JOURNAL OF DIFFERENTIAL EQUATIONS **156**, 309–327 (1999)
Article ID jdeq.1998.3610, available online at http://www.idealibrary.com on IDEAL®

Almost Periodicity of Mild Solutions of Inhomogeneous Periodic Cauchy Problems*

Charles J. K. Batty

St. John's College, Oxford OX1 3JP, England
E-mail: charles.batty@sjc.ox.ac.uk

and

Walter Hutter and Frank Räbiger

Universität Tübingen, Mathematisches Institut, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail: wahu@michelangelo.mathematik.uni-tuebingen.de, frra@michelangelo.mathematik.uni-tuebingen.de

Received June 15, 1998; revised November 4, 1998

We consider a mild solution u of a well-posed, inhomogeneous, Cauchy problem, $\dot{u}(t) = A(t)\,u(t) + f(t)$, on a Banach space X, where $A(\cdot)$ is periodic. For a problem on $\mathbf{R}^+$, we show that u is asymptotically almost periodic if f is asymptotically almost periodic, u is bounded, uniformly continuous and totally ergodic, and the spectrum of the monodromy operator V contains only countably many points of the unit circle. For a problem on $\mathbf{R}$, we show that a bounded, uniformly continuous solution u is almost periodic if f is almost periodic and various supplementary conditions are satisfied. We also show that there is a unique bounded solution subject to certain spectral assumptions on V, f and u.

Key Words: inhomogeneous; periodic; Cauchy problem; evolution family; almost periodic; countable; spectrum; monodromy operator; totally ergodic.

1. INTRODUCTION

For a well-posed Cauchy problem

$$\dot{u}(t) = A(t)\,u(t) \quad (t \geqslant 0), \quad u(0) = x \in X, \tag{1.1}$$

* We are grateful to the referee for reading the first version of this paper very carefully and pointing out several defects.

0022-0396/99 $30.00

on a complex Banach space X with (unbounded) linear operators $A(t)$ ($t \in \mathbf{R}^+$), the solutions of (1.1) lead to an *evolution family* $\mathcal{U} = \{U(t, s) : t \geqslant s \geqslant 0\}$ in the space $\mathcal{L}(X)$ of bounded linear operators on X, i.e.,

(1) $U(t, t) = I$, $U(t, r)\, U(r, s) = U(t, s)$ for $t \geqslant r \geqslant s$ in $\mathbf{R}^+$,

(2) $\{(t, s) \in \mathbf{R}^+ \times \mathbf{R}^+ : t \geqslant s\} \to \mathcal{L}(X) : (t, s) \mapsto U(t, s)$ is strongly continuous,

(3) there are constants $M \geqslant 1$ and $\omega \in \mathbf{R}$ such that $\|U(t, s)\| \leqslant Me^{\omega(t-s)}$, for $t \geqslant s$ in $\mathbf{R}^+$.

We refer to [11], [18], [26] for conditions implying the existence of an evolution family. For a function $f: \mathbf{R}^+ \to X$ a *mild solution* of the inhomogeneous Cauchy problem

$$\dot{u}(t) = A(t)\, u(t) + f(t) \quad (t \geqslant 0),$$

is defined by

$$u(t) = U(t, 0)\, u(0) + \int_0^t U(t, r) f(r)\, dr \quad (t \geqslant 0). \tag{1.2}$$

When the Cauchy problem (1.1) is periodic, i.e., there exists $q > 0$ such that $A(t + q) = A(t)$ for $t \in \mathbf{R}^+$, the corresponding evolution family $\mathcal{U}$ is periodic in the following sense

$$U(t + q, s + q) = U(t, s) \quad (t \geqslant s \geqslant 0). \tag{1.3}$$

In the present paper we study the asymptotic behaviour of an individual mild solution u depending on properties of the inhomogeneity f. We shall be concerned with q-periodic evolution families according to the above definition, without assuming the existence of a related Cauchy problem. In particular, we deduce almost periodicity properties of the function u from almost periodicity properties of the inhomogeneity f in conjunction with spectral conditions on the monodromy operator $V = U(q, 0)$ of the evolution family $\mathcal{U}$. Vũ [28, Theorem 3.2] showed that a bounded, uniformly continuous solution u is asymptotically almost periodic when $f \equiv 0$, assuming that V is power-bounded, the intersection of the spectrum $\sigma(V)$ of V with the unit circle Γ is countable, and γV is mean-ergodic for every $\gamma \in \Gamma$. Other results in this area have been obtained by Ruess and Summers [23] and Kreulich [13]. In Section 2, we shall generalise Vũ's result to inhomogeneous problems on $\mathbf{R}^+$. Our approach is based on a modification of a factorisation technique developed in [2], where the corresponding question was discussed in the autonomous situation.

Given a q-periodic evolution family $\mathscr{U} = \{U(t, s) : t \geqslant s \geqslant 0\}$, there is an extension to an evolution family $\mathscr{U} = \{U(t, s) : t \geqslant s \in \mathbf{R}\}$ such that

$$U(t+q, s+q) = U(t, s) \quad (t \geqslant s \in \mathbf{R}).$$

Given a function f: $\mathbf{R} \to X$, one can consider solutions of the equation

$$u(t) = U(t, s)\, u(s) + \int_s^t U(t, r)\, f(r)\, dr \quad (t \geqslant s \in \mathbf{R}) \tag{1.4}$$

as corresponding to complete mild solutions of the inhomogeneous Cauchy problem on $\mathbf{R}$:

$$\dot{u}(t) = A(t)\, u(t) + f(t) \quad (t \in \mathbf{R}).$$

If f is almost periodic and the evolution family has a Floquet representation, Vũ [28, Theorems 4.2 and 4.5] showed firstly that a bounded, uniformly continuous, totally ergodic solution u of (1.4) is almost periodic if $\sigma(V) \cap \Gamma$ is countable, and secondly that there is a unique almost periodic solution subject to some other conditions when there is an absence of resonance between V and f. The latter result has also been proved by Naito and Nguyen [16] without assuming a Floquet representation, but instead assuming that $t \mapsto U(t+q, t)$ is norm-continuous. In Section 3, we shall prove such results without assuming the existence of a Floquet representation or any norm-continuity. We shall also give a periodic version of a recent result of Arendt and Schweiker [3].

2. SOLUTIONS ON THE HALF-LINE

We begin by recalling some notation and terminology from [2].

Let $\mathrm{BUC}(\mathbf{R}^+, X)$ be the space of all bounded, uniformly continuous functions from $\mathbf{R}^+$ to a complex Banach space X. Let $\mathscr{S} = \{S(t) : t \geqslant 0\}$ be the $\mathscr{C}_0$-semigroup of translations on $\mathrm{BUC}(\mathbf{R}^+, X)$ given by $(S(t)\, f)(s) = f(s+t)$. Denote by D the generator of $\mathscr{S}$. Consider the quotient space

$$Y_0 = \mathrm{BUC}(\mathbf{R}^+, X)/C_0(\mathbf{R}^+, X),$$

and let $\pi_0 : \mathrm{BUC}(\mathbf{R}^+, X) \to Y_0$ be the quotient map, so

$$\|\pi_0 f\| = \inf \{ \|f - g\| : g \in C_0(\mathbf{R}^+, X)\} = \limsup_{t \to \infty} \|f(t)\|.$$

Then $\mathscr{S}$ induces a $\mathscr{C}_0$-group $\mathscr{S}_0 = \{S_0(t): t \in \mathbf{R}\}$ on Y_0 consisting of isometries. A closed subspace $\mathscr{F}$ of $\mathrm{BUC}(\mathbf{R}^+, X)$ is said to be *translation-biinvariant* if $\mathscr{F} = \{f \in \mathrm{BUC}(\mathbf{R}^+, X) : S(t)\, f \in \mathscr{F}\}$ for each $t \geqslant 0$, or, equivalently, if $\mathscr{F}$ contains $C_0(\mathbf{R}^+, X)$ and $\mathscr{F}_0 := \mathscr{F}/C_0(\mathbf{R}^+, X) \subseteq Y_0$ is

$\mathscr{S}_0$-invariant. A discussion of these properties, and some examples, can be found in [2].

For $\eta \in \mathbf{R}$, let

$$M_\eta(u) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{-i\eta t} S(t)\, u\, dt$$

if this exists in BUC($\mathbf{R}^+, X$). When $M_\eta(u)$ exists, there exists $x \in X$ such that $M_\eta(u)(t) = e^{i\eta t} x$ for all t. As in [2], we say that u is *uniformly ergodic* at $i\eta$ if $M_\eta(u)$ exists, and that u is *totally ergodic* if u is uniformly ergodic at every point of $i\mathbf{R}$.

For later use, we mention the following lemma, the proof of which is given in [1, Lemma 2.2] (see also [6, Theorem 2.2]).

LEMMA 2.1. *Let A be the generator of a $\mathscr{C}_0$-group $\mathscr{W}$ of isometries on the Banach space Z. Let $z \in Z$, $\xi \in \mathbf{R}$, and suppose that there exist a neighbourhood G of $i\xi$ in $\mathbf{C}$ and a holomorphic function h: $G \to Z$ such that $h(\lambda) = R(\lambda, A)\, z$ whenever $\lambda \in G$ and* $\operatorname{Re} \lambda > 0$. *Then $i\xi \in \rho(A_z)$, where A_z is the generator of the restriction of $\mathscr{W}$ to the closed linear span of $\{W(t)\, z$: $t \in \mathbf{R}\}$ in Z.*

For the remainder of this section, $\mathscr{U}$ will be a q-periodic evolution family, so that property (1.3) holds. Denote by V the *monodromy operator* $U(q, 0)$ of $\mathscr{U}$. For notational convenience we set $U_s(t, r) = U(t+s, r+s)$ and $V_s = U_s(q, 0)$ for $t \geqslant r$ and $s \in \mathbf{R}^+$. Furthermore, we define $f_s(t) = f(t+s)$ for $s, t \geqslant 0$ and $f \in$ BUC($\mathbf{R}^+, X$). Note that $V_{s+q} = V_s$ and $\sigma(V_s) \backslash \{0\}$ is independent of $s \in \mathbf{R}^+$ [10, Proposition 6.3] and that for $\lambda \in \rho(V)$ the mapping $s \mapsto R(\lambda, V_s)$ from $\mathbf{R}^+$ into $\mathscr{L}(X)$ is q-periodic and strongly continuous [21, proof of Proposition 12].

LEMMA 2.2. *Let $\mathscr{F}$ be a closed subspace of* BUC($\mathbf{R}^+, X$), *let $f \in \mathscr{F}$, and suppose that*

(1) *f has relatively compact range,*

(2) *if $B \in \mathscr{L}(X)$ and $n \in \mathbf{Z}$, then $s \mapsto \exp(2\pi i n s/q)\, Bf(s)$ belongs to $\mathscr{F}$.*

Let T: $\mathbf{R}^+ \to \mathscr{L}(X)$ be strongly continuous and q-periodic. Then $s \mapsto T(s)\, f(s)$ belongs to $\mathscr{F}$.

Proof. Let T_n be the nth Cesàro mean of the Fourier series of T, so T_n is a q-periodic trigonometric polynomial with values in $\mathscr{L}(X)$ and $\|T_n(s)\| \leqslant \sup_{0 \leqslant t \leqslant q} \|T(t)\|$. By Fejér's Theorem, $T_n(s)\, x \to T(s)\, x$, uniformly for $s \geqslant 0$, for each $x \in X$. Since f has relatively compact range, $T_n(s)\, f(s) \to T(s)\, f(s)$ uniformly for $s \geqslant 0$. By (2), $T_n(\cdot)\, f(\cdot) \in \mathscr{F}$, so $T(\cdot)\, f(\cdot) \in \mathscr{F}$. ∎

LEMMA 2.3. *Let $\mathscr{U} = \{U(t, s) : t \geqslant s \geqslant 0\}$ be a q-periodic evolution family on the Banach space X and let $\mathscr{F}$ be a closed translation-invariant subspace of* $\mathrm{BUC}(\mathbf{R}^+, X)$. *Let $f \in \mathscr{F}$, $e^{\lambda q} \in \rho(V)$, and define*

$$w_f(s) = \int_0^q U_s(q, r)\, f_s(r)\, dr, \qquad r_{\lambda, f}(s) = R(e^{\lambda q}, V_s)\, w_f(s) \tag{2.1}$$

for $s \geqslant 0$. Suppose that

(1) *f has relatively compact range,*

(2) *if $g \in \mathscr{F}$, $B \in \mathscr{L}(X)$ and $n \in \mathbf{Z}$, then $s \mapsto \exp(2\pi i n s/q)\, Bg(s)$ belongs to $\mathscr{F}$.*

Then $w_f \in \mathscr{F}$ and $r_{\lambda, f} \in \mathscr{F}$.

Proof. Replacing $\mathscr{F}$ by the subspace of all functions in $\mathscr{F}$ with relatively compact range, we may assume that every function in $\mathscr{F}$ has relatively compact range.

For $r \in [0, q]$ and $s \geqslant 0$, let

$$R_f(r)(s) = U_s(q, r)\ f_s(r).$$

By Lemma 2.2, $R_f(r) \in \mathscr{F}$.

Let $r \in [0, q]$ and $v \geqslant 0$ such that $r + v \in [0, q]$. Then

$$\begin{aligned} &\| U_s(q, r+v)\, f_s(r+v) - U_s(q, r)\, f_s(r) \| \\ &\qquad \leqslant \| U_s(q, r+v) \| (\| f_s(r+v) - f_s(r) \| + \| (I - U_s(r+v, r))\, f_s(r) \|) \\ &\qquad \to 0 \end{aligned}$$

as $v \to 0$, uniformly for $s \geqslant 0$, since f has relatively compact range and $\mathscr{U}$ is q-periodic and strongly continuous. Thus R_f is continuous from the right. A similar argument leads to continuity of R_f from the left. Hence, $w_f = \int_0^q R_f(r)\, dr \in \mathscr{F}$. Since $s \mapsto R(e^{\lambda q}, V_s)$ is strongly continuous and q-periodic, $r_{\lambda, f} \in \mathscr{F}$, by Lemma 2.2. ∎

We are now in a position to formulate the main result of this section.

THEOREM 2.4. *Let $\mathscr{U} = \{U(t, s) \colon t \geqslant s \geqslant 0\}$ be a q-periodic evolution family on the Banach space X, and suppose that $\sigma(V) \cap \Gamma$ is countable. Let $f \in \mathrm{BUC}(\mathbf{R}^+, X)$ with relatively compact range, and suppose that $u \in \mathrm{BUC}(\mathbf{R}^+, X)$ satisfies* (1.2) *and that u is uniformly ergodic at $i\eta$ whenever $e^{i\eta q} \in \sigma(V) \cap \Gamma$. Let $\mathscr{F}$ be a closed, translation-biinvariant subspace of* $\mathrm{BUC}(\mathbf{R}^+, X)$, *satisfying the following conditions*:

(1) $f \in \mathscr{F}$,

(2) $M_\eta(u) \in \mathscr{F}$ *whenever* $e^{i\eta q} \in \sigma(V) \cap \Gamma$,

(3) *if* $g \in \mathscr{F}$, $B \in \mathscr{L}(X)$ *and* $n \in \mathbf{Z}$, *then* $s \mapsto \exp(2\pi ins/q)\, Bg(s)$ *belongs to* $\mathscr{F}$.

Then $u \in \mathscr{F}$.

Proof. Let $G := \{\lambda \in \mathbf{C} : e^{\lambda q} \in \rho(V)\}$. For $\lambda \in G$, let $r_{\lambda, f}$ be defined by (2.1), and let

$$H(\lambda)(s) := \int_0^q e^{-\lambda t} e^{\lambda q} R(e^{\lambda q}, V_{s+t})\, u_s(t)\, dt \quad (s \geqslant 0).$$

By Lemma 2.3, $r_{\lambda, f} \in \mathscr{F}$. By the Dominated Convergence Theorem, H maps G into the space $C^b(\mathbf{R}^+, X)$ of bounded, continuous functions from $\mathbf{R}^+$ to X. Moreover, H is locally bounded, and for each $s \geqslant 0$, the map $\lambda \mapsto H(\lambda)(s)$ is holomorphic. It follows from Cauchy's Integral Formula that $H : G \to C^b(\mathbf{R}^+, X)$ is continuous, and from Morera's Theorem that H is holomorphic. We shall show below that H actually takes values in $\mathrm{BUC}(\mathbf{R}^+, X)$.

We first establish a local description of the resolvent of the generator D of the translation semigroup $\mathscr{S}$. Note that

$$u_s(t) = U_s(t, 0)\, u(s) + \int_0^t U_s(t, r)\ f_s(r)\, dr$$

for $s, t \geqslant 0$, and hence

$$u_s(t+q) - V_{s+t} u_s(t) = \int_t^{t+q} U_s(t+q, r)\ f_s(r)\, dr = \int_0^q U_{s+t}(q, r)\ f_{s+t}(r)\, dr.$$

Let $\operatorname{Re} \lambda > 0$. Then $R(\lambda, D)$ exists and is given by the Laplace transform of the semigroup $\mathscr{S}$, so

$$(R(\lambda, D)\, u)(s) = \int_0^\infty e^{-\lambda t} (S(t)\, u)(s)\, dt = \int_0^\infty e^{-\lambda t} u_s(t)\, dt \quad (s \geqslant 0).$$

Assume, in addition, that $\lambda \in G$. Then, for $s \geqslant 0$,

$$\begin{aligned}
(R(\lambda, D)\, u)(s) &= \int_0^\infty e^{-\lambda t} R(e^{\lambda q}, V_{s+t})(e^{\lambda q} - V_{s+t})\, u_s(t)\, dt \\
&= H(\lambda)(s) + \int_0^\infty e^{-\lambda t} R(e^{\lambda q}, V_{s+t})(u_s(t+q) - V_{s+t} u_s(t))\, dt \\
&= H(\lambda)(s) + \int_0^\infty e^{-\lambda t} R(e^{\lambda q}, V_{s+t}) \int_0^q U_{s+t}(q, r)\ f_{s+t}(r)\, dr\, dt
\end{aligned}$$

$$= H(\lambda)(s) + \int_0^\infty e^{-\lambda t} r_{\lambda, f}(s+t)\, dt$$

$$= H(\lambda)(s) + (R(\lambda, D)\, r_{\lambda, f})(s). \tag{2.2}$$

It follows that

$$H(\lambda) = R(\lambda, D)\, u - R(\lambda, D)\, r_{\lambda, f} \in \mathrm{BUC}(\mathbf{R}^+, X)$$

whenever $\operatorname{Re} \lambda > 0$ and $\lambda \in G$. By analytic continuation, $H(\lambda) \in \mathrm{BUC}(\mathbf{R}^+, X)$ whenever λ belongs to the union G_0 of the connected components of G which intersect $\{\lambda \in \mathbf{C} : \operatorname{Re} \lambda > 0\}$.

The group $\mathscr{S}_0$ on $Y_0 = \mathrm{BUC}(\mathbf{R}^+, X)/C_0(\mathbf{R}^+, X)$ induces a $\mathscr{C}_0$-group $\mathscr{S}_{\mathscr{F}}$ on

$$Y_{\mathscr{F}} := \mathrm{BUC}(\mathbf{R}^+, X)/\mathscr{F} = Y_0/\mathscr{F}_0,$$

and $S_{\mathscr{F}}(t)\, \pi_{\mathscr{F}} f = \pi_{\mathscr{F}} S(t) f$ for all $f \in \mathrm{BUC}(\mathbf{R}^+, X)$, where $\pi_{\mathscr{F}}: \mathrm{BUC}(\mathbf{R}^+, X) \to Y_{\mathscr{F}}$ is the quotient map. Let $D_{\mathscr{F}}$ be the generator of $\mathscr{S}_{\mathscr{F}}$. Since $\mathscr{F}$ contains $r_{\lambda, f}$ and is translation-invariant, $R(\lambda, D)\, r_{\lambda, f} \in \mathscr{F}$, so

$$R(\lambda, D_{\mathscr{F}})\, \pi_{\mathscr{F}} u = \pi_{\mathscr{F}} R(\lambda, D)\, u = \pi_{\mathscr{F}}(H(\lambda))$$

whenever $\operatorname{Re} \lambda > 0$ and $\lambda \in G$. This shows that $\lambda \mapsto R(\lambda, D_{\mathscr{F}})\, \pi_{\mathscr{F}} u$ has a holomorphic extension to a map $g: G_0 \to Y_{\mathscr{F}} = \mathrm{BUC}(\mathbf{R}^+, X)/\mathscr{F}$, given by $g(\lambda) = \pi_{\mathscr{F}}(H(\lambda))$. Let $Z_{\mathscr{F}, u}$ be the closed linear span of $\{S_{\mathscr{F}}(t)\, \pi_{\mathscr{F}} u : t \in \mathbf{R}\}$ in $Y_{\mathscr{F}}$. By Lemma 2.1,

$$\sigma(D_{\mathscr{F}, u}) \subseteq \{\lambda \in i\mathbf{R} : e^{\lambda q} \in \sigma(V)\}$$

where $D_{\mathscr{F}, u}$ is the generator of the restriction of $\mathscr{S}_{\mathscr{F}}$ to $Z_{\mathscr{F}, u}$. In particular, $\sigma(D_{\mathscr{F}, u})$ is countable.

To finish the proof, suppose that $u \notin \mathscr{F}$. Then $Z_{\mathscr{F}, u}$ is non-zero and therefore the spectrum $\sigma(D_{\mathscr{F}, u})$ is nonempty [15, Lemma 7.6, p. 91]. Since $\sigma(D_{\mathscr{F}, u})$ is countable and closed in $i\mathbf{R}$, it has an isolated point $i\eta$. By [9, Theorem 8.16], this point is an eigenvalue. So, there is a non-zero $z \in Z_{\mathscr{F}, u}$ such that $S_{\mathscr{F}}(t)\, z = e^{i\eta t} z$ for all $t \in \mathbf{R}$. From assumption (2), we know that

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{-i\eta t} S(t)\, u_s\, dt = e^{i\eta s} M_\eta(u) \in \mathscr{F}.$$

Applying $\pi_{\mathscr{F}}$, taking linear combinations and interchanging limits, it follows that

$$0 \neq z = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{-i\eta t} S_{\mathscr{F}}(t)\, z\, dt = 0.$$

This contradiction proves the result. ∎

Remark 2.5. 1. The formula (2.2) for $R(\lambda, D)\,u$ is valid whenever $\mathscr{U}$ is q-periodic and u satisfies (1.2). It is based on the variation of constants formula for u and can be written explicitly as

$$(R(\lambda, D)\,u)(s) = \int_0^q e^{-\lambda t} e^{\lambda q} R(e^{\lambda q}, V_{s+t})\, u_s(t)\, dt$$

$$+ \int_0^\infty e^{-\lambda t} R(e^{\lambda q}, V_{s+t}) \int_0^q U_{s+t}(q, r)\; f_{s+t}(r)\, dr\, dt. \qquad (2.3)$$

Moreover, $(R(\lambda, D)\,u)(s)$ is the Laplace transform of u_s and if $\eta \in \mathbf{R}$ the existence of the Cesàro mean $M_\eta(u)$ is equivalent (for $u \in \mathrm{BUC}(\mathbf{R}^+, X)$) to the existence of the Abel mean $\lim_{\alpha \downarrow 0} \alpha R(\alpha + i\eta, D)\, u$ in $\mathrm{BUC}(\mathbf{R}^+, X)$. It follows that if $e^{i\eta q} \in \rho(V)$ and $r_{i\eta, f}$ is uniformly ergodic at $i\eta$, then u is uniformly ergodic at $i\eta$ and $M_\eta(u) = M_\eta(r_{i\eta, f})$. An application of Lemma 2.3 shows that $r_{i\eta, f}$ is uniformly ergodic at $i\eta$ if f has relatively compact range and f is uniformly ergodic at $i\eta'$ whenever $\eta' - \eta \in (2\pi/q)\,\mathbf{Z}$.

2. In Theorem 2.4, the conditions (1) and (3) and the assumption that f has relatively compact range can be replaced by the assumption that $r_{\lambda, f} \in \mathscr{F}$ whenever $e^{\lambda q} \in \rho(V)$.

When $f \equiv 0$, we obtain the following corollary, which is a generalization of a result of Vũ [28, Theorem 3.2], as mentioned in the introduction. Note that the ergodicity condition on V assumed in that theorem implies that $U(\cdot, 0)\, x$ is totally ergodic for all $x \in X$.

Corollary 2.6. *Let $\mathscr{U} = \{U(t, s) : t \geqslant s \geqslant 0\}$ be a q-periodic evolution family on the Banach space X, and suppose that $\sigma(V) \cap \Gamma$ is countable. Let $x \in X$ and consider the function $u = U(\cdot, 0)\, x : \mathbf{R}^+ \to X$. Assume that $u \in \mathrm{BUC}(\mathbf{R}^+, X)$ and that u is uniformly ergodic at $i\eta$ whenever $e^{i\eta q} \in \sigma(V) \cap \Gamma$. Let $\mathscr{F}$ be a closed, translation-biinvariant subspace of $\mathrm{BUC}(\mathbf{R}^+, X)$ and suppose that $\mathscr{F}$ contains $M_\eta(u)$ whenever $e^{i\eta q} \in \sigma(V) \cap \Gamma$. Then $u \in \mathscr{F}$.*

We are now in a position to discuss almost periodicity properties of u. Let $\mathrm{AAP}(\mathbf{R}^+, X)$ be the space of all asymptotically almost periodic functions from $\mathbf{R}^+$ to X, so

$$\mathrm{AAP}(\mathbf{R}^+, X) = C_0(\mathbf{R}^+, X) \oplus \overline{\mathrm{span}}\{e^{i\eta\,\cdot}x \colon \eta \in \mathbf{R}, x \in X\}.$$

See [12, Theorem 9.3] for other characterisations of $\mathrm{AAP}(\mathbf{R}^+, X)$.

Proposition 2.7. (Asymptotic almost periodicity). *Let $\mathscr{U} = \{U(t, s) : t \geqslant s \geqslant 0\}$ be a q-periodic evolution family on the Banach space X and suppose that $\sigma(V) \cap \Gamma$ is countable. Let $f \in \mathrm{AAP}(\mathbf{R}^+, X)$, and consider a func-*

tion u which satisfies (1.2). *Assume that* $u \in \mathrm{BUC}(\mathbf{R}^+, X)$ *and that u is uniformly ergodic at* $i\eta$ *whenever* $e^{i\eta q} \in \sigma(V) \cap \Gamma$. *Then* $u \in \mathrm{AAP}(\mathbf{R}^+, X)$.

Proof. This follows from Theorem 2.4 with $\mathcal{F} = \mathrm{AAP}(\mathbf{R}^+, X)$. ▮

Remark 2.8. Proposition 2.7 remains valid if $\mathrm{AAP}(\mathbf{R}^+, X)$ is replaced by the space $\mathrm{WRC}(\mathbf{R}^+, X)$ of all Eberlein-weakly almost periodic functions with relatively compact range. See [22], [24] for further properties of these functions, and [13] for results related to this version of Proposition 2.7.

PROPOSITION 2.9. (Stability). *Let* $\mathcal{U} = \{U(t, s) : t \geqslant s \geqslant 0\}$ *be a q-periodic evolution family on the Banach space X, and suppose that* $\sigma(V) \cap \Gamma$ *is countable. Let* $f \in \mathrm{AAP}(\mathbf{R}^+, X)$, *and consider a function u which satisfies* (1.2). *Assume that* $u \in \mathrm{BUC}(\mathbf{R}^+, X)$ *and that one of the following two conditions is satisfied*:

(1) $f \in C_0(\mathbf{R}^+, X)$, *u is uniformly ergodic at* $i\eta$ *and* $M_\eta(u) = 0$ *whenever* $e^{i\eta q} \in \sigma(V) \cap \Gamma$;

(2) *u is totally ergodic and* $M_\eta(u) = 0$ *for all* $\eta \in \mathbf{R}$.

Then $u \in C_0(\mathbf{R}^+, X)$.

Proof. In case (1), this follows from Theorem 2.4 with $\mathcal{F} = C_0(\mathbf{R}^+, X)$. In case (2), it follows from Proposition 2.7 and the fact that any asymptotically almost periodic function whose means are all 0 belongs to $C_0(\mathbf{R}^+, X)$ (see [14, p. 24]). ▮

Now consider the space $\mathrm{AP}_q(\mathbf{R}^+, X)$ of all asymptotically q-periodic functions from $\mathbf{R}^+$ to X, as in [25, Section 6]. Thus

$$\begin{aligned} \mathrm{AP}_q(\mathbf{R}^+, X) &= C_0(\mathbf{R}^+, X) \oplus \{g \in \mathrm{BUC}(\mathbf{R}^+, X) : g \text{ is } q\text{-periodic}\} \\ &= C_0(\mathbf{R}^+, X) \oplus \overline{\mathrm{span}}\{e^{i(2\pi n/q)\cdot} x : n \in \mathbf{Z}, x \in X\} \\ &= \{f \in \mathrm{AAP}(\mathbf{R}^+, X) : M_\eta(f) = 0 \text{ unless } \eta \in (2\pi/q)\,\mathbf{Z}\}. \end{aligned}$$

PROPOSITION 2.10. (Asymptotic periodicity). *Let* $\mathcal{U} = \{U(t, s) : t \geqslant s \geqslant 0\}$ *be a q-periodic evolution family on the Banach space X, and suppose that* $\sigma(V) \cap \Gamma \subseteq \{1\}$. *Let* $f \in \mathrm{AP}_q(\mathbf{R}^+, X)$, *and consider a function u which satisfies* (1.2). *Assume that* $u \in \mathrm{BUC}(\mathbf{R}^+, X)$ *and that u is uniformly ergodic at* $i\eta$ *whenever* $\eta \in (2\pi/q)\,\mathbf{Z}$. *Then* $u \in \mathrm{AP}_q(\mathbf{R}^+, X)$.

Proof. This follows from Theorem 2.4 with $\mathcal{F} = \mathrm{AP}_q(\mathbf{R}^+, X)$. ▮

In applications, it is not easy to check whether individual solutions u have means $M_\eta(u)$. We shall show in Proposition 2.13 that this condition is automatically satisfied when $\sigma(V) \cap \Gamma$ consists only of poles of the resolvent of V. For this, we follow the approach in [28] via the sequences $(u(nq))_{n \geqslant 0}$, and we shall need the following two preliminary results, the first of which is similar to [12, Theorem 9.7]. Recall that a sequence (x_n) in X is said to be *asymptotically almost periodic* if $x_n = y_n + a_n$, where $\|y_n\| \to 0$ as $n \to \infty$, and for every $\varepsilon > 0$ there exist $\gamma_r \in \Gamma$ and $b_r \in X$ $(r = 1, 2, ..., m)$ such that $\|a_n - \sum_{r=1}^{m} \gamma_r^n b_r\| < \varepsilon$ for all n.

PROPOSITION 2.11. *Let $\mathscr{U} = \{U(t, s): t \geqslant s \geqslant 0\}$ be a q-periodic evolution family on the Banach space X. Let $f \in \mathrm{BUC}(\mathbf{R}^+, X)$, and consider a function u which satisfies* (1.2).

(1) *If $f \in C_0(\mathbf{R}^+, X)$ and $\|u(nq)\| \to 0$ as $n \to \infty$, then $u \in C_0(\mathbf{R}^+, X)$.*

(2) *If $f \in \mathrm{AAP}(\mathbf{R}^+, X)$ and $(u(nq))_{n \geqslant 0}$ is asymptotically almost periodic, then $u \in \mathrm{AAP}(\mathbf{R}^+, X)$.*

Proof. Note that the function u is continuous. For $0 \leqslant s \leqslant q$,

$$u(nq + s) = U(s, 0)\, u(nq) + \int_0^s U(s, r)\; f(nq + r)\, dr.$$

Now (1) follows immediately.

For (2), suppose that the sequence $(u(nq))_{n \geqslant 0}$ and the function f are both asymptotically almost periodic. By considering trigonometric polynomials approximating the almost periodic parts, it is straightforward to establish the following property of simultaneous ε-almost periods (see [12, Corollary 2.3 and pp. 163–164]). For any $\varepsilon > 0$, there exist non-negative integers M and l such that, for all non-negative integers k, there exists $m \in \{k, k+1, \cdots, k+l\}$ such that

$$\|u((n+m)\, q) - u(nq)\| < \varepsilon \text{ whenever } n \geqslant M, \text{ and}$$

$$\|f(t + mq) - f(t)\| < \varepsilon \text{ whenever } t \geqslant Mq.$$

For such m, and for $n \geqslant M$ and $0 \leqslant s \leqslant q$,

$$\begin{aligned}
&\|u(nq + s + mq) - u(nq + s)\| \\
&\quad = \|U(s, 0)(u((n+m)\, q) - u(nq)) + \int_0^s U(s, r)(f(nq + r + mq) - f(nq + r))\, dr\| \\
&\quad \leqslant C\|u((n+m)\, q) - u(nq)\| + Cq \sup_{t \geqslant nq} \|f(t + mq) - f(t)\| \\
&\quad \leqslant C(1 + q)\, \varepsilon,
\end{aligned}$$

where $C = \sup_{0 \leqslant r \leqslant s \leqslant q} \|U(s, r)\|$. Now mq is a $C(1+q)\, \varepsilon$-almost period for u. Thus $u \in \mathrm{AAP}(\mathbf{R}^+, X)$. ∎

With essentially the same proof as Proposition 2.11 (2), one can show that if $f \in \mathrm{AP}(\mathbf{R}, X)$, u is a solution of (1.4) on $\mathbf{R}$, and $(u(nq))_{n \in \mathbf{Z}}$ is an almost periodic sequence, then $u \in \mathrm{AP}(\mathbf{R}, X)$. This clarifies a question of Vũ [28, p. 411].

The following is a discrete analogue of a result given in [1, Theorem 5.2] for homogeneous autonomous Cauchy problems.

PROPOSITION 2.12. *Let $T \in \mathscr{L}(X)$ and suppose that $\sigma(T) \cap \Gamma$ consists only of poles of the resolvent of T. Let $x \in X$, and suppose that $(T^n x)_{n \geqslant 0}$ is bounded. Then $(T^n x)$ is an asymptotically almost periodic sequence.*

Proof. Let $h_m(\lambda) = \sum_{n=0}^{\infty} \lambda^n T^{n+m} x$ ($|\lambda| < 1$). When $\lambda^{-1} \in \rho(T)$, $h_m(\lambda) = (I - \lambda T)^{-1} T^m x$. If $\lambda \in \Gamma$ is a singular point of h_0, then λ^{-1} is a pole of the resolvent of T. Arguing as in [1, Theorem 5.2], the assumption that $(T^n x)$ is bounded implies that $\lim_{r \nearrow 1} (1-r) h_m(r\lambda)$ exists, uniformly for $m \geqslant 0$. The result now follows from [7, Theorem 6.1]. ∎

Now we are able to give an analogue for homogeneous periodic problems of the result of [1, Theorem 5.2].

PROPOSITION 2.13. *Let $\mathscr{U} = \{U(t, s) : t \geqslant s \geqslant 0\}$ be a q-periodic evolution family on the Banach space X, and suppose that $\sigma(V) \cap \Gamma$ consists only of poles of the resolvent of V. Let $x \in X$ and suppose that the function $u := U(\cdot, 0)\, x : \mathbf{R}^+ \to X$ is bounded. Then $u \in \mathrm{AAP}(\mathbf{R}^+, X)$.*

Proof. Consider the sequence $(u(nq))_{n \geqslant 0}$. Since $u(nq) = V^n x$, it follows from Proposition 2.12 that the sequence is asymptotically almost periodic. By Proposition 2.11, $u \in \mathrm{AAP}(\mathbf{R}^+, X)$. ∎

3. SOLUTIONS ON THE LINE

Now we turn to solutions of (1.4) on $\mathbf{R}$. In this context, $\mathscr{U} = \{U(t, s) : t \geqslant s \in \mathbf{R}\}$ will be a q-periodic evolution family on $\mathbf{R}$ with monodromy operator $V = U(q, 0)$, and we shall use the same notation as in Section 2, with $\mathbf{R}^+$ replaced by $\mathbf{R}$ and variables such as s and t taking any value in $\mathbf{R}$. We shall consider subspaces $\mathscr{F}$ of $\mathrm{BUC}(\mathbf{R}, X)$ which are invariant under the $\mathscr{C}_0$-group $\mathscr{S}$ of translations on $\mathbf{R}$. The analogues of Lemmas 2.2 and 2.3 hold, *mutatis mutandis*. We now state the analogue of Theorem 2.4.

THEOREM 3.1. *Let $\mathscr{U} = \{U(t, s) : t \geqslant s \in \mathbf{R}\}$ be a q-periodic evolution family on the Banach space X, and suppose that $\sigma(V) \cap \Gamma$ is countable. Let $f \in \mathrm{BUC}(\mathbf{R}, X)$ with relatively compact range, and suppose that $u \in \mathrm{BUC}(\mathbf{R}, X)$ satisfies* (1.4) *and that u is uniformly ergodic at $i\eta$ whenever $e^{i\eta q} \in \sigma(V) \cap \Gamma$.*

Let $\mathcal{F}$ *be a closed, translation-invariant subspace of* BUC($\mathbf{R}$, X), *satisfying the following conditions*:

(1) $f \in \mathcal{F}$,

(2) $M_\eta(u) \in \mathcal{F}$ *whenever* $e^{i\eta q} \in \sigma(V) \cap \Gamma$,

(3) *if* $g \in \mathcal{F}$, $B \in \mathcal{L}(X)$ *and* $n \in \mathbf{Z}$, *then* $s \mapsto \exp(2\pi ins/q)\, Bg(s)$ *belongs to* $\mathcal{F}$.

Then $u \in \mathcal{F}$.

Proof. The proof is very similar to Theorem 2.4. The analogues of Lemmas 2.2 and 2.3 hold, the equality (2.2) now holds for all $s \in \mathbf{R}$, and one works with the C_0-group $S_{\mathcal{F}}$ on $Y_{\mathcal{F}} := \mathrm{BUC}(\mathbf{R}, X)/\mathcal{F}$, induced by translations. ∎

To give more concrete results, we shall need the notion of the *spectrum* of a function $f \in L^\infty(\mathbf{R}, X)$:

$$\mathrm{sp}(f) = \{\xi \in \mathbf{R} : \text{for all } \varepsilon > 0 \text{ there exists } \phi \in L^1(\mathbf{R}) \\ \text{such that } \mathrm{supp}(\hat{\phi}) \subseteq (\xi - \varepsilon, \xi + \varepsilon) \text{ and } \phi * f \neq 0\}.$$

Here, $\hat{\phi}$ denotes the Fourier transform of ϕ and $\phi * f$ is the convolution of ϕ and f. There are several alternative definitions of $\mathrm{sp}(f)$. In particular, $\mathrm{sp}(f)$ is the support of the Fourier transform of the vector-valued distribution associated with f, and it coincides with the Carleman spectrum [20, Proposition 0.5]. If $f \in \mathrm{BUC}(\mathbf{R}, X)$ and D_f is the generator of the restriction of $\mathcal{S}$ to the closed linear span of $\{S(t) f : t \in \mathbf{R}\}$ in BUC($\mathbf{R}$, X), then $\sigma(D_f) = i\,\mathrm{sp}(f)$ (see [1, Section 2], [27, Section 3]).

For a closed subset Λ of $\mathbf{R}$, let $L^\infty_\Lambda(\mathbf{R}, X)$ be the space of all functions $f \in L^\infty(\mathbf{R}, X)$ such that $\mathrm{sp}(f) \subseteq \Lambda$. A simple argument in harmonic analysis shows that $f \in L^\infty_\Lambda(\mathbf{R}, X)$ if and only if $\phi * f = 0$ whenever $\phi \in L^1(\mathbf{R})$ and $\mathrm{supp}(\hat{\phi}) \cap \Lambda$ is empty (for $f \in \mathrm{BUC}(\mathbf{R}, X)$, this can also be seen by observing that the latter property coincides with Arveson's definition of spectrum and spectral subspaces [4, p. 225], [9, p. 206] and the generator of the restriction D_Λ to this subspace also satisfies $\sigma(D_\Lambda) = i\Lambda$ [9, Theorem 8.19]). It is almost immediate from this or the definition of the spectrum that $\mathrm{sp}(\phi * f) \subseteq \mathrm{supp}(\hat{\phi}) \cap \mathrm{sp}(f)$ for any $\phi \in L^1(\mathbf{R})$ and $f \in L^\infty(\mathbf{R}, X)$ (see [8, Lemma 3.2.38] and [20, Proposition 0.6]).

We shall also need notation for the following q-periodic version of the spectrum:

$$\Sigma_q(f) = \overline{\mathrm{sp}(f) + (2\pi/q)\,\mathbf{Z}} \subseteq \mathbf{R}.$$

For a closed, translation-invariant subspace $\mathscr{F}$ of $\mathrm{BUC}(\mathbf{R}, X)$, we let $\mathrm{sp}_{\mathscr{F}}(f)$ be the $\mathscr{F}$-spectrum of f, as in [5, Section 4], [25, Section 3]:

$$\mathrm{sp}_{\mathscr{F}}(f) = \{\xi \in \mathbf{R} : \text{for all } \varepsilon > 0 \text{ there exists } \phi \in L^1(\mathbf{R}) \text{ such that } \mathrm{supp}(\hat{\phi}) \subseteq (\xi - \varepsilon, \xi + \varepsilon) \text{ and } \phi * f \notin \mathscr{F}\}.$$

Then $i\,\mathrm{sp}_{\mathscr{F}}(f) = \sigma(D_{\mathscr{F}, f})$, where $D_{\mathscr{F}, f}$ is the generator of the restriction of the group $\mathscr{S}_{\mathscr{F}}$ to the closed linear span of $\{S_{\mathscr{F}}(t)\, \pi_{\mathscr{F}} f : t \in \mathbf{R}\}$ in $Y_{\mathscr{F}} = \mathrm{BUC}(\mathbf{R}, X)/\mathscr{F}$ [1, proof of Theorem 3.4].

The following general result relates the spectra of the various components of the equation (1.4). An analogous result for autonomous problems on $\mathbf{R}^+$ is given in [2, Proposition 3.1].

PROPOSITION 3.2. *Let $\mathscr{U} = \{U(t, s) : t \geqslant s \in \mathbf{R}\}$ be a q-periodic evolution family on the Banach space X. Let $f \in \mathrm{BUC}(\mathbf{R}, X)$ with relatively compact range, and suppose that $u \in \mathrm{BUC}(\mathbf{R}, X)$ is a solution of* (1.4). *Then*

$$\mathrm{sp}(u) \subseteq \{\eta \in \mathbf{R} : e^{i\eta q} \in \sigma(V)\} \cup \Sigma_q(f).$$

Proof. Let $G = \{\lambda \in \mathbf{C} : e^{\lambda q} \in \rho(V)\}$ and

$$\mathscr{F} = \{g \in \mathrm{BUC}(\mathbf{R}, X) : \mathrm{sp}(g) \subseteq \Sigma_q(f)\}.$$

By the analogue of Lemma 2.3, $w_f \in \mathscr{F}$ and $r_{\lambda, f} \in \mathscr{F}$ whenever $\lambda \in G$. Define $F\colon G \to \mathscr{F}$ by $F(\lambda) = r_{\lambda, f}$. Then F is holomorphic.

Let $D^{\mathscr{F}}$ be the generator of the translation group on $\mathscr{F}$, so $\sigma(D^{\mathscr{F}}) = i\Sigma_q(f)$ [9, Theorem 8.19]. Equation (2.2) in Theorem 2.4 gives

$$R(\lambda, D)\, u = H(\lambda) + R(\lambda, D_{\mathscr{F}})\, r_{\lambda, f}$$

whenever $\lambda \in G$ and $\mathrm{Re}\,\lambda > 0$. The term $R(\lambda, D^{\mathscr{F}})\, r_{\lambda, f}$ has a holomorphic $\mathscr{F}$-valued extension to $G \backslash i\Sigma_q(f)$ given by the same formula. In the proof of Theorem 2.4 it is shown that H is a holomorphic function from G_0 to $\mathrm{BUC}(\mathbf{R}, X)$, where G_0 is the union of the connected components of G which intersect $\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 0\}$. It follows that $R(\lambda, D)\, u$ has a holomorphic extension to a map from $G_0 \backslash i\Sigma_q(f)$ into $\mathrm{BUC}(\mathbf{R}, X)$. Since $\mathrm{sp}(u) = -i\sigma(D_u)$, Lemma 2.1 now gives the result. ∎

Let $\mathrm{AP}(\mathbf{R}, X)$ denote the space of all almost periodic functions from $\mathbf{R}$ to X, so

$$\mathrm{AP}(\mathbf{R}, X) = \overline{\mathrm{span}}\{e^{i\eta\cdot} x : \eta \in \mathbf{R}, x \in X\}.$$

For further properties of these functions, see [12], [14]. The following result answers a question of Vũ who proved cases (2) and (3) under the additional assumption that $\mathscr{U}$ has a Floquet representation [28, Theorem 4.2].

THEOREM 3.3. (Almost periodicity). *Let* $\mathscr{U}=\{U(t,s): t \geqslant s \in \mathbf{R}\}$ *be a q-periodic evolution family on the Banach space X and suppose that* $\sigma(V) \cap \Gamma$ *is countable. Let* $f \in \mathrm{AP}(\mathbf{R}, X)$, *and suppose that* $u \in \mathrm{BUC}(\mathbf{R}, X)$ *is a solution of* (1.4) *on* $\mathbf{R}$ *and that one of the following conditions holds*:

(1) *u is uniformly ergodic at* $i\eta$ *whenever* $e^{i\eta q} \in \sigma(V) \cap \Gamma$,

(2) *u has relatively weakly compact range*,

(3) *X does not contain* c_0.

Then $u \in \mathrm{AP}(\mathbf{R}, X)$.

Proof. In case (1), the result follows from Theorem 3.1 with $\mathscr{F}=\mathrm{AP}(\mathbf{R}, X)$.

In general, the proofs of Lemmas 2.2 and 2.3 and Theorem 2.4 show that the map $\lambda \mapsto R(\lambda, D_{\mathbf{AP}})\, \pi_{\mathbf{AP}} u$ $(\operatorname{Re} \lambda > 0)$ has a holomorphic extension near $i\eta$ whenever $e^{i\eta q} \in \rho(V)$. Since $\mathrm{sp}_{\mathbf{AP}}(u) = -i\sigma(D_{\mathbf{AP},u})$ this together with Lemma 2.1 yields $\mathrm{sp}_{\mathbf{AP}}(u) \subseteq \{\eta \in \mathbf{R} : e^{i\eta q} \in \sigma(V)\}$, which is countable. Each of the three cases now follows from [25, Theorem 3.11] (see also [1, Remark 3.3 and Corollary 3.5]). ∎

Of the three alternative conditions in Theorem 3.3, the first is hard to verify in applications, and the other two are specific to the cases of almost periodic functions and a few other special classes. In the remainder of this section we shall give some results of more general applicability, in which we assume that $\sigma(V)$ and $\mathrm{sp}(f)$ are out of phase. The first is an adaptation to the periodic case of a result of [3] for the autonomous case.

THEOREM 3.4. *Let* $\mathscr{U}=\{U(t,s): t \geqslant s \in \mathbf{R}\}$ *be a q-periodic evolution family on the Banach space X, and suppose that* $\sigma(V) \cap \Gamma$ *is finite. Let* $f \in \mathrm{AP}(\mathbf{R}, X)$, *and suppose that* $\sigma(V)$ *contains no accumulation points of* $\{e^{i\eta q} : \eta \in \mathrm{sp}(f)\}$. *Let* $u \in \mathrm{BUC}(\mathbf{R}, X)$ *be a solution of* (1.4) *on* $\mathbf{R}$. *Then* $u \in \mathrm{AP}(\mathbf{R}, X)$.

Proof. By Proposition 3.2,

$$\mathrm{sp}(u) \subseteq \{\eta \in \mathbf{R} : e^{i\eta q} \in \sigma(V)\} \cup \Sigma_q(f).$$

By [3, Proposition 3.4], $\mathrm{sp}_{\mathbf{AP}}(u)$ consists only of accumulation points of $\mathrm{sp}(u)$, so the assumptions imply that each point of $\mathrm{sp}_{\mathbf{AP}}(u)$ is an accumulation point of $\Sigma_q(f)$.

The argument of Theorem 2.4 shows that

$$\mathrm{sp}_{\mathbf{AP}}(u) = -i\sigma(D_{\mathbf{AP},u}) \subseteq \{\eta \in \mathbf{R} : e^{i\eta q} \in \sigma(V)\}.$$

It follows from the assumptions that $\mathrm{sp}_{\mathbf{AP}}(u)$ is empty, so $u \in \mathrm{AP}(\mathbf{R}, X)$, by [25, Proposition 3.1]. ∎

Remark 3.5. Theorem 3.4 remains valid if the space $\mathrm{AP}(\mathbf{R}, X)$ is replaced throughout by any closed, translation-invariant subspace of $\mathrm{BUC}(\mathbf{R}, X)$ containing $\mathrm{AP}(\mathbf{R}, X)$ and consisting only of functions with relatively compact range.

We shall show in Theorem 3.8 that if f has relatively compact range and $\sigma(V)$ and $\mathrm{sp}(f)$ are out of phase, then there is a unique solution u of (1.4) with $\mathrm{sp}(u) \subseteq \Sigma_q(f)$. This situation has been studied by Vũ [28] and Naito and Nguyen [16], under some supplementary conditions, and by Vũ and Schüler [29] in the autonomous case. Our proof is based on that of [16], but first we require two lemmas.

LEMMA 3.6. *Let Λ be a closed subset of $\mathbf{R}$ such that $\eta + (2\pi n/q) \in \Lambda$ whenever $\eta \in \Lambda$ and $n \in \mathbf{Z}$. Let $f \in L^\infty_\Lambda(\mathbf{R}, X)$, and let $T\colon \mathbf{R} \to \mathscr{L}(X)$ be strongly continuous and q-periodic. Then the spectrum of $s \mapsto T(s)\, f(s)$ is contained in Λ.*

Proof. As in Lemma 2.2, Fejér's Theorem provides a uniformly bounded sequence (T_n) of q-periodic trigonometric polynomials with values in $\mathscr{L}(X)$ such that $T_n(s)\, x \to T(s)\, x$, uniformly in s, for each $x \in X$. It follows from the assumption on Λ that the spectrum of $s \mapsto T_n(s)\, f(s)$ is contained in Λ. Thus, if $\phi \in L^1(\mathbf{R})$ and $(\mathrm{supp}\, \hat{\phi}) \cap \Lambda$ is empty, then

$$0 = \int_{-\infty}^{\infty} \phi(t-s)\, T_n(s)\; f(s)\, ds \to \int_{-\infty}^{\infty} \phi(t-s)\, T(s)\; f(s)\, ds,$$

as $n \to \infty$, by the Dominated Convergence Theorem. Thus, $\int_{-\infty}^{\infty} \phi(t-s)\, T(s)\, f(s)\, ds = 0$ for all such ϕ. This proves the result. ∎

We shall say that a (norm-)closed, translation-invariant subspace Z of $L^\infty(\mathbf{R}, X)$ is *convolution-invariant* if $\phi * f \in Z$ whenever $\phi \in L^1(\mathbf{R})$ and $f \in Z$. Note that $\phi * f$ is always bounded and uniformly continuous. Examples of convolution-invariant spaces include:

(1) norm-closed translation-invariant subspaces of $\mathrm{BUC}(\mathbf{R}, X)$;

(2) translation-invariant subspaces of $L^\infty(\mathbf{R}, X)$ which are weakly closed for the natural duality between $L^\infty(\mathbf{R}, X)$ and $L^1(\mathbf{R}, X^*)$;

(3) $L^\infty_\Lambda(\mathbf{R}, X)$ for any closed subset Λ of $\mathbf{R}$.

LEMMA 3.7. *Let Λ be a closed subset of* $\mathbf{R}$, *and let Z be a closed, translation-invariant, convolution-invariant subspace of $L^\infty_\Lambda(\mathbf{R}, X)$ containing the functions $s \mapsto e^{i\eta s}x$ ($\eta \in \Lambda$, $x \in X$). Define $W: Z \to Z$ by*

$$(Wf)(s) = f(s-q).$$

Then $\sigma(W)$ is the closure of $\{e^{-i\eta q} : \eta \in \Lambda\}$.

Proof. Since $e^{i\eta \cdot} x$ is an eigenvector of W with eigenvalue $e^{-i\eta q}$, it is immediate that $\overline{\{e^{-i\eta q} : \eta \in \Lambda\}} \subseteq \sigma(W)$.

Now let $Z_c = Z \cap \mathrm{BUC}(\mathbf{R}, X)$ and $W_c = S(-q)|_{Z_c}$. Since the spectrum of the generator of the restriction of $\mathcal{S}$ to Z_c is $i\Lambda$ [9, Theorem 8.19], it follows from the Weak Spectral Mapping Theorem for $\mathscr{C}_0$-groups [15, Theorem 7.4, p. 91] that $\sigma(W_c) = \overline{\{e^{-i\eta q}: \eta \in \Lambda\}}$. Moreover, if $\mu \in \rho(W_c) \cap \Gamma$, there exists a C^2-function ϕ on Γ such that $\phi(z) = (\mu - z)^{-1}$ for all z in a neighbourhood N of $\sigma(W_c) \cap \Gamma$ in Γ. Let (a_n) be the sequence of Fourier coefficients of ϕ, and $b_n = \mu a_n - a_{n-1}$. Then $(a_n) \in \ell^1(\mathbf{Z})$ and

$$\sum_{n=-\infty}^{\infty} b_n z^n = (\mu - z) \sum_{n=-\infty}^{\infty} a_n z^n = 1$$

whenever $z \in N$. It follows from the spectral theory of invertible isometries (for example, [19, Corollary 8.1.11] and [8, Lemma 3.2.38]) that

$$\sum_{n=-\infty}^{\infty} a_n (W_c)^n (\mu I_{Z_c} - W_c) = \sum_{n=-\infty}^{\infty} b_n (W_c)^n = I_{Z_c}. \tag{3.1}$$

Now, consider $f \in Z$. For any $\phi \in L^1(\mathbf{R})$, $\phi * f \in Z_c$, so (3.1) gives

$$\sum_{n=-\infty}^{\infty} a_n(\mu W^n(\phi * f) - W^{n+1}(\phi * f)) = \phi * f,$$

and the series converges uniformly for $\|\phi\|_1 \leqslant 1$. As ϕ runs through an approximate identity of $L^1(\mathbf{R})$, $W^n(\phi * f) \to W^n f$ in norm if f is uniformly continuous; pointwise if f is continuous; or in the sense of vector-valued distributions if f is measurable (that is, $\int_{\mathbf{R}} \psi W^n(\phi * f) \to \int_{\mathbf{R}} \psi W^n f$ for every $\psi \in C_c^\infty(\mathbf{R})$). It follows that

$$\sum_{n=-\infty}^{\infty} a_n W^n(\mu I - W) f = f$$

for all $f \in Z$. Thus $\mu I - W$ has the inverse $\sum_{n=-\infty}^{\infty} a_n W^n$, and $\mu \in \rho(W)$. ∎

THEOREM 3.8. *Let $\mathscr{U} = \{U(t,s) : t \geqslant s \in \mathbf{R}\}$ be a q-periodic evolution family on the Banach space X. Let $f \in C^b(\mathbf{R}, X)$, and suppose that $\sigma(V) \cap \overline{\{e^{i\eta q} : \eta \in \mathrm{sp}(f)\}}$ is empty. Then:*

(1) *There is at most one* $u \in C^b(\mathbf{R}, X)$ *such that* (*a*) *u is a solution of* (1.4) *on* $\mathbf{R}$, *and* (*b*) $\mathrm{sp}(u) \subseteq \Sigma_q(f)$.

(2) *Let* $\mathscr{F}$ *be a closed, translation-invariant subspace of* $\mathrm{BUC}(\mathbf{R}, X)$ *such that* $s \mapsto \exp(2\pi ins/q)\, Bg(s)$ *belongs to* $\mathscr{F}$ *whenever* $g \in \mathscr{F}$, $B \in \mathscr{L}(X)$ *and* $n \in \mathbf{Z}$. *Suppose that* $f \in \mathscr{F}$ *and f has relatively compact range. Then there exists* $u \in \mathscr{F}$ *satisfying* (*a*) *and* (*b*) *above, and u has relatively compact range.*

Proof. We consider the evolution semigroup $\{T(t) : t \geqslant 0\}$ defined on $C^b(\mathbf{R}, X)$ by

$$(T(t)\, g)(s) = U(s, s-t)\, g(s-t) \quad (t \geqslant 0, s \in \mathbf{R},\ g \in C^b(\mathbf{R}, X)).$$

This semigroup may not be strongly continuous. Let

$$E_c = \{g \in C^b(\mathbf{R}, X) : \|T(t)\, g - g\| \to 0 \text{ as } t \to 0+\}.$$

Then the evolution semigroup restricts to a C_0-semigroup on E_c, whose generator will be denoted by L. A standard argument shows that E_c contains all $g \in C^b(\mathbf{R}, X)$ with relatively compact range.

Let $u \in C^b(\mathbf{R}, X)$. A simple calculation shows that u satisfies (1.4) if and only if

$$(T(t)\, u)(s) - u(s) = -\int_0^t (T(r)\, f)(s)\, dr.$$

This implies that $u \in E_c$. Furthermore, if $f \in E_c$, then u satisfies (1.4) if and only if $u \in D(L)$ and $Lu = -f$ (cf. [17, Lemma 1.1] or [16, Lemma 2]).

Let $\{S(t) : t \in \mathbf{R}\}$ be the translation group on $C^b(\mathbf{R}, X)$. Define $\hat{V}$: $C^b(\mathbf{R}, X) \to C^b(\mathbf{R}, X)$ by

$$(\hat{V}g)(s) = V_s g(s).$$

Note that

$$T(q) = \hat{V}S(-q) = S(-q)\, \hat{V}. \tag{3.2}$$

(1) Let

$$Z = \{g \in C^b(\mathbf{R}, X) : \mathrm{sp}(g) \subseteq \Sigma_q(f)\} = C^b(\mathbf{R}, X) \cap L^\infty_{\Sigma_q(f)}(\mathbf{R}, X).$$

Then Z is convolution-invariant, and invariant under $T(q)$, $S(-q)$ and $\hat{V}$, by Lemma 3.6. Let $\lambda \in \rho(V) \backslash \{0\}$, so $\lambda \in \rho(V_s)$ for all s. Given $g \in Z$, let $v(s) = R(\lambda, V_s)\, g(s)$. Then $v \in Z$ by Lemma 3.6, and $(\lambda I - \hat{V})\, v = g$. Thus $\lambda I - \hat{V}$ maps Z onto Z. Moreover, $(\lambda I - \hat{V})$ is injective, since each $\lambda I_X - V_s$ is injective. This shows that $\sigma(\hat{V}|_Z) \subseteq \sigma(V) \cup \{0\}$.

By Lemma 3.7,

$$\sigma(S(-q)|_Z) = \{e^{-i\eta q} : \eta \in \Sigma_q(f)\} = \overline{\{e^{-i\eta q} : \eta \in \mathrm{sp}(f)\}}.$$

By restricting (3.2) to Z, it follows that

$$\sigma(T(q)|_Z) \subseteq \{\lambda_1 \lambda_2 : \lambda_1 \in \sigma(\hat{V}|_Z), \lambda_2 \in \sigma(S(-q)|_Z)\}.$$

Our assumptions imply that $1 \in \rho(T(q)|_Z)$.

If u and v are solutions of (1.4) in Z, then

$$(T(q)\,u)(s) - u(s) = (T(q)\,v)(s) - v(s) = -\int_0^t (T(r)\,f)(s)\,dr,$$

so $(T(q) - I)(u - v) = 0$. Since $(T(q) - I)|_Z$ is injective, it follows that $u = v$.

(2) Now, let

$$Z = \{g \in \mathscr{F} : g \text{ has relatively compact range, } \operatorname{sp}(g) \subseteq \Sigma_q(f)\}.$$

Under the assumptions of (2), $f \in Z \subseteq E_c$, and Z is invariant under $T(t)$, by Lemma 2.2. Let L_Z be the generator of the C_0-semigroup $\{T(t)|_Z : t \geqslant 0\}$, so L_Z is the restriction of L to $D(L) \cap Z$. Arguing as in (1) above, but using Lemma 2.2 in place of Lemma 3.6, we see that $1 \in \rho(T(q)|_Z)$. Since $\{e^{\lambda q}: \lambda \in \sigma(L_Z)\} \subseteq \sigma(T(q)|_Z)$, it follows that $0 \in \rho(L_Z)$. Hence there is exactly one $u \in Z$ such that $u \in D(L_Z)$ and $L_Z u = -f$. This proves (2). ∎

By taking $\mathscr{F} = \mathrm{AP}(\mathbf{R}, X)$ in Theorem 3.8 (2), we obtain the following corollary, which answers a question of Vũ [28], who proved the result under the assumption that V has a Floquet representation. The result has also been proved by Naito and Nguyen [16], under the assumption that $t \mapsto V_t$ is norm-continuous.

COROLLARY 3.9. *Let $\mathscr{U} = \{U(t, s) : t \geqslant s \in \mathbf{R}\}$ be a q-periodic evolution family on the Banach space X. Let $f \in \mathrm{AP}(\mathbf{R}, X)$, and suppose that $\sigma(V) \cap \overline{\{e^{i\eta q} : \eta \in \operatorname{sp}(f)\}}$ is empty. Then there is a unique $u \in C^b(\mathbf{R}, X)$ such that (a) u is a solution of (1.4) on $\mathbf{R}$, and (b) $\operatorname{sp}(u) \subseteq \Sigma_q(f)$. Moreover, $u \in \mathrm{AP}(\mathbf{R}, X)$.*

REFERENCES

1. W. Arendt and C. J. K. Batty, Almost periodic solutions of first and second order Cauchy problems, *J. Differential Equations* **137** (1997), 363–383.
2. W. Arendt and C. J. K. Batty, Asymptotic almost periodic solutions of inhomogeneous Cauchy problems on the half-line, *Bull. London Math. Soc.* (to appear).
3. W. Arendt and S. Schweiker, Discrete spectrum and almost periodicity, *Taiwanese J. Math.* (to appear).
4. W. Arveson, On groups of automorphisms of operator algebras, *J. Functional Anal.* **15** (1974), 217–243.

5. B. Basit, Some problems concerning different types of vector-valued almost periodic functions, *Dissertationes Math.* (*Rozprawy Mat.*) **338** (1995).
6. C. J. K. Batty, J. van Neerven, and F. Räbiger, Local spectra and individual stability of uniformly bounded $\mathscr{C}_0$-semigroups, *Trans. Amer. Math. Soc.* **350** (1998), 2071–2085.
7. C. J. K. Batty, J. van Neerven, and F. Räbiger, Tauberian theorems and stability of solutions of the abstract Cauchy problem, *Trans. Amer. Math. Soc.* **350** (1998), 2087–2103.
8. O. Bratteli and D. W. Robinson, "Operator Algebras and Quantum Statistical Mechanics, I," Springer-Verlag, Berlin, 1979.
9. E. B. Davies, "One-Parameter Semigroups," Academic Press, London, 1980.
10. D. Daners and P. Koch Medina, "Abstract Evolution Equations, Periodic Problems and Applications," Pitman Research Notes in Math. Series, Vol. 279, Longman, Harlow/New York, 1992.
11. H. O. Fattorini, "The Cauchy Problem," Addison–Wesley, London, 1983.
12. A. M. Fink, "Almost Periodic Differential Equations," Lecture Notes in Math., Vol. 377, Springer-Verlag, Berlin, 1974.
13. J. Kreulich, Eberlein-weakly almost-periodic solutions of evolution equations in Banach spaces, *Differential Integral Equations* **9** (1996), 1005–1027.
14. B. M. Levitan and V. V. Zhikov, "Almost Periodic Functions and Differential Equations," Cambridge Univ. Press, Cambridge, UK, 1982.
15. R. Nagel (Ed.), "One-Parameter Semigroups of Positive Operators," Lecture Notes in Math., Vol. 1184, Springer-Verlag, Berlin, 1986.
16. T. Naito and Nguyen Van Minh, Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations, *J. Differential Equations* **152** (1999), 358–376.
17. Nguyen Van Minh, F. Räbiger, and R. Schnaubelt, Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line, *Integral Equations Operator Theory* **32** (1998), 332–353.
18. A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.
19. G. K. Pedersen, "C^*-algebras and Their Automorphism Groups," Academic Press, London, 1979.
20. J. Prüss, "Evolutionary Integral Equations and Applications," Birkhäuser, Basel, 1993.
21. R. T. Rau, Hyperbolic evolution semigroups on vector valued function spaces, *Semigroup Forum* **48** (1994), 107–118.
22. W. M. Ruess and W. H. Summers, Integration of asymptotically almost periodic functions and weak almost periodicity, *Dissertationes Math.* (*Rozprawy Mat.*) **279** (1989).
23. W. M. Ruess and W. H. Summers, Weak almost periodicity and the strong ergodic limit theorem for periodic evolution systems, *J. Functional Anal.* **94** (1990), 177–195.
24. W. M. Ruess and W. H. Summers, Ergodic theorems for semigroups of operators, *Proc. Amer. Math. Soc.* **114** (1992), 423–432.
25. W. M. Ruess and Vũ Quôc Phóng, Asymptotically almost periodic solutions of evolution equations in Banach spaces, *J. Differential Equations* **122** (1995), 282–301.
26. H. Tanabe, "Equations of Evolution," Pitman, London, 1979.
27. Vũ Quôc Phóng, On the spectrum, complete trajectories, and asymptotic stability of linear semi-dynamical systems, *J. Differential Equations* **105** (1993), 30–45.
28. Vũ Quôc Phóng, Stability and almost periodicity of trajectories of periodic processes, *J. Differential Equations* **115** (1995), 402–415.
29. Vũ Quôc Phóng and E. Schüler, The operator equation $AX - XB = C$, admissibility, and asymptotic behaviour of differential equations, *J. Differential Equations* **145** (1998), 394–419.