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Lyapunov's Second Method for Random Dynamical Systems

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The method of Lyapunov functions (Lyapunov's second or direct method) was originally developed for studying the stability of a fixed point of an autonomous or non-autonomous differential equation. It was then extended from fixed points to sets, from differential equations to dynamical systems and to stochastic differential equations. We go one step further and develop Lyapunov's second method for random dynamical systems and random sets, together with matching notions of attraction and stability. As a consequence, Lyapunov functions will also be random. Our test is that the extension be coherent in the sense that it reduces to the deterministic theory in case the noise is absent, and that we can prove that a random set is asymptotically stable if and only if it has a Lyapunov function. Several examples are treated, including the stochastic Lorenz system.

Key Words: Lyapunov function; Lyapunov's second method; dynamical system; random dynamical system; stochastic differential equation; stability; stochastic stability; attractor; random attractor.

1. INTRODUCTION

One of the basic tasks of the theory of differential equations and dynamical systems is to study the qualitative, asymptotic, long-term behavior of solutions/orbits, in particular their "stability", i.e. the behavior of "perturbed" orbits relative to a reference orbit in the course of time.

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In his seminal 1892 thesis [17] at Kharkov University, M. A. Lyapunov (Liapounoff) proposed two methods (named by himself the *first* and *second* method) to study stability. These are

- *Lyapunov's First Method*: The method of linearization of the nonlinear equation along an orbit, the study of the resulting non-autonomous linear *variational equation* by means of *characteristic numbers* (exponential growth rates of solutions, today known as *Lyapunov exponents*), and the transfer of stability from the linear to the nonlinear equation.
- *Lyapunov's Second Method*: The method of Lyapunov functions, i.e. of scalar functions on the state space which decrease along orbits. The biggest advantage of this method is that it is "direct", as "decrease along orbits" can be infinitesimally expressed by means of the generator, hence one does not need to solve the equation explicitly. The biggest drawback is that there is no general method for obtaining Lyapunov functions.

Lyapunov's first method turned out to be rather difficult to implement, mainly due to the fact that the behavior of a non-autonomous linear system $\dot{x} = A(t)\,x$ in $\mathbb{R}^d$ can be incredibly complicated. For an account see the monograph [3] by Bylov, Vinograd, Grobman and Nemytskii (which, unfortunately, was never translated into English).

Lyapunov's first method was, however, filled with new life in 1968 when Oseledets [20] proved his celebrated *multiplicative ergodic theorem*. For (random) dynamical systems under an invariant measure this theorem establishes the existence of Lyapunov exponents as limits and can be used to conclude nonlinear stability from linear stability. A systematic account of the theory of nonlinear random dynamical systems based on Lyapunov's first method through the multiplicative ergodic theorem is given by Arnold [1].

In contrast to his first method, Lyapunov's second method turned out to be very successful from the beginning, in particular in numerous applied problems. Early systematic accounts in the West were given by the *Springer Grundlehren* volumes of Hahn [12] in 1967 and of Bhatia and Szegö [2] (developing Lyapunov's second method for dynamical systems) in 1970, both of which are still classical references.

Our contribution consists of proposing a simultaneous generalization of the second method for topological dynamical systems and for stochastic differential equations to a general class of systems now known by the name *random dynamical systems*. This class comprises practically all systems under the influence of randomness which are presently of interest, in particular random and stochastic differential and difference equations.

As random dynamical systems have attractors which typically are *random* sets (see e.g. Crauel and Flandoli [8, 9], Flandoli and Schmalfuss [11], Schmalfuss [21, 22], and for a brief survey [1, Chap. 9]), it is natural to define stability and attractivity also for random sets. As a consequence, the Lyapunov functions for random dynamical systems will be random functions as well.

The main conceptual problem is to find definitions of stability, attractor (hence of asymptotic stability) and Lyapunov function which are matching in the sense that they allow to prove that a random set is asymptotically stable if and only if it has a Lyapunov function.

Random dynamical systems are formally skew-product flows—but only in the measurable category. In particular, any topology and continuity is stripped-off from the driving or base flow, so that we consider approaches working with topological and continuity assumptions on the base flow and/or on the cocycle in its dependence on the base flow (e.g. by Hale [13, Chap. 3] or Kloeden [15, 16]) not the final answer to our problem.

The structure of this paper is as follows: In Section 2 we will briefly review the second method for topological dynamical systems.

In Section 3 we will recall some basic facts about random dynamical systems and random sets.

Section 4 will be devoted to the concept of asymptotic stability of a random set under a random dynamical system.

Our paradigmatic example will turn out to be $\dot{x} = \alpha x - x^2$, where α is disturbed by white noise. It will be treated in Section 5.

The general concept of a Lyapunov function for a random dynamical system will be presented in Section 6 which also contains our main and only theorem, stating indeed that a random set is asymptotically stable if and only if it has a Lyapunov function.

Section 7 contains two further examples: the Lorenz system with multiplicative white noise, and the affine stochastic differential equation.

2. SECOND METHOD FOR DYNAMICAL SYSTEMS

We briefly recall the following basic stability and attraction definitions for topological dynamical systems.

Let X be a topological space. A continuous mapping $\varphi: \mathbb{R} \times X \to X$, $(t, x) \mapsto \varphi(t, x)$, is called a *topological dynamical system* if the family $\varphi(t, \cdot) = \varphi(t): X \to X$ of self-mappings of X satisfies the flow properties $\varphi(0) = \mathrm{id}_X$, $\varphi(t+s) = \varphi(t) \circ \varphi(s)$ for all $t, s \in \mathbb{R}$, where "$\circ$" denotes composition of mappings. It follows that all mappings $\varphi(t)$ are homeomorphisms of X, and $\varphi(t)^{-1} = \varphi(-t)$.

2.1. Definition. Let φ be a topological dynamical system on a locally compact metric space X. Let A be a nonempty compact subset of X which is *invariant* under φ, i.e. for which $\varphi(t)\, A = A$ for all $t \in \mathbb{R}$.

(i) *Stability:* A is called *stable* under φ if every neighborhood[1] U of A contains a *forward invariant*[2] *neighborhood* V of A, i.e. for which $\varphi(t) V \subset V$ for all $t \geqslant 0$.

(The following more common definition is equivalent to ours: For all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varphi(t)\, U_\delta(A) \subset U_\varepsilon(A) \qquad \text{for all} \quad t \geqslant 0,$$

where $U_\varepsilon(A) := \{x \in X : d(x, A) < \varepsilon\}$, $d(x, A) := \inf_{y \in A} d(x, y)$, is the open ε-neighborhood of A.)

(ii) *Attractor:* A is a (global) *attractor* of φ if

$$\lim_{t \to \infty} d(\varphi(t, x), A) = 0 \qquad \text{for all} \quad x \in X.$$

(iii) *Asymptotic stability:* A is (globally) *asymptotically stable* if it is stable and is an attractor.

We quote the following best known result on asymptotic stability for further use (see Bhatia and Szegö [2, Theorem V.2.12]).

2.2. THEOREM. *Let φ be a topological dynamical system on a locally compact space X, and let A be a nonvoid compact set which is invariant under φ.*

Then A is asymptotically stable if and only if there exists a Lyapunov function for A, i.e. a function $V: X \to \mathbb{R}^+$ such that

(i) *V is continuous,*

(ii) *V is uniformly unbounded, i.e. for all $C > 0$ there exists a compact set $K \subset X$ such that $V(x) \geqslant C$ for all $x \notin K$,*

(iii) *V is positive-definite, i.e. $V(x) = 0$ if $x \in A$, and $V(x) > 0$ if $x \notin A$,*

(iv) *V is strictly decreasing along orbits of φ, i.e. $V(\varphi(t, x)) < V(x)$ for $x \notin A$ and $t > 0$.*

If A is asymptotically stable, V can be chosen to satisfy

$$V(\varphi(t, x)) = e^{-t} V(x) \qquad \textit{for all} \quad x \in X, t \in \mathbb{R}.$$

3. RANDOM DYNAMICAL SYSTEMS AND RANDOM SETS

The concept of a random dynamical system is an extension of the deterministic concept and reduces to it if the noise is absent. It is tailor-made

[1] U is called a neighborhood of a set A if U contains an open set which contains A.

[2] We prefer the more dynamic and self-explanatory notion "forward/backward invariant" to the traditional "positively/negatively invariant".

to treat many interesting systems which are under the influence of some "randomness" from the point of view of dynamical systems, such as random or stochastic difference and differential equations. For a comprehensive study see the monograph [1].

Here is a formal definition, where for the sake of not overburdening the presentation we restrict ourselves to the case of a state space $\mathbb{R}^d$ and continuous two-sided time $\mathbb{R}$.

3.1. DEFINITION (Random Dynamical System (RDS)). A *(topological) random dynamical system*, shortly denoted by φ, consists of two ingredients:

(i) A model of the noise, namely a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(t, \omega) \mapsto \theta_t \omega$ is a measurable flow which leaves $\mathbb{P}$ invariant, i.e. $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$. For simplicity we also assume that θ is ergodic under $\mathbb{P}$, meaning that a θ-invariant set has probability 0 or 1.

(ii) A model of the system perturbed by noise, namely a *cocycle* φ over θ, i.e. a measurable mapping $\varphi\colon \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $(t, \omega, x) \mapsto \varphi(t, \omega, x)$, such that $(t, x) \mapsto \varphi(t, \omega, x)$ is continuous for all $\omega \in \Omega$ and the family $\varphi(t, \omega, \cdot) = \varphi(t, \omega) : \mathbb{R}^d \to \mathbb{R}^d$ of random self-mappings of $\mathbb{R}^d$ satisfies the *cocycle property*:

$$\varphi(0, \omega) = \mathrm{id}_{\mathbb{R}^d}, \quad \varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \qquad \text{for all} \quad t, s \in \mathbb{R}, \omega \in \Omega. \tag{3.1}$$

Note that a cocycle reduces to a flow in case ω is absent. The quantifier "for all $\omega \in \Omega$" in equations and inequalities is henceforth often omitted.

To emphasize the flow point of view we often write $\varphi(t, \omega)\, x$ in place of $\varphi(t, \omega, x)$.

It follows from (3.1) that $\varphi(t, \omega)$ is a homeomorphism of $\mathbb{R}^d$, and

$$\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega).$$

Combining the two ingredients θ and φ as $\Theta_t(\omega, x) := (\theta_t \omega, \varphi(t, \omega)\, x)$ gives the corresponding *skew product flow* on $\Omega \times \mathbb{R}^d$. Skew product flows have been extensively studied in the context of topological dynamics, in which case Ω is a topological space and $(t, \omega) \mapsto \theta_t \omega$ and $(t, \omega, x) \mapsto \varphi(t, \omega, x)$ are continuous. For recent accounts see Chicone and Latushkin [5] or Colonius and Kliemann [6] and the references therein.

We stress, however, that the skew-product flow corresponding to an RDS is only measurable with respect to ω, and the continuity assumptions just mentioned would exclude very relevant applications such as stochastic differential equations.

The following notion of a set-valued random variable (measurable multifunction) will be crucial for what follows.

3.2. DEFINITION (Random Set). (i) A function $\omega \mapsto M(\omega)$ taking values in the non-empty closed/compact subsets of $\mathbb{R}^d$ is called a *random closed/compact set* if $\omega \mapsto d(x, M(\omega))$ is measurable for each $x \in \mathbb{R}^d$, where $d(x, M) := \inf_{y \in M} \|x - y\|$. We will often supress the ω argument of M.

(ii) A function $\omega \mapsto U(\omega)$ taking values in the non-empty open subsets of $\mathbb{R}^d$ is called a *random open set* if $\omega \mapsto U(\omega)^c$ is a random closed set, where U^c denotes the complement of U.

The property of M being a random closed set is slightly stronger than graph $M := \{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in M(\omega)\}$ being $\mathscr{F} \otimes \mathscr{B}^d$ measurable and $M(\omega)$ being closed, and is equivalent to it if $\mathscr{F}$ is $\mathbb{P}$-complete. M is a random compact set if and only if it is a random variable with values in $\mathscr{K}(\mathbb{R}^d)$, the space of non-empty compact subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric d_H.

Recall that if A and B are non-empty closed sets, the Hausdorff semimetric $d(A \mid B)$ is defined by

$$d(A \mid B) := \sup_{x \in A} d(x, B), \qquad d(x, B) := \inf_{y \in B} d(x, y) = \inf_{y \in B} \|x - y\|,$$

while

$$d_H(A, B) := d(A \mid B) + d(B \mid A)$$

denotes the Hausdorff metric, which makes $\mathscr{K}(\mathbb{R}^d)$ a Polish space.

We refer to Castaing and Valadier [4], Crauel [7, Chap. 3] and Arnold [1] for a proof of the following basic facts.

3.3. LEMMA. (i) *If M is a random closed set in $\mathbb{R}^d$, then so is $\overline{M^c}$, the closure of M^c.*

(ii) *If M is a random open set, then $\bar{M}$ is a random closed set.*

(iii) *If M is a random closed set, then* int M, *the interior of M, is a random open set.*

(iv) *If $(M_n)_{n \in \mathbb{N}}$ is a sequence of random compact sets with non-void intersection, then $\bigcap_{n \in \mathbb{N}} M_n$ is a random compact set.*

(v) *If $(M_n)_{n \in \mathbb{N}}$ is a sequence of random compact sets and if the set $M := \overline{\bigcup_{n \in \mathbb{N}} M_n}$ is compact, then M is a random compact set.*

(vi) *If M is a random compact set and $f: \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a function such that $f(\omega, \cdot)$ is continuous for all ω and $f(\cdot, x)$ is measurable for all x, then $\omega \mapsto f(\omega, M(\omega))$ is a random compact set.*

(vii) *Measurable selection theorem: A function* $\omega \mapsto M(\omega)$ *taking values in the nonempty closed subsets of* $\mathbb{R}^d$ *is a random closed set if and only if there exists a sequence* $(X_n)_{n \in \mathbb{N}}$ *of measurable maps* $X_n \colon \Omega \to \mathbb{R}^d$ *such that*

$$M(\omega) = \overline{\{X_n(\omega) : n \in \mathbb{N}\}} \qquad \text{for all} \quad \omega \in \Omega.$$

In particular, a random closed set admits a measurable selection, i.e. there exists a random variable $X \colon \Omega \to \mathbb{R}^d$ *for which* $X(\omega) \in M(\omega)$ *for all* $\omega \in \Omega$.

3.4. Definition (Invariance of Random Set). A random set M is called *forward invariant* under the RDS φ if $\varphi(t, \omega)\, M(\omega) \subset M(\theta_t \omega)$ for all $t \geqslant 0$. It is called *invariant* if $\varphi(t, \omega)\, M(\omega) = M(\theta_t \omega)$ for all $t \in \mathbb{R}$.

4. ASYMPTOTIC STABILITY OF RANDOM SETS

In this section we will develop notions of stability and attractivity of a random set under an RDS whose justification will be (i) that they are found in our prototypical example (Section 5), (ii) that they can serve, together with an appropriate definition of Lyapunov function given in Section 6, in a basic second method theorem for RDS (see Theorem 6.5), and (iii) that they reduce to the deterministic notions in case the RDS is deterministic.

Recall that a sequence (X_n) of random variables converges to a random variable X *in probability*, in symbols $\mathbb{P}\text{-}\lim_{n \to \infty} X_n = X$, if

$$\lim_{n \to \infty} \mathbb{P}\{\omega \colon |X_n(\omega) - X(\omega)| > \varepsilon\} = 0 \qquad \text{for all} \quad \varepsilon > 0.$$

Convergence in probability is weaker than ω-wise or $\mathbb{P}$-a.s. convergence. The space of $\mathbb{R}^d$-valued random variables endowed with the topology of convergence in probability can be completely metrized e.g. by using the metric

$$\rho(X, Y) := \inf\{\varepsilon > 0 : \mathbb{P}\{|X - Y| \geqslant \varepsilon\} < \varepsilon\}$$

(see e.g. Dudley [10, Sect. 9.2]).

Also note the crucial technical fact that if X is a random variable and A and B are random closed sets, the expressions $d(X, B)$ and $d(A \mid B)$ are random variables since by Lemma 3.3(vii) the sup and inf over an uncountable set can be replaced by the one over an exhausting countable set. In particular, the sets $\{\omega \colon X(\omega) \in B(\omega)\} = \{\omega \colon d(X(\omega), B(\omega)) = 0\}$ and $\{\omega \colon A(\omega) \subset B(\omega)\} = \{\omega \colon d(A(\omega) \mid B(\omega)) = 0\}$ are measurable.

4.1. DEFINITION (Stability, Attractor for RDS). Let φ be an RDS and A be a random compact set which is invariant under φ.

(i) A is called *stable* under φ if for any $\varepsilon > 0$ there exists a random compact set C which is a neighborhood of A (i.e. $C(\omega)$ is a neighborhood of $A(\omega)$ for all ω) such that

- $\mathbb{P}\{d(C \mid A) \geqslant \varepsilon\} < \varepsilon$, i.e. C is ε-close to A in the metric of convergence in probability,
- $\varphi(t, \omega)\, C(\omega) \subset C(\theta_t \omega)$ for all $t \geqslant 0$, i.e. C is forward invariant under φ.

(ii) A is called a (global) *attractor* of φ if for any random variable X

$$\mathbb{P}\text{-}\lim_{t \to \infty} d(\varphi(t, \cdot)\, X(\cdot), A(\theta_t \cdot)) = 0.$$

(iii) A is called (globally) *asymptotically stable* if it is stable and is an attractor.

4.2. *Remark.* (i) An inspection of the example in Section 5 tells us that we need to allow the neighborhood $C(\omega)$ of $A(\omega)$ in the stability definition to be possibly far away from $A(\omega)$ on an ω-set of small probability. Hence the use of the distance in probability. We claim that an ω-wise ε-δ definition of stability with a constant or random ε and δ will fail to serve our purpose.

(ii) An RDS is a *non-autonomous* system in the way described by the cocycle property (3.1). It hence matters whether we define asymptotic properties like attractivity (a) by going from $-t$ to 0 or (b) by going from 0 to t, and then letting $t \to \infty$. The choice (a) offers itself as the mathematically natural one for the following reason: While t is moving, the quantity in question, $d(\varphi(t, \theta_{-t}\omega)\, X(\theta_{-t}\omega), A(\omega))$, is always studied at time 0, where typically ω-wise convergence can be expected. The choice (b), in contrast, seems to be physically natural, but considers quantities, namely $d(\varphi(t, \omega)$ $X(\omega), A(\theta_t\omega))$, which are moving with t forever, being responsible for the fact that they often do not converge ω-wise.

The choice of the weaker mode of convergence in probability symmetrizes the situation and makes the two approaches equivalent since

$$\mathbb{P}\{d(\varphi(t, \theta_{-t} \cdot)\, X(\theta_{-t} \cdot), A(\cdot)) \geqslant \varepsilon\} = \mathbb{P}\{d(\varphi(t, \cdot)\, X(\cdot), A(\theta_t \cdot)) \geqslant \varepsilon\},$$

due to the fact that $\mathbb{P}$ is invariant under θ_t.

Random attractors defined by the choice (a) are known as "pullback attractors" and were introduced and studied by Crauel and Flandoli [8, 9], Flandoli and Schmalfuss [11] and Schmalfuss [21, 22], among others.

The concept of attractor given in Definition 4.1(ii) was introduced and studied by Ochs [19] under the name "weak attractor".

The next lemma shows that we can easily construct an invariant set once we have a forward invariant set.

4.3. LEMMA. *Suppose that C is a forward invariant random compact set. Then*

$$A(\omega) := \bigcap_{t \geqslant 0} \varphi(t, \theta_{-t}\omega)\, C(\theta_{-t}\omega) \tag{4.1}$$

is an invariant random compact set. Moreover, A is attracting C in the "pull-back" sense, i.e.

$$\lim_{t \to \infty} d(\varphi(t, \theta_{-t}\omega)\, C(\theta_{-t}\omega) \mid A(\omega)) = 0, \tag{4.2}$$

hence also in the sense of convergence in probability,

$$\mathbb{P}\text{-}\lim_{t \to \infty} d(\varphi(t, \cdot\,)\, C(\,\cdot\,) \mid A(\theta_t \cdot\,)) = 0.$$

Proof. $A(\omega)$ is the intersection of a decreasing sequence of compact sets contained in $C(\omega)$, hence is non-void and compact.

Using the elementary fact that if (C_n) is a decreasing sequence of compact sets and f is continuous then $\bigcap_n f(C_n) = f(\bigcap_n C_n)$, the cocycle property and the monotonicity of $(\varphi(t, \theta_{-t}\omega, C(\theta_{-t}\omega)))$ we obtain for any $T \in \mathbb{R}$

$$\begin{aligned}
\varphi(T, \omega)\, A(\omega) &= \bigcap_{t \geqslant 0} \varphi(T, \omega, \varphi(t, \theta_{-t}\omega, C(\theta_{-t}\omega))) \\
&= \bigcap_{t \geqslant 0} \varphi(T+t, \theta_{-t-T}(\theta_T\omega), C(\theta_{-t-T}(\theta_T\omega))) \\
&= \bigcap_{t \geqslant T} \varphi(t, \theta_{-t}(\theta_T\omega), C(\theta_{-t}(\theta_T\omega))) \\
&= \bigcap_{t \geqslant 0} \varphi(t, \theta_{-t}(\theta_T\omega), C(\theta_{-t}(\theta_T\omega))) = A(\theta_T\omega),
\end{aligned}$$

proving the invariance of A.

To see that A is a random compact set note that if C is a random compact set then so are $\omega \mapsto C(\theta_{-t}\omega)$ and $\omega \mapsto \varphi(t, \theta_{-t}\omega)\, C(\theta_{-t}\omega)$ for each fixed t by Lemma 3.3(vi). Now replace the decreasing intersection over $\mathbb{R}^+$ in (4.1) by the intersection over $\mathbb{N}$ and use Lemma 3.3(iv) to conclude that A is a random compact set.

(4.2) follows directly from (4.1), again by using the fact that the intersection is decreasing. ∎

For deterministic topological dynamical systems φ on a locally compact space, an asymptotically stable set A is automatically a uniform attractor, i.e. for any compact set K and any neighborhood U of A there is a finite time $T > 0$ such that $\varphi(t)\, K \subset U$ for all $t \geqslant T$ (see [2, 1.30.2]).

We have a similar statement in the random case.

4.4. Proposition (Uniform Attraction). *Let the invariant random compact set A be an attractor of φ, and assume that C is a forward invariant random compact neighborhood of A (if A is asymptotically stable then for any $\varepsilon > 0$ such a C exists with $\mathbb{P}\{d(C \mid A) \geqslant \varepsilon\} < \varepsilon$). Then there exists a θ-invariant set $\bar{\Omega}$ of full measure such that for any $\omega \in \bar{\Omega}$ and any random compact set D there exists a $T(\omega)$ such that*

$$\varphi(t, \omega)\, D(\omega) \subset \operatorname{int} C(\theta_t\omega) \qquad \textit{for all} \quad t \geqslant T(\omega). \tag{4.3}$$

Proof. Note first that if C is a forward invariant random compact neighborhood of A and since $\varphi(t, \omega)$ is a homeomorphism, $\operatorname{int} C$ is a random open neighborhood of A which is also forward invariant.

Define

$$E(\omega) := \bigcap_{n \in \mathbb{N}} (E_n(\omega) \cup A(\omega)),$$

where

$$E_n(\omega) := \varphi(-n, \theta_n\omega)(\operatorname{int} C(\theta_n\omega))^c, \qquad n \in \mathbb{N},$$

is the set of states which have not entered $\operatorname{int} C$ until time n. Thus $E(\omega) \setminus A(\omega)$ is the set of states which never enter $\operatorname{int} C(\theta_t\omega)$.

We need to make sure that E is a random closed set. The fibers $E(\omega)$ are clearly closed. Further, by the fact that $(\operatorname{int} C)^c = \overline{C^c}$, $(\operatorname{int} C)^c$ is a random closed set by Lemma 3.3(i), hence so is E_n and thus $E_n \cup A$ for each n. The sequence $E_n \cup A$ is decreasing to $E \neq \varnothing$, thus

$$d(x, E(\omega)) = \sup_{n \in \mathbb{N}} d(x, E_n(\omega) \cup A(\omega)) < \infty$$

is measurable for any x. Hence E is a random closed set.

Define

$$\bar{\Omega} := \{\omega \in \Omega : \forall x \in \mathbb{R}^d,\ \varphi(t, \omega, x) \text{ eventually enters the set int } C(\theta_t \omega)\}.$$

The set $\bar{\Omega}$ is measurable as $\bar{\Omega} = \{\omega : d(E(\omega) \mid A(\omega)) = 0\}$.

$\bar{\Omega}$ is also invariant. Indeed, for any $\omega \in \bar{\Omega}$, $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$ the cocycle property yields

$$\varphi(t, \theta_s \omega, x) = \varphi(t+s, \omega, \varphi(-s, \theta_s \omega, x)),$$

so $\omega \in \bar{\Omega}$ if and only $\theta_s \omega \in \bar{\Omega}$.

We prove that $\mathbb{P}(\bar{\Omega}) = 1$ by constructing a contradiction to the assumption $\mathbb{P}(\bar{\Omega}) = 1 - \eta$ with $\eta > 0$.

Since E is a random closed set and int $C(\omega)$ is a neighborhood of $A(\omega)$,

$$F(\omega) := \begin{cases} E(\omega) \setminus A(\omega), & \omega \in \bar{\Omega}^c, \\ A(\omega), & \omega \in \bar{\Omega}, \end{cases}$$

is a random closed set. There is thus a random variable $X\colon \Omega \to \mathbb{R}^d$ for which $X(\omega) \in F(\omega)$, hence $X(\omega) \in E(\omega) \setminus A(\omega)$ for $\omega \in \bar{\Omega}^c$. This implies

$$\mathbb{P}\{\omega\colon \varphi(n, \omega, X(\omega)) \notin \text{int } C(\theta_n \omega)\} \geqslant \mathbb{P}(\bar{\Omega}^c) \geqslant \eta > 0 \qquad \text{for all} \quad n \in \mathbb{N} \tag{4.4}$$

by assumption.

For random closed sets B and C

$$\delta(B(\omega), C(\omega)) := \inf\{d(x, y) : x \in B(\omega),\ y \in C(\omega)\}$$

is a random variable, by the countable exhaustion property. Since C is a random neighborhood of A there is an $\varepsilon > 0$ for which

$$\mathbb{P}\{\omega\colon \delta((\text{int } C(\omega))^c, A(\omega)) < \varepsilon\} < \frac{\eta}{2}.$$

Since A is an attractor, there exists for this ε and η an $N > 0$ for which

$$\mathbb{P}\left\{\omega\colon d(\varphi(N, \omega, X(\omega)), A(\theta_N \omega)) > \frac{\varepsilon}{2}\right\} < \frac{\eta}{2}.$$

Now with

$$\Omega_1 := \{\delta((\text{int } C(\theta_N \cdot))^c, A(\theta_N \cdot)) \geqslant \varepsilon\},$$

$$\Omega_2 := \left\{d(\varphi(N, \cdot, X(\cdot)), A(\theta_N \cdot)) \leqslant \frac{\varepsilon}{2}\right\}$$

and

$$\Omega_3 := \{\varphi(N, \cdot, X(\cdot)) \in \operatorname{int} C(\theta_N \cdot)\}$$

we have

$$\Omega_1 \cap \Omega_2 \subset \Omega_3,$$

implying

$$\begin{aligned}
&\mathbb{P}\{\varphi(N, \cdot, X(\cdot)) \notin \operatorname{int} C(\theta_N \cdot)\} \\
&\quad \leqslant \mathbb{P}\{\delta((\operatorname{int} C(\theta_N \cdot))^c, A(\theta_N \cdot)) < \varepsilon\} + \mathbb{P}\left\{d(\varphi(N, \cdot, X(\cdot)), A(\theta_N \cdot)) > \frac{\varepsilon}{2}\right\} \\
&\quad < \frac{\eta}{2} + \frac{\eta}{2} = \eta,
\end{aligned}$$

contradicting (4.4) and completing the proof of $\mathbb{P}(\bar{\Omega}) = 1$.

Now fix some $\omega \in \bar{\Omega}$ and consider an arbitrary random compact set D. The homeomorphism property of $\varphi(t, \omega, \cdot)$ entails that for each $x \in \mathbb{R}^d$ there exists a neighborhood $U(\omega, x)$ of x and a finite $\tau(\omega, x)$ for which $\varphi(\tau(\omega, x), \omega)\, U(\omega, x) \subset \operatorname{int} C(\theta_{\tau(\omega, x)}\omega)$. As finitely many of these neighborhoods suffice to cover $D(\omega)$ it follows that $\varphi(t, \omega)\, D(\omega) \subset \operatorname{int} C(\theta_t \omega)$ after finite time. This completes the proof. ∎

As a consequence of the last proposition we are able to prove that a stable attractor is unique.

4.5. COROLLARY. *Let A be an invariant random compact set for the RDS φ. If A is asymptotically stable, then any other invariant random compact set A' satisfies $A'(\omega) \subset A(\omega)$ on an invariant ω set of full measure.*

In particular, a stable random attractor of an RDS is unique $\mathbb{P}$-a.s.

Proof. Suppose that $A' \not\subset A$ with positive probability, thus there exist $\varepsilon > 0$ and $\varepsilon' > 0$ such that

$$\mathbb{P}\{d(A' \mid A) > \varepsilon'\} = \varepsilon. \tag{4.5}$$

Since A is stable we can find a forward invariant neighborhood C of A with $\mathbb{P}\{d(C \mid A) > \varepsilon'\} < \varepsilon/2$. Now use Proposition 4.4 for this C and $D = A'$, resulting in the existence of a deterministic $T > 0$ for which

$$\mathbb{P}\{\varphi(T, \cdot)\, A'(\cdot) \not\subset C(\theta_T \omega)\} = \mathbb{P}\{A' \not\subset C\} = \mathbb{P}\{d(A' \mid C) > 0\} < \varepsilon/2.$$

Putting things together and recalling that $d(A' \mid A) \leqslant d(A' \mid C) + d(C \mid A)$ we obtain

$$\mathbb{P}\{d(A' \mid A) > \varepsilon'\} < \varepsilon,$$

contradicting (4.5) and proving that $\mathbb{P}\{d(A' \mid A) = 0\} = 1$.

The invariance of the set $\{\omega : A'(\omega) \subset A(\omega)\}$ follows from the fact that $A'(\omega) \subset A(\omega)$ implies $\varphi(t, \omega)\, A'(\omega) = A'(\theta_t \omega) \subset \varphi(t, \omega)\, A(\omega) = A(\theta_t \omega)$ for all $t \in \mathbb{R}$. ∎

5. A PROTOTYPICAL EXAMPLE

The family $\dot{x} = \alpha x - x^2$ undergoes a transcritical bifurcation at $\alpha = 0$. At the bifurcation point, $x = 0$ is asymptotically stable though with subexponential rate of attraction.

We now consider this family under parametric white noise, i.e. α is replaced by $\alpha + \sigma \xi(t)$, where $\xi(t)$ stands for white noise and $\sigma > 0$ is a strength parameter. As we are only interested in the critical case $\alpha = 0$ and the picture is qualitatively the same for any value of the strength parameter σ we put $\sigma = 1$ and consider the scalar stochastic differential equation

$$dx = -x^2\, dt + x \circ dW = (x/2 - x^2)\, dt + x\, dW \tag{5.1}$$

with two-sided time $\mathbb{R}$, where W is a standard Wiener process and $\circ$ means Stratonovich stochastic integral.

To put a stochastic differential equation in the framework of RDS, we model white noise as a metric dynamical system as follows: Let Ω be the space of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ which satisfy $\omega(0) = 0$, let $\mathscr{F}$ be the Borel sigma-algebra induced by the compact-open topology of Ω, and let $\mathbb{P}$ be the Wiener measure on $(\Omega, \mathscr{F})$, i.e. the distribution on $\mathscr{F}$ of a standard Wiener process with two-sided time. The shift θ_t is defined by $\theta_t \omega(s) := \omega(t+s) - \omega(t)$. Then $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system "driving" the stochastic differential equation (5.1), and $W_t(\omega) = \omega(t)$. We refer to [1] for details.

(5.1) is solved by a C^∞ RDS φ over θ which can be explicitly given by

$$\varphi(t, \omega, x) := \frac{x e^{W_t(\omega)}}{1 + x \int_0^t e^{W_s(\omega)}\, ds}. \tag{5.2}$$

This RDS is marred by the fact that it is *local*, i.e. $\varphi(t, \omega) : D_t(\omega) \to R_t(\omega)$ are local C^∞ diffeomorphisms with domain $D_t(\omega)$ and range $R_t(\omega)$, and the cocycle property holds whenever both sides make sense. The concept of a local RDS takes into account that solutions of stochastic differential equations may explode in finite time.

However, if we restrict the cocycle φ to the invariant subset $\mathbb{R}^+$ of the state space $\mathbb{R}$, a quick inspection of (5.2) tells us that $D_t(\omega) = \mathbb{R}^+$ and

$$R_t(\omega) = [0, r_t(\omega)), \qquad r_t(\omega) := \frac{e^{W_t(\omega)}}{\int_0^t e^{W_s(\omega)}\, ds},$$

for all $t > 0$, i.e. φ does not explode forwards in time. In particular, everything said and proved in the previous sections remains valid for $\varphi\,|_{\mathbb{R}^+}$.

Furthermore, $A = \{0\}$ is an invariant set of φ whose stability and attractivity under φ will now be studied. For further details on invariant measures, Lyapunov exponents etc. see [1, 9.3.7].

We first check in what sense $\{0\}$ is an attractor. We claim that it is *not* attracting ω-wise forwards in time. Indeed, considering the φ-invariant set $X = (0, \infty)$, the boundary points 0 and ∞ are both natural (repelling) boundaries for the diffusion process $\varphi(t, \cdot, x)$ for $t \to \infty$, implying that it is (null) recurrent, whence

$$\liminf_{t \to \infty} \varphi(t, \omega, x) = 0 \qquad \text{and} \qquad \limsup_{t \to \infty} \varphi(t, \omega, x) = \infty$$

for any (arbitrarily small!) $x \in (0, \infty)$, $\mathbb{P}$-a.s. In particular, $\{0\}$ is not "attracting in probability" in the sense of Khasminskii [14, Chap. V], i.e. we do *not* have

$$\lim_{x \to 0} \mathbb{P}\{\lim_{t \to \infty} \varphi(t, \cdot, x) = 0\} = 1.$$

However, we have for any random variable $X\colon \Omega \to (0, \infty)$

$$\begin{aligned}\varphi(t, \theta_{-t}\omega, X(\theta_{-t}\omega)) &= \frac{X(\theta_{-t}\omega)\, e^{-W_{-t}(\omega)}}{1 + X(\theta_{-t}\omega)\, e^{-W_{-t}(\omega)} \int_{-t}^0 e^{W_s(\omega)}\, ds} \\ &\leqslant \frac{1}{\int_{-t}^0 e^{W_s(\omega)}\, ds} \to 0, \ \mathbb{P}\text{-a.s.}\end{aligned}$$

Here we have used that

$$\lim_{t \to \infty} \int_{-t}^0 e^{W_s(\omega)}\, ds = \infty, \qquad \mathbb{P}\text{-a.s.}$$

(cf. [1, Lemma 2.3.41]). Hence $A = \{0\}$ is a random pull-back attractor attracting all sets of the form $[0, X(\omega)]$.

This implies that $\varphi(t, \cdot, X(\cdot)) \to 0$ in probability as $t \to \infty$ for any $X: \Omega \to (0, \infty)$, hence $A = \{0\}$ is an attractor in the sense of Definition 4.1(ii).

We now study the stability of $\{0\}$. First consider $\{0\}$ as a pullback attractor and note that

$$c(\omega, x) := \sup_{t \geqslant 0} \varphi(t, \theta_{-t}\omega, x) = \sup_{t \geqslant 0} \frac{xe^{-W_{-t}(\omega)}}{1 + xe^{-W_{-t}(\omega)} \int_{-t}^{0} e^{W_s(\omega)}\, ds} \tag{5.3}$$

is a finite random variable as the function of which the sup is taken is continuous with respect to t and tends to 0 as $t \to \infty$.

Below we prove that $\mathbb{P}\text{-}\lim_{x \to 0} c(\cdot, x) = 0$, which is equivalent to the fact that for any $\varepsilon > 0$

$$\lim_{x \to 0} \mathbb{P}\{\sup_{t \geqslant 0} |\varphi(t, \theta_{-t}\omega, x)| > \varepsilon\} = 0, \tag{5.4}$$

i.e. we have convergence in probability of the ω-wise $\sup_{t \geqslant 0}$ to 0 as in Khasminskii's definition of stability for stochastic differential equations (see below), the only difference being that here the ω-wise sup is taken of the orbit $\varphi(t, \theta_{-t}\omega, x)$ obtained by starting progressively earlier and evaluated at time 0.

Hence (5.4) offers itself as the definition of the stability of a pull-back attractor which, to our knowledge, has never been defined.

If we consider, in contrast, the sup of the orbit forwards in time, we obtain, due to the recurrence of the one-point motion (see above),

$$\sup_{t \geqslant 0} \varphi(t, \omega, x) = \infty, \qquad \mathbb{P}\text{-a.s.},$$

hence $\{0\}$ is *not* "stable in probability" in the sense of Khasminskii [14, Chap. V], i.e. we do *not* have

$$\lim_{x \to 0} \mathbb{P}\{\sup_{t \geqslant 0} |\varphi(t, \cdot, x)| > \varepsilon\} = 0 \qquad \text{for all} \quad \varepsilon > 0.$$

Khasminskii's definition thus turns out to be too strong for our case.

We claim that $\{0\}$ is stable in the sense of our Definition 4.1(i). To verify this, just note that the random compact neighborhood $C(\omega, \delta) = [0, c(\omega, \delta)]$, where c is defined in (5.3) and $\delta > 0$ is fixed, is forward invariant by definition since

$$C(\omega, \delta) = [0, c(\omega, \delta)] = \overline{\bigcup_{t \geqslant 0} \varphi(t, \theta_{-t}\omega, [0, \delta])}$$

is the closure of the forward orbit of $[0, \delta]$ under φ.

We prove that for any $\varepsilon > 0$ we can choose δ such that

$$\mathbb{P}\{d(C \mid \{0\}) \geqslant \varepsilon\} = \mathbb{P}\{c(\cdot, \delta) \geqslant \varepsilon\} < \varepsilon.$$

First note that for any $T > 0$

$$c(\omega, \delta) \leqslant \frac{\delta}{\min(Y_T(\omega), \delta \int_{-T}^{0} e^{W_s(\omega)}\, ds)},$$

where

$$Y_T(\omega) := \inf_{[0,T]} e^{W_{-t}(\omega)}$$

is a positive random variable.

Now fix $\varepsilon > 0$ and choose T so large that

$$\mathbb{P}\left\{\frac{1}{\int_{-T}^{0} e^{W_s}\, ds} < \varepsilon\right\} \geqslant 1 - \frac{\varepsilon}{2}$$

and then choose $\delta = \delta(\varepsilon)$ so small that

$$\mathbb{P}\left\{\delta < \frac{Y_T}{\int_{-T}^{0} e^{W_s}\, ds}\right\} \geqslant 1 - \frac{\varepsilon}{2}.$$

Then

$$\mathbb{P}\left\{\frac{\delta}{\min(Y_T, \delta \int_{-T}^{0} e^{W_s}\, ds)} = \frac{1}{\int_{-T}^{0} e^{W_s}\, ds}\right\} \geqslant 1 - \frac{\varepsilon}{2},$$

Thus

$$\mathbb{P}\{c(\cdot, \delta) \geqslant \varepsilon\} < \varepsilon;$$

i.e. $C = [0, c(\cdot, \delta)]$ is an ε-neighborhood of $\{0\}$ in probability satisfying

$$\varphi(t, \omega)\, C(\omega) \subset C(\theta_t \omega) \qquad \text{for all} \quad t \geqslant 0,$$

where $C(\theta_t \omega)$ is capable of expanding sometimes in order to contain the infinitely many and arbitrarily large occasional ω-wise outbursts of $\varphi(t, \omega)\, C(\omega)$ due to recurrence.

Altogether we have proved that $A = \{0\}$ is

(i) a pullback attractor which is stable in the sense of (5.3),

(ii) neither stable nor attracting in probability in the sense of Khasminskii,

(iii) asymptotically stable in the sense of Definition 4.1.

This example thus constitutes a first justification of our concepts of stability and attractivity.

6. LYAPUNOV FUNCTIONS FOR RANDOM DYNAMICAL SYSTEMS

6.1. DEFINITION (Lyapunov function). Let φ be an RDS in $\mathbb{R}^d$ and A be a random compact set which is invariant under φ. A function $V: \Omega \times \mathbb{R}^d \to \mathbb{R}^+$ is called a *Lyapunov function* for A (under φ) if it has the following properties:

(i) $\omega \mapsto V(\omega, x)$ is measurable for each $x \in \mathbb{R}^d$, and $x \mapsto V(\omega, x)$ is continuous for each $\omega \in \Omega$;

(ii) V is *uniformly unbounded*, i.e. $\lim_{\|x\| \to \infty} V(\omega, x) = \infty$ for all ω;

(iii) V is *positive-definite*, i.e. $V(\omega, x) = 0$ for $x \in A(\omega)$, and $V(\omega, x) > 0$ for $x \notin A(\omega)$;

(iv) V is *strictly decreasing along orbits* of φ, i.e.

$$V(\theta_t\omega, \varphi(t, \omega, x)) < V(\omega, x) \qquad \text{for all} \quad t > 0 \text{ and } x \notin A(\omega). \tag{6.1}$$

Property (i) of Definition 6.1 implies that $(\omega, x) \mapsto V(\omega, x)$ is measurable (cf. [4, Theorem III.14]).

Property (6.1) seems to be "incorrect" in the sense that it compares the value of V in the fiber over time t with the one in the fiber over time 0. However, writing

$$V(\Theta_t(\omega, x)) = V(\theta_t\omega, \varphi(t, \omega, x)) = C(t, \omega, x)\, V(\omega, x),$$

the flow property of Θ implies that C is a cocycle over Θ with values in the multiplicative group $\mathbb{R}_*^+$ of positive reals, and (6.1) is "correctly" read as $C(t, \omega, x) < \mathscr{E}(t, \omega, x)$, where $\mathscr{E}(t, \omega, x) \equiv 1$ is the trivial cocycle in $\mathbb{R}_*^+$ over Θ. The choice $C(t, \omega, x) = e^{-t}$ will be made below in Theorem 6.5.

Let us first show that Lyapunov functions ensure the uniqueness of invariant random compact sets in the following sense.

6.2. PROPOSITION. *Suppose there exists a Lyapunov function for A. Then any other invariant random compact set A' satisfies $A'(\omega) \subset A(\omega)$ on a θ-invariant ω set of full measure.*

Proof. Suppose we have $A'(\omega) \not\subset A(\omega)$ with positive probability. Consider

$$v(\omega) := \sup_{x \in A'(\omega)} V(\omega, x).$$

The function $v \geqslant 0$ is measurable because A' is a random compact set, and by assumption $v > 0$ with positive probability. By (6.1) for all $t > 0$ and all ω

$$v(\omega) \geqslant \sup_{x \in A'(\omega)} V(\theta_t \omega, \varphi(t, \omega, x)) = \sup_{y \in A'(\theta_t \omega)} V(\theta_t \omega, y) = v(\theta_t \omega)$$

and $v(\omega) > v(\theta_t \omega)$ with positive probability, which is in contradiction to the invariance of $\mathbb{P}$. Hence $v(\omega) = 0$, equivalently $A'(\omega) \subset A(\omega)$ $\mathbb{P}$-a.s., and the invariance of this set follows as in the proof of Corollary 4.5. ∎

The following two lemmas will play a key role in the proof of our main theorem.

6.3. LEMMA. *Let V be a Lyapunov function for A. Then for any $\delta > 0$*

$$C_\delta(\omega) := \overline{V^{-1}(\omega, [0, \delta))} = \overline{\{x: V(\omega, x) < \delta\}}$$

is a forward invariant random compact set.

Proof. (i) We first verify that C_δ is a random compact set: First note that $V^{-1}(\omega, [0, \delta))$ is open since $V(\omega, \cdot)$ is continuous and $C_\delta(\omega) \subset V^{-1}(\omega, [0, \delta])$ is compact since $V(\omega, \cdot)$ is uniformly unbounded.

Now $A(\omega) = V^{-1}(\omega, \{0\})$ is a random compact set. It can thus be countably exhausted by Lemma 3.3(vii), i.e. there are random variables $(X_{0,n})_{n \in \mathbb{N}}$ such that $A(\omega) = \overline{\{X_{0,n}(\omega): n \in \mathbb{N}\}}$.

Let $B_r(a) := \{x \in \mathbb{R}^d : \|x - a\| \leqslant r\}$ be the closed ball with center a and radius $r > 0$ and define

$$A_m(\omega) := \overline{\bigcup_{n \in \mathbb{N}} B_m(X_{0,n}(\omega))}, \qquad m \in \mathbb{N}.$$

$A_m(\omega)$ is compact, hence by Lemma 3.3(v) A_m is a random compact set. There are thus random variables $(X_{m,n})_{n \in \mathbb{N}}$ for which

$$A_m(\omega) = \overline{\{X_{m,n}(\omega) : n \in \mathbb{N}\}}.$$

We claim that those $X_{m,n}(\omega)$ which are elements of $V^{-1}(\omega, [0, \delta))$ are dense in $V^{-1}(\omega, [0, \delta)) \cap A_m(\omega)$. Indeed, let $x \in V^{-1}(\omega, [0, \delta)) \cap A_m(\omega)$.

As $V^{-1}(\omega, [0, \delta))$ is open, x has a neighborhood which is contained in $V^{-1}(\omega, [0, \delta))$ and which necessarily contains one of the $X_{m,n}(\omega)$ since they are dense in $A_m(\omega)$.

Now define the random variables

$$Y_{m,n}(\omega) := \begin{cases} X_{m,n}(\omega), & X_{m,n}(\omega) \in V^{-1}(\omega, [0, \delta)), \\ X_{0,n}(\omega), & X_{m,n}(\omega) \notin V^{-1}(\omega, [0, \delta)). \end{cases}$$

Collecting the above,

$$\begin{aligned} \overline{V^{-1}(\omega, [0, \delta)) \cap A_m(\omega)} &= \overline{\{X_{m,n}(\omega) \colon n \in \mathbb{N},\, X_{m,n}(\omega) \in V^{-1}(\omega, [0, \delta))\}} \\ &= \overline{\{Y_{m,n}(\omega) \colon n \in \mathbb{N}\}}. \end{aligned}$$

This proves that $\overline{V^{-1}(\omega, [0, \delta)) \cap A_m(\omega)}$ is a random compact set for each $m \in \mathbb{N}$, whence finally

$$\overline{V^{-1}(\omega, [0, \delta))} = \bigcup_{m \in \mathbb{N}} \overline{V^{-1}(\omega, [0, \delta)) \cap A_m(\omega)}$$

is a random compact set by Lemma 3.3(v).

(ii) We now prove that C_δ is forward invariant: For $x \in C_\delta(\omega)$ we have $0 \leqslant V(\omega, x) \leqslant \delta$, hence $0 \leqslant V(\theta_t\omega, \varphi(t, \omega, x)) < \delta$ for any $t > 0$ by Definition 4.1(iv). This even says that $\varphi(t, \omega, x) \in V^{-1}(\theta_t\omega, [0, \delta)) \subset \operatorname{int} C_\delta(\theta_t\omega)$ for any $t > 0$. ∎

6.4. Lemma. *Let V be a Lyapunov function for A and φ. Denote for any fixed $\varepsilon > 0$ by $B_\varepsilon(A(\omega)) := \{x \colon d(x, A(\omega)) \leqslant \varepsilon\}$ the closed ε-neighborhood of $A(\omega)$. Then for any $\varepsilon > 0$ and $\varepsilon' > 0$ there exists a $\delta > 0$ such that*

(i)

$$\mathbb{P}\{\overline{V^{-1}(\cdot, [0, \delta))} \subset B_\varepsilon(A(\cdot))\} \geqslant 1 - \varepsilon',$$

(ii)

$$\mathbb{P}\{X(\cdot) \in B_\varepsilon(A(\cdot))\} \geqslant 1 - \varepsilon'$$

for any random variable X satisfying $\mathbb{P}\{V(\cdot, X(\cdot)) \leqslant \delta/2\} \geqslant 1 - \varepsilon'/2$.

Proof. (i) (a) First fix ω. Then for any $\varepsilon > 0$ we can find a $\delta = \delta(\omega)$ such that $d(y, A(\omega)) \leqslant \varepsilon$ provided $V(\omega, y) \leqslant \delta$. In the contrary case we would have an $\varepsilon_0 > 0$ and a sequence (y_n) with $V(\omega, y_n) \leqslant 1/n$ such that $d(y_n, A(\omega)) > \varepsilon_0$. Since V is uniformly unbounded the preimages

$V^{-1}(\omega, [0, 1/n])$ are compact. We can select a subsequence $(y_{n'})$ converging to y_0 such that $d(y_0, A(\omega)) \geqslant \varepsilon_0$. But then $V(\omega, y_0) > 0$ since V is positive-definite which is in contradiction to the fact that $V(\omega, y_0) \leqslant 1/n$ for any $n \in \mathbb{N}$.

(b) By step (a), by a standard argument and by the fact that $\overline{V^{-1}(\omega, [0, \delta))} \subset V^{-1}(\omega, [0, \delta])$ we can find for any $\varepsilon > 0$ and $\varepsilon' > 0$ a nonrandom $\delta > 0$ such that

$$\mathbb{P}\{\overline{V^{-1}(\cdot, [0, \delta))} \subset B_\varepsilon(A(\cdot))\} \geqslant 1 - \varepsilon'/2. \tag{6.2}$$

This proves (i).

(ii) Take the choice of δ from (i) and assume that X is a random variable for which

$$\mathbb{P}\{X \notin \overline{V^{-1}(\cdot, [0, \delta))}\} \leqslant \mathbb{P}\{V(\cdot, X(\cdot)) > \delta/2\} < \varepsilon'/2. \tag{6.3}$$

Since

$$\{X \notin B_\varepsilon(A)\} \subset \{X \notin \overline{V^{-1}(\cdot, [0, \delta))}\} \cup \{\overline{V^{-1}(\cdot, [0, \delta))} \not\subset B_\varepsilon(A)\},$$

the result follows using (6.2) and (6.3). ∎

Here is the main result of this paper which gives the final justification of our concepts. It reduces to the deterministic Theorem 2.2 if the noise is absent. We used the proof of the deterministic theorem as some guideline for ours.

6.5. THEOREM. *Let φ be an RDS in $\mathbb{R}^d$ and let A be a random compact set which is invariant under φ. Then A is asymptotically stable if and only if there exists a Lyapunov function for A.*

If A is asymptotically stable, then the Lyapunov function can be chosen to satisfy

$$V(\theta_t\omega, \varphi(t, \omega, x)) = e^{-t} V(\omega, x) \qquad \textit{for all} \quad t \in \mathbb{R},\ x \in \mathbb{R}^d \tag{6.4}$$

on a θ-invariant ω set of full measure.

Proof. As the proof is rather involved we split it into three parts:

Part 1: *Existence of Lyapunov function implies asymptotic stability.*

(i) A is stable: Take an arbitrary $\varepsilon > 0$ and choose $\delta > 0$ according to Lemma 6.4 with $\varepsilon' = \varepsilon$. We know from Lemma 6.3 that

$$C_\delta(\omega) := \overline{V^{-1}(\omega, [0, \delta))} = \overline{\{x: V(\omega, x) < \delta\}}$$

is a forward invariant random compact set which clearly is a neighborhood of A. Since $d(C_\delta(\omega) \mid A(\omega)) > \varepsilon$ if and only if $C_\delta(\omega) \not\subset B_\varepsilon(A(\omega))$, it follows with the above choice of δ that $\mathbb{P}\{d(C_\delta \mid A) \geqslant \varepsilon\} < \varepsilon$. Hence A is stable.

(ii) A is attractive: Let X be any random variable. Since V is uniformly unbounded we can choose for any $\varepsilon > 0$ an $N > 0$ such that

$$\mathbb{P}\{X(\cdot) \in C_N(\cdot)\} \geqslant 1 - \frac{\varepsilon}{2}.$$

Again by Lemma 6.3, C_N is a forward invariant random compact set which is a neighborhood of A.

By Lemma 4.3

$$A'_N(\omega) := \bigcap_{t \geqslant 0} \varphi(t, \theta_{-t}\omega)\, C_N(\theta_{-t}\omega)$$

is an invariant random compact set. Moreover, A'_N is attracting C_N in the "pull-back" sense, i.e.

$$\lim_{t \to \infty} d(\varphi(t, \theta_{-t}\omega, C_N(\theta_{-t}\omega)) \mid A'_N(\omega)) = 0, \tag{6.5}$$

implying that for all $t \geqslant T(\varepsilon, N)$

$$\mathbb{P}\{d(\varphi(t, \cdot, C_N(\cdot)) \mid A'_N(\theta_t \cdot)) > \varepsilon\} < \varepsilon/2.$$

Since $A(\omega) \subset A'_N(\omega)$, Proposition 6.2 entails that $A = A'_N$ for any N on an invariant set of full measure. Using

$$\begin{aligned} &d(\varphi(t, \omega, X(\omega)), A(\theta_t\omega)) \\ &\qquad \leqslant d(\varphi(t, \omega, X(\omega)), \varphi(t, \omega, C_N(\omega))) + d(\varphi(t, \omega, C_N(\omega)) \mid A(\theta_t\omega)) \end{aligned}$$

and the fact that $X(\omega) \in C_N(\omega)$ if and only $\varphi(t, \omega, X(\omega)) \in \varphi(t, \omega, C_N(\omega))$ it follows that

$$\mathbb{P} - \lim_{t \to \infty} d(\varphi(t, \cdot)\, X(\cdot), A(\theta_t \cdot)) = 0;$$

hence A is attracting.

Remark. (6.5) says that the existence of a Lyapunov function implies that A is a random pull-back attractor, attracting all sets of the form $C_N(\omega)$, $N > 0$, hence all random variables taking values in one of these sets. *Arbitrary* random variables, however, are in general attracted only in the sense of convergence in probability.

Part 2: Existence of absorbing set implies existence of Lyapunov function. This statement is of independent interest. We hence formulate it as a proposition.

6.6. PROPOSITION. *If A is an attractor of φ and if there exists a forward invariant random compact set C which is a random neighborhood of A and has the additional property that*

$$\varphi(t, \omega, x) \in C(\theta_t \omega) \Rightarrow \varphi(s, \omega, x) \in \operatorname{int} C(\theta_s \omega) \qquad \textit{for all} \quad s > t, \tag{6.6}$$

then there exists a Lyapunov function satisfying (6.4) *on an invariant set of full measure.*

In particular, A is asymptotically stable.

We now present the proof of this proposition in five steps:

(i) We define the first entrance time of $\varphi(t, \omega, x)$ into $C(\theta_t \omega)$ as follows:

$$\tau(\omega, x) := \begin{cases} \inf \{t \in \mathbb{R} : \varphi(t, \omega, x) \in C(\theta_t \omega)\}, & x \notin A(\omega), \\ -\infty, & x \in A(\omega). \end{cases}$$

By property (6.6) and the forward invariance of C and $\operatorname{int} C$, $\tau(\omega, x)$ is the unique time for which $\varphi(t, \omega, x)$ is outside of $C(\theta_t \omega)$ for all $t < \tau(\omega, x)$ and inside of $\operatorname{int} C(\theta_t \omega)$ for all $t > \tau(\omega, x)$.

For $x \notin A(\omega)$, $\tau(\omega, x)$ is finite on an invariant set of full measure. Indeed, if $x \notin C(\omega)$ we have $0 \leqslant \tau(\omega, x) < \infty$ by Proposition 4.4. If $x \in C(\omega) \setminus A(\omega)$ we have $-\infty < \tau(\omega, x) \leqslant 0$. In the contrary case $\varphi(-t, \omega, x)$ would never leave $C(\theta_{-t} \omega)$. But

$$\begin{aligned} 0 < d(x, A(\omega)) &= d(\varphi(t, \theta_{-t} \omega, \varphi(-t, \omega, x)), A(\omega)) \\ &\leqslant d(\varphi(t, \theta_{-t} \omega, C(\theta_{-t} \omega)), A(\omega)) \to 0 \qquad \text{as} \quad t \to \infty \end{aligned}$$

by Lemma 4.3 which is a contradiction.

(ii) $\omega \mapsto \tau(\omega, x)$ is measurable for each $x \in \mathbb{R}^d$. Indeed, since C is a random compact set, $\omega \mapsto d(x, C(\omega))$ is measurable, hence so is $\omega \mapsto d(\varphi(t, \omega, x), C(\theta_t \omega))$ for any $t \in \mathbb{R}$. This implies that

$$\{\omega : \tau(\omega, x) \geqslant t\} = \bigcap_{s < t,\, s \in \mathbb{Q}} \{\omega : d(\varphi(s, \omega, x), C(\theta_s \omega)) > 0\}$$

is measurable.

(iii) $\tau(\omega,\cdot)\colon A(\omega)^c \to \mathbb{R}$ is continuous for each ω from the invariant set of full measure from above. Let $\varepsilon > 0$. By (6.6) $\varphi(\tau(\omega, x)+\varepsilon, \omega, x) \in \operatorname{int} C(\theta_{\tau(\omega, x)+\varepsilon}\omega)$, hence there exists a neighborhood $U_\omega^1(x)$ of x such that for $y \in U_\omega^1(x)$

$$\tau(\omega, y) \leqslant \tau(\omega, x)+\varepsilon.$$

On the other hand, $\varphi(\tau(\omega, x)-\varepsilon, \omega, x) \in C(\theta_{\tau(\omega, x)-\varepsilon}\omega)^c$ which is open. Hence there exists a neighborhood $U_\omega^2(x)$ of x such that for $y \in U_\omega^2(x)$

$$\tau(\omega, y) \geqslant \tau(\omega, x)-\varepsilon,$$

altogether

$$\tau(\omega, x)-\varepsilon \leqslant \tau(\omega, y) \leqslant \tau(\omega, x)+\varepsilon$$

for $y \in U_\omega^1(x) \cap U_\omega^2(x)$, which proves continuity of $\tau(\omega,\cdot)$.

(iv) We now prove that $\tau(\omega, x) \to -\infty$ as $x \to A(\omega)$, $x \notin A(\omega)$. Combined with (iii) this establishes the continuity of $x \mapsto \tau(\omega, x)$ on all of $\mathbb{R}^d$, which in turn together with (ii) assures the measurability of $(\omega, x) \mapsto \tau(\omega, x)$.

Suppose there exists a sequence (x_n) with $x_n \to A(\omega)$ such that $\tau(\omega, x_n)$ is bounded from below. Hence there is a converging subsequence $x_{n'} \to \bar{x} \in A(\omega)$ for which $\tau(\omega, x_{n'}) \to T(\omega) > -\infty$.

By the continuity of $(t, x) \mapsto \varphi(t, \omega, x)$ for each fixed ω and the invariance of A

$$\varphi(T-1, \omega, x_{n'}) \to \varphi(T-1, \omega, \bar{x}) \in A(\theta_{T-1}\omega).$$

On the other hand we have

$$\varphi(T-1, \omega, x_{n'}) \notin C(\theta_{T-1}\omega)$$

for sufficiently large n'. But this is a contradiction because $C(\theta_{T-1}\omega)$ is a neighborhood of $A(\theta_{T-1}\omega)$ by assumption.

(v) We now prove that

$$V(\omega, x) := e^{\tau(\omega, x)}$$

is a Lyapunov function with the additional property (6.4).

We just have proved that V satisfies properties (i) and (iii) of Definition 6.1. Property (iv) will follow from (6.4) which we will show next.

By the definition of τ and the cocycle property

$$\begin{aligned}\tau(\theta_t\omega, \varphi(t, \omega, x)) &= \inf\{s \in \mathbb{R} : \varphi(s, \theta_t\omega, \varphi(t, \omega, x)) \in C(\theta_{s+t}\omega)\}\\ &= \inf\{s \in \mathbb{R} : \varphi(s+t, \omega, x) \in C(\theta_{s+t}\omega)\}\\ &= \tau(\omega, x)-t,\end{aligned}$$

proving (6.4).

There remains to be shown that $V(\omega, \cdot)$ is uniformly unbounded. Due to Proposition 4.4 and the forward invariance of C the compact sets

$$D_n(\omega) := \varphi(-n, \theta_n\omega, C(\theta_n\omega))$$

of those x whose orbits have entered C at time n tend to $\mathbb{R}^d$ as $n \to \infty$. For $x \notin D_n(\omega)$ we have $\tau(\omega, x) \geqslant n$, thus $V(\omega, x) \geqslant e^n$ outside the compact set $D_n(\omega)$, proving uniform unboundedness.

This completes the proof of Proposition 6.6 and of Part 2.

Part 3: *Asymptotic stability implies existence of forward invariant random compact set with property* (6.6).

(i) Since A is stable, there is a forward invariant random compact set C_1 which is a neighborhood of A. The problem is that C_1 might not have the crucial property (6.6) which allowed us to prove the continuity of τ and hence of V. It is thus necessary to pass to an improved forward invariant set via constructing a Lyapunov function of C_1 and φ and taking the preimage of $[0, 1)$.

For this purpose define

$$V_1(\omega, x) := \sup_{t \geqslant 0} d(\varphi(t, \omega, x), C_1(\theta_t\omega)). \tag{6.7}$$

First note that $V_1(\omega, x) = 0$ if and only if $x \in C_1(\omega)$ due to the forward invariance of C_1. Also, since $V_1(\omega, x) \geqslant d(x, C_1(\omega)) \to \infty$ as $\|x\| \to \infty$, V_1 is uniformly unbounded.

Secondly, we have for each fixed $s > 0$ by the cocycle property

$$\begin{aligned}V_1(\theta_s\omega, \varphi(s, \omega, x)) &= \sup_{t \geqslant 0} d(\varphi(t+s, \omega, x), C_1(\theta_{t+s}\omega))\\ &= \sup_{t \geqslant s} d(\varphi(t, \omega, x), C_1(\theta_t\omega))\\ &\leqslant V_1(\omega, x),\end{aligned}$$

i.e. V_1 is decreasing along orbits.

We now show that $V_1(\omega, x)$ is finite. By Proposition 4.4 for each $x \in \mathbb{R}^d$ there is a $T = T(\omega, x) < \infty$ for which $\varphi(t, \omega, x) \in C_1(\theta_t\omega)$ for all $t \geqslant T$, implying that

$$V_1(\omega, x) = \sup_{0 \leqslant t \leqslant T} d(\varphi(t, \omega, x), C_1(\theta_t\omega)). \tag{6.8}$$

On the other hand, forward invariance of C_1 gives

$$C_1(\theta_t\omega) \subset \varphi(t-T, \theta_T\omega)\, C_1(\theta_T\omega) \qquad \text{for all} \quad 0 \leqslant t \leqslant T.$$

Since the image of $[0, T] \times C_1(\theta_T\omega)$ under the continuous map $(t, x) \mapsto \varphi(t-T, \theta_T\omega, x)$ is compact, (6.8) yields that $V_1(\omega, x) < \infty$.

We now prove that $\omega \mapsto V_1(\omega, x)$ is measurable for each $x \in \mathbb{R}^d$. Since for each fixed $t \geqslant 0$ and $x \in \mathbb{R}^d$, $\omega \mapsto d(\varphi(t, \omega, x), C_1(\theta_t\omega))$ is measurable, it suffices to make sure that we can replace the $\sup_{t \geqslant 0}$ by $\sup_{t \geqslant 0,\, t \in \mathbb{Q}}$ in (6.7).

Clearly

$$\sup_{t \geqslant 0} d(\varphi(t, \omega, x), C_1(\theta_t\omega)) \geqslant \sup_{t \geqslant 0,\, t \in \mathbb{Q}} d(\varphi(t, \omega, x), C_1(\theta_t\omega)).$$

We will prove that

$$\sup_{t \geqslant 0} d(\varphi(t, \omega, x), C_1(\theta_t\omega)) \leqslant \sup_{t \geqslant 0,\, t \in \mathbb{Q}} d(\varphi(t, \omega, x), C_1(\theta_t\omega)). \tag{6.9}$$

Choose an arbitrary $s > 0$. It can be easily checked that the continuity of $(t, x) \mapsto \varphi(t, \omega, x)$ entails that $t \mapsto \varphi(t-s, \theta_s\omega, C_1(\theta_s\omega))$ is continuous in the Hausdorff metric d_H.

By the forward invariance of C_1 for $0 \leqslant t \leqslant s$

$$C_1(\theta_t\omega) \subset \varphi(t-s, \theta_s\omega)\, C_1(\theta_s\omega);$$

hence

$$d(\varphi(t, \omega, x), \varphi(t-s, \theta_s\omega)\, C_1(\theta_s\omega)) \leqslant d(\varphi(t, \omega, x), C_1(\theta_t\omega)),$$

and, due to continuity,

$$\lim_{t \to s} d(\varphi(t, \omega, x), \varphi(t-s, \theta_s\omega)\, C_1(\theta_s\omega)) = d(\varphi(s, \omega, x), C_1(\theta_s\omega)).$$

For each $\varepsilon > 0$ and $s > 0$ there is thus a rational $r \leqslant s$ for which

$$d(\varphi(r, \omega, x), C_1(\theta_r \omega)) \geqslant d(\varphi(s, \omega, x), C_1(\theta_s \omega)) - \varepsilon,$$

from which (6.9) follows.

We next show that $x \mapsto V_1(\omega, x)$ is continuous for every ω. This will be a consequence of the fact that a stable attractor A is *uniformly* attracting. Indeed, by Proposition 4.4 for any closed ball $B_\varepsilon(x)$ with center x there exists a $T(\omega)$ such that

$$\varphi(t, \omega)\, B_\varepsilon(x) \subset \operatorname{int} C_1(\theta_t \omega) \qquad \text{for all} \quad t \geqslant T(\omega).$$

By the triangle inequality for any $y \in B_\varepsilon(x)$

$$|V_1(\omega, x) - V_1(\omega, y)| \leqslant \sup_{0 \leqslant t \leqslant T(\omega)} \|\varphi(t, \omega, x) - \varphi(t, \omega, y)\| \to 0$$

as $y \to x$ since T is independent of $y \in B_\varepsilon(x)$.

As a result, $(\omega, x) \mapsto V_1(\omega, x)$ is measurable.

(ii) As V_1 might not be *strictly* decreasing along orbits, we consider instead

$$V_2(\omega, x) := \int_0^\infty e^{-t} V_1(\theta_t \omega, \varphi(t, \omega, x))\, dt + V_1(\omega, x)\,.$$

Since the integrand is dominated by $e^{-t} V_1(\omega, x)$ for all t, the integral makes sense, and $V_1(\omega, x) \leqslant V_2(\omega, x) \leqslant 2V_1(\omega, x)$, proving that V_2 is uniformly unbounded since V_1 is.

Furthermore, V_2 is measurable since V_1 is, and $x \mapsto V_2(\omega, x)$ is continuous by the continuity of V_1 and $\varphi(t, \omega, \cdot)$ and the dominated convergence theorem.

Clearly $x \in C_1(\omega)$ if and only if $V_2(\omega, x) = 0$.

Further, the fact that V_1 is decreasing along orbits clearly implies that V_2 is also decreasing along orbits.

To show that V_2 is strictly decreasing we have to rule out $V_2(\theta_s \omega, \varphi(s, \omega, x)) = V_2(\omega, x) > 0$ for some (ω, x) with $x \notin C_1(\omega)$ for fixed $s > 0$. Observe that in this case we must have, by the monotonicity of V_1,

$$V_1(\theta_t \omega, \varphi(t, \omega, x)) = V_1(\omega, x) > 0 \qquad \text{for all} \qquad 0 \leqslant t \leqslant s, \tag{6.10}$$

and

$$V_1(\theta_{s+t} \omega, \varphi(s+t, \omega, x)) = V_1(\theta_t \omega, \varphi(t, \omega, x)) \qquad \text{for Leb-almost all} \quad t \geqslant 0$$

(due to monotonicity, the exceptional set is even at most countable). Thus, using the fact that the shift of a set of full Lebesgue measure by ns and the intersection of countably many sets of full measure have full measure,

$$V_1(\theta_{ns+t}\omega, \varphi(ns+t, \omega, x)) = V_1(\theta_t\omega, \varphi(t, \omega, x)) \tag{6.11}$$

for all $n \in \mathbb{N}$ and for Leb-almost all $t \geqslant 0$.

There thus exists a $\tau \geqslant 0$ for which both (6.10) as well as (6.11) hold, i.e. we have

$$V_1(\theta_{ns+\tau}\omega, \varphi(ns+\tau, \omega, x)) = V_1(\omega, x) > 0 \qquad \text{for all} \quad n \in \mathbb{N}. \tag{6.12}$$

On the other hand, since C_1 absorbs all orbits in finite time, there exists a $T < \infty$ for which

$$\varphi(ns+\tau, \omega, x) \in C_1(\theta_{ns+\tau}\omega) \qquad \text{for all} \quad n \geqslant T,$$

hence

$$V_1(\theta_{ns+\tau}\omega, \varphi(ns+\tau, \omega, x)) = 0 \qquad \text{for all} \quad n \geqslant T,$$

contradicting (6.12).

As a result, V_2 is strictly decreasing along orbits which start at points $x \notin C_1(\omega)$.

(iii) Define

$$C_2(\omega) := \overline{V_2^{-1}(\omega, [0, 1))}.$$

A slight modification of the proof of Lemma 6.3 yields that this is a forward invariant random compact set. Since $A \subset C_1 \subset C_2$, C_2 is a random neighborhood of A. Finally, since V_2 is strictly decreasing along orbits starting outside C_1, $\varphi(t, \omega, x) \in C_2(\theta_t\omega)$ (hence $0 \leqslant V_2(\theta_t\omega, \varphi(t, \omega, x)) \leqslant 1$) implies (using once more the cocycle property) that $0 \leqslant V_2(\theta_s\omega, \varphi(s, \omega, x)) < 1$, thus $\varphi(s, \omega, x) \in \operatorname{int} C_2(\theta_s\omega) = V_2^{-1}(\theta_s\omega, [0, 1))$ for all $s > t$. Hence C_2 satisfies (6.6).

This terminates the proof of Part 3 and hence of the theorem. ∎

In order not to overburden this paper, the case where A is not necessarily invariant, the theory of local attractors and the infinitesimal form of "decreasing along orbits" will be dealt with elsewhere.

We also claim that in case φ is solving a stochastic differential equation and A is a deterministic set, we can recover the classical stochastic stability theory based on the theory of Markov processes (see e.g. Khasminskii [14, Chap. V] or Mao [18]) from our approach by conditioning our random Lyapunov function "on the past of the noise".

7. FURTHER EXAMPLES

The Stochastic Lorenz System

We consider the Lorenz system under multiplicative white noise in $\mathbb{R}^3$, described by the Stratonovich stochastic differential equation

$$dx = (f + Bx + F(x))\, dt + \sigma x \circ dW, \tag{7.1}$$

where

$$f = \begin{pmatrix} 0 \\ 0 \\ -b(r+s) \end{pmatrix}, \quad B = \begin{pmatrix} -s & s & 0 \\ -s & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \quad F(x) = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix},$$

s, r, b and σ are positive constants and f is an external force. For the deterministic version of this equation ($\sigma = 0$) see Temam [23, p. 34].

Due to the fact that $\langle F(x), x\rangle = 0$ we have

$$d\,\|x\|^2 = 2\langle f + Bx, x\rangle\, dt + 2\sigma\,\|x\|^2 \circ dW, \tag{7.2}$$

implying that the solution of (7.1) does not explode in finite positive or negative time, and (7.1) generates an RDS φ in the sense of Definition 3.1.

Schmalfuss [22] proved that for any value of the above parameters φ has a random attractor A in the sense of Definition 4.1(ii).

We are now going to investigate its stability.

By the fact that $\langle Bx, x\rangle \leqslant -c_1\,\|x\|^2$, where $c_1 = \min(1, b, s)$, the drift in (7.2) can be estimated by

$$2\langle f + Bx, x\rangle \leqslant -2c_1\,\|x\|^2 + 2\,\|f\|\,\|x\| < -c_1\,\|x\|^2 + c_2, \tag{7.3}$$

where $c_2 > \|f\|^2/c_1$.

By the comparison principle for scalar stochastic differential equations, the solution of (7.2) for $t \geqslant 0$ is hence dominated by the solution of the affine equation

$$dz = (-c_1 z + c_2)\, dt + 2\sigma z \circ dW$$

which is known to have a unique stationary solution with random initial value

$$z_0(\omega) = c_2 \int_{-\infty}^{0} \exp(c_1 t - 2\sigma W_t(\omega))\, dt$$

(see e.g. [1, 5.6]). The closed ball $C(\omega)$ with center zero and radius $z_0(\omega)$ defines a forward invariant random compact set for φ (see [22]). By the fact that the second inequality in (7.3) is strict, C satisfies property (6.6). By Proposition 6.6, there exists a Lyapunov function for A with the additional property (6.4). As a consequence, A is asymptotically stable.

Affine Random Dynamical Systems

Affine RDS (i.e. RDS for which $\varphi(t, \omega)$ takes values in the affine group) are generated by affine random or stochastic differential equations (see [1, 5.6] for a detailed study). For brevity we only consider the stochastic case, i.e. we study the affine stochastic differential equation in $\mathbb{R}^d$ given by

$$dx = \sum_{j=0}^{m} (A_j x + b_j) \circ dW^j, \qquad \circ\, dW^0 := dt, \tag{7.4}$$

where $A_j \in \mathbb{R}^{d \times d}$ and $b_j \in \mathbb{R}^d$, $j = 0, \ldots, m$. (7.4) generates an affine RDS φ which can be represented by the variation of constants formula as

$$\varphi(t, \omega, x) = \Phi(t, \omega) \left(x + \sum_{j=0}^{m} \int_0^t \Phi(s, \omega)^{-1}\, b_j \circ dW_s^j(\omega) \right), \tag{7.5}$$

where Φ is the linear RDS (fundamental matrix) generated by the corresponding linear stochastic differential equation

$$dx = \sum_{j=0}^{m} A_j x \circ dW^j.$$

Suppose that the top Lyapunov exponent λ of Φ is negative,

$$\lambda := \lim_{t \to \infty} \frac{1}{t} \log \|\Phi(t, \omega)\| < 0.$$

Then

$$x_0(\omega) := \sum_{j=0}^{m} \int_{-\infty}^{0} \Phi(t, \omega)^{-1}\, b_j \circ dW_t^j(\omega)$$

is the initial value of the unique stationary solution of φ, i.e. for which $\varphi(t, \omega, x_0(\omega)) = x_0(\theta_t \omega)$.

Clearly $A(\omega) = \{x_0(\omega)\}$ is a φ-invariant random compact set. It is an attractor since for any random variable X

$$\varphi(t, \omega, X(\omega)) - x_0(\theta_t \omega) = \Phi(t, \omega)(X(\omega) - x_0(\omega)) \to 0$$

as $t \to \infty$ even $\mathbb{P}$-a.s. and exponentially fast due to $\lambda < 0$.

We claim that $\{x_0\}$ is stable with corresponding Lyapunov function

$$V(\omega, x) := \|x_0(\omega) - x\|_{\kappa,\,\omega}\,,$$

where $\|\cdot\|_{\kappa,\,\omega}$ is a random Lyapunov norm (see [1, 4.3]) which makes Φ contracting, i.e. such that

$$\|\Phi(t, \omega)\|_{\kappa,\,\omega,\,\theta_t\omega} \leqslant e^{(\lambda+\kappa)\,t} \qquad \text{for} \quad t \geqslant 0, \tag{7.6}$$

provided the constant $\kappa > 0$ is chosen such that $\lambda + \kappa < 0$, which we assume.

It is clear that V has the properties (i), (ii) and (iii) of Definition 6.1. V is also strictly decreasing along orbits since by (7.6)

$$\begin{aligned} V(\theta_t\omega, \varphi(t, \omega, x)) &= \|\Phi(t, \omega)(x_0(\omega) - x)\|_{\kappa,\,\omega} \\ &\leqslant \|\Phi(t, \omega)\|_{\kappa,\,\omega,\,\theta_t\omega}\, V(\omega, x) \\ &\leqslant e^{(\lambda+\kappa)\,t}\, V(\omega, x) < V(\omega, x) \qquad \text{for all} \quad t > 0. \end{aligned}$$

Consequently, V is a Lyapunov function for A, hence $A = \{x_0\}$ is asymptotically stable by Theorem 6.5.

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