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Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators

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A B S T R A C T

For singular elliptic fully-nonlinear operators we prove that the eigenfunctions corresponding to the principal eigenvalues in bounded domains are simple. The proof uses in particular a regularity result obtained in the first part of the paper which is of independent interest.

1. Introduction

The concept of principal eigenvalue for boundary value problems of elliptic operators has been extended, in the last decades, to quasi-nonlinear and fully-nonlinear equations (somehow "abusing" the name of eigenvalue) see [1,2,26,28,29,22,7,23], etc. In all the cases we know, two features of the operators are requested, *homogeneity* and *ellipticity*.

The meta-definition of these principal eigenvalues could be the following: Given a zero order and odd operator H with the same homogeneity than the second order elliptic operator F, and given a domain Ω, λ is a principal eigenvalue if there exists a nontrivial solution of constant sign of the problem

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$$\begin{cases} F(x, \nabla u, D^2u) + \lambda H(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

not surprisingly, that function will be called an "eigenfunction".

Of course when F is a second order linear elliptic operator and $H(u) = u$, the principal eigenvalue is just the first eigenvalue in the "classical" sense, it is well known that it is both *simple* and *isolated*.

It is interesting to notice that when the operator is not "odd" with respect to the Hessian, there may be two principal eigenvalues, one corresponding to a positive eigenfunction and one corresponding to a negative eigenfunction. This is the case when F is one of the Pucci operators e.g. for some $0 < a < A$

$$F(D^2u) := \mathcal{M}^+_{a,A}(D^2u) := \sup_{aI \leqslant M \leqslant AI} Tr(MD^2u)$$

(see [10]). In these cases, or in general when the operator is uniformly elliptic and homogenous of degree 1, the principal eigenvalues have been proved to be simple and isolated (see [29,22,27]).

When the operator is "quasi-linear" but in divergence form, for example in the case of the p-Laplacian, it is well known that the principal eigenvalue can be defined through the Rayleigh quotient and it was proved, independently by Anane [1] and by Ôtani and Teshima [26], that it is simple and isolated (see also [25]). The variational structure plays a key role there. On the other hand, for the ∞-Laplacian (see [23]), the question of the simplicity of the principal eigenvalue is still open.

The cases treated in this paper concern operators that have the "homogeneity" of the p-Laplacian, but are "fully-nonlinear" and hence are not variational. In previous works, we proved the existence of the principal eigenvalues for this large class of operators and many features related to them [6–9]. The inspiration for these definitions and results was the acclaimed work of Berestycki, Nirenberg and Varadhan [5] where the eigenvalue for linear elliptic operators in general bounded domains was defined through the maximum principle.

The main questions left open in our previous works were: Are these eigenvalues "simple"? Are they "isolated"?

We shall now proceed to describe the results obtained in this note but for the sake of comprehension we shall do it for an operator that exemplifies well the cases treated here (the general conditions and hypothesis will be given in the next section). For some $0 \geqslant \alpha > -1$, and some Hölder continuous function h of exponent $1 + \alpha$, let

$$F[u] := F(x, \nabla u, D^2u) = |\nabla u|^\alpha \mathcal{M}^+_{a,A}(D^2u) + h(x) \cdot \nabla u |\nabla u|^\alpha.$$

Suppose that Ω is a bounded, smooth domain of $\mathbb{R}^N$. Then we can define

$$\lambda^+ := \sup\{\lambda;\ \exists\phi > 0,\ \phi \in \mathcal{C}(\overline{\Omega}),\ F[\phi] + \lambda\phi^{1+\alpha} \leqslant 0 \text{ in } \Omega\},$$
$$\lambda^- := \sup\{\lambda;\ \exists\phi < 0,\ \phi \in \mathcal{C}(\overline{\Omega}),\ F[\phi] + \lambda\phi|\phi|^\alpha \geqslant 0 \text{ in } \Omega\}.$$

It is clear that the inequalities are meant in the "viscosity sense" adapted to these non-smooth operators (see the next section for a precise definition).

For a much more general context and for any $\alpha > -1$ we proved in [7] that these are well defined. For any $\lambda < \lambda^+$ the maximum principle holds i.e. if u is a viscosity sub-solution of

$$\begin{cases} F[u] + \lambda u|u|^\alpha = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{1}$$

then $u \leqslant 0$ in Ω. Furthermore, for any $\lambda < \min\{\lambda^+, \lambda^-\}$ and any continuous f there exists a solution of

$$\begin{cases} F[u]+\lambda u|u|^{\alpha}=f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

which is Lipschitz continuous.

Of course we also proved that there exist ϕ^+ and ϕ^- respectively positive and negative eigenfunctions in the sense that e.g. for $\lambda=\lambda^+$ there exists $\phi^+>0$ viscosity solution of (1).

One of the question we answer here is, if $\psi>0$ is another solution of (1) with $\lambda=\lambda^+$, is it true that there exists $t>0$ such that $\phi^+=t\psi$? The answer is yes for $\alpha\in(-1,0]$, for any domain Ω such that $\partial\Omega$ has only one connected component. When $\partial\Omega$ has two connected components we can prove the result when $N=2$.

Let us recall that for any $\alpha>-1$ and any N we proved in [8] the simplicity of the eigenvalue for radial solutions.

It is clear that these results are somehow equivalent to a "strong comparison principle" i.e. it is equivalent to know that if two solutions are one above the other in some open set $\mathcal{O}$ and they "touch" at some point of $\mathcal{O}$ then they coincide in $\mathcal{O}$. In Proposition 4.4, we prove such a result when at least one of the solutions has the gradient bounded away from zero in $\mathcal{O}$.

This restriction implies that in order to apply Proposition 4.4 we need to know that there is some subset $\mathcal{O}$ of Ω where this condition is satisfied. Naturally the Hopf's lemma, together with a C^1 regularity, is the convenient ingredient since it guarantees that the gradient is bounded away from zero in a neighborhood of $\partial\Omega$. This explains why we start by proving a $\mathcal{C}^{1,\beta}$ regularity result, which is interesting in itself, but mainly it is essential in the proof of the simplicity of the principal eigenvalue.

The regularity result is proved through a fixed point theorem. Let us mention that in general the tools to prove regularity are the Alexandroff–Bakelman–Pucci (ABP) inequality and some "sublinearity" of the operator. In a recent paper Davila, Felmer and Quaas have proved ABP [17] for fully-nonlinear operators, but in this case it does not seem to be useful to prove $\mathcal{C}^{1,\beta}$ regularity because the difference of a sub- and super-solution may not be a sub-solution of an elliptic equation. It is interesting to remark that Imbert [21] used ABP to prove a local Hölder regularity.

The dimensional restriction is due to the fact that when there are two connected components of the boundary of Ω, we are led to use Sard's theorem. A famous counterexample of Whitney shows that Sard's theorem does not hold if the functions are only C^1. It seems that the least regularity that can be asked is $C^{N-1,1}$ (see e.g. [4]), and since we cannot hope for a better regularity than $\mathcal{C}^{2,\beta}$ with $\beta\in(0,1)$ we have to require that $N=2$.

When the boundary is connected, it is enough to know that the solutions are $\mathcal{C}^{1,\beta}$; this is the regularity we obtain for the solutions, when α is negative.

Other important results concerning eigenvalues are given as a consequence of simplicity. In particular we prove that there are no eigenfunctions that change sign for $\lambda=\lambda^{\pm}$. Further results include the strict monotonicity of the eigenvalue with respect to the inclusion of domains. And finally that the eigenvalues are isolated.

The paper is organized as follows: In the next section we give the precise hypothesis concerning the operator and we recall the known results concerning singular operators. In section three we prove the $C^{1,\beta}$ regularity of the solutions. Section four is devoted to the strong comparison principle and simplicity of the principal eigenvalues. We end the paper with other properties of the eigenvalues.

2. Assumptions and known results

Let Ω be a bounded $\mathcal{C}^2$ domain of $\mathbb{R}^N$. Let us recall what we mean by *viscosity solutions*, adapted to our context.

Definition 2.1. Let g be a continuous function on $\Omega\times\mathbb{R}$, then v, continuous on $\overline{\Omega}$ is called a viscosity super-solution (respectively sub-solution) of $F(x,\nabla u,D^2u)=g(x,u)$ if for all $x_0\in\Omega$:

- Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which $v = c \in \mathbb{R}$ and $0 \leqslant g(x, c)$, for all $x \in B(x_0, \delta)$ (respectively $0 \geqslant g(x, c)$ for all $x \in B(x_0, \delta)$).
- Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v - \varphi$ has a local minimum (respectively local maximum) at x_0 and $\nabla\varphi(x_0) \neq 0$, one has

$$F\big(x_0, \nabla\varphi(x_0), D^2\varphi(x_0)\big) \leqslant g\big(x_0, v(x_0)\big)$$

(respectively

$$F\big(x_0, \nabla\varphi(x_0), D^2\varphi(x_0)\big) \geqslant g\big(x_0, v(x_0)\big)).$$

A viscosity solution is a function which is both a super-solution and a sub-solution.

Remark 2.2. When F is continuous in p, and $F(x, 0, 0) = 0$, this definition is equivalent to the classical definition of viscosity solutions, as in the User's guide [16].

We now state the assumptions satisfied by the operator F. Let S be the set of $N \times N$ symmetric matrices, and let $\alpha \in (-1, 0)$. Then F, defined on $\Omega \times \mathbb{R}^N \setminus \{0\} \times S$, is given by

$$F(x, p, M) = |p|^\alpha \big(\tilde{F}(x, M) + h(x) \cdot p\big). \tag{3}$$

On $\tilde{F}$ we suppose

(F) $\tilde{F}(x, tM) = t\tilde{F}(x, M)$ for any $t \in \mathbb{R}^+$, and there exist $A \geqslant a > 0$ such that for any $M \in S$ and any $N \in S$ such that $N \geqslant 0$

$$a\, tr(N) \leqslant \tilde{F}(x, M + N) - \tilde{F}(x, M) \leqslant A\, tr(N). \tag{4}$$

Furthermore $(x, M) \mapsto \tilde{F}(x, M)$ is continuous.

(J) $\tilde{F}$ is Hölder continuous in x and there exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$ and $\zeta \in \mathbb{R}^+$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $(x, y) \in \mathbb{R}^N$, $x \neq y$

$$\tilde{F}(x, X) - \tilde{F}(y, -Y) \leqslant \omega\big(\zeta|x - y|^2\big).$$

On h we suppose that:

(H) h is Hölder continuous of exponent $1 + \alpha$.

Remark 2.3. Since $\alpha < 0$, if $\tilde{F}$ satisfies condition (J), then so does F i.e. for X, Y and ζ as above

$$F\big(x, \zeta(x - y), X\big) - F\big(y, \zeta(x - y), -Y\big) \leqslant \omega\big(\zeta|x - y|^2\big).$$

Example 2.4. We suppose that h satisfies (H).

(1) Let $0 < a < A$ and $\mathcal{M}^+_{a,A}(M)$ be the Pucci's operator $\mathcal{M}^+_{a,A}(M) = A\,tr(M^+) - a\,tr(M^-)$ where $M^\pm$ are the positive and negative part of M, and $\mathcal{M}^-_{a,A}(M) = -\mathcal{M}^+_{a,A}(-M)$. Then F defined as

$$F(x, p, M) = |p|^\alpha \left(\mathcal{M}^\pm_{a,A}(M) + h(x)\cdot p\right)$$

satisfies the assumptions.

(2) Let $B(x)$ be some matrix with Lipschitz coefficients, which is invertible for all $x \in \Omega$. Let us consider $A(x) = B^\star B(x)$ and the operator $F(x, p, M) = |p|^\alpha (tr(A(x)(M)) + h(x)\cdot p)$, then $\tilde{F}$ satisfies (F) and (J), arguing as in [6, Example 2.4].

In the whole paper, we shall suppose that F is as in (3) and we suppose that the conditions (F), (J) and (H) are satisfied. We begin to recall some of the results obtained in previous papers which will be needed in this article.

Theorem 2.5. *(See [6].) Suppose that c is a continuous, bounded function that satisfies $c \leqslant 0$.*
Suppose that f_1 and f_2 are continuous and bounded and that u and v satisfy

$$F\left(x, \nabla u, D^2u\right) + c(x)|u|^\alpha u \geqslant f_1 \quad \text{in } \Omega,$$
$$F\left(x, \nabla v, D^2v\right) + c(x)|v|^\alpha v \leqslant f_2 \quad \text{in } \Omega,$$
$$u \leqslant v \quad \text{on } \partial\Omega.$$

If $f_2 < f_1$ or if $c < 0$ in Ω and $f_2 \leqslant f_1$ then

$$u \leqslant v \quad \text{in } \Omega.$$

Using the existence of sub- and super-solutions constructed with the aid of the distance function to $\partial\Omega$, together with Perron's method adapted to our context, this comparison theorem allows to prove the following existence's result:

Theorem 2.6. *(See [7].) Suppose that $c \leqslant 0$ and f is continuous and bounded, then there exists a continuous solution to*

$$\begin{cases} F\left(x, \nabla u, D^2u\right) + c(x)|u|^\alpha u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

If $f \leqslant 0$ then $u \geqslant 0$. If $f \geqslant 0$, $u \leqslant 0$. If $c < 0$ in Ω, the solution is unique.

Remark 2.7. (See [7].) The Hopf principle asserts that if $\mathcal{O}$ is a smooth bounded domain, and u is a solution of

$$F\left(x, \nabla u, D^2u\right) \leqslant 0 \quad \text{in } \mathcal{O} \tag{6}$$

such that $u > c$ inside $\mathcal{O}$ and $u(\bar{x}) = c$ at some boundary point of $\mathcal{O}$, then

$$\liminf_{h\to 0^+} \frac{u(\bar{x} - h\vec{\nu}(\bar{x})) - u(\bar{x})}{h} > 0,$$

where $\vec{\nu}(\bar{x})$ denotes the unit outer normal of $\mathcal{O}$ at $\bar{x}$.

In particular this implies that a nonconstant super-solution of (6) has no interior minimum.

We also recall some regularity results.

Proposition 2.8. *(See [7].) Let f be a continuous function in $\overline{\Omega}$. Let u be a viscosity non-negative bounded solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Then there exists a constant $C > 0$, $C = C(a, A, |f|_\infty, |h|_\infty, |u|_\infty)$ such that

$$\left|u(x) - u(y)\right| \leqslant C|x - y|$$

for any $(x, y) \in \overline{\Omega}^2$.

This proposition implies compactness for bounded sequences of solutions, this will be used in Section 3.

We shall also need in the proof of Theorem 4.1 the following comparison principle.

Theorem 2.9. *(See [8].) Suppose that c is continuous and bounded and that c is positive on $\overline{\Omega}$. Let u and v be respectively positive continuous super- and sub-solutions of*

$$|\nabla u|^\alpha \left(\tilde{F}\left(x, D^2 u\right) + h(x).\nabla u\right) + c(x)u^{1+\alpha} = 0 \quad \text{in } \Omega.$$

(1) *If $u \geqslant v > 0$ on $\partial\Omega$ then $u \geqslant v$ in Ω.*
(2) *If $u > v$ on $\partial\Omega$ then $u > v$ on $\overline{\Omega}$.*

3. Regularity

In this section, we establish that the solutions of (7) are $\mathcal{C}^{1,\beta}$ for some $\beta \in (0, 1)$; this will be a consequence of the known regularity results in the case $\alpha = 0$. Recall that we are under the assumption that $\alpha \leqslant 0$.

We expect that some $C^{1,\beta}$ regularity of the solutions be true for more general operators i.e. operators that are only homogeneous and singularly elliptic but not necessarily of the form given in (3). As an example, in the second subsection we illustrate this for a class of operators which does not satisfy (3) and is somehow close to the ∞-Laplacian, though not as degenerate.

3.1. Regularity result

To prove the regularity results announced (which will be stated precisely in Corollary 3.3), we remark that the solution of some convenient Dirichlet problem can be obtained as a fixed point of some operator acting in $\mathcal{C}_0(\Omega) \cap W^{1,\infty}$.

We define $\mathcal{C}_0(\Omega)$ as the space of continuous functions on $\overline{\Omega}$ which are zero on the boundary, and $\mathcal{L}(\Omega) := \{u \in \mathcal{C}_0(\Omega);\ u \text{ is Lipschitz}\}$ with the norm

$$|u|_{\mathcal{L}} := \sup_{x, y \in \Omega,\ x \neq y} \frac{|u(x) - u(y)|}{|x - y|}.$$

Let $g : \mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$ be a continuous and decreasing function such that $l \mapsto lg(|l|)$ is increasing on $\mathbb{R}^+$.

Proposition 3.1. *Let $f \in L^\infty(\Omega)$, let T_ϵ be the operator $\mathcal{L} \to \mathcal{L}$ such that $T_\epsilon u = v$, where v is the unique solution of*

$$\begin{cases} \tilde{F}\big(x, D^2 v\big) + h(x) \cdot \nabla v = \dfrac{f + \epsilon g(|u|)u}{g(|\nabla u|)} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

There exists $\epsilon_0 = \epsilon_0(a, A, \Omega)$ such that for any $\epsilon < \epsilon_0$, T_ϵ has a fixed point in $\mathcal{L}$.

Remark 3.2. The solution is taken in the sense of L^p viscosity solutions, see [13,32].

Before giving the proof of Proposition 3.1 we shall prove the main result of this section i.e.

Corollary 3.3. *Suppose that $\tilde{F}$, h satisfy respectively* (F), (J) *and* (H) *and that $f \in \mathcal{C}(\overline{\Omega})$. Let u be a solution, in the sense of Definition* 2.1, *of*

$$\begin{cases} |\nabla u|^\alpha \big(\tilde{F}\big(x, D^2 u\big) + h(x) \cdot \nabla u\big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \in \mathcal{C}^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

If $\tilde{F}$ is convex and f is $\mathcal{C}^\beta$ for some $\beta \in (0,1)$ then there exists $\gamma \in (0,1)$ such that $u \in \mathcal{C}^{2,\gamma}(\overline{\Omega})$.

Proof. We prove first the announced regularity result for the equation

$$\begin{cases} |\nabla u|^\alpha \big(\tilde{F}\big(x, D^2 u\big) + h(x) \cdot \nabla u\big) - \epsilon |u|^\alpha u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

when ϵ is small enough.

Fix $\delta > 0$ and let v_δ be a fixed point of T_ϵ with $g : l \mapsto (|l|^2 + \delta^2)^{\frac{\alpha}{2}}$. We now prove that it is the unique solution, in the sense of Definition 2.1, of

$$\begin{cases} \big(|\nabla u|^2 + \delta^2\big)^{\frac{\alpha}{2}} \big(\tilde{F}\big(x, D^2 u\big) + h(x) \cdot \nabla u\big) - \epsilon\big(\delta^2 + |u|^2\big)^{\frac{\alpha}{2}} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

We detail the proof that v_δ is a super-solution, an analogous procedure gives that it is also a sub-solution. First observe that since $\tilde{F}$ satisfies (F), by standard regularity results $v_\delta \in \mathcal{C}^{1,\beta}(\Omega)$ for all $\beta < 1$ (see Evans [18], Caffarelli [12], Caffarelli and Cabré [14] and Winter [32]; see also [15] for precise a priori bounds under our conditions).

According to Definition 2.1 of viscosity solution, we have to distinguish wether v_δ is locally constant or not. Let $\bar{x} \in \Omega$.

If for some $c_0 \in \mathbb{R}$, $v_\delta(x) = c_0$ for x in a neighborhood of $\bar{x}$, from the equation

$$-\epsilon\big(\delta^2 + c_0^2\big)^{\frac{\alpha}{2}} c_0 \leqslant f,$$

which is the same condition required by Definition 2.1.

On the other hand, suppose that there exists φ in $\mathcal{C}^2$ such that $(v_\delta - \varphi)(x) \geqslant (v_\delta - \varphi)(\bar{x}) = 0$, with $\nabla\varphi(\bar{x}) \neq 0$. Since v_δ is $\mathcal{C}^{1,\beta}$, $\nabla\varphi(\bar{x}) = \nabla v_\delta(\bar{x})$ hence using the fact that v_δ is a super-solution of the fixed point equation:

$$\tilde F\big(\bar x, D^2\varphi(\bar x)\big) + h(\bar x)\cdot\nabla\varphi(\bar x) \leqslant \big(f(\bar x) + \epsilon\big(\big|v_\delta(\bar x)\big|^2+\delta^2\big)^{\frac{\alpha}{2}} v_\delta(\bar x)\big)\big(|\nabla v_\delta|^2(\bar x)+\delta^2\big)^{-\frac{\alpha}{2}}$$
$$= \big(f(\bar x) + \epsilon\big(\big|\varphi(\bar x)\big|^2+\delta^2\big)^{\frac{\alpha}{2}} \varphi(\bar x)\big)\big(|\nabla \varphi|^2(\bar x)+\delta^2\big)^{\frac{-\alpha}{2}}$$

i.e.

$$\big(|\nabla\varphi|^2(\bar x)+\delta^2\big)^{\frac{\alpha}{2}}\big(\tilde F\big(\bar x, D^2\varphi(\bar x)\big) + h(\bar x)\cdot\nabla\varphi(\bar x)\big) - \epsilon\big(\big|\varphi(\bar x)\big|^2+\delta^2\big)^{\frac{\alpha}{2}}\varphi(\bar x) \leqslant f.$$

Both the requirements of Definition 2.1 are satisfied and v_δ is a super-solution of (9).

Proceeding similarly for the sub-solutions we have obtained that v_δ is a solution of (9). Furthermore there exists C depending only on the structural constants of $\tilde F$ and h such that

$$\|v_\delta\|_{\mathcal{C}^{1,\beta}(\Omega)} \leqslant C\big(|f|_\infty + \epsilon|v_\delta|_\infty^{1+\alpha}\big)\big(|\nabla v_\delta|_\infty^2+\delta^2\big)^{-\frac{\alpha}{2}}$$
$$\leqslant C\big(|f|_\infty + \epsilon|v_\delta|_\infty^{1+\alpha}\big)\big(|\nabla v_\delta|_\infty^2+1\big)^{-\frac{\alpha}{2}}.$$

By Proposition 2.8, v_δ is Lipschitz, with Lipschitz constants independent of δ; this implies both that the bounds on the $\mathcal{C}^{1,\beta}$ norm do not depend on δ and that we can pass to the limit in Eq. (9).

Stability of viscosity solutions implies that, for $\delta\to 0$, v_δ converges to the unique solution of

$$\begin{cases} |\nabla u|^\alpha\big(\tilde F\big(x, D^2u\big) + h(x)\cdot\nabla u\big) - \epsilon|u|^\alpha u = f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Proving in that way that $u\in\mathcal{C}^{1,\beta}(\overline{\Omega})$ for some β and satisfies

$$\|u\|_{\mathcal{C}^{1,\beta}(\Omega)} \leqslant C\big(|f|_\infty + \epsilon|u|_\infty^{1+\alpha}\big)|\nabla u|_\infty^{-\alpha}.$$

The case $\epsilon = 0$ is obtained by writing Eq. (8) as

$$|\nabla u|^\alpha\big(\tilde F\big(x, D^2u\big) + h(x)\cdot\nabla u\big) - \epsilon|u|^\alpha u = f - \epsilon|u|^\alpha u,$$

and, since $u\in L^\infty$, in the previous estimates we just replace f with $f-\epsilon|u|^\alpha u$.

When $\tilde F$ is convex or concave since $x\mapsto|x|^{-\alpha}$ is Hölder continuous and there exists $\beta\in(0,1)$ such that $u\in\mathcal{C}^{2,\beta}(\Omega)$. □

Remark 3.4. The result of Corollary 3.3 can be extended to solutions of

$$\begin{cases} |\nabla u|^\alpha\big(\tilde F\big(x, D^2u\big) + h(x)\cdot\nabla u\big) + g(x,u) = f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous and bounded.

Proof of Proposition 3.1. Clearly $\mathcal{L}(\Omega)$, equipped with the norm $|.|_{\mathcal{L}}$ is a Banach space. It is well known (see [16]) that for any $k\in L^\infty$ there exists a unique solution v of

$$\begin{cases} \tilde F\big(x, D^2v\big) + h(x)\cdot\nabla v = k & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega.\end{cases}$$

Furthermore v is Lipschitz continuous and there exists some constant c which depends only on the structural data such that

$$|v|_{\mathcal{L}} \leqslant c|k|_\infty.$$

Let ϵ be small enough in order that $\epsilon c\ d_\Omega < 1$ where d_Ω is the diameter of Ω.

Let G be the inverse function of $l \mapsto lg(l)$. Let $d = \max\{d_\Omega, 1\}$, $f_0 = \frac{c|f|_\infty}{1-\epsilon dc}$ and

$$\mathcal{B} = \big\{u \in \mathcal{L}(\Omega),\ \text{such that } |u|_{\mathcal{L}} \leqslant G(f_0)\big\}.$$

$\mathcal{B}$ is a convex and compact subset of $\mathcal{L}$. And observe that $|u|_\infty \leqslant |u|_{\mathcal{L}} d_\Omega$. We use the fact that $l \mapsto lg(|l|)$ is increasing and g is decreasing to obtain that, if $u \in \mathcal{B}$,

$$\begin{aligned}
\big|T_\epsilon(u)\big|_{\mathcal{L}} &\leqslant c\frac{|f|_\infty + \epsilon\ d_\Omega G(f_0)g(d_\Omega G(f_0))}{g(G(f_0))} \\
&\leqslant c\frac{|f|_\infty + \epsilon\ dG(f_0)g(G(f_0))}{g(G(f_0))} \\
&\leqslant c\big(|f|_\infty + \epsilon d\ f_0\big)\frac{G(f_0)}{f_0} = G(f_0).
\end{aligned}$$

So that $T_\epsilon(B) \subset B$, T_ϵ is continuous and compact, and Schauder's fixed point theorem implies the result. □

3.2. Other operators

We now present an example for which the results of the previous section can be extended to even though it does not satisfy the previous assumptions.

Let $q \geqslant 0$ and $\alpha \in (-1, 0]$, we define the operator $\mathcal{F} : \mathbb{R}^N \setminus \{0\} \times S \to \mathbb{R}$ by

$$\mathcal{F}(p, M) = |p|^\alpha \left(tr\, M + q\left\langle \frac{Mp}{|p|}, \frac{p}{|p|}\right\rangle\right) := |p|^\alpha \tilde{F}(M, p).$$

As this operator is not defined on points where the gradient is zero, we consider viscosity solutions as in Definition 2.1.

Proposition 3.5. *Suppose that u is a viscosity solution of*

$$\begin{cases} \mathcal{F}\big(\nabla u, D^2 u\big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

with $f \in \mathcal{C}(\overline{\Omega})$. Then there exists $\beta = \beta(q, N) \in (0, 1)$ such that u is in $C^{1,\beta}(\overline{\Omega})$.

Proof. Recall that the $(q+2)$-Laplacian is defined by

$$\Delta_{q+2} u = |\nabla u|^q \left(\Delta u + \frac{q}{|\nabla u|^2}\big\langle D^2 u(\nabla u), \nabla u\big\rangle\right).$$

We first prove a fixed point property. For $\epsilon > 0$ and $\delta > 0$ let T_{lap} be the map defined on $\mathcal{L}$ such that $T_{lap} u = v$ is the solution of

$$\begin{cases} \Delta_{q+2}(v) = \left(f + \epsilon\left(|u|^2 + \delta^2\right)^{\frac{\alpha}{2}} u\right)\left(|\nabla u|^2 + \delta^2\right)^{\frac{q-\alpha}{2}} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Recall that by regularity results for the $(q+2)$-Laplacian (e.g. [19,31,24]), there exist $\beta \in (0,1)$ and $C_{lap} > 0$ such that

$$\left|T_{lap}(u)\right|_{\mathcal{C}^{1,\beta}} \leqslant C_{lap}\left(|f|_\infty + \epsilon\left(|u|_\infty^2 + \delta^2\right)^{\frac{\alpha}{2}} |u|_\infty\right)^{\frac{1}{q+1}} \left(|\nabla u|_\infty^2 + \delta^2\right)^{\frac{q-\alpha}{2(q+1)}}. \tag{11}$$

Let l_ϵ be a solution of

$$l_\epsilon^{q+1} = C_{lap}\left(|f|_\infty + \epsilon\left((d_\Omega l_\epsilon)^2 + \delta^2\right)^{\frac{\alpha}{2}} d_\Omega l_\epsilon\right)\left(l_\epsilon^2 + \delta^2\right)^{\frac{q-\alpha}{2}},$$

which exists for $\epsilon < \epsilon_0$ for some ϵ_0 small enough. We define the closed convex compact set in $\mathcal{L}(\Omega)$

$$\mathcal{B} = \left\{u \in \mathcal{L}(\Omega),\ |u|_{\mathcal{L}} \leqslant l_\epsilon\right\}.$$

Then $T_{lap}\mathcal{B} \subset \mathcal{B}$ and by the Schauder fixed point theorem there exists $u_\delta \in \mathcal{C}^{1,\beta}$ a solution of

$$\begin{cases} \Delta_{q+2}(u) = \left(f + \epsilon\left(|u|^2 + \delta^2\right)^{\frac{\alpha}{2}} u\right)\left(|\nabla u|^2 + \delta^2\right)^{\frac{q-\alpha}{2}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As before, we can show that u_δ is a solution in the sense of Definition 2.1 of

$$\begin{cases} \left(|\nabla u|^2 + \delta^2\right)^{\frac{\alpha-q}{2}} \Delta_{q+2}(u) - \epsilon\left(|u|^2 + \delta^2\right)^{\frac{\alpha}{2}} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Arguing as in the proof of Hölder's and Lipschitz's regularity in [7], one can prove some uniform Lipschitz estimates on u_δ. Then it is clear from the estimates in (11) that the $C^{1,\beta}$ norm of u_δ is independent of δ; letting δ go to zero, u_δ converges to u a solution of

$$\begin{cases} |\nabla u|^\alpha\left(\Delta u + q\left\langle D^2u\left(\frac{\nabla u}{|\nabla u|}\right), \frac{\nabla u}{|\nabla u|}\right\rangle\right) - \epsilon|u|^\alpha u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

When $\epsilon = 0$, the regularity is obtained by writing Eq. (10) under the form

$$\begin{cases} |\nabla u|^\alpha\left(\Delta u + q\left\langle D^2u\left(\frac{\nabla u}{|\nabla u|}\right), \frac{\nabla u}{|\nabla u|}\right\rangle\right) - \epsilon|u|^\alpha u = f - \epsilon|u|^\alpha u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad \square$$

4. Strong comparison principle and uniqueness of the first eigenfunction

Let $c(x)$ be a continuous and bounded function. As in the introduction we define

$$\lambda^+ := \sup\left\{\lambda \in \mathbb{R};\ \exists\phi > 0,\ \phi \in \mathcal{C}(\overline{\Omega}),\ |\nabla\phi|^\alpha\left(\tilde{F}\left(x, D^2\phi\right) + h(x)\cdot\nabla\phi\right) + c(x)\phi^{\alpha+1} \leqslant 0 \text{ in } \Omega\right\},$$

$$\lambda^- := \sup\left\{\lambda \in \mathbb{R};\ \exists\phi < 0,\ \phi \in \mathcal{C}(\overline{\Omega}),\ |\nabla\phi|^\alpha\left(\tilde{F}\left(x, D^2\phi\right) + h(x)\cdot\nabla\phi\right) + c(x)|\phi|^\alpha\phi \geqslant 0 \text{ in } \Omega\right\}.$$

We prove the uniqueness result for λ^+, the changes to bring for λ^- being obvious.

Theorem 4.1. *Suppose that Ω is a bounded regular domain such that*

- *either $\partial\Omega$ is connected*;
- *or, if $N = 2$ and $\tilde{F}$ is concave or convex, $\partial\Omega$ has at most two connected components.*

Suppose that $c(x) + \lambda^+ > 0$ in Ω.

If ψ and φ are two positive eigenfunctions for the eigenvalue λ^+, there exists $t > 0$ such that $\psi \equiv t\varphi$.

Remark 4.2. When the boundary has two connected components our proof relies on Sard's theorem which holds true for $\mathcal{C}^{N-1,1}$ functions (see [4]) this explains the conditions $N = 2$ and the convexity of $\tilde{F}$.

Remark 4.3. When $\alpha > 0$ and $\partial\Omega$ is connected, the result still holds, if one knows that the gradient of one of the eigenfunction is bounded away from zero in a neighborhood of the boundary. For this it would be sufficient to have a $\mathcal{C}^1$ regularity result near the boundary.

In order to prove Theorem 4.1 we shall need a few results concerning comparison principle and applications of Hopf's principle. We begin by a strong comparison principle inside Ω for sub- and super-solutions u and v which coincide at one point $\bar{x}$ inside Ω, where $|\nabla u|(\bar{x}) \neq 0$, or $|\nabla v|(\bar{x}) \neq 0$:

Proposition 4.4. *Suppose that u and v are respectively nonnegative C^1 solutions of*

$$F\big(x, \nabla u, D^2 u\big) \leqslant f \quad \text{in } \Omega,$$
$$F\big(x, \nabla v, D^2 v\big) \geqslant g \quad \text{in } \Omega,$$

with $f \leqslant g$. Suppose that $\mathcal{O}$ is an open connected subset of Ω, such that

(1) *$u \geqslant v$ in $\mathcal{O}$,*
(2) *either $|\nabla u(x)| \neq 0$, or $|\nabla v(x)| \neq 0$ in $\mathcal{O}$.*

Then either $u \equiv v$ or $u > v$ in $\mathcal{O}$.

Proof. Either $u \equiv v$ or there exists $x_0 \in \mathcal{O}$ such that $u(x_0) > v(x_0)$.

Suppose by contradiction that there exists some point x_1 such that $u(x_1) = v(x_1)$ which can be chosen in such a way that, for $R = |x_1 - x_0|$, $u > v$ in $B(x_0, R)$ and x_1 is the only point in the closure of that ball on which u and v coincide.

Without loss of generality, one can assume that $B(x_0, \frac{3R}{2}) \subset \mathcal{O}$ and $\nabla v \neq 0$ in $\mathcal{O}$.

We shall prove that there exist constants $c > 0$ and $\delta > 0$ such that

$$u \geqslant v + \delta\big(e^{-c|x-x_0|} - e^{\frac{-3cR}{2}}\big) \equiv v + w \quad \text{in } \frac{R}{2} \leqslant |x - x_0| = r \leqslant \frac{3R}{2}.$$

This will contradict the fact that $u(x_1) = v(x_1)$.

Let $\delta \leqslant \min_{|x-x_0|=\frac{R}{2}}(u - v)$, so that

$$u \geqslant v + w \quad \text{on } \partial\left(B\left(x_0, \frac{3R}{2}\right) \setminus \overline{B\left(x_0, \frac{R}{2}\right)}\right).$$

Let φ be some test function for v from above, by the hypothesis on v there exists L_1 and L_2 such that $L_1 \leqslant |\nabla\varphi| \leqslant L_2$ in the annulus. A simple calculation on w implies that, if $c \geqslant \frac{1}{a}(\frac{2A(N-1)}{R} + 2|h|_\infty)$ then

$$\begin{aligned}
&|\nabla\varphi+\nabla w|^\alpha\cdot\big(\tilde F\big(x,D^2\varphi+D^2w\big)+h(x)\cdot(\nabla\varphi+\nabla w)\big)\\
&\quad\geqslant|\nabla\varphi+\nabla w|^\alpha\big(\tilde F\big(x,D^2\varphi\big)+h(x)\cdot\nabla\varphi\big)+|\nabla\varphi+\nabla w|^\alpha\big(\mathcal M^-\big(D^2w\big)+h(x)\cdot\nabla w\big)\\
&\quad\geqslant|\nabla\varphi+\nabla w|^\alpha\frac{g}{|\nabla\varphi|^\alpha}+|\nabla\varphi+\nabla w|^\alpha\left(ac^2-Ac\left(\frac{N-1}{r}\right)-|h|_\infty c\right)\delta e^{-cr}\\
&\quad\geqslant|\nabla\varphi+\nabla w|^\alpha\frac{g}{|\nabla\varphi|^\alpha}+|\nabla\varphi+\nabla w|^\alpha\frac{ac^2}{2}\delta e^{-cr}.
\end{aligned}$$

We also impose that $\delta<\frac{RL_1e}{16}$, which implies in particular that $|\nabla w|\leqslant\frac{|\nabla\varphi|}{8}$.

We now use the inequalities

$$\big||\nabla\varphi+\nabla w|^\alpha-|\nabla\varphi|^\alpha\big|\leqslant|\alpha||\nabla w||\nabla\varphi|^{\alpha-1}\left(\frac12\right)^{\alpha-1}\leqslant\frac{|\nabla\varphi|^\alpha}{2}$$

to get

$$\begin{aligned}
&|\nabla\varphi+\nabla w|^\alpha\big(\tilde F\big(x,D^2\varphi+D^2w\big)+h(x)\cdot(\nabla\varphi+\nabla w)\big)\\
&\quad\geqslant g-|g|_\infty|\nabla\varphi|^{-1}|\alpha|2^{1-\alpha}c\delta e^{-cr}+L_2^\alpha\frac{ac^2}{4}\delta e^{-cr}.
\end{aligned}$$

It is now enough to choose

$$c=\sup\left\{\frac{2(\frac{2A(N-1)}{R}+|h|_\infty)}{a},\frac{2^{4-\alpha}|g|_\infty}{aL_1L_2^\alpha}\right\}$$

to finally obtain

$$|\nabla\varphi+\nabla w|^\alpha\big(\tilde F\big(x,D^2\varphi+D^2w\big)+h(x)\cdot(\nabla\varphi+\nabla w)\big)\geqslant f+\frac{ac^2\delta L_2^\alpha e^{-cr}}{8}$$

i.e.

$$F\big(x,\nabla(v+w),D^2(v+w)\big)>F\big(x,\nabla u,D^2u\big).$$

By the comparison principle, Theorem 2.5,

$$u\geqslant v+w$$

in the annulus $B(x_0,\frac{3R}{2})\setminus\overline{B(x_0,\frac R2)}$ which is the desired contradiction. This ends the proof of Proposition 4.4. □

A consequence of Proposition 4.4 is the following: Let $\partial_\nu u$ be the normal derivative $\nabla u\cdot\vec\nu$ where $\vec\nu$ is the unit outer normal to $\partial\Omega$. Then

Proposition 4.5. *Suppose that Ω is a smooth bounded domain of $\mathbb R^N$ and $f\leqslant g$. Let u and v be respectively nonnegative $\mathcal C^1(\overline\Omega)$ solutions of*

$$\begin{cases} F\big(x, \nabla u, D^2 u\big) \leqslant f & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

$$\begin{cases} F\big(x, \nabla v, D^2 v\big) \geqslant g & in\ \Omega, \\ v = 0 & on\ \partial\Omega. \end{cases}$$

If $u \geqslant v$ in Ω and if there exists $\bar{x} \in \partial\Omega$ such that $\partial_\nu u(\bar{x}) = \partial_\nu v(\bar{x})$ then there exists $\epsilon > 0$ such that

$$u \equiv v \quad in\ \Omega \setminus \overline{\Omega}_\epsilon$$

where Ω_ϵ is the set of points of Ω whose distance to the connected component of the boundary which contains $\bar{x}$ is greater than ϵ.

Proof. By Hopf principle in Remark 2.7, $|\nabla u| > 0$ and $|\nabla v| > 0$ on $\partial\Omega$ then the regularity of u and v implies that there exists $\epsilon > 0$, L_1 and L_2 such that $L_1 \leqslant |\nabla u|, |\nabla v| \leqslant L_2$ in $\Omega \setminus \overline{\Omega}_\epsilon$.

If there exists a point x_1 of $\Omega \setminus \overline{\Omega}_\epsilon$ such that $u(x_1) = v(x_1)$, we have nothing to prove, using Proposition 4.4. So we can suppose by contradiction that there exists a ball $B \subset \Omega$ which is tangent to $\partial\Omega$ in $\bar{x}$, with $u > v$ in B. Let x_2 be the center of B, and let $R = |\bar{x} - x_2|$.

Let $w = \delta(e^{-c|x-x_2|} - e^{-c|x_2-\bar{x}|})$, where δ is chosen such that $\delta \leqslant \inf_{|x-x_2|=\frac{R}{2}}(u - v)$. Reasoning as in Proposition 4.4 we get that there exists c such that

$$u \geqslant v + w \quad \text{in } B(x_2, R) \setminus \overline{B\left(x_2, \frac{R}{2}\right)}.$$

This implies in particular that

$$\left|\partial_\nu u(\bar{x})\right| \geqslant \left|(\partial_\nu v + \partial_\nu w)(\bar{x})\right|.$$

Since $|(\partial_\nu v + \partial_\nu w)(\bar{x})| > |\partial_\nu v(\bar{x})|$ this contradicts the hypotheses and ends the proof of Proposition 4.5. $\square$

Remark 4.6. Observe that Proposition 4.5 holds under the weaker condition that only one of the functions u or v satisfies the boundary condition say e.g. v; as long as there is $\bar{x} \in \partial\Omega$ where they are both zero and where $\partial_\nu u(\bar{x}) = \partial_\nu v(\bar{x})$. Indeed, in order to apply Proposition 4.4 in $\Omega \setminus \Omega_\varepsilon$ it is enough that only one of the functions satisfies $|\nabla v| > 0$.

In the sequel we shall need the following well-known result:

Lemma 4.7. *Suppose that O is an open bounded set. There exists some point on ∂O where ∂O satisfies the interior sphere condition.*

See [11] for more complete results on that property.

Proof of Theorem 4.1. Let $d(x)$ denote the distance to the boundary of Ω. Suppose that ψ and φ are two positive eigenfunctions and let $\Gamma = \sup \frac{\psi}{\varphi}$ and $\gamma = \inf \frac{\psi}{\varphi}$. These extrema are well defined and positive because, using Hopf lemma and the comparison principle, there exist c_1 and c_2 such that in a neighborhood of $\partial\Omega$:

$$c_1 d(x) \leqslant \psi(x), \varphi(x) \leqslant c_2 d(x)$$

(see [7] for the details). We need to prove that $\gamma = \Gamma$.

We begin to observe that Γ is "achieved" on the boundary, in the sense that e.g. there exists a sequence $(x_n)_n$ that converges to $\bar{x} \in \partial\Omega$, such that $\frac{\psi}{\varphi}(x_n) \to \Gamma$:

Indeed, suppose not, then, there exists an open set Ω', $\Omega' \Subset \Omega$ such that on $\Omega \setminus \overline{\Omega}'$

$$\frac{\psi}{\varphi} \leqslant \Gamma - \epsilon$$

for some $\epsilon > 0$. Since $\psi > 0$ in $\overline{\Omega}'$, the comparison principle in Theorem 2.9 implies that $\psi \leqslant (\Gamma - \epsilon)\varphi$ in Ω' and finally in all Ω, and this contradicts the definition of the upper bound.

In particular one has $\partial_{\vec{\nu}}\psi(\bar{x}) = \Gamma\partial_{\vec{\nu}}\varphi(\bar{x})$ and by Corollary 3.3 ψ and φ are $\mathcal{C}^{1,\beta}$. The hypotheses of Proposition 4.5 are satisfied by ψ and $\Gamma\varphi$ therefore $\psi = \Gamma\varphi$ in a neighborhood of the component of $\partial\Omega$ that contains $\bar{x}$.

Proceeding analogously for γ, we obtain that there exists $\underline{x} \in \partial\Omega$ where the infimum is "achieved". Then $\frac{\psi}{\varphi} = \gamma$ on a neighborhood of the connected component of $\partial\Omega$ that contains $\underline{x}$.

When $\partial\Omega$ is connected this implies that $\Gamma = \gamma$ and it ends the proof in that case.

We are now in the hypothesis that $N = 2$, $\tilde{F}$ is convex or concave and $\partial\Omega$ has two connected components. In that case, there exist Ω_1 and Ω_2, simply connected, smooth and bounded, such that $\Omega_1 \Subset \Omega_2$ and $\Omega = \Omega_2 \setminus \overline{\Omega_1}$.

By the previous reasoning, we have obtained that

$$\psi \equiv \Gamma_1\varphi \quad \text{in a neighborhood of } \partial\Omega_1,$$

and

$$\psi \equiv \Gamma_2\varphi \quad \text{in a neighborhood of } \partial\Omega_2,$$

with $\{\Gamma_1, \Gamma_2\} = \{\Gamma, \gamma\}$.

For $i = 1, 2$, let

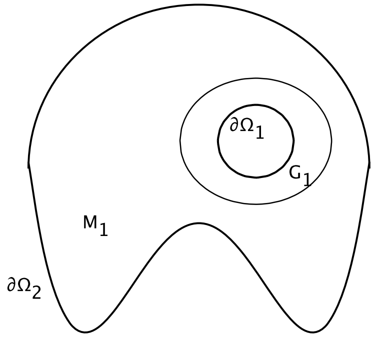
$$A_i \text{ be the connected component of } \bigl\{x,\ \psi(x) = \Gamma_i\varphi(x),\ \nabla\psi(x) \neq 0 \text{ or } \nabla\varphi(x) \neq 0\bigr\}$$

whose boundary contains $\partial\Omega_i$.

A_i is open. Indeed if $x_1 \in A_1$ then $\psi(x_1) = \Gamma_1\varphi(x_1)$ and there is a neighborhood of x_1, say N_{x_1}, where either $\nabla\psi \neq 0$ or $\nabla\varphi \neq 0$. Hence, using Proposition 4.4, $\psi = \Gamma_1\varphi$ in N_{x_1} and then $N_{x_1} \subset A_1$.

Observe that if $\partial A_1 \cap \partial\Omega_2 \neq \emptyset$ or $\partial A_2 \cap \partial\Omega_1 \neq \emptyset$, this ends the proof of Theorem 4.1, since it implies that $\Gamma = \gamma$. We then suppose that these intersections are empty.

Let $G_i := \overline{A_i}$; $K_i := \partial G_i \cap \Omega$, and $M_i := \Omega \setminus G_i$.



Let us note that K_i satisfies that, for all $x \in K_i$, $\psi(x) = \Gamma_i \varphi(x)$, $\nabla\psi(x) = 0$ and $\nabla\varphi(x) = 0$. To see this, let us observe that $\partial G_i \subset \partial A_i$ since G_i is closed, then it is sufficient to prove that these equalities are true on the boundary of A_i intersected with Ω:

The first equality is true by continuity and the others two because otherwise one would have a point $x \in K_i$, with $\psi(x) = \Gamma_i \varphi(x)$, $\nabla\psi(x) \neq 0$ or $\nabla\varphi(x) \neq 0$, and then by Proposition 4.4, there would exist an open neighborhood of x on which all these facts still hold, and x would belong to A_i, this contradicts the notion of boundary.

Moreover $\partial M_1 = \partial\Omega_2 \cup K_1$ (and $\partial M_2 = \partial\Omega_1 \cup K_2$).

Indeed note that $\overline{M_1} \cap \partial\Omega_1 = \emptyset$ since A_1 contains an open neighborhood of $\partial\Omega_1$, then $\partial M_1 \subset (\overline{\Omega} \cap \partial G_1) \cup (\overline{C(G_1)} \cap \partial\Omega) \subset (\Omega \cap \partial G_1) \cup \partial\Omega_1 \cup \partial\Omega_2$ and since $\partial M_1 \cap \partial\Omega_1 = \emptyset$, $\partial M_1 \subset \partial\Omega_2 \cup K_1$. To prove the reverse inclusion, let $x \in K_1$, then for all $r > 0$, $B(x,r) \cap (\mathbb{R}^N \setminus G_1) \neq \emptyset$. Taking r such that $B(x,r) \subset \Omega$ one gets $x \in \overline{M_1}$. On the other hand $x \notin M_1$, since if $x \in M_1$ there exists some ball $B(x,\epsilon)$ included in M_1, and $x \in K_1$ implies $B(x,\epsilon) \cap G_1 \neq \emptyset$, a contradiction.

It is clear that $\partial\Omega_2$ is included in $\overline{M_1}$, and it has no point of M_1 since M_1 is an open subset of Ω.

Let us admit for a while the following three claims,

Claim 1. *M_i is connected for $i = 1, 2$.*

Claim 2. *∂G_i has at most two connected components i.e. K_i is connected.*

Claim 3. $\partial(M_1 \cap M_2) = K_1 \cup K_2$.

Let us finish the proof. Using the fact that both K_i are connected, since $\psi \in \mathcal{C}^{2,\beta}$ we can apply Sard's theorem to conclude that there exist two constants c_i such that $\psi_{|K_i} = c_i$ for $i = 1$ and $i = 2$.

Suppose that there exists one point $\bar{x} \in M = M_1 \cap M_2$ such that $\psi(\bar{x}) < \min(c_1, c_2)$. Then ψ would have a local minimum inside M, a contradiction with Hopf principle (Remark 2.7). Then the minimum is achieved on the boundary of M, suppose to fix the ideas that $c_1 \leqslant c_2$. Now take a ball in M where $\psi > c_1$ that touches K_1 at some point x_1 (Lemma 4.7) by Hopf's principle $\nabla\psi(x_1) \neq 0$, this contradicts the fact that for all $x \in K_i$, $\nabla\psi(x) = 0$.

Now since ψ cannot be locally constant we have proved that $M = \emptyset$, then $\Gamma_1 = \Gamma_2$ or equivalently $\Gamma = \gamma$. This ends the proof of Theorem 4.1 provided we prove the claims.

Proof of Claim 1. To fix the ideas we consider the case $i = 1$.

Let us recall that $\partial M_1 = (\partial G_1 \cap \Omega) \cup \partial\Omega_2$. Suppose that M_1 has at least two connected components, $M_{1,1}$, $M_{1,2}$, necessarily one of them, say $M_{1,2}$, has a boundary which contains $\partial\Omega_2$. Then $\partial M_{1,1} \subset K_1 = \Omega \cap \partial G_1$, so $M_{1,1}$ is a connected open set on the boundary of which ψ and ϕ have their gradient equal to zero.

We prove that $M_{1,1}$ is simply connected.

If not there exists some open regular domain O', $\overline{O'} \subset \Omega_2$, with $\partial O' \subset M_{1,1}$, O' is not included in $M_{1,1}$, and $\partial M_{1,1}$ is not included in O'. This implies that O' is not included in M_1. Indeed, if $O' \subset M_1$, since M_{11} is connected, this implies that $O' \subset M_{11}$. Let $x \in O' \setminus M_1$, then $x \in G_1$, hence $G_1 \cap O' \neq \emptyset$.

Either $G_1 \subset O'$, then $\partial M_{1,1} \subset K_1 \subset O'$ and the contradiction is in the choice of O'.

Or $G_1 \cap C(O') \neq \emptyset$, and this contradicts the fact that $\partial O' \subset M_{1,1}$. We have obtained that $M_{1,1}$ is simply connected.

By Sard's theorem there exists a constant $c_{1,1}$ such that $\psi = c_{1,1}$ on $\partial M_{1,1}$. Then one gets a contradiction with Hopf's principle. Indeed, either there exists a minimum inside $M_{1,1}$ and this contradicts Remark 2.7, or $c_{1,1}$ is a minimum for ψ and taking some point on the boundary of $M_{1,1}$ which possesses the interior sphere condition, one gets once more a contradiction with Hopf's principle.

We have obtained that $M_{1,1} = \emptyset$ and M_1 is connected. □

Proof of Claim 2. We prove this claim by establishing that $G_1 \cup \Omega_1$ is simply connected. Let O be a simply connected open set, such that $\overline{O} \subset \Omega_2$, $\partial O \subset G_1 \cup \Omega_1$. We need to prove that $O \subset G_1 \cup \Omega_1$.

If $O \subset \Omega_1$ it is true. If not, $O \cap G_1 \neq \emptyset$ and if $O \subset G_1$ the result holds, so we assume both that $O \cap G_1 \neq \emptyset$ and $O \cap C(G_1) \neq \emptyset$. Then $O \cap M_1 \neq \emptyset$. Since M_1 is a connected set which meets O we claim that $\partial O \cap M_1 \neq \emptyset$. Indeed, if not $M_1 \subset O$. Then one would have $\overline{M_1} \subset \overline{O} \subset \Omega_2$, a contradiction since $\partial \Omega_2 \subset \overline{M_1}$. Then $\partial O \cap M_1 \neq \emptyset$, which implies that $M_1 \cap (G_1 \cup \Omega_1) \neq \emptyset$, once more a contradiction.

To prove that K_2 is a connected set, one proves proceeding as above that $M_2 \cup \Omega_1$ is simply connected. □

Proof of Claim 3. A first inclusion can be obtained as follows

$$\begin{aligned}\partial(M_1 \cap M_2) &= \overline{M_1 \cap M_2} \setminus (M_1 \cap M_2) \\ &\subset \overline{M_1} \cap \overline{M_2} \cap \big((\Omega \setminus M_1) \cup (\Omega \setminus M_2)\big) \\ &\subset (\partial M_1 \cap \overline{M_2}) \cup (\partial M_2 \cap \overline{M_1}) \\ &\subset K_1 \cup K_2.\end{aligned}$$

To prove the reverse inclusion, we recall that, by hypothesis, $G_1 \cap G_2 = \emptyset$.

Then $K_1 \subset G_1 \subset M_2$. Let $x \in K_1$ and $r > 0$. We need to prove that $B(x,r) \cap (M_1 \cap M_2) \neq \emptyset$. Since M_2 is open there exists a ball $B(x,\epsilon) \subset M_2$. One can assume that $\epsilon < r$. Since $x \in K_1 \subset \partial M_1$, $B(x,\varepsilon) \cap \overline{M_1} \neq \emptyset$ and also $B(x,\epsilon) \cap M_1 \neq \emptyset$. Finally,

$$B(x,\varepsilon) \cap M_1 \cap M_2 \subset B(x,r) \cap M_1 \cap M_2, \quad \text{hence } x \in \overline{M_1 \cap M_2}.$$

$x \in K_1$ implies $x \notin M_1$ hence $x \notin M_1 \cap M_2$ i.e. $K_1 \subset \partial(M_1 \cap M_2)$. In the same manner $K_2 \subset \partial(M_1 \cap M_2)$. □

We have also obtained the following strong comparison principle:

Theorem 4.8. *Suppose that $N = 2$, that Ω and $\partial\Omega$ are connected. Suppose that f is a continuous function on $\overline{\Omega}$, $f \leqslant 0$, and f not identically zero. Suppose that u and v are respectively super- and sub-solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume in addition that $u \in \mathcal{C}^1(\overline{\Omega}) \cap C^{1,1}(\Omega)$, and that $u \geqslant v$ in Ω. Then either $u > v$ inside Ω or $u \equiv v$.

Remark 4.9. The symmetric result holds for $f \geqslant 0$, assuming that the sub-solution $v \in \mathcal{C}^1(\overline{\Omega}) \cap C^{1,1}(\Omega)$.

Proof. First we remark that by the strong maximum principle $u > 0$ since $f \not\equiv 0$.

Then by Hopf's principle $\partial_{\vec{\nu}} u < 0$ on the boundary.

Using Proposition 4.5 there exists a neighborhood of the boundary where either $u \equiv v$ or $u > v$. In that last case one can conclude by using the comparison principle in Theorem 2.9 and $u > v$ in Ω.

If $u \equiv v$ on a neighborhood of $\partial\Omega$, as in the previous proof, let

$$A \text{ be the connected component of } \big\{x,\ u(x) = v(x),\ \nabla u(x) \neq 0\big\},$$

whose boundary contains $\partial\Omega$. Let $G := \overline{A}$, $K := \partial G \cap \Omega$, $M = \Omega \setminus G$.

We want to prove that $M = \emptyset$. Suppose not, proceeding as in the proof of Theorem 4.1, it can be proved that M and K are connected, hence $\partial M = K$. Furthermore, $\nabla u = \nabla v = 0$ on K.

Then, using Sard's theorem, u is constant on K and, since M cannot contain a local minimum of u, the minimum of u is achieved on the boundary i.e. on K; this contradicts the Hopf principle on some point where the interior sphere condition is satisfied, since $\nabla u = 0$ on K. Finally $M = \emptyset$ and $u \equiv v$ in Ω. $\square$

5. Further results about the principal eigenvalues

In all this section we suppose that $c(x) + \lambda^{\pm} > 0$ on $\overline{\Omega}$.

5.1. Properties concerning the dependence on the domain of the eigenvalues

The next two theorems concern the strict monotonicity of the principal eigenvalues with respect to the domain inclusion. We state them in the case of the eigenvalue $\lambda^+(\Omega)$, the symmetric results hold for $\lambda^-(\Omega)$ with obvious changes. In the sequel, when no ambiguities arise, we shall write λ^+ without writing the dependence on the domain.

Theorem 5.1. *Suppose that Ω is a $\mathcal{C}^2$ bounded domain in $\mathbb{R}^N$. If u is a positive solution of*

$$F\big(x, \nabla u, D^2 u\big) + \big(c(x) + \lambda^+\big)u^{1+\alpha} = 0 \quad \text{in } \Omega,$$

then there exists $(\partial\Omega)'$ a connected component of $\partial\Omega$ such that

$$u = 0 \quad \text{on } (\partial\Omega)'.$$

In particular if $\partial\Omega$ is connected, there exists $t \in \mathbb{R}$ such that $u = t\phi$ where ϕ is a positive eigenfunction corresponding to λ^+.

Proof. To begin with, let us note that u must be zero somewhere on $\partial\Omega$. Suppose not, i.e. $u > 0$ on $\overline{\Omega}$. Using the hypothesis $c(x) + \lambda^+ > 0$, this implies that there exists $\varepsilon > 0$ and $\lambda' > \lambda^+$ such that $u_\varepsilon := u - \varepsilon > 0$ is a solution of

$$F\big(x, \nabla u_\varepsilon, D^2 u_\varepsilon\big) + \big(c(x) + \lambda'\big)u_\varepsilon^{1+\alpha} \leqslant 0,$$

which contradicts the definition of λ^+.

Let ϕ be an eigenfunction associated to λ^+, so that $\phi > 0$ in Ω and $\phi = 0$ on $\partial\Omega$. We now consider $\tau = \sup \frac{\phi}{u}$. Reasoning again as in Theorem 4.1, τ is finite and positive and it must be achieved at least on one point of the boundary in the sense that there exists a sequence $(x_n)_n$, $x_n \in \Omega$ which converges to $\bar{x} \in \partial\Omega$, with $\frac{\phi(x_n)}{u(x_n)} \to \tau$.

Hopf principle and the regularity result imply that $|\nabla\phi| \geqslant L_1 > 0$, on a neighborhood of the boundary. Let us denote by $\mathcal{V}$ the neighborhood of the connected component of $\partial\Omega$ that contains $\bar{x}$. Hence, by Proposition 4.5, in $\mathcal{V}$ either $\tau u \equiv \phi$ or $\tau u > \phi$. We shall see that the first case holds and this will end the proof.

Since $\partial\Omega$ is $\mathcal{C}^2$, there exists a ball denoted $B_{\bar{x}}(r_1)$ such that $B_{\bar{x}}(r_1) \subset \Omega$ and $\overline{B_{\bar{x}}(r_1)} \cap \partial\Omega = \{\bar{x}\}$. Moreover r_1 can be chosen such that $\bigcup_{\bar{x}\in\partial\Omega} B_x(r_1) \subset \mathcal{V}$. Using Remark 4.6, $\phi \equiv \tau u$ in $B_{\bar{x}}(r_1)$ and hence in $\mathcal{V}$. $\square$

It is an immediate consequence of the definition of the principal eigenvalues, that if $\Omega' \subset \Omega$ then $\lambda^{\pm}(\Omega') \geqslant \lambda^{\pm}(\Omega)$. In the next theorem we shall prove that the monotonicity is strict.

Theorem 5.2. *Let Ω be a $\mathcal{C}^2$ domain with a connected boundary, and Ω' a subdomain of Ω.*

Suppose that $\partial\Omega$ is not included in $\partial\Omega'$ or, for $N = 2$ and $\tilde{F}$ convex or concave, suppose only that $\overline{\Omega'} \neq \overline{\Omega}$. Then

$$\lambda^{\pm}\big(\Omega'\big) > \lambda^{\pm}(\Omega).$$

Proof. Let us begin with the first case i.e. we suppose that there exists $x_0 \in \partial\Omega$ such that $x_0 \notin \partial\Omega'$. Then, there exists $\delta > 0$, such that $B(x_0, \delta) \subset \mathbb{R}^N \setminus \overline{\Omega'}$. One can choose δ small enough in order that $\partial(\Omega \setminus \overline{B(x_0, \delta)})$ is connected.

Let Ω'' be a smooth domain whose boundary is connected, and such that

$$\Omega' \subset \Omega'' \subset \Omega \setminus \overline{B(x_0, \delta)}.$$

We know that

$$\lambda^+(\Omega') \geqslant \lambda^+(\Omega'') \geqslant \lambda^+(\Omega \setminus \overline{B(x_0, \delta)}) \geqslant \lambda^+(\Omega).$$

Suppose by contradiction that $\lambda^+(\Omega') = \lambda^+(\Omega)$; this implies that $\lambda^+(\Omega'') = \lambda^+(\Omega) := \lambda^+$.

Let $\phi > 0$ be an eigenfunction corresponding to λ^+ in Ω. The above assumptions imply that it satisfies

$$F(x, \nabla\phi, D^2\phi) + (c(x) + \lambda^+)\phi^{1+\alpha} = 0 \quad \text{in } \Omega'',$$

then, using Theorem 5.1, this implies that $\phi = 0$ on $\partial\Omega''$. This contradicts the fact that $\phi > 0$ in $\partial\Omega'' \cap \Omega$ and it ends the first case.

We are left to prove the case $\partial\Omega \cap \partial\Omega' = \partial\Omega$ and $N = 2$.

We argue by contradiction and suppose that $\lambda^+(\Omega') = \lambda^+(\Omega)$. We can assume that Ω' is smooth by replacing Ω' with some smooth subdomain Ω'' which contains Ω' and is such that $\partial\Omega'' \cap \partial\Omega = \partial\Omega$. Arguing as before, $\lambda^+(\Omega'') = \lambda(\Omega)$.

For simplicity we rename this set Ω', and we consider some positive eigenfunction ϕ (respectively ϕ') for Ω (respectively for Ω'). By hypothesis they satisfy the same equation in Ω'. Let

$$\tau = \sup_{\Omega'} \frac{\phi'}{\phi},$$

proceeding as in the proof of Theorem 4.1, one proves that τ is bounded and achieved on $\partial\Omega$.

Using Proposition 4.5, there exists a closed connected neighborhood of $\partial\Omega$, say G, such that $\phi' \equiv \tau\phi$ in G, and

$$\nabla\phi = \nabla\phi' = 0 \quad \text{in } K := \partial G \cap \Omega.$$

Hence using Sard's theorem there exists some constant c such that

$$\phi = c, \ \phi' = \tau c \quad \text{in } K.$$

This of course leads to a contradiction, because either $\phi < c$ somewhere in $\Omega \setminus G$ and then ϕ would have a local minimum, and this contradicts Remark 2.7. Or $\phi \geqslant c$ in $\Omega \setminus G$ and then by Hopf's lemma and Lemma 4.7, there is a point where $\nabla\phi \neq 0$ in K, this is again a contradiction. This ends the proof. □

In recent papers, when $\alpha = 0$, Armstrong [3], Felmer, Quaas, Sirakov [30,20] have studied some sort of "Fredholm" alternative in order to establish for which functions f there exists a solution of the Dirichlet problem when $\lambda^- \leqslant \lambda \leqslant \lambda^+$.

Here we just present a nonexistence result at resonance. To fix the ideas, in the next theorem we suppose $\lambda^+ < \lambda^-$, with obvious symmetric results in the other case.

In that hypothesis, if $f \leqslant 0$, using the minimum principle below λ^-, any solution u of the equation

$$\begin{cases} F\big(x, \nabla u, D^2 u\big) + \big(c(x) + \lambda^+\big) u^{1+\alpha} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is nonnegative.

Theorem 5.3. *Suppose that $\partial\Omega$ is connected.*

Case 1. $f \geqslant 0$ *in* Ω $(f \not\equiv 0)$.

If $N = 2$ and $\tilde{F}$ is convex or concave or if $f > 0$ somewhere near the boundary, then there are no solutions of

$$\begin{cases} F\big(x, \nabla u, D^2 u\big) + \big(c(x) + \lambda^+\big) |u|^\alpha u = f & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{12}$$

Case 2. $f \leqslant 0$ *in* Ω $(f \not\equiv 0)$.

If $N = 2$ and $\tilde{F}$ is convex or concave or if $f < 0$ somewhere near the boundary, then there are no solutions of

$$\begin{cases} F\big(x, \nabla u, D^2 u\big) + \big(c(x) + \lambda^-\big) |u|^\alpha u = f & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{13}$$

Proof. We begin by Case 2 and hence we suppose that u is a solution of (13). If $u \geqslant 0$ then, according to Hopf's principle, since $f \leqslant 0$, u cannot have an inside minimum and $u > 0$. But this contradicts the definition of λ^+. Then there exists some point $x_0 \in \Omega$ where $u(x_0) < 0$. Let $-\varphi^-$ be some normalized eigenfunction for λ^-, with $\varphi^- > 0$. $-u^-$ is a super-solution of the equation in Ω and we define $\Gamma = \sup \frac{u^-}{\varphi^-}$. Then $\Gamma > 0$ and $u^- \leqslant \Gamma \varphi^-$.

Reasoning as in Theorem 4.1 we can prove that Γ is achieved on $\partial\Omega$. Furthermore the strong comparison principle in Proposition 4.5 implies that $-u^- = -\Gamma\varphi^-$ in a neighborhood of the boundary.

In particular this implies that $f \equiv 0$ in a neighborhood of $\partial\Omega$, this ends the proof in one of the hypotheses of Case 2. In the other hypothesis, i.e. $N = 2$, Theorem 4.8 in its form with $f \geqslant 0$ implies $u^- \equiv \Gamma\varphi^-$ in Ω. This would imply $f \equiv 0$ which is a contradiction.

Suppose now that v is a solution of (12), by the minimum principle, since $\lambda^+ < \lambda^-$, $v \geqslant 0$ in Ω. Since $f \not\equiv 0$, there exists x_0 such that $v(x_0) > 0$. Let ϕ be some positive eigenfunction corresponding to λ^+, and $\gamma = \inf\{t\colon\ v \leqslant t\phi\}$, proceeding as above one can prove that $\gamma > 0$ and γ is "achieved" on the boundary. This implies that $v \equiv \gamma\phi$ in a neighborhood of the boundary. This gives a contradiction if $f < 0$ somewhere near the boundary.

Furthermore, in the case $N = 2$, we can conclude using Theorem 4.8 that $v \equiv \gamma\phi$ everywhere, a contradiction. $\square$

5.2. Further properties

We want to prove that we can recover some other of the standard properties of the eigenvalues for linear elliptic equations. We consider the Dirichlet problem

$$\begin{cases} F\big(x, \nabla u, D^2 u\big) + \big(c(x) + \lambda\big)|u|^\alpha u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Nontrivial solutions of (14) will be called eigenfunctions.

We now recall the following result, which is an easy consequence of the definitions of $\lambda^\pm$ and the maximum and the minimum principle (see [7,9]).

Theorem 5.4. *Suppose that Ω is a bounded $\mathcal{C}^2$ domain*:

(1) *If $\lambda > \lambda_1 = \sup(\lambda^+(\Omega), \lambda^-(\Omega))$, then every nontrivial solution of* (14) *changes sign in Ω.*
(2) *For any λ between λ^+ and λ^- there are no nontrivial solutions of* (14)*.*

As a consequence of the results of the previous section, we now have further results regarding the signs of the eigenfunctions.

Theorem 5.5. *Suppose that Ω is a $\mathcal{C}^2$ bounded domain and suppose that one of the following hypotheses holds*

1. *$\lambda^+ = \lambda^-$ and $\partial\Omega$ is connected,*
2. *$N = 2$ and $\tilde{F}$ is convex or concave, $\lambda^+ = \lambda^-$, $\partial\Omega$ has at most two connected components,*
3. *$N = 2$ and $\tilde{F}$ is convex or concave, $\partial\Omega$ is connected,*

then, any eigenfunction corresponding to $\lambda = \lambda^\pm$ is of constant sign.

An application of the previous result is the following

Corollary 5.6. *Under the hypotheses of Theorem* 5.5*, the eigenvalues λ^+ or λ^- are isolated, i.e. there exists $\delta > 0$ such that for any $\lambda \in (\lambda^\pm, \lambda^\pm + \delta)$, the solutions of* (14) *are trivial.*

Proof of Theorem 5.5. In the whole proof, φ will denote the positive eigenfunction corresponding to $\lambda^+(\Omega)$, and we suppose by contradiction that there exists ψ, another eigenfunction, that changes sign and $\Omega^+ = \{x \in \Omega;\ \psi(x) > 0\}$ and $\Omega^- = \{x \in \Omega;\ \psi(x) < 0\}$.

We begin with Case 1 i.e. $\lambda_1 = \lambda^+ = \lambda^-$ and $\partial\Omega$ is connected. Clearly, for any connected component of Ω^+, denoted by $\tilde{\Omega}^+$, respectively for any connected component of Ω^- denoted by $\tilde{\Omega}^-$:

$$\lambda^+\big(\tilde{\Omega}^+\big) = \lambda^-\big(\tilde{\Omega}^-\big) = \lambda_1.$$

If $\partial\tilde{\Omega}^+ \cap \partial\Omega \neq \partial\Omega$, using Theorem 5.2 one would have $\lambda^+(\tilde{\Omega}^+) > \lambda^+(\Omega)$. But, on the other hand, if $\partial\tilde{\Omega}^+ \cap \partial\Omega = \partial\Omega$, then $\partial\tilde{\Omega}^- \cap \partial\Omega \neq \partial\Omega$, and then the contradiction is given by the fact that it would imply $\lambda^-(\tilde{\Omega}^-) > \lambda^-(\Omega)$.

We now suppose to be in the second case i.e. $\Omega = \Omega_2 \setminus \overline{\Omega}_1$ where for $i = 1, 2$, Ω_i is a simply connected open set, with $\Omega_1 \Subset \Omega_2$.

If $\partial\Omega^+ \cap \partial\Omega_i \neq \emptyset$ then reasoning as in the proof of Theorem 5.1, $\partial\Omega^+ \cap \partial\Omega = \partial\Omega_i$. The same is true for Ω^-, hence one can assume without loss of generality that $\partial\Omega^+ \cap \partial\Omega = \partial\Omega_1$ and $\partial\Omega^- \cap \partial\Omega = \partial\Omega_2$, finally $\partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega \neq \emptyset$.

Since $\lambda^+ = \lambda^-$, $-\varphi$ is an eigenfunction corresponding to λ^-. Reasoning as in the proof of Theorem 4.1, we can define $\Gamma := \sup \frac{\psi}{\varphi}$ and it is "achieved" on $\partial\Omega_1$, while $\gamma := \sup \frac{-\psi}{\varphi}$ is "achieved" on $\partial\Omega_2$. Thanks to Proposition 4.5 $\psi = \Gamma\varphi$ on a neighborhood of $\partial\Omega_1$ and $\psi = -\gamma\varphi$ on a neighborhood of $\partial\Omega_2$.

We can also define as in the proof of Theorem 4.1 M_1, K_1, M_2, K_2 with $\nabla\psi = \nabla\varphi = 0$ on K_1 and K_2, and K_1 and K_2 are connected sets in Ω. Since $\varphi > 0$ in $M_1 \cap M_2$, using Sard's theorem $c_i = \varphi_{|K_i}$. Again reasoning as in Theorem 4.1, the minimum of φ is either on K_1 or K_2 where its gradient is zero, a contradiction with Hopf principle.

We consider the last case, i.e. $N = 2$, $\lambda^+ \neq \lambda^-$ and $\partial\Omega$ is connected. It is clear, using the maximum principle, that the result is true for the smallest of the two eigenvalues. We suppose to fix the ideas that $\lambda^- < \lambda^+$ and we prove that every eigenfunction corresponding to λ^+ is positive.

Since ψ is a positive eigenfunction in Ω^+, $\lambda^+(\Omega^+) = \lambda^+(\Omega)$, in the same manner $\lambda^-(\Omega^-) = \lambda^-(\Omega)$; but, in dimension 2, this contradicts Theorem 5.2. □

Proof of Corollary 5.6. The result needs only to be proved for $\lambda_1 = \sup(\lambda^+, \lambda^-)$. Suppose by contradiction that there exists a sequence of eigenvalues $(\lambda_n)_n$, such that $\lambda_n \to \lambda_1$, $\lambda_n > \lambda_1$.

Let $(u_n)_n$ be a sequence of solutions of (14) with $\lambda = \lambda_n$ such that $|u_n|_\infty = 1$. This implies that the Lipschitz norm is uniformly bounded with respect to n (see Proposition 2.8).

Then $(u_n)_n$ is relatively compact and, up to a subsequence, it converges in $\mathcal{C}(\overline{\Omega})$ towards a solution u of (14) with $\lambda = \lambda_1$.

By Theorem 5.5 u must be either positive or negative. Hence for any $K = \overline{\Omega}_1 \subset \Omega$ and for n large enough u_n has constant sign in K.

Without loss of generality we can suppose that $\lambda_1 = \lambda^+$ and $u > 0$ in Ω and hence u_n is positive in K.

We choose Ω_1 a regular subset in Ω such that $\lambda^-(\Omega \setminus \overline{\Omega}_1) > \lambda_n$. By the minimum principle, since

$$u_n \geqslant 0 \quad \text{in } \partial(\Omega \setminus \overline{\Omega}_1),$$

one gets that $u_n \geqslant 0$ in $\Omega \setminus \overline{\Omega}_1$, then u_n is positive in Ω, and this contradicts Theorem 5.4. □

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