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Long-time asymptotics for the short pulse equation

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Abstract

In this paper, we analyze the long-time behavior of the solution of the initial value problem (IVP) for the short pulse (SP) equation. As the SP equation is a completely integrable system, which posses a Wadati–Konno–Ichikawa (WKI)-type Lax pair, we formulate a 2×2 matrix Riemann–Hilbert problem to this IVP by using the inverse scattering method. Since the spectral variable k is the same order in the WKI-type Lax pair, we construct the solution of this IVP parametrically in the new scale (y, t), whereas the original scale (x, t) is given in terms of functions in the new scale, in terms of the solution of this Riemann–Hilbert problem. However, by employing the nonlinear steepest descent method of Deift and Zhou for oscillatory Riemann–Hilbert problems, we can get the explicit leading order asymptotic of the solution of the short pulse equation in the original scale (x, t) as time t goes to infinity.

Keywords: Riemann–Hilbert problem; Short pulse equation; Initial value problem; Long-time asymptotics

1. Introduction

The present work is devoted to the study of the long-time asymptotic behavior of the short pulse (SP) equation formulated on the whole line,

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1.1a}$$

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where $u(x,t)$ is a real-valued function, which represents the magnitude of the electric field, while the subscripts t and x denote partial differentiations, with the initial value data

$$u(x, t = 0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.1b}$$

and assuming that $u_0(x)$ lies in Schwartz space.

The SP equation was proposed in [1] by Schäfer and Wayne to describe the propagation of ultra-short optical pulses in silica optical fibers. Usually, in nonlinear optics, the nonlinear Schrödinger (NLS) equation was always used to model the slowly varying wave trains. As the pulse duration shortens, however, the NLS equation becomes less accurate, the SP equation provides an increasingly better approximation to the corresponding solution of the Maxwell equations [2]. For the details of physical background, see [1] and references therein.

Actually, the SP equation appeared first as one of Rabelo's equations which describe pseudospherical surfaces, possessing a zero-curvature representation, in [3]. Recently, the Wadati–Konno–Ichikawa (WKI) type Lax pair of the SP equation was rediscovered in [4] (see the following (2.1a)). The integrable properties of SP equation like bi-Hamiltonian structure and the conservation laws were studied in [5,6]. The loop-soliton solutions the short pulse equation was found in [7]. The connection between the short pulse equation and the sine-Gordon equation through the hodograph transformation was found by Matsuno, and thus, multi-soliton solutions including multi-loop and multi-breather ones were given in [8]. And a lot of generalizations of the SP equation, such as vector SP equation, discretizations of SP equation, complex SP equation,and so on, were studied in [9–11] and references therein.

The local well-posedness in H^2 (which denotes the usual Sobolev space) and non-existence of smooth traveling wave solutions were shown in [1], and global well-posedness of small solutions was proved in [12] for SP equation in H^2 by using conservation laws. In [13], Liu, Pelinovsky and Sakovich showed the blow-up result for the SP equation for large data.

The purpose of this paper is to analyze the long-time asymptotic behavior of the SP equation. Due to the SP equation admits a Lax pair, the inverse scattering transform method can be used to solve the initial value problem for the SP equation. Here, we relate the inverse scattering problem to a 2×2-matrix Riemann–Hilbert problem. The most important advantage of formulating the initial value problem (1.1a)–(1.1b) as a Riemann–Hilbert problem is that the long-time asymptotic behavior of the solution of the initial value problem can be analyzed by employing the nonlinear steepest descent method introduced by Deift and Zhou [14]. This method has previous applied to many integrable equations, such as the NLS equation [15], the Sine–Gordon equation [16], the KdV equation [17], the Fokas–Lenells equation [18], the Camassa–Holm equation [19] and so on.

Recently, this approach has been applied to the so-called short-wave approximations of integrable equations, which themselves are integrable, such as the modified Hunter–Saxton (mHS) equation [20] and the Ostrovsky–Vakhnenko (OV) equation [21], which can be viewed as the short-wave limit of the Camassa–Holm and Degasperis–Procesi equations, respectively. The SP equation considered in this paper can be viewed as the short-wave limit of the modified Camassa–Holm equation [22] (see, also [23]). All of the short-wave limit equations named above have the common feature that their solutions can be extracted from the development of the solutions of the respective Riemann–Hilbert problems at $k \to 0$. Although the long-time asymptotic analysis of (1.1a) is in many ways similar to those of integrable equations, it also presents some distinctive features: *(1)* **The spectral variable k is the same order in the Lax pair**, it firstly has to be transformed by introduction of a matrix $G(x,t)$ (see the following equation (2.12)) to arrive at a Riemann–Hilbert problem with the appropriate boundary condition at infinity. This is made possible since the conservation law (2.15) holds. *(2)* **The solution $u(x,t)$ of the SP equation is constructed from a 2×2-matrix Riemann–Hilbert problem in terms of the order $O(k)$ as** $k \to 0$, which is different from the mHS equation in [20] and the OV equation in [21], where they construct the solution from a vector Riemann–Hilbert problem.

Remark 1.1. We thank the reviewers for pointing out the recent paper [23]. The authors of [23] studied the SP equation using the Riemann–Hilbert approach, analyzing the long-time asymptotics, the soliton solutions and other results. The analysis of the SP equation by the authors of [23] and the present work were completed independently. In fact, we follow the original paper of Deift and Zhou [14] devoted to the long-time asymptotic behavior of the solution of the SP equation and we give a detailed description of the asymptotic procedure, but the authors of [23] followed the more recent interpretation of this approach by Lenells, see [25], and include fewer explicit estimates than we do.

The main results of this paper are summarized by the following theorems:

Theorem 1.2. *Let $u_0(x)$ satisfy the initial value (1.1b) and be such that no discrete spectrum is present. Let ε be any small positive number, then for $\xi = \frac{x}{t} > \varepsilon$, the solution $u(x,t)$ of the initial value problem (1.1a)–(1.1b) has rapid decay, as $t \to \infty$.*

Theorem 1.3. *Let $u_0(x)$ satisfy the hypotheses of Theorem 1.2. For $\xi = \frac{x}{t} < -\varepsilon$, ε be any small positive number, the solution $u(x,t)$ of the initial value problem (1.1a)–(1.1b) equals*

$$u(x,t) = \sqrt{\frac{-4\nu(\kappa_0)}{\kappa_0 t}} \sin\{\frac{t}{\kappa_0} + \nu(\kappa_0)\ln(\frac{4t}{\kappa_0}) + \phi(\kappa_0)\} + O\left(\frac{\ln(t)}{t}\right), \quad \text{as } t \to \infty, \tag{1.2}$$

where

$$\kappa_0 = \frac{1}{\sqrt{-4\xi}} \tag{1.3}$$

and $\nu(\kappa_0)$ is defined as (5.67) replaced k_0 with κ_0, $\phi(\kappa_0)$ is defined as (5.124).

Remark 1.4. The sectors of different asymptotic behavior match, as $\varepsilon \to 0$, through the fast decay. Indeed, as $\frac{x}{t} \to 0^-$, then $\kappa_0 \to \infty$ and $\nu(\kappa_0) \to 0$ and thus the amplitude in (1.2) decays faster.

Organization of the paper: In section 2, since the associated Lax pair of SP equation (1.1a) has singularities at $k = 0$ and $k = \infty$, we perform the spectral analysis to deal with the two singularities, respectively. In section 3, we formulate the associated Riemann–Hilbert problem in an alternative space variable y instead of the original space variable x. Hence, we can reconstruct the solution $u(x,t)$ parameterized from the solution of the Riemann–Hilbert problem via the asymptotic behavior of the spectral variable at $k = 0$. In section 4 and 5, we can also obtain the asymptotic relation between y and x when analyzing the vector Riemann–Hilbert problem by using the nonlinear steepest descent method. Hence, we can calculate the leading order asymptotic behavior of the solution $u(x,t)$ and prove the main results of this paper, i.e., Theorem 1.2 and 1.3, respectively.

2. Spectral analysis

The beginning of the long-time asymptotic analysis is to formulate the initial value problem for the SP equation to a Riemann–Hilbert problem. It depends on short pulse equation admits a WKI type Lax pair,

$$\Psi_x = U(x,t,k)\Psi, \tag{2.1a}$$

$$\Psi_t = V(x,t,k)\Psi, \tag{2.1b}$$

where

$$U = ikU_1. \tag{2.2}$$

and

$$V = \frac{ik}{2}u^2U_1 + \frac{1}{4ik}\sigma_3 - \frac{iu}{2}\sigma_2, \tag{2.3}$$

with

$$U_1 = \begin{pmatrix} 1 & u_x \\ u_x & -1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.4}$$

Usually, we only use the x-part of Lax pair to analyze the initial value problem for the integrable equations by inverse scattering transform method. The t-part of Lax pair is only used to determine the time evolution of the scattering data. However, from the Lax pair (2.1), we know that there are singularities at $k=\infty$ and $k=0$. In order to construct the solution $u(x,t)$ of the SP equation (1.1a), we need use the t-part or using the expansion of the eigenfunction as spectral parameter $k \to 0$. Hence, in the following we use two different transformations to analyze these two singularities ($k=\infty$ and $k=0$), respectively.

2.1. For $k=0$

2.1.1. The closed one-form

Introducing the following transformation

$$\Psi(x,t,k) = \mu^0(x,t,k)e^{(ikx+\frac{t}{4ik})\sigma_3}, \tag{2.5}$$

then we get the Lax pair of μ^0

$$\begin{cases} \mu_x^0 - ik[\sigma_3,\mu^0] = V_1^0\mu^0, \\ \mu_t^0 - \frac{1}{4ik}[\sigma_3,\mu^0] = V_2^0\mu^0, \end{cases} \tag{2.6}$$

where

$$V_1^0 = \begin{pmatrix} 0 & iku_x \\ iku_x & 0 \end{pmatrix}, \quad V_2^0 = \begin{pmatrix} \frac{ik}{2}u^2 & \frac{ik}{2}u^2u_x - \frac{u}{2} \\ \frac{ik}{2}u^2u_x + \frac{u}{2} & -\frac{ik}{2}u^2 \end{pmatrix} \tag{2.7}$$

Letting $\hat{A}$ denotes the operator which acts on a 2×2 matrix X by $\hat{A}X = [A, X]$, then the Lax pair of μ^0 (2.6) can be written as

$$d(e^{-(ikx+\frac{t}{4ik})\hat{\sigma}_3}\mu^0) = W^0(x,t,k), \tag{2.8}$$

where $W^0(x,t,k)$ is the closed one-form defined by

$$W^0(x,t,k) = e^{-(ikx+\frac{t}{4ik})\hat{\sigma}_3}(V_1^0 dx + V_2^0 dt)\mu^0. \tag{2.9}$$

2.1.2. The Jost functions μ_j^0

We define two eigenfunctions $\{\mu_j^0\}_{j=1}^2$ of (2.6) by the Volterra integral equations,

$$\mu_1^0(x,t,k) = \mathbb{I} + \int_{-\infty}^{x} e^{ik(x-y)\hat{\sigma}_3} V_1^0(y,t,k)\mu_1^0(y,t,k)dy, \tag{2.10a}$$

$$\mu_2^0(x,t,k) = \mathbb{I} - \int_{x}^{+\infty} e^{ik(x-y)\hat{\sigma}_3} V_1^0(y,t,k)\mu_2^0(y,t,k)dy. \tag{2.10b}$$

Proposition 2.1 *(Analytic property). From the above definition, we find that the functions $\{\mu_j^0\}_1^2$ are bounded and analytic properties as following:*

- *$[\mu_1^0]_1(x,t,k)$ is bounded and analytic in D_2, $[\mu_1^0]_2(x,t,k)$ is in D_1;*
- *$[\mu_2^0]_1(x,t,k)$ is bounded and analytic in D_1, $[\mu_2^0]_2(x,t,k)$ is in D_2.*

where $[\mu_j]_i$ denotes the i-th column of μ_j, D_1 denotes the upper-half plane and D_2 denotes the lower-half plane of the complex k-sphere.

Proposition 2.2 *(Asymptotic property). The functions $\mu_j^0(x,t,k)$ have the expansions in powers of k, for $k \to 0$,*

$$\mu_j^0(x,t,k) = \mathbb{I} + iu(x,t)\sigma_1 k + [-\frac{u^2}{2}\mathbb{I} + i(u^2u_x - 2u_t)\sigma_2]k^2 + O(k^3). \tag{2.11}$$

2.2. For $k=\infty$

2.2.1. The closed one-form

Define a 2×2 matrix-value function $G(x,t)$ as

$$G(x,t) = \sqrt{\frac{\sqrt{m}+1}{2\sqrt{m}}}\begin{pmatrix} 1 & -\frac{\sqrt{m}-1}{u_x} \\ \frac{\sqrt{m}-1}{u_x} & 1 \end{pmatrix}, \tag{2.12}$$

where m is a function of (x,t) defined by

$$m = 1 + u_x^2. \tag{2.13}$$

Remark 2.3. Notice that when $u_x \to 0$, the nominator $\sqrt{m}-1$ is a high order infinitesimal than denominator u_x. So, the matrix function $G(x,t)$ is well-defined.

Define

$$p(x,t,k) = x - \int_x^\infty (\sqrt{m(x',t)}-1)dx' - \frac{t}{4k^2}. \tag{2.14}$$

As we can write the SPE (1.1a) into the conservation law form:

$$(\sqrt{m})_t = \frac{1}{2}(u^2\sqrt{m})_x, \quad m = 1+u_x^2, \tag{2.15}$$

we get

$$p_x = \sqrt{m}, \quad p_t = \frac{1}{2}u^2\sqrt{m} - \frac{1}{4k^2}. \tag{2.16}$$

And introducing a transformation

$$\Psi(x,t,k) = G(x,t)\mu(x,t,k)e^{ikp(x,t,k)\sigma_3} \tag{2.17}$$

then we find the Lax pair equations

$$\begin{cases} \mu_x - ikp_x[\sigma_3,\mu] = V_1\mu, \\ \mu_t - ikp_t[\sigma_3,\mu] = V_2\mu, \end{cases} \tag{2.18}$$

where

$$V_1 = \frac{iu_{xx}}{2m}\sigma_2, \tag{2.19a}$$

$$V_2 = \frac{1}{4ik}(\frac{1}{\sqrt{m}}-1)\sigma_3 + \frac{iu^2u_{xx}}{4m}\sigma_2 - \frac{1}{4ik}\frac{u_x}{\sqrt{m}}\sigma_1, \tag{2.19b}$$

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Then the equations in (2.18) can be written in differential form as

$$d(e^{-ikp(x,t,k)\hat{\sigma}_3}\mu) = W(x,t,k), \tag{2.20}$$

where $W(x,t,k)$ is the closed one-form defined by

$$W = e^{-ikp(x,t,k)\hat{\sigma}_3}(V_1dx + V_2dt)\mu. \tag{2.21}$$

2.2.2. The Jost functions μ_j

We define two eigenfunctions $\{\mu_j\}_1^2$ of (2.18) by the Volterra integral equations

$$\mu_1(x,t,k) = \mathbb{I} + \int_{-\infty}^{x} e^{ik[p(x,t,k)-p(y,t,k)]\hat{\sigma}_3} V_1(y,t,k)\mu_1(y,t,k)dy, \tag{2.22a}$$

$$\mu_2(x,t,k) = \mathbb{I} - \int_{x}^{+\infty} e^{ik[p(x,t,k)-p(y,t,k)]\hat{\sigma}_3} V_1(y,t,k)\mu_2(y,t,k)dy. \tag{2.22b}$$

Proposition 2.4. *(Analytic property) From the above definition, we find that the functions $\{\mu_j\}_1^2$ are bounded and analytic properties as following:*

- *$[\mu_1]_1(x,t,k)$ is bounded and analytic in D_2, $[\mu_1]_2(x,t,k)$ is in D_1;*
- *$[\mu_2]_1(x,t,k)$ is bounded and analytic in D_1, $[\mu_2]_2(x,t,k)$ is in D_2.*

Proposition 2.5 *(Large k property). The matrix functions $\mu_j(x,t,k)$ also satisfy the asymptotic condition*

$$\mu_j(x,t,k) = \mathbb{I} + \frac{D_1(x,t)}{k} + O(\frac{1}{k^2}), \quad k \to \infty, \tag{2.23}$$

where $\mathbb{I}$ is an 2×2 identity matrix, and the off-diagonal entries of the matrix $D_1(x,t)$ are

$$D_{12}(x,t) = \frac{i}{4}\frac{u_{xx}}{m\sqrt{m}}, \quad D_{21}(x,t) = \frac{i}{4}\frac{u_{xx}}{m\sqrt{m}}. \tag{2.24}$$

2.2.3. The scattering matrix $S(k)$

Because the eigenfunctions $\mu_1(x,t,k)$ and $\mu_2(x,t,k)$ are both the solutions of the Lax pair (2.18), they are related by a matrix $S(k)$ which is independent of the variable (x,t).

$$\mu_1(x,t,k) = \mu_2(x,t,k)e^{ikp(x,t,k)\hat{\sigma}_3} S(k). \tag{2.25}$$

By the definition of $\mu_j(x,t,k)$, $j = 1,2$ (2.22), the matrix $S(k)$ has the form

$$S(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ -\overline{b(\bar{k})} & a(k) \end{pmatrix}. \tag{2.26}$$

The function $a(k)$ can be computed by

$$a(k) = \det([\mu_2]_1, [\mu_1]_2), \tag{2.27}$$

where $\det(A)$ means the determinate of a matrix A. We can know that $a(k)$ is analytic in D_1.

Assumption 2.6. In this paper, we assume that the initial value $u_0(x)$ is chosen such that $a(k)$ has no zero (usually, we can assume $u_0(x)$ has small norm).

2.3. *The relation between $\mu_j(x,t,k)$ and $\mu_j^0(x,t,k)$*

As usual, we use the eigenfunctions μ_j to define the matrix $M(x,t,k)$ (see (3.1)) which is used to formulate a Riemann–Hilbert problem. However, in order to construct the solution $u(x,t)$ from the associated Riemann–Hilbert problem, we need the asymptotic behavior of μ_j as $k \to 0$. So, we need relate the eigenfunctions $\mu_j(x,t,k)$ to $\mu_j^0(x,t,k)$.

Note that the eigenfunctions $\mu(x,t,k)$ and $\mu^0(x,t,k)$ being related to the same Lax pair (2.1), must be related to each other as

$$\mu_j(x,t,k) = G^{-1}(x,t)\mu_j^0(x,t,k)e^{(ikx+\frac{t}{4ik})\sigma_3}C_j(k)e^{-ikp(x,t,k)\sigma_3} \tag{2.28}$$

with $C_j(k)$ independent of x and t. Evaluating (2.28) as $x \to \pm\infty$ gives

$$C_1(k) = e^{-ikc\sigma_3}, \qquad C_2(k) = \mathbb{I}, \tag{2.29}$$

where $c = \int_{-\infty}^{+\infty}(\sqrt{m(x,t)} - 1)dx$ is a quantity conserved under the dynamics governed by (1.1a).

Proposition 2.7. *The functions $\mu_j(x,t,k)$ and $\mu_j^0(x,t,k)$ are related as follows:*

$$\mu_1(x,t,k) = G^{-1}(x,t)\mu_1^0(x,t,k)e^{-ik\int_{-\infty}^{x}(\sqrt{m(x',t)}-1)dx'\sigma_3}, \tag{2.30a}$$

$$\mu_2(x,t,k) = G^{-1}(x,t)\mu_2^0(x,t,k)e^{ik\int_{x}^{+\infty}(\sqrt{m(x',t)}-1)dx'\sigma_3}. \tag{2.30b}$$

Proposition 2.8. *The proposition (2.7) together with (2.27) allows expressing the expansions in powers of k of $a(k)$ at $k = 0$,*

$$a(k) = 1 + ikc - \frac{c^2}{2}k^2 + O(k^3). \tag{2.31}$$

3. The Riemann–Hilbert problem for SP equation

Let us define

$$M(x,t,k) = \begin{cases} \begin{pmatrix} [\mu_2]_1 & \frac{[\mu_1]_2}{a(k)} \end{pmatrix}, & k \in D_1, \\ \begin{pmatrix} \frac{[\mu_1]_1}{\overline{a(\bar{k})}} & [\mu_2]_2 \end{pmatrix}, & k \in D_2. \end{cases} \tag{3.1}$$

From the definition (3.1) and (2.22), we can deduce $M(x,t,k)$ satisfies the symmetry condition

$$\overline{M(x,t,\bar{k})} = M(x,t,-k) = \sigma_2 M(x,t,k)\sigma_2. \tag{3.2}$$

And $M(x,t,k)$ satisfies the following Riemann–Hilbert problem (P. 1–P. 2 in [24]):

- Jump condition: The two limiting values

$$M_\pm(x,t,k) = \lim_{\varepsilon\to 0} M_\pm(x,t,k \pm i\varepsilon), \quad k \in \mathbb{R}, \tag{3.3}$$

  are related by

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \quad k \in \mathbb{R}, \tag{3.4}$$

  where

$$J(x,t,k) = e^{ikp(x,t,k)\hat{\sigma}_3} J_0(k) \tag{3.5}$$

  here

$$J_0(k) = \begin{pmatrix} 1 & r(k) \\ \overline{r(k)} & 1+|r(k)|^2 \end{pmatrix} \tag{3.6}$$

  with $r(k) = \frac{b(k)}{a(k)}$.
- Normalize condition as $k \to \infty$

$$M(x,t,k) = \mathbb{I} + O(\frac{1}{k}). \tag{3.7}$$

In order to get the information of the solution $u(x,t)$, we should consider the asymptotic behavior of $M(x,t,k)$ as $k \to 0$, that is,

$$M(x,t,k) = G^{-1}(x,t)\left[\mathbb{I} + k(ic_+\sigma_3 + iu\sigma_1) + k^2[-\frac{c_+^2+u^2}{2}\mathbb{I} + i(uc_+ - 2u_t + u^2u_x)\sigma_2] + O(k^3)\right], \tag{3.8}$$

where

$$c_+ = \int_x^{+\infty} (\sqrt{m(x',t)} - 1)dx'. \tag{3.9}$$

Equation (3.8) shows that the matrix-valued function $M(x,t,k)$ contains all necessary information for reconstructing the solution of the initial value problem of (1.1a)–(1.1b) in terms of the solution of a matrix-valued Riemann–Hilbert problem.

However, the jump relation (3.5) cannot be used immediately for recovering the solution of SP equation (1.1a)–(1.1b). Since, in the representation of the jump matrix $e^{ikp(x,t,k)\hat{\sigma}_3}J_0(k)$ the factor $J_0(k)$ is indeed given in terms of the known initial data $u_0(x)$ but $p(x,t,k)$ is not, it involves $m(x,t)$ which is unknown (and, in fact, is to be reconstructed).

To overcome this, we introduce the new (time-dependent) scale

$$y(x,t) = x - \int_x^{+\infty} (\sqrt{m(x',t)} - 1)dx' = x - c_+(x,t), \tag{3.10}$$

in terms of which the jump matrix becomes explicit. The price to pay for this, however, is that the solution of the initial problem can be given only implicitly, or parametrically: it will be given in terms of functions in the new scale, whereas the original scale will also be given in terms of functions in the new scale.

By the definition of the new scale $y(x,t)$, we define

$$\tilde{M}(y,t,k) = M(x(y,t),t,k), \tag{3.11}$$

then we can obtain the Riemann–Hilbert problem for $\tilde{M}(y,t,k)$ as follows:

- Analyticity: $\tilde{M}(y,t,k)$ is analytic in the two open half-planes D_1 and D_2, and continuous up to the boundary $k \in \mathbb{R}$.
- Jump condition: The two limiting values

$$\tilde{M}_+(y,t,k) = \tilde{M}_-(y,t,k)\tilde{J}(y,t,k), \quad k \in \mathbb{R}, \tag{3.12a}$$

  where the jump matrix is

$$\tilde{J}(y,t,k) = e^{i(ky-\frac{t}{4k})\hat{\sigma}_3} J_0(k) \tag{3.12b}$$

  with $J_0(k)$ is defined as (3.6).
- Normalization:

$$\tilde{M}(y,t,k) \to \mathbb{I}, \quad k \to \infty. \tag{3.13}$$

Theorem 3.1. *Let $\tilde{M}(y,t,k)$ satisfies the above conditions, then this Riemann–Hilbert problem has a unique solution. And the solution $u(x,t)$ of the initial value problem (1.1a)–(1.1b) can be expressed, in parametric form, in terms of the solution of this Rieamnn–Hilbert problem:*

$$u(x,t) = u(y(x,t),t), \tag{3.14a}$$

where

$$x(y,t) = y + \lim_{k\to 0} \frac{\left((\tilde{M}(y,t,0))^{-1}\tilde{M}(y,t,k)\right)_{11} - 1}{ik} \tag{3.14b}$$

$$u(y,t) = \lim_{k\to 0} \frac{\left((\tilde{M}(y,t,0))^{-1}\tilde{M}(y,t,k)\right)_{21}}{ik} \tag{3.14c}$$

Proof. Since the jump matrix $\tilde{J}(y,t,k)$ is a Hermitian matrix, then the Riemann–Hilbert problem of $\tilde{M}(y,t,k)$ indeed has a solution. Furthermore, the Riemann–Hilbert problem has only one solution because of the normalize condition.

The statements of the solution $u(x,t)$ is following from the asymptotic formula (3.8). □

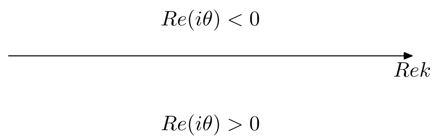

Fig. 1. The signs of $\mathrm{Re}\, i\theta$ in the k-plane in the case $\tilde{\xi} > 0$.

4. Long-time asymptotics: fast decaying region $\xi = \frac{x}{t} > \varepsilon > 0$, Proof of Theorem 1.2

In this section, we employ the nonlinear steepest descent method introduced by Deift and Zhou [14] to analyze the long-time asymptotic behavior of the solution $u(x,t)$ of the initial value problem (1.1a)–(1.1b).

The key feature of the method is the deformation of the original Riemann–Hilbert problem according to the signature table for the phase function θ in jump matrix $\tilde{J}$ written in the form

$$\tilde{J}(y,t,k) = e^{it\theta(\tilde{\xi},k)\hat{\sigma}_3} J_0(k), \tag{4.1}$$

where

$$\theta(\tilde{\xi},k) = \tilde{\xi}k - \frac{1}{4k}, \tag{4.2}$$

$$\tilde{\xi} = \frac{y}{t}. \tag{4.3}$$

The signature table is the distribution of signs of $\mathrm{Im}\theta(\tilde{\xi},k)$ in the k-plane,

$$\mathrm{Im}\theta(\tilde{\xi},k) = k_2[\tilde{\xi} + \frac{1}{4(k_1^2+k_2^2)}], \tag{4.4}$$

where k_1 and k_2 are the real and image part of k, respectively.

Under the condition $\tilde{\xi} > \varepsilon$ for any $\varepsilon > 0$, then we have $\mathrm{Im}\theta(\tilde{\xi},k) > 0$ and $\mathrm{Im}\theta(\tilde{\xi},k) < 0$, as $k_2 = \mathrm{Im}k > 0$ and $k_2 = \mathrm{Im}k < 0$, respectively, see Fig. 1.

This suggests the use of the following factorization of the jump matrix for all $k \in \mathbb{R}$:

$$\tilde{J}(y,t,k) = \begin{pmatrix} 1 & 0 \\ \overline{r(k)}e^{-2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & r(k)e^{2it\theta} \\ 0 & 1 \end{pmatrix} \tag{4.5}$$

Let $\overline{r(k)}$ as a Fourier transform with respect to θ,

$$\begin{aligned} \overline{r(k)}e^{-2it\theta} &= \tfrac{e^{-2it\theta}}{\sqrt{2\pi}(k-i)^2} \textstyle\int_{-\infty}^{\infty} e^{is\theta(k)}\hat{g}(s)ds \\ &= \tfrac{e^{-2it\theta}}{\sqrt{2\pi}(k-i)^2} \textstyle\int_{t}^{\infty} e^{is\theta(k)}\hat{g}(s)ds + \tfrac{e^{-2it\theta}}{\sqrt{2\pi}(k-i)^2} \textstyle\int_{-\infty}^{t} e^{is\theta(k)}\hat{g}(s)ds \\ &= e^{-2it\theta(k)}h_I(k) + e^{-2it\theta(k)}h_{II}(k), \end{aligned} \tag{4.6}$$

where

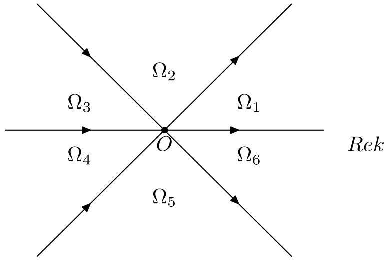

Fig. 2. The contour $\tilde{\Sigma}$ in the k-plane as $\tilde{\xi} > 0$.

$$\hat{g}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-is\theta(k)} g(\theta) d\theta,$$
$$g(\theta) = \overline{r(k(\theta))}(k(\theta) - i)^2.$$

Here $e^{-2it\theta(k)} h_{II}(k)$ has an analytic continuation to the lower half-plane and decays exponentially in $L^1 \cap L^\infty(\Sigma \cap \{k|\mathrm{Im}k < 0\})$, as $t \to \infty$, while $e^{-2it\theta(k)} h_I(k)$ decays rapidly in $L^1 \cap L^\infty(\mathbb{R})$, as $t \to \infty$.

Introducing the following the following transformation:

$$\tilde{M}^{(1)}(y,t,k) = \begin{cases} \tilde{M}(y,t,k) \begin{pmatrix} 1 & -\overline{h_{II}(\bar{k})} e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \Omega_1 \cup \Omega_3, \\ \tilde{M}(y,t,k) \begin{pmatrix} 1 & 0 \\ h_{II}(k) e^{-2it\theta} & 1 \end{pmatrix}, & k \in \Omega_4 \cup \Omega_6, \\ \tilde{M}(y,t,k), & k \in \Omega_2 \cup \Omega_5, \end{cases} \tag{4.7}$$

where Ω_j, $j = 1, 2, \ldots, 6$ are shown in Fig. 2. We obtain the new Riemann–Hilbert problem for $\tilde{M}^{(1)}(y,t,k)$,

$$\begin{cases} \tilde{M}^{(1)}_+(y,t,k) = \tilde{M}^{(1)}_-(y,t,k) \tilde{J}^{(1)}(y,t,k), & k \in \tilde{\Sigma}, \\ \tilde{M}^{(1)}(y,t,k) \to \mathbb{I}, & k \to \infty. \end{cases} \tag{4.8}$$

where

$$\tilde{J}^{(1)}(y,t,k) = \begin{cases} \begin{pmatrix} 1 & \overline{h_{II}(\bar{k})} e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \tilde{\Sigma} \cap D_1, \\ \begin{pmatrix} 1 & 0 \\ h_{II}(k) e^{-2it\theta} & 1 \end{pmatrix}, & k \in \tilde{\Sigma} \cap D_2, \\ \begin{pmatrix} 1 & 0 \\ h_I(k) e^{-2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \overline{h_I(\bar{k})} e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \mathbb{R}. \end{cases} \tag{4.9}$$

Theorem 4.1. *As $t \to \infty$, the solution $u(x,t)$ of the initial value problem (1.1a)–(1.1b) decays rapidly in the range $\xi > \varepsilon$ for any $\varepsilon > 0$.*

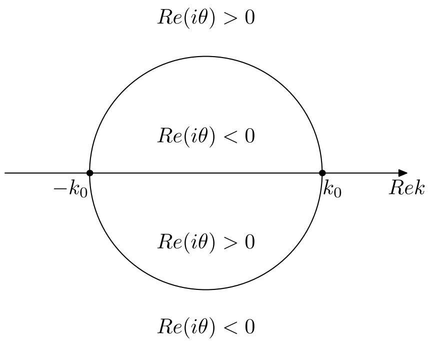

Fig. 3. The signs of Re$i\theta$ in the k-plane in the case $\tilde{\xi} < 0$.

Proof. The above transformation reduces the Riemann–Hilbert problem for $\tilde{M}^{(1)}(y,t,k)$ to that with exponentially decaying in t to the identity matrix jump matrix. Since this Riemann–Hilbert problem is holomorphic, its solution decays fast to $\mathbb{I}$ and consequently $\tilde{u}(y,t)$ decays fast to 0 while y approaches fast x and thus the domain $\tilde{\xi} > \varepsilon$ and $\xi > \varepsilon$ coincide asymptotically. □

5. Long-time asymptotics: oscillation region $\xi < -\varepsilon < 0$, Proof of Theorem 1.3

If $\tilde{\xi} < -\varepsilon$ for any $\varepsilon > 0$, let k_0 be defined by

$$k_0 = \sqrt{\frac{-1}{4\tilde{\xi}}}, \tag{5.1}$$

then the signature table is shown as Fig. 3,

This suggests the use of the following factorizations of the jump matrix $\tilde{J}(y,t,k)$:

$$\tilde{J}(y,t,k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \overline{r(k)}e^{-2it\theta} & 1 \end{pmatrix}\begin{pmatrix} 1 & r(k)e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & |k| < k_0, \\ \begin{pmatrix} 1 & \frac{r(k)}{1+|r(k)|^2}e^{2it\theta} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{1+|r(k)|^2} & 0 \\ 0 & 1+|r(k)|^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\overline{r(k)}}{1+|r(k)|^2}e^{-2it\theta} & 1 \end{pmatrix}, & |k| > k_0. \end{cases} \tag{5.2}$$

Then we need make some appropriate sequence of deformations of this Riemann–Hilbert problem.

5.1. The conjugate transformation

The aim of the first transformation involves the removal of the diagonal factor in (5.2) for $|k| > k_0$.

Introducing a scalar function $\delta(k)$ which satisfies the following scalar Riemann–Hilbert problem

$$\begin{cases} \delta_+(k) = \delta_-(k)(1+|r(k)|^2), & |k| > k_0, \\ \qquad\;\; = \delta_-(k) = \delta(k), & |k| < k_0, \\ \delta(k) \to 1, & k \to \infty. \end{cases} \tag{5.3}$$

Then the function $\delta(k)$ is given by

$$\delta(k) = e^{\frac{1}{2\pi i}(\int_{-\infty}^{-k_0} + \int_{k_0}^{+\infty})\frac{\ln(1+|r(s)|^2)}{s-k}ds}. \tag{5.4}$$

The conjugate transformation

$$\tilde{M}^{(1)}(y,t,k) = \tilde{M}(y,t,k)\delta(k)^{\sigma_3}, \tag{5.5}$$

yields the Riemann–Hilbert problem for $\tilde{M}^{(1)}(y,t,k)$

$$\begin{cases} \tilde{M}_+^{(1)}(y,t,k) = \tilde{M}_-^{(1)}(y,t,k)\tilde{J}^{(1)}(y,t,k), & k \in \mathbb{R}, \\ \tilde{M}^{(1)}(y,t,k) \to \mathbb{I}, & k \to \infty, \end{cases} \tag{5.6a}$$

where

$$\tilde{J}^{(1)}(y,t,k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \overline{r(k)}\delta^2 e^{-2it\theta} & 1 \end{pmatrix}\begin{pmatrix} 1 & r(k)\delta^{-2}e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & |k| < k_0, \\ \begin{pmatrix} 1 & \frac{r(k)}{1+|r(k)|^2}\delta_-^{-2}e^{2it\theta} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\overline{r(k)}}{1+|r(k)|^2}\delta_+^2 e^{-2it\theta} & 1 \end{pmatrix}, & |k| > k_0. \end{cases} \tag{5.6b}$$

Now, let us come back to the solution $u(x,t)$. From (5.4) it follows that

$$\delta(k) = \delta_0 + k\delta_1 + O(k^2) = 1 - \frac{ik}{\pi}\int_{k_0}^{\infty}\frac{\ln(1+|r(s)|^2)}{s^2}ds + O(k^2). \tag{5.7}$$

If we write

$$\tilde{M}(y,t,k) = \tilde{M}_0(y,t) + k\tilde{M}_1(y,t) + O(k^2), \quad k \to 0, \tag{5.8}$$

and

$$\tilde{M}^{(1)}(y,t,k) = \tilde{M}_0^{(1)}(y,t) + k\tilde{M}_1^{(1)}(y,t) + O(k^2), \quad k \to 0, \tag{5.9}$$

then from the transformation (5.5) we obtain

$$\tilde{M}_0(y,t) = \tilde{M}_0^{(1)}(y,t), \quad \tilde{M}_1(y,t) = \tilde{M}_1^{(1)}(y,t) - \tilde{M}_0^{(1)}(y,t)\delta_1\sigma_3. \tag{5.10}$$

Hence, we have

$$u(x,t) = -i\left[(\tilde{M}_0^{(1)})^{-1}\tilde{M}_1^{(1)}\right]_{21}, \tag{5.11a}$$

$$c_+ = -i\left(\left[(\tilde{M}_0^{(1)})^{-1}\tilde{M}_1^{(1)}\right]_{11} - \delta_1\right) \tag{5.11b}$$

5.2. Analytic extension

For the convenience of the notation, we transverse the direction of the component $|k| < k_0$ of the jump contour $\mathbb{R}$ for the Riemann–Hilbert problem for $\tilde{M}^{(1)}(y,t,k)$. Then, the jump matrix $\tilde{J}^{(1)}(y,t,k)$ becomes

$$\tilde{J}^{(1)}(y,t,k) = \begin{cases} \begin{pmatrix} 1 & -r(k)\delta^{-2}e^{2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\overline{r(k)}\delta^2 e^{-2it\theta} & 1 \end{pmatrix}, & |k| < k_0, \\ \begin{pmatrix} 1 & \frac{r(k)}{1+|r(k)|^2}\delta_-^{-2}e^{2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\overline{r(k)}}{1+|r(k)|^2}\delta_+^2 e^{-2it\theta} & 1 \end{pmatrix}, & |k| > k_0. \end{cases} \tag{5.12}$$

Denoting some contours:

$$L_0 = \{k | k = k_0\lambda e^{-i\frac{\pi}{4}}, 0 \le k \le \frac{1}{\sqrt{2}}\} \cup \{k | k = k_0\lambda e^{i\frac{3\pi}{4}}, 0 \le k \le \frac{1}{\sqrt{2}}\}, \tag{5.13a}$$

$$L_1 = \{k | k = k_0 + k_0\lambda e^{-i\frac{3\pi}{4}}, -\infty < \lambda < \frac{1}{\sqrt{2}}\}, \quad L_{1\varepsilon} = \{k | k = k_0 + k_0\lambda e^{-i\frac{3\pi}{4}}, \varepsilon < \lambda < \frac{1}{\sqrt{2}}\} \tag{5.13b}$$

$$L_2 = \{k | k = -k_0 + k_0\lambda e^{-i\frac{\pi}{4}}, -\infty < \lambda < \frac{1}{\sqrt{2}}\}, \quad L_{2\varepsilon} = \{k | k = -k_0 + k_0\lambda e^{-i\frac{\pi}{4}}, \varepsilon < \lambda < \frac{1}{\sqrt{2}}\}. \tag{5.13c}$$

Proposition 5.1. *Let*

$$\rho(k) = \begin{cases} -\overline{r(k)}, & |k| < k_0, \\ \frac{\overline{r(k)}}{1+|r(k)|^2}, & |k| > k_0. \end{cases} \tag{5.14}$$

Then $\rho(k)$ has a decomposition

$$\rho(k) = h_I(k) + (h_{II}(k) + R(k)), \tag{5.15}$$

where $h_I(k)$ is small and $h_{II}(k)$ has an analytic continuation to L and L_0. For example, if $k > k_0$, $h_{II}(k)$ of the function $\rho(k)$ has an analytic continuation to the $L_1 \cap \mathrm{Im}k > 0$. And $R(k)$ is piecewise rational ($R(k) = 0$, if $k \in L_0$) function.

And let M be a positive constant, as $k_0 < M$, $R(k), h_I(k), h_{II}(k)$ satisfy

$$|e^{-2it\theta(k)}h_I(k)| \le \frac{c}{(1+|k|^2)t^l}, \quad k \in \mathbb{R}, \tag{5.16a}$$

$$|e^{-2it\theta(k)}\frac{h_I(k)}{k^j}| \le \frac{c}{(1+|k|^2)t^l}, \quad 0 < |k| < \frac{|k_0|}{2}, j = 1, 2, \tag{5.16b}$$

$$|e^{-2it\theta(k)}h_{II}(k)| \le \frac{c}{(1+|k|^2)t^l}, \quad k \in L = Ł_1 \cup L_2, \tag{5.16c}$$

$$|e^{-2it\theta(k)}h_{II}(k)| \le ce^{-t\frac{1}{8k_0}}, \quad k \in L_0, \tag{5.16d}$$

$$|e^{-2it\theta(k)}\frac{h_{II}(k)}{k^j}| \le ce^{-t\frac{1}{8k_0}}, \quad k \in L_0,\ j = 1, 2, \tag{5.16e}$$

and

$$|e^{-2it\theta(k)}R(k)| \le ce^{-\frac{\varepsilon^2}{2k_0}t}, \quad k \in L_\varepsilon = L_{1\varepsilon} \cup L_{2\varepsilon}. \tag{5.16f}$$

for arbitrary natural number l, for sufficiently large constants c, for some fixed positive constant M.

Proof. When $\frac{k_0}{2} < |k| < k_0$ and $|k| > k_0$ for $k \in \mathbb{R}$, the proof is similar to P. 310–318 in [14]. Here, we prove the case $0 < |k| < \frac{|k_0|}{2}$, which has some differences, we just consider $0 < k < \frac{k_0}{2}$, the case for $-\frac{k_0}{2} < k < 0$ is similar.

Define

$$\begin{cases} \rho(\theta) &= \rho(k(\theta)),\ \ \theta < \theta(\frac{k_0}{2}), \\ &= 0, \qquad\qquad \theta \ge \theta(\frac{k_0}{2}). \end{cases} \tag{5.17}$$

We claim that $\rho(\theta) \in H^j(-\infty < \theta < \infty)$ for any nonnegative integer j.

By Fourier inversion,

$$\rho(\theta(k)) = \int_{-\infty}^{\infty} e^{is\theta(k)}\hat{\rho}(s)\bar{d}s, \quad 0 < k < \frac{k_0}{2}, \tag{5.18}$$

where

$$\hat{\rho}(s) = \int_{-\infty}^{\theta(\frac{k_0}{2})} e^{-is\theta(k)}\rho(\theta(k))\bar{d}\theta(k). \tag{5.19}$$

Then,

$$\begin{aligned} &\int_{-\infty}^{\theta(\frac{k_0}{2})} \left|\left(\frac{d}{d\theta}\right)^j \rho(\theta(k))\right|^2 |\bar{d}\theta(k)| \\ &= \int_0^{\frac{k_0}{2}} \left|\left(\frac{4k^2k_0^2}{k_0^2-k^2}\frac{d}{dk}\right)^j \rho(k)\right|^2 |\frac{k_0^2-k^2}{4k^2k_0^2}|\bar{d}k \le C < \infty, \end{aligned} \tag{5.20}$$

for any nonnegative integer j, $0 < k_0 < M$, since $r(k) \to 0$ rapidly, as $k \to 0$.

Hence

$$\int_{-\infty}^{\infty} (1+s^2)^j |\hat{\rho}(s)|^2 \bar{d}s \le C, \tag{5.21}$$

for any nonnegative integer j.

Split

$$\begin{aligned}\rho(k) &= \int_t^\infty e^{is\theta(k)}\hat{\rho}(s)\bar{d}s + \int_{-\infty}^t e^{is\theta(k)}\hat{\rho}(s)\bar{d}s \\ &= h_I(k) + h_{II}(k).\end{aligned} \tag{5.22}$$

Then, for $0 < k < \frac{k_0}{2}$ and any positive integer j, we obtain,

$$\begin{aligned}|e^{-2it\theta(k)}h_I(k)| &\le \int_t^\infty |\hat{\rho}|\bar{d}s \\ &\le (\int_t^\infty (1+s^2)^{-j}\bar{d}s)^{\frac{1}{2}}(\int_t^\infty (1+s^2)^j|\hat{\rho}(s)|^2\bar{d}s)^{\frac{1}{2}} \\ &\le \frac{c}{t^{j-\frac{1}{2}}}.\end{aligned} \tag{5.23}$$

Consider the contour $l_2 : k(u) = uk_0e^{-i\frac{\pi}{4}}, 0 < u < \frac{1}{\sqrt{2}}$. Since $\mathrm{Re}i\theta(k)$ is positive on this contour, h_{II} has an analytic continuation to contour l_2.

On the contour l_2,

$$\begin{aligned}|e^{-2it\theta(k)}h_{II}(k)| &\le e^{-t\mathrm{Re}i\theta(k)}\int_{-\infty}^t e^{(s-t)\mathrm{Re}i\theta(k)}|\hat{\rho}(k)|\bar{d}s \\ &\le e^{-t\mathrm{Re}i\theta(k)}(\int_{-\infty}^t (1+s^2)^{-1}\bar{d}s)^{\frac{1}{2}}(\int_{-\infty}^t (1+s^2)|\hat{\rho}(k)|^2\bar{d}s)^{\frac{1}{2}},\end{aligned} \tag{5.24}$$

where

$$\mathrm{Re}i\theta(k) = \frac{\sqrt{2}}{8k_0}(\frac{1}{u} - u) \ge \frac{1}{8k_0}, \tag{5.25}$$

for $0 < u \le \frac{1}{\sqrt{2}}$.

Thus, we obtain,

$$|e^{-2it\theta(k)}h_{II}(k)| \le ce^{-\frac{t}{8k_0}}. \quad \square \tag{5.26}$$

5.3. *Second Transformation*

The main purpose of this subsection is to reformulate the Riemann–Hilbert problem for $\tilde{M}^{(1)}(y,t,k)$ (5.6) as an equivalent Riemann–Hilbert problem on the augmented contour $\Sigma = L \cup \bar{L} \cup L_0 \cup \bar{L}_0 \cup \mathbb{R}$ (see Fig. 4).

According to the above analytic extension of $\rho(k)$, we can write the jump matrix $\tilde{J}^{(1)}(y,t,k)$ as

$$\tilde{J}^{(1)}(y,t,k) = b_-^{-1}(y,t,k)b_+(y,t,k), \tag{5.27a}$$

where

$$b_-(y,t,k) = \begin{pmatrix} 1 & -\overline{h_I(\bar{k})}\delta_-^{-2}e^{2it\theta} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\overline{h_{II} + R}(\bar{k})\delta_-^{-2}e^{2it\theta} \\ 0 & 1 \end{pmatrix} = b_-^R(y,t,k)b_-^I(y,t,k), \tag{5.27b}$$

and

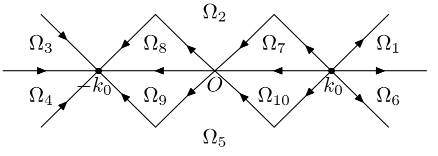

Fig. 4. The jump contour Σ for $\tilde{M}^{(2)}(y,t,k)$.

$$b_-(y,t,k)=\begin{pmatrix}1 & 0\\ h_I(k)\delta_+^2e^{-2it\theta} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ (h_{II}+R)(k)\delta_+^2e^{-2it\theta} & 1\end{pmatrix}=b_+^R(y,t,k)b_+^I(y,t,k). \tag{5.27c}$$

We make a transformation as

$$\tilde{M}^{(2)}(y,t,k)=\tilde{M}^{(1)}(y,t,k)T(y,t,k), \tag{5.28}$$

where

$$T(y,t,k)=\begin{cases}(b_+^I(y,t,k))^{-1}, & k\in\Omega_1\cup\Omega_3\cup\Omega_9\cup\Omega_{10},\\ (b_-^I(y,t,k))^{-1}, & k\in\Omega_4\cup\Omega_6\cup\Omega_7\cup\Omega_8,\\ \mathbb{I}, & k\in\Omega_2\cup\Omega_5.\end{cases} \tag{5.29}$$

with the regions $\{\Omega_j\}_{j=1}^{10}$ defined as Fig. 4.

Then, $\tilde{M}^{(2)}(y,t,k)$ satisfies the following Riemann–Hilbert problem,

$$\begin{cases}\tilde{M}_+^{(2)}(y,t,k)=\tilde{M}_-^{(2)}(y,t,k)\tilde{J}^{(2)}(y,t,k),\\ \tilde{M}^{(2)}(y,t,k)\to\mathbb{I}, \quad k\to\infty,\end{cases} \tag{5.30a}$$

where

$$\tilde{J}^{(2)}(y,t,k)=(b_-^{(2)}(y,t,k))^{-1}b_+^{(2)}(y,t,k)=\begin{cases}(b_-^R(y,t,k))^{-1}b_+^R(y,t,k), & k\in\mathbb{R},\\ b_+^I(y,t,k), & k\in\Sigma\cap\mathrm{Im}k>0,\\ (b_-^I(y,t,k))^{-1}, & k\in\Sigma\cap\mathrm{Im}k<0.\end{cases} \tag{5.30b}$$

Proposition 5.2. *The reflection coefficient $r(k)=O(k^3)$ as $k\to 0$.*

Proof. A direct calculation following from (2.31) and from the identity $|r(k)|^2=\frac{1}{|a(k)|^2}-1$. $\square$

Now, let us come back to the considered problem in this paper again. The solution $u(y,t)$ is related to the solution of the Riemann–Hilbert problem evaluated at $k=0$, it may be affected by this transformation. However, due to the above fact, the second transformation turns out not to affect the terms in the expansion of the solution of the Riemann–Hilbert problem at $k=0$ at least up to the terms of order $O(k^2)$ and thus it does not really affect $u(y,t)$.

So, if we write

$$\tilde{M}^{(2)}(y,t,k) = \tilde{M}_0^{(2)}(y,t) + k\tilde{M}_1^{(2)}(y,t) + O(k^2), \quad k \to 0, \tag{5.31}$$

then we have

$$u(x,t) = -i\left[(\tilde{M}_0^{(2)})^{-1}\tilde{M}_1^{(2)}\right]_{21}, \tag{5.32a}$$

$$c_+ = -i\left(\left[(\tilde{M}_0^{(2)})^{-1}\tilde{M}_1^{(2)}\right]_{11} - \delta_1\right) \tag{5.32b}$$

Set

$$\omega_\pm^{(2)}(y,t,k) = \pm(b_\pm^{(2)}(y,t,k) - \mathbb{I}), \quad \omega = \omega_+^{(2)} + \omega_-^{(2)}, \tag{5.33}$$

and let $\mu^{(2)}(y,t,k)$ be the solution of the singular integral equation $\mu^{(2)} = \mathbb{I} + C_\omega \mu^{(2)}$, here C_ω is defined as $C_\omega f = C_+(f\omega_-) + C_-(f\omega_+)$, with $C_\pm$ denote the Cauchy operator, then

$$\tilde{M}^{(2)}(y,t,k) = \mathbb{I} + \frac{1}{2\pi i}\int_\Sigma \frac{\mu^{(2)}(y,t,\eta)\omega(y,t,\eta)}{\eta - k}d\eta, \quad k \in \mathbb{C}\backslash\Sigma \tag{5.34}$$

is the solution of Riemann–Hilbert problem (5.30), see the P. 323 in [14].

Expanding the integral (5.34) around $k = 0$, we have

$$\tilde{M}_0^{(2)}(y,t) = \mathbb{I} + \frac{1}{2\pi i}\int_\Sigma \frac{\mu^{(2)}(y,t,\eta)\omega(y,t,\eta)}{\eta}d\eta, \tag{5.35a}$$

$$\tilde{M}_1^{(2)}(y,t) = \frac{1}{2\pi i}\int_\Sigma \frac{\mu^{(2)}(y,t,\eta)\omega(y,t,\eta)}{\eta^2}d\eta \tag{5.35b}$$

Remark 5.3. Since, $\omega(y,t,k)$ decays rapidly at $k = 0$, the integral (5.35) are nonsingular.

5.4. Reduction to the cross

Let ω^e be a sum of four terms

$$\omega^e = \omega^a + \omega^b + \omega^c + \omega^d. \tag{5.36}$$

We then have the following:

$$\begin{aligned} &\omega^a = \omega \text{ is supported on the } \mathbb{R} \text{ and consists of terms of type } h_I(k) \text{ and } \overline{h_I(k)}.\\ &\omega^b = \omega \text{ is supported on the } L \cup \bar{L} \text{ and consists of terms of type } h_{II}(k) \text{ and } \overline{h_{II}(\bar{k})}.\\ &\omega^c = \omega \text{ is supported on the } L_\varepsilon \cup \bar{L}_\varepsilon \text{ and consists of terms of type } R(k) \text{ and } \overline{R(\bar{k})}.\\ &\omega^d = \omega \text{ is supported on the } L_0 \cup \bar{L}_0. \end{aligned} \tag{5.37}$$

Fig. 5. The jump contour $\Sigma^{(3)}$ for $\tilde{M}^{(3)}(y,t,k)$.

Set $\omega' = \omega - \omega^e$. Then, $\omega' = 0$ on $\Sigma^{(2)} \setminus \Sigma^{(3)}$. Thus, ω' is supported on $\Sigma^{(3)}$ (Fig. 5) with contribution to ω from rational terms R and $\bar{R}$.

Proposition 5.4. *For $0 < k_0 < M$, we have*

$$||\omega^a||_{L^1\cap L^2\cap L^\infty(\mathbb{R})} \leq \frac{c}{t^l}, \tag{5.38a}$$

$$||\frac{\omega^a}{k^j}||_{L^1\cap L^2\cap L^\infty(|k|<k_0)} \leq \frac{c}{t^l},\ j=1,2 \tag{5.38b}$$

$$||\omega^b||_{L^1(L\cup\bar{L})\cap L^2(L\cup\bar{L})\cap L^\infty(L\cup\bar{L})} \leq \frac{c}{t^l}, \tag{5.38c}$$

$$||\omega^c||_{L^1(L_\varepsilon\cup\bar{L}_\varepsilon)\cap L^2(L_\varepsilon\cup\bar{L}_\varepsilon)\cap L^\infty(L_\varepsilon\cup\bar{L}_\varepsilon)} \leq ce^{-\frac{\varepsilon^2}{2k_0}t}, \tag{5.38d}$$

$$||\omega^d||_{L^1(L_0\cup\bar{L}_0)\cap L^2(L_0\cup\bar{L}_0)\cap L^\infty(L_0\cup\bar{L}_0)} \leq ce^{-\frac{t}{8k_0}}, \tag{5.38e}$$

$$||\frac{\omega^d}{k^j}||_{L^1(L_0\cup\bar{L}_0)\cap L^2(L_0\cup\bar{L}_0)\cap L^\infty(L_0\cup\bar{L}_0)} \leq ce^{-\frac{t}{8k_0}},\ j=1,2 \tag{5.38f}$$

Moreover,

$$||\omega'||_{L^2(\Sigma^{(3)})} \leq \frac{c}{t^{\frac{1}{4}}}, \qquad ||\omega'||_{L^1(\Sigma^{(3)})} \leq \frac{c}{t^{\frac{1}{2}}} \tag{5.39}$$

Proof. Consequence of Proposition 5.1, and analogous calculations as in lemma 2.13 of [14]. $\square$

Proposition 5.5. *As $t \to \infty$ and $0 < k_0 < M$, $||(1 - C_{\omega'})^{-1}||_{L^2(\Sigma^{(2)})}$ exists and is uniformly bounded, and $||(1 - C_\omega)^{-1}||_{L^2(\Sigma^{(2)})} \leq C$ is equivalent to $||(1 - C_{\omega'})^{-1}||_{L^2(\Sigma^{(2)})} \leq C$.*

Proof. The existence of the operator $(1 - C_{\omega'})^{-1}$ is followed similar to [14], P. 324. And the equivalence is the consequence of the following inequality, $||C_\omega - C_{\omega'}||_{L^2(\Sigma^{(2)})} \leq c||\omega^e||_{L^2(\Sigma^{(2)})}$, the fact that $||\omega^e||_{L^2(\Sigma^{(2)})} \leq \frac{c}{t^l}$, and the second resolvent identity. $\square$

Proposition 5.6. *If $||(1 - C_{\omega'})^{-1}||_{L^2(\Sigma^{(2)})} \leq C$, then for arbitrary positive integer l, as $t \to \infty$ such that $0 < k_0 < M$,*

$$\int_\Sigma \frac{((\mathbb{I} - C_\omega)^{-1}\mathbb{I})(\eta)\omega(x,t,\eta)}{\eta^j} d\eta = \int_\Sigma \frac{((\mathbb{I} - C_{\omega'})^{-1}\mathbb{I})(\eta)\omega'(x,t,\eta)}{\eta^j} d\eta + O(\frac{c}{t^l}),\ j=1,2. \tag{5.40}$$

Proof. From the second resolvent identity, one can derive the following expression (see equation (2.27) in [14]),

$$\begin{aligned}\int_\Sigma \frac{((1-C_\omega)^{-1}\mathbb{I})\omega}{\eta^j}d\eta &= \int_\Sigma \frac{((1-C_{\omega'})^{-1}\mathbb{I})\omega'}{\eta^j}d\eta + \int_\Sigma \frac{\omega^e}{\eta^j}d\eta \\ &\quad + \int_\Sigma \frac{((1-C_{\omega'})^{-1}(C_{\omega^e}\mathbb{I}))\omega}{\eta^j}d\eta \\ &\quad + \int_\Sigma \frac{((1-C_{\omega'})^{-1}(C_{\omega'}\mathbb{I}))\omega^e}{\eta^j}d\eta \\ &\quad + \int_\Sigma \frac{((1-C_{\omega'})^{-1}C_{\omega^e}(1-C_\omega)^{-1})(C_\omega\mathbb{I})\omega}{\eta^j}d\eta \\ &= \int_\Sigma \frac{((1-C_{\omega'})^{-1}\mathbb{I})\omega'}{\eta^j}d\eta + I + II + III + IV.\end{aligned} \tag{5.41}$$

For $0 < k_0 < M$, from Proposition (5.4) it follows that,

$$\begin{aligned}|I| &\le ||\tfrac{\omega^a}{k^j}||_{L^1(\mathbb{R})} + ||\tfrac{\omega^b}{k^j}||_{L^1(L\cup\bar{L})} + ||\tfrac{\omega^c}{k^j}||_{L^1(L_\varepsilon\cup\bar{L}_\varepsilon)} + ||\tfrac{\omega^d}{k^j}||_{L^1(L_0\cup\bar{L}_0)} \\ &\le ct^{-l},\end{aligned} \tag{5.42}$$

$$\begin{aligned}|II| &\le ||(1-C_{\omega'})^{-1}||_{L^2(\Sigma)}||(C_{\omega^e}\mathbb{I})||_{L^2(\Sigma)}||\tfrac{\omega}{k^j}||_{L^2(\Sigma)} \\ &\le c||\omega^e||_{L^2(\Sigma^{(2)})}(||\omega^e||_{L^2(\Sigma)} + ||\omega'||_{L^2(\Sigma)}) \\ &\le ct^{-l}(ct^{-l}+c) \le ct^{-l},\end{aligned} \tag{5.43}$$

$$\begin{aligned}|III| &\le ||(1-C_{\omega'})^{-1}||_{L^2(\Sigma)}||(C_{\omega'}\mathbb{I})||_{L^2(\Sigma)}||\tfrac{\omega^e}{k^j}||_{L^2(\Sigma)} \\ &\le ct^{-l},\end{aligned} \tag{5.44}$$

$$\begin{aligned}|IV| &\le ||(1-C_{\omega'})^{-1}C_{\omega^e}(1-C_\omega)^{-1})(C_\omega\mathbb{I})||_{L^2(\Sigma)}||\tfrac{\omega}{k^j}||_{L^2(\Sigma)} \\ &\le ||(1-C_{\omega'})^{-1}||_{L^2(\Sigma)}||C_{\omega^e}||_{L^2(\Sigma)}||(1-C_\omega)^{-1}||_{L^2(\Sigma)}||(C_\omega\mathbb{I})||_{L^2(\Sigma)}||\tfrac{\omega}{k^j}||_{L^2(\Sigma)} \\ &\le c||C_{\omega^e}||_{L^2(\Sigma^{(2)})}||(C_\omega\mathbb{I})||_{L^2(\Sigma)}||\tfrac{\omega}{k^j}||_{L^2(\Sigma^{(2)})} \\ &\le c||\omega^e||_{L^2(\Sigma)}||\tfrac{\omega}{k^j}||^2_{L^2(\Sigma)} \\ &\le ct^{-l}.\end{aligned} \tag{5.45}$$

Hence,

$$|I + II + III + IV| \le ct^{-l}. \tag{5.46}$$

Applying these estimates to equation (5.41), we can obtain equation (5.40). □

Following the method in [14] (P. 328–P. 330), we apply the lemma 2.56 in [14] to the case $u = \omega'$, $\Sigma_{12} = \Sigma$ and $\Sigma_1 = \Sigma^{(3)}$. From identity (2.58) in [14], we get the following proposition, which shows that the integral region can be changed from Σ to $\Sigma^{(3)}$ without alternating the Riemann–Hilbert problem.

Proposition 5.7.

$$\int_{\Sigma} \frac{((\mathbb{I}-C_{\omega'})^{-1}\mathbb{I})(\eta)\omega'(x,t,\eta)}{\eta^j} d\eta = \int_{\Sigma^{(3)}} \frac{((\mathbb{I}-C_{\omega'})^{-1}\mathbb{I})(\eta)\omega'(x,t,\eta)}{\eta^j} d\eta. \tag{5.47}$$

Set

$$L' = L \backslash L_\varepsilon.$$

Then, $\Sigma^{(3)} = L' \cup \bar{L}'$. On $\Sigma^{(3)}$, set $\mu' = (1^{\Sigma^{(3)}} - C_{\omega'}^{\Sigma^{(3)}})^{-1}\mathbb{I}$. Then,

$$M^{(3)}(x,t,k) = \mathbb{I} + \int_{\Sigma^{(3)}} \frac{\mu'(\xi)\omega'(\xi)}{\xi - k} \frac{d\xi}{2\pi i} \tag{5.48}$$

solves the Riemann–Hilbert problem

$$\begin{cases} M_+^{(3)}(x,t,k) = M_-^{(3)}(x,t,k) J^{(3)}(x,t,k), & k \in \Sigma^{(3)}, \\ M^{(3)} \to \mathbb{I}, & k \to \infty, \end{cases} \tag{5.49}$$

where

$$\omega' = \omega'_+ + \omega'_-, \tag{5.50}$$

$$b'_\pm = \mathbb{I} \pm \omega'_\pm, \tag{5.51}$$

$$J^{(3)}(x,t,k) = (b'_-)^{-1} b'_+ \tag{5.52}$$

Hence, we have the representation of the solution is as follows,

Theorem 5.8. *As $t \to \infty$,*

$$\begin{aligned} iu(y,t) \;=\; & \left(\mathbb{I} + \tfrac{1}{2\pi i}\int_{\Sigma^{(3)}} \tfrac{(\mathbb{I}-C_{\omega'})^{-1}\mathbb{I}(\eta)\omega'(\eta)}{\eta} d\eta + O(t^{-l})\right)_{11} \\ & \cdot \left(\tfrac{1}{2\pi i}\int_{\Sigma^{(3)}} \tfrac{(\mathbb{I}-C_{\omega'})^{-1}\mathbb{I}(\eta)\omega'(\eta)}{\eta^2} d\eta + O(t^{-l})\right)_{21} \\ & - \left(\tfrac{1}{2\pi i}\int_{\Sigma^{(3)}} \tfrac{(\mathbb{I}-C_{\omega'})^{-1}\mathbb{I}(\eta)\omega'(\eta)}{\eta} d\eta + O(t^{-l})\right)_{21} \\ & \cdot \left(\tfrac{1}{2\pi i}\int_{\Sigma^{(3)}} \tfrac{(\mathbb{I}-C_{\omega'})^{-1}\mathbb{I}(\eta)\omega'(\eta)}{\eta^2} d\eta + O(t^{-l})\right)_{11}, \end{aligned} \tag{5.53a}$$

and

$$\begin{aligned} ic_+(y,t) \;=\; & \left(\mathbb{I} + \tfrac{1}{2\pi i}\int_{\Sigma^{(3)}} \tfrac{(\mathbb{I}-C_{\omega'})^{-1}\mathbb{I}(\eta)\omega'(\eta)}{\eta} d\eta + O(t^{-l})\right)_{22} \\ & \cdot \left(\tfrac{1}{2\pi i}\int_{\Sigma^{(3)}} \tfrac{(\mathbb{I}-C_{\omega'})^{-1}\mathbb{I}(\eta)\omega'(\eta)}{\eta^2} d\eta + O(t^{-l})\right)_{11} \\ & - \left(\tfrac{1}{2\pi i}\int_{\Sigma^{(3)}} \tfrac{(\mathbb{I}-C_{\omega'})^{-1}\mathbb{I}(\eta)\omega'(\eta)}{\eta} d\eta + O(t^{-l})\right)_{12} \\ & \cdot \left(\tfrac{1}{2\pi i}\int_{\Sigma^{(3)}} \tfrac{(\mathbb{I}-C_{\omega'})^{-1}\mathbb{I}(\eta)\omega'(\eta)}{\eta^2} d\eta + O(t^{-l})\right)_{21} - \delta_1. \end{aligned} \tag{5.53b}$$

5.5. Separate out the contributions of the two crosses

Using the estimates of the Proposition 5.4 and the similar method in [14], P. 330–331, we can separate out the contributions of the two crosses in $\Sigma^{(3)}$ to the solution $u(y,t)$ in formula (5.53a). Let the contour $\Sigma^{(3)} = \Sigma^{A'} \cup \Sigma^{B'}$ and write

$$\omega' = \omega^{A'} + \omega^{B'}, \tag{5.54}$$

where

$$\begin{aligned} &\omega^{A'}(k) = 0, \quad \text{for } k \in \Sigma_{B'}, \\ &\omega^{B'}(k) = 0, \quad \text{for } k \in \Sigma_{A'}. \end{aligned} \tag{5.55}$$

Proposition 5.9.

$$\begin{aligned} &||C_{\omega^{B'}}^{\Sigma^{(3)}} C_{\omega^{A'}}^{\Sigma^{(3)}}||_{L^2(\Sigma^{(3)})} = ||C_{\omega^{A'}}^{\Sigma^{(3)}} C_{\omega^{B'}}^{\Sigma^{(3)}}||_{L^2(\Sigma^{(3)})} \le \frac{C(k_0)}{\sqrt{t}}, \\ &||C_{\omega^{B'}}^{\Sigma^{(3)}} C_{\omega^{A'}}^{\Sigma^{(3)}}||_{L^\infty \to L^2(\Sigma^{(3)})}, \quad ||C_{\omega^{A'}}^{\Sigma^{(3)}} C_{\omega^{B'}}^{\Sigma^{(3)}}||_{L^\infty \to L^2(\Sigma^{(3)})} \le \frac{C(k_0)}{t^{3/4}}. \end{aligned} \tag{5.56}$$

Proof. Since

$$\omega_+^{B'}(\eta)\omega_+^{A'}(\xi) = 0, \quad \omega_-^{B'}(\eta)\omega_-^{A'}(\xi) = 0, \quad \text{for } \eta, \xi \in \Sigma^{(3)}, \tag{5.57a}$$

and

$$C_{\omega^{A'}}^{\Sigma^{(3)}} C_{\omega^{B'}}^{\Sigma^{(3)}} \phi = C_+\left((C_-\phi\omega_+^{B'})\omega_-^{A'}\right) + C_-\left((C_+\phi\omega_-^{B'})\omega_+^{A'}\right) \tag{5.57b}$$

Here we estimate the first term and the second term is similar.

Since C_- is bounded in $L^2(\Sigma^{(3)})$, and the Proposition 5.4, we have

$$\begin{aligned} &||C_+\left((C_-\phi\omega_+^{B'})\omega_-^{A'}\right)(k)||_{L^2(\Sigma^{(3)})} \\ &= ||\int_{\Sigma^{A'}}\left(\int_{\Sigma^{B'}}\phi(\xi)\omega_+^{B'}(\xi)\tfrac{d\xi}{(\xi-\eta)_-}\right)\omega_-^{A'}(\eta)\tfrac{d\eta}{(\eta-k)_+}||_{L^2(\Sigma^{(3)})} \\ &\le c||\omega_-^{A'}||_{L^2(\Sigma^{A'})}\sup_{\eta\in\Sigma^{A'}}|\int_{\Sigma^{B'}}\phi(\xi)\omega_+^{B'}(\xi)\tfrac{d\xi}{\xi-\eta}| \\ &\le \tfrac{c}{k_0}||\omega_-^{A'}||_{L^2(\Sigma^{A'})}||\omega_+^{B'}||_{L^2(\Sigma^{B'})}||\phi||_{L^2(\Sigma^{(3)})} \\ &\le C(k_0)t^{-1/2}||\phi||_{L^2(\Sigma^{(3)})}, \end{aligned} \tag{5.58}$$

where

$$\operatorname{dist}(\Sigma^{A'}, \Sigma^{B'}) > k_0. \tag{5.59}$$

Thus, we have

$$||C_{\omega^{A'}}^{\Sigma^{(3)}} C_{\omega^{B'}}^{\Sigma^{(3)}}||_{L^2(\Sigma^{(3)})} \le \frac{C(k_0)}{\sqrt{t}}. \tag{5.60}$$

On the other hand.

$$\begin{aligned}
&||C_+\left((C_-\phi\omega_+^{B'})\omega_-^{A'}\right)(k)||_{L^2(\Sigma^{(3)})}\\
&=||\int_{\Sigma^{A'}}\left(\int_{\Sigma^{B'}}\phi(\xi)\omega_+^{B'}(\xi)\frac{d\xi}{(\xi-\eta)_-}\right)\omega_-^{A'}(\eta)\frac{d\eta}{(\eta-k)_+}||_{L^2(\Sigma^{(3)})}\\
&\le c||\omega_-^{A'}||_{L^2(\Sigma^{A'})}\sup_{\eta\in\Sigma^{A'}}|\int_{\Sigma^{B'}}\phi(\xi)\omega_+^{B'}(\xi)\frac{d\xi}{\xi-\eta}|\\
&\le \frac{c}{k_0}||\omega_-^{A'}||_{L^2(\Sigma^{A'})}||\omega_-^{B'}||_{L^1(\Sigma^{B'})}||\phi||_{L^\infty(\Sigma^{(3)})}\\
&\le C(k_0)t^{-1/2}t^{-1/4}||\phi||_{L^\infty(\Sigma^{(3)})}.
\end{aligned}\tag{5.61}$$

Thus, we have

$$||C_{\omega^{A'}}^{\Sigma^{(3)}}C_{\omega^{B'}}^{\Sigma^{(3)}}||_{L^\infty\to L^2(\Sigma^{(3)})}\le\frac{C(k_0)}{t^{3/4}}.\quad\square\tag{5.62}$$

Using the identity

$$\begin{aligned}
&\left(\mathbb{I}-C_{\omega^{A'}}^{\Sigma^{(3)}}-C_{\omega^{B'}}^{\Sigma^{(3)}}\right)\left(\mathbb{I}+C_{\omega^{A'}}^{\Sigma^{(3)}}(\mathbb{I}-C_{\omega^{A'}}^{\Sigma^{(3)}})^{-1}+C_{\omega^{B'}}^{\Sigma^{(3)}}(\mathbb{I}-C_{\omega^{B'}}^{\Sigma^{(3)}})^{-1}\right)\\
&=\mathbb{I}-C_{\omega^{B'}}^{\Sigma^{(3)}}C_{\omega^{A'}}^{\Sigma^{(3)}}(\mathbb{I}-C_{\omega^{A'}}^{\Sigma^{(3)}})^{-1}-C_{\omega^{A'}}^{\Sigma^{(3)}}C_{\omega^{B'}}^{\Sigma^{(3)}}(\mathbb{I}-C_{\omega^{B'}}^{\Sigma^{(3)}})^{-1}
\end{aligned}\tag{5.63}$$

and Proposition 5.4, we show that as $t\to\infty$,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Sigma^{(3)}}\frac{(\mathbb{I}-C_{\omega'})^{-1}\mathbb{I}(\eta)\omega'(\eta)}{\eta^j}d\eta\\
&=\frac{1}{2\pi i}\int_{\Sigma^{A'}}\frac{(\mathbb{I}-C_{\omega^{A'}})^{-1}\mathbb{I}(\eta)\omega^{A'}(\eta)}{\eta^j}d\eta\\
&+\frac{1}{2\pi i}\int_{\Sigma^{B'}}\frac{(\mathbb{I}-C_{\omega^{B'}})^{-1}\mathbb{I}(\eta)\omega^{B'}(\eta)}{\eta^j}d\eta+O(\frac{C(k_0)}{t}),\quad j=1,2,
\end{aligned}\tag{5.64}$$

where $C(k_0)$ is a constant dependent on k_0.

Then, using the lemma 2.56 in [14], we obtain

Proposition 5.10. *As $t\to\infty$,*

$$\begin{aligned}
iu(y,t)=&\left(\mathbb{I}+\frac{1}{2\pi i}\int_{\Sigma^{A'}}\frac{(\mathbb{I}-C_{\omega^{A'}})^{-1}\mathbb{I}(\eta)\omega^{A'}(\eta)}{\eta}d\eta+\frac{1}{2\pi i}\int_{\Sigma^{B'}}\frac{(\mathbb{I}-C_{\omega^{B'}})^{-1}\mathbb{I}(\eta)\omega^{B'}(\eta)}{\eta}d\eta+O(t^{-l})\right)_{11}\\
&\cdot\left(\frac{1}{2\pi i}\int_{\Sigma^{A'}}\frac{(\mathbb{I}-C_{\omega^{A'}})^{-1}\mathbb{I}(\eta)\omega^{A'}(\eta)}{\eta^2}d\eta+\frac{1}{2\pi i}\int_{\Sigma^{B'}}\frac{(\mathbb{I}-C_{\omega^{B'}})^{-1}\mathbb{I}(\eta)\omega^{B'}(\eta)}{\eta^2}d\eta+O(t^{-l})\right)_{21}\\
&-\left(\frac{1}{2\pi i}\int_{\Sigma^{A'}}\frac{(\mathbb{I}-C_{\omega^{A'}})^{-1}\mathbb{I}(\eta)\omega^{A'}(\eta)}{\eta}d\eta+\frac{1}{2\pi i}\int_{\Sigma^{B'}}\frac{(\mathbb{I}-C_{\omega^{B'}})^{-1}\mathbb{I}(\eta)\omega^{B'}(\eta)}{\eta}d\eta+O(t^{-l})\right)_{21}\\
&\cdot\left(\frac{1}{2\pi i}\int_{\Sigma^{A'}}\frac{(\mathbb{I}-C_{\omega^{A'}})^{-1}\mathbb{I}(\eta)\omega^{A'}(\eta)}{\eta^2}d\eta+\frac{1}{2\pi i}\int_{\Sigma^{B'}}\frac{(\mathbb{I}-C_{\omega^{B'}})^{-1}\mathbb{I}(\eta)\omega^{B'}(\eta)}{\eta^2}d\eta+O(t^{-l})\right)_{11},
\end{aligned}\tag{5.65a}$$

and

$$
\begin{aligned}
ic_+(y,t) = &\left(\mathbb{I} + \tfrac{1}{2\pi i}\int_{\Sigma^{A'}} \frac{(\mathbb{I}-C_{\omega^{A'}})^{-1}\mathbb{I}(\eta)\omega^{A'}(\eta)}{\eta}d\eta + \tfrac{1}{2\pi i}\int_{\Sigma^{B'}} \frac{(\mathbb{I}-C_{\omega^{B'}})^{-1}\mathbb{I}(\eta)\omega^{B'}(\eta)}{\eta}d\eta + O(t^{-l})\right)_{22} \\
&\cdot\left(\tfrac{1}{2\pi i}\int_{\Sigma^{A'}} \frac{(\mathbb{I}-C_{\omega^{A'}})^{-1}\mathbb{I}(\eta)\omega^{A'}(\eta)}{\eta^2}d\eta + \tfrac{1}{2\pi i}\int_{\Sigma^{B'}} \frac{(\mathbb{I}-C_{\omega^{B'}})^{-1}\mathbb{I}(\eta)\omega^{B'}(\eta)}{\eta^2}d\eta + O(t^{-l})\right)_{11} \\
&-\left(\tfrac{1}{2\pi i}\int_{\Sigma^{A'}} \frac{(\mathbb{I}-C_{\omega^{A'}})^{-1}\mathbb{I}(\eta)\omega^{A'}(\eta)}{\eta}d\eta + \tfrac{1}{2\pi i}\int_{\Sigma^{B'}} \frac{(\mathbb{I}-C_{\omega^{B'}})^{-1}\mathbb{I}(\eta)\omega^{B'}(\eta)}{\eta}d\eta + O(t^{-l})\right)_{12} \\
&\cdot\left(\tfrac{1}{2\pi i}\int_{\Sigma^{A'}} \frac{(\mathbb{I}-C_{\omega^{A'}})^{-1}\mathbb{I}(\eta)\omega^{A'}(\eta)}{\eta^2}d\eta + \tfrac{1}{2\pi i}\int_{\Sigma^{B'}} \frac{(\mathbb{I}-C_{\omega^{B'}})^{-1}\mathbb{I}(\eta)\omega^{B'}(\eta)}{\eta^2}d\eta + O(t^{-l})\right)_{21} \\
&-\delta_1.
\end{aligned}
\tag{5.65b}
$$

5.6. *The scaling transformation*

In order to reduce the Riemann–Hilbert problem for $\tilde{M}^{(3)}(y,t,k)$, as $t \to \infty$, to a model Riemann–Hilbert problem whose solution can be given explicitly in terms of parabolic cylinder functions, see [14], the leading term of the factor $\delta(k)e^{-it\theta(k)}$ as $k \to \pm k_0$ is to be evaluated.

First, we extend the crosses $\Sigma^{A'}$ and $\Sigma^{B'}$ to contours $\hat{\Sigma}^{A'}$ and $\hat{\Sigma}^{B'}$ by zero extension. Thus, the corresponding functions $\hat{\omega}^{A'}$ and $\hat{\omega}^{B'}$ are well-defined by zero extension of the functions $\omega^{A'}$ and $\omega^{B'}$, too. Then, we denote Σ^A and Σ^B as the contours $\{k = k_0\lambda e^{\pm\frac{\pi i}{4}}, -\infty < \lambda < \infty\}$ oriented as $\hat{\Sigma}^{A'}$ and $\hat{\Sigma}^{B'}$, respectively.

For k near k_0,

$$\delta(k) = \left(\frac{k-k_0}{k+k_0}\right)^{-i\nu(k_0)} e^{\chi(k)}, \tag{5.66}$$

where

$$\nu(k_0) = \nu = -\frac{1}{2\pi}\ln(1+|r(k_0)|^2), \tag{5.67}$$

$$\chi(k) = -\frac{1}{2\pi i}\Big(\int_{-\infty}^{-k_0} + \int_{k_0}^{+\infty}\Big)\ln|k-s|d\ln(1+|r(s)|^2). \tag{5.68}$$

And

$$\theta(k) = -\frac{1}{2k_0} - \frac{1}{4k_0^3}(k-k_0)^2 + \frac{1}{4\eta^4}(k-k_0)^3, \quad \eta \text{ lies between } k_0 \text{ and } k. \tag{5.69}$$

Then, introducing the scaling operator by

$$(N_A f)(k) = f\Big(k_0 + \frac{k}{\sqrt{k_0^{-3}t}}\Big), \tag{5.70}$$

the factor $\delta(k)e^{-it\theta(k)}$ can be scaled as

$$(N_A\delta e^{-it\theta})(k) = \delta_A^0\delta_A^1, \tag{5.71}$$

where

$$\delta_A^0 = (\frac{4t}{k_0})^{\frac{i\nu(k_0)}{2}} e^{\chi(k_0)} e^{\frac{it}{2k_0}}, \tag{5.72a}$$

$$\delta_A^1 = k^{-i\nu(k_0)} e^{\frac{ik^2}{4}} \left(\frac{2k_0}{2k_0 + \frac{k}{\sqrt{k_0^3 t}}} \right)^{-i\nu(k_0)} e^{\chi(k_0 + \frac{k}{\sqrt{k_0^{-3} t}}) - \chi(k_0)} e^{-\frac{ik^3}{4\eta^4 k_0^{-9/2} t^{1/2}}}. \tag{5.72b}$$

Here $k^{-i\nu(k_0)}$ is cut along $(0, \infty)$.

Form the definition of $\chi(k)$, we know that $\chi(k_0)$ is purely imaginary, thus $|\delta_A^0| = 1$. Define

$$\Delta_A^0 = (\delta_A^0)^{-\sigma_3}, \quad \tilde{\Delta}_A^0 \phi = \phi \Delta_A^0 \tag{5.73}$$

We have

$$C_{\hat{\omega}^{A'}} = N_A^{-1} (\Delta_A^0)^{-1} A \tilde{\Delta}_A^0 N_A, \tag{5.74}$$

where the operator $A : L^2(\Sigma^A) \to L^2(\Sigma^A)$ is given by

$$A\phi = C_{(\Delta_A^0)^{-1}(N_A \hat{\omega}^{A'})\Delta_A^0} \phi = C_+(\phi (\Delta_A^0)^{-1} (N_A \hat{\omega}_-^{A'}) \Delta_A^0) + C_-(\phi (\Delta_A^0)^{-1} (N_A \hat{\omega}_+^{A'}) \Delta_A^0) \tag{5.75}$$

On the part $\{k = k_0 \lambda e^{\frac{i\pi}{4}}, -\varepsilon < \lambda < \varepsilon\}$ of Σ^A,

$$(\Delta_A^0)^{-1} (N_A \hat{\omega}_+^{A'}) \Delta_A^0 = \begin{pmatrix} 0 & 0 \\ R(k_0 + \frac{k}{\sqrt{k_0^{-3} t}})(\delta_A^1)^2 & 1 \end{pmatrix}, \tag{5.76}$$

otherwise, $(\Delta_A^0)^{-1} (N_A \hat{\omega}_+^{A'}) \Delta_A^0 = 0$.

Similarly, on the part $\{k = k_0 \lambda e^{-\frac{i\pi}{4}}, -\varepsilon < \lambda < \varepsilon\}$ of Σ^A,

$$(\Delta_A^0)^{-1} (N_A \hat{\omega}_-^{A'}) \Delta_A^0 = \begin{pmatrix} 0 & -\overline{R}(k_0 + \frac{k}{\sqrt{k_0^{-3} t}})(\delta_A^1)^{-2} \\ 0 & 1 \end{pmatrix}, \tag{5.77}$$

otherwise, $(\Delta_A^0)^{-1} (N_A \hat{\omega}_-^{A'}) \Delta_A^0 = 0$.

By the definition of $R(k)$, we have

$$R(k_0+) = \lim_{\mathrm{Re}k > k_0} R(k) = \frac{\overline{r(k_0)}}{1 + |r(k_0)|^2}, \tag{5.78}$$

and

$$R(k_0-) = \lim_{\mathrm{Re}k < k_0} R(k) = -\overline{r(k_0)}. \tag{5.79}$$

Proposition 5.11. *As $t \to \infty$, let β be a fixed small number, $0 < 2\beta < 1$, then for $k \in \{k = k_0\lambda e^{\frac{i\pi}{4}}, -\varepsilon < \lambda < \varepsilon\}$,*

$$||R(k_0 + \frac{k}{\sqrt{k_0^{-3}t}})(\delta_A^1)^2 - R(k_0\pm)k^{-2i\nu(k_0)}e^{\frac{ik^2}{2}}||_{L^1\cap L^\infty} \le C(k_0)|e^{\frac{i\beta^2k^2}{2}}|\left(\frac{\ln(t)}{\sqrt{t}}\right). \tag{5.80}$$

Proof. See the appendix A. $\square$

Now, we have,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Sigma^{A'}}\frac{((\mathbb{I}_{A'}-C_{\omega^{A'}}^{\Sigma^{A'}})^{-1}\mathbb{I})(\eta)\omega^{A'}(\eta)}{\eta}d\eta\\
&=\frac{1}{2\pi i}\int_{\hat{\Sigma}^{A'}}\frac{((\mathbb{I}_{\hat{A}'}-C_{\hat{\omega}^{A'}}^{\hat{\Sigma}^{A'}})^{-1}\mathbb{I})(\eta)\hat{\omega}^{A'}(\eta)}{\eta}d\eta\\
&=\frac{1}{2\pi i}\int_{\hat{\Sigma}^{A'}}\frac{(N_A^{-1}(\tilde{\Delta}_A^0)^{-1}(\mathbb{I}_A-A)^{-1}\tilde{\Delta}_A^0N_A\mathbb{I})(\eta)\hat{\omega}^{A'}(\eta)}{\eta}d\eta\\
&=\frac{1}{2\pi i}\int_{\hat{\Sigma}^{A'}}\frac{(\mathbb{I}_A-A)^{-1}\Delta_A^0((\eta-k_0)\sqrt{k_0^{-3}t})(\Delta_A^0)^{-1}(N_A\hat{\omega}^{A'})((\eta-k_0)\sqrt{k_0^{-3}t})}{\eta}d\eta\\
&=\frac{1}{2\pi i}\frac{1}{\sqrt{k_0^{-3}t}}\int_{\Sigma^A}\frac{(\mathbb{I}_A-A)^{-1}\Delta_A^0(\eta)(\Delta_A^0)^{-1}(N_A\hat{\omega}^{A'})(\eta)}{\frac{\eta}{\sqrt{k_0^{-3}t}}+k_0}d\eta\\
&=\frac{1}{2\pi i}\frac{1}{\sqrt{k_0^{-3}t}}\Delta_A^0\left(\int_{\Sigma^A}\frac{(\mathbb{I}_A-A)^{-1}\mathbb{I}(\eta)\omega^A(\eta)}{\frac{\eta}{\sqrt{k_0^{-3}t}}+k_0}d\eta\right)(\Delta_A^0)^{-1}
\end{aligned}\tag{5.81a}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Sigma^{A'}}\frac{((\mathbb{I}_{A'}-C_{\omega^{A'}}^{\Sigma^{A'}})^{-1}\mathbb{I})(\eta)\omega^{A'}(\eta)}{\eta^2}d\eta\\
&=\frac{1}{2\pi i}\int_{\hat{\Sigma}^{A'}}\frac{((\mathbb{I}_{\hat{A}'}-C_{\hat{\omega}^{A'}}^{\hat{\Sigma}^{A'}})^{-1}\mathbb{I})(\eta)\hat{\omega}^{A'}(\eta)}{\eta^2}d\eta\\
&=\frac{1}{2\pi i}\int_{\hat{\Sigma}^{A'}}\frac{(N_A^{-1}(\tilde{\Delta}_A^0)^{-1}(\mathbb{I}_A-A)^{-1}\tilde{\Delta}_A^0N_A\mathbb{I})(\eta)\hat{\omega}^{A'}(\eta)}{\eta^2}d\eta\\
&=\frac{1}{2\pi i}\int_{\hat{\Sigma}^{A'}}\frac{(\mathbb{I}_A-A)^{-1}\Delta_A^0((\eta-k_0)\sqrt{k_0^{-3}t})(\Delta_A^0)^{-1}(N_A\hat{\omega}^{A'})((\eta-k_0)\sqrt{k_0^{-3}t})}{\eta^2}d\eta\\
&=\frac{1}{2\pi i}\frac{1}{\sqrt{k_0^{-3}t}}\int_{\Sigma^A}\frac{(\mathbb{I}_A-A)^{-1}\Delta_A^0(\eta)(\Delta_A^0)^{-1}(N_A\hat{\omega}^{A'})(\eta)}{\left(\frac{\eta}{\sqrt{k_0^{-3}t}}+k_0\right)^2}d\eta\\
&=\frac{1}{2\pi i}\frac{1}{\sqrt{k_0^{-3}t}}\Delta_A^0\left(\int_{\Sigma^A}\frac{(\mathbb{I}_A-A)^{-1}\mathbb{I}(\eta)\omega^A(\eta)}{\left(\frac{\eta}{\sqrt{k_0^{-3}t}}+k_0\right)^2}d\eta\right)(\Delta_A^0)^{-1}
\end{aligned}\tag{5.81b}$$

By the Proposition 5.11, we have

$$\int_{\Sigma^A}\frac{(\mathbb{I}_A-A)^{-1}\mathbb{I}(\eta)\omega^A(\eta)}{\frac{\eta}{\sqrt{k_0^{-3}t}}+k_0}d\eta = \int_{\Sigma^A}\frac{(\mathbb{I}_A-A)^{-1}\mathbb{I}(\eta)\omega^{A^0}(\eta)}{\frac{\eta}{\sqrt{k_0^{-3}t}}+k_0}d\eta + O\left(C(k_0)\frac{\ln(t)}{\sqrt{t}}\right), \tag{5.82a}$$

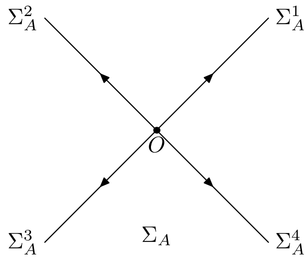

Fig. 6. The Σ^A.

$$\int_{\Sigma^A} \frac{(\mathbb{I}_A - A)^{-1}\mathbb{I}(\eta)\omega^A(\eta)}{\left(\frac{\eta}{\sqrt{k_0^{-3}t}} + k_0\right)^2} d\eta = \int_{\Sigma^A} \frac{(\mathbb{I}_A - A)^{-1}\mathbb{I}(\eta)\omega^{A^0}(\eta)}{\left(\frac{\eta}{\sqrt{k_0^{-3}t}} + k_0\right)^2} d\eta + O\left(C(k_0)\frac{\ln(t)}{\sqrt{t}}\right). \tag{5.82b}$$

Here (see Fig. 6)

$$\omega^{A^0} = \begin{cases} \omega_+^{A^0} = \begin{pmatrix} 0 & 0 \\ \frac{\overline{r(k_0)}}{1+|r(k_0)|^2} k^{-2i\nu(k_0)} e^{\frac{ik^2}{2}} & 0 \end{pmatrix}, & k \in \Sigma_A^1, \\ \omega_+^{A^0} = \begin{pmatrix} 0 & 0 \\ -\overline{r(k_0)} k^{-2i\nu(k_0)} e^{\frac{ik^2}{2}} & 0 \end{pmatrix}, & k \in \Sigma_A^3, \\ \omega_-^{A^0} = \begin{pmatrix} 0 & -r(k_0) k^{2i\nu(k_0)} e^{-\frac{ik^2}{2}} \\ 0 & 0 \end{pmatrix}, & k \in \Sigma_A^2, \\ \omega_+^{A^0} = \begin{pmatrix} 0 & \frac{r(k_0)}{1+|r(k_0)|^2} k^{2i\nu(k_0)} e^{-\frac{ik^2}{2}} \\ 0 & 0 \end{pmatrix}, & k \in \Sigma_A^4. \end{cases} \tag{5.83}$$

Define

$$\tilde{M}^{A_0}(k) = \mathbb{I} + \frac{1}{2\pi i} \int_{\Sigma^A} \frac{((\mathbb{I}_A - A^0)^{-1}\mathbb{I})(\eta)\omega^{A^0}(\eta)}{\eta - k} d\eta, \tag{5.84}$$

then $\tilde{M}^{A_0}(k)$ satisfies the Riemann–Hilbert problem

$$\begin{cases} \tilde{M}_+^{A^0}(k) = \tilde{M}_-^{A^0}(k)\tilde{J}^{A_0}(k), & k \in \Sigma^A, \\ \tilde{M}^{A^0}(k) \to \mathbb{I}, & k \to \infty, \end{cases} \tag{5.85}$$

where

$$\tilde{J}^{A_0}(k) = (b_-^{A^0})^{-1}(k) b_+^{A^0}(k) = (\mathbb{I} - \omega_-^{A^0})^{-1}(\mathbb{I} + \omega_+^{A^0}). \tag{5.86}$$

If

$$\tilde{M}^{A^0}(k) = \mathbb{I} - \frac{\tilde{M}_1^{A^0}}{k} + O(k^{-2}), \quad k \to \infty, \tag{5.87}$$

then

$$\begin{aligned} &\tfrac{1}{2\pi i}\tfrac{1}{\sqrt{k_0^{-3}t}}\int_{\Sigma^A} \tfrac{(\mathbb{I}_A - A)^{-1}\mathbb{I}(\eta)\omega^A(\eta)}{\frac{\eta}{\sqrt{k_0^{-3}t}} + k_0} d\eta \\ &= \tfrac{1}{2\pi i}\int_{\Sigma^A} \tfrac{(\mathbb{I}_A - A)^{-1}\mathbb{I}(\eta)\omega^A(\eta)}{\eta + k_0\sqrt{k_0^{-3}t}} d\eta \\ &= \tilde{M}^{A^0}(-k_0\sqrt{k_0^{-3}t}) - \mathbb{I} \\ &= \tfrac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} + O(t^{-1}\ln t), \quad t \to \infty \end{aligned} \tag{5.88a}$$

and

$$\begin{aligned} &\tfrac{1}{2\pi i}\tfrac{1}{\sqrt{k_0^{-3}t}}\int_{\Sigma^A} \tfrac{(\mathbb{I}_A - A)^{-1}\mathbb{I}(\eta)\omega^A(\eta)}{\left(\frac{\eta}{\sqrt{k_0^{-3}t}} + k_0\right)^2} d\eta \\ &= \tfrac{\sqrt{k_0^{-3}t}}{2\pi i}\int_{\Sigma^A} \tfrac{(\mathbb{I}_A - A)^{-1}\mathbb{I}(\eta)\omega^A(\eta)}{\left(\eta + k_0\sqrt{k_0^{-3}t}\right)^2} d\eta \\ &= \tfrac{d\tilde{M}^{A^0}}{dk}\Big|_{k=-k_0\sqrt{k_0^{-3}t}} \\ &= \tfrac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} + O(t^{-1}\ln t), \quad t \to \infty. \end{aligned} \tag{5.88b}$$

Remark 5.12. Similarly for k near $-k_0$. The scaling operator is

$$(N_B f)(k) = f(-k_0 + \frac{k}{\sqrt{k_0^{-3}t}}) \tag{5.89}$$

and

$$(N_B \delta e^{-it\theta})(k) = \delta_B^0 \delta_B^1, \to \tilde{\delta}\tilde{k}^{-i\nu(k_0)} e^{\frac{i\tilde{k}^2}{4}}, \quad as \quad t \to \infty, \tag{5.90}$$

where

$$\delta_B^0 = (\frac{4t}{k_0})^{-\frac{i\nu(-k_0)}{2}} e^{\chi(-k_0)} e^{-\frac{it}{2k_0}}, \tag{5.91a}$$

$$\delta_B^1 = (-k)^{i\nu(k_0)} e^{-\frac{ik^2}{4}} \left(\frac{-2k_0}{-2k_0 + \frac{k}{\sqrt{k_0^3 t}}}\right)^{i\nu(k_0)} e^{\chi(-k_0 + \frac{k}{\sqrt{k_0^{-3}t}}) - \chi(-k_0)} e^{-\frac{ik^3}{4\eta^4 k_0^{-9/2} t^{1/2}}}. \tag{5.91b}$$

If

$$\tilde{M}^{B^0}(k) = \mathbb{I} - \frac{\tilde{M}_1^{B^0}}{k} + O(k^{-2}), \quad k \to \infty, \tag{5.92}$$

then

$$\begin{aligned}
&\frac{1}{2\pi i}\frac{1}{\sqrt{k_0^{-3}t}}\int_{\Sigma^B}\frac{(\mathbb{I}_B - B)^{-1}\mathbb{I}(\eta)\omega^B(\eta)}{\frac{\eta}{\sqrt{k_0^{-3}t}} - k_0}d\eta \\
&= \frac{1}{2\pi i}\int_{\Sigma^B}\frac{(\mathbb{I}_B - B)^{-1}\mathbb{I}(\eta)\omega^A(\eta)}{\eta - k_0\sqrt{k_0^{-3}t}}d\eta \\
&= \tilde{M}^{B^0}(k_0\sqrt{k_0^{-3}t}) - \mathbb{I} \\
&= -\frac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O(t^{-1}\ln t), \quad t \to \infty
\end{aligned} \tag{5.93a}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\frac{1}{\sqrt{k_0^{-3}t}}\int_{\Sigma^B}\frac{(\mathbb{I}_B - B)^{-1}\mathbb{I}(\eta)\omega^B(\eta)}{\left(\frac{\eta}{\sqrt{k_0^{-3}t}} - k_0\right)^2}d\eta \\
&= \frac{\sqrt{k_0^{-3}t}}{2\pi i}\int_{\Sigma^B}\frac{(\mathbb{I}_B - B)^{-1}\mathbb{I}(\eta)\omega^B(\eta)}{\left(\eta - k_0\sqrt{k_0^{-3}t}\right)^2}d\eta \\
&= \left.\frac{d\tilde{M}^{B^0}}{dk}\right|_{k = k_0\sqrt{k_0^{-3}t}} \\
&= \frac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O(t^{-1}\ln t), \quad t \to \infty.
\end{aligned} \tag{5.93b}$$

5.7. Model Riemann–Hilbert problem

Consider the Riemann–Hilbert problem (5.85) for $\tilde{M}^{A^0}$, we introduce a transformation (Fig. 7)

$$\tilde{\Phi}^{A^0}(k) = \tilde{M}^{A^0}(k)\Phi^T(k), \tag{5.94}$$

where

$$\Phi^T(k) = \begin{cases}
k^{i\nu(k_0)\sigma_3}e^{-\frac{ik^2}{4}\hat{\sigma}_3}\begin{pmatrix} 1 & 0 \\ \frac{r(k_0)}{1+|r(k_0)|^2} & 1 \end{pmatrix}, & k \in \Omega_1^e, \\
k^{i\nu(k_0)\sigma_3}e^{-\frac{ik^2}{4}\hat{\sigma}_3}\begin{pmatrix} 1 & r(k_0) \\ 0 & 1 \end{pmatrix}, & k \in \Omega_3^e, \\
k^{i\nu(k_0)\sigma_3}e^{-\frac{ik^2}{4}\hat{\sigma}_3}\begin{pmatrix} 1 & 0 \\ -r(k_0) & 1 \end{pmatrix}, & k \in \Omega_4^e, \\
k^{i\nu(k_0)\sigma_3}e^{-\frac{ik^2}{4}\hat{\sigma}_3}\begin{pmatrix} 1 & -\frac{r(k_0)}{1+|r(k_0)|^2} \\ 0 & 1 \end{pmatrix}, & k \in \Omega_6^e, \\
k^{i\nu(k_0)\sigma_3}, & k \in \Omega_2^e \cup \Omega_5^e.
\end{cases} \tag{5.95}$$

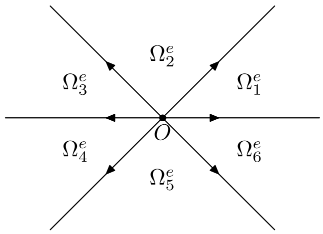

Fig. 7. The regions Ω_j^e, $j = 1, 2, \ldots, 6$.

Then, $\tilde{\Phi}^{A^0}(k)$ satisfies the Riemann–Hilbert problem

$$\begin{cases} \tilde{\Phi}_+^{A^0}(k) = \tilde{\Phi}_-^{A^0}(k) e^{-\frac{ik^2}{4}\hat{\sigma}_3} \tilde{J}_\Phi^{A^0}(k), & k \in \mathbb{R}, \\ \tilde{\Phi}^{A^0}(k) \to k^{i\nu(k_0)\sigma_3}, & k \to \infty, \end{cases} \tag{5.96}$$

where

$$\tilde{J}_\Phi^{A^0}(k) = \begin{cases} \begin{pmatrix} 1 & r(k_0) \\ \overline{r(k_0)} & 1 + |r(k_0)|^2 \end{pmatrix}, & k > 0, \\ \begin{pmatrix} 1 + |r(k_0)|^2 & -r(k_0) \\ -\overline{r(k_0)} & 1 \end{pmatrix}, & k < 0, \end{cases} \tag{5.97}$$

the contour $\mathbb{R}$ oriented from the original to ∞ and $-\infty$.

Hence, if we reorient the contour $\mathbb{R}$ from $-\infty$ to ∞, we get the jump matrix

$$\tilde{J}_\Phi^{A^0}(k) = \begin{pmatrix} 1 & r(k_0) \\ \overline{r(k_0)} & 1 + |r(k_0)|^2 \end{pmatrix}, \quad k \in \mathbb{R}. \tag{5.98}$$

Let

$$\tilde{M}^{model}(k) = \tilde{\Phi}^{A^0}(k) e^{-\frac{ik^2}{4}\sigma_3}, \tag{5.99}$$

then we have the model Riemann–Hilbert problem

$$\tilde{M}_+^{model}(k) = \tilde{M}_-^{model}(k) \tilde{J}(k_0), \quad k \in \mathbb{R}, \tag{5.100}$$

where

$$\tilde{J}(k_0) = \tilde{J}_\Phi^{A^0}(k) = \begin{pmatrix} 1 & r(k_0) \\ \overline{r(k_0)} & 1 + |r(k_0)|^2 \end{pmatrix}. \tag{5.101}$$

So, we have

$$\left(\frac{\partial \tilde{M}^{model}}{\partial k} + \frac{ik}{2}\sigma_3 \tilde{M}^{model}\right)(\tilde{M}^{model})^{-1} = -\frac{i}{2}[\sigma_3, \tilde{M}^{A^0}] + O(k^{-1}), \tag{5.102}$$

since $(\frac{\partial \tilde{M}^{model}}{\partial k} + \frac{ik}{2}\sigma_3 \tilde{M}^{model})(\tilde{M}^{model})^{-1}$ is entire,

$$\frac{\partial \tilde{M}^{model}}{\partial k} + \frac{ik}{2}\sigma_3 \tilde{M}^{model} = \beta \tilde{M}^{model}, \tag{5.103}$$

here

$$\beta = -\frac{i}{2}[\sigma_3, \tilde{M}^{A^0}] = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix}. \tag{5.104}$$

Thus, we have

$$(\tilde{M}^{A^0})_{12} = i\beta_{12}, \quad (\tilde{M}^{A^0})_{21} = -i\beta_{21}. \tag{5.105}$$

Let us consider $\mathrm{Im} k > 0$, denote $\tilde{M}^{model}(k)$ by $\tilde{M}^{+}(k)$, we have

$$\frac{\partial \tilde{M}_{11}^{+}}{\partial k} + \frac{ik}{2}\tilde{M}_{11}^{+} = \beta_{12}\tilde{M}_{21}^{+} \tag{5.106a}$$

and

$$\frac{\partial \tilde{M}_{21}^{+}}{\partial k} - \frac{ik}{2}\tilde{M}_{21}^{+} = \beta_{21}\tilde{M}_{11}^{+}, \tag{5.106b}$$

so

$$\frac{\partial^2 \tilde{M}_{11}^{+}}{\partial k^2} = (-\frac{k^2}{4} - \frac{i}{2} + \beta_{12}\beta_{21})\tilde{M}_{11}^{+} \tag{5.107}$$

Setting

$$\tilde{M}_{11}^{+} = g(e^{-\frac{3i\pi}{4}}k), \tag{5.108}$$

we have the parabolic cylinder equation

$$\frac{\partial^2 g}{\partial \xi^2} + (\frac{1}{2} - \frac{\xi^2}{4} + a)g = 0, \tag{5.109}$$

where $a = i\beta_{12}\beta_{21}$.

Then,

$$\tilde{M}_{11}^{+}(k) = c_1 D_a(e^{-\frac{3i\pi}{4}}k) + c_2 D_a(-e^{-\frac{3i\pi}{4}}k), \tag{5.110}$$

where $D_a(z)$ denotes the parabolic cylinder function.

As $z \to \infty$, we have the asymptotic formula [26], P. 327,

$$\begin{aligned} D_a(z) &= z^a e^{-\frac{z^2}{4}} \left(1 + O(z^{-2})\right), \quad |\arg z| < \tfrac{3\pi}{4}, \\ &= z^a e^{-\frac{z^2}{4}} \left(1 + O(z^{-2})\right) - \tfrac{\sqrt{2\pi}}{\Gamma(-a)} e^{a\pi i} z^{-a-1} e^{\frac{z^2}{4}} \left(1 + O(z^{-2})\right), \quad \tfrac{\pi}{4} < \arg z < \tfrac{5\pi}{4}, \\ &= z^a e^{-\frac{z^2}{4}} \left(1 + O(z^{-2})\right) - \tfrac{\sqrt{2\pi}}{\Gamma(-a)} e^{-a\pi i} z^{-a-1} e^{\frac{z^2}{4}} \left(1 + O(z^{-2})\right), \quad -\tfrac{5\pi}{4} < \arg z < -\tfrac{\pi}{4} \end{aligned} \tag{5.111}$$

We have

$$a = i\nu(k_0), \tag{5.112}$$

so

$$\nu(k_0) = \beta_{12}\beta_{21}. \tag{5.113}$$

Thus, for $\mathrm{Im}k > 0$,

$$\begin{aligned} \tilde{M}_{11}^{+}(k) &= e^{-\frac{3\pi\nu(k_0)}{4}} D_a(e^{-\frac{3\pi i}{4}} k), \\ \tilde{M}_{21}^{+}(k) &= \tfrac{1}{\beta_{12}} e^{-\frac{3\pi\nu(k_0)}{4}} (\partial_k D_a(e^{-\frac{3\pi i}{4}} k)) + \tfrac{ik}{2} D_a(e^{-\frac{3\pi i}{4}} k). \end{aligned} \tag{5.114}$$

Similarly, for $\mathrm{Im}k < 0$, we have

$$\begin{aligned} \tilde{M}_{11}^{-}(k) &= e^{\frac{\pi\nu(k_0)}{4}} D_a(e^{\frac{\pi i}{4}} k), \\ \tilde{M}_{21}^{-}(k) &= \tfrac{1}{\beta_{12}} e^{\frac{\pi\nu(k_0)}{4}} (\partial_k D_a(e^{\frac{\pi i}{4}} k)) + \tfrac{ik}{2} D_a(e^{\frac{\pi i}{4}} k). \end{aligned} \tag{5.115}$$

Since

$$(\tilde{M}_{-}^{model})^{-1} \tilde{M}_{+}^{model} = \begin{pmatrix} 1 & r(k_0) \\ \overline{r(k_0)} & 1 + |r(k_0)|^2 \end{pmatrix}, \tag{5.116}$$

we have

$$\begin{aligned} \overline{r(k_0)} &= \tilde{M}_{11}^{-}\tilde{M}_{21}^{+} - \tilde{M}_{21}^{-}\tilde{M}_{11}^{+} \\ &= \tfrac{1}{\beta_{12}} e^{-\frac{\pi\nu(k_0)}{2}} \mathrm{Wr}(D_a(e^{\frac{i\pi}{4}} k), D_a(e^{-\frac{3\pi i}{4}} k)) \\ &= \frac{\sqrt{2\pi} e^{\frac{i\pi}{4}} e^{-\frac{\pi\nu(k_0)}{2}}}{\beta_{12}\Gamma(-a)}, \end{aligned} \tag{5.117}$$

where $\mathrm{Wr}(f, g)$ denotes the Wronskian of f, g and $\Gamma(\cdot)$ is the Euler Gamma function.
Hence, we have

$$\beta_{12} = \frac{\sqrt{2\pi} e^{\frac{i\pi}{4}} e^{-\frac{\pi\nu(k_0)}{2}}}{\overline{r(k_0)}\Gamma(-i\nu(k_0))}, \tag{5.118}$$

and

$$\beta_{21} = \frac{\nu(k_0)}{\beta_{12}} = -\frac{\sqrt{2\pi} e^{-\frac{i\pi}{4}} e^{-\frac{\pi\nu(k_0)}{2}}}{r(k_0)\Gamma(i\nu(k_0))}, \tag{5.119}$$

since $|\Gamma(i\nu(k_0))|^2 = \frac{\pi}{\nu(k_0)\sinh(\pi\nu(k_0))}$.

5.8. The asymptotic behavior of the solution $u(x,t)$

Remind as $t \to \infty$, the representation (5.53) of the solution $u(y,t)$ and the computation results (5.81) and (5.93), we have

$$\begin{aligned} iu(y,t) = & \left(\mathbb{I} + \frac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} - \frac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O\left(C(k_0)\frac{\ln t}{t}\right)\right)_{11} \\ & \cdot \left((\delta_A^0)^2 \frac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} + (\delta_B^0)^2 \frac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O\left(C(k_0)\frac{\ln t}{t}\right)\right)_{21} \\ & - \left((\delta_A^0)^2 \frac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} - (\delta_B^0)^2 \frac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O\left(C(k_0)\frac{\ln t}{t}\right)\right)_{21} \\ & \cdot \left(\frac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} + \frac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O\left(C(k_0)\frac{\ln t}{t}\right)\right)_{11}, \end{aligned} \tag{5.120a}$$

and

$$\begin{aligned} ic_+(y,t) = & \left(\mathbb{I} + \frac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} - \frac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O\left(C(k_0)\frac{\ln t}{t}\right)\right)_{22} \\ & \cdot \left(\frac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} + \frac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O\left(C(k_0)\frac{\ln t}{t}\right)\right)_{11} \\ & - \left((\delta_A^0)^{-2} \frac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} - (\delta_B^0)^{-2} \frac{1}{k_0\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O\left(C(k_0)\frac{\ln t}{t}\right)\right)_{12} \\ & \cdot \left((\delta_A^0)^2 \frac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{A^0} + (\delta_B^0)^2 \frac{1}{k_0^2\sqrt{k_0^{-3}t}}\tilde{M}_1^{B^0} + O\left(C(k_0)\frac{\ln t}{t}\right)\right)_{21}, \\ & - \delta_1. \end{aligned} \tag{5.120b}$$

Notice that we have

$$\delta_B^0 = \overline{\delta_A^0}, \tag{5.121}$$

as $\chi(-k_0) = -\chi(k_0) = \overline{\chi(k_0)}$.

And from the symmetry conditions (3.2), we get

$$\tilde{M}_1^{A^0} = -\overline{\tilde{M}_1^{B^0}}. \tag{5.122}$$

Hence, a direct computation shows that,

$$u(x,t)=\sqrt{\frac{-4\nu(\kappa_0)}{\kappa_0 t}}\sin\{\frac{t}{\kappa_0}+\nu(\kappa_0)\ln(\frac{4t}{\kappa_0})+\phi(\kappa_0)\}+O\left(\frac{\ln(t)}{t}\right),\quad \text{as } t\to\infty, \tag{5.123}$$

where

$$\phi(\kappa_0)=\frac{\pi}{4}-\arg r(\kappa_0)-\arg\Gamma(i\nu(\kappa_0))+\frac{1}{\pi}(\int_{-\infty}^{\kappa_0}+\int_{\kappa_0}^{\infty})\ln|\kappa_0-s|d\ln(1+|r(s)|^2)+2\kappa_0\Delta, \tag{5.124}$$

here

$$\Delta=\frac{1}{\pi}\int_{\kappa_0}^{\infty}\frac{\ln(1+|r(s)|^2)}{s^2}ds, \tag{5.125}$$

and κ_0 is defined as (1.3).

This finishes the proof of Theorem 1.3.

Acknowledgments

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Appendix A. Proof of the Proposition 5.11

Proof. Write

$$\begin{aligned}
&R\left(\frac{k}{\sqrt{k_0^{-3}t}}+k_0\right)(\delta_A^1(k))^2-R(k_0\pm)k^{-2i\nu}e^{i\frac{k^2}{2}}\\
&=e^{i\frac{\beta}{2}k^2}e^{i\frac{\beta}{2}k^2}R\left(\frac{k}{\sqrt{k_0^{-3}t}}+k_0\right)k^{-2i\nu}e^{i(1-2\beta)\frac{k^2}{2}(1-\frac{k}{(1-2\beta)\eta^4\sqrt{k_0^{-9}t}})}\\
&\times\left(\frac{2k_0}{2k_0+\frac{k}{\sqrt{k_0^{-3}t}}}\right)^{-2i\nu}e^{2\left(\chi(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)-\chi(k_0)\right)}-e^{i\frac{\beta}{2}k^2}e^{i\frac{\beta}{2}k^2}R(k_0\pm)k^{-2i\nu}e^{i(1-2\beta)\frac{k^2}{2}}
\end{aligned} \tag{A.1}$$

and divide it into six terms

$$R\left(\frac{k}{\sqrt{k_0^{-3}t}}+k_0\right)(\delta_A^1(k))^2-R(k_0\pm)k^{-2i\nu}e^{i\frac{k^2}{2}}=e^{i\beta\frac{k^2}{2}}(I+II+III+IV) \tag{A.2}$$

where

$$I=e^{i(1-\beta)\frac{k^2}{2}}k^{-2i\nu}[R(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)-R(k_0\pm)]$$
$$II=e^{i\beta\frac{k^2}{2}}k^{-2i\nu}R(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)\left(e^{i(1-2\beta)\frac{k^2}{2}(1-\frac{k}{(1-2\beta)\eta^4\sqrt{k_0^{-9}t}})}-e^{i(1-2\beta)\frac{k^2}{2}}\right)$$
$$III=e^{i\beta\frac{k^2}{2}}k^{-2i\nu}R(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)e^{i(1-2\beta)\frac{k^2}{2}(1-\frac{k}{(1-2\beta)\eta^4\sqrt{k_0^{-9}t}})}\left(\left(\frac{2k_0}{2k_0+\frac{k}{\sqrt{k_0^{-3}t}}}\right)^{-2i\nu}-1\right)$$
$$IV=e^{i\beta\frac{k^2}{2}}k^{-2i\nu}R(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)e^{i(1-2\beta)\frac{k^2}{2}(1-\frac{k}{(1-2\beta)\eta^4\sqrt{k_0^{-9}t}})}\left(\frac{2k_0}{2k_0+\frac{k}{\sqrt{k_0^{-3}t}}}\right)^{-2i\nu}$$
$$\left(e^{2\left(\chi(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)-\chi(k_0)\right)}-1\right)$$

Note that $|e^{i\beta\frac{k^2}{2}}|=e^{-\frac{\beta\lambda^2k_0^2}{2}}$ and $|k^{-2i\nu}|=e^{2\nu\arg k}\leq C$, where C is a constant which is independent of k, for $k=k_0\lambda e^{\frac{i\pi}{4}}$, $-\varepsilon<\lambda<\varepsilon$. The terms I, II, III and IV can be estimated as follows,

$$\begin{aligned}|I| &\leq |k^{-2i\nu}|\cdot|e^{i\beta\frac{k^2}{2}}|\cdot|\frac{k}{\sqrt{k_0^{-3}t}}|\cdot||\partial_kR(k)||_{L^\infty}\\ &\leq \frac{C}{\sqrt{t}},\end{aligned}$$

$$\begin{aligned}|II| &\leq |k^{-2i\nu}|\cdot|e^{i\beta\frac{k^2}{2}}|\cdot||R||_{L^\infty}\cdot\left|\frac{d}{ds}e^{i(1-2\beta)\frac{k^2}{2}(1-s\frac{k}{(1-2\beta)\eta^4\sqrt{k_0^{-9}t}})}\right|, \quad 0<s<1\\ &\leq \frac{C}{\sqrt{t}}\end{aligned}$$

To estimate III, we write

$$\begin{aligned}|III| &\leq |k^{-2i\nu}|\cdot|e^{i\beta\frac{k^2}{2}}|\cdot||R||_{L^\infty}\cdot|e^{i(1-2\beta)\frac{k^2}{2}(1-\frac{k}{(1-2\beta)\eta^4\sqrt{k_0^{-9}t}})}|\cdot|\int_1^{1+\sqrt{\frac{k_0}{4t}}k}2i\nu\xi^{2i\nu-1}d\xi|\\ &\leq \frac{C}{\sqrt{t}},\end{aligned}$$

as $|\xi^{2i\nu-1}|\leq c$ for $\xi=1+s\sqrt{\frac{k_0}{4t}}k, 0\leq s\leq 1$.

The estimate for IV is as follows,

$$|IV| \leq C\sup_{0\leq s\leq 1}|e^{2\left(\chi(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)-\chi(k_0)\right)}|\cdot\left|2e^{i\beta\frac{k^2}{2}}(\chi(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)-\chi(k_0))\right|$$

now let us show how to control $\left|e^{i\beta\frac{k^2}{2}}(\chi(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)-\chi(k_0))\right|$.

$$\begin{aligned}
&\left|e^{i\beta\frac{k^2}{2}}(\chi(\frac{k}{\sqrt{k_0^{-3}t}}+k_0)-\chi(k_0))\right|\\
&=\left|\frac{e^{i\beta\frac{k^2}{2}}}{2\pi}\int_{-\infty}^{-k_0}\ln\frac{k_0-s+\frac{k}{\sqrt{k_0^{-3}t}}}{k_0-s}d\ln(1+|r(s)|^2)+\int_{k_0}^{\infty}\ln\frac{s-k_0-\frac{k}{\sqrt{k_0^{-3}t}}}{s-k_0}d\ln(1+|r(s)|^2)\right|\\
&=\left|\frac{e^{i\beta\frac{k^2}{2}}}{2\pi}\int_{-\infty}^{-1}\ln(1+\frac{\sqrt{\frac{k_0}{t}}k}{1-s})d\ln(1+|r(sk_0)|^2)+\int_1^{\infty}\ln(1-\frac{\sqrt{\frac{k_0}{t}}k}{s-1})d\ln(1+|r(sk_0)|^2)\right|\\
&=|IV_1+IV_2|
\end{aligned}$$

where

$$IV_1=\frac{e^{i\beta\frac{k^2}{2}}}{2\pi}\int_{-\infty}^{-1}(g(s)-g(1))\ln(1+\frac{\sqrt{\frac{k_0}{t}}k}{1-s})ds+\int_1^{\infty}(g(s)-g(1))\ln(1-\frac{\sqrt{\frac{k_0}{t}}k}{s-1})ds$$

$$IV_2=\frac{e^{i\beta\frac{k^2}{2}}}{2\pi}\int_{-\infty}^{-1}g(1)\ln(1+\frac{\sqrt{\frac{k_0}{t}}k}{1-s})ds+\int_1^{\infty}g(1)\ln(1-\frac{\sqrt{\frac{k_0}{t}}k}{s-1})ds$$

here $g(s)=\partial_s\ln(1+|r(sk_0)|^2)$.

Then, using the Lipschitz condition $|\ln(1+a)|\leq|a|$, we have

$$\begin{aligned}
|IV_1|&\leq\left|\frac{e^{i\beta\frac{k^2}{2}}}{2\pi}\right|\cdot\int_{-\infty}^{-1}\sqrt{\frac{k_0}{t}}|k|\cdot|\frac{g(s)-g(1)}{s-1}|ds+\left|\frac{e^{i\beta\frac{k^2}{2}}}{2\pi}\right|\cdot\int_1^{\infty}\sqrt{\frac{k_0}{t}}|k|\cdot|\frac{g(s)-g(1)}{s-1}|ds\\
&\leq Ct^{-1/2}.
\end{aligned}$$

Notice that $g(s)$ is rapidly decay as $s\to\infty$, so the above integral is well-defined. And notice $|ke^{i\beta\frac{k^2}{2}}|$ and $\partial_s g$ are bounded.

$$\begin{aligned}
|IV_2|&\leq\left|\frac{e^{i\beta\frac{k^2}{2}}}{2\pi}\right|\cdot\left|\int_2^{\infty}\ln(1-\frac{k_0k^2}{s^2t})ds\right|\\
&=\left|\frac{e^{i\beta\frac{k^2}{2}}}{2\pi}\right|\cdot\left|(\int_2^{L}+\int_L^{\infty})\ln(1-\frac{k_0k^2}{s^2t})ds\right|,
\end{aligned}$$

where L is a big-enough positive constant. Since the above infinity integral is well-defined, the second integral is very small. And integral by parts shows that the first integral becomes

$$\int_2^L \ln(1-\frac{k_0k^2}{s^2t})ds = \left((s-\sqrt{\frac{k_0}{t}}k)\ln(s-\sqrt{\frac{k_0}{t}}k)+(s+\sqrt{\frac{k_0}{t}}k)\ln(s+\sqrt{\frac{k_0}{t}}k)-2s\ln s\right)\Bigg|_2^L$$

so we have

$$|IV_2| \le C\frac{\log t}{\sqrt{t}}.$$

Then, we can get the estimate of (5.80). This finishes the proof of the Proposition 5.11. □

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