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On the solvability of an indefinite nonlinear Kirchhoff equation via associated eigenvalue problems

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Abstract

We study the non-existence, existence and multiplicity of positive solutions to the following nonlinear Kirchhoff equation:

$$\begin{cases} -M\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)\Delta u+\mu V(x)\,u=Q(x)\,|u|^{p-2}u+\lambda f(x)\,u \text{ in } \mathbb{R}^N, \\ u\in H^1\left(\mathbb{R}^N\right), \end{cases}$$

where $N\geq 3, 2<p<2^*:=\frac{2N}{N-2}$, $M(t)=at+b$ $(a,b>0)$, the potential V is a nonnegative function in $\mathbb{R}^N$ and the weight function $Q\in L^\infty\left(\mathbb{R}^N\right)$ with changes sign in $\overline{\Omega}:=\{V=0\}$. We mainly prove the existence of at least two positive solutions in the cases that (i) $2<p<\min\left\{4,2^*\right\}$ and $0<\lambda<\left[1-2\left[(4-p)/4\right]^{2/p}\right]\lambda_1(f_\Omega)$; (ii) $p\geq 4, \lambda\geq\lambda_1(f_\Omega)$ and near $\lambda_1(f_\Omega)$ for $\mu>0$ sufficiently large, where $\lambda_1(f_\Omega)$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ with weight function $f_\Omega:=f|_{\overline{\Omega}}$, whose corresponding positive principal eigenfunction is denoted by ϕ_1. Furthermore, we also investigated the non-existence and existence of positive solutions if a,λ belongs to different intervals.

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1. Introduction

In this paper we are concerned the following nonlinear Kirchhoff equation:

$$\begin{cases} -M\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)\Delta u + \mu V(x)\, u = g(x,u) \text{ in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1.1}$$

where $N \geq 3$, $g \in \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ being continuous, $M(s) = as + b$ $(a, b > 0)$ and the parameter $\mu > 0$. We assume that the potential function V satisfies the following conditions:

(V_1) V is a nonnegative continuous function on $\mathbb{R}^N$;
(V_2) there exists $c > 0$ such that the set $\{V < c\} := \left\{x \in \mathbb{R}^N : V(x) < c\right\}$ is nonempty and has finite Lebesgue measure;
(V_3) $\Omega = \operatorname{int}\left\{x \in \mathbb{R}^N : V(x) = 0\right\}$ is nonempty bounded domain and has a smooth boundary with $\overline{\Omega} = \left\{x \in \mathbb{R}^N : V(x) = 0\right\}$.

The hypotheses $(V_1) - (V_3)$ imply that μV represents a potential well whose depth is controlled by μ. μV is called a steep potential well if μ is sufficiently large and one expects to find solutions which localize near its bottom Ω. This problem has found much interest after being first introduced by Bartsch and Wang [9] in the study of the existence of positive solutions for nonlinear Schrödinger equations and has been attracting much attention, see [3,7,8,33,38] and the references therein.

Kirchhoff type equations, of the form similar to Equation (1.1), originate from physics. Indeed, if we set $V(x) \equiv 0$ and replace $\mathbb{R}^N$ by a bounded domain $\Omega \subset \mathbb{R}^N$ in Equation (1.1), then it becomes the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -\left(a\int_\Omega |\nabla u|^2 dx + b\right)\Delta u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt} - \left(a\int_\Omega |\nabla u|^2 dx + b\right)\Delta u = g(x,u), \tag{1.3}$$

where u denotes the displacement, g is the external force and b is the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Equation (1.3) was first proposed by Kirchhoff [23] in 1883 to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations. It is notable that Equation (1.3) is often referred to as being nonlocal because of the presence of the integral over the domain Ω.

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After the pioneering work by Pohozaev [28] and Lions [24], the qualitative analysis of nontrivial solutions for the nonlinear Kirchhoff type equations, similar to Equation (1.1), has begun to receive much attention in recent years. We refer the reader to [2,12,15,16,18–22,26,29–32,34,37,39] and the references therein.

Let us briefly comment on some of the things that are relevant to our work. In [30], the authors introduced the steep potential well V to the Kirchhoff type equations. When the potential V satisfies the hypotheses $(V_1)-(V_3)$, the following results were obtained.
(i) $N \geq 3$: if $0 < a < a^*$ and $\mu > 0$ sufficiently large, then Equation (1.1) has at least one positive solution, when $g(x,u)$ is asymptotically linear at infinity on u and $b\lambda_1^{(1)} < 1$;
(ii) $N = 3$: if $0 < a < \lambda_1^{(3)}$ and $\mu > 0$ sufficiently large, then Equation (1.1) has at least one positive solution, when $g(x,u)$ is asymptotically 3-linear at infinity on u;
(iii) $N = 3$: for any $a > 0$ and $\mu > 0$ sufficiently large, Equation (1.1) has at least one positive solution, when $g(x,u)$ is asymptotically 4-linear at infinity on u,
where

$$\lambda_1^{(k)} = \inf\left\{ (\int_\Omega |\nabla u|^2 dx)^{\frac{k+1}{2}} : u \in H_0^1(\Omega), \int_\Omega q|u|^{k+1} dx = 1 \right\}$$

and q is a bounded function on $\bar{\Omega}$ with $q^+ \not\equiv 0$. After that, Xie and Ma [39] obtained the existence and concentration of positive solutions for Equation (1.1) with $N = 3$ when potential V satisfies conditions $(V_1)-(V_3)$ and nonlinearity g satisfies the following conditions:

(G_1) there exists $\rho > 4$ such that $0 < \rho G(x,u) \leq g(x,u)u$ for $u > 0$, where $G(x,u) = \int_0^u g(x,s)ds$;
(G_2) $\frac{G(x,u)}{u^3}$ is increasing for $u > 0$.

In our recent papers [29,32], we concluded that when $N \geq 3$ and $g(x,u)$ is superlinear and subcritical on u, the geometric structure of the functional J related to Equation (1.1) is known to have a global minimum and a mountain pass, owing to the fourth power of the nonlocal term. By using the standard variational methods, two different positive solutions can be found, since some embedding inequalities are proved with the help of the fact of $2^* := \frac{2N}{N-2} \leq 4$.

In simple terms, when $g(x,u) = Q(x)|u|^{p-2}u$ and $Q \in L^\infty(\mathbb{R}^N)$ is sign-changing, the current progress through the above literature is as follows:

(I) $N = 3$ and $4 < p < 6$: for any $a > 0$ and $\mu > 0$ sufficiently large, Equation (1.1) has at least one positive solution;
(II) $N = 3$ and $2 < p \leq 4$: for $a > 0$ small enough and $\mu > 0$ sufficiently large, Equation (1.1) has at least one positive solution;
(III) $N \geq 4$ and $2 < p < 2^*$: for $a > 0$ small enough and $\mu > 0$ sufficiently large, Equation (1.1) has at least two positive solution.

Motivated by these findings, we now extend the analysis to the Kirchhoff type equation with combination of a superlinear term and a linear term, that is $g(x,u) = Q(x)|u|^{p-2}u + \lambda f(x)u$. Our intension here is to illustrate the difference in the solution behavior which arises from the consideration of the nonlocal and eigenvalue problem effects. The problem we consider is thus

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$$\begin{cases} -M\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)\Delta u+\mu V(x)\,u=Q(x)\,|u|^{p-2}\,u+\lambda f(x)\,u \text{ in } \mathbb{R}^N, \\ u\in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{$E_{\mu,\lambda}$}$$

where $N\geq 3, 2<p<2^*:=\frac{2N}{N-2}$, $M(t)=at+b\ (a,b>0)$ and the parameters $\mu,\lambda>0$. We are interested in the case the weight functions f and Q satisfying $\{f>0\}\cap\Omega$ and $\{Q>0\}\cap\Omega$ has the positive Lebesgue measures which is why we call indefinite nonlinear Kirchhoff equation in the title.

To go further, let us give some notations first. For the sake of simplicity, we always assume that $b=1$ in Equation $(E_{\mu,\lambda})$. Let $D^{1,2}\left(\mathbb{R}^N\right)$ be the completing of $C_0^\infty\left(\mathbb{R}^N\right)$ with respect to the norm $\|u\|_{D^{1,2}}^2=\int_{\mathbb{R}^N}|\nabla u|^2\,dx$. Denote by S_p, $S_p(\Omega)$ and S the best constants for the embeddings of $H^1(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$, $H_0^1(\Omega)$ in $L^p(\Omega)$ and $D^{1,2}(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$, respectively. We denote a strong convergence by "$\to$" and a weak convergence by "$\rightharpoonup$".

Now, we give the variational setting for Equation $\left(E_{\mu,\lambda}\right)$. Let

$$X=\left\{u\in H^1\left(\mathbb{R}^N\right)\ :\int_{\mathbb{R}^N}Vu^2dx<\infty\right\}$$

be equipped with the inner product and norm

$$\langle u,v\rangle=\int_{\mathbb{R}^N}\nabla u\nabla v+Vuvdx,\ \ \|u\|=\langle u,u\rangle^{1/2}.$$

For $\mu>0$, we also need the following inner product and norm

$$\langle u,v\rangle_\mu=\int_{\mathbb{R}^N}\nabla u\nabla v+\mu Vuvdx,\ \ \|u\|_\mu=\langle u,u\rangle_\mu^{1/2}.$$

It is clear that $\|\cdot\|\leq\|\cdot\|_\mu$ for $\mu\geq 1$ and set $X_\mu=\left(X,\|\cdot\|_\mu\right)$.

Note that $u\in X_\mu$ is a solution of Equation $(E_{\mu,\lambda})$ if for any $v\in X_\mu$ there holds

$$M\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)\int_{\mathbb{R}^N}\nabla u\nabla v+\mu\int_{\mathbb{R}^N}V(x)\,uv=\int_{\mathbb{R}^N}\left(Q(x)\,|u|^{p-2}\,uv+\lambda f(x)\,uv\right)dx.$$

And u is called a positive solution if u is a solution and $u>0$ in $\mathbb{R}^N$.

It is well known that Equation $\left(E_{\mu,\lambda}\right)$ is variational, and its solutions correspond to the critical point of the energy functional $J_{\mu,\lambda}:X_\mu\to\mathbb{R}$

$$J_{\mu,\lambda}(u)=\frac{a}{4}\|u\|_{D^{1,2}}^4+\frac{1}{2}\|u\|_\mu^2-\frac{1}{p}\int_{\mathbb{R}^N}Q\,|u|^p\,dx-\frac{\lambda}{2}\int_{\mathbb{R}^N}fu^2dx,$$

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where $\|u\|_\mu = \left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \mu V u^2\right) dx\right]^{1/2}$ is the standard norm in X_μ and X_μ is a subspace of $H^1\left(\mathbb{R}^N\right)$ (see below). Thus, if u is a critical point of $J_{\mu,\lambda}$ on X_μ, then u is a solution of Equation $\left(E_{\mu,\lambda}\right)$.

Assume the following hypotheses (D):

(D_1) $f \in L^{N/2}\left(\mathbb{R}^N\right)$ which $f^+ := \max\{f, 0\} \not\equiv 0$ in Ω;
(D_2) $Q \in L^\infty\left(\mathbb{R}^N\right)$ which $Q^+ := \max\{Q, 0\} \not\equiv 0$ in Ω.

Remark 1.1. Since $\{f > 0\} \cap \Omega$ has a positive Lebesgue measure, we can assume that $\lambda_1(f_\Omega)$ denote the positive principal eigenvalue of the problem

$$-\Delta u(x) = \lambda f_\Omega(x) u(x) \text{ for } x \in \Omega;\; u(x) = 0 \text{ for } x \in \partial\Omega, \tag{1.4}$$

where f_Ω is a restriction of f on $\overline{\Omega}$. Clearly, $\lambda_1(f_\Omega)$ has a corresponding positive principal eigenfunction ϕ_1 with $\int_\Omega f_\Omega \phi_1^2 dx = 1$ and $\int_\Omega |\nabla \phi_1|^2 dx = \lambda_1(f_\Omega)$.

We now summarize our main results as follows.

Theorem 1.1. *Suppose that $N = 3, 4 < p < 6$ and conditions $(V_1)-(V_3)$ and $(D_1)-(D_2)$ hold. Then for each $a > 0$ and $0 < \lambda < \lambda_1(f_\Omega)$, Equation $\left(E_{\mu,\lambda}\right)$ has a positive solution u_μ^- satisfying $J_{\mu,\lambda}\left(u_\mu^-\right) > 0$ for $\mu > 0$ sufficiently large.*

Theorem 1.2. *Suppose that $N = 3$, $4 < p < 6$, conditions $(V_1)-(V_3)$ and $(D_1)-(D_2)$ hold and $\int_\Omega Q\phi_1^p dx < 0$. Then for each $a > 0$ there exists δ_0 such that for every $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \delta_0$, Equation $\left(E_{\mu,\lambda}\right)$ has at least two positive solutions u_μ^- and u_μ^+ satisfying $J_{\mu,\lambda}\left(u_\mu^+\right) < 0 < J_{\mu,\lambda}\left(u_\mu^-\right)$ for $\mu > 0$ sufficiently large.*

To consider the case $N = 3$ and $p = 4$, we need the following maximum problem

$$\Gamma_0 := \sup_{u \in X} \frac{\int_{\mathbb{R}^3} Q|u|^4 dx}{\|u\|_{D^{1,2}}^4} > 0.$$

Then we have the following results.

Theorem 1.3. *Suppose that $N = 3$, $p = 4$ and conditions $(V_1)-(V_3)$ and $(D_1)-(D_2)$ hold. Then we have the following results.*
(i) For each $0 < a < \Gamma_0$ and $0 < \lambda < \lambda_1(f_\Omega)$, Equation $\left(E_{\mu,\lambda}\right)$ has a positive solution u_μ^- satisfying $J_{\mu,\lambda}\left(u_\mu^-\right) > 0$ for $\mu > 0$ sufficiently large.
(ii) If $\Gamma_0 < \infty$, then for each $a \ge \Gamma_0$ and $0 < \lambda < \lambda_1(f_\Omega)$, Equation $\left(E_{\mu,\lambda}\right)$ does not admit nontrivial solution for $\mu > 0$ sufficiently large.
(iii) If $\Gamma_0 < \infty$, then for each $a > \Gamma_0$ and $\lambda \ge \lambda_1(f_\Omega)$, Equation $\left(E_{\mu,\lambda}\right)$ has a positive solution u_μ^+ satisfying $J_{\mu,\lambda}\left(u_\mu^+\right) < 0$ for $\mu > 0$ sufficiently large.
(iv) If $\Gamma_0 < \infty$ and Γ_0 is not attained, then for $a = \Gamma_0$ and $\lambda \ge \lambda_1(f_\Omega)$, Equation $\left(E_{\mu,\lambda}\right)$ has a positive solution u_μ^+ satisfying $J_{\mu,\lambda}\left(u_\mu^+\right) < 0$ for $\mu > 0$ sufficiently large.

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Theorem 1.4. *Suppose that $N = 3$, $p = 4$ and conditions $(V_1) - (V_3)$ and $(D_1) - (D_2)$ hold. Then for each $\lambda_1^{-2}(f_\Omega)\int_\Omega Q\phi_1^4 dx < a < \Gamma_0$ there exists δ_0 such that for every $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \delta_0$, Equation $(E_{\mu,\lambda})$ has two positive solutions u_μ^- and u_μ^+ satisfying $J_{\mu,\lambda}(u_\mu^+) < 0 < J_{\mu,\lambda}(u_\mu^-)$ for $\mu > 0$ sufficiently large.*

Remark 1.2. When $4 < p < 6$, by the hypothesis of Theorem 1.2, in order to obtain the multiplicity of the positive solution for Equation $(E_{\mu,\lambda})$, the weight function Q must be sign-changing in Ω, but when $p = 4$, the weight function Q can be positive in Ω from the hypothesis of Theorem 1.4.

To consider the case $2 < p < \min\{4, 2^*\}$, we first show that the non-existence of solutions.

Theorem 1.5. *Suppose that $N \ge 4$, $2 < p < 2^*$ and conditions $(V_1) - (V_3)$ and $(D_1) - (D_2)$ hold. Then for each $0 < \lambda < \lambda_1(f_\Omega)$ there exists*

$$0 < \overline{\mathbf{A}}_\lambda < \frac{1}{2}\left(\frac{(4-p)\lambda_1(f_\Omega)}{p(\lambda_1(f_\Omega)-\lambda)}\right)^{(4-p)/(p-2)}\left(\frac{\|Q\|_\infty |\{V < c\}|^{\frac{2^*-p}{2^*}}}{S^p}\right)^{2/(p-2)}$$

such that for every $a > \overline{\mathbf{A}}_\lambda$, Equation $(E_{\mu,\lambda})$ does not admit nontrivial solution for $\mu > 0$ sufficiently large.

To prove the existence of positive solution, we need the following conditions:

(D_3) There exist two numbers $c_*, R_* > 0$ such that

$$|x|^{p-2} Q(x) \le c_* [V(x)]^{4-p} \text{ for all } |x| > R_*.$$

(D_4) $|\{V < c\}|^{(6-p)/6} \le \frac{S^p Q_{\Omega,\min}}{S_p^p(\Omega)\|Q\|_\infty}$, where $Q_{\Omega,\min} = \inf_{x\in\overline{\Omega}} Q(x) > 0$.

Then we have the following results.

Theorem 1.6. *Suppose that $N = 3$, $2 < p < 4$ and conditions $(V_1) - (V_3)$ and $(D_1) - (D_3)$ hold. Then we have the following results.*
(i) There exists $a_0 > 0$ such that for every $0 < a < a_0$ and $0 < \lambda < \lambda_1(f_\Omega)$, Equation $(E_{\mu,\lambda})$ has a positive solution u_μ^+ satisfying $J_{\mu,\lambda}(u_\mu^+) < 0$ for $\mu > 0$ sufficiently large.
(ii) For each $\lambda \ge \lambda_1(f_\Omega)$ and $a > 0$, Equation $(E_{\mu,\lambda})$ has a positive solution u_μ^+ satisfying $J_{\mu,\lambda}(u_\mu^+) < 0$ for $\mu > 0$ sufficiently large.

Theorem 1.7. *Suppose that $N \ge 4$, $2 < p < 2^*$ and conditions $(V_1) - (V_3)$ and $(D_1) - (D_2)$ hold. Then we have the following results.*
(i) There exists $a_0 > 0$ such that for every $0 < a < a_0$ and $0 < \lambda < \lambda_1(f_\Omega)$, Equation $(E_{\mu,\lambda})$ has a positive solution u_μ^+ satisfying $J_{\mu,\lambda}(u_\mu^+) < 0$ for $\mu > 0$ sufficiently large.
(ii) For each $a > 0$ and $\lambda \ge \lambda_1(f_\Omega)$, Equation $(E_{\mu,\lambda})$ has a positive solution u_μ^+ satisfying $J_{\mu,\lambda}(u_\mu^+) < 0$ for $\mu > 0$ sufficiently large.

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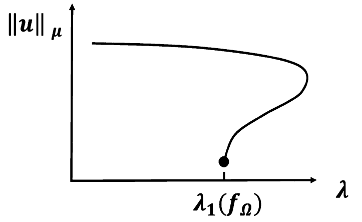

(a) $a>0$ and $\int_\Omega Q\phi_1^p dx<0$.

Fig. 1. Bifurcation diagram for Theorems 1.1 and 1.2.

Theorem 1.8. *Suppose that $N \geq 3$, $2<p<\min\{4,2^*\}$ and conditions $(V_1)-(V_3)$, $(D_1)-(D_2)$ and (D_4) hold. Then there exists $a_0>0$ such that for every $0<a<a_0$ and $0<\lambda<\left[1-2\left(\frac{4-p}{4}\right)^{2/p}\right]\lambda_1(f_\Omega)$, Equation $\left(E_{\mu,\lambda}\right)$ has a positive solution u_μ^- satisfying $J_{\mu,\lambda}\left(u_\mu^-\right)>0$ for $\mu>0$ sufficiently large.*

Combining the Theorems 1.6, 1.8 results, we have the following multiplicity result.

Corollary 1.9. *Suppose that $N=3$, $2<p<4$ and conditions $(V_1)-(V_3)$ and $(D_1)-(D_4)$ hold. Then there exists $a_0>0$ such that for every $0<a<a_0$ and $0<\lambda<\left[1-2\left(\frac{4-p}{4}\right)^{2/p}\right]\lambda_1(f_\Omega)$, Equation $\left(E_{\mu,\lambda}\right)$ has two positive solutions u_μ^- and u_μ^+ satisfying $J_{\mu,\lambda}\left(u_\mu^+\right)<0<J_{\mu,\lambda}\left(u_\mu^-\right)$ for $\mu>0$ sufficiently large.*

Combining the Theorems 1.7, 1.8 results, we have the following multiplicity result.

Corollary 1.10. *Suppose that $N \geq 4$, $2<p<2^*$ and conditions $(V_1)-(V_3)$, $(D_1)-(D_2)$ and (D_4) hold. Then there exists $a_0>0$ such that for every $0<a<a_0$ and $0<\lambda<\left[1-2\left(\frac{4-p}{4}\right)^{2/p}\right]\lambda_1(f_\Omega)$, Equation $\left(E_{\mu,\lambda}\right)$ has two positive solutions u_μ^- and u_μ^+ satisfying $J_{\mu,\lambda}\left(u_\mu^+\right)<0<J_{\mu,\lambda}\left(u_\mu^-\right)$ for $\mu>0$ sufficiently large.*

In order to make the above theoretical results more intuitive, the bifurcation diagrams of positive solutions concerning with the ranges of constant p, a, λ is shown.

(I) $4<p<6$.

(II) $p=4$.

(III) $2<p<\min\{4,2^*\}$.

We illustrate the finding of Theorems 1.1-1.8 graphically in Figs. 1-3 with different values of a, p and λ. These figures depict how the number of positive solutions of u changes with the parameter λ under certain conditions. Subgraphs show the bifurcation diagram of the positive solution of u when a is in different ranges, respectively.

Remark 1.3. In Fig. 3 (a), the part marked with a question mark is not covered in this article, it is indicated by a dotted line that the exact number of positive solutions is unknown.

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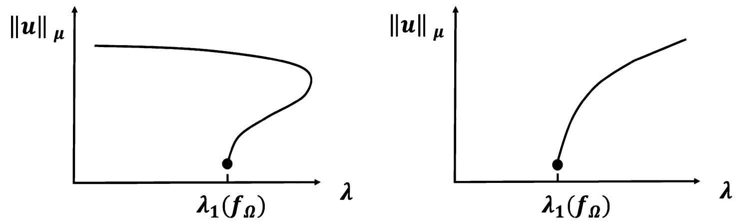
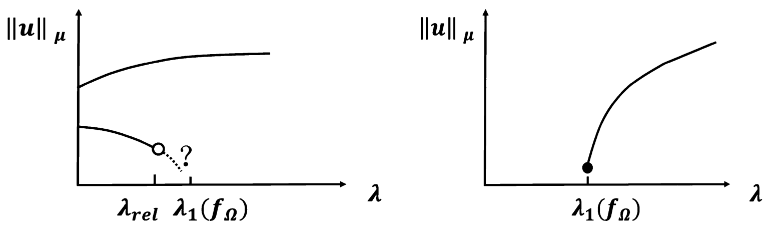


(a) $a_{rel} < a < \Gamma_0$. (b) $a \geq \Gamma_0$.

Fig. 2. Bifurcation diagrams for Theorems 1.3 and 1.4, where $a_{rel} := \max\{0, \lambda_1^{-2}(f_\Omega) \int_\Omega Q\phi^4 dx\}$.



(a) $0 < a < a_0$. (b) $\overline{\mathbf{A}}_\lambda < a < +\infty$.

Fig. 3. Bifurcation diagrams for Theorems 1.7 (i) and 1.8 (also for Theorems 1.6 and 1.8) on (a) and for Theorems 1.5 and 1.7 (ii) on (b), where $\lambda_{rel} := \left[1 - 2\left(\frac{4-p}{4}\right)^{2/p}\right] \lambda_1(f_\Omega)$.

To study the main Theorems, we shall establish their result by considering minimization on two distinct components of the Nehari manifold corresponding to Equation $(E_{\mu,\lambda})$. We are likewise interested in the conditions of M and g that subsequently gives rise to the non-existence and existence of positive solutions. Our focus here, however, is on a given set of M and g so that it is possible to examine in detail the number of solutions admitted subject to the variations of parameters imbedded in these functions. A similar analysis has been carried out on other elliptic equations with interesting results. Amann and Lopez-Gomez [1], Binding et al. [4,5], and Brown and Zhang [10], for example, studied the following semilinear boundary value problem:

$$\begin{cases} -\Delta u = \lambda f_\Omega(x) u + b(x) |u|^{p-2} u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $\lambda > 0$ is a real parameter, $2 < p < 2^*$ and $f_\Omega, b : \overline{\Omega} \to \mathbb{R}$ are smooth functions which change sign in $\overline{\Omega}$. In [4,5] by using variational methods, in Brown and Zhang [10] by using Nehari manifold and fibrering maps, and in Amann and Lopez-Gomez [1] by using global bifurcation theory. The existence and multiplicity results can be summarized as follows. It is known that

(A) there exists a positive solution to Equation (1.5) whenever $0 < \lambda < \lambda_1(f_\Omega)$;
(B) if $\int_\Omega b\phi_1^p dx < 0$, there exists $\delta_0 > 0$ such that Equation (1.5) has at least two positive solutions whenever $\lambda_1(f_\Omega) < \lambda < \lambda_1(f_\Omega) + \delta_0$.

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Results (A) and (B) can be understood in term of global bifurcation theory as the sign of $\int_\Omega b\phi_1^p dx$ determines the direction of bifurcation from the branch of zero solutions at the bifurcation point at $\lambda = \lambda_1(f_\Omega)$ so that bifurcation is to the left when $\int_\Omega b\phi_1^p dx > 0$ and to the right when $\int_\Omega b\phi_1^p dx < 0$; the corresponding bifurcation diagrams are shown in Fig. 1 of [10]. Furthermore, some who's been done for this type of problem in $\mathbb{R}^N$. We are only aware of the works Chabrowski and Costa [11] and Costa and Tehrani [13] which also studied the existence and multiplicity of positive solutions for Schrödinger type equations in $\mathbb{R}^N$

$$-\Delta_p u = \lambda \widehat{f}(x) u + \widetilde{Q}(x) |u|^{p-2} u \text{ in } \mathbb{R}^N, \tag{1.6}$$

where λ is a real parameter and $p < q < Np/(N-p)$ and $1 < p < N$. The functions $\widetilde{f}$ and $\widetilde{Q}$ denote sign-changing potentials such that $\widetilde{f} \in L^{N/p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\widetilde{Q} \in L^\infty(\mathbb{R}^N)$. Let $\lambda_1\left(\widetilde{f}\right)$ denote the lowest positive eigenvalue of $-\Delta_p$ and let $\varphi_1 > 0$ be the associated eigenfunction. When $p = 2$, under a slightly more general assumption on the nonlinearity appearing on the right-hand side of (1.6), some results are obtained in [13] by using the Mountain-Pass Theorem and variational methods. However, in order to apply their result to Equation (1.6) they needed a "thickness" condition on the set $\Omega_o = \left\{x : \widetilde{Q}(x) = 0\right\}$. [11] by using Nehari manifold and fibrering maps which under a limits condition $\lim_{|x|\to\infty} \widetilde{Q}(x) = \widetilde{Q}_\infty < 0$. Their main result is almost the same as in results (A) and (B) above. However, the principal eigenvalue and eigenfunction are replaced by the problem $-\Delta u(x) = \lambda \widetilde{f}(x) u(x)$ for $x \in \mathbb{R}^N$.

The approach to Equation $\left(E_{\mu,\lambda}\right)$ has been inspired by the papers of [10,11] without any condition on Ω_o or $\lim_{|x|\to\infty} Q(x) = Q_\infty < 0$. Moreover, since Equation $\left(E_{\mu,\lambda}\right)$ is on $\mathbb{R}^N$, its variational setting is characterized by a lack of compactness. To overcome this difficulty we apply a simplified version of the steep well method of [9] and concentration compactness principle of [25]. Furthermore, the first eigenvalue of problem $-\Delta u + \mu V(x) u = \lambda f(x) u$ in $\mathbb{R}^N$ is less than $\lambda_1(f_\Omega)$, which indicates that the original method at [10,11] cannot be directly applied, thus we provide an approximation estimate of eigenvalue to prove our main results.

The plan of the paper is as follows. In Section 2, we discuss the Nehari manifold and examine carefully the connection between the Nehari manifold and the fibrering maps. In Section 3, we establish the non-emptiness of submanifolds and the proofs of the main theorems are given in the remaining sections. In section 4, we discuss the Nehari manifold when $4 < p < 6$. In particular, we prove that Theorems 1.1, 1.2. In Section 5, we discuss the case when $p = 4$ and prove that Theorems 1.3, 1.4. In section 6, we discuss the case when $p < 4$ and prove that Theorems 1.6, 1.7 and 1.8.

2. Preliminaries

It follows from conditions (V_1) and (V_2) and similar to the argument in [30], one has

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx \le \left(1 + S^{-2} \left|\{V < c\}\right|^{\frac{2}{N}}\right) \|u\|_\mu^2$$

for all $\mu \ge \mu_0 := \frac{S^2}{c} \left|\{V < c\}\right|^{-\frac{2}{N}}$, which implies that the imbedding $X_\mu \hookrightarrow H^1(\mathbb{R}^N)$ is continuous. Moreover, for any $r \in [2, 2^*]$, there holds

$$\int_{\mathbb{R}^N} |u|^r \, dx \leq S^{-r} \left|\{V < c\}\right|^{\frac{2^*-r}{2^*}} \|u\|_\mu^r \text{ for } \mu \geq \mu_0. \tag{2.1}$$

Because the energy functional $J_{\mu,\lambda}$ is not bounded below on X_μ, it is useful to consider the functional on the Nehari manifold (see [27])

$$\mathbf{N}_{\mu,\lambda} = \left\{ u \in X_\mu \setminus \{0\} : \left\langle J'_{\mu,\lambda}(u), u \right\rangle = 0 \right\}.$$

Thus, $u \in \mathbf{N}_{\mu,\lambda}$ if and only if

$$a \|u\|_{D^{1,2}}^4 + \|u\|_\mu^2 = \int_{\mathbb{R}^N} Q |u|^p \, dx + \lambda \int_{\mathbb{R}^N} f u^2 dx.$$

Note that $\mathbf{N}_{\mu,\lambda}$ contains every nonzero solution of Equation $\left(E_{\mu,\lambda}\right)$. It is useful to understand $\mathbf{N}_{\mu,\lambda}$ in terms of the stationary points of mappings of the form $h_u(t) = J_{\mu,\lambda}(tu)(t > 0)$. Such a map is known as the fibrering map. It was introduced by Drábek and Pohozaev [14], and further discussed by Brown and Zhang [10]. It is clear that, if u is a local minimizer of $J_{\mu,\lambda}$, then h_u has a local minimum at $t = 1$. Thus, $tu \in \mathbf{N}_{\mu,\lambda}$ if and only if $h'_u(t) = 0$ for $u \in X \setminus \{0\}$. Thus points in $\mathbf{N}_{\mu,\lambda}$ correspond to stationary points of the maps h_u and so it is natural to divide $\mathbf{N}_{\mu,\lambda}$ into three subsets $\mathbf{N}^+_{\mu,\lambda}$, $\mathbf{N}^-_{\mu,\lambda}$ and $\mathbf{N}^0_{\mu,\lambda}$ corresponding to local minima, local maxima and points of inflexion of fibrering maps. We have

$$h'_u(t) = at^3 \|u\|_{D^{1,2}}^4 + t \left(\|u\|_\mu^2 - \lambda \int_{\mathbb{R}^N} f u^2 dx \right) - t^{p-1} \int_{\mathbb{R}^N} Q |u|^p \, dx$$

and

$$h''_u(t) = 3at^2 \|u\|_{D^{1,2}}^4 + \left(\|u\|_\mu^2 - \lambda \int_{\mathbb{R}^N} f u^2 dx \right) - (p-1) t^{p-2} \int_{\mathbb{R}^N} Q |u|^p \, dx.$$

Hence if we define

$$\begin{aligned} \mathbf{N}^+_{\mu,\lambda} &= \left\{ u \in \mathbf{N}_{\mu,\lambda} : h''_u(1) > 0 \right\}; \\ \mathbf{N}^0_{\mu,\lambda} &= \left\{ u \in \mathbf{N}_{\mu,\lambda} : h''_u(1) = 0 \right\}; \\ \mathbf{N}^-_{\mu,\lambda} &= \left\{ u \in \mathbf{N}_{\mu,\lambda} : h''_u(1) < 0 \right\}, \end{aligned}$$

which indicates that for $u \in \mathbf{N}_{\mu,\lambda}$, we have $h'_u(1) = 0$ and $u \in \mathbf{N}^+_{\mu,\lambda}, \mathbf{N}^0_{\mu,\lambda}, \mathbf{N}^-_{\mu,\lambda}$ if $h''_u(1) > 0, h''_u(1) = 0, h''_u(1) < 0$, respectively. Note that for all $u \in \mathbf{N}_{\mu,\lambda}$,

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$$\begin{aligned}
h_u''(1) &= -(p-2)\left(\|u\|_\mu^2 - \lambda \int\limits_{\mathbb{R}^N} f u^2 dx\right) - a(p-4)\|u\|_{D^{1,2}}^4 \\
&= 2a\|u\|_{D^{1,2}}^4 - (p-2)\int\limits_{\mathbb{R}^N} Q|u|^p dx \\
&= -2\left(\|u\|_\mu^2 - \lambda \int\limits_{\mathbb{R}^N} f u^2 dx\right) - (p-4)\int\limits_{\mathbb{R}^N} Q|u|^p dx. \qquad (2.2)
\end{aligned}$$

Now, we define

$$\Lambda_\mu^+ = \left\{ u \in X : \|u\|_\mu = 1, \|u\|_\mu^2 - \lambda \int\limits_{\mathbb{R}^N} f u^2 dx > 0 \right\}$$

and Λ_μ^- and Λ_μ^0 similarly by replacing $>$ by $<$ and $=$ respectively. We also define

$$\Theta_\mu^+(p) = \left\{ u \in X : \|u\|_\mu = 1, \Phi_p(u) > 0 \right\}$$

and $\Theta_\mu^-(p)$ and $\Theta_\mu^0(p)$ analogously, where

$$\Phi_p(u) = \begin{cases} \int_{\mathbb{R}^N} Q|u|^p dx & \text{for } 2 < p < 2^* \text{ and } p \neq 4, \\ \int_{\mathbb{R}^N} Q|u|^p dx - a\|u\|_{D^{1,2}}^4 & \text{for } p = 4. \end{cases}$$

Thus, if $u \in \Lambda_\mu^+ \cap \Theta_\mu^+(p)$ and $p \geq 4$, $h_u(t) > 0$ for t small and positive but $h_u(t) \to -\infty$ as $t \to \infty$; also $h_u(t)$ has a unique (maximum) stationary point $t_{\max}(u)$ and $t_{\max}(u)u \in \mathbf{N}_{\mu,\lambda}^-$. Similarly, if $u \in \Lambda_\mu^- \cap \Theta_\mu^-(p)$ and $2 < p < 2^*$, $h_u(t) < 0$ for t small and positive, $h_u(t) \to \infty$ as $t \to \infty$ and $h_u(t)$ has a unique minimum $t_{\min}(u)$ so that $t_{\min}(u)u \in \mathbf{N}_{\mu,\lambda}^+$. Finally, if $u \in \Lambda_\mu^+ \cap \Theta_\mu^-(p)$, h_u is strictly increasing for all $t > 0$. Thus, we have the following results.

Lemma 2.1. *Suppose that $N = 3$ and $4 < p < 6$. If $\overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p) = \emptyset$ and $u \in X_\mu \backslash \{0\}$, then*
(i) a multiple of u lies in $\mathbf{N}_{\mu,\lambda}^-$ if and only if $\frac{u}{\|u\|_\mu}$ lies in $\Lambda_\mu^+ \cap \Theta_\mu^+(p)$;
(ii) a multiple of u lies in $\mathbf{N}_{\mu,\lambda}^+$ if and only if $\frac{u}{\|u\|_\mu}$ lies in $\Lambda_\mu^- \cap \Theta_\mu^-(p)$;
(iii) when $u \in \Lambda_\mu^+ \cap \Theta_\mu^-(p)$, no multiple of u lies in $\mathbf{N}_{\mu,\lambda}$.

Lemma 2.2. *Suppose that $N = 3$ and $p = 4$. If $u \in X_\mu \backslash \{0\}$, then*
(i) a multiple of u lies in $\mathbf{N}_{\mu,\lambda}^-$ if and only if $\frac{u}{\|u\|_\mu}$ lies in $\Lambda_\mu^+ \cap \Theta_\mu^+(p)$;
(ii) a multiple of u lies in $\mathbf{N}_{\mu,\lambda}^+$ if and only if $\frac{u}{\|u\|_\mu}$ lies in $\Lambda_\mu^- \cap \Theta_\mu^-(p)$;
(iii) when $u \in \overline{\Lambda_\mu^+} \cap \overline{\Theta_\mu^-(p)}$ or $\overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+(p)}$, no multiple of u lies in $\mathbf{N}_{\mu,\lambda}$.

Lemma 2.3. *Suppose that $N \geq 3$ and $2 < p < \min\{4, 2^*\}$. If $u \in X_\mu \backslash \{0\}$, then*
(i) if $\frac{u}{\|u\|_\mu}$ lies in $\overline{\Lambda_\mu^-} \cap \Theta_\mu^+(p)$ or $\Lambda_\mu^- \cap \overline{\Theta_\mu^-}(p)$, then a multiple of u lies in $\mathbf{N}_{\mu,\lambda}^+$;

(ii) when $u \in \Lambda_{\mu}^{+} \cap \Theta_{\mu}^{-}(p)$, no multiple of u lies in $\mathbf{N}_{\mu,\lambda}$.

The following Lemma shows that minimizers on $\mathbf{N}_{\mu,\lambda}$ are critical points for $J_{\mu,\lambda}$ in X_{μ}.

Lemma 2.4. *Suppose that u_0 is a local minimizer for $J_{\mu,\lambda}$ on $\mathbf{N}_{\mu,\lambda}$ and that $u_0 \notin \mathbf{N}_{\mu,\lambda}^{0}$. Then $J_{\mu,\lambda}'(u_0) = 0$ in X_{μ}^{-1}.*

Proof. The proof of Lemma 2.4 is essentially same as that in Brown and Zhang [10, Theorem 2.3] (or see Binding et al. [4]), so we omit it here. □

Finally, we investigate the compactness condition for the functional $J_{\mu,\lambda}$. Here we call that a C^1-functional $J_{\mu,\lambda}$ satisfies Palais-Smale condition at level β ((PS)$_\beta$-condition for short) in $\mathbf{N}_{\mu,\lambda}$, if any sequence $\{u_n\} \subset \mathbf{N}_{\mu,\lambda}$ is uniformly bounded which satisfy $J_{\mu,\lambda}(u_n) = \beta + o(1)$ and $J_{\mu,\lambda}'(u_n) = o(1)$ has a convergent subsequence.

Proposition 2.5. *Suppose that conditions $(V_1) - (V_2)$ and $(D_1) - (D_2)$ hold. Then there exists $\widehat{D}_0 \in \mathbb{R}$ independent of μ such that $J_{\mu,\lambda}$ satisfies (PS)$_\beta$–condition in $\mathbf{N}_{\mu,\lambda}$ with $\beta < \widehat{D}_0$ for $\mu > 0$ sufficiently large.*

Proof. Let $\{u_n\} \subset \mathbf{N}_{\mu,\lambda}$ be a (PS)$_\beta$–sequence for $J_{\mu,\lambda}$ with $\beta < \widehat{D}_0$. Since $\{u_n\} \subset X_\mu$ is uniformly bounded, i.e., there exists $d_0 > 0$ such that

$$\|u_n\|_\mu < d_0. \tag{2.3}$$

Then there exist a subsequence $\{u_n\}$ and u_0 in X_μ such that

$$\begin{aligned} &u_n \rightharpoonup u_0 \text{ weakly in } X_\mu; \\ &u_n \to u_0 \text{ strongly in } L_{loc}^r(\mathbb{R}^N) \text{ for } 2 \le r < 2^*. \end{aligned}$$

Then by condition (D_1),

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f u_n^2 dx = \int_{\mathbb{R}^N} f u_0^2 dx. \tag{2.4}$$

Now, we prove that $u_n \to u_0$ strongly in X_μ. Let $v_n = u_n - u_0$. By (2.3) one has

$$\|u_0\|_\mu \le \liminf_{n\to\infty} \|u_n\|_\mu \le d_0,$$

leading to

$$\|v_n\|_\mu = \|u_n - u_0\|_\mu \le 2d_0. \tag{2.5}$$

It follows from the condition (V_1) that

$$\int_{\mathbb{R}^N} v_n^2 dx = \int_{\{V \ge c\}} v_n^2 dx + \int_{\{V<c\}} v_n^2 dx \le \frac{1}{\mu c} \|v_n\|_\mu^2 + o(1),$$

which implies that

$$
\begin{aligned}
\int_{\mathbb{R}^N} |v_n|^p \, dx &\le \left(\frac{1}{\mu c} \|v_n\|_\mu^2 \right)^{\frac{2^*-p}{2^*-2}} \left(S^{-2^*} \|v_n\|_{D^{1,2}}^{2^*} \right)^{\frac{p-2}{2^*-2}} + o(1) \\
&\le \left(\frac{1}{\mu c} \right)^{\frac{(2^*-p)(N-2)}{4}} S^{-\frac{N(p-2)}{2}} \|v_n\|_\mu^p + o(1),
\end{aligned} \tag{2.6}
$$

where we have used the Hölder and Sobolev inequalities. Moreover, by Brezis-Lieb Lemma [6] and condition (D_2), we have

$$
\int_{\mathbb{R}^N} Q |v_n|^p \, dx = \int_{\mathbb{R}^N} Q |u_n|^p \, dx - \int_{\mathbb{R}^N} Q |u_0|^p \, dx + o(1). \tag{2.7}
$$

Since the sequence $\{u_n\}$ is bounded in X_μ, there exists a constant $A > 0$ such that

$$
\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \to A \text{ as } n \to \infty.
$$

It indicates that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$
\begin{aligned}
o(1) &= \left\langle J'_{\mu,\lambda}(u_n), \varphi \right\rangle \\
&\to \int_{\mathbb{R}^N} \nabla u_0 \nabla \varphi dx + \int_{\mathbb{R}^N} \mu V u_0 \varphi dx + aA \int_{\mathbb{R}^N} \nabla u_0 \nabla \varphi dx \\
&\quad - \int_{\mathbb{R}^N} f u_0 \varphi dx - \int_{\mathbb{R}^N} Q |u_0|^{p-2} u_0 \varphi dx \text{ as } n \to \infty,
\end{aligned}
$$

which shows that

$$
\|u_0\|_\mu^2 + aA \|u_0\|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} f u_0^2 dx - \int_{\mathbb{R}^N} Q |u_0|^p \, dx = 0. \tag{2.8}
$$

Note that

$$
\|u_n\|_\mu^2 + a \|u_n\|_{D^{1,2}}^4 - \int_{\mathbb{R}^N} f u_n^2 dx - \int_{\mathbb{R}^N} Q |u_n|^p \, dx = 0. \tag{2.9}
$$

Then by (2.4) and (2.6)–(2.9) one has

$$
\begin{aligned}
o(1) &= \|v_n\|_\mu^2 + a\|u_n\|_{D^{1,2}}^4 - aA\|u_0\|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} Q|v_n|^p\,dx \\
&= \|v_n\|_\mu^2 + a\|u_n\|_{D^{1,2}}^2 \left(\|u_n\|_{D^{1,2}}^2 - \|u_0\|_{D^{1,2}}^2\right) - \int_{\mathbb{R}^N} Q|v_n|^p\,dx \\
&= \|v_n\|_\mu^2 + a\|u_n\|_{D^{1,2}}^2 \|v_n\|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} Q|v_n|^p\,dx. \qquad (2.10)
\end{aligned}
$$

It follows from (2.1), (2.5), (2.6), (2.10) and condition (D_2) that

$$
\begin{aligned}
o(1) &= \|v_n\|_\mu^2 + a\|u_n\|_{D^{1,2}}^2 \|v_n\|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} Q|v_n|^p\,dx \\
&\geq \|v_n\|_\mu^2 - \|Q\|_\infty \left(\int_{\mathbb{R}^N} |v_n|^p dx\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^N} |v_n|^p dx\right)^{\frac{2}{p}} \\
&\geq \left[1 - \|Q\|_\infty \left[\frac{(2d_0)^{\frac{p}{2}} |\{V < c\}|^{\frac{2^*-p}{2^*}}}{S^p}\right]^{\frac{p-2}{p}} \left(\frac{1}{\mu c}\right)^{\frac{(2^*-p)(N-2)}{2p}} S^{-\frac{N(p-2)}{p}}\right] \|v_n\|_\mu^2 + o(1),
\end{aligned}
$$

which implies that $v_n \to 0$ strongly in X_μ for $\mu > 0$ sufficiently large. Consequently, this completes the proof. □

3. Non-emptiness of submanifolds

First, we need the following result.

Theorem 3.1. *Let $\mu_n \to \infty$ as $n \to \infty$ and $\{v_n\} \subset X$ with $\|v_n\|_{\mu_n} \leq c_0$ for some $c_0 > 0$. Then for every $\mu > 0$ there exist subsequence $\{v_n\}$ and $v_0 \in H_0^1(\Omega)$ such that $v_n \rightharpoonup v_0$ in X_μ and $v_n \to v_0$ in $L^r(\mathbb{R}^N)$ for all $2 \leq r < 2^*$.*

Proof. Since $\|v_n\|_\mu \leq \|v_n\|_{\mu_n} \leq c_0$ for n sufficiently large. We may assume that there exists $v_0 \in X$ such that

$$
\begin{aligned}
& v_n \rightharpoonup v_0 \text{ in } X_\mu, \\
& v_n \to v_0 \text{ a.e. in } \mathbb{R}^N, \\
& v_n \to v_0 \text{ in } L_{loc}^r\left(\mathbb{R}^N\right) \text{ for } 2 \leq r < 2^*.
\end{aligned}
$$

By Fatou's Lemma, we have

$$
\int_{\mathbb{R}^N} V v_0^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V v_n^2 dx \leq \liminf_{n\to\infty} \frac{\|v_n\|_{\mu_n}^2}{\mu_n} = 0,
$$

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this implies that $\int_{\mathbb{R}^N} V v_0^2 dx = 0$ or $v_0 = 0$ a.e. in $\mathbb{R}^N \setminus \overline{\Omega}$ and $v_0 \in H_0^1(\Omega)$ by condition (V_3). We now show that $v_n \to v_0$ in $L^p(\mathbb{R}^N)$. Suppose on the contrary. Then by Lions vanishing lemma (see [25, Lemma I.1] or [36, Lemma 1.21]), there exist $d_0 > 0$, $R_0 > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\int\limits_{B(x_n,R_0)} (v_n - v_0)^2 \, dx \geq d_0.$$

Moreover, $x_n \to \infty$, and hence, $\left|B(x_n, R_0) \cap \left\{x \in \mathbb{R}^N : V < c\right\}\right| \to 0$. By the Hölder inequality, we have

$$\int\limits_{B(x_n,R_0)\cap\{V<c\}} (v_n - v_0)^2 \, dx \to 0.$$

Consequently,

$$\begin{aligned} c_0 \ \geq \ \|v_n\|_{\mu_n}^2 \geq \mu_n c & \int\limits_{B(x_n,R_0)\cap\{V\geq c\}} v_n^2 dx = \mu_n c \int\limits_{B(x_n,R_0)\cap\{V\geq c\}} (v_n - v_0)^2 \, dx \\ &= \mu_n c \left(\int\limits_{B(x_n,R_0)} (v_n - v_0)^2 \, dx - \int\limits_{B(x_n,R_0)\cap\{V<c\}} (v_n - v_0)^2 \, dx \right) \\ &\to \infty, \end{aligned}$$

which a contradiction. Thus, $v_n \to v_0$ in $L^r(\mathbb{R}^N)$ for all $2 \leq r < 2^*$. This completes the proof. □

Next, we consider the following eigenvalue problem

$$-\Delta u(x) + \mu V(x) u(x) = \lambda f(x) u(x) \text{ for } x \in \mathbb{R}^N. \tag{3.1}$$

We can approach this problem by a direct method and attempt to obtain nontrivial solutions of problem (3.1) as relative minima of the functional

$$I_\mu(u) = \frac{1}{2} \int\limits_{\mathbb{R}^N} |\nabla u|^2 + \mu V u^2 dx,$$

on the unit sphere in $\mathbb{B} = \left\{u \in X : \int_{\mathbb{R}^N} f u^2 dx = 1\right\}$. Equivalently, we may seek to minimize a quotient as follows

$$\widetilde{\lambda}_{1,\mu}(f) = \inf_{u \in X\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + \mu V u^2 dx}{\int_{\mathbb{R}^N} f u^2 dx}. \tag{3.2}$$

Then, by (2.1),

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + \mu V u^2 dx}{\int_{\mathbb{R}^N} f u^2 dx} \geq \frac{S^2}{\|f\|_\infty |\{V < c\}|^{\frac{2}{3}}} \text{ for all } \mu \geq \mu_0,$$

this implies that $\widetilde{\lambda}_{1,\mu}(f) \geq \frac{S^2}{\|f\|_\infty |\{V<c\}|^{\frac{2}{3}}} > 0$. Moreover, by condition (V_3),

$$\inf_{u \in X\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + \mu V u^2 dx}{\int_{\mathbb{R}^N} f u^2 dx} \leq \inf_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + \mu V u^2 dx}{\int_{\mathbb{R}^N} f u^2 dx} = \inf_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega f_\Omega u^2 dx},$$

which indicates that $\widetilde{\lambda}_{1,\mu}(f) \leq \lambda_1(f)$ for all $\mu \geq \mu_0$. Then we have the following results.

Lemma 3.2. *For each $\mu \geq \mu_0$ there exists a positive function $\varphi_\mu \in X$ with $\int_{\mathbb{R}^N} f\varphi_\mu^2 dx = 1$ such that*

$$\widetilde{\lambda}_{1,\mu}(f) = \int_{\mathbb{R}^N} \left|\nabla\varphi_\mu\right|^2 + \mu V \varphi_\mu^2 dx < \lambda_1(f_\Omega).$$

Furthermore, $\widetilde{\lambda}_{1,\mu}(f) \to \lambda_1^-(f_\Omega)$ and $\varphi_\mu \to \phi_1$ as $\mu \to \infty$, where ϕ_1 is positive principal eigenfunction of problem (1.4).

Proof. Let $\{u_n\} \subset X$ with $\int_{\mathbb{R}^N} f u_n^2 dx = 1$ be a minimizing sequence of (3.2), that is

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + \mu V u_n^2 dx \to \widetilde{\lambda}_{1,\mu}(f) \text{ as } n \to \infty.$$

Since $\widetilde{\lambda}_{1,\mu}(f) \leq \lambda_1(f_\Omega)$ for all $\mu \geq \mu_0$, there exists $C_0 > 0$ independent of μ such that $\|u_n\|_\mu \leq C_0$. Thus, there exist a subsequence $\{u_n\}$ and $\varphi_\mu \in X$ such that

$$\begin{aligned} &u_n \rightharpoonup \varphi_\mu \text{ in } X_\mu, \\ &u_n \to \varphi_\mu \text{ a.e. in } \mathbb{R}^N, \\ &u_n \to \varphi_\mu \text{ in } L^r_{loc}\left(\mathbb{R}^N\right) \text{ for } 2 \leq r < 2^*. \end{aligned}$$

Moreover, by condition (D_1),

$$\int_{\mathbb{R}^N} f u_n^2 dx \to \int_{\mathbb{R}^N} f\varphi_\mu^2 dx = 1.$$

Now we show that $u_n \to \varphi_\mu$ in X_μ. Suppose on the contrary. Then

$$\int_{\mathbb{R}^N} \left|\nabla\varphi_\mu\right|^2 + \mu V \varphi_\mu^2 dx < \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \mu V u_n^2 dx = \widetilde{\lambda}_{1,\mu}(f),$$

which is impossible. Thus, $u_n \to \varphi_\mu$ in X_μ, which implies that $\int_{\mathbb{R}^N} f\varphi_\mu^2 dx = 1$ and $\int_{\mathbb{R}^N} \left|\nabla\varphi_\mu\right|^2 + \mu V\varphi_\mu^2 dx = \widetilde{\lambda}_{1,\mu}(f)$. Since $\left|\varphi_\mu\right| \in X$ and

$$\widetilde{\lambda}_{1,\mu}(f) = \int_{\mathbb{R}^N} \left|\nabla\varphi_\mu\right|^2 + \mu V\varphi_\mu^2 dx = \int_{\mathbb{R}^N} \left|\nabla\left|\varphi_\mu\right|\right|^2 + \mu V\left|\varphi_\mu\right|^2 dx,$$

by the maximum principle, we may assume that φ_μ is positive eigenfunction of problem (3.1). Moreover, by the Harnack inequality due to Trudinger [35], we must have $\widetilde{\lambda}_{1,\mu}(f) < \lambda_1(f_\Omega)$. Now, by the definition of $\widetilde{\lambda}_{1,\mu}(f)$, there holds $\widetilde{\lambda}_{1,\mu_1}(f) \leq \widetilde{\lambda}_{1,\mu_2}(f)$ for $\mu_1 < \mu_2$. Hence, for any sequence $\mu_n \to \infty$, let $\varphi_n := \varphi_{\mu_n}$ be the minimizer of $\widetilde{\lambda}_{1,\mu_n}(f)$. Then $\int_{\mathbb{R}^N} f\varphi_n^2 dx = 1$ and

$$\widetilde{\lambda}_{1,\mu_n}(f) = \int_{\mathbb{R}^N} |\nabla\varphi_n|^2 + \mu_n V\varphi_n^2 dx < \lambda_1(f_\Omega),$$

that

$$\widetilde{\lambda}_{1,\mu_n}(f) \to d_0 \leq \lambda_1(f_\Omega) \text{ for some } d_0 > 0$$

and

$$\|\varphi_n\| \leq \|\varphi_n\|_{\mu_n} \leq \sqrt{\lambda_1(f_\Omega)}, \text{ for } n \text{ sufficiently large.}$$

Thus, by Theorem 3.1, we may assume that for every $\mu > 0$ there exists $\varphi_0 \in H_0^1(\Omega)$ such that $\varphi_n \rightharpoonup \varphi_0$ in X_μ and $\varphi_n \to \varphi_0$ in $L^r\left(\mathbb{R}^N\right)$ for all $2 \leq r < 2^*$. Then

$$\int_\Omega |\nabla\varphi_0|^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla\varphi_n|^2 + \mu_n V\varphi_n^2 dx = d_0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f\varphi_n^2 dx = \int_\Omega f_\Omega\varphi_0^2 dx = 1.$$

Since $d_0 \leq \lambda_1(f_\Omega)$ and $\lambda_1(f_\Omega)$ is positive principal eigenvalue of problem (1.4). Thus, we must have $\int_\Omega |\nabla\varphi_0|^2 dx = \lambda_1(f_\Omega)$ and $\varphi_0 = \phi_1$ a positive principal eigenfunction of problem (1.4), which completes the proof. □

By Lemma 3.2, for each $0 < \lambda < \lambda_1(f_\Omega)$ there exists $\overline{\mu}_0(\lambda) \geq \mu_0$ with $\overline{\mu}_0(\lambda) \to \infty$ as $\lambda \to \lambda_1^-(f_\Omega)$ such that for every $\mu > \overline{\mu}_0(\lambda)$, there holds $\lambda < \widetilde{\lambda}_{1,\mu}(f) < \lambda_1(f_\Omega)$, which indicates that

$$\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2 dx \geq \frac{\widetilde{\lambda}_{1,\mu}(f) - \lambda}{\widetilde{\lambda}_{1,\mu}(f)}\|u\|_\mu^2 \text{ for all } u \in X_\mu. \tag{3.3}$$

Moreover, it is easy to show that

$$h_u''(1) = -(p-2)\left(\|u\|_\mu^2 - \lambda \int_{\mathbb{R}^N} fu^2 dx\right) - (p-4)\|u\|_{D^{1,2}}^4 < 0$$

for all $4 \le p < 6$ and $u \in \mathbf{N}_{\mu,\lambda}$. Furthermore, we have the following results.

Lemma 3.3. *Suppose that $N = 3, 4 \le p < 6$ and $\Gamma_0 = \infty$ (if $p = 4$). Then for each $a > 0$ and $0 < \lambda < \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^-$ and $\mathbf{N}_{\mu,\lambda}^- = \{t_{\max}(u)u : u \in \Theta_\mu^+(p)\}$ for $\mu > 0$ sufficiently large.*

Proof. By (3.3), $\Lambda_\mu^+ \neq \emptyset$ and $\Lambda_\mu^- \cup \Lambda_\mu^0 = \emptyset$, this implies that the submanifolds $\mathbf{N}_{\mu,\lambda}^+$ and $\mathbf{N}_{\mu,\lambda}^0$ are empty and

$$\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^- = \{t_{\max}(u)u : u \in \Theta_\mu^+(p)\}$$

for $\mu > 0$ sufficiently large. This completes the proof. □

Lemma 3.4. *Suppose that $N = 3$, $p = 4$ and $\Gamma_0 < +\infty$ (if $p = 4$). Then we have the following results.*
(i) For each $0 < a < \Gamma_0$ and $0 < \lambda < \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^-$ and $\mathbf{N}_{\mu,\lambda}^- = \{t_{\max}(u)u : u \in \Theta_\mu^+(p)\}$ for $\mu > 0$ sufficiently large.
(ii) For each $a \ge \Gamma_0$ and $0 < \lambda < \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \emptyset$ for $\mu > 0$ sufficiently large.
(iii) For each $a > \Gamma_0$ and $\lambda \ge \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^+$ and $\mathbf{N}_{\mu,\lambda}^+ = \{t_{\min}(u)u : u \in \Theta_\mu^-(p)\}$ for $\mu > 0$ sufficiently large.
(iv) If Γ_0 is not attained and $a = \Gamma_0$, then for each $\lambda \ge \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^+$ and $\mathbf{N}_{\mu,\lambda}^+ = \{t_{\min}(u)u : u \in \Theta_\mu^-(p)\}$ for $\mu > 0$ sufficiently large.

Proof. (i) The proof is almost the same as Lemma 3.3, and we omit it here.
(ii) Since

$$\int_{\mathbb{R}^3} Q|u|^4 dx \le \Gamma_0 \|u\|_{D^{1,2}}^4 \text{ for all } u \in X,$$

we can obtain

$$\Phi_p(u) = \int_{\mathbb{R}^3} Q|u|^4 dx - a\|u\|_{D^{1,2}}^4 \le 0 \tag{3.4}$$

for all $a \ge \Gamma_0$ and $u \in X$, this implies that $\Theta_\mu^+(p) = \emptyset$. Moreover, $\Lambda_\mu^- \cup \Lambda_\mu^0 = \emptyset$ for $\mu > 0$ sufficiently large, by Lemma 2.2, $\mathbf{N}_{\mu,\lambda} = \emptyset$ for $\mu > 0$ sufficiently large.
(iii) By Lemma 3.2, there exists a positive function $\varphi_\mu \in X$ such that $\int_{\mathbb{R}^3} f\varphi_\mu^2 dx = 1$ and

$$\widetilde{\lambda}_{1,\mu}(f) = \int_{\mathbb{R}^3} |\nabla\varphi_\mu|^2 + \mu V\varphi_\mu^2 dx < \lambda_1(f_\Omega) \text{ for } \mu > 0 \text{ sufficiently large.}$$

If $\lambda \geq \lambda_1(f_\Omega)$, then

$$\int_{\mathbb{R}^3} |\nabla\varphi_\mu|^2 + \mu V\varphi_\mu^2 dx - \lambda \int_{\mathbb{R}^3} f\varphi_\mu^2 dx = \widetilde{\lambda}_{1,\mu}(f) - \lambda < 0, \tag{3.5}$$

and so $\varphi_\mu \in \Lambda_\mu^-$ for $\mu > 0$ sufficiently large. Moreover, $\Phi_p(u) < 0$ for all $a > \Gamma_0$ and $u \in X \setminus \{0\}$, this implies that $\Lambda_\mu^- \cap \Theta_\mu^-(p) \neq \emptyset$ and $\Theta_\mu^+(p) \cup \Theta_\mu^0(p) = \emptyset$. Then by Lemma 2.2,

$$\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^+ = \{t_{\min}(u)u : u \in \Theta_\mu^-(p)\}$$

for $\mu > 0$ sufficiently large.
(iv) The proof is essentially same as that in part (iii), so we omit it here. □

If $\lambda \geq \lambda_1(f_\Omega)$, then by (3.5), $\varphi_\mu \in \Lambda_\mu^-$ for $\mu > 0$ sufficiently large. Moreover, if $\Phi_p(\phi_1) < 0$, then by Lemma 3.2, $\Phi_p(\varphi_\mu) < 0$ for $\mu > 0$ sufficiently large, this implies that $\varphi_\mu \in \Lambda_\mu^- \cap \Theta_\mu^-(p)$ and so $\mathbf{N}_{\mu,\lambda}^+ \neq \emptyset$ for $\mu > 0$ sufficiently large. Thus, as well shall see, $\mathbf{N}_{\mu,\lambda}$ may consist of two distinct components in this case which makes it possible to prove the existence of at least two positive solutions by showing that $J_{\mu,\lambda}$ has an appropriate minimizer on each component.

Moreover, if $\lambda \geq \lambda_1(f_\Omega)$, then roughly speaking $\|u\|_\mu^2 \leq \lambda \int_{\mathbb{R}^3} fu^2 dx$ if and only if u is almost a multiple of ϕ_1 for μ sufficiently large. Thus, if $\phi_1 \in \Theta_\mu^-(p)$, it should follow that $\overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p) = \emptyset$ for $\mu > 0$ sufficiently large. This is made precise in the following lemma and we show subsequently that $\overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p) = \emptyset$ is an important condition for establishing the existence of minimizers.

Theorem 3.5. *Suppose that $N = 3, 4 \leq p < 6$ and $\Phi_p(\phi_1) < 0$. Then for each $a > 0$ there exists $\delta_0 > 0$ such that for every $\lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_0$, there holds $\overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p) = \emptyset$ for $\mu > 0$ sufficiently large.*

Proof. Suppose that the result is false. Then there exist sequences $\{\mu_n\}$, $\{\lambda_n\}$ and $\{u_n\} \subset X$ with $\lambda_n \to \lambda_1^+(f_\Omega)$ and $\mu_n \to \infty$ such that $\|u_n\|_{\mu_n} = 1$ and

$$\|u_n\|_{\mu_n}^2 - \lambda_n \int_{\mathbb{R}^3} fu_n^2 dx \leq 0, \quad \Phi_p(u_n) \geq 0. \tag{3.6}$$

By Theorem 3.1, we may assume that for every $\mu > 0$ there exists $u_0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ in X_μ and $u_n \to u_0$ in $L^r(\mathbb{R}^3)$ for all $2 \leq r < 6$. This implies that

$$\lim_{n\to\infty} \lambda_n \int_{\mathbb{R}^3} fu_n^2 dx = \lambda_1(f_\Omega) \int_{\mathbb{R}^3} fu_0^2 dx \geq 1 \tag{3.7}$$

Now, we show that $\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_\Omega |\nabla u_0|^2 dx$. Suppose on the contrary. Then by (3.6) and (3.7),

$$\int_{\Omega} \left(|\nabla u_0|^2 - \lambda_1 (f_\Omega) f_\Omega u_0^2 \right) dx = \int_{\mathbb{R}^3} \left(|\nabla u_0|^2 - \lambda_1 (f_\Omega) f u_0^2 \right) dx$$

$$< \liminf_{n \to \infty} \left(\|u_n\|_{\mu_n}^2 - \lambda_n \int_{\mathbb{R}^3} f u_n^2 dx \right) \leq 0,$$

which is impossible. Hence $\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_\Omega |\nabla u_0|^2 dx$. And we get

$$\lim_{n\to\infty} \Phi_p(u_n) = \Phi_p(u_0).$$

It follows that

$$(i) \int_\Omega (|\nabla u_0|^2 - \lambda_1(f_\Omega) f_\Omega u_0^2) dx \leq 0, \quad (ii) \Phi_p(u_0) \geq 0.$$

But (i) implies that $u_0 = k\phi_1$ for some k and then (ii) implies that $k = 0$ which is impossible as $\lambda_1(f_\Omega) \int_{\mathbb{R}^3} f u_0^2 dx \geq 1$. Thus, there exists $\delta_0 > 0$ and $\widehat{\mu}_0 \geq \mu_0$ such that $\overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p) = \emptyset$ for all $\lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_0$ and $\mu > \widehat{\mu}_0$. Moreover, if $\mathbf{N}_{\mu,\lambda}^0 \neq \emptyset$, then there exists $u_0 \in \mathbf{N}_{\mu,\lambda}^0$ such that $\frac{u_0}{\|u_0\|_\mu} \in \Lambda_\mu^0 \cap \Theta_\mu^0(p) \subset \overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p) = \emptyset$ which is impossible. Therefore, $\mathbf{N}_{\mu,\lambda}^0 = \emptyset$ for all $\lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_0$ and for $\mu > 0$ sufficiently large. This completes the proof. □

Lemma 3.6. *Suppose that $N = 3, 4 < p < 6$ and $\int_\Omega Q\phi_1^p dx < 0$. Let $\delta_0 > 0$ be as in Theorem 3.5. Then for each $a > 0$ and $\lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_0$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^- \cup \mathbf{N}_{\mu,\lambda}^+$ for $\mu > 0$ sufficiently large. Furthermore, $\mathbf{N}_{\mu,\lambda}^\pm$ are nonempty sets for $\mu > 0$ sufficiently large.*

Proof. Since $\int_\Omega Q\phi_1^p dx < 0$, by Lemma 3.2, there exists a positive function $\varphi_\mu \in X$ such that

$$\widetilde{\lambda}_{1,\mu}(f) = \int_{\mathbb{R}^3} |\nabla \varphi_\mu|^2 + \mu V \varphi_\mu^2 dx < \lambda_1(f_\Omega)$$

and

$$\varphi_\mu \to \phi_1 \text{ as } \mu \to \infty.$$

Hence, for $\mu > 0$ large enough,

$$\int_{\mathbb{R}^3} Q|\varphi_\mu|^p dx < 0,$$

this implies that $\varphi_\mu \in \Theta_\mu^-(p)$ for $\mu > 0$ sufficiently large. Moreover, $\|\varphi_\mu\|_\mu^2 < \lambda \int_{\mathbb{R}^3} f\varphi_\mu^2 dx$ for all $\lambda \geq \lambda_1(f_\Omega)$, we have $\varphi_\mu \in \Lambda_\mu^- \cap \Theta_\mu^-(p)$ for $\mu > 0$ sufficiently large. Next, by conditions (D_1) and (D_2), we have $\Lambda_\mu^+ \cap \Theta_\mu^+(p) \neq \emptyset$. Thus, by Lemma 2.1 $(i), (ii)$, $\mathbf{N}_{\mu,\lambda}^\pm \neq \emptyset$. Moreover,

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by Theorem 3.5, $\overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p) = \emptyset$ for $\mu > 0$ sufficiently large. Then by Lemma 2.1 (iii) this implies that $\mathbf{N}_{\mu,\lambda}^0 = \emptyset$ or $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^- \cup \mathbf{N}_{\mu,\lambda}^+$ for $\mu > 0$ sufficiently large. This completes the proof. □

Lemma 3.7. *Suppose that $N = 3$, $p = 4$ and $\Phi_p(\phi_1) < 0$. Then there exists $\delta_0 > 0$ such that for every $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \delta_0$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^- \cup \mathbf{N}_{\mu,\lambda}^+$ for $\mu > 0$ sufficiently large. Furthermore, $\mathbf{N}_{\mu,\lambda}^\pm$ are nonempty sets for $\mu > 0$ sufficiently large.*

Proof. The proof is almost the same as Lemma 3.6, and we omit it here. □

Next, we consider the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u = \lambda f_\Omega u + Q_\Omega |u|^{p-2} u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{$E_{\lambda,\Omega}$}$$

where $N \ge 3$, $2 < p < \min\{4, 2^*\}$, $0 \le \lambda < \lambda_1(f_\Omega)$ and Q_Ω is a restriction of Q on $\overline{\Omega}$.

It is well-known that for each $0 \le \lambda < \lambda_1(f_\Omega)$, Equation $(E_{\lambda,\Omega})$ has positive ground state solution $w_{\lambda,\Omega}$ such that

$$\inf_{u \in \mathbf{N}_{\lambda,\Omega}^\infty} J_{\lambda,\Omega}^\infty(u) = J_{\lambda,\Omega}^\infty(w_{\lambda,\Omega}) = \alpha_{\lambda,Q}^\infty(\Omega), \tag{3.8}$$

and

$$\int_\Omega |\nabla w_{\lambda,\Omega}|^2 dx - \lambda \int_\Omega f_\Omega w_{\lambda,\Omega}^2 dx = \int_\Omega Q_\Omega w_{\lambda,\Omega}^p dx = \frac{2p}{p-2} \alpha_{\lambda,Q}^\infty(\Omega) > 0 \tag{3.9}$$

where $J_{\lambda,\Omega}^\infty$ is the energy functional of Equation $(E_{\lambda,\Omega})$ in $H_0^1(\Omega)$ in the form

$$J_{\lambda,\Omega}^\infty(u) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega f_\Omega u^2 dx\right) - \frac{1}{p}\int_\Omega Q_\Omega |u|^p dx,$$

and

$$\mathbf{N}_{\lambda,\Omega}^\infty = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \left\langle \left(J_{\lambda,\Omega}^\infty\right)'(u), u \right\rangle = 0\}.$$

Clearly $\alpha_{\lambda,Q}^\infty(\Omega) \le \alpha_{0,Q}^\infty(\Omega) \le \frac{p-2}{2p}\left(\frac{S_p^p(\Omega)}{Q_{\Omega,\min}}\right)^{2/(p-2)}$ for all $0 < \lambda < \lambda_1(f_\Omega)$. Then we have the following nonemptiness and properties of submanifolds $\mathbf{N}_{\mu,\lambda}^+$ and $\mathbf{N}_{\mu,\lambda}^-$.

Theorem 3.8. *Suppose that $N \ge 3$ and $2 < p < \min\{4, 2^*\}$. Then we have the following results.*
(i) Let $0 < \lambda < \lambda_1(f_\Omega)$ and let $w_{\lambda,\Omega}$ be the ground state positive solution of Equation $(E_{\lambda,\Omega})$. If conditions $(V_1), (V_3), (D_1), (D_2)$ and (D_4) hold, then there exists $a_0 > 0$ independent of λ, μ such that for every $0 < a < a_0$, there exist two positive constants t_a^- and t_a^+ satisfying

$$1 < t_a^- < \left(\frac{2}{4-p}\right)^{1/(p-2)} < t_a^+$$

and

$$t_a^- \to 1;\, t_a^+ \to \infty \text{ as } a \to 0^+$$

such that $t_a^\pm w_{\lambda,\Omega} \in \mathbf{N}_{\mu,\lambda}^\pm$. *Furthermore, if* $0 < \lambda < \left[1 - 2\left(\frac{4-p}{4}\right)^{2/p}\right]\lambda_1(f_\Omega)$, *then*

$$J_{\mu,\lambda}\left(t_a^- w_\Omega\right) < \frac{p-2}{4p}\left(\frac{\widetilde{\lambda}_{1,\mu}(f) - \lambda}{\widetilde{\lambda}_{1,\mu}(f)}\right)^{p/(p-2)}\left(\frac{2S_p^p(\Omega)}{Q_{\Omega,\min}(4-p)}\right)^{2/(p-2)}$$

for $\mu > 0$ *sufficiently large.*
(ii) *Let* $\lambda \geq \lambda_1(f_\Omega)$ *and let* ϕ_1 *be positive principal eigenfunction of* $-\Delta$ *in* $H_0^1(\Omega)$ *with weight function* $f_\Omega := f|_{\overline{\Omega}}$. *Then for each* $a > 0$ *there exists* $t_a^+ > 0$ *such that* $t_a^+\phi_1 \in \mathbf{N}_{\mu,\lambda}^+$ *and*

$$J_{\mu,\lambda}\left(t_a^+\phi_1\right) = \inf_{t\geq 0} J_{\mu,\lambda}(t\phi_1) < 0.$$

In particular, $\mathbf{N}_{\mu,\lambda}^+$ *is nonempty and* $\inf_{u\in\mathbf{N}_{\mu,\lambda}^+} J_{\mu,\lambda}(u) < J_{\mu,\lambda}\left(t_a^+\phi_1\right)$.

Proof. (i) Define

$$m(t) = t^{-2}\left(\|w_{\lambda,\Omega}\|_\mu^2 - \lambda\int_\Omega f_\Omega w_{\lambda,\Omega}^2 dx\right) - t^{p-4}\int_\Omega Q_\Omega |w_{\lambda,\Omega}|^p dx \text{ for } t > 0.$$

Clearly, $tw_{\lambda,\Omega} \in \mathbf{N}_{\mu,\lambda}$ if and only if $m(t) + a\left(\int_\Omega |\nabla w_{\lambda,\Omega}|^2 dx\right)^2 = 0$. Since $\|w_{\lambda,\Omega}\|_\mu^2 - \lambda\int_\Omega f_\Omega w_{\lambda,\Omega}^2 dx > 0$, similar to the arguments of [32, Lemmas 2.4 and 2.5], there exist two positive constants t_a^- and t_a^+ satisfying

$$1 < t_a^- < \left(\frac{2}{4-p}\right)^{1/(p-2)} < t_a^+$$

and

$$t_a^- \to 1;\, t_a^+ \to \infty \text{ as } a \to 0^+$$

such that $t_a^\pm w_\Omega \in \mathbf{N}_{\mu,\lambda}^\pm$,

$$J_{\mu,\lambda}\left(t_a^- w_{\lambda,\Omega}\right) = \sup_{0\leq t\leq t_a^+} J_{\mu,\lambda}\left(tw_{\lambda,\Omega}\right)$$

and

$$J_{\mu,\lambda}\left(t_a^+ w_{\lambda,\Omega}\right) = \inf_{t\geq t_a^-} J_{\mu,\lambda}\left(t w_{\lambda,\Omega}\right) = \inf_{t\geq 0} J_{\mu,\lambda}\left(t w_{\lambda,\Omega}\right) < 0. \tag{3.10}$$

Furthermore, we get

$$\begin{aligned} J_{\mu,\lambda}\left(t_a^- w_{\lambda,\Omega}\right) &= \frac{\left(t_a^-\right)^2}{4}\left(\|w_{\lambda,\Omega}\|_\mu^2 - \lambda\int_{\mathbb{R}^N} f w_{\lambda,\Omega}^2 dx\right) - \frac{(4-p)\left(t_a^-\right)^p}{4p}\int_{\mathbb{R}^N} Q\left|w_{\lambda,\Omega}\right|^p dx \\ &= \frac{p}{2(p-2)}\left[\left(t_a^-\right)^2 - \frac{(4-p)\left(t_a^-\right)^p}{p}\right]\alpha_{\lambda,Q}^{\infty}(\Omega) \\ &< \frac{1}{4}\left[\left(t_a^-\right)^2 - \frac{(4-p)\left(t_a^-\right)^p}{p}\right]\left(\frac{S_p^p(\Omega)}{Q_{\Omega,\min}}\right)^{2/(p-2)} \\ &\to \frac{p-2}{2p}\left(\frac{S_p^p(\Omega)}{Q_{\Omega,\min}}\right)^{2/(p-2)} \quad \text{as } a\to 0^+. \end{aligned} \tag{3.11}$$

Since $0<\lambda<\left[1-2\left(\frac{4-p}{4}\right)^{2/p}\right]\lambda_1\left(f_\Omega\right)$, by Lemma 3.2 and (3.11), we can conclude that there exists $a_0>0$ independent of λ,μ such that for every $0<a<a_0$,

$$J_{\mu,\lambda}\left(t_a^- w_\lambda\right) < \frac{p-2}{4p}\left(\frac{\widetilde{\lambda}_{1,\mu}(f)-\lambda}{\widetilde{\lambda}_{1,\mu}(f)}\right)^{p/(p-2)}\left(\frac{2S_p^p(\Omega)}{Q_{\Omega,\min}(4-p)}\right)^{2/(p-2)}$$

for $\mu>0$ sufficiently large.

(ii) Since $\lambda\geq\lambda_1\left(f_\Omega\right)$, we have $\|\phi_1\|_\mu^2-\lambda\int_\Omega f_\Omega\phi_1^2dx\leq 0$, this implies that $\frac{\phi_1}{\|\phi_1\|_\mu}\in\overline{\Lambda_\mu^-}$. Then by Lemma 2.3 (i), for each $a>0$ there exists $t_a^+>0$ such that $t_a^+\phi_1\in\mathbf{N}_{\mu,\lambda}^+$. Moreover, $h_{\phi_1}'(t)<0$ for all $t\in\left(0,t_a^+\right)$ and $h_{\phi_1}'(t)>0$ for all $t>t_a^+$, which leads to

$$J_{\mu,\lambda}\left(t_a^+\phi_1\right) = \inf_{t\geq 0} J_{\mu,\lambda}(t\phi_1) < 0.$$

This completes the proof. □

4. The case when $N=3$ and $4<p<6$

4.1. The subcase: $\lambda<\lambda_1\left(f_\Omega\right)$

We need the following results.

Lemma 4.1. *Suppose that $4<p<6$. Then for each $0<\lambda<\lambda_1\left(f_\Omega\right)$ there exists $\overline{\mu}_0(\lambda)\geq\mu_0$ with $\overline{\mu}_0(\lambda)\to\infty$ as $\lambda\to\lambda_1^-\left(f_\Omega\right)$ such that for every $\mu>\overline{\mu}_0(\lambda)$, the energy functional $J_{\mu,\lambda}$ is coercive and bounded below on $\mathbf{N}_{\mu,\lambda}^-$. Furthermore,*

$$\inf_{u\in \mathbf{N}^{-}_{\mu,\lambda}} J_{\mu,\lambda}(u) \geq \frac{1}{4}\left(\frac{\widetilde{\lambda}_{1,\mu}(f)-\lambda}{\widetilde{\lambda}_{1,\mu}(f)}\right)^{p/(p-2)}\left(\frac{S^p}{\|Q\|_\infty |\{V<c\}|^{\frac{6-p}{6}}}\right)^{2/(p-2)} > 0 \tag{4.1}$$

for all $u \in \mathbf{N}^{-}_{\mu,\lambda}$.

Proof. By (2.1) and (3.3), for each $\mu > \overline{\mu}_0(\lambda)$ and $u \in \mathbf{N}^{-}_{\mu,\lambda}$, we obtain

$$\begin{aligned}\frac{\widetilde{\lambda}_{1,\mu}(f)-\lambda}{\widetilde{\lambda}_{1,\mu}(f)}\|u\|_\mu^2 &\leq \|u\|_\mu^2 - \lambda\int_{\mathbb{R}^3} f u^2 dx + a\|u\|^4_{D^{1,2}}\\ &= \int_{\mathbb{R}^3} Q|u|^p dx \leq \|Q\|_\infty |\{V<c\}|^{\frac{6-p}{6}} S^{-p}\|u\|_\mu^p,\end{aligned}$$

which indicates that

$$\|u\|_\mu \geq \left(\frac{S^p\left(\widetilde{\lambda}_{1,\mu}(f)-\lambda\right)}{\widetilde{\lambda}_{1,\mu}(f)\|Q\|_\infty |\{V<c\}|^{\frac{6-p}{6}}}\right)^{1/(p-2)}.$$

Thus,

$$\begin{aligned}J_{\mu,\lambda}(u) &\geq \frac{1}{4}\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^3} f u^2 dx\right) \geq \frac{\widetilde{\lambda}_{1,\mu}(f)-\lambda}{4\widetilde{\lambda}_{1,\mu}(f)}\|u\|_\mu^2\\ &\geq \frac{1}{4}\left(\frac{\widetilde{\lambda}_{1,\mu}(f)-\lambda}{\widetilde{\lambda}_{1,\mu}(f)}\right)^{p/(p-2)}\left(\frac{S^p}{\|Q\|_\infty |\{V<c\}|^{\frac{6-p}{6}}}\right)^{2/(p-2)} > 0,\end{aligned}$$

this implies that the energy functional $J_{\mu,\lambda}$ is coercive and bounded below on $\mathbf{N}^{-}_{\mu,\lambda}$. This completes the proof. □

We now show that there exists a minimizer on $\mathbf{N}^{-}_{\mu,\lambda}$ which is a critical point of $J_{\mu,\lambda}(u)$ and so a nontrivial solution of Equation $\left(E_{\mu,\lambda}\right)$. First, we define

$$\theta_{a,\lambda}(\Omega) = \inf_{u\in \mathbf{M}_\lambda(\Omega)} J_{\mu,\lambda}|_{H_0^1(\Omega))}(u),$$

where

$$\mathbf{M}_\lambda(\Omega) = \{u \in H_0^1(\Omega)\cap H^1(\mathbb{R}^3) : \left\langle J'_{\mu,\lambda}|_{H_0^1(\Omega)}(u), u\right\rangle = 0\}.$$

Note that

$$J_{\mu,\lambda}|_{H_0^1(\Omega)}(u) = \frac{a}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega Q|u|^p dx - \frac{\lambda}{2}\int_\Omega f_\Omega u^2 dx,$$

a restriction of $J_{\mu,\lambda}$ on $H_0^1(\Omega)$, and $\theta_{a,\lambda}(\Omega)$ independent of μ. Since $\max\{Q,0\} \not\equiv 0$ in Ω, $\mathbf{M}_\lambda(\Omega) \neq \emptyset$. Thus, similar to the argument of (4.1), we can conclude that $J_{\mu,\lambda}|_{H_0^1(\Omega)}$ is bounded below on $\mathbf{M}_\lambda(\Omega)$. Moreover, $H_0^1(\Omega) \subset X_\mu$ for all $\mu > 0$, one can see that

$$0 < \eta \le \inf_{u\in\mathbf{N}_{\mu,\lambda}^-} J_{\mu,\lambda}(u) \le \theta_{a,\lambda}(\Omega) \text{ for all } \mu \ge \overline{\mu}_0(\lambda). \tag{4.2}$$

Taking $D_{a,\lambda} > \theta_{a,\lambda}(\Omega)$. Then we have

$$\inf_{u\in\mathbf{N}_{\mu,\lambda}^-} J_{\mu,\lambda}(u) \le \theta_{a,\lambda}(\Omega) < D_{a,\lambda} \text{ for all } \mu \ge \overline{\mu}_0(\lambda). \tag{4.3}$$

We are now ready to prove Theorem 1.1: When $0 < \lambda < \lambda_1(f_\Omega)$. By Lemma 4.1, (4.3) and the Ekeland variational principle [17], for each $\mu > \overline{\mu}_0(\lambda)$ there exists a minimizing sequence $\{u_n\} \subset \mathbf{N}_{\mu,\lambda}^-$ such that

$$D_{a,\lambda} > \lim_{n\to\infty} J_{\mu,\lambda}(u_n) = \inf_{u\in\mathbf{N}_{\mu,\lambda}^-} J_{\mu,\lambda}(u) > 0 \text{ and } J'_{\mu,\lambda}(u_n) = o(1).$$

Since $\inf_{u\in\mathbf{N}_{\mu,\lambda}^-} J_{\mu,\lambda}(u) < D_{a,\lambda}$, again using Lemma 4.1, there exists $c_{a,\lambda} > 0$ such that $\|u_n\|_\mu \le c_{a,\lambda}$. By Proposition 2.5, there exist a subsequence $\{u_n\}$ and $u_0 \in X$ such that $J'_{\mu,\lambda}(u_0) = 0$ and $u_n \to u_0$ strongly in X_μ for $\mu > 0$ sufficiently large, which implies that $J_{\mu,\lambda}$ has minimizer u_0 in $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^-$ for μ sufficiently large. Since $J_{\mu,\lambda}(u_0) = J_{\mu,\lambda}(|u_0|)$, by Lemma 2.4, we may assume that u_0 is a positive solution of Equation $(E_{\mu,\lambda})$ such that $J_{\mu,\lambda}(u_0) = \inf_{u\in\mathbf{N}_{\mu,\lambda}^-} J_{\mu,\lambda}(u) > 0$.

4.2. *The subcase: $\lambda \ge \lambda_1(f_\Omega)$*

By Theorem 3.5, for each $a > 0$ there exists $\delta_0 > 0$ such that for every $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \delta_0$, there holds $\overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p) = \emptyset$ for $\mu > 0$ sufficiently large, it is possible to obtain more information about the nature of the Nehari manifold as follows.

Lemma 4.2. *Suppose that $4 < p < 6$ and $\int_\Omega Q\phi_1^p dx < 0$. Let $\delta_0 > 0$ be as in Theorem 3.5. Then for every $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \delta_0$, we have the following results.*
(i) There exists $c_0 > 0$ such that $\|u\|_\mu \ge c_0$ for all $u \in \mathbf{N}_{\mu,\lambda}^-$ and $\mu > 0$ sufficiently large.
(ii) $\mathbf{N}_{\mu,\lambda}^-$ and $\mathbf{N}_{\mu,\lambda}^+$ are separated for $\mu > 0$ sufficiently large, i.e., $\overline{\mathbf{N}_{\mu,\lambda}^-} \cap \overline{\mathbf{N}_{\mu,\lambda}^+} = \emptyset$ for $\mu > 0$ sufficiently large.
(iii) $\mathbf{N}_{\mu,\lambda}^+$ is uniform bounded for $\mu > 0$ sufficiently large.

Proof. (i) Suppose on the contrary. Then there exist $\{\mu_n\} \subset \mathbb{R}^+$ and $u_n \in \mathbf{N}_{\mu_n,\lambda}^-$ such that $\mu_n \to \infty$ and $\|u_n\|_{\mu_n} \to 0$. Hence, by (2.1),

$$\begin{aligned} 0 &< \|u_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f u_n^2 dx \\ &< \|u_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f u_n^2 dx + a\|u_n\|_{D^{1,2}}^4 = \int_{\mathbb{R}^3} Q|u_n|^p dx \to 0 \text{ as } n\to\infty. \end{aligned} \tag{4.4}$$

Let $v_n = \frac{u_n}{\|u_n\|_{\mu_n}}$. Then, by Theorem 3.1, for every $\mu > 0$ there exist subsequence $\{v_n\}$ and $v_0 \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v_0 \text{ in } X_\mu;\ v_n \to v_0 \text{ in } L^r\left(\mathbb{R}^3\right) \text{ for all } 2 \le r < 6. \tag{4.5}$$

Thus, by (4.4) and (4.5),

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f v_n^2 dx = \int_{\mathbb{R}^3} f v_0^2 dx \tag{4.6}$$

and

$$\|v_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f v_n^2 dx \to 0 \text{ as } n \to \infty. \tag{4.7}$$

Moreover, by (4.6), (4.7), $v_0 \in H_0^1(\Omega)$ and Fatou's Lemma, we can obtain that

$$0 = \lim_{n\to\infty} \left(\|v_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f v_n^2 dx \right) = 1 - \lambda \int_{\mathbb{R}^3} f v_0^2 dx,$$

and for every $\mu > 0$

$$\|v_0\|_\mu^2 - \lambda \int_{\mathbb{R}^3} f v_0^2 dx \le \liminf_{n\to\infty} \left(\|v_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f v_n^2 dx \right) = 0,$$

this implies that $v_0 \ne 0$ and $\frac{v_0}{\|v_0\|_\mu} \in \overline{\Lambda_\mu^-}$ for all $\mu > 0$. Since $\int_{\mathbb{R}^3} Q|v_n|^p dx > 0$ and $v_n \to v_0$ in $L^p\left(\mathbb{R}^3\right)$, it follows that $\int_{\mathbb{R}^3} Q|v_0|^p dx \ge 0$ and so $\frac{v_0}{\|v_0\|_\mu} \in \overline{\Theta_\mu^+}(p)$ for all $\mu > 0$. Hence, $\frac{v_0}{\|v_0\|_\mu} \in \overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p)$ for all $\mu > 0$, which a contradiction. Thus, $0 \notin \overline{\mathbf{N}_{\mu,\lambda}^-}$ for $\mu > 0$ sufficiently large. Moreover, by Lemma 3.6, $\overline{\mathbf{N}_{\mu,\lambda}^-} \subset \mathbf{N}_{\mu,\lambda}^- \cup \{0\}$. Since $0 \notin \overline{\mathbf{N}_{\mu,\lambda}^-}$, it follows that $\overline{\mathbf{N}_{\mu,\lambda}^-} = \mathbf{N}_{\mu,\lambda}^-$, i.e., $\mathbf{N}_{\mu,\lambda}^-$ is closed.
(ii) By Theorem 3.5 and part (i),

$$\begin{aligned}\overline{\mathbf{N}_{\mu,\lambda}^-} \cap \overline{\mathbf{N}_{\mu,\lambda}^+} &\subseteq \mathbf{N}_{\mu,\lambda}^- \cap (\mathbf{N}_{\mu,\lambda}^+ \cup \mathbf{N}_{\mu,\lambda}^0) = \mathbf{N}_{\mu,\lambda}^- \cap (\mathbf{N}_{\mu,\lambda}^+ \cup \emptyset) \\ &= (\mathbf{N}_{\mu,\lambda}^- \cap \mathbf{N}_{\mu,\lambda}^+) \cup (\mathbf{N}_{\mu,\lambda}^- \cap \emptyset) = \emptyset,\end{aligned}$$

and so $\mathbf{N}_{\mu,\lambda}^-$ and $\mathbf{N}_{\mu,\lambda}^+$ are separated for $\mu > 0$ sufficiently large.
(iii) Suppose on the contrary. Then there exist sequences $\{\mu_n\} \subset \mathbb{R}^+$ and $u_n \in \mathbf{N}_{\mu_n,\lambda}^+$ such that $\mu_n \to \infty$ and $\|u_n\|_{\mu_n} \to \infty$ as $n \to \infty$. Clearly,

$$\|u_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f u_n^2 dx + a \|u_n\|_{D^{1,2}}^4 = \int_{\mathbb{R}^3} Q|u_n|^p dx < 0. \tag{4.8}$$

Let $v_n = \frac{u_n}{\|u_n\|_{\mu_n}}$. Then by Theorem 3.1, we may assume that for every $\mu > 0$ there exists $v_0 \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v_0 \text{ in } X_\mu; \, v_n \to v_0 \text{ in } L^r\left(\mathbb{R}^3\right) \text{ for all } 2 \le r < 6.$$

Thus,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f v_n^2 dx = \int_{\mathbb{R}^3} f v_0^2 dx \tag{4.9}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} Q\,|v_n|^p\,dx = \int_{\mathbb{R}^3} Q\,|v_0|^p\,dx. \tag{4.10}$$

Moreover, by Fatou's Lemma,

$$\int_{\mathbb{R}^3} |\nabla v_0|^2 dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 dx. \tag{4.11}$$

Dividing (4.8) by $\|u_n\|_{\mu_n}^2$ gives

$$\|v_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f v_n^2 dx + a\|u_n\|_{\mu_n}^2 \|v_n\|_{D^{1,2}}^4 = \|u_n\|_{\mu_n}^{p-2} \int_{\mathbb{R}^3} Q|v_n|^p dx < 0. \tag{4.12}$$

Since

$$\lim_{n\to\infty} \left(\|v_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f v_n^2 dx \right) = 1 - \lambda \lim_{n\to\infty} \int_{\mathbb{R}^3} f v_n^2 dx = 1 - \lambda \int_{\mathbb{R}^3} f v_0^2 dx$$

and $\|u_n\|_{\mu_n} \to \infty$, it obtains that $\int_{\mathbb{R}^3} Q|v_0|^p dx = 0$ and $\int_{\mathbb{R}^3} f v_0^2 dx > 0$ from the conclusions (4.10) and (4.12). Moreover, by $v_0 \in H_0^1(\Omega)$, (4.9) and (4.11), for every $\mu > 0$,

$$\|v_0\|_\mu^2 - \lambda \int_{\mathbb{R}^3} f v_0^2 dx \le \liminf_{n\to\infty} \left(\|v_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f v_n^2 dx \right) \le 0.$$

We now show that $v_n \to v_0$ in X_μ. Suppose on the contrary. Then

$$\begin{aligned}\|v_0\|_\mu^2 - \lambda \int\limits_{\mathbb{R}^3} f v_0^2 dx &= \int\limits_{\mathbb{R}^3} |\nabla v_0|^2 dx - \lambda \int\limits_{\mathbb{R}^3} f v_0^2 dx \\ &< \liminf_{n\to\infty} \left(\|v_n\|_{\mu_n}^2 - \lambda \int\limits_{\mathbb{R}^3} f v_n^2 dx \right) \leq 0,\end{aligned}$$

since $\int_{\mathbb{R}^3} V v_0^2 dx = 0$. Hence $\frac{v_0}{\|v_0\|_\mu} \in \overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p)$ which is impossible. Since $v_n \to v_0$ in X_μ, then $\|v_0\|_\mu = 1$. Hence $v_0 \in \Theta_\mu^0(p)$ and so $v_0 \in \overline{\Theta_\mu^+}(p)$. Moreover,

$$\|v_0\|_\mu^2 - \lambda \int\limits_{\mathbb{R}^3} f v_0^2 dx = \lim_{n\to\infty} \left(\|v_n\|_\mu^2 - \lambda \int\limits_{\mathbb{R}^3} f v_n^2 dx \right) \leq 0,$$

and so $v_0 \in \overline{\Lambda_\mu^-}$. Thus, $v_0 \in \overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p)$ which is impossible. Therefore, we can conclude that $\mathbf{N}_{\mu,\lambda}^+$ is uniform bounded for $\mu > 0$ sufficiently large. This completes the proof. □

When $\mathbf{N}_{\mu,\lambda}^+$ and $\mathbf{N}_{\mu,\lambda}^-$ are separated and $\mathbf{N}_{\mu,\lambda}^0 = \emptyset$, any non-zero minimizer for $J_{\mu,\lambda}$ on $\mathbf{N}_{\mu,\lambda}^+$ (or on $\mathbf{N}_{\mu,\lambda}^-$) is also a local minimizer on $\mathbf{N}_{\mu,\lambda}$ and so will be a critical point for $J_{\mu,\lambda}$ on $\mathbf{N}_{\mu,\lambda}$ and a nontrivial solution of Equation $\left(E_{\mu,\lambda}\right)$. Since $\int_{\mathbb{R}^3} Q\phi_1^p dx < 0$, we can obtain that $\Lambda_\mu^- \cap \Theta_\mu^-(p) \neq \emptyset$ for all $\mu > 0$. Furthermore, we have the following result.

Theorem 4.3. *Suppose that $4 < p < 6$ and $\int_\Omega Q\phi_1^p dx < 0$. Then for each $\lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_0$, there exists a minimizer of $J_{\mu,\lambda}(u)$ on $\mathbf{N}_{\mu,\lambda}^+$ for $\mu > 0$ sufficiently large.*

Proof. By Lemmas 3.6 and 4.2, $\mathbf{N}_{\mu,\lambda}^+ \neq \emptyset$ and $\mathbf{N}_{\mu,\lambda}^+$ is uniformly bounded for $\mu > 0$ sufficiently large. Then there exists $C_{a,\lambda} > 0$ such that $\|u\|_\mu \leq C_{a,\lambda}$ for all $u \in \mathbf{N}_{\mu,\lambda}^+$. Hence, making use of (2.1), for $u \in \mathbf{N}_{\mu,\lambda}^+$ we have

$$\begin{aligned} J_{\mu,\lambda}(u) &\geq -\frac{a}{4}\|u\|_\mu^4 - \frac{(p-2)\|Q\|_\infty}{2p} \int\limits_{\mathbb{R}^3} |u|^p dx \\ &\geq -\frac{a}{4} C_{a,\lambda}^4 - \frac{(p-2)\|Q\|_\infty}{2pS^p} |\{V < c\}|^{\frac{6-p}{6}} C_{a,\lambda}^p. \end{aligned} \tag{4.13}$$

Thus, $J_{\mu,\lambda}$ is bounded from below on $\mathbf{N}_{\mu,\lambda}^+$ and so $\inf_{u\in\mathbf{N}_{\mu,\lambda}^+} J_{\mu,\lambda}(u)$ is finite. Since $\int_\Omega Q\phi_1^p dx < 0$ and $\int_\Omega |\nabla\phi_1|^2 dx - \lambda\int_\Omega f_\Omega\phi_1^2 dx < 0$, which indicates that the function $h_{\phi_1}(t) = J_{\mu,\lambda}(t\phi_1)$ have $t_0^+ > 0$ and $\kappa_0 < 0$ are independent of μ such that $t_0^+\phi_1 \in \mathbf{N}_{\mu,\lambda}^+$ and

$$\inf_{0<t<\infty} h_{\phi_1}(t) = h_{\phi_1}\left(t_0^+\right) = \kappa_0 < 0.$$

This implies that

$$\inf_{u\in\mathbf{N}^{+}_{\mu,\lambda}} J_{\mu,\lambda}(u) \leq \kappa_0 < 0 \text{ for } \mu > 0 \text{ sufficiently large.} \tag{4.14}$$

Then by the Ekeland variational principle [17], there exists a minimizing sequence $\{u_n\} \subset \mathbf{N}^{+}_{\mu,\lambda}$ such that

$$\lim_{n\to\infty} J_{\mu,\lambda}(u_n) = \inf_{u\in\mathbf{N}^{+}_{\mu,\lambda}} J_{\mu,\lambda}(u) \leq \kappa_0 \text{ and } J'_{\mu,\lambda}(u_n) = o(1).$$

Since $\|u_n\|_\mu \leq C_{a,\lambda}$. Thus, by Proposition 2.5, there exist a subsequence $\{u_n\}$ and $u_0 \in X$ such that $J'_{\mu,\lambda}(u_0) = 0$ and $u_n \to u_0$ strongly in X_μ for $\mu > 0$ sufficiently large, which implies that $J_{\mu,\lambda}$ has minimizer u_0 in $\mathbf{N}^{+}_{\mu,\lambda}$ for μ sufficiently large, and so

$$J_{\mu,\lambda}(u_0) = \lim_{n\to\infty} J_{\mu,\lambda}(u_n) = \inf_{u\in\mathbf{N}^{+}_{\mu,\lambda}} J_{\mu,\lambda}(u) \leq \kappa_0 < 0,$$

which implies that u_0 is a minimizer on $\mathbf{N}^{+}_{\mu,\lambda}$ for $\mu > 0$ sufficiently large. □

We now turn our attention to $\mathbf{N}^{-}_{\mu,\lambda}$. Since

$$J_{\mu,\lambda}(u) = \frac{1}{4}\left(\|u\|^2_\mu - \lambda\int_{\mathbb{R}^3} f u^2 dx\right) + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} Q|u|^p\, dx > 0 \text{ for all } u \in \mathbf{N}^{-}_{\mu,\lambda}, \tag{4.15}$$

we have $\inf_{u\in\mathbf{N}^{-}_{\mu,\lambda}} J_{\mu,\lambda}(u) \geq 0$ for all $\mu > 0$. Since $\max\{Q, 0\} \not\equiv 0$ in Ω, similar to the arguments in (4.3), there exists $\overline{D}_{a,\lambda} > 0$ independent of μ such that $\inf_{u\in\mathbf{N}^{-}_{\mu,\lambda}} J_{\mu,\lambda}(u) < \overline{D}_{a,\lambda}$ and the set

$$\left\{J_{\mu,\lambda} < \overline{D}_{a,\lambda}\right\} := \left\{u \in \mathbf{N}^{-}_{\mu,\lambda} : J_{\mu,\lambda}(u) < \overline{D}_{a,\lambda}\right\} \neq \emptyset \text{ for } \mu > 0 \text{ sufficiently large.}$$

Furthermore, we have the following results.

Lemma 4.4. *Suppose that $4 < p < 6$ and $\int_\Omega Q\phi_1^p dx < 0$. Then for each $\lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_0$, we have the following results.*
(i) There exists $C_{a,\lambda} > 0$ such that $\|u\|_\mu \leq C_{a,\lambda}$ for all $u \in \left\{J_{\mu,\lambda} < \overline{D}_{a,\lambda}\right\}$ and for $\mu > 0$ sufficiently large.
(ii) We have

$$\inf_{u\in\mathbf{N}^{-}_{\mu,\lambda}} J_{\mu,\lambda}(u) = \inf_{u\in\left\{J_{\mu,\lambda} < \overline{D}_{a,\lambda}\right\}} J_{\mu,\lambda}(u) > 0$$

for $\mu > 0$ sufficiently large.

Proof. (i) Suppose on the contrary. Then there exist a sequence $\{\mu_n\} \subset \mathbb{R}^+$ with $\mu_n \to \infty$ and a sequence $u_n \in \left\{J_{\mu_n,\lambda} < \overline{D}_{a,\lambda}\right\}$ such that $\|u_n\|_{\mu_n} \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_{\mu_n}}$. Then by

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Theorem 3.1, we may assume that for every $\mu > 0$ there exist subsequence $\{v_n\}$ and $v_0 \in H_0^1(\Omega)$ such that $v_n \rightharpoonup v_0$ in X_μ and $v_n \to v_0$ in $L^r(\mathbb{R}^3)$ for all $2 \le r < 6$. Then

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} Q|v_n|^p dx = \int_{\mathbb{R}^3} Q|v_0|^p dx \tag{4.16}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f v_n^2 dx = \int_{\mathbb{R}^3} f v_0^2 dx. \tag{4.17}$$

Dividing (4.15) by $\|u_n\|_{\mu_n}^2$ gives

$$\frac{J_{\mu_n,\lambda}(u_n)}{\|u_n\|_{\mu_n}^2} = \frac{1}{4}\left(1 - \lambda \int_{\mathbb{R}^3} f v_n^2 dx\right) + \left(\frac{1}{4} - \frac{1}{p}\right) \|u_n\|_{\mu_n}^{p-2} \int_{\mathbb{R}^3} Q\,|v_n|^p\, dx. \tag{4.18}$$

Since $\|u_n\|_{\mu_n} \to +\infty$ and $\frac{J_{\mu_n,\lambda}(u_n)}{\|u_n\|_{\mu_n}^2} \to 0$, by (4.16)–(4.18), we have that $\int_{\mathbb{R}^3} Q|v_n|^p dx \to 0$ and so $\int_{\mathbb{R}^3} Q|v_0|^p dx = 0$. We now show that for each $\mu > 0$, we have $v_n \to v_0$ in X_μ. Suppose otherwise, then by (4.17) and (4.18), there exists $\mu > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla v_0|^2 dx - \lambda \int_{\mathbb{R}^3} f v_0^2 dx &= \|v_0\|_\mu^2 - \lambda \int_{\mathbb{R}^3} f v_0^2 dx \\ &< \liminf_{n\to\infty}\left(\|v_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f v_n^2 dx\right) = 0.\end{aligned}$$

Thus, $v_0 \ne 0$ and

$$\frac{v_0}{\|v_0\|_\mu} = \frac{v_0}{\left(\int_\Omega |\nabla v_0|^2 dx\right)^{1/2}} \in \overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p),$$

which is impossible. Hence $v_n \to v_0$ in X_μ. It follows that $\|v_0\|_\mu = 1$, $\int_{\mathbb{R}^3} V v_0^2 dx = 0$ and

$$\|v_0\|_\mu^2 - \lambda \int_{\mathbb{R}^3} f v_0^2 dx + a\,\|v_0\|_{D^{1,2}}^4 = \int_{\mathbb{R}^3} Q|v_0|^p dx = 0.$$

Thus, for every $\mu > 0$, there holds $v_0 \in \Lambda_\mu^0 \cap \Theta_\mu^0(p)$ which is impossible as $\overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p) = \emptyset$. Hence there exists $C_{a,\lambda} > 0$ such that $\|u\|_\mu \le C_{a,\lambda}$ for all $u \in \left\{J_{\mu,\lambda} < \overline{D}_{a,\lambda}\right\}$ and for $\mu > 0$ sufficiently large.
(ii) Since $\inf_{u\in \mathbf{N}_{\mu,\lambda}^-} J_{\mu,\lambda}(u) \ge 0$, by Lemma 4.2 and the Ekeland variational principle [17], there exists a minimizing sequence $\{u_n\} \subset \left\{J_{\mu,\lambda} < \overline{D}_{a,\lambda}\right\} \subset \mathbf{N}_{\mu,\lambda}^-$ such that

$$\lim_{n\to\infty} J_{\mu,\lambda}(u_n) = \inf_{u\in \mathbf{N}^-_{\mu,\lambda}} J_{\mu,\lambda}(u) \text{ and } J'_{\mu,\lambda}(u_n) = o(1).$$

By part (i), there exists $C_{a,\lambda} > 0$ such that $\|u_n\|_\mu \le C_{a,\lambda}$. Then by Proposition 2.5, and so there exist a subsequence $\{u_n\}$ and $u_0 \in \mathbf{N}^-_{\mu,\lambda}$ such that $u_n \to u_0$ in X_μ and $J'_{\mu,\lambda}(u_0) = 0$ for $\mu > 0$ sufficiently large. If $\inf_{u\in \mathbf{N}^-_{\mu,\lambda}} J_{\mu,\lambda}(u) = 0$, then

$$J_{\mu,\lambda}(u_0) = \lim_{n\to\infty} J_{\mu,\lambda}(u_n) = \inf_{u\in \mathbf{N}^-_{\mu,\lambda}} J_{\mu,\lambda}(u) = 0.$$

It then follows exactly as in the proof in part (i) that $\frac{u_0}{\|u_0\|_\mu} \in \Lambda^0_\mu \cap \Theta^0_\mu(p)$ which is impossible as $\overline{\Lambda^-_\mu} \cap \overline{\Theta^+_\mu}(p) = \emptyset$. This completes the proof. □

Theorem 4.5. *Suppose that $4 < p < 6$ and $\int_\Omega Q\phi_1^p dx < 0$. Then for each $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \delta_0$, there exists a minimizer of $J_{\mu,\lambda}(u)$ on $\mathbf{N}^-_{\mu,\lambda}$ for $\mu > 0$ sufficiently large.*

Proof. By Lemmas 4.2, 4.4 and the Ekeland variational principle [17], for each $\mu > \overline{\mu}_0(\lambda)$ there exists a minimizing sequence $\{u_n\} \subset \left\{J_{\mu,\lambda} < \overline{D}_{a,\lambda}\right\} \subset \mathbf{N}^-_{\mu,\lambda}$ such that

$$\lim_{n\to\infty} J_{\mu,\lambda}(u_n) = \inf_{u\in \mathbf{N}^-_{\mu,\lambda}} J_{\mu,\lambda}(u) > 0 \text{ and } J'_{\mu,\lambda}(u_n) = o(1).$$

Since $\inf_{u\in \mathbf{N}^-_{\mu,\lambda}} J_{\mu,\lambda}(u) < \overline{D}_{a,\lambda}$, by Lemma 4.4 (i), there exists a positive constant $C_{a,\lambda}$ independent of μ such that $\|u_n\|_\mu \le C_{a,\lambda}$. Thus, by Proposition 2.5, there exist a subsequence $\{u_n\}$ and $u_0 \in X$ such that $J'_{\mu,\lambda}(u_0) = 0$ and $u_n \to u_0$ strongly in X_μ for $\mu > 0$ sufficiently large, which implies that $J_{\mu,\lambda}$ has minimizer u_0 in $\in \mathbf{N}^-_{\mu,\lambda}$ for μ sufficiently large, and so

$$J_{\mu,\lambda}(u_0) = \lim_{n\to\infty} J_{\mu,\lambda}(u_n) = \inf_{u\in \mathbf{N}^-_{\mu,\lambda}} J_{\mu,\lambda}(u) < \overline{D}_{a,\lambda},$$

which implies that u_0 is a minimizer on $\mathbf{N}^-_{\mu,\lambda}$. This completes the proof. □

We are now ready to prove Theorem 1.2: By Theorems 4.3 and 4.5, there exists $\delta_0 > 0$ such that, when $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \delta_0$, $J_{\mu,\lambda}$ has minimizers in each of $\mathbf{N}^+_{\mu,\lambda}$ and $\mathbf{N}^-_{\mu,\lambda}$ for μ sufficiently large. Since $J_{\mu,\lambda}(u) = J_{\mu,\lambda}(|u|)$, we may assume that these minimizers are positive. Moreover, by Lemma 3.6 we may assume that $\mathbf{N}^+_{\mu,\lambda}$ and $\mathbf{N}^-_{\mu,\lambda}$ are separated and $\mathbf{N}^0_{\mu,\lambda} = \emptyset$. It follows that the minimizers are local minimizers in $\mathbf{N}_{\mu,\lambda}$ which do not lie in $\mathbf{N}^0_{\mu,\lambda}$ and so by Lemma 2.4, they are positive solutions of Equation $\left(E_{\mu,\lambda}\right)$.

5. The case when $N = 3$ and $p = 4$

We are now ready to prove Theorem 1.3: (i) When $0 < \lambda < \lambda_1(f_\Omega)$. Similar to the argument of proofs in Lemma 4.1 and Theorem 1.1, $J_{\mu,\lambda}$ has minimizer u_0 in $\mathbf{N}_{\mu,\lambda} = \mathbf{N}^-_{\mu,\lambda}$ for μ sufficiently large. Since $J_{\mu,\lambda}(u) = J_{\mu,\lambda}(|u|)$, by Lemma 2.4, we may assume that u_0 is a positive solution of Equation $\left(E_{\mu,\lambda}\right)$ such that $J_{\mu,\lambda}(u^-_\mu) = \inf_{u\in \mathbf{N}^-_{\mu,\lambda}} J_{\mu,\lambda}(u) > 0$.

(ii) Since $\Gamma_0 < \infty$, by Lemma 3.4 (ii) for each $a \geq \Gamma_0$ and $0 < \lambda < \lambda_1(f_\Omega)$, we have $\mathbf{N}_{\mu,\lambda} = \emptyset$ for μ sufficiently large, this implies that for each $a \geq \Gamma_0$ and $0 < \lambda < \lambda_1(f_\Omega)$, Equation $\left(E_{\mu,\lambda}\right)$ does not admit nontrivial solution.

(iii) Since $\Gamma_0 < \infty$, by Lemma 3.4 (iii), for each $a > \Gamma_0$ and $\lambda \geq \lambda_1(f_\Omega)$, we have $\mathbf{N}_{\mu,\lambda} = \mathbf{N}^+_{\mu,\lambda}$ and $\mathbf{N}^+_{\mu,\lambda} = \{t_{\min}(u)u : u \in \Theta^-_\mu(p)\}$ for $\mu > 0$ sufficiently large. Now, we will show that $\mathbf{N}^+_{\mu,\lambda}$ is uniform bounded for $\mu > 0$ sufficiently large. Suppose on the contrary. Then there exist sequences $\{\mu_n\} \subset \mathbb{R}$ and $u_n \in \mathbf{N}^+_{\mu_n,\lambda}$ such that $\mu_n \to \infty$ and $\|u_n\|_{\mu_n} \to \infty$ as $n \to \infty$. Clearly,

$$\|u_n\|^2_{\mu_n} - \lambda \int_{\mathbb{R}^3} f u_n^2 dx = \int_{\mathbb{R}^3} Q|u_n|^4 dx - a\|u_n\|^4_{D^{1,2}} < 0. \tag{5.1}$$

Let $v_n = \frac{u_n}{\|u_n\|_{\mu_n}}$. Then by Theorem 3.1, we may assume that for every $\mu > 0$ there exists $v_0 \in H^1_0(\Omega)$ such that

$$v_n \rightharpoonup v_0 \text{ in } X_\mu;\ v_n \to v_0 \text{ in } L^r\left(\mathbb{R}^3\right) \text{ for all } 2 \leq r < 6.$$

Thus,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f v_n^2 dx = \int_{\mathbb{R}^3} f v_0^2 dx \tag{5.2}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} Q|v_n|^p dx = \int_{\mathbb{R}^3} Q|v_0|^p dx. \tag{5.3}$$

Moreover, by Fatou's Lemma,

$$\int_{\mathbb{R}^3} |\nabla v_0|^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 dx. \tag{5.4}$$

Dividing (5.1) by $\|u_n\|^2_{\mu_n}$ gives

$$\|v_n\|^2_{\mu_n} - \lambda \int_{\mathbb{R}^3} f v_n^2 dx = \|u_n\|^2_{\mu_n}\left(\int_{\mathbb{R}^3} Q|v_n|^4 dx - a\|v_n\|^4_{D^{1,2}}\right) < 0. \tag{5.5}$$

Since

$$\lim_{n\to\infty}\left(\|v_n\|^2_{\mu_n} - \lambda \int_{\mathbb{R}^3} f v_n^2 dx\right) = 1 - \lambda \lim_{n\to\infty} \int_{\mathbb{R}^3} f v_n^2 dx = 1 - \lambda \int_{\mathbb{R}^3} f v_0^2 dx \tag{5.6}$$

and $\|u_n\|_{\mu_n} \to \infty$, by (5.3)–(5.6), it obtains that $\int_{\mathbb{R}^3} Q|v_0|^4 dx - a\|v_0\|^4_{D^{1,2}} \geq 0$ and $\int_{\mathbb{R}^3} f v_0^2 dx > 0$. Moreover, by $v_0 \in H_0^1(\Omega)$, (5.2) and (5.5), for every $\mu > 0$,

$$\|v_0\|^2_\mu - \lambda \int_{\mathbb{R}^3} f v_0^2 dx = \int_{\mathbb{R}^3} |\nabla v_0|^2 dx - \lambda \int_{\mathbb{R}^3} f v_0^2 dx$$
$$\leq \liminf_{n\to\infty} \left(\|v_n\|^2_{\mu_n} - \lambda \int_{\mathbb{R}^3} f v_n^2 dx \right) \leq 0.$$

We now show that $v_n \to v_0$ in X_μ. Suppose on the contrary. Then

$$\|v_0\|^2_\mu - \lambda \int_{\mathbb{R}^3} f v_0^2 dx = \int_{\mathbb{R}^3} |\nabla v_0|^2 dx - \lambda \int_{\mathbb{R}^3} f v_0^2 dx$$
$$< \liminf_{n\to\infty} \left(\|v_n\|^2_{\mu_n} - \lambda \int_{\mathbb{R}^3} f v_n^2 dx \right) \leq 0,$$

since $\int_{\mathbb{R}^3} V v_0^2 dx = 0$. Hence $\frac{v_0}{\|v_0\|_\mu} \in \overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p)$ which is impossible. Since $v_n \to v_0$ in X_μ, then $\|v_0\|_\mu = 1$. Hence $v_0 \in \Theta_\mu^0(p)$ and so $v_0 \in \overline{\Theta_\mu^+}(p)$. Moreover,

$$\|v_0\|^2_\mu - \lambda \int_{\mathbb{R}^3} f v_0^2 dx = \lim_{n\to\infty} \left(\|v_n\|^2_\mu - \lambda \int_{\mathbb{R}^3} f v_n^2 dx \right) \leq 0,$$

and so $v_0 \in \overline{\Lambda_\mu^-}$. Thus, $v_0 \in \overline{\Lambda_\mu^-} \cap \overline{\Theta_\mu^+}(p)$ which is impossible. Therefore, we can conclude that $\mathbf{N}^+_{\mu,\lambda}$ is uniform bounded for $\mu > 0$ sufficiently large. Then similar to the argument of proof in Theorem 4.3, $J_{\mu,\lambda}$ has minimizer u_μ^+ in $\mathbf{N}_{\mu,\lambda} = \mathbf{N}^+_{\mu,\lambda}$ for μ sufficiently large such that $J_{\mu,\lambda}(u_\mu^+) < 0$. Since $J_{\mu,\lambda}(u_\mu^+) = J_{\mu,\lambda}(|u_\mu^+|)$, by Lemma 2.4, we may assume that u_μ^+ is a positive solution of Equation $(E_{\mu,\lambda})$.

(iv) The proof is essentially same as that in part (iii), so we omit it here.

We are now ready to prove Theorem 1.4: Since $\lambda_1^{-2}(f_\Omega)\int_\Omega Q\phi_1^4 dx < a < \Gamma_0$,

$$\Phi_p(\phi_1) = \int_\Omega Q|\phi_1|^p dx - a\left(\int_\Omega |\nabla\phi_1|^2 dx\right)^2 < 0 \text{ for } p = 4.$$

By Lemma 3.7, there exists $\delta_0 > 0$ such that for every $\lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_0$, $\mathbf{N}^\pm_{\mu,\lambda}$ are nonempty sets and $\mathbf{N}_{\mu,\lambda} = \mathbf{N}^+_{\mu,\lambda} \cup \mathbf{N}^-_{\mu,\lambda}$ for $\mu > 0$ sufficiently large. Then similar to the argument of proof in Theorem 1.2, Equation $(E_{\mu,\lambda})$ has two positive solutions u_μ^- and u_μ^+ satisfying $J_{\mu,\lambda}(u_\mu^+) < 0 < J_{\mu,\lambda}(u_\mu^-)$ for $\mu > 0$ sufficiently large.

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6. The case when $N \geq 3$ and $2 < p < \min\{4, 2^*\}$

6.1. The proof of Theorem 1.5

We are now ready to prove Theorem 1.5: For $0 < \lambda < \lambda_1(f)$ and $u \in X\setminus\{0\}$, we know that $tu \in \mathbf{N}_\mu^0$ if and only if $h'_{tu}(1) = h''_{tu}(1) = 0$, i.e., the following system of equations is satisfied:

$$\begin{cases} at^3 \|u\|_{D^{1,2}}^4 + t\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) - t^{p-1}\int_{\mathbb{R}^N} Q|u|^p\,dx = 0, \\ 3at^2\|u\|_{D^{1,2}}^4 + \left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) - (p-1)t^{p-2}\int_{\mathbb{R}^N} Q|u|^p\,dx = 0. \end{cases} \tag{6.1}$$

By solving the system (6.1) with respect to the variables t and a, we have

$$t(u) = \left(\frac{2\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right)}{(4-p)\int_{\mathbb{R}^N} Q|u|^p\,dx}\right)^{1/(p-2)}$$

and

$$a(u) = \frac{p-2}{4-p}\left(\frac{4-p}{2}\right)^{2/(p-2)} \overline{A}_\lambda(u),$$

where

$$\overline{A}_\lambda(u) = \frac{\left(\int_{\mathbb{R}^N} Q|u|^p dx\right)^{2/(p-2)}}{\|u\|_{D^{1,2}}^4\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right)^{(4-p)/(p-2)}}. \tag{6.2}$$

We conclude that $a(u)$ is the unique parameter $a > 0$ for which the fibrering map h_u has a critical point with second derivative zero at $t(u)$. Hence, if $a > a(u)$, then h_u is increasing on $(0, \infty)$ and has no critical point. Moreover, for $0 < \lambda < \lambda_1(f)$, we define

$$\overline{\mathbf{A}}_\lambda = \frac{p-2}{4-p}\left(\frac{4-p}{p}\right)^{2/(p-2)} \sup_{u\in X\setminus\{0\}} \overline{A}_\lambda(u). \tag{6.3}$$

Note that by (3.3) and the Hölder and Sobolev inequalities,

$$\begin{aligned} \overline{A}_\lambda(u) &\leq \frac{\left(\|Q\|_\infty\left(\int_{\{V\geq c\}} u^2dx + \int_{\{V<c\}} u^2dx\right)^{\frac{2^*-p}{2^*-2}}\left(\frac{\|u\|_{D^{1,2}}^{2^*}}{S^{2^*}}\right)^{\frac{p-2}{2^*-2}}\right)^{2/(p-2)}}{\|u\|_{D^{1,2}}^4\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right)^{(4-p)/(p-2)}} \\ &\leq \left(\frac{\widetilde{\lambda}_{1,\mu}(f)}{\widetilde{\lambda}_{1,\mu}(f)-\lambda}\right)^{(4-p)/(p-2)}\left(\frac{\|Q\|_\infty |\{V<c\}|^{\frac{2^*-p}{2^*}}\|u\|_{D^{1,2}}^{\frac{N(p-2)}{2}}\|u\|_\mu^{\frac{(N-2)(2^*-p)}{2}}}{S^p\|u\|_{D^{1,2}}^{2(p-2)}\|u\|_\mu^{4-p}}\right)^{2/(p-2)} \\ &\leq \left(\frac{\widetilde{\lambda}_{1,\mu}(f)}{\widetilde{\lambda}_{1,\mu}(f)-\lambda}\right)^{(4-p)/(p-2)}\left(\frac{\|Q\|_\infty|\{V<c\}|^{\frac{2^*-p}{2^*}}}{S^p}\right)^{2/(p-2)} \quad \text{for al } \mu > \overline{\mu}_0(\lambda), \end{aligned}$$

which implies that for each $0<\lambda<\lambda_1(f)$,

$$\overline{\mathbf{A}}_\lambda < \frac{1}{2}\left(\frac{(4-p)\lambda_1(f_\Omega)}{p(\lambda_1(f_\Omega)-\lambda)}\right)^{(4-p)/(p-2)}\left(\frac{\|Q\|_\infty |\{V<c\}|^{\frac{2^*-p}{2^*}}}{S^p}\right)^{2/(p-2)},$$

for $\mu>0$ sufficiently large. Hence, the energy functional $J_{\mu,\lambda}$ has no any nontrivial critical points for $a>\overline{\mathbf{A}}_\lambda$ for $\mu>0$ sufficiently large. Consequently, we complete the proof.

6.2. *The proofs of Theorems 1.6, 1.7*

First, we define

$$\alpha^+_{\mu,\lambda} = \inf_{u\in\mathbf{N}^+_{\mu,\lambda}} J_{\mu,\lambda}(u).$$

Then we have the following results.

Proposition 6.1. *Suppose that $N=3$, $2<p<4$ and conditions $(V_1)-(V_3)$ and $(D_1)-(D_3)$ hold. Then the following statements are true.*
(i) For each $\lambda>0$ and $a>0$, we have $\mathbf{N}^+_{\mu,\lambda}$ is uniformly bounded for $\mu>0$ sufficiently large;
(ii) For each $\lambda>0$ and $a>0$, there exist two numbers $d_0, D_0>0$ such that

$$\inf_{u\in\mathbf{N}^-_{\mu,\lambda}\cup\mathbf{N}^0_{\mu,\lambda}} J_{\mu,\lambda}(u)\geq 0 > -d_0 > \alpha^+_{\mu,\lambda} > -D_0 \text{ for } \mu>0 \text{ sufficiently large.}$$

Proof. (i) Let $u\in\mathbf{N}^+_{\mu,\lambda}$. Then by (2.2) and the Hölder and Sobolev inequalities,

$$\|u\|^2_\mu < \frac{a(4-p)}{(p-2)}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 + \frac{\lambda\|f\|_{L^{3/2}}}{S^2}\int_{\mathbb{R}^3}|\nabla u|^2dx. \tag{6.4}$$

Moreover, using the Sobolev, Hölder and Hardy inequalities, condition (D_3) and (6.4) gives

$$\begin{aligned}
1 &= \frac{\int_{\mathbb{R}^3}Q|u|^p dx + \lambda\int_{\mathbb{R}^3}fu^2dx}{\|u\|^2_\mu + a\|u\|^4_{D^{1,2}}} < \frac{\int_{\mathbb{R}^3}Q|u|^p dx + \lambda\int_{\mathbb{R}^3}fu^2dx}{a\|u\|^4_{D^{1,2}}}\\
&= \frac{1}{aS^{\frac{3(p-2)}{2}}}\left[\frac{\int_{|x|>R_*}Q^{\frac{4}{6-p}}u^2dx + \|Q\|_\infty^{\frac{4}{6-p}}\left|B_{R_*}(0)\right|^{\frac{2}{3}}S^{-2}\int_{\mathbb{R}^3}|\nabla u|^2dx}{\|u\|_{D^{1,2}}^{\frac{2(14-3p)}{6-p}}}\right]^{\frac{6-p}{4}} + \frac{\lambda\|f\|_{L^{3/2}}}{aS^2\|u\|^2_{D^{1,2}}}\\
&\leq \frac{1}{aS^{\frac{3(p-2)}{2}}}\left[\frac{\int_{|x|>R_*}\left(Vu^2\right)^{\frac{2(4-p)}{6-p}}\left(\frac{|u|}{|x|}\right)^{\frac{2(p-2)}{6-p}}dx}{\|u\|_{D^{1,2}}^{\frac{2(14-3p)}{6-p}}} + \frac{\|Q\|_\infty^{\frac{4}{6-p}}\left|B_{R_*}(0)\right|^{\frac{2}{3}}S^{-2}}{\|u\|_{D^{1,2}}^{\frac{4(4-p)}{6-p}}}\right]^{\frac{6-p}{4}} + \frac{\lambda\|f\|_{L^{3/2}}}{aS^2\|u\|^2_{D^{1,2}}}
\end{aligned}$$

$$\leq \frac{1}{aS^{\frac{3(p-2)}{2}}}\left[\overline{C}_0\left(\frac{\int_{|x|>R_*} V(x)u^2dx}{\|u\|_{D^{1,2}}^4}\right)^{\frac{2(4-p)}{6-p}}+\frac{\|Q\|_\infty^{\frac{4}{6-p}}\left|B_{R_*}(0)\right|^{\frac{2}{3}}S^{-2}}{\|u\|_{D^{1,2}}^{\frac{4(4-p)}{6-p}}}\right]^{\frac{6-p}{4}}+\frac{\lambda\|f\|_{L^{3/2}}}{aS^2\|u\|_{D^{1,2}}^2}$$

$$<\frac{1}{aS^{\frac{3(p-2)}{2}}}\left[\overline{C}_0\left(\frac{a(4-p)}{\mu(p-2)}+\frac{\lambda\|f\|_{L^{3/2}}}{\mu S^2\|u\|_{D^{1,2}}^2}\right)^{\frac{2(4-p)}{6-p}}+\frac{\|Q\|_\infty^{\frac{4}{6-p}}\left|B_{R_*}(0)\right|^{\frac{2}{3}}}{S^2\|u\|_{D^{1,2}}^{\frac{4(4-p)}{6-p}}}\right]^{\frac{6-p}{4}}+\frac{\lambda\|f\|_{L^{3/2}}}{aS^2\|u\|_{D^{1,2}}^2}$$

where $\overline{C}_0$ is the sharp constant of Caffarelli-Kohn-Nirenberg inequality. This implies that there exists a constant $d_1>0$, dependent on a and λ such that

$$\int_{\mathbb{R}^3}|\nabla u|^2dx\leq d_1 \text{ for all } u\in\mathbf{N}_{\mu,\lambda}^+ \text{ and for } \mu>0 \text{ sufficiently large.} \tag{6.5}$$

Thus, by (6.4) and (6.5), we have

$$\|u\|_\mu^2<\frac{a(4-p)}{p-2}d_1^2+\frac{\lambda\|f\|_{L^{3/2}}}{S^2}d_1 \text{ for all } u\in\mathbf{N}_{\mu,\lambda}^+.$$

(ii) By Theorem 3.8 (ii), there exists $d_0>0$ such that $\alpha_{\mu,\lambda}^+<-d_0:=J_{\mu,\lambda}\left(t_a^+\phi_1\right)$. Next, we prove that there exist constants $D_0,\mu_2>0$ such that

$$\alpha_{\mu,\lambda}^+>-D_0 \text{ for all } \mu\geq\mu_2 \text{ and } a>0.$$

Let $u\in\mathbf{N}_{\mu,\lambda}^+$. Similar to (6.4), we obtain

$$\int_{\mathbb{R}^3}fu^2dx\leq\frac{\lambda\|f\|_{L^{3/2}}}{S^2}\int_{\mathbb{R}^3}|\nabla u|^2dx$$

and

$$\int_{\mathbb{R}^3}Q|u|^pdx\leq\frac{\overline{C}_0^{\frac{6-p}{4}}}{S^{\frac{3(p-2)}{2}}}\left(\frac{a(4-p)}{2\lambda(p-2)}\right)^{\frac{4-p}{2}}\|u\|_{D^{1,2}}^4+\frac{\|Q\|_\infty\left|B_{R_*}(0)\right|^{\frac{6-p}{6}}}{S^p}\|u\|_{D^{1,2}}^p.$$

Using the above inequalities gives

$$J_{\mu,\lambda}(u)=\frac{1}{2}\left(\|u\|_\mu^2-\lambda\int_{\mathbb{R}^3}fu^2dx\right)+\frac{a}{4}\|u\|_{D^{1,2}}^4-\frac{1}{p}\int_{\mathbb{R}^3}Q|u|^pdx$$

$$>\left[\frac{a}{4}-\frac{\overline{C}_0^{\frac{6-p}{4}}}{S^{\frac{3(p-2)}{2}}}\left(\frac{a(4-p)}{2\lambda(p-2)}\right)^{\frac{4-p}{2}}\right]\|u\|_{D^{1,2}}^4-\frac{\|Q\|_\infty\left|B_{R_*}(0)\right|^{\frac{6-p}{6}}}{S^p}\|u\|_{D^{1,2}}^p$$

$$-\frac{\lambda \|f\|_{L^{3/2}}}{2S^2}\|u\|^2_{D^{1,2}}.$$

This implies that there exists a constant $D_{a,\lambda} > 0$ such that $\alpha^+_{\mu,\lambda} > -D_{a,\lambda}$ for $\mu > 0$ sufficiently large. Moreover, for $u \in \mathbf{N}^-_{\mu,\lambda} \cup \mathbf{N}^0_{\mu,\lambda}$, by (2.2),

$$\begin{aligned} J_{\mu,\lambda}(u) &= \frac{1}{4}\left(\|u\|^2_\mu - \lambda \int_{\mathbb{R}^3} fu^2 dx\right) - \frac{4-p}{4p}\int_{\mathbb{R}^3} Q|u|^p dx \\ &\geq \frac{(4-p)(p-2)}{8p}\int_{\mathbb{R}^3} Q|u|^p dx > 0. \end{aligned}$$

Therefore,

$$\inf_{u \in \mathbf{N}^-_{\mu,\lambda} \cup \mathbf{N}^0_{\mu,\lambda}} J_{\mu,\lambda}(u) \geq 0 > -d_0 > \alpha^+_{\mu,\lambda} > -D_{a,\lambda},$$

for $\mu > 0$ sufficiently large. This completes the proof. □

Proposition 6.2. *Suppose that $N \geq 4$, $2 < p < 2^*$ and conditions $(V_1)-(V_3)$ and $(D_1)-(D_2)$ hold. Then the following statements are true.*
(i) For each $\lambda > 0$ and $a > 0$, we have $\mathbf{N}^+_{\mu,\lambda}$ is uniformly bounded for $\mu > 0$ sufficiently large;
(ii) For each $\lambda > 0$ and $a > 0$, there exist two numbers $d_0, D_0 > 0$ such that

$$\inf_{u \in \mathbf{N}^-_{\mu,\lambda} \cup \mathbf{N}^0_{\mu,\lambda}} J_{\mu,\lambda}(u) \geq 0 > -d_0 > \alpha^+_{\mu,\lambda} > -D_0 \text{ for } \mu > 0 \text{ sufficiently large.}$$

Proof. (i) Let $u \in \mathbf{N}^+_{\mu,\lambda}$. Then by (2.2) and the Hölder and Sobolev inequalities,

$$\|u\|^2_\mu < \frac{a(4-p)}{(p-2)}\|u\|^4_{D^{1,2}} + \frac{\lambda \|f\|_{L^{N/2}}}{S^2}\int_{\mathbb{R}^N} |\nabla u|^2 dx. \tag{6.6}$$

Moreover, using the Sobolev and Hölder inequalities and (6.6) gives

$$\begin{aligned} 1 &= \frac{\int_{\mathbb{R}^N} Q|u|^p dx + \lambda \int_{\mathbb{R}^N} fu^2 dx}{\|u\|^2_\mu + a\|u\|^4_{D^{1,2}}} < \frac{\int_{\mathbb{R}^N} Q|u|^p dx + \lambda \int_{\mathbb{R}^N} fu^2 dx}{a\|u\|^4_{D^{1,2}}} \\ &\leq \frac{\|Q\|_\infty \left(\frac{1}{\mu c}\|u\|^2_\mu + \frac{|\{V<c\}|^{2/N}}{S^2}\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{2p-N(p-2)}{4}}}{aS^{N(p-2)/2}\|u\|^{\frac{8-N(p-2)}{2}}_{D^{1,2}}} + \frac{\lambda \|f\|_{L^{N/2}}}{aS^2\|u\|^2_{D^{1,2}}} \\ &< \frac{\|Q\|_\infty \left[\frac{a(4-p)}{\mu c(p-2)}\|u\|^4_{D^{1,2}} + \left(\frac{\lambda \|f\|_{L^{N/2}}}{\mu c S^2} + \frac{|\{V<c\}|^{2/N}}{S^2}\right)\|u\|^2_{D^{1,2}}\right]^{\frac{2p-N(p-2)}{4}}}{aS^{N(p-2)/2}\|u\|^{\frac{8-N(p-2)}{2}}_{D^{1,2}}} + \frac{\lambda \|f\|_{L^{N/2}}}{aS^2\|u\|^2_{D^{1,2}}}. \end{aligned}$$

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Since

$$\frac{8-N(p-2)}{2} \geq 2p - N(p-2) \text{ for } N \geq 4,$$

this implies that there exists a constant $d_1 > 0$, dependent on a and λ such that

$$\|u\|_{D^{1,2}} \leq d_1 \text{ for all } u \in \mathbf{N}^+_{\mu,\lambda} \text{ and for } \mu > 0 \text{ sufficiently large.} \tag{6.7}$$

Thus, by (6.6) and (6.7), we have

$$\|u\|^2_\mu < \frac{a(4-p)}{p-2} d_1^4 + \frac{\lambda \|f\|_{L^{N/2}}}{S^2} d_1^2 \text{ for all } u \in \mathbf{N}^+_{\mu,\lambda}.$$

(ii) The proof is essentially same as that in Proposition 6.1 (ii), so we omit it here. □

We are now ready to prove Theorem 1.6: (i) By the Ekeland variational principle [17], Lemma 6.4 and Proposition 6.1, for each $0 < \lambda < \lambda_1(f_\Omega)$ and $0 < a < a_0$ there exists a bounded sequence $\{u_n\} \subset \mathbf{N}^+_{\mu,\lambda}$ such that

$$J_{\mu,\lambda}(u_n) = \alpha^+_{\mu,\lambda} + o(1) \text{ and } J'_{\mu,\lambda}(u_n) = o(1) \text{ in } X^{-1}_\mu.$$

It follows from Propositions 2.5, 6.1 that $J_{\mu,\lambda}$ satisfies the $(PS)_{\alpha^+_{\mu,\lambda}}$–condition in $\mathbf{N}^+_{\mu,\lambda}$ for $\mu > 0$ sufficiently large. Thus, there exist a subsequence $\{u_n\}$ and $u^+_{\mu,\lambda} \in \mathbf{N}^+_{\mu,\lambda}$ such that $u_n \to u^+_{\mu,\lambda}$ strongly in X_μ for $\mu > 0$ sufficiently large. Note that $\alpha^+_{\mu,\lambda} = J_{\mu,\lambda}\left(u^+_{\mu,\lambda}\right) < 0$. Hence, $u^+_{\mu,\lambda} \in \mathbf{N}^+_{\mu,\lambda}$ is a minimizer for $J_{\mu,\lambda}$ on $\mathbf{N}^+_{\mu,\lambda}$. Since $|u^+_{\mu,\lambda}| \in \mathbf{N}^+_{\mu,\lambda}$ and $J_{\mu,\lambda}\left(|u^+_{\mu,\lambda}|\right) = J_{\mu,\lambda}\left(u^+_{\mu,\lambda}\right) = \alpha^+_{\mu,\lambda} < 0$, one can see that $u^+_{\mu,\lambda}$ is a positive solution for Equation $(E_{\mu,\lambda})$ by Lemma 6.3.
(ii) By the Ekeland variational principle [17], Theorem 3.8 (ii) and Proposition 6.1, for each $a > 0$ and $\lambda \geq \lambda_1(f_\Omega)$ there exists a bounded sequence $\{u_n\} \subset \mathbf{N}^+_{\mu,\lambda}$ with $J_{\mu,\lambda}(u_n) < -d_0 < \inf_{u \in \mathbf{N}^-_{\mu,\lambda} \cup \mathbf{N}^0_{\mu,\lambda}} J_{\mu,\lambda}(u)$ such that

$$J_{\mu,\lambda}(u_n) = \alpha^+_{\mu,\lambda} + o(1) \text{ and } J'_{\mu,\lambda}(u_n) = o(1) \text{ in } X^{-1}_\mu.$$

By Proposition 2.5, we can establish a compactness conclusion for $\{u_n\}$, this means that there exist a subsequence $\{u_n\}$ and $u^+_{\mu,\lambda} \in \mathbf{N}^+_{\mu,\lambda}$ such that $u_n \to u^+_{\mu,\lambda}$ strongly in X_μ for $\mu > 0$ sufficiently large. In fact that $u^+_{\mu,\lambda}$ is a positive solution for Equation $(E_{\mu,\lambda})$.

We are now ready to prove Theorem 1.7: The proof is essentially same as that in Theorem 1.6, so we omit it here.

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6.3. The proof of Theorem 1.8

Note that $u \in \mathbf{N}_{\mu,\lambda}$ if and only if $a\|u\|_{D^{1,2}}^4 + \|u\|_\mu^2 = \int_{\mathbb{R}^N} Q|u|^p dx + \lambda \int_{\mathbb{R}^N} fu^2 dx$. It follows from Lemma 3.2, (3.3) and the Sobolev inequality that

$$\begin{aligned}\frac{\widetilde{\lambda}_{1,\mu}(f)-\lambda}{\widetilde{\lambda}_{1,\mu}(f)}\|u\|_\mu^2 &\le \|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx + a\|u\|_{D^{1,2}}^4 = \int_{\mathbb{R}^N} Q|u|^p dx \\ &\le \|Q\|_\infty S^{-p}|\{V<c\}|^{\frac{6-p}{6}}\|u\|_\mu^p \text{ for all } u\in\mathbf{N}_{\mu,\lambda} \text{ and } \mu>\overline{\mu}_0(\lambda).\end{aligned}$$

Thus, it leads to

$$\frac{\widetilde{\lambda}_{1,\mu}(f)}{\widetilde{\lambda}_{1,\mu}(f)-\lambda}\int_{\mathbb{R}^N} Q|u|^p dx \ge \|u\|_\mu^2 \ge \left(\frac{S^p\left(\widetilde{\lambda}_{1,\mu}(f)-\lambda\right)}{\widetilde{\lambda}_{1,\mu}(f)\|Q\|_\infty|\{V<c\}|^{\frac{2^*-p}{2^*}}}\right)^{2/(p-2)} \tag{6.8}$$

for all $u\in\mathbf{N}_{\mu,\lambda}$ and $\mu>\overline{\mu}_0(\lambda)$. Moreover, by (2.2) and (6.8),

$$\begin{aligned}J_{\mu,\lambda}(u) &= \frac{1}{4}\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) - \frac{4-p}{4p}\int_{\mathbb{R}^N} Q|u|^p dx \\ &\ge \frac{(p-2)\left(\widetilde{\lambda}_{1,\mu}(f)-\lambda\right)}{4p\widetilde{\lambda}_{1,\mu}(f)}\|u\|_\mu^2 \\ &\ge \frac{p-2}{4p}\left(\frac{S^p}{\|Q\|_\infty|\{V<c\}|^{\frac{2^*-p}{2^*}}}\right)^{2/(p-2)}\left(\frac{\widetilde{\lambda}_{1,\mu}(f)-\lambda}{\widetilde{\lambda}_{1,\mu}(f)}\right)^{p/(p-2)} \text{ for all } u\in\mathbf{N}^-_{\mu,\lambda}.\end{aligned}$$

Hence, the following statement is true.

Lemma 6.3. *Suppose that $2<p<\min\{4,2^*\}$ and condition (V_1) hold. Then $J_{\mu,\lambda}$ is coercive and bounded below on $\mathbf{N}^-_{\mu,\lambda}$. Furthermore, for all $u\in\mathbf{N}^-_{\mu,\lambda}$, there holds*

$$J_{\mu,\lambda}(u) > d_\mu := \frac{(p-2)K^p(\mu)}{4p}\left(\frac{S^p}{\|Q\|_\infty|\{V<c\}|^{\frac{2^*-p}{2^*}}}\right)^{2/(p-2)},$$

where $K(\mu) := \left(\frac{\widetilde{\lambda}_{1,\mu}(f)-\lambda}{\widetilde{\lambda}_{1,\mu}(f)}\right)^{1/(p-2)} \le \left(\frac{\lambda_1(f_\Omega)-\lambda}{\lambda_1(f_\Omega)}\right)^{1/(p-2)}$ for all $\mu\ge\mu_0$.

Let $C(p) := \left(\frac{2S_p^p(\Omega)}{Q_{\Omega,\min}(4-p)}\right)^{2/(p-2)}$. Then for any $u\in\mathbf{N}_{\mu,\lambda}$ with $J_{\mu,\lambda}(u) < \frac{p-2}{4p}C(p)K^p(\mu)$, we deduce that

$$\begin{aligned}\frac{p-2}{4p}C(p)K^p(\mu) > J_{\mu,\lambda}(u) \\ &= \frac{1}{2}\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) + \frac{a}{4}\|u\|_{D^{1,2}}^4 - \frac{1}{p}\int_{\mathbb{R}^N} Q|u|^p dx \\ &\geq \frac{(p-2)\left(\widetilde{\lambda}_{1,\mu}(f)-\lambda\right)}{2p\widetilde{\lambda}_{1,\mu}(f)}\|u\|_\mu^2 - \frac{a(4-p)}{4p}\|u\|_{D^{1,2}}^4 \\ &\geq \frac{(p-2)\left(\widetilde{\lambda}_{1,\mu}(f)-\lambda\right)}{2p\widetilde{\lambda}_{1,\mu}(f)}\|u\|_\mu^2 - \frac{a(4-p)}{4p}\|u\|_\mu^4. \end{aligned} \tag{6.9}$$

It implies that if $0 < a < C^{-1}(p)$, then there exist two positive numbers $\widehat{D}_1(\mu)$ and $\widehat{D}_2(\mu)$ satisfying

$$0 < \widehat{D}_1(\mu) < C(p)K(\mu) < \widehat{D}_2(\mu)$$

such that

$$\|u\|_\mu < \widehat{D}_1(\mu) \text{ or } \|u\|_\mu > \widehat{D}_2(\mu). \tag{6.10}$$

Thus, we have

$$\begin{aligned}\mathbf{N}_{\mu,\lambda}\left[\frac{p-2}{4p}C(p)K^p(\mu)\right] &= \left\{u \in \mathbf{N}_{\mu,\lambda} \mid J_{\mu,\lambda}(u) < \frac{p-2}{4p}C(p)K^p(\mu)\right\} \\ &= \mathbf{N}_{\mu,\lambda}^{(1)} \cup \mathbf{N}_{\mu,\lambda}^{(2)},\end{aligned} \tag{6.11}$$

where

$$\mathbf{N}_{\mu,\lambda}^{(1)} := \left\{u \in \mathbf{N}_{\mu,\lambda}\left[\frac{p-2}{4p}C(p)K^p(\mu)\right] : \|u\|_\mu < \widehat{D}_1(\mu)\right\}$$

and

$$\mathbf{N}_{\mu,\lambda}^{(2)} := \left\{u \in \mathbf{N}_{\mu,\lambda}\left[\frac{p-2}{4p}C(p)K^p(\mu)\right] : \|u\|_\mu > \widehat{D}_2(\mu)\right\}.$$

For $0 < a < a_0 := \frac{p-2}{4-p}C^{-1}(p)$, we further have

$$\|u\|_\mu < \widehat{D}_1(\mu) < C^{1/2}(p)K(\mu) \text{ for all } u \in \mathbf{N}_{\mu,\lambda}^{(1)} \tag{6.12}$$

and

$$\|u\|_\mu > \widehat{D}_2(\mu) > C^{1/2}(p)K(\mu) \text{ for all } u \in \mathbf{N}_{\mu,\lambda}^{(2)}. \tag{6.13}$$

Using (2.2), (6.10), condition (D_4) and the Sobolev inequality gives

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$$\begin{aligned}
h_u''(1) &= -2\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) + (4-p)\int_{\mathbb{R}^N} Q|u|^pdx \\
&\le \frac{-2\left(\widetilde{\lambda}_{1,\mu}(f)-\lambda\right)}{\widetilde{\lambda}_{1,\mu}(f)}\|u\|_\mu^2 + \frac{\|Q\|_\infty(4-p)\left|\{V<c\}\right|^{(2^*-p)/2^*}}{S^p}\|u\|_\mu^p \\
&< 0 \text{ for all } u \in \mathbf{N}_{\mu,\lambda}^{(1)}.
\end{aligned}$$

By (6.9), one has

$$\begin{aligned}
\frac{1}{4}\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) - \frac{4-p}{4p}\int_{\mathbb{R}^N} Q|u|^pdx = J_{\mu,\lambda}(u) &< \frac{p-2}{4p}C(p)K^p(\mu) \\
&< \frac{p-2}{4p}\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right)
\end{aligned}$$

for all $u \in \mathbf{N}_{\mu,\lambda}^{(2)}$, which implies that

$$2\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) < (4-p)\int_{\mathbb{R}^N} Q|u|^pdx \text{ for all } u \in \mathbf{N}_{\mu,\lambda}^{(2)}. \tag{6.14}$$

Applying (2.2) and (6.14) leads to

$$h_u''(1) = -2\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) + (4-p)\int_{\mathbb{R}^N} Q|u|^pdx > 0 \text{ for all } u \in \mathbf{N}_{\mu,\lambda}^{(2)}.$$

Moreover, by Theorem 3.8 (i), there exist $t_a^\pm > 0$ such that $t_a^- w_{\lambda,\Omega} \in \mathbf{N}_{\mu,\lambda}^{(1)}$ and $t_a^+ w_{\lambda,\Omega} \in \mathbf{N}_{\mu,\lambda}^{(2)}$. Namely, $\mathbf{N}_{\mu,\lambda}^{(i)}$ are nonempty. Hence, we obtain the following result.

Lemma 6.4. *Suppose that* $2 < p < \min\{4, 2^*\}$ *and conditions* $(V_1) - (V_3), (D_1) - (D_2)$ *and* (D_4) *hold. Then there exists* $a_0 > 0$ *such that for every* $0 < a < a_0$ *and* $0 < \lambda < \left[1 - 2\left(\frac{4-p}{4}\right)^{2/p}\right]\lambda_1(f_\Omega)$, $\mathbf{N}_{\mu,\lambda}^{(1)} \subset \mathbf{N}_{\mu,\lambda}^{-}$ *and* $\mathbf{N}_{\mu,\lambda}^{(2)} \subset \mathbf{N}_{\mu,\lambda}^{+}$ *are* C^1 *nonempty sub-manifolds. Furthermore, each local minimizer of the functional* $J_{\mu,\lambda}$ *in the sub-manifolds* $\mathbf{N}_{\mu,\lambda}^{(1)}$ *and* $\mathbf{N}_{\mu,\lambda}^{(2)}$ *is a critical point of* $J_{\mu,\lambda}$ *in* X_μ.

Define

$$\alpha_{\mu,\lambda}^- = \inf_{u\in\mathbf{N}_{\mu,\lambda}^{(1)}} J_{\mu,\lambda}(u) = \inf_{u\in\mathbf{N}_{\mu,\lambda}} J_{\mu,\lambda}(u).$$

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It follows from Lemma 6.3 and (6.11) that

$$0 < d_\mu < \alpha^-_{\mu,\lambda} < \frac{p-2}{4p} C(p) K^p(\mu) \le \frac{p-2}{4p} C(p) \left(\frac{\lambda_1(f_\Omega) - \lambda}{\lambda_1(f_\Omega)} \right)^{p/(p-2)}. \tag{6.15}$$

By the Ekeland variational principle [17], there exists a sequence $\{u_n\} \subset \mathbf{N}^{(1)}_{\mu,\lambda}$ such that

$$J_{\mu,\lambda}(u_n) = \alpha^-_{\mu,\lambda} + o(1) \text{ and } J'_{\mu,\lambda}(u_n) = o(1) \text{ in } X^{-1}_\mu. \tag{6.16}$$

We are now ready to prove Theorem 1.8: By (6.10), (6.15), (6.16) and Proposition 2.5, for each $0 < a < a_0$ we can obtain that $J_{\mu,\lambda}$ satisfies the $(\mathrm{PS})_{\alpha^-_{\mu,\lambda}}$–condition in X_μ for $\mu > 0$ sufficiently large. Thus, there exist a subsequence $\{u_n\}$ and $u^-_{\mu,\lambda} \in X_\mu$ such that $u_n \to u^-_{\mu,\lambda}$ strongly in X_μ for $\mu > 0$ sufficiently large. Hence, $u^-_{\mu,\lambda}$ is a minimizer for $J_{\mu,\lambda}$ on $\mathbf{N}^{(1)}_{\mu,\lambda}$. Note that

$$0 < \alpha^-_{\mu,\lambda} = J_{\mu,\lambda}\left(u^-_{\mu,\lambda}\right) < \frac{p-2}{4p} C(p) \left(\frac{\lambda_1(f_\Omega) - \lambda}{\lambda_1(f_\Omega)} \right)^{p/(p-2)},$$

which implies that $u^-_{\mu,\lambda} \in \mathbf{N}^{(1)}_{\mu,\lambda}$. Since $|u^-_{\mu,\lambda}| \in \mathbf{N}^{(1)}_{\mu,\lambda}$ and $J_{\mu,\lambda}\left(\left|u^-_{\mu,\lambda}\right|\right) = J_{\mu,\lambda}\left(u^-_{\mu,\lambda}\right) = \alpha^-_{\mu,\lambda}$, one can see that $u^-_{\mu,\lambda}$ is a positive solution for Equation $(E_{\mu,\lambda})$ by Lemma 6.3.

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References

[1] H. Amann, J. Lopez-Gomez, A priori bounds and multiple solutions for superlinear indefinite elliptic problems, J. Differ. Equ. 146 (1998) 336–374.
[2] C.O. Alves, G.M. Figueiredo, Multi-bump solutions for a Kirchhoff-type problem, Adv. Nonlinear Anal. 5 (2016) 1–26.
[3] C.O. Alves, A.B. Nobrega, Multi-bump solutions for Choquard equation with deepening potential well, Calc. Var. Partial Differ. Equ. 55 (2016) 48.
[4] P.A. Binding, P. Drábek, Y.X. Huang, On Neumann boundary value problems for some quasilinear elliptic equations, Electron. J. Differ. Equ. 5 (1997) 1–11.
[5] P.A. Binding, P. Drábek, Y.X. Huang, Existence of multiple solutions of critical quasilinear elliptic Neumann problems, Nonlinear Anal. 42 (2000) 613–629.
[6] H. Brezis, E. Lieb, A relation between point convergence of functions and convergence of functionals, Proc. Am. Math. Soc. 88 (1983) 486–490.
[7] T. Bartsch, A. Pankov, Z.-Q. Wang, Nonlinear Schrödinger equations with steep potential well, Commun. Contemp. Math. 3 (2001) 549–569.
[8] T. Bartsch, Z. Tang, Multibump solutions of nonlinear Schrödinger equations with steep potential well and indefinite potential, Discrete Contin. Dyn. Syst., Ser. A 33 (2013) 7–26.
[9] T. Bartsch, Z.-Q. Wang, Existence and multiplicity results for superlinear elliptic problems on $\mathbb{R}^N$, Commun. Partial Differ. Equ. 20 (1995) 1725–1741.

[10] K.J. Brown, Y. Zhang, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, J. Differ. Equ. 193 (2003) 481–499.
[11] J. Chabrowski, D.G. Costa, On a class of Schrödinger-type equations with indefinite weight functions, Commun. Partial Differ. Equ. 33 (2008) 1368–1394.
[12] C. Chen, Y. Kuo, T.F. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, J. Differ. Equ. 250 (2011) 1876–1908.
[13] D.G. Costa, H. Tehrani, Existence of positive solutions for a class of indefinite elliptic problems in $\mathbb{R}^N$, Calc. Var. Partial Differ. Equ. 13 (2001) 159–189.
[14] P. Drábek, S.I. Pohozaev, Positive solutions for the p-Laplacian: application of the fibrering method, Proc. R. Soc. Edinb. 127 (1997) 703–726.
[15] Y. Deng, S. Peng, W. Shuai, Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$, J. Funct. Anal. 269 (2015) 3500–3527.
[16] Y. Deng, W. Shuai, Sign-changing multi-bump solutions for Kirchhoff-type equations in $\mathbb{R}^3$, Discrete Contin. Dyn. Syst., Ser. A 38 (2018) 3139–3168.
[17] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Springer, 1990.
[18] G.M. Figueiredo, N. Ikoma, J.R.S. Júnior, Existence and concentration result for the Kirchhoff type equations with general nonlinearities, Arch. Ration. Mech. Anal. 213 (2014) 931–979.
[19] Z. Guo, Ground states for Kirchhoff equations without compact condition, J. Differ. Equ. 259 (2015) 2884–2902.
[20] X. He, W. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$, J. Differ. Equ. 252 (2012) 1813–1834.
[21] N. Ikoma, Existence of ground state solutions to the nonlinear Kirchhoff type equations with potentials, Discrete Contin. Dyn. Syst. 35 (2015) 943–966.
[22] H. Jia, X. Luo, Existence and concentrating behavior of solutions for Kirchhoff type equations with steep potential well, J. Math. Anal. Appl. 467 (2018) 893–915.
[23] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.
[24] J.L. Lions, On some questions in boundary value problems of mathematical physics, in: Contemporary Developments in Continuum Mechanics and Partial Differential Equations, in: North-Holl. Math. Stud., vol. 30, North-Holland, Amsterdam, New York, 1978, pp. 284–346.
[25] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part I, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 1 (1984) 109–145.
[26] G. Li, H. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, J. Differ. Equ. 257 (2014) 566–600.
[27] Z. Nehari, On a class of nonlinear second-order differential equations, Trans. Am. Math. Soc. 95 (1960) 101–123.
[28] S.I. Pohozaev, A certain class of quasilinear hyperbolic equations, Mat. Sb. (N.S.) 96 (138) (1975) 152–166, 168 (in Russian).
[29] J. Sun, Y.H. Cheng, T.F. Wu, Z. Feng, Positive solutions of a superlinear Kirchhoff type equation in $\mathbb{R}^N$ $(N \geq 4)$, Commun. Nonlinear Sci. Numer. Simul. 71 (2019) 141–160.
[30] J. Sun, T.F. Wu, Ground state solutions for an indefinite Kirchhoff type problem with steep potential well, J. Differ. Equ. 256 (2014) 1771–1792.
[31] J. Sun, T.F. Wu, Existence and multiplicity of solutions for an indefinite Kirchhoff-type equation in bounded domains, Proc. R. Soc. Edinb., Sect. A 146 (2016) 435–448.
[32] J. Sun, T.F. Wu, Steep potential well may help Kirchhoff type equations to generate multiple solutions, Nonlinear Anal. 190 (2020) 111609.
[33] C. Stuart, H. Zhou, Global branch of solutions for nonlinear Schrödinger equations with deepening potential well, Proc. Lond. Math. Soc. 92 (2006) 655–681.
[34] X. Tang, S. Chen, Ground state solutions of Nehari–Pohozaev type for Kirchhoff-type problems with general potentials, Calc. Var. Partial Differ. Equ. 56 (2017) 110.
[35] N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Commun. Pure Appl. Math. 20 (1967) 721–747.
[36] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.
[37] Y. Wu, Y. Huang, Z. Liu, Kirchhoff type problems with potential well and indefinite potential, Electron. J. Differ. Equ. 2016 (178) (2016) 1–13.
[38] Z. Wang, H. Zhou, Positive solutions for nonlinear Schrödinger equations with deepening potential well, J. Eur. Math. Soc. 11 (2009) 545–573.
[39] Q. Xie, S. Ma, Existence and concentration of positive solutions for Kirchhoff-type problems with a steep well potential, J. Math. Anal. Appl. 431 (2015) 1210–1223.

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