calderón–Zygmund theory and the higher integrability of solutions, see [23,25,27].

The novelty of this paper will be the local Hölder continuity of the spatial gradient of solutions to the parabolic obstacle problem with subquadratic growth, provided the coefficients and the obstacle are regular enough. For this aim, we will use the concept of localizable solutions to derive (without any approximation arguments and additional assumptions) a as general as possible local Hölder continuity result for the spatial gradient of the solution. In [26] we established the local Hölder continuity of the spatial gradient of solutions to the parabolic obstacle problem with superquadratic growth. This paper closes the gap, such that the local Hölder continuity is valid for every $\frac{2n}{n+2} < p < \infty$. Here, we want to highlight that we use the same approach as in [26] but the proof is often more difficult as in [26], since there arise factors which has to treat very carefully to prevent that our needed Hölder estimates blow up. Furthermore, we are able to prove the local $C^{0;1,\frac{1}{2}}$-regularity of the localizable solution to the parabolic obstacle problem.

The history of $C^{1,\alpha}$-regularity starts in the elliptic setting with the fundamental regularity result of De Giorgi [15] and Nash [39] for solutions of linear elliptic equations. Later on, the result of De Giorgi and Nash was generalized amongst others by Ladyzhenskaya and Uraltseva [35]. Further elliptic problems mainly with subquadratic growth are discussed in [10]. Moreover, the Hölder regularity of solutions to elliptic obstacle problems has been proved by Choe [11], as well as the Hölder continuity of solutions to elliptic obstacle problems with $p(x)$-growth is given by Eleuteri and Habermann [21,22], see also [40]. Furthermore, the everywhere $C^{1,\alpha}$-regularity to parabolic problems goes back to fundamental achievement of DiBendetto and Friedman in [17,18], see also [16]. Moreover, DiBenedetto, Gianazza and Vespri proved the Hölder continuity of solutions to quasi-linear parabolic equations via Harnack's inequality [19,20]. In the context of Hölder regularity for p-Laplacian systems we should also refer [32,37,38]. Finally, we want to call attention to the results of Choe in [12] and Struwe and Vivaldi [43]. Choe proved the Hölder regularity of the spatial gradient of solutions to parabolic obstacle problems in the case $p = 2$, while Struwe and Vivaldi proved the Hölder continuity of bounded weak solutions of quasi-linear parabolic variational inequalities in the case $p = 2$. Note also that Kuusi, Mingione and Nyström proved the $C^{0,\alpha}$-regularity to parabolic obstacle problems in the case $p \geq 2$, see [34].

In the following, $\Omega \subset \mathbb{R}^n$ denotes a bounded domain of dimension $n \geq 2$ and we write $\Omega_T := \Omega \times (0,T)$ for the space–time cylinder over Ω with height $T > 0$. Moreover, u_t respectively $\partial_t u$ denotes the partial derivate with respect to time and Du denotes the one with respect to the spatial variable. The aim of this work is to show that the spatial gradient of solutions to the parabolic obstacle problem are locally Hölder continuous in the interior of Ω_T, i.e. there exists a Hölder exponent $\alpha \in (0,1)$, such that

$$Du \in C^{0;\alpha,\frac{\alpha}{2}}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n).$$

This means that the spatial gradient of the solution admits a representative being α-Hölder continuous with respect to the spatial variables and $\frac{\alpha}{2}$-Hölder continuous with respect to the time.

1.1. General assumptions

First of all, we introduce the data on which our parabolic obstacle problem depends. For the **boundary data** we shall assume, that

$$g \in C^0([0,T];L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)) \quad \text{and} \quad \partial_t g \in L^{p'}(0,T;W^{-1,p'}(\Omega)), \tag{1}$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate to p. Furthermore, we consider an **obstacle function** $\psi : \Omega_T \to \mathbb{R}$

$$\psi \in C^0([0,T];L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)) \quad \text{and} \quad \partial_t \psi \in L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{2}$$

Moreover, we assume that the boundary data satisfy the following condition

$$g \geq \psi \text{ a.e. on } \partial\Omega \times (0,T) \quad \text{and} \quad u_0 \geq \psi \text{ a.e. on } \Omega, \tag{3}$$

where the initial values $u_0 = u(\cdot, 0) \in L^2(\Omega)$ are given and the first inequality is to be understood in the $W^{1,p}$-sense, i.e. $(\psi - g)_+ \in L^p(0,T;W_0^{1,p}(\Omega))$. The **inhomogeneities** of the variational inequality will be determined by functions

$$F \in L^p(\Omega_T, \mathbb{R}^n) \quad \text{and} \quad f \in L^{p'}(\Omega_T). \tag{4}$$

Next, we define the **function space** for the solution to the obstacle problem

$$\mathcal{K}_{\psi,g}(\Omega_T) := \left\{ u \in C^0([0,T];L^2(\Omega)) \cap [g + L^p(0,T;W_0^{1,p}(\Omega))] : u \geq \psi \text{ a.e. on } \Omega_T \right\}.$$

Notice that $\mathcal{K}_{\psi,g}(\Omega_T)$ is non-empty due to the **compatibility condition** $g \geq \psi$ a.e. on the lateral boundary $\partial\Omega \times (0,T)$ from (3). Moreover, we introduce a function $a : \Omega_T \to \mathbb{R}$ satisfying

$$\nu \leq a(z) \leq L \quad \text{and} \quad |a(z_1) - a(z_2)| \leq \omega(d_{\mathcal{P}}(z_1, z_2)) \tag{5}$$

for every $z, z_1, z_2 \in \Omega_T$ and for some structure constants $0 < \nu \leq 1 \leq L$. The **parabolic distance** between two points $z_1 = (x_1, t_1)$ and $z_2 = (x_2, t_2)$ in $\mathbb{R}^{n+1}$ is given by $d_{\mathcal{P}}(z_1, z_2) := \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}$. Finally, we assume that the **modulus of continuity** ω is a Hölder-modulus, i.e. $\omega : [0,\infty) \to [0,1]$ is a concave, non-decreasing function with $\lim_{\rho \downarrow 0} \omega(\rho) = 0 = \omega(0)$ and satisfies

$$\omega(\rho) \leq L\rho^\sigma, \quad \text{for some } \sigma \in (0,1] \text{ and any } \rho \in (0,1]. \tag{6}$$

1.1.1. The parabolic obstacle problem

Now, we are in the situation to specify our parabolic obstacle problem, i.e. we consider the **weak formulation of the variational inequality**

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$$\int_0^T \langle \partial_t w, w-u\rangle_\Omega \,\mathrm{d}t + \int_{\Omega_T} a(z)|Du|^{p-2}Du\cdot D(w-u)\,\mathrm{d}z + \frac{1}{2}\|w(\cdot,0)-u_0\|^2_{L^2(\Omega)} \geq \int_{\Omega_T} |F|^{p-2}F\cdot D(w-u) + f(w-u)\,\mathrm{d}z. \tag{7}$$

Here, $u \in \mathcal{K}_{\psi,g}(\Omega_T)$ denotes a solution, which solves the variational inequality (7) for all testing functions

$$w \in \mathcal{K}'_{\psi,g}(\Omega_T) := \left\{ w \in \mathcal{K}_{\psi,g}(\Omega_T) : \partial_t w \in L^{p'}(0,T; W^{-1,p'}(\Omega)) \right\}$$

and $\langle \cdot,\cdot\rangle_\Omega$ denotes the pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

1.1.2. The concept of localizable solutions

The concept of localizable solutions goes back to Scheven [41] and the idea is the following: In general, we consider solutions that might not necessarily satisfy $\partial_t u \in L^{p'}(0,T; W^{-1,p'}(\Omega))$. In this case, the weak formulation (7) does not seem to be the most suitable notion of solution since it is not of local nature. More precisely, for a given parabolic cylinder $\mathcal{O}_I = \mathcal{O}\times(t_1,t_2)$, it is not a priori clear that the restriction $u\big|_{\mathcal{O}_I}$ of a solution u to (7) again satisfies a variational inequality on $\mathcal{O}_I$. Even more, it is not clear that if the space $\mathcal{K}'_{\psi,u}(\mathcal{O}_I)$ of admissible comparison maps is not empty. In fact, it is not evident from the formulation (7) that there exists any map that agrees with u on the lateral boundary of $\mathcal{O}_I$ and at the same time possesses a time derivative in the distributional space $L^{p'}(t_1,t_2; W^{-1,p'}(\mathcal{O}))$, which would be necessary for the construction of suitable comparison maps. These considerations motivate the following new concept of a weak solution to a parabolic obstacle problem.

Definition 1. We say that $u \in \mathcal{K}_{\psi,g}(\Omega_T)$ is a **localizable solution** of the weak formulation (7) of the obstacle problem if for every parabolic cylinder $\mathcal{O}_I = \mathcal{O}\times I \subset \Omega_T$, where $\mathcal{O} = \tilde{\mathcal{O}}\cap\Omega$ with a Lipschitz regular domain $\tilde{\mathcal{O}} \subset \mathbb{R}^n$ and time interval $I = (t_1,t_2) \subset (0,T)$, the following two conditions hold.

i) The map u satisfies the extension property, i.e. there holds $\mathcal{K}'_{\psi,u}(\mathcal{O}_I) \neq \emptyset$
ii) for all comparison maps $w \in \mathcal{K}'_{\psi,u}(\mathcal{O}_I)$, there holds

$$\int_{t_1}^{t_2} \langle \partial_t w, w-u\rangle_{\mathcal{O}} \,\mathrm{d}t + \int_{\mathcal{O}_I} a(z)|Du|^{p-2}Du\cdot D(w-u)\,\mathrm{d}z + \frac{1}{2}\|w(\cdot,t_1)-u_0\|^2_{L^2(\mathcal{O})} \geq \int_{\mathcal{O}_I} |F|^{p-2}F\cdot D(w-u) + f(w-u)\,\mathrm{d}z.$$

Note that the existence of localizable solutions to the variational inequality (7) was proved in [41].

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1.1.3. Parabolic Hölder space

Here, we introduce the parabolic Hölder spaces. Therefore, we define for two points $z_1 = (x_1, t_1)$ and $z_2 = (x_2, t_2)$ in $\mathbb{R}^{n+1}$ the following $\varrho(z_1, z_2) := |x_1 - x_2| + \sqrt{|t_1 - t_2|}$. Note that $d_{\mathcal{P}}(z_1, z_2) \le \varrho(z_1, z_2)$. Moreover, we define the σ**th-parabolic Hölder seminorm** of $f : \Omega_T \to \mathbb{R}$ by

$$[f]_{C^{0;\sigma,\frac{\sigma}{2}}(\Omega_T)} := \sup_{z_1, z_2 \in \Omega_T, z_1 \neq z_2} \frac{|f(z_1) - f(z_2)|}{\varrho(z_1, z_2)^{\sigma}},$$

where $\sigma \in (0, 1)$ denotes the Hölder exponent. Then, $f \in C^{0;\sigma,\frac{\sigma}{2}}(\Omega_T)$ if the norm $\|f\|_{C^{0;\sigma,\frac{\sigma}{2}}(\Omega_T)} := \|f\|_{C^0(\Omega_T)} + [f]_{C^{0;\sigma,\frac{\sigma}{2}}(\Omega_T)}$ is finite. By $C^{k;\sigma,\frac{\sigma}{2}}(\Omega_T)$, we denote the space of all function, which satisfy

$$\|f\|_{C^{k;\sigma,\frac{\sigma}{2}}(\Omega_T)} := \sum_{|\beta| \le k} \|D^{\beta} f\|_{C^0(\Omega_T)} + \sum_{|\beta| = k} [D^{\beta} f]_{C^{0,\sigma,\frac{\sigma}{2}}(\Omega_T)} < \infty.$$

1.1.4. Parabolic Morrey space

Finally, we define the parabolic Morrey space.

Definition 2. With $q \ge 1$, $\theta \in [0, n+2]$ and $Q \subset \mathbb{R}^{n+1}$ being a bounded open set, a measurable map $v : Q \to \mathbb{R}^k$, $k \ge 1$ belongs to the parabolic Morrey space $L^{q,\theta}(Q, \mathbb{R}^k)$ if and only if

$$\|v\|^q_{L^{q,\theta}(Q,\mathbb{R}^k)} := \sup_{z_0 \in Q,\ 0 < \rho < \mathrm{diam}(Q)} \rho^{\theta - (n+2)} \int_{Q \cap Q_\rho(z_0)} |v|^q \, \mathrm{d}z < \infty,$$

where $Q_\rho(z_0)$ denotes the symmetric parabolic cylinder $B_\rho(x_0) \times (t_0 - \rho^2, t_0 + \rho^2)$.

1.2. Statement of the result

The main result reads as follows.

Theorem 1. *Let $\frac{2n}{n+2} < p < 2$, $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\sigma \in (0, 1)$. Suppose that the assumptions (1)–(6) are valid and that $u \in \mathcal{K}_{\psi,g}(\Omega_T)$ is a localizable solution – see Definition 1 – of the variational inequality (7). Moreover, let $Q_{\mathfrak{R}}(\mathfrak{z}_o) \subseteq \Omega_T$ for some $\mathfrak{R} > 0$ and assume that*

$$F \in C^{0;\sigma,\frac{\sigma}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o)),\ \psi \in C^{1;\sigma,\frac{\sigma}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o)) \text{ and } \partial_t \psi,\ f \in L^{p',(1-\sigma)p'}(Q_{\mathfrak{R}}(\mathfrak{z}_o)). \tag{8}$$

Then, there exists a Hölder exponent $\alpha = \alpha(n, p, \nu, L, \sigma) \in (0, \sigma)$, such that

$$Du \in C^{0;\alpha,\frac{\alpha}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o), \mathbb{R}^n).$$

Remark 1. The reason for considering localizable solution is on the one hand based on the fact that there is a more general existence result available as mentioned in the introduction and in [41]. On the other hand we have to establish local estimates and to this aim we have to compare the solution of (7) with weak solution of a certain boundary value problem. Therefore we have to

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ensure that there is a admissible comparison function available. If we consider a solution of (7) as in [5], which was not known to be localizable, we have to use the approximation procedure from [5] on the whole domain Ω_T, which requires to impose unnatural conditions on the regularity of the boundary data for proving regularity in the interior, please cf. again [41]. Hence, we use the concept of localizable solutions to prove our main result.

Moreover, we should to mention how the Hölder exponents arise in this context. The main aim of the proof of Theorem 1 (similar to the degenerate case as in [26]) is to state also in the singular case a (scaling invariant) Campanato type estimate of the form

$$\fint_{Q_\rho(z_0)} |Du - (Du)_{Q_\rho(z_0)}| \,\mathrm{d}z \le c\rho^\alpha$$

on a standard symmetric parabolic cylinders $Q_\rho(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0 + \rho^2)$ with center in $z_0 = (x_0, t_0) \in \Omega_T$. Now, the result of Da Prato in [14] is available and from this parabolic version of Campanato's integral characterization of continuous functions, we can conclude that Du is Hölder continuous with respect to the standard parabolic metric locally on Ω_T, i.e. the spatial gradient of the solution admits a representative being α-Hölder continuous with respect to the spatial variables and $\frac{\alpha}{2}$-Hölder continuous with respect to time, please see the end of the proof of Theorem 1 and also e.g. [4,13,44].

Furthermore, the question on global Hölder regularity in this context is to the knowledge of the author still an open problem, but a very interesting project for the future. □

Moreover, from Theorem 1 it follows that $u \in C^{0;1,\frac{1}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_\circ), \mathbb{R})$.

Theorem 2. *Under the assumption of Theorem 1 the localizable solution u of the variational inequality (7) is locally Lipschitz continuous, i.e.*

$$u \in C^{0;1,\frac{1}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_\circ), \mathbb{R}).$$

Plan of the paper. Now, we briefly describe the strategy of the proof to our main result and the technical novelties of the paper. We start with certain comparison arguments, such that we deduce a comparison estimate between the localizable solution u of the variational inequality and the solution v_0 of some associated Cauchy–Dirichlet problem. Then, for such parabolic p-Laplacian equations the regularity results of DiBenedetto and Friedman [16–18] are available, yielding that the spatial gradient of the comparison function v_0 is Hölder continuous. Moreover, we will use the fact that the solution u can be compared to the solution of a Cauchy–Dirichlet problem and the a priori estimates of DiBenedetto for v_0 should in principle be transferable to the solution u itself. Therefore, we will utilize Lemma 7, from which we gain a Lipschitz bound and an excess decay estimate for the comparison function v_0. Further, we will derive from the comparison estimate between u and v_0, a Lipschitz bound and an excess decay estimate of v_0, both from Lemma 7 and an excess decay estimate for the solution u on the intrinsic cylinder $Q_\rho^{(\lambda)}(z_0)$. Finally, we will derive a Campanato type estimate for the spatial gradient Du, leading to the desired $C^{1,\alpha}$-regularity. Moreover, we will establish a Poincaré type estimate for localizable solutions to proof the Lipschitz regularity of the localizable solution u.

2. Preliminaries and notations

2.1. Intrinsic geometry

First, we introduce symmetric parabolic cylinders with center in $z_0 = (x_0, t_0) \in \Omega_T$ of the form

$$Q_\rho(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0 + \rho^2),$$

where $(t_0 - \rho^2, t_0 + \rho^2) \subset (0, T)$ and $B_\rho(x_0) \subset \Omega$ denotes a ball with radius $\rho > 0$ and center x_0. To obtain the relevant (scaling invariant) local estimates we will use, in order to re-balance the non-homogeneity of parabolic problems, certain scaled cylinders, the so-called **intrinsic cylinders** of the form

$$Q_\rho^{(\lambda)}(z_0) := B_\rho(x_0) \times \Lambda_\rho^{(\lambda)}(t_0), \text{ where } \Lambda_\rho^{(\lambda)}(t_0) := \left(t_0 - \lambda^{2-p}\rho^2, t_0 + \lambda^{2-p}\rho^2\right),$$

where $\lambda > 0$. The reason for such scaled cylinder is based on the fact (explained by the easiest problem), that a multiple $c \cdot u$ of a solution to $\partial_t u - \operatorname{div}(|Du|^{p-2}Du) = 0$ is no longer a solution, except $c \in \{0, 1\}$, $p = 2$ or $u \equiv 0$. Such kind of intrinsic cylinders were introduced in the pioneering work of DiBenedetto and Friedman [18]. The delicate aspect in this technique relies in the fact that the cylinders will be constructed in such a way, that the scaling parameter $\lambda > 0$ and the average of $|Du|^p$ over $Q_\rho^{(\lambda)}(z_0)$ are coupled in the following way:

$$⨍_{Q_\rho^{(\lambda)}(z_0)} |Du|^p \, dz \approx \lambda^p.$$

On such intrinsic cylinders, i.e. when $|Du|$ is comparable to λ in the above sense, the parabolic p-Laplacian equation $\partial_t u - \operatorname{div}(|Du|^{p-2}Du) = 0$ behaves in a certain sense like $\partial_t u - \lambda^{p-2}\Delta u = 0$. Therefore, using intrinsic cylinders of the type $Q_\rho^{(\lambda)}(z_0)$ we can re-balance the occurring multiplicative factor λ^{p-2}, which has the same effect as re-scaling u in time by a factor λ^{2-p}.

2.2. Technical tools

Before we are able to prove the comparison estimates, we have to mention some useful auxiliary material. The following classical lemma will be a useful tool to treat the time-part of our variational inequality. This lemma is stated e.g. in [42, Chapter III, Proposition 1.2].

Lemma 3. *Let $p > \frac{2n}{n+2}$. Then,*

$$W_p(0, T) := \left\{ v \in L^p(0, T; W_0^{1,p}(\Omega)) : v_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) \right\}$$

is contained in $C^0([0, T]; L^2(\Omega))$. Moreover, if $u \in W_p(0, T)$ then $t \mapsto \|u(\cdot, t)\|^2_{L^2(\Omega)}$ is absolutely continuous on $[0, T]$,

$$\frac{d}{dt}\int_\Omega |u(\cdot,t)|^2\,\mathrm{d}x = 2\,\langle u_t(\cdot,t), u(\cdot,t)\rangle_\Omega\,, \quad \textit{for a.e. } t\in[0,T],$$

and there is a constant c for which $\|u\|_{C^0([0,T];L^2(\Omega))} \le c\|u\|_{W_p(0,T)}$ holds for every $u \in W_p(0,T)$.

Here, we cite a useful tool when dealing with p-growth problems. The first lemma we will need, is stated by Hamburger [30, Lemma 2.1] in the following version.

Lemma 4. *There exists a positive constant, depending on $\mathfrak{p} > -1$, such that for all $A, B \in \mathbb{R}^k$ with $A \ne B$, we have*

$$\begin{aligned}\frac{1}{c}(\mu^2+|A|^2+|B|^2)^{\frac{\mathfrak{p}}{2}}|A-B| &\le |(\mu^2+|A|^2)^{\frac{\mathfrak{p}}{2}}A-(\mu^2+|B|^2)^{\frac{\mathfrak{p}}{2}}B| \\ &\le c(\mu^2+|A|^2+|B|^2)^{\frac{\mathfrak{p}}{2}}|A-B|\end{aligned}$$

with $\mu \ge 0$.

This lemma was established for the case $\mathfrak{p} \ge 0$ in [29, Lemma 2.2] and in the case $0 > \mathfrak{p} > -1$ in [30, Lemma 2.1]. Since $p > \frac{2n}{n+2}$, we are able to choose $\mathfrak{p} = p-2 > -1$. Moreover, we choose $\mu = 0$ and $k = n \ge 2$. This allows us to infer from Lemma 4 (cf. [28, Lemma 2.2] in the case $p > 2$ and [31, Lemma 2] in the case $1 < p < 2$) the next lemma.

Lemma 5. *There exists a constant $c := c(n,p)$, such that for any $A, B \in \mathbb{R}^n$, there holds*

$$(|A|^2+|B|^2)^{\frac{p-2}{2}}|A-B|^2 \le c\left(|A|^{p-2}A-|B|^{p-2}B\right)\cdot(A-B), \quad \textit{where } A \ne B.$$

Furthermore, we conclude from Lemma 4 that the assumption (8) implies that $|F|^{p-2}F$ and $|D\psi|^{p-2}D\psi$ are Hölder continuous on $Q_{\mathfrak{R}}(\mathfrak{z}_o)$ with Hölder exponent $\sigma(p-1)$. Indeed, we have

$$\begin{aligned}&|\,|F(z_1)|^{p-2}F(z_1)-|F(z_2)|^{p-2}F(z_2)\,| \\ &\le c(|F(z_1)|^2+|F(z_2)|^2)^{\frac{p-2}{2}}|F(z_1)-F(z_2)| \\ &\le c(|F(z_1)|+|F(z_2)|)^{p-2}(|F(z_1)|+|F(z_2)|)^{2-p}|F(z_1)-F(z_2)|^{p-1} \\ &\le c(p)\|F\|^{p-1}_{C^{0;\sigma,\frac{\sigma}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o))}\varrho(z_1,z_2)^{\sigma(p-1)}\end{aligned}$$

for all $z_1, z_2 \in Q_{\mathfrak{R}}(\mathfrak{z}_o)$ with $z_1 \ne z_2$. Thus, we have

$$|F|^{p-2}F,\ |D\psi|^{p-2}D\psi \in C^{0;\sigma_1,\frac{\sigma_1}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o)) \tag{9}$$

$\sigma_1 := \sigma(p-1)$ with and

$$\Big||F(z_1)|^{p-2}F(z_1)-|F(z_2)|^{p-2}F(z_2)\Big|\le c(p)\|F\|^{p-1}_{C^{0;\sigma_1,\frac{\sigma_1}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o))}\varrho(z_1,z_2)^{\sigma_1},$$

$$\Big||D\psi(z_1)|^{p-2}D\psi(z_1)-|D\psi(z_2)|^{p-2}D\psi(z_2)\Big|\le c(p)\|D\psi\|^{p-1}_{C^{0;\sigma_1,\frac{\sigma_1}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o))}\varrho(z_1,z_2)^{\sigma_1} \tag{10}$$

for all $z_1,z_2\in Q_{\mathfrak{R}}(\mathfrak{z}_o)$ with $z_1\neq z_2$.

Comparison principle. Moreover, we will need the following comparison lemma, which can be found in [5].

Lemma 6. *Suppose that $\psi,w\in C^0([0,T];L^2(\Omega))\cap L^p(0,T;W^{1,p}(\Omega))$ satisfy (in the weak sense)*

$$\begin{cases}\partial_t\psi-\operatorname{div}(a(z)|D\psi|^{p-2}D\psi)\ \le\ \partial_t w-\operatorname{div}(a(z)|Dw|^{p-2}Dw) & \text{in } \Omega_T,\\ \qquad\qquad\qquad\qquad\psi\ \le\ w & \text{on } \partial_{\mathcal{P}}\Omega_T,\end{cases}$$

where (5) is valid. Then, there holds $\psi\le w$ a.e. on Ω_T.

2.3. A priori estimate

A final tool we need, is an a priori estimate for solutions to the parabolic standard p-Laplacian equation. This result will be useful, since in the next section we will compare several times, such that we derive a comparison estimate between the solutions of our obstacle problem and the solution to the parabolic standard p-Laplacian equation. Then, we can apply the following a priori estimate, which is a consequence of the $C^{1,\alpha}$-regularity of DiBenedetto and Friedman [17,18] and was stated in this form by Bögelein and Duzaar in [4, Lemma 2], and exploit the conclusion of the next lemma to the solution of the variational inequality (7). The a priori estimate reads as follows.

Lemma 7. *Suppose that $\frac{2n}{n+2}<p<2$, $\tilde a\in[\nu,L]$, $c_*\ge 1$, $t_1<t_2$ and $\mathcal{O}\subseteq\mathbb{R}^n$. Let $v_0\in C^0([t_1,t_2],L^2(\mathcal{O}))\cap L^p(t_1,t_2;W^{1,p}(\mathcal{O}))$ be a weak solution of*

$$\partial_t v_0-\operatorname{div}\Big(\tilde a|Dv_0|^{p-2}Dv_0\Big)=0\quad\text{in}\quad\mathcal{O}\times[t_1,t_2)$$

and assume that

$$\fint_{Q_R^{(\lambda)}(z_0)}|Dv_0|^p\le c_*\lambda^p \tag{11}$$

for some intrinsic cylinder $Q_R^{(\lambda)}(z_0)\subseteq\mathcal{O}_I$, where $I=(t_1,t_2)$ with center $z_0\in\mathcal{O}_I$, radius $R>0$ and scaling factor $\lambda>0$. Then, there exists $\alpha_0=\alpha_0(n,p,\nu,L)\in(0,1)$ and a constant $\mu_0=\mu_0(n,p,\nu,L,c_*)\ge 1$ and $R_s\in[0,R_0]$, where $R_0:=\frac{1}{2}\mu_0^{(p-2)/2}R$, such that the following assertions hold: If $R_s>0$, then for any $0<r\le R_0$ there exists a scaling factor μ, such that there holds: $Q_r^{(\lambda\mu)}(z_0)\subseteq Q_{\frac{R}{2}}^{(\lambda)}(z_0)$ and

$$\mu_0 \left(\frac{\max\{r, R_s\}}{R_0} \right)^{\alpha_0} \le \mu \le 2\mu_0 \left(\frac{\max\{r, R_s\}}{R_0} \right)^{\alpha_0} \tag{12}$$

and

$$\sup_{Q_r^{(\lambda\mu)}(z_0)} |Dv_0| \le \lambda\mu. \tag{13}$$

Moreover, we have

$$\fint_{Q_r^{(\lambda\mu)}(z_0)} |Dv_0 - (Dv_0)_{Q_r^{(\lambda\mu)}(z_0)}|^p \,\mathrm{d}z \le c\mu_0^2 \mu^{p-2}\lambda^p \left(\frac{r}{R_0} \right)^{2\alpha_0}, \tag{14}$$

where $c = c(n, p, \nu, L)$. In the case $R_s = 0$ the inequality (12) holds with $R_s = 0$. Moreover, we have (13) and (14). We note that R_s cannot be explicitly computed and might depend on z_0 and the solution v_0 itself.

2.4. Comparison estimates

Let $\rho \in (0, 1]$, $\lambda \ge 1$ and $Q_{2\rho}^{(\lambda)}(z_0) \subset Q_{2\mathfrak{s}}(z_0) \subset Q_{\mathfrak{R}}(\mathfrak{z}_0) \subset \Omega_T$ with $\mathfrak{s} := \lambda^{\frac{2-p}{2}}\rho$. In this section, we will compare the localizable solution $u \in \mathcal{K}_{\psi,g}(\Omega_T)$ – see Definition 1 – of the variational inequality (7) and the weak solution $v \in C^0(\Lambda_{2\rho}^{(\lambda)}(t_0); L^2(B_{2\rho}(x_0))) \cap L^p(\Lambda_{2\rho}^{(\lambda)}(t_0); W^{1,p}(B_{2\rho}(x_0)))$ with $\partial_t v \in L^{p'}(\Lambda_{2\rho}^{(\lambda)}(t_0); W^{-1,p'}(B_{2\rho}(x_0)))$ of the following boundary value problem

$$\begin{cases} \partial_t v - \operatorname{div}(a(z)|Dv|^{p-2}Dv) = \partial_t \psi - \operatorname{div}(a(z)|D\psi|^{p-2}D\psi) & \text{in } Q_{2\rho}^{(\lambda)}(z_0), \\ v = u & \text{on } \partial_{\mathcal{P}} Q_{2\rho}^{(\lambda)}(z_0). \end{cases} \tag{15}$$

Here, we have to mention that by the definition of a localizable solution $u \in \mathcal{K}_{\psi,g}(\Omega_T)$, we know that there exists a function with $\mathcal{K}'_{\psi,u}(Q_{2\rho}^{(\lambda)}(z_0)) \neq \emptyset$ (cf. Definition 1), i.e. a function in $\mathcal{K}_{\psi,u}(Q_{2\rho}^{(\lambda)}(z_0))$ with boundary datum u, which possess a time derivate in $L^{p'}(\Lambda_{2\rho}^{(\lambda)}(t_0); W^{-1,p'}(B_{2\rho}(x_0)))$. Therefore, we are allowed to use u as a boundary datum. Then, the existence of such a solution v follows from classical functional analytic methods results, see e.g. [36,42]. Next, we will compare v to the weak solution $v_0 \in C^0(\Lambda_\rho^{(\lambda)}(t_0); L^2(B_\rho(x_0))) \cap L^p(\Lambda_\rho^{(\lambda)}(t_0); W^{1,p}(B_\rho(x_0)))$ with $\partial_t v_0 \in L^{p'}(\Lambda_\rho^{(\lambda)}(t_0); W^{-1,p'}(B_\rho(x_0)))$ of

$$\begin{cases} \partial_t v_0 - \operatorname{div}(a(z_0)|Dv_0|^{p-2}Dv_0) = 0 & \text{in } Q_\rho^{(\lambda)}(z_0), \\ v_0 = v & \text{on } \partial_{\mathcal{P}} Q_\rho^{(\lambda)}(z_0). \end{cases} \tag{16}$$

Notice that we consider a function $a(z)$, which satisfies (5). Finally, we will derive a comparison estimate between the solution u and the weak solution v_0. We start with comparison estimate between u and v.

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Lemma 8. *Let* $\lambda \geq 1$, $p \in (\frac{2n}{n+2}, 2)$, $n \geq 2$, $\sigma \in (0,1)$ *and* $Q_{2\rho}^{(\lambda)}(z_0) \subset Q_{2\mathfrak{s}}(z_0) \subset Q_{\mathfrak{R}}(\mathfrak{z}_o)$ *with* $\mathfrak{s} := \lambda^{\frac{2-p}{2}}\rho$ *and* $\rho \in (0,1]$. *Assume that the assumptions (1)–(6) are in force. Moreover, assume that the inhomogeneities* F, f *and the obstacle function* ψ *satisfy the additional Hölder regularity assumptions (8). Furthermore, suppose that the map*

$$v \in C^0(\Lambda_{2\rho}^{(\lambda)}(t_0); L^2(B_{2\rho}(x_0))) \cap L^p(\Lambda_{2\rho}^{(\lambda)}(t_0); W^{1,p}(B_{2\rho}(x_0)))$$

with $\partial_t v \in L^{p'}(\Lambda_{2\rho}^{(\lambda)}(t_0); W^{-1,p'}(B_{2\rho}(x_0)))$ *solves the Cauchy–Dirichlet problem (15) and* $u \in \mathcal{K}_{\psi,v}(Q_{2\rho}^{(\lambda)}(z_0))$ *is a localizable solution of the variational inequality*

$$\begin{aligned}&\int_{\Lambda_{2\rho}^{(\lambda)}(t_0)} \langle \partial_t w, w-u\rangle_{B_{2\rho}(x_0)}\, \mathrm{d}t \\ &+ \int_{Q_{2\rho}^{(\lambda)}(z_0)} a(z)|Du|^{p-2}Du \cdot D(w-u)\, \mathrm{d}z + \frac{1}{2}\|(w-u)(\cdot,t_1)\|^2_{L^2(B_{2\rho}(x_0))} \\ &\geq \int_{Q_{2\rho}^{(\lambda)}(z_0)} |F|^{p-2}F\cdot D(w-u) + f\cdot(w-u)\, \mathrm{d}z\end{aligned} \tag{17}$$

for all comparison functions $w \in \mathcal{K}'_{\psi,v}(Q_{2\rho}^{(\lambda)}(z_0))$, *where* $t_1 := t_0 - \lambda^{2-p}(2\rho)^2$. *Then, there exists a constant* $c = c(n,p,\nu,L,\Psi)$ *with*

$$\begin{aligned}\Psi &:= \|F\|_{C^{0;\sigma_1,\frac{\sigma_1}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o))} + \|D\psi\|_{C^{0;\sigma_1,\frac{\sigma_1}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o))} + \|D\psi\|_{L^\infty(Q_{\mathfrak{R}}(\mathfrak{z}_o))} \\ &\quad + \|f\|_{L^{p',(1-\sigma)p'}(Q_{\mathfrak{R}}(\mathfrak{z}_o))} + \|\partial_t\psi\|_{L^{p',(1-\sigma)p'}(Q_{\mathfrak{R}}(\mathfrak{z}_o))},\end{aligned} \tag{18}$$

where $\sigma_1 = \sigma(p-1)$, *such that the energy estimate*

$$\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv|^p\, \mathrm{d}z \leq c\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p\, \mathrm{d}z + \lambda^{\frac{2-p}{2}n}\rho^\sigma\right) \tag{19}$$

holds and furthermore, the comparison estimate

$$\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv-Du|^p\, \mathrm{d}z \leq c(\lambda^{\frac{2-p}{2}n}\rho^\sigma)^{(p-1)}\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p\, \mathrm{d}z + \lambda^{\frac{2-p}{2}n}\rho^\sigma\right)^{2-p} \tag{20}$$

is valid.

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Proof. The proof is divided in several steps. We start with

Step 1: Preliminary studies. First, we consider the weak formulation of (15) and choose the admissible test function $\varphi = v - u \in L^p(\Lambda_{2\rho}^{(\lambda)}(t_0); W_0^{1,p}(B_{2\rho}(x_0)))$, since $L^{p'} - W^{-1,p'}$ is the dual of $L^p - W_0^{1,p}$. Hence, we have

$$\begin{aligned}\int\limits_{\Lambda_{2\rho}^{(\lambda)}(t_0)} \langle \partial_t v, v-u\rangle_{B_{2\rho}(x_0)}\,\mathrm{d}t + \int\limits_{Q_{2\rho}^{(\lambda)}(z_0)} a(z)|Dv|^{p-2}Dv\cdot D(v-u)\,\mathrm{d}z \\ = \int\limits_{Q_{2\rho}^{(\lambda)}(z_0)} \partial_t\psi\cdot(v-u)\,\mathrm{d}z + \int\limits_{Q_{2\rho}^{(\lambda)}(z_0)} a(z)|D\psi|^{p-2}D\psi\cdot D(v-u)\,\mathrm{d}z.\end{aligned} \tag{21}$$

Note that by the comparison principle of Lemma 6, the solution v satisfies the obstacle constraint $v \geq \psi$ a.e. on $Q_{2\rho}^{(\lambda)}(z_0)$. Thus, $w := v \in \mathcal{K}'_{\psi,v}(Q_{2\rho}^{(\lambda)}(z_0))$ is admissible as comparison function in the variational inequality (17). Moreover, we note that $v(\cdot,t_1) = u(\cdot,t_1)$ with $t_1 = t_0 - \lambda^{2-p}(2\rho)^2$, so we have

$$\begin{aligned}\int\limits_{\Lambda_{2\rho}^{(\lambda)}(t_0)} \langle \partial_t v, v-u\rangle_{B_{2\rho}(x_0)}\,\mathrm{d}t + \int\limits_{Q_{2\rho}^{(\lambda)}(z_0)} a(z)|Du|^{p-2}Du\cdot D(v-u)\,\mathrm{d}z \\ \geq \int\limits_{Q_{2\rho}^{(\lambda)}(z_0)} |F|^{p-2}F\cdot D(v-u) + f\cdot(v-u)\,\mathrm{d}z.\end{aligned}$$

By subtracting this inequality from (21), we can conclude

$$\begin{aligned}&\int\limits_{Q_{2\rho}^{(\lambda)}(z_0)} a(z)\Big(|Dv|^{p-2}Dv - |Du|^{p-2}Du\Big)\cdot D(v-u)\,\mathrm{d}z \\ &\leq -\int\limits_{Q_{2\rho}^{(\lambda)}(z_0)} \Big(|F|^{p-2}F - (|F|^{p-2}F)_{Q_{2\rho}^{(\lambda)}(z_0)}\Big)\cdot D(v-u) + f\cdot(v-u)\,\mathrm{d}z \\ &+ \int\limits_{Q_{2\rho}^{(\lambda)}(z_0)} \Big([a(z)-a(z_0)]|D\psi|^{p-2}D\psi\Big)\cdot D(v-u) + \partial_t\psi\cdot(v-u)\,\mathrm{d}z \\ &+ \int\limits_{Q_{2\rho}^{(\lambda)}(z_0)} a(z_0)\Big[|D\psi|^{p-2}D\psi - (|D\psi|^{p-2}D\psi)_{Q_{2\rho}^{(\lambda)}(z_0)}\Big]\cdot D(v-u)\,\mathrm{d}z,\end{aligned}$$

since $v - u = 0$ on $\partial_{\mathcal{P}} Q_{2\rho}^{(\lambda)}(z_0)$. Using several times Hölder's inequality and $(5)_1$ to the right-hand side of the previous estimate, we get

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$$
\begin{aligned}
&\fint_{Q_{2\rho}^{(\lambda)}(z_0)} a(z)\left(|Dv|^{p-2}Dv-|Du|^{p-2}Du\right)\cdot D(v-u)\,\mathrm{d}z \\
&\le \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} \left||F|^{p-2}F-(|F|^{p-2}F)_{Q_{2\rho}^{(\lambda)}(z_0)}\right|^{p'}\mathrm{d}z+\fint_{Q_{2\rho}^{(\lambda)}(z_0)}|a(z)-a(z_0)|^{p'}|D\psi|^p\,\mathrm{d}z\right. \\
&+L\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)}\left||D\psi|^{p-2}D\psi-(|D\psi|^{p-2}D\psi)_{Q_{2\rho}^{(\lambda)}(z_0)}\right|^{p'}\mathrm{d}z\right) \qquad (22)\\
&\left.+c(n,p)\rho^{p'}\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)}|f|^{p'}+|\partial_t\psi|^{p'}\,\mathrm{d}z\right)\right)^{\frac{1}{p'}}\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)}|D(v-u)|^p\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&=\left(I_{(22)}+II_{(22)}+III_{(22)}+IV_{(22)}\right)^{\frac{1}{p'}}\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)}|D(v-u)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}
\end{aligned}
$$

with the obvious labeling and where we also utilized the Poincaré inequality slicewise to derive the last term on the right-hand side.

Step 2: Estimates for $I_{(22)}-IV_{(22)}$. The terms $I_{(22)}$ and $III_{(22)}$ can be bounded by (10). Moreover, we will apply

$$\varrho(z_1,z_2)^{\sigma(p-1)p'}\le\left(4\rho+2\rho\sqrt{\lambda^{2-p}}\right)^{\sigma p}\le 4^p([1+\lambda^{\frac{2-p}{2}}]\rho)^{\sigma p}\le c(p)\lambda^{\frac{2-p}{2}\sigma p}\rho^{\sigma}$$

with $\varrho(z_1,z_2)=|x_1-x_2|+\sqrt{|t_1-t_2|}$ for all $z_1,z_2\in Q_{2\rho}^{(\lambda)}(z_0)$. This yields

$$I_{(22)}+III_{(22)}\le c\varrho(z_1,z_2)^{\sigma_1 p'}\le c\lambda^{\frac{2-p}{2}\sigma p}\rho^{\sigma}, \tag{23}$$

with a constant $c=c(p,L,\|F\|_{C^{0;\sigma_1,\frac{\sigma_1}{2}}},\|D\psi\|_{C^{0;\sigma_1,\frac{\sigma_1}{2}}})$, where $\sigma_1=\sigma(p-1)$. Next, we estimate $II_{(22)}$ from above by (5), (18) and the fact that the modulus of continuity is concave and non-decreasing, which yields

$$|a(z_1)-a(z_2)|\le\omega(d_{\mathcal{P}}(z_1,z_2))\le\omega(4\rho+\sqrt{4\lambda^{2-p}\rho^2})\le 4\omega(\lambda^{\frac{2-p}{2}}\rho)\le c\lambda^{\frac{2-p}{2}}\rho^{\sigma}$$

for all $z_1,z_2\in Q_{2\rho}^{(\lambda)}(z_0)$ with $c=c(L)$. Moreover, we have

$$II_{(22)}\le 6^{p'}\omega(\lambda^{\frac{2-p}{2}}\rho)^{p'}\fint_{Q_{2\rho}^{(\lambda)}(z_0)}|D\psi|^p\,\mathrm{d}z\le c(p)\omega(\lambda^{\frac{2-p}{2}}\rho)\|D\psi\|^p_{L^\infty(Q_{2\rho}^{(\lambda)}(z_0))}\le c\lambda^{\frac{2-p}{2}}\rho^{\sigma} \tag{24}$$

with a constant $c = c(p, L, \|D\psi\|_{L^\infty})$. Finally, we have to bound $IV_{(22)}$. This yields

$$\begin{aligned}
IV_{(22)} &\le \frac{c(n,p)\rho^{p'}}{|Q_{2\rho}^{(\lambda)}(z_0)|}\int_{Q_{2\mathfrak{s}}} |f|^{p'} + |\partial_t\psi|^{p'}\,\mathrm{d}z = \frac{c(n,p)\rho^{p'}}{|Q_{2\rho}^{(\lambda)}(z_0)|}\int_{Q_{2\mathfrak{s}}\cap Q_{\mathfrak{R}}(\mathfrak{z}_o)} |f|^{p'} + |\partial_t\psi|^{p'}\,\mathrm{d}z \\
&\le \frac{c(n,p)\rho^{p'}}{|Q_{2\rho}^{(\lambda)}(z_0)|}\left(2\lambda^{\frac{2-p}{2}}\rho\right)^{(n+2)-(1-\sigma)p'}\left[\|f\|^{p'}_{L^{p',(1-\sigma)p'}(Q_{\mathfrak{R}}(\mathfrak{z}_o))} + \|\partial_t\psi\|^{p'}_{L^{p',(1-\sigma)p'}(Q_{\mathfrak{R}}(\mathfrak{z}_o))}\right] \\
&= c\lambda^{\frac{2-p}{2}(n-(1-\sigma)p')}\rho^{p'-(1-\sigma)p'} = c\lambda^{\frac{2-p}{2}(n-(1-\sigma)p')}\rho^{\sigma p'} < c\lambda^{\frac{2-p}{2}n}\rho^{\sigma},
\end{aligned} \tag{25}$$

with a constant $c = c(n, p, \|f\|_{L^{p',(1-\sigma)p'}}, \|\partial_t\psi\|_{L^{p',(1-\sigma)p'}})$, where

$$Q_{2\rho}^{(\lambda)}(z_0) \subset Q_{2\mathfrak{s}}(z_0) = Q_{2\mathfrak{s}}(z_0) \cap Q_{\mathfrak{R}}(\mathfrak{z}_o), \quad \mathfrak{s} = \lambda^{\frac{2-p}{2}}\rho$$

and $|Q_{2\rho}^{(\lambda)}(z_0)| = c(n)\rho^{n+2}\lambda^{2-p}$.

Step 3: Energy estimate. We start with the proof of the energy estimate of Dv. Therefore, we make the following calculation

$$\begin{aligned}
&\fint_{Q_{2\rho}^{(\lambda)}(z_0)} a(z)\left(|Dv|^{p-2}Dv - |Du|^{p-2}Du\right)\cdot D(v-u)\,\mathrm{d}z \\
&= \fint_{Q_{2\rho}^{(\lambda)}(z_0)} a(z)|Dv|^p + a(z)|Du|^p\,\mathrm{d}z \\
&\quad - \fint_{Q_{2\rho}^{(\lambda)}(z_0)} a(z)\left(|Dv|^{p-2}Dv\cdot Du + |Du|^{p-2}Du\cdot Dv\right)\mathrm{d}z.
\end{aligned}$$

Using this, the fact that $a(z)|Du|^p \ge 0$, (5) and Hölder's inequality, we gain from (22) the following

$$\begin{aligned}
\nu \fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv|^p\,\mathrm{d}z &\le L \fint_{Q_{2\rho}^{(\lambda)}(z_0)} \left(|Dv|^{p-1}|Du| + |Du|^{p-1}|Dv|\right)\mathrm{d}z \\
&\quad + \left(I_{(22)} + II_{(22)} + III_{(22)} + IV_{(22)}\right)^{\frac{1}{p'}}\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |D(v-u)|^p\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\le L\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv|^p\,\mathrm{d}z\right)^{\frac{1}{p'}}\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p\,\mathrm{d}z\right)^{\frac{1}{p}}
\end{aligned}$$

$$+ L \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p \, \mathrm{d}z \right)^{\frac{1}{p'}} \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv|^p \, \mathrm{d}z \right)^{\frac{1}{p}}$$

$$+ \left(I_{(22)} + II_{(22)} + III_{(22)} + IV_{(22)} \right)^{\frac{1}{p'}} \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |D(v-u)|^p \, \mathrm{d}z \right)^{\frac{1}{p}}.$$

The next step is to estimate the previous inequality by Young's inequality, (23), (24) and (25) from above. This yields for any $\varepsilon > 0$

$$\begin{aligned}
\nu \fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv|^p \, \mathrm{d}z \le{}& \left(\frac{p-1}{p} + \frac{1}{p} \right) \varepsilon \fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv|^p \, \mathrm{d}z \\
&+ \left(\frac{p-1}{p} L^{p'} \varepsilon^{\frac{1}{1-p}} \frac{1}{p} L^p \varepsilon^{1-p} \right) \fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p \, \mathrm{d}z \\
&+ \frac{4\varepsilon}{p} \fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv - Du|^p \, \mathrm{d}z + \varepsilon^{\frac{1}{1-p}} c \lambda^{\frac{2-p}{2} n} \rho^\sigma \\
\le{}& \left(\frac{4 \cdot 2^{p-1}}{p} + 1 \right) \varepsilon \fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv|^p \, \mathrm{d}z + c_\varepsilon \left[\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p \, \mathrm{d}z + \lambda^{\frac{2-p}{2} n} \rho^\sigma \right]
\end{aligned}$$

with a constant $c_\varepsilon = c_\varepsilon(n, p, L, \Psi, \varepsilon)$, where we also used that $\rho \le 1$, $\lambda \ge 1$, $n \ge 2$ and $\frac{2n}{n+2} < p < 2$. Then, choosing $\varepsilon > 0$, such that $\varepsilon \le \frac{\nu p}{2(2^{p+1}+p)}$, we gain the following

$$\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv|^p \le c \fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p \, \mathrm{d}z + c \lambda^{\frac{2-p}{2} n} \rho^\sigma$$

with a constant $c = c(n, p, \nu, L, \Psi)$. Thus, we have shown the energy estimate (19).

Step 4: Comparison estimate. Next, we will conclude the desired comparison estimate (20). Therefore, we apply Lemma 5 to the left-hand side of (22) and (23)–(25) to the right-hand side to infer that

$$\frac{\nu}{c} \fint_{Q_{2\rho}^{(\lambda)}(z_0)} (|Dv|^2 + |Du|^2)^{\frac{p-2}{2}} |Dv - Du|^2 \, \mathrm{d}z \le \left(c \lambda^{\frac{2-p}{2} n} \rho^\sigma \right)^{\frac{1}{p'}} \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |D(v-u)|^p \, \mathrm{d}z \right)^{\frac{1}{p}}.$$

Moreover, using Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, we have

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$$\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv-Du|^p \,\mathrm{d}z = \fint_{Q_{2\rho}^{(\lambda)}(z_0)} (|Dv|^2+|Du|^2)^{\frac{p(p-2)}{4}}|Dv-Du|^p(|Dv|^2+|Du|^2)^{\frac{p(2-p)}{4}} \,\mathrm{d}z$$

$$\le \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} (|Dv|^2+|Du|^2)^{\frac{p-2}{2}}|Dv-Du|^2 \,\mathrm{d}z\right)^{\frac{p}{2}} \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} (|Dv|^2+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}z\right)^{\frac{2-p}{2}}.$$

Then, we bound the second term on the right-hand side by the energy estimate (19) and the utilize the next to last estimate. This yields

$$\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv-Du|^p \,\mathrm{d}z$$

$$\le c\left(\left(\lambda^{\frac{2-p}{2}n}\rho^\sigma\right)^{\frac{1}{p'}}\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |D(v-u)|^p \,\mathrm{d}z\right)^{\frac{1}{p}}\right)^{\frac{p}{2}} \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p \,\mathrm{d}z + \lambda^{\frac{2-p}{2}n}\rho^\sigma\right)^{\frac{2-p}{2}}$$

with a constant $c = c(n, p, \nu, L, \Psi)$. Moreover, we can infer that

$$\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv-Du|^p \,\mathrm{d}z$$

$$\le c\left(\lambda^{\frac{2-p}{2}n}\rho^\sigma\right)^{\frac{p-1}{2}}\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |D(v-u)|^p \,\mathrm{d}z\right)^{\frac{1}{2}} \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p \,\mathrm{d}z + \lambda^{\frac{2-p}{2}n}\rho^\sigma\right)^{\frac{2-p}{2}}$$

with a constant $c = c(n, p, \nu, L, \Psi)$. Finally using Cauchy–Schwarz inequality. This yields the desired comparison estimate (20). Thus the lemma is proved. □

Now, we derive the comparison estimate between v and v_0. This reads as follows.

Lemma 9. *Let $\lambda \ge 1$, $p \in (\frac{2n}{n+2}, 2)$, $n \ge 2$, $\sigma \in (0,1)$ and $Q_{2\rho}^{(\lambda)}(z_0) \subset Q_{2\mathfrak{s}}(z_0) \subset Q_{\mathfrak{R}}(\mathfrak{z}_0)$ with $\mathfrak{s} := \lambda^{\frac{2-p}{2}}\rho$ and $\rho \in (0,1]$. Assume that the assumptions (1)–(5) and (6) are in force. Moreover, assume that the obstacle function ψ satisfies the assumption (8). Furthermore, suppose that the maps $v \in C^0(\Lambda_{2\rho}^{(\lambda)}(t_0); L^2(B_{2\rho}(x_0))) \cap L^p(\Lambda_{2\rho}^{(\lambda)}(t_0); W^{1,p}(B_{2\rho}(x_0)))$ with $\partial_t v \in L^{p'}(\Lambda_{2\rho}^{(\lambda)}(t_0); W^{-1,p'}(B_{2\rho}(x_0)))$ and $v_0 \in C^0(\Lambda_{\rho}^{(\lambda)}(t_0); L^2(B_\rho(x_0))) \cap L^p(\Lambda_{\rho}^{(\lambda)}(t_0); W^{1,p}(B_\rho(x_0)))$ with $\partial_t v_0 \in L^{p'}(\Lambda_{\rho}^{(\lambda)}(t_0); W^{-1,p'}(B_\rho(x_0)))$ are weak solutions of the boundary value problems (15) and (16). Then, there exists a constant c depending on n, p, ν, L, $\|D\psi\|_{C^{0;\sigma_1,\frac{\sigma_1}{2}}}$, $\|D\psi\|_{L^\infty}$, $\|\partial_t\psi\|_{L^{p',(1-\sigma)p'}}$ and number $\sigma_1 = \sigma(p-1)$, such that the energy estimate*

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$$\fint_{Q_\rho^{(\lambda)}(z_0)} |Dv_0|^p \,\mathrm{d}z \le c \left(\fint_{Q_\rho^{(\lambda)}(z_0)} |Dv|^p \,\mathrm{d}z + \lambda^{\frac{2-p}{2}n} \rho^\sigma \right) \tag{26}$$

holds and furthermore, the comparison estimate

$$\fint_{Q_\rho^{(\lambda)}(z_0)} |Dv_0 - Dv|^p \,\mathrm{d}z \le c\lambda^{\frac{2-p}{2}} \rho^\sigma \left(\fint_{Q_\rho^{(\lambda)}(z_0)} |Dv|^p \,\mathrm{d}z \right) + c\lambda^{\frac{2-p}{2}n} \rho^\sigma \tag{27}$$

is valid.

Proof. The proof is divided in several steps. The approach is similar to the proof of Lemma 8.

Step 1: Preliminary studies. We start by considering the weak formulation of (15) with the admissible choice of the test function $\varphi = v_0 - v \in L^p(\Lambda_\rho^{(\lambda)}(t_0); W_0^{1,p}(B_\rho(x_0)))$, since $L^{p'} - W^{-1,p'}$ is the dual of $L^p - W_0^{1,p}$. Thus, we get the following weak formulation of (16):

$$\int_{\Lambda_\rho^{(\lambda)}(t_0)} \langle \partial_t v_0, v_0 - v\rangle_{B_\rho(x_0)} \,\mathrm{d}t + \int_{Q_\rho^{(\lambda)}(z_0)} a(z_0)|Dv_0|^{p-2}Dv_0 \cdot D(v_0 - v) \,\mathrm{d}z = 0. \tag{28}$$

Then, we subtract the weak formulation of (15) from (28). This yields

$$\begin{aligned} &\int_{\Lambda_\rho^{(\lambda)}(t_0)} \langle \partial_t (v_0 - v), v_0 - v\rangle_{B_\rho(x_0)} \,\mathrm{d}t \\ &+ \int_{Q_\rho^{(\lambda)}(z_0)} (a(z_0)|Dv_0|^{p-2}Dv_0 - a(z)|Dv|^{p-2}Dv) \cdot D(v_0 - v) \,\mathrm{d}z \\ &= - \int_{Q_\rho^{(\lambda)}(z_0)} \partial_t \psi \cdot (v_0 - v) \,\mathrm{d}z - \int_{Q_\rho^{(\lambda)}(z_0)} a(z)|D\psi|^{p-2}D\psi \cdot D\,(v_0 - v)\,\mathrm{d}z. \end{aligned}$$

Observing the first term on the right-hand side of the previous equation, we can infer by Lemma 3 that

$$\begin{aligned} &\int_{\Lambda_\rho^{(\lambda)}(t_0)} \langle \partial_t (v_0 - v), v_0 - v\rangle_{B_\rho(x_0)} \,\mathrm{d}t \\ &= \frac{1}{2} \int_{\Lambda_\rho^{(\lambda)}(t_0)} \frac{d}{dt} \int_{B_\rho(x_0)} |(v_0 - v)(\cdot, t)|^2 \,\mathrm{d}x \,\mathrm{d}t \\ &= \frac{1}{2} \int_{B_\rho(x_0)} |(v_0 - v)(\cdot, t_2)|^2 \,\mathrm{d}x - \frac{1}{2} \int_{B_\rho(x_0)} |(v_0 - v)(\cdot, t_1)|^2 \,\mathrm{d}x \end{aligned}$$

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$$= \frac{1}{2} \int\limits_{B_\rho(x_0)} |(v_0 - v)(\cdot, t_2)|^2 \,\mathrm{d}x \geq 0$$

with $t_1 = t_0 - \lambda^{2-p}\rho^2$ and $t_2 = t_0 + \lambda^{2-p}\rho^2$, since $v_0(\cdot, t_1) = v(\cdot, t_1)$. Therefore, we have

$$\begin{aligned}
&\int\limits_{Q_\rho^{(\lambda)}(z_0)} a(z_0)\left(|Dv_0|^{p-2}Dv_0 - |Dv|^{p-2}Dv\right)\cdot D(v_0 - v)\,\mathrm{d}z \\
&\quad \leq \int\limits_{Q_\rho^{(\lambda)}(z_0)} a(z_0)\left||D\psi|^{p-2}D\psi - \left(|D\psi|^{p-2}D\psi\right)_{Q_\rho^{(\lambda)}(z_0)}\right|\cdot |D(v_0 - v)|\,\mathrm{d}z \\
&\quad + \int\limits_{Q_\rho^{(\lambda)}(z_0)} |\partial_t\psi|\cdot|v_0 - v| + |a(z) - a(z_0)|\left(|Dv|^{p-1} + |D\psi|^{p-1}\right)\cdot|D(v_0 - v)|\,\mathrm{d}z.
\end{aligned}$$

Using (5) and the properties of $\omega(\cdot) \in [0, 1]$, i.e. the concavity and that $\omega(\cdot)$ is non-decreasing, we have $|a(z_1) - a(z_2)| \leq 2\omega(\lambda^{\frac{2-p}{2}}\rho)$ for all $z_1, z_2 \in Q_\rho^{(\lambda)}(z_0)$. Then, applying Hölder's inequality and the Poincaré inequality slicewise to the last estimate, we gain

$$\begin{aligned}
&\nu \int\limits_{Q_\rho^{(\lambda)}(z_0)} \left(|Dv_0|^{p-2}Dv_0 - |Dv|^{p-2}Dv\right)\cdot D(v_0 - v)\,\mathrm{d}z \\
&\leq \left(c(n,p)\rho^{p'} \int\limits_{Q_\rho^{(\lambda)}(z_0)} |\partial_t\psi|^{p'}\,\mathrm{d}z + 2\omega(\lambda^{\frac{2-p}{2}}\rho) \int\limits_{Q_\rho^{(\lambda)}(z_0)} \left(|Dv|^p + |D\psi|^p\right)\,\mathrm{d}z\right. \\
&\left. + L \int\limits_{Q_\rho^{(\lambda)}(z_0)} \left||D\psi|^{p-2}D\psi - \left(|D\psi|^{p-2}D\psi\right)_{Q_\rho^{(\lambda)}(z_0)}\right|^{p'}\,\mathrm{d}z\right)^{\frac{1}{p'}} \left(\int\limits_{Q_\rho^{(\lambda)}(z_0)} |D(v_0 - v)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}.
\end{aligned}$$

Next, we conclude by the additional Hölder assumption in (8), i.e. $\psi \in C^{1;\sigma,\frac{\sigma}{2}}(\Omega_T)$, which implies $|D\psi|^{p-2}D\psi$ is Hölder continuous with Hölder exponent $\sigma_1 = \sigma(p-1)$ [cf. (9), (10) and Step 2 of the preceding proof], that

$$\begin{aligned}
&\nu \fint\limits_{Q_\rho^{(\lambda)}(z_0)} \left(|Dv_0|^{p-2}Dv_0 - |Dv|^{p-2}Dv\right)\cdot D(v_0 - v)\,\mathrm{d}z \\
&\leq c\left(\lambda^{\frac{2-p}{2}n}\rho^\sigma + \omega(\lambda^{\frac{2-p}{2}}\rho) \fint\limits_{Q_\rho^{(\lambda)}(z_0)} |Dv|^p\,\mathrm{d}z\right)^{\frac{1}{p'}} \left(\fint\limits_{Q_\rho^{(\lambda)}(z_0)} |D(v_0 - v)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}
\end{aligned} \tag{29}$$

with a constant $c = c(n, p, L, \|D\psi\|_{C^{0;\sigma_1,\frac{\sigma_1}{2}}}, \|D\psi\|_{L^\infty}, \|\partial_t\psi\|_{L^{p',(1-\sigma)p'}})$, since $\rho \in (0,1]$, $\sigma \in (0,1)$, $\lambda \geq 1$, $p, p' > 1$ and where we finally divided the resulting estimate by $|Q_\rho^{(\lambda)}(z_0)|$. Notice that (29) is the starting point from which we first infer the energy estimate of Dv_0 and then, we consider again (29) to deduce the comparison estimate between Dv_0 and Dv.

Step 2: Energy estimate. We start by the same calculation as in the previous proof, i.e. we get from (29) the following inequality

$$\begin{aligned}
&\nu \fint_{Q_\rho^{(\lambda)}(z_0)} |Dv_0|^p \,\mathrm{d}z \\
&\leq L \fint_{Q_\rho^{(\lambda)}(z_0)} |Dv_0|^{p-1}|Dv| + |Dv|^{p-1}|Dv_0| \,\mathrm{d}z \\
&\quad + c\left(\lambda^{\frac{2-p}{2}n}\rho^\sigma + \omega(\lambda^{\frac{2-p}{2}}\rho) \fint_{Q_\rho^{(\lambda)}(z_0)} |Dv|^p \,\mathrm{d}z\right)^{\frac{1}{p'}} \left(\fint_{Q_\rho^{(\lambda)}(z_0)} |D(v_0 - v)|^p \,\mathrm{d}z\right)^{\frac{1}{p}}
\end{aligned}$$

with a constant $c = c(n, p, L, \|D\psi\|_{C^{0;\sigma_1,\frac{\sigma_1}{2}}}, \|D\psi\|_{L^\infty}, \|\partial_t\psi\|_{L^{p',(1-\sigma)p'}})$, where we used that $a(z)|Dv|^p \geq 0$ to estimate the left-hand side from above. Then, we use Young's inequality several times to conclude for any $\varepsilon > 0$

$$\begin{aligned}
\nu \fint_{Q_\rho^{(\lambda)}(z_0)} |Dv_0|^p \,\mathrm{d}z &\leq \left(\frac{p-1}{p} + \frac{1}{p}\right)\varepsilon \fint_{Q_\rho^{(\lambda)}(z_0)} |Dv_0|^p \,\mathrm{d}z + \frac{1}{p}\varepsilon \fint_{Q_\rho^{(\lambda)}(z_0)} |D(v_0 - v)|^p \,\mathrm{d}z \\
&\quad + c\left(\varepsilon^{\frac{1}{1-p}} + \varepsilon^{1-p} + \omega(\lambda^{\frac{2-p}{2}}\rho)\varepsilon^{\frac{1}{1-p}}\right) \fint_{Q_\rho^{(\lambda)}(z_0)} |Dv|^p \,\mathrm{d}z + c\varepsilon^{\frac{1}{1-p}}\lambda^{\frac{2-p}{2}n}\rho^\sigma \\
&\leq \left(1 + \frac{2^{p-1}}{p}\right)\varepsilon \fint_{Q_\rho^{(\lambda)}(z_0)} |Dv_0|^p \,\mathrm{d}z + c_\varepsilon \fint_{Q_\rho^{(\lambda)}(z_0)} |Dv|^p \,\mathrm{d}z + c_\varepsilon \lambda^{\frac{2-p}{2}n}\rho^\sigma,
\end{aligned}$$

with two constants $c = c(n, p, L, \|D\psi\|_{C^{0;\sigma_1,\frac{\sigma_1}{2}}}, \|D\psi\|_{L^\infty}, \|\partial_t\psi\|_{L^{p',(1-\sigma)p'}})$ and

$$c_\varepsilon = c_\varepsilon(\varepsilon, n, p, L, \|D\psi\|_{C^{0;\sigma_1,\frac{\sigma_1}{2}}}, \|D\psi\|_{L^\infty}, \|\partial_t\psi\|_{L^{p',(1-\sigma)p'}}),$$

where we also used that $\omega \in [0,1]$. Choosing $\varepsilon > 0$, such that $\left(1 + 2^{p-1}/p\right)\varepsilon \leq \frac{\nu}{2}$, e.g. $\varepsilon \leq \frac{\nu}{2^{p+1}}$. Then, re-absorbing the first term on the right-hand side to the left-hand side and multiplying the resulting estimate by $\frac{2}{\nu}$, we get (26).

Step 3: Comparison estimate. Now, we come back to (29) and infer the comparison estimate. By Lemma 5, we can estimate the left-hand side of the (29) from below, such that

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$$\begin{aligned}
&\fint_{Q_\rho^{(\lambda)}(z_0)} (|Dv_0|^2+|Dv|^2)^{\frac{p-2}{2}}|Dv_0-Dv|^2\,\mathrm{d}z \\
&\le c\left(\omega(\lambda^{\frac{2-p}{2}}\rho)\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv|^p\,\mathrm{d}z+\lambda^{\frac{2-p}{2}n}\rho^\sigma\right)^{\frac{1}{p'}}\left(\fint_{Q_\rho^{(\lambda)}(z_0)}|D(v_0-v)|^p\,\mathrm{d}z\right)^{\frac{1}{p}},
\end{aligned}\tag{30}$$

where we used the properties of $\omega(\cdot)$ and (6). Next, we need the following estimate

$$\begin{aligned}
&\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv_0-Dv|^p\,\mathrm{d}z \\
&=\fint_{Q_\rho^{(\lambda)}(z_0)}(|Dv_0|^2+|Dv|^2)^{\frac{p(p-2)}{4}}|Dv_0-Dv|^p(|Dv_0|^2+|Dv|^2)^{\frac{p(2-p)}{4}}\,\mathrm{d}z \\
&\le\left(\fint_{Q_\rho^{(\lambda)}(z_0)}(|Dv_0|^2+|Dv|^2)^{\frac{p-2}{2}}|Dv_0-Dv|^2\,\mathrm{d}z\right)^{\frac{p}{2}}\left(\fint_{Q_\rho^{(\lambda)}(z_0)}(|Dv_0|^2+|Dv|^2)^{\frac{p}{2}}\,\mathrm{d}z\right)^{\frac{2-p}{2}},
\end{aligned}$$

where we used Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$. Then, we utilize the energy estimate (26), (30), (8) [cf. (23)–(25)], $p-2\in(-\frac{4}{n+2},0)$, $p-1\in(\frac{n-2}{n+2},1)$, $\lambda\ge1$, $\sigma,\rho\in(0,1)$ and the properties of the modulus of continuity $\omega(\cdot)$, mainly (6) to bound the last estimate from above. This yields

$$\begin{aligned}
&\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv_0-Dv|^p\,\mathrm{d}z \\
&\le\left(\fint_{Q_\rho^{(\lambda)}(z_0)}(|Dv_0|^2+|Dv|^2)^{\frac{p-2}{2}}|Dv_0-Dv|^2\right)^{\frac{p}{2}}\left(\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv_0|^p+|Dv|^p\,\mathrm{d}z\right)^{\frac{2-p}{2}} \\
&\le c\left[\left(\omega(\lambda^{\frac{2-p}{2}}\rho)\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv|^p\,\mathrm{d}z+\lambda^{\frac{2-p}{2}n}\rho^\sigma\right)^{\frac{1}{p'}}\left(\fint_{Q_\rho^{(\lambda)}(z_0)}|D(v_0-v)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}\right]^{\frac{p}{2}} \\
&\quad\times\left[\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv|^p\,\mathrm{d}z+\lambda^{\frac{2-p}{2}n}\rho^\sigma\right]^{\frac{2-p}{2}}
\end{aligned}$$

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$$\leq c\left[\rho^{\sigma\frac{p-1}{p}}\left(\lambda^{\frac{2-p}{2}}\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv|^p\,\mathrm{d}z+\lambda^{\frac{2-p}{2}n}\right)^{\frac{p-1}{p}}\left(\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv|^p\,\mathrm{d}z+\lambda^{\frac{2-p}{2}n}\right)^{\frac{2-p}{p}}\right]^{\frac{p}{2}}$$

$$\times\left[\fint_{Q_\rho^{(\lambda)}(z_0)}|D(v_0-v)|^p\,\mathrm{d}z\right]^{\frac{1}{2}}$$

$$\leq c\left(\rho^{\sigma(p-1)}\right)^{\frac{1}{2}}\left[\lambda^{\frac{2-p}{2}}\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv|^p\,\mathrm{d}z+\lambda^{\frac{2-p}{2}n}\right]^{\frac{1}{2}}\left[\fint_{Q_\rho^{(\lambda)}(z_0)}|D(v_0-v)|^p\,\mathrm{d}z\right]^{\frac{1}{2}},$$

where we used that $\lambda^{\frac{2-p}{2}}\geq 1$. Finally, utilizing Cauchy–Schwarz inequality and summarizing the terms as in the previous proof, we derive at (27) for any $\frac{2n}{n+2}<p<2$. This completes the proof. □

Corollary 10. *Under the assumptions of Lemma 8 and Lemma 9, there exists a constant $c=c(n,p,\nu,L,\Psi)$ with Ψ introduced in (18), such that the energy estimate*

$$\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv_0|^p\,\mathrm{d}z\leq c\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)}|Du|^p\,\mathrm{d}z+\lambda^{\frac{2-p}{2}n}\rho^\sigma\right).\tag{31}$$

holds and furthermore, the comparison estimate

$$\begin{aligned}\fint_{Q_\rho^{(\lambda)}(z_0)}|Du-Dv_0|^p\,\mathrm{d}z&\leq c\lambda^{\frac{2-p}{2}}\rho^\sigma\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)}|Du|^p\,\mathrm{d}z\right)\\&\quad+c\left(\lambda^{\frac{2-p}{2}n}\rho^\sigma\right)^{(p-1)}\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)}|Du|^p\,\mathrm{d}z\right)^{2-p}+c\lambda^{\frac{2-p}{2}(n+1)}\rho^\sigma\end{aligned}\tag{32}$$

is valid.

Proof. First of all, we consider the following

$$\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv_0|^p\,\mathrm{d}z\leq c\left(\fint_{Q_\rho^{(\lambda)}(z_0)}|Dv|^p\,\mathrm{d}z+\lambda^{\frac{2-p}{2}n}\rho^\sigma\right)\leq c\left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)}|Dv|^p\,\mathrm{d}z+\lambda^{\frac{2-p}{2}n}\rho^\sigma\right)$$

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where we used (26) and $\frac{|Q_{2\rho}^{(\lambda)}|}{|Q_{\rho}^{(\lambda)}|} = \frac{\alpha_n (2\rho)^n 2(2\rho)^2 \lambda^{2-p}}{\alpha_n \rho^n 2\rho^2 \lambda^{2-p}} = 2^{n+2}$. Then, we infer the energy estimate (31) from the previous estimate by using the energy estimate (19). Next, we consider

$$\begin{aligned} \fint_{Q_\rho^{(\lambda)}(z_0)} |Du - Dv_0|^p \, \mathrm{d}z &\le 2^{(n+p+1)} \fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du - Dv|^p \, \mathrm{d}z + 2^{p-1} \fint_{Q_\rho^{(\lambda)}(z_0)} |Dv - Dv_0|^p \, \mathrm{d}z \\ &=: I + II \end{aligned}$$

with the obvious choice of I and II. By the comparison estimate (27) and the energy estimate (19) we can estimate the term II from above, as follows

$$\begin{aligned} II &\overset{(27)}{\le} c\lambda^{\frac{2-p}{2}} \rho^\sigma \left(\fint_{Q_\rho^{(\lambda)}(z_0)} |Dv|^p \, \mathrm{d}z \right) + c\lambda^{\frac{2-p}{2} n} \rho^\sigma \\ &\overset{(19)}{\le} c\lambda^{\frac{2-p}{2}} \rho^\sigma \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p \, \mathrm{d}z \right) + c\lambda^{\frac{2-p}{2}(n+1)} \rho^\sigma \end{aligned}$$

with a constant $c = c(n, p, \nu, L, \Psi)$, since $\omega(\cdot) \in (0, 1)$ and $\lambda \ge 1$. The term I we can bound by the comparison estimate (20). This yields

$$\begin{aligned} I &\le c \left(\lambda^{\frac{2-p}{2} n} \rho^\sigma \right)^{(p-1)} \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p \, \mathrm{d}z + \lambda^{\frac{2-p}{2} n} \rho^\sigma \right)^{2-p} \\ &\le c \left(\lambda^{\frac{2-p}{2} n} \rho^\sigma \right)^{(p-1)} \left(\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^p \, \mathrm{d}z \right)^{2-p} + c\lambda^{\frac{2-p}{2} n} \rho^\sigma \end{aligned}$$

with a constant $c = c(n, p, \nu, L, \Psi)$. Combining the last estimates, we get the desired comparison estimate (32). This completes the proof. □

3. Proof of the main result

Proof of Theorem 1. First of all, let $R \in (0, R_*]$ with $R_* \in (0, 1]$ to be chosen later. Then, we consider a parabolic cylinder $Q_R(z_0) \subseteq Q_{R_*}(z_0) \subset Q_{\mathfrak{R}}(\mathfrak{z}_0) \subset \Omega_T$. Next, we define

$$M := \left(\fint_{Q_R(z_0)} |Du|^p \, \mathrm{d}z + 1 \right) \ge 1. \tag{33}$$

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Furthermore, we set $R_i := \theta^i R$ and $\lambda_i := \delta^i$ for $i \in \mathbb{N}_0$, where $\theta \in (0, \frac{1}{4}]$ and $\delta \geq 4$. Later on, we shall fix the parameter θ and δ, such that the constraint

$$\theta \leq \frac{1}{\delta^2} \tag{34}$$

holds. This implies $\theta \leq \frac{1}{4}\delta^{\frac{p-2}{2}}$, since $\theta \leq \frac{1}{\delta^2} \leq \frac{1}{4}\delta^{-1} \leq \frac{1}{4}\delta^{p-2} \leq \frac{1}{4}\delta^{\frac{p-2}{2}}$. Moreover, we define a sequence of nested cylinders

$$Q_{R_i}^{(\lambda_i)}(z_0) = B_{R_i}(x_0) \times (t_0 - \lambda_i^{2-p} R_i^2, t_0 + \lambda_i^{2-p} R_i^2),$$

such that

$$Q_{R_i}^{(\lambda_i)}(z_0) \subseteq Q_{\frac{1}{4}R_{i-1}}^{(\lambda_{i-1})}(z_0) \subseteq Q_{R_{i-1}}^{(\lambda_{i-1})}(z_0) \subseteq \cdots \subseteq Q_{R_1}^{(\lambda_1)}(z_0) \subseteq Q_R(z_0), \tag{35}$$

where $R_i = \theta^i R = \theta R_{i-1} \leq \frac{1}{4} R_{i-1} \leq R_{i-1}$ and

$$\lambda_i^{2-p} R_i^2 = \lambda_i^{2-p} \theta^2 R_{i-1}^2 \leq \lambda_{i-1}^{2-p} (\frac{1}{4} R_{i-1})^2 \leq \lambda_{i-1}^{2-p} R_{i-1}^2,$$

since $\theta \in (0, \frac{1}{4}]$ and $\theta \leq \frac{1}{4}\delta^{\frac{p-2}{2}}$. The rest of the proof is divided in several steps. We start with

Step 1. An almost L^∞-estimate. In the first part of the proof of Theorem 1, we want to infer that

$$\fint_{Q_{R_i}^{(\lambda_i)}(z_0)} |Du|^p \, \mathrm{d}z \leq M \lambda_i^p \quad \text{for all } i \in \mathbb{N}_0. \tag{36}$$

This will be achieved by induction. In the case $i = 0$ the estimate (36) is trivially satisfied by (33), since $R_0 = R$, $\lambda_0 = 1$ and therefore $Q_{R_i}^{(\lambda_i)}(z_0) = Q_R(z_0)$. Now, we assume that (36) is in force for some $i \geq 0$ and infer from this assumption that (36) is valid for $i + 1$. Thus, we construct comparison functions v_i and $v_{0,i}$, such that $v_i \in C^0(\Lambda_{R_i}^{(\lambda_i)}(t_0); L^2(B_{R_i}(x_0))) \cap L^p(\Lambda_{R_i}^{(\lambda_i)}(t_0); W^{1,p}(B_{R_i}(x_0)))$ denotes the unique solution of the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t v_i - \operatorname{div}(a(z_0)|Dv_i|^{p-2} Dv_i) = \partial\psi - \operatorname{div}(a(z)|D\psi|^{p-2} D\psi) & \text{in } Q_{R_i}^{(\lambda)}(z_0), \\ v_i = u & \text{on } \partial_{\mathcal{P}} Q_{R_i}^{(\lambda)}(z_0). \end{cases}$$

and the map $v_{0,i} \in C^0(\Lambda_{\frac{1}{2}R_i}^{(\lambda_i)}(t_0); L^2(B_{\frac{1}{2}R_i}(x_0))) \cap L^p(\Lambda_{\frac{1}{2}R_i}^{(\lambda_i)}(t_0); W^{1,p}(B_{\frac{1}{2}R_i}(x_0)))$ denotes the unique solution of the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t v_{0,i} - \operatorname{div}(a(z_0)|Dv_{0,i}|^{p-2} Dv_{0,i}) = 0 & \text{in } Q_{\frac{1}{2}R_i}^{(\lambda_i)}(z_0), \\ v_{0,i} = v_i & \text{on } \partial_{\mathcal{P}} Q_{\frac{1}{2}R_i}^{(\lambda_i)}(z_0). \end{cases} \tag{37}$$

Next, we can infer from (31) the following energy estimate, where we use (λ_i, R_i) instead of (λ, ρ). This yields

$$\fint_{Q^{(\lambda_i)}_{\frac{1}{2}R_i}(z_0)} |Dv_{0,i}|^p \,\mathrm{d}z \le c\left(\fint_{Q^{(\lambda_i)}_{R_i}(z_0)} |Du|^p \,\mathrm{d}z\right) + c\lambda_i^p R_i^\sigma,$$

since $\frac{2-p}{2}n < \frac{2-\frac{2n}{n+2}}{2}n = \frac{\frac{2n+4-2n}{n+2}}{2}n = \frac{2n}{n+2} < p$ and $\lambda_i \ge 1$. Utilizing (36)$_i$, then we get

$$\fint_{Q^{(\lambda_i)}_{\frac{1}{2}R_i}(z_0)} |Dv_{0,i}|^p \,\mathrm{d}z \le cM\lambda_i^p + c\lambda_i^p R_i^\sigma \le cM\lambda_i^p$$

with a constant $c = c(n, p, \nu, L, \Psi)$, since $R_i \in (0, R]$ and $M, \lambda_i \ge 1$. Therefore, we can obtain that

$$\fint_{Q^{(\lambda_i)}_{\frac{1}{2}R_i}(z_0)} |Dv_{0,i}|^p \,\mathrm{d}z \le c\lambda_i^p \tag{38}$$

with a constant $c = c(n, p, \nu, L, \Psi, M)$. Since $v_{0,i}$ is a solution of the parabolic p-Laplacian equation, we can apply the sup-estimates from [16, Chapter VIII, Theorem 5.2] for the case $\frac{2n}{n+2} < p < 2$ (in the subquadratic case we have to take into account also [16, Chapter VIII, Remark 5.4]). Together with (38) the sup-estimates leads us to

$$\sup_{Q^{(\lambda_i)}_{\frac{1}{4}R_i}(z_0)} |Dv_{0,i}| \le c\lambda_i^{1-\frac{2p}{p(n+2)-2n}} \left(\fint_{Q^{(\lambda_i)}_{\frac{1}{2}R_i}(z_0)} |Dv_{0,i}|^p \,\mathrm{d}z\right)^{\frac{2}{p(n+2)-2n}} + c\lambda_i \le c\lambda_i \tag{39}$$

with a constant $c = c(n, p, \nu, L, \Psi, M)$. In the following, we will use the comparison estimate (32) to infer

$$\begin{aligned}\fint_{Q^{(\lambda_i)}_{\frac{1}{2}R_i}(z_0)} |Du - Dv_{0,i}|^p \,\mathrm{d}z &\le c\left(\lambda_i^{\frac{2-p}{2}} R_i^\sigma\right)\left(\fint_{Q^{(\lambda_i)}_{R_i}(z_0)} |Du|^p \,\mathrm{d}z\right) \\ &\quad + c\lambda_i^{\frac{2-p}{2}(n+1)} R_i^\sigma + c\left(\lambda_i^{\frac{2-p}{2}n} R_i^\sigma\right)^{(p-1)} \left(\fint_{Q^{(\lambda_i)}_{R_i}(z_0)} |Du|^p \,\mathrm{d}z\right)^{2-p}\end{aligned} \tag{40}$$

with a constant $c = c(n, p, \nu, L, \Psi)$. Next, we utilize (39) and (40) to deduce

$$
\begin{aligned}
\fint_{Q^{(\lambda_{i+1})}_{R_{i+1}}(z_0)} |Du|^p \,\mathrm{d}z &\le 2^{p-1} \fint_{Q^{(\lambda_{i+1})}_{R_{i+1}}(z_0)} |Dv_{0,i}|^p \,\mathrm{d}z + 2^{p-1} \fint_{Q^{(\lambda_{i+1})}_{R_{i+1}}(z_0)} |Du - Dv_{0,i}|^p \,\mathrm{d}z \\
&\le 2^{p-1} \left(\sup_{Q^{(\lambda_i)}_{\frac14 R_i}(z_0)} |Dv_{0,i}|^p + \fint_{Q^{(\lambda_{i+1})}_{R_{i+1}}(z_0)} |Du - Dv_{0,i}|^p \,\mathrm{d}z \right) \\
&\overset{(39)}{\le} c\lambda_i^p + \frac{2^{p-1}}{(2\theta)^{n+2}} \left(\frac{\lambda_{i+1}}{\lambda_i}\right)^{p-2} \fint_{Q^{(\lambda_i)}_{\frac12 R_i}(z_0)} |Du - Dv_{0,i}|^p \,\mathrm{d}z \\
&\overset{(40)}{\le} c\lambda_i^p + c\frac{2^{p-1}}{(2\theta)^{n+2}} \left(\frac{\lambda_{i+1}}{\lambda_i}\right)^{p-2} \left[\left(\lambda_i^{\frac{2-p}{2}} R_i^\sigma\right) \left(\fint_{Q^{(\lambda_i)}_{R_i}(z_0)} |Du|^p \,\mathrm{d}z \right) \right. \\
&\quad \left. + \left(\lambda_i^{\frac{2-p}{2}n} R_i^\sigma\right)^{(p-1)} \left(\fint_{Q^{(\lambda_i)}_{R_i}(z_0)} |Du|^p \,\mathrm{d}z \right)^{2-p} + \lambda_i^{\frac{2-p}{2}(n+1)} R_i^\sigma \right]
\end{aligned}
$$

with a constant $c = c(n, p, \nu, L, \Psi)$, where we used

$$
\frac{|Q^{(\lambda_i)}_{\frac12 R_i}(z_0)|}{|Q^{(\lambda_{i+1})}_{R_{i+1}}(z_0)|} = \frac{\alpha_n \left(\frac{R_i}{2}\right)^n}{\alpha_n (R_{i+1})^n} \cdot \frac{2\lambda_i^{2-p} \left(\frac{R_i}{2}\right)^2}{2\lambda_{i+1}^{2-p} (R_{i+1})^2} = \left(\frac{\frac{R_i}{2}}{2\theta \frac{R_i}{2}}\right)^{n+2} \left(\frac{\lambda_i}{\lambda_{i+1}}\right)^{2-p}.
$$

Thus, we deduce that

$$
\fint_{Q^{(\lambda_{i+1})}_{R_{i+1}}(z_0)} |Du|^p \,\mathrm{d}z \le c\lambda_i^p + c\frac{2^{p-1}}{(2\theta)^{n+2}} \left(\frac{\lambda_{i+1}}{\lambda_i}\right)^{p-2} \left[\lambda_i^{\frac{2+p}{2}} + \lambda_i^{p(p-1)+p(2-p)} + \lambda^2 \right] R_i^\sigma M,
$$

where we used the induction assumption (36) for i, $\frac{2-p}{2}(n+1) < 2$ and $R_i \in (0, 1]$. Further, we conclude that

$$
\begin{aligned}
\fint_{Q^{(\lambda_{i+1})}_{R_{i+1}}(z_0)} |Du|^p \,\mathrm{d}z &\le c\lambda_i^p + \frac{2^{p-1}}{(2\theta)^{n+2}} \lambda_{i+1}^{p-2} \lambda_i^2 c M R_i^\sigma = c \left(\delta^{-p} + \frac{2^{p-1}}{(2\theta)^{n+2}} \left(\frac{\lambda_i}{\lambda_{i+1}}\right)^2 R_i^\sigma \right) \lambda_{i+1}^p \\
&\le c \left[\delta^{-p} + \frac{2^{p-1}}{2^{n+2}} \left(\theta^{-(n+2)} \delta^{-2}\right) \theta^{i\sigma} R^\sigma \right] \lambda_{i+1}^p,
\end{aligned}
$$

since $\frac{2+p}{2} \le 2$, $p(p-1) + p(2-p) = p < 2$ and $\lambda_{i+1}^{p-2} \lambda_i^2 = \lambda_{i+1}^p \left(\frac{\lambda_i}{\lambda_{i+1}}\right)^2$. By the definition of λ_i, i.e. $\lambda_i = \delta^i$ and (34) we get

$$\fint_{Q_{R_{i+1}}^{(\lambda_{i+1})}(z_0)} |Du|^p \,\mathrm{d}z \le \left[2^{-p} + 2^{p-n-5}\theta^{-(n+2)}R^\sigma\right]\lambda_{i+1}^p \le \left[2^{-p} + 2^{-5}\right]\lambda_{i+1}^p.$$

In the last step, we used that δ and R_* can be chosen to satisfy

$$\delta \ge 2c \quad \text{and} \quad 2^{p-n}R_*^\sigma \le \theta^{(n+2)} \tag{41}$$

with a constant $c = c(n, p, \nu, L, \Psi, M)$. This yields

$$\fint_{Q_{R_{i+1}}^{(\lambda_{i+1})}(z_0)} |Du|^p \,\mathrm{d}z \le \left[2^{-p} + 2^{-5}\right]\lambda_{i+1}^p \le M\lambda_{i+1}^p.$$

This proves that (36) is also valid for $i + 1$ and therefore (36) holds for any $i \in \mathbb{N}_0$.

Step 2. A Campanato type estimate. Here, we want to derive a Campanato type estimate by means of the Lemma 7. In order to apply Lemma 7, we have to ensure that the requirements are valid. First of all, we should notice that the bound (11) from Lemma 7 is similar to the bound (38) from the preceding step of this proof. Thus, we have to align (11) with (38). Therefore, let M be the constant defined in (33). This guarantees us the validity of the energy bound (38). Then, we denote by $c_* = c_*(n, p, \nu, L, M)$ the constant appearing to (38). Next, by $\mu_0 = \mu_0(n, p, \nu, L, c_*) \ge 1$ and $\alpha_0 = \alpha_0(n, p, \nu, L) \in (0, 1)$ we denote the constants introduced in Lemma 7. Finally, we have to choose the radius R from Lemma 7 equal to $\frac{1}{2}R_i$, where R_i is the radius from the beginning of this proof. Now, the estimate (11) with these choices is equivalent to the energy bound (38). Then, we define

$$\xi := \frac{n + 2 + 2\alpha_0}{n + 2 + 2\alpha_0 + \sigma} \tag{42}$$

and consider radii ρ satisfying

$$0 < \rho \le \left(\frac{1}{4}\mu_0^{\frac{p-2}{2}} R\right)^{\frac{2}{\xi}}. \tag{43}$$

With $r_i := \left(\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i\right)^{\frac{1}{\xi}}$ we choose $i \in \mathbb{N} \cup \{0\}$, such that

$$(\mu_0\lambda_{i+1}r_{i+1}^{\alpha_0(1-\xi)})^{\frac{2-p}{2}} r_{i+1} < \rho \le (\mu_0\lambda_i r_i^{\alpha_0(1-\xi)})^{\frac{2-p}{2}} r_i. \tag{44}$$

These particular choices are possible, we have

$$(\mu_0\lambda_0 r_0^{\alpha_0(1-\xi)})^{\frac{2-p}{2}} r_0 \ge r_0^2 = \left(\frac{1}{4}\mu_0^{\frac{p-2}{2}} R\right)^{\frac{2}{\xi}} \ge \rho$$

and $(\lambda_i r_i^{\alpha_0(1-\xi)})^{\frac{2-p}{2}} r_i$ tends to zero as $i \to \infty$. Now, we are in the situation to apply Lemma 7. Thus, we consider again (as in Step 1) on a fixed intrinsic cylinder $Q_{\frac{1}{2}R_i}^{(\lambda_i)}(z_0)$ the comparison function $v_{0,i}$ defined in (37). Then, the energy estimate (38) for $v_{0,i}$ reads as follows

$$\fint_{Q^{(\lambda_i)}_{\frac{1}{2}R_i}(z_0)} |Dv_{0,i}|^p \,\mathrm{d}z \le c_* \lambda_i^p.$$

Therefore, the assumption (11) of Lemma 7 is valid for $v_{0,i}$ on the intrinsic cylinders $Q^{(\lambda_i)}_{\frac{1}{2}R_i}(z_0)$. This allows us to conclude the existence of $R_{s,i} \in [0, \frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i]$, such that there holds: If $R_{s,i} > 0$, then for any $0 < r \le \frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i$, there exists a scaling factor μ, such that $Q_r^{(\lambda_i\mu)}(z_0) \subseteq Q^{(\lambda_i)}_{\frac{1}{2}R_i}(z_0)$ and

$$\mu_0 \left(\frac{\max\{r, R_{s,i}\}}{\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i} \right)^{\alpha_0} \le \mu \le 2\mu_0 \left(\frac{\max\{r, R_{s,i}\}}{\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i} \right)^{\alpha_0} \tag{45}$$

and

$$\sup_{Q_r^{(\lambda_i\mu)}(z_0)} |Dv_{0,i}| \le \lambda_i \mu \tag{46}$$

hold. Moreover, we have from (14) the following estimate

$$\fint_{Q_r^{(\lambda_i\mu)}(z_0)} |Dv_{0,i} - (Dv_{0,i})_{Q_r^{(\lambda_i\mu)}(z_0)}|^p \,\mathrm{d}z \le c\mu_0^2 \mu^{p-2} \lambda_i^p \left(\frac{r}{\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i} \right)^{2\alpha_0} \tag{47}$$

for any $0 < r < \frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i$ and a constant $c = c(n, p, \nu, L)$. In the case $R_{s,i} = 0$, we have that (45) holds with $R_{s,i} = 0$. Furthermore, (46) and (47) hold also in this case. As mentioned in Lemma 7, the radii $R_{s,i}$ might depends on z_0 and the solution $v_{i,0}$. Now, we want to derive an excess decay estimate for the solution u similar to the one for $v_{0,i}$ from (46). We start with

$$\begin{aligned}
\fint_{Q_{r_i}^{(\lambda_i\mu)}(z_0)} |Du - (Du)_{Q_{r_i}^{(\lambda_i\mu)}(z_0)}|^p \,\mathrm{d}z &\le 2 \cdot 3^{p-1} \fint_{Q_{r_i}^{(\lambda_i\mu)}(z_0)} |Du - Dv_{0,i}|^p \,\mathrm{d}z \\
&\quad + 3^{p-1} \fint_{Q_{r_i}^{(\lambda_i\mu)}(z_0)} |Dv_{0,i} - (Dv_{0,i})_{Q_{r_i}^{(\lambda_i\mu)}(z_0)}|^p \,\mathrm{d}z \\
&\le 3^p (I + II)
\end{aligned}$$

with the obvious labeling for I and II, where we used Hölder's inequality for the second to last estimate. Here, we want to utilize the comparison estimate (40) to estimate I from above. This is possible, since $r_i = (\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i)^{\frac{1}{\xi}} \le \frac{1}{2} R_i$. Finally, the have to divide the resulting comparison estimate by $|Q^{(\lambda_i)}_{R_i}(z_0)|$. Now, we apply (40) to I as follows

$$I \le \mu^{p-2}\left(\frac{R_i}{2r_i}\right)^{n+2} \fint_{Q^{(\lambda_i)}_{\frac{1}{2}R_i}(z_0)} |Du - Dv_{0,i}|^p \, dz$$
$$\le c\mu^{p-2}\left(\frac{R_i}{r_i}\right)^{n+2} \lambda_i^2 M R_i^\sigma = c\mu^{p-2}\left(\frac{R_i}{r_i}\right)^{n+2} \lambda_i^2 R_i^\sigma$$

with a constant $c = c(n, p, \nu, L, \Psi, M)$, where we used (36). To estimate II we use the excess decay estimate for $v_{0,i}$ from (47) to obtain

$$II \le c\mu_0^2 \mu^{p-2} \lambda_i^p \left(\frac{r_i}{\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i}\right)^{2\alpha_0}$$

with a constant $c = c(n, p, \nu, L)$, where we replaced r by r_i in (47). Therefore, we have the following excess decay estimate for the solution u of the variational inequality (7):

$$\fint_{Q^{(\lambda_i\mu)}_{r_i}(z_0)} |Du - (Du)_{Q^{(\lambda_i\mu)}_{r_i}(z_0)}|^p \, dz \le c\mu^{p-2}\left(\frac{R_i}{r_i}\right)^{n+2} \lambda_i^2 R_i^\sigma + c\mu_0^2\mu^{p-2}\lambda_i^p \left(\frac{r_i}{\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i}\right)^{2\alpha_0}$$
$$= c\mu^{p-2}\lambda_i^2 \left[\left(\frac{R_i}{r_i}\right)^{n+2} R_i^\sigma + \mu_0^2 \left(\frac{r_i}{\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i}\right)^{2\alpha_0}\right],$$

where $c = c(n, p, \nu, L, \Psi, M)$. Finally, we want to conclude an excess decay estimate for the solution u on a standard cylinder $Q_\rho(z_0)$ with radius $\rho > 0$. Therefore, we have to use the restriction $Q_\rho(z_0) \subseteq Q^{(\lambda_i\mu)}_{r_i}(z_0)$. This allows us to enlarge the domain of integration in the excess functional from $Q_\rho(z_0)$ to $Q^{(\lambda_i\mu)}_{r_i}(z_0)$ and subsequently, we apply the quasi-minimality of the mean value for the integrability exponent p. This yields

$$\begin{aligned}
&\fint_{Q_\rho(z_0)} |Du - (Du)_{Q_\rho(z_0)}|^p \, dz \\
&\le c\frac{|Q^{(\lambda_i\mu)}_{r_i}(z_0)|}{|Q_\rho(z_0)|} \fint_{Q^{(\lambda_i\mu)}_{r_i}(z_0)} |Du - (Du)_{Q^{(\lambda_i\mu)}_{r_i}(z_0)}|^p \, dz \\
&\le c(\lambda_i\mu)^{2-p}\left(\frac{r_i}{\rho}\right)^{n+2} \mu^{p-2}\lambda_i^2 \left[\left(\frac{R_i}{r_i}\right)^{n+2} R_i^\sigma + \mu_0^2\left(\frac{r_i}{\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i}\right)^{2\alpha_0}\right] \\
&\le c\lambda_i^2 \left[\left(\frac{R_i}{\rho}\right)^{n+2} R_i^\sigma + \mu_0^2\left(\frac{r_i}{\rho}\right)^{n+2}\left(\frac{r_i}{\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i}\right)^{2\alpha_0}\right]
\end{aligned} \tag{48}$$

with a constant $c = c(n, p, \nu, L, \Psi, M)$. Here, we utilize the bound of ρ^{-1} from (44), i.e.

$$\frac{1}{\rho} < \left[(\mu_0 \lambda_{i+1} r_{i+1}^{\alpha_0(1-\xi)})^{\frac{2-p}{2}} r_{i+1} \right]^{-1} = \left[(\mu_0 \lambda_{i+1} r_{i+1}^{\alpha_0(1-\xi)})^{\frac{2-p}{2}} \theta^{\frac{1}{\xi}} r_i \right]^{-1}$$
$$\leq \left[(\mu_0 \delta \lambda_i r_i^{\alpha_0(1-\xi)})^{\frac{2-p}{2}} \theta^3 r_i \right]^{-1}.$$

Furthermore, we use the definitions of r_i and R_i to conclude that

$$\fint_{Q_\rho(z_0)} |Du - (Du)_{Q_\rho(z_0)}|^p \, dz$$
$$\leq c \lambda_i^{\frac{4-(n+2)(2-p)}{2}} \left[\left(\frac{R_i}{r_i^{1+\alpha_0(1-\xi)\frac{2-p}{2}}} \right)^{n+2} R_i^\sigma + r_i^{\alpha_0(1-\xi)(n+2)\frac{2-p}{2}} \left(\frac{r_i}{\frac{1}{4}\mu_0^{\frac{p-2}{2}} R_i} \right)^{2\alpha_0} \right]$$
$$\leq c \lambda_i^2 \left[r_i^{-(1-\xi)(n+2)(1+\alpha_0\frac{2-p}{2})+\xi\sigma} + r_i^{-\alpha_0(1-\xi)(n+2)\frac{2-p}{2}+2\alpha_0(1-\xi)} \right]$$
$$\leq c \lambda_i^2 r_i^{\alpha_0(1-\xi)[2-(n+2)\frac{2-p}{2}]}$$

with a constant $c = c(n, p, \nu, L, \Psi, M, \delta, \mu_0, \theta)$, where we again used (42). Notice that

$$\alpha_0(1-\xi) \left[2 - (n+2)\frac{2-p}{2} \right] \geq 0,$$

since $p > \frac{2n}{n+2}$. Finally, we get (similar to the case $p \geq 2$, cf. [26]) the following estimate

$$\fint_{Q_\rho(z_0)} |Du - (Du)_{Q_\rho(z_0)}|^p \, dz \leq c r_i^{\frac{1}{2}\alpha_0(1-\xi)[2-(n+2)\frac{2-p}{2}]} \left[\delta^2 \theta^{\frac{1}{2}\alpha_0\frac{1-\xi}{\xi}[2-(n+2)\frac{2-p}{2}]} \right]^i$$
$$\leq c \rho^{\frac{1}{4}\alpha_0(1-\xi)[2-(n+2)\frac{2-p}{2}]} \left[\delta^2 \theta^{\frac{1}{2}\alpha_0\frac{1-\xi}{\xi}[2-(n+2)\frac{2-p}{2}]} \right]^i$$

with a constant $c = c(n, p, \nu, L, \Psi, M, \delta, \mu_0, \theta)$, where we used $r_i \leq c\rho^{\frac{1}{2}}$ for the last estimate, which follows again from (44), since

$$\rho > (\mu_0 \lambda_{i+1} r_{i+1}^{\alpha_0(1-\xi)})^{\frac{2-p}{2}} r_{i+1} = (\mu_0 \lambda_i \delta)^{\frac{2-p}{2}} \theta^{\frac{1}{\xi}[\alpha_0(1-\xi)\frac{2-p}{2}+1]} r_i^{[\alpha_0(1-\xi)\frac{2-p}{2}+1]}$$
$$\geq \theta^{\frac{1}{\xi}[\alpha_0(1-\xi)\frac{2-p}{2}+1]} r_i^{[\alpha_0(1-\xi)\frac{2-p}{2}+1]} = c r_i^{[\alpha_0(1-\xi)\frac{2-p}{2}+1]} \geq c r_i^2$$

with a constant $c = c(n, p, \alpha_0, \xi)$, since $\frac{2n}{n+2} < p < 2$, $\mu_0, \lambda_i, \delta \geq 1$, $r_i \in (0, 1)$ and $[\alpha_0(1-\xi)\frac{2-p}{2} + 1] \leq 2$. Finally, we derive at

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$$\fint_{Q_\rho(z_0)} |Du - (Du)_{Q_\rho(z_0)}|^p \, dz \le c\rho^{\frac{1}{4}\alpha_0(1-\xi)[2-(n+2)\frac{2-p}{2}]} \tag{49}$$

with a constant $c = c(n, p, \nu, L, \|F\|_{C^{0,\sigma,\frac{\sigma}{2}}}, \|D\psi\|_{C^{0,\sigma,\frac{\sigma}{2}}}, M, \delta, \mu_0, \theta)$, provided we assume

$$\theta^{\frac{1}{2}\alpha_0\frac{1-\xi}{\xi}[2-(n+2)\frac{2-p}{2}]} \le \delta^{-2}. \tag{50}$$

Step 3: Adjusting the parameter. At this stage, we have to fix the constants. First of all, we have to choose δ according to (41). Therefore, we have $\delta = \delta(n, p, \nu, L, \Psi, M)$. Next, we fix θ in such a way that (34) and (50) are valid. By this choice, we determinate the dependencies of θ, i.e. θ depends on α_0, ξ, δ. Therefore $\theta = \theta(n, p, \nu, L, \Psi, M, \sigma)$, since α_0, ξ, δ depend on n, p, ν, L, Ψ, M, σ. Thus, we have proved the existence of an exponent $\alpha_1 = \alpha_1(n, p, \nu, L, \sigma) := \frac{1}{4}\alpha_0(1-\xi)[2-(n+2)\frac{2-p}{2}]$, such that

$$\fint_{Q_\rho(z_0)} |Du - (Du)_{Q_\rho(z_0)}|^p \, dz \le c\rho^{\alpha_1},$$

holds with a constant $c = c(n, p, \nu, L, \Psi, M, \sigma)$, whenever $Q_{\mathfrak{R}}(\mathfrak{z}_o) \Subset \Omega_T$ and ρ satisfies (41) and (43). Then, Hölder's inequality implies that

$$\fint_{Q_\rho(z_0)} |Du - (Du)_{Q_\rho(z_0)}| \, dz \le c\rho^{\frac{\alpha_1}{p}}.$$

From the parabolic version of Campanato's integral characterization of continuous functions proved by Da Prato in [14], we can conclude that Du is Hölder continuous with respect to the parabolic metric with Hölder exponent $\alpha = \frac{\alpha_1}{p}$ locally on Ω_T, i.e. $Du \in C^{0;\alpha,\frac{\alpha}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o), \mathbb{R}^n)$ for a Hölder exponent $\alpha = \alpha(n, p, \nu, L, \sigma) \in (0, \sigma)$. Thus, the proof is finished. □

4. Poincaré type estimate for localizable solutions of parabolic obstacle problems

Since a localizable solution u of the weak formulation of the variational inequality (7) does not admit the necessary regularity properties for an immediate application of a Poincaré inequality for maps defined on $\mathbb{R}^{n+1}$, our last aim is to prove a Poincaré type estimate for localizable solutions of our parabolic obstacle problem (7). Then, we are able to prove that $u \in C^{0;1,\frac{1}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_o), \mathbb{R})$ under the assumption of Theorem 1. The Poincaré type estimate we will prove by a comparison argument. Therefore, we start with a Poincaré type estimate for the weak solution of (15). For this goal we refer to [3, Lemma 5.1]. Here, the authors proved a Poincaré type estimate on intrinsic cylinders $Q_\rho^{(\lambda)}(z_0)$ for the weak solution of the following parabolic problem with nonstandard growth

$$\partial_t v - \operatorname{div}(a(z)|Dv|^{p(z)-2}Dv) = \operatorname{div}(|F|^{p(z)-2}F) \quad \text{in } \Omega_T.$$

The proof would be the same in our case – up to some modifications. The obvious difference is that we consider standard p-growth instead of $p(z)$-growth and we have to replace $\operatorname{div}(|F|^{p(z)-2}F)$ by $\partial_t\psi - \operatorname{div}(a(z)|D\psi|^{p-2}D\psi)$. Thus, we get in (51) instead of $|F|^{p(\cdot)}$ the terms $|D\psi|^p$ and $|\partial\psi|^{p'}$. Moreover, we need this estimate on symmetric cylinders $Q_\rho(z_0)$. Therefore, the term λ^{2-p} does not occur. Thus, the statement reads as follows

Lemma 11. *Let $Q_\rho(z_0) \subset \Omega_T$ be a symmetric parabolic cylinder with radius $0 < \rho \le 1$. Suppose that the map $v \in C^0([t_0-\rho^2, t_0+\rho^2]; L^2(B_\rho(x_0))) \cap L^p(t_0-\rho^2, t_0+\rho^2; W^{1,p}(B_\rho(x_0)))$ with $\partial_t v \in L^{p'}(t_0-\rho^2, t_0+\rho^2; W^{-1,p'}(B_\rho(x_0)))$ is a weak solution of the following inhomogeneous parabolic equation*

$$\partial_t v - \operatorname{div}(a(z)|Dv|^{p-2}Dv) = \partial_t\psi - \operatorname{div}(a(z)|D\psi|^{p-2}D\psi) \quad in\ Q_\rho(z_0),$$

under the assumptions (1)–(5). Then, there holds for all $\vartheta \in [1, p]$

$$\fint_{Q_\rho(z_0)} \left|\frac{v-(v)_{Q_\rho(z_0)}}{\rho}\right|^\vartheta dz \le c \fint_{Q_\rho(z_0)} |Dv|^\vartheta\, dz + c\left(\fint_{Q_\rho(z_0)} |Dv|^{p-1} + |D\psi|^{p-1} + |\partial_t\psi|\, dz\right)^\vartheta \tag{51}$$

with a constant $c = c(n, p, L, \vartheta)$.

Now, we are in the situation to prove the Poincaré type estimate for localizable solutions of (7).

Lemma 12. *Suppose that $u \in \mathcal{K}_{\psi,g}(\Omega_T)$ is a localizable solution – cf. Definition 1 – of the variational inequality (7) under the assumptions (1)–(5), (6) and (8). Furthermore, let $Q_\rho(z_0) \subset Q_{\mathfrak{R}}(\mathfrak{z}_o) \subset \Omega_T$ be a parabolic cylinder with radius $0 < \rho \le 1$. Then, there holds for all $\vartheta \in [1, p]$*

$$\fint_{Q_\rho(z_0)} \left|\frac{u-(u)_{Q_{2\rho}(z_0)}}{\rho}\right|^\vartheta dz \le c \fint_{Q_\rho(z_0)} |Du|^\vartheta\, dz + c\left(\fint_{Q_\rho(z_0)} |Du|^p + 1\, dz\right)^\vartheta \tag{52}$$

with a constant $c = c(n, p, \nu, L, \Psi, \vartheta)$.

Proof. First of all, we define the map

$$v \in C^0([t_0-\rho^2, t_0+\rho^2]; L^2(B_\rho(x_0))) \cap L^p(t_0-\rho^2, t_0+\rho^2; W^{1,p}(B_\rho(x_0)))$$

with $\partial_t v \in L^{p'}(t_0-\rho^2, t_0+\rho^2; W^{-1,p'}(B_\rho(x_0)))$ to be the weak solution of the following parabolic equation

$$\begin{cases} \partial_t v - \operatorname{div}(a(z)|Dv|^{p-2}Dv) = \partial_t\psi - \operatorname{div}(a(z)|D\psi|^{p-2}D\psi) & \text{in } Q_\rho(z_0), \\ v = u & \text{on } \partial_{\mathcal{P}} Q_\rho(z_0) \end{cases}$$

under the assumptions (1)–(5). We start with the following

$$
\begin{aligned}
&\fint_{Q_\rho(z_0)} \left|\frac{u-(u)_{Q_\rho(z_0)}}{\rho}\right|^{\vartheta} \mathrm{d}z \\
&\le c(\vartheta) \fint_{Q_\rho(z_0)} \left|\frac{u-v}{\rho}\right|^{\vartheta} \mathrm{d}z + c(\vartheta) \fint_{Q_\rho(z_0)} \left|\frac{(v)_{Q_{2\rho}^{(\lambda)}(z_0)}-(u)_{Q_\rho(z_0)}}{\rho}\right|^{\vartheta} \mathrm{d}z \\
&+ c(\vartheta) \fint_{Q_\rho(z_0)} \left|\frac{v-(v)_{Q_\rho(z_0)}}{\rho}\right|^{\vartheta} \mathrm{d}z =: I + II + III
\end{aligned}
$$

with the obvious labeling. Here, we estimate I by the Poincaré inequality slicewise. This yields

$$
I \le c(n,\vartheta) \fint_{Q_\rho(z_0)} |Du - Dv|^{\vartheta}\, \mathrm{d}z \le c(n,\vartheta) \fint_{Q_\rho(z_0)} |Du|^{\vartheta} + |Dv|^{\vartheta}\, \mathrm{d}z.
$$

Then, we use Hölder's inequality (if $\vartheta < p$) and the energy estimate (19) on the symmetric cylinder $Q_\rho(z_0)$. Thus, we

$$
\begin{aligned}
I &\le c(n,\vartheta) \fint_{Q_\rho(z_0)} |Du|^{\vartheta}\, \mathrm{d}z + c(n,\vartheta) \left(\fint_{Q_\rho(z_0)} |Dv|^{p}\, \mathrm{d}z \right)^{\frac{\vartheta}{p}} \\
&\le c \fint_{Q_\rho(z_0)} |Du|^{\vartheta}\, \mathrm{d}z + c \left(\fint_{Q_\rho(z_0)} |Du|^{p}\, \mathrm{d}z + \rho^{\sigma} \right)^{\frac{\vartheta}{p}}
\end{aligned}
$$

with a constant $c = c(n, p, \nu, L, \Psi, \vartheta)$. Moreover, the term II is bounded by I. The last term, we estimate from above by (51) and finally by the energy estimate (19) again on the symmetric cylinder $Q_{2\rho}(z_0)$. This yields

$$
III \le c \left(\fint_{Q_\rho(z_0)} |Du|^{p} + 1\, \mathrm{d}z \right)^{\vartheta}.
$$

Combining the last estimates and using $\rho \le 1$, we arrive at (52). This completes the proof. □

5. Proof of Theorem 2

Finally, we are able to prove Theorem 2.

Proof of Theorem 2. In the proof of Theorem 1, we have shown that $Du \in C^{0;\alpha,\frac{\alpha}{2}}(Q_{\mathfrak{R}}(\mathfrak{z}_0), \mathbb{R}^n)$ for a Hölder exponent $\alpha = \alpha(n, p, \nu, L, \sigma) \in (0, \sigma)$. Notice that in particular this implies that Du is locally bounded, i.e. $Du \in L^{\infty}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n)$. Next, we apply this result to conclude that u is

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locally $C^{0;1,\frac{1}{2}}$ continuous. Therefore, we have to use that $Q_\rho(z_0) \subset Q_{\mathfrak{R}}(z_0)$ and by the Poincaré type estimate (52), we get

$$\fint_{Q_\rho(z_0)} \left| \frac{u - (u)_{Q_\rho(z_0)}}{\rho} \right| \mathrm{d}z \leq c$$

with a constant $c = c(n, p, \nu, L, \Psi, \|Du\|_{L^\infty})$. Finally, from the parabolic version of Campanato's integral characterization of continuous functions proved by Da Prato in [14], we can conclude that u is locally $C^{0;1,\frac{1}{2}}$ continuous in $Q_{\mathfrak{R}}(\mathfrak{z}_0)$. $\square$

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