J. Differential Equations 250 (2011) 2737–2778

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Martingale solutions and Markov selection of stochastic 3D Navier–Stokes equations with jump ☆

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ARTICLE INFO

Article history:
Received 17 January 2010
Revised 28 December 2010
Available online 1 February 2011

MSC:
primary 60H15, 60J75
secondary 60G44, 60G55

Keywords:
3D Navier–Stokes equation
Martingale solutions
Markov selection
Lévy processes

ABSTRACT

In this paper, we study the existence of martingale solutions of stochastic 3D Navier–Stokes equations with jump, and following Flandoli and Romito (2008) [7] and Goldys et al. (2009) [8], we prove the existence of Markov selections for the martingale solutions.

1. Introduction

Let D be a bounded open domain of R^3 with regular boundary ∂D, consider Newtonian fluid described by the stochastic 3-dimensional Navier–Stokes equation on D,

$$\frac{\partial u(t,x)}{\partial t} - \nu \Delta u(t,x) + \big(u(t,x)\cdot\nabla\big)u(t,x) = -\nabla p(t,x) + f(t,x) + G(u,\xi)(t,x), \tag{1.1}$$

with the incompressibility condition

☆ The work was supported by the Key Laboratory of Random Complex Structures and Data Science, AMSS, Chinese Academy of Sciences, 973 Project (2011CB808000), Science Fund for Creative Research Groups (10721101), the NSFC (Nos. 10671197, 11071008).

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.01.018

$$\operatorname{div} u(t,x) = 0, \quad t \in [0,\infty),\ x \in D, \tag{1.2}$$

the boundary condition

$$u(t,x) = 0, \quad t \in [0,\infty),\ x \in \partial D, \tag{1.3}$$

and the initial condition

$$u(0,x) = u_0(x), \quad x \in D. \tag{1.4}$$

The fluid is described by the velocity field $u = u(t,x)$ and the pressure field $p = p(t,x)$. The parameter $\nu > 0$ is the kinematic viscosity. Here G is an operator acting on noise and solution. When the process $\xi(t,x)$ is a Brownian motion, the stochastic equation (1.1) has been studied by many authors, see [3,5,6,8]. It is known that there exists a global solution of the martingale problem for this case; and also the Markov selections for the martingale solution, see [5,7,8].

Up to our knowledge, there have no results as the $\xi(t,x)$ is a Lévy noise. In this paper, we prove that there exist Martingale solutions of stochastic 3D Navier–Stokes equations with jump, and then we prove that there exist Markov selections for the martingale solutions.

We consider the usual abstract form of Eqs. (1.1)–(1.4). Let $\mathscr{D}^\infty$ be the space of infinitely differentiable 3-dimensional vector fields $u(x)$ on D with compact support strictly contained in D, satisfying $\operatorname{div} u(x) = 0$. Denote by V_α the closure of $\mathscr{D}^\infty$ in the Soblev space $[H^\alpha(D)]^3$, for $\alpha \geqslant 0$, and in particular

$$H = V_0, \qquad V = V_1.$$

Denote by $|\cdot|_H$ and $\langle\cdot,\cdot\rangle_{H,H}$ the norm and inner product in H. Identifying H with its dual space H', and let V'_α the dual space of V_α, we have $V_\alpha \subset H = H' \subset V'_\alpha$ with continuous injections. Denote the dual pairing between V_α and V'_α by $\langle\cdot,\cdot\rangle_{V_\alpha,V'_\alpha}$.

Let $D(A) = [H^2(D)]^3 \cap V$, and define the linear operator $A : D(A) \subset H \to H$ as $Au = -P\Delta u$, where P is the projection from $[L^2(D)]^3$ to H. Since V coincides with $D(A^{1/2})$, we can endow V with the norm $\|u\|_V = |A^{1/2}u|_H$. The operator A is positive selfadjoint with compact resolvent. Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ be the eigenvalues of A, and $e_1, e_2, \ldots$ be the corresponding eigenvectors, which form a complete orthonormal system in H. We remark that $\|u\|_V^2 \geqslant \lambda_1 |u|_H^2$.

Remark 1.1. Note that $D(A) = \{u = \sum_{i=1}^\infty u_i \cdot e_i \in H : \sum_{i=1}^\infty \lambda_i^2 u_i^2 < \infty\}$, we may endow $D(A)$ with the inner product

$$\langle u, v\rangle_{D(A)} = \sum_{i=1}^\infty \lambda_i^2 u_i v_i,$$

$u_i = \langle u, e_i\rangle_H$, $v_i = \langle v, e_i\rangle_H$ for every $u, v \in D(A)$. So $D(A)$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{D(A)}$ and $\{\frac{e_i}{\lambda_i}\}_{i\in N}$ is a complete orthonormal system of D(A). For the dual space of $D(A)$, $D(A)' = \{u = \sum_{i=1}^\infty u_i \cdot e_i\colon \sum_{i=1}^\infty \frac{u_i^2}{\lambda_i^2} < \infty\}$ and endow $D(A)'$ with the inner product $\langle u, v\rangle_{D(A)'} = \sum_{i=1}^\infty \frac{u_i v_i}{\lambda_i^2}$, $D(A)'$ is a Hilbert space. For every $u \in D(A)$, $v \in D(A)'$, $\langle u, v\rangle_{D(A),D(A)'} = \sum_{i=1}^\infty u_i v_i$, and if $v \in H$, we have $\langle u, v\rangle_{D(A),D(A)'} = \langle u, v\rangle_H$. It will be convenient to use fractional powers of the operator A, as well as their domains $D(A^\alpha)$ for $\alpha \in R$.

Define the bilinear operator $B(u,v): V \times V \to (V \cap [L^2(D)]^3)'$ as

$$\langle B(u,v), z\rangle = \int_D z(x) \cdot \big(u(x) \cdot \nabla\big) v(x)\, dx, \quad z \in V \cap \big[L^2(D)\big]^3.$$

From the incompressibility condition, $\langle B(u,v), v\rangle = 0$, $\langle B(u,v), z\rangle = -\langle B(u,z), v\rangle$. By [11], there exists $\beta > 1$, B can be extended to a continuous operator

$$B : H \times H \to D\big(A^{-\beta}\big)$$

and

$$\big|\langle w, B(u,v)\rangle\big| \leqslant C|u|_H |v|_H \|w\|_{A^\beta}. \tag{1.5}$$

Eqs. (1.1)–(1.4) have the abstract form as a stochastic evolution equation:

$$\begin{cases} du(t) + \nu Au(t)\, dt + B\big(u(t), u(t)\big)\, dt = f(t)\, dt + \displaystyle\int_{|x|_K \leqslant 1} F\big(u(t-), x\big)\widetilde{N}_p(dt, dx), \\ u(0) = u_0. \end{cases} \tag{1.6}$$

In this article, we assume that

(i) $u_0 \in H$ and $f \in L^2([0,\infty); V')$.
(ii) $p = p(t)$, $t \in D_p$ is a stationary $\mathcal{F}_t$-Poisson point process of the class (QL) on a measurable space K, with compensator $t\lambda(U)$. $\lambda(dx)$ is the characteristic measure of p satisfying $\int_K |x|_K^2 \wedge 1\lambda(dx) < \infty$. $N_p(dt, dx)$ is the counting measure defined as follows:

$$N_p\big((0,t] \times U\big) = \#\big\{s \in D_p;\ s \leqslant t,\ p(s) \in U\big\}$$

for $t > 0$, $U \in \mathscr{B}(K)$, where D_p is the domain of p, $\widetilde{N}_p(dt, dx) = N_p(dt, dx) - dt\lambda(dx)$.
(iii) $F(\cdot,\cdot)$ is measurable function from $H \times K$ to H.

2. Preliminaries

Let (E, r) denote a metric space. Denote by $\mathscr{B}$ the Borel σ-field of (E, r), $Pr(E)$ the set of all probability measures on $(E, \mathscr{B})$. Let $D_E[0,\infty)$ be the space of right continuous functions from $[0,\infty)$ into (E, r) having left limits, with the Skorokhod metric $d(\cdot,\cdot)$ (see in Chapter 3 of [4] for the details).

Lemma 2.1. *Assume $\{x, y, x_n\} \subset D_E[0,\infty)$ and $\theta \in E$ $f_T(y) = \sup_{t\in[0,T]} r(y(t), \theta)$. If $\lim_{n\to\infty} d(x_n, x) = 0$, then for any $T \in \{t\colon\ x(t-) = x(t)\}$,*

$$\lim_{n\to\infty} f_T(x_n) = f_T(x).$$

Proof. By [4], if $\lim_{n\to\infty} d(x_n, x) = 0$, then for any sequence $t_n \in [0,\infty)$, $t \geqslant 0$, and $\lim_{n\to\infty} t_n = t$,

$$\lim_{n\to\infty} r\big(x_n(t_n), x(t)\big) \wedge r\big(x_n(t_n), x(t-)\big) = 0 \tag{2.1}$$

and x has at most countably many points of discontinuity. We can select a countable subset $Q \subset \{s \in [0,T]:\ x(s) = x(s-)\}$ such that Q is dense in $[0,T]$ and $T \in Q$. For every $s \in Q$, $\lim_{n\to\infty} r(x_n(s), x(s)) = 0$. So for any $\varepsilon > 0$,

$$\varepsilon + \lim_{n\to\infty} \inf_n \sup_{s\in Q} r\big(x_n(s), \theta\big) \geqslant \sup_{s\in Q} r\big(x(s), \theta\big).$$

Since $f_T(y) = \sup_{s\in Q} r(y(s), \theta)$, we have

$$\varepsilon + \lim_{n\to\infty} \inf_n f_T(x_n) \geqslant f_T(x). \tag{2.2}$$

On the other hand, from (2.1), for any fixed $t \in [0,T]$, and any $\varepsilon > 0$, there exists $\delta_t > 0$ and N_t, such that for any $|s-t| \leqslant \delta_t$ and $n \geqslant N_t$, $r(x_n(s), x(t)) \wedge r(x_n(s), x(t-)) \leqslant \varepsilon$, thus $r(x_n(s), \theta) \leqslant r(x(t), \theta) \vee r(x(t-), \theta) + \varepsilon$. Therefore there exists $\{A_i = \{|t_i - s| \leqslant \delta_{t_i}\}, 1 \leqslant i \leqslant N\}_{i\in\{1,2,\dots,N\}}$, for $n \geqslant \max\{N_{t_i}\}_{i\in\{1,2,\dots,N\}}$,

$$\sup_{s\in[0,T]} r\big(x_n(s), \theta\big) \leqslant \max_{1\leqslant i\leqslant N} r\big(x(t_i), \theta\big) \vee r\big(x(t_i-), \theta\big) + \varepsilon \leqslant \sup_{s\in[0,T]} r\big(x(t), \theta\big) + \varepsilon. \tag{2.3}$$

This implies $\lim_{n\to\infty} f_T(x_n) = f_T(x)$. $\square$

For $\xi \in D_R[0,\infty)$ let

$$\begin{aligned} \triangle\xi(s) &= \xi(s) - \xi(s-), \\ U(\xi) &= \big\{u > 0:\ \big|\triangle\xi(s)\big| = u \text{ for some } s\big\}, \\ U_T^{u_0}(\xi) &= \big\{u > 0:\ \big|\triangle\xi(s)\big| = u > u_0 \text{ for some } s \in [0,T]\big\}. \end{aligned}$$

Then $U(\xi)$, the collection of all jump size of ξ, is at most countable. Let $U^c(\xi)$ be the complement of $U(\xi)$. For $u > 0$, let

$$t^0(\xi,u) = 0, \qquad t^p(\xi,u) = \inf\big\{t > t^{p-1}(\xi,u):\ \big|\triangle\xi(t)\big| > u\big\},$$

$t^p(\xi,u)$ is the p-th jump time of ξ with the norm of jump size greater than u. Because $\xi \in D_R[0,\infty)$, $\lim_{p\to\infty} t^p(\xi,u) = \infty$. Set $p_T(\xi) = \max\{p:\ t^p(\xi,u) \leqslant T\}$, it is easy to see that $p_T(\xi) < \infty$.

Lemma 2.2. *Suppose g is a continuous function from R to R with $g(x) = 0$, $x \in [-u,u]$ for some positive constant u. For any fixed $T > 0$, set*

$$G_T(\xi) = \sum_{s\leqslant T} g\big(\triangle\xi(s)\big), \quad \xi \in D_R[0,\infty).$$

If ξ is continuous at T, then $G_T(\cdot)$ is continuous at ξ.

Proof. Suppose $\lim_{n\to\infty} \xi_n = \xi$ in $D_R[0,\infty)$. Let $0 < t_1 < t_2 < \cdots < t_m < T$ be total points with $|\triangle\xi(t)| \geqslant u$, $t \in [0,T]$. There exists $\epsilon > 0$, satisfying

$$u - \epsilon \notin U(\xi), \qquad [u-\epsilon, u) \cap U_T^{u/2}(\xi) = \emptyset.$$

Because for any $d > 0$, the set $\{t \in [0,T]\colon |\triangle\xi(t)| \geqslant d\}$ only has finite elements, so there exists $\delta > 0$, such that $[u-\delta, u) \cap U_T^{u/2}(\xi) = \emptyset$. Since $U(\xi)$ is at most countable, $[u-\delta, u) \cap U^c(\xi) \neq \emptyset$. Choose $\epsilon > 0$, such that $u - \epsilon \in [u-\delta, u) \cap U^c(\xi)$.

Applying Theorem 15.30 in [16], we know that $t^i(\xi, u-\epsilon)$ and $\triangle\xi(t^i(\xi, u-\epsilon))$ is continuous in $D_R[0,\infty)$ at ξ. Note that $t^{n+1}(\xi, u-\epsilon) > T$, let $\delta = \frac{t^{n+1}(\xi,u-\epsilon)-T}{2} \wedge \frac{T-t^n(\xi,u-\epsilon)}{2}$, there exists N, such that

$$\left|t^i(\xi_{m'}, u-\epsilon) - t^i(\xi, u-\epsilon)\right| \leqslant \frac{\delta}{2}, \quad m' > N,\ i = 1, 2, \ldots, n+1.$$

So

$$G_T(\xi_{m'}) = \sum_{i=1}^{n} g\bigl(\triangle\xi_{m'}\bigl(t^i(\xi_{m'}, u-\epsilon)\bigr)\bigr).$$

Since $t^i(\xi, u-\epsilon) = t_i$, G_T is continuous at ξ. $\square$

Lemma 2.3. *Let X be a Polish space, $P_n \xrightarrow{w} P$, $\varphi_n, \varphi : X \to R$ be measurable. If there exists $X' \in \mathscr{B}(X)$ with $P(X') = 1$ such that for $x \in X'$, $x_n \to x$ in X, $\varphi_n(x_n) \to \varphi(x)$ and $P_n[|\varphi_n|^{1+\varepsilon}] \leqslant C$ for some $\varepsilon,\ C > 0$, then*

(1) $P[|\varphi|] < \infty$ *and* $P_n[\varphi_n] \to P[\varphi]$,
(2) $P[|\varphi|^{1+\varepsilon}] \leqslant C$.

Proof. The proof of (1) is similar to the argument as in [8]. We only prove (2). By (1) we have

$$P\bigl[|\varphi|^{1+\delta}\bigr] = \lim_{n\to\infty} P_n\bigl[|\varphi|^{1+\delta}\bigr], \quad \text{for } \delta \in [0, \varepsilon).$$

From Hölder's inequality

$$P_n\bigl[|\varphi|^{1+\delta}\bigr] \leqslant \bigl(P_n\bigl[|\varphi|^{1+\varepsilon}\bigr]\bigr)^{(1+\delta)/(1+\varepsilon)} \leqslant C^{(1+\delta)/(1+\varepsilon)},$$

and by Fatou lemma

$$P\bigl[|\varphi|^{1+\varepsilon}\bigr] \leqslant C. \qquad \square$$

Remark 2.1. By [4], if (E, r) is separable and complete, then $(D_E[0,\infty), d)$ is a Polish space.

To get our main results, we need to prove the tightness in vector valued Skorokhod space. The Aldous criterion for tightness is a sufficient condition for proving the tightness, refer to [9,10]. In [13,14], by using this criterion, one can get the tightness in Skorokhod space once the energy inequality is proved, thus martingale problem is formulated on vector valued Skorokhod space and with Gaussian as the limit measure. We use the same Aldous criterion to get the tightness in vector valued Skorokhod space, see Lemmas 2.4 and 2.5. Also we prove the energy inequality.

Lemma 2.4. *(See [1].) Let E be a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. For an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ in E, define the function $r_N^2 : E \to R^+$ by*

$$r_N^2(x) = \sum_{k\geqslant N+1} \langle x, e_k\rangle^2, \quad N \in \mathbb{N}.$$

Let D be a total and closed under addition subset of E. Then the sequence $\{X_n\}_{n\in\mathbb{N}}$ of stochastic processes with trajectories in $D_E[0,\infty)$ is tight iff it is D-weakly tight and for every $\epsilon > 0$ and $t > 0$,

$$\lim_{N\to\infty}\lim_{n\to\infty}\sup P\big(r_N^2\big(X_n(s)\big) > \epsilon \text{ for some } s \in [0,t]\big) = 0. \tag{2.4}$$

Remark 2.2. If there exist positive C and a sequence λ_n satisfying $\lambda_n \to \infty$ such that

$$E^P\Big(\sup_{s\in[0,t]} r_N^2\big(X_n(s)\big)\Big) \leqslant \frac{C}{\lambda_N},$$

then (2.4) holds.

Let $\{X_n\}$ be a sequence of random elements of $D_R[0,T]$, and $\{\tau_n, \delta_n\}$ be such that:

(a) For each n, τ_n is a stopping time with respect to the natural σ-fields, and takes only finitely many values.
(b) For each n, the constant $\delta_n \in [0,T]$ satisfy $\delta_n \to 0$ as $n \to \infty$.

We introduce the following condition on $\{X_n\}$: for each sequence $\{\tau_n, \delta_n\}$ in (a), (b),

$$\text{(A)} \quad X_n(\tau_n + \delta_n) - X_n(\tau_n) \to 0, \quad \text{in probability}.$$

For $f \in D_R[0,T]$, let $J(f)$ denote the maximum of the jump $|f(t) - f(t-)|$.

Lemma 2.5. *(See [2].) Suppose that $\{X_n\}_{n\in\mathbb{N}}$ satisfies (A), and either $\{X_n(0)\}$ and $\{J(X_n)\}$ are tight on the line; or $\{X_n(t)\}$ is tight on the line for each $t \in [0,T]$, then $\{X_n\}$ is tight in $D_R[0,T]$.*

3. Existence of martingale solution of Eq. (1.6)

We divide this section into additive Lévy and multiplicative noise parts. Instead of martingale representation theorems like wiener processes as in [6] etc., we use the Lévy–Khinchin formula for additive Lévy noise. For the multiplicative noise, this method fall to use and we use martingale character.

3.1. Additive Lévy noise

Eq. (1.6) has the following form:

$$du(t) + \big[\nu Au(t) + B\big(u(t), u(t)\big)\big]\,dt = \sum_{i=1}^{\infty} \sigma_i e_i\,dL_i(t) \tag{3.1}$$

where $\{L_i(t)\}_{i\in N}$ are independent Lévy processes defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$,

$$L_i(t) = \int_0^{t+}\int_{|x|\leqslant 1} x\widetilde{N}_i(dt,dx), \quad \widetilde{N}_i(dt,dx) = N_i(dt,dx) - dt\lambda(dx).$$

$\{N_i(dt,dx)\}_{i\in N}$ are independent Poisson random measure with $\int_{|x|\leqslant 1} x^2\lambda(dx) < \infty$.

3.2. Finite dimensional models

Let $H_n = span\{e_i, 1 \leqslant i \leqslant n\}$, π_n is the orthogonal projection of H on H_n, and A_n is the restriction of A to H_n. $B_n(\cdot,\cdot): H_n \times H_n \to H_n$ the continuous bilinear operator defined as

$$\big\langle B_n(u,v), w\big\rangle = \big\langle B(u,v), w\big\rangle$$

for every $u, v, w \in H_n$, $B_n(u,v) = \pi_n B(u,v)$.

Consider the equation on H_n:

$$dX_t^n + \big[\nu A_n X_t^n + B_n\big(X_t^n, X_t^n\big)\big]\,dt = \sum_{i=1}^n \sigma_i e_i\, dL_i(t). \tag{3.2}$$

Let $(X_t^n)_{t\geqslant 0}$ be the $RCLL$ adapted solution of Eq. (3.2). In the following, $C(\cdot)$ means a positive constant with dependent only on the elements in the bracket.

Theorem 3.1. *Assume $E|X_0^n|_H^2 < \infty$, then for every $T > 0$,*

$$E\left(\sup_{t\in[0,T]}\big|X_t^n\big|_H^2 + 2\nu\int_0^T \big\|X_s^n\big\|_V^2\,ds\right) \leqslant C\left(E\big|X_0^n\big|_H^2, T, \sum_{i=1}^n \sigma_i^2, \int_{|x|\leqslant 1} x^2\lambda(dx)\right), \tag{3.3}$$

$$\begin{aligned}\big|X_t^n\big|_H^2 + 2\nu\int_0^t \big\|X_s^n\big\|_V^2\,ds &= \big|X_0^n\big|_H^2 + 2\int_0^t \left\langle X^n(s-), \sum_{i=1}^n \sigma_i e_i\, dL_i(s)\right\rangle_H \\ &\quad + \sum_{i=1}^n \int_0^t \int_{|x|\leqslant 1} \sigma_i^2 x^2\, dN_i(ds,dx),\end{aligned} \tag{3.4}$$

$$E\big|X_t^n\big|_H^2 + 2\nu E\int_0^t \big\|X_s^n\big\|_V^2\,ds = E\big|X_0^n\big|_H^2 + t\sum_{i=1}^n \sigma_i^2 \int_{|x|\leqslant 1} x^2\lambda(dx). \tag{3.5}$$

Proof. From Eq. (3.2) and Itó formula (refer to [12]), we have

$$\begin{aligned}\big|X_t^n\big|_H^2 &= \big|X^n(0)\big|_H^2 - 2\nu\int_0^t \big\langle A_n X_s^n, X_s^n\big\rangle_H\,ds - 2\int_0^t \big\langle B_n\big(X_s^n, X_s^n\big), X_s^n\big\rangle_H\,ds \\ &\quad + 2\int_0^t \left\langle X^n(s-), \sum_{i=1}^n \sigma_i e_i\, dL_i(s)\right\rangle_H + \sum_{i=1}^n \int_0^t \int_{|x|\leqslant 1} \sigma_i^2 x^2\, dN_i(ds,dx).\end{aligned}$$

Since $\langle B_n(x,x), x\rangle = 0$ for every $x \in H_n$,

$$\left|X_t^n\right|_H^2 + 2\nu \int_0^t \left\|X_s^n\right\|_V^2 ds = \left|X^n(0)\right|_H^2 + 2\int_0^t \left\langle X^n(s-), \sum_{i=1}^n \sigma_i e_i \, dL_i(s)\right\rangle_H$$

$$+ \sum_{i=1}^n \int_0^t \int_{|x|\leqslant 1} \sigma_i^2 x^2 \, dN_i(ds, dx).$$

Set $M_t^n = 2\int_0^t \langle X^n(s-), \sum_{i=1}^n \sigma_i e_i \, dL_i(s)\rangle_H$, since for $T > 0$,

$$\sum_{i=1}^n E\left(\int_0^T \int_{|x|\leqslant 1} \langle X^n(s), \sigma_i e_i x\rangle_H^2 \lambda(dx)\, ds\right) < \infty, \tag{3.6}$$

M_t^n is a square integrable martingale.

Define the stopping time $\tau_m = \inf\{t \geqslant 0\colon\ |X_t^n|_H^2 \geqslant m\}$, then

$$\left|X_{t\wedge\tau_m}^n\right|_H^2 \leqslant \left|X^n(0)\right|_H^2 + M_{t\wedge\tau_m}^n + \sum_{i=1}^n \int_0^{t\wedge\tau_m} \int_{|x|\leqslant 1} \sigma_i^2 x^2 \, dN_i(ds, dx),$$

and thus

$$E\left|X_{t\wedge\tau_m}^n\right|_H^2 \leqslant E\left|X^n(0)\right|_H^2 + t\left(\sum_{i=1}^n \sigma_i^2\right) \int_{|x|\leqslant 1} x^2 \lambda(dx).$$

From this inequality and the monotone convergence theorem, we get

$$E\int_0^T \left|X_t^n\right|_H^2 dt \leqslant T\left(E\left|X_0^n\right|_H^2 + T\left(\sum_{i=1}^n \sigma_i^2\right) \int_{|x|\leqslant 1} x^2 \lambda(dx)\right).$$

Thus

$$\sum_{i=1}^n E\left(\int_0^T \int_{|x|\leqslant 1} \langle X^n(s), \sigma_i e_i x\rangle^2 \lambda(dx)\, ds\right) = \sum_{i=1}^n E\left(\int_0^T \langle X^n(s), e_i\rangle^2 \cdot \int_{|x|\leqslant 1} \sigma_i^2 x^2 \lambda(dx)\, ds\right)$$

$$\leqslant E\left(\int_0^T \left|X^n(s)\right|_H^2 ds\right) \cdot \left(\sum_{i=1}^n \sigma_i^2\right) \int_{|x|\leqslant 1} x^2 \lambda(dx)$$

$$< \infty,$$

so M_t^n is now a square integrable martingale. The (3.4) and (3.5) are follows.

For the inequality (3.3), since

$$\left|X_t^n\right|_H^2 \leqslant \left|X_0^n\right|_H^2 + \left|M^n(t)\right| + \sum_{i=1}^n \int_0^t \int_{|x|\leqslant 1} \sigma_i^2 x^2 N_i(ds, dx),$$

we have

$$\sup_{t\in[0,T]}\left|X_t^n\right|_H^2 \leqslant \left|X_0^n\right|_H^2 + 1 + \sup_{t\in[0,T]}\left|M^n(t)\right|^2 + \sum_{i=1}^n \int_0^T \int_{|x|\leqslant 1} \sigma_i^2 x^2 N_i(ds,dx)$$

and

$$\begin{aligned}
E\sup_{t\in[0,T]}\left|X_t^n\right|_H^2 &\leqslant E\left|X_0^n\right|_H^2 + 1 + E\sup_{t\in[0,T]}\left|M^n(t)\right|^2 + T\left(\sum_{i=1}^n \sigma_i^2\right)\int_{|x|\leqslant 1} x^2\lambda(dx) \\
&\leqslant E\left|X_0^n\right|_H^2 + 1 + C\sum_{i=1}^n E\int_0^T\int_{|x|<1}\left\langle X^n(s-),\sigma_i e_i x\right\rangle^2 N_i(ds,dx) + T\left(\sum_{i=1}^n \sigma_i^2\right)\int_{|x|\leqslant 1} x^2\lambda(dx) \\
&\leqslant E\left|X_0^n\right|_H^2 + 1 + CE\left(\int_0^T\left|X^n(s)\right|_H^2\,ds\right)\cdot\left(\sum_{i=1}^n \sigma_i^2\right)\int_{|x|\leqslant 1} x^2\lambda(dx) \\
&\quad + T\left(\sum_{i=1}^n \sigma_i^2\right)\int_{|x|\leqslant 1} x^2\lambda(dx) \\
&\leqslant E\left|X_0^n\right|_H^2 + 1 + T\left(\sum_{i=1}^n \sigma_i^2\right)\int_{|x|\leqslant 1} x^2\lambda(dx) \\
&\quad + CT\left(E\left|X^n(0)\right|_H^2 + T\left(\sum_{i=1}^n \sigma_i^2\right)\int_{|x|\leqslant 1} x^2\lambda(dx)\right)\left(\sum_{i=1}^n \sigma_i^2\right)\int_{|x|\leqslant 1} x^2\lambda(dx) \\
&< \infty.
\end{aligned}$$

(3.5) implies the second part of the bound (3.3). □

Corollary 3.1. *Let $\tau \geqslant 0$ be a stopping time and $(X_t^n)_{t\geqslant 0}$ be a càdlàg adapted processes that P-a.s. satisfy* (3.2) *for $t \in [0,\tau(\omega)]$. Assume $E|X_0^n|_H^2 < \infty$. Then for $T > 0$,*

$$E\left(\sup_{t\in[0,T]}\left|X_{t\wedge\tau}^n\right|_H^2 + 2\nu\int_0^T \left\|X_{s\wedge\tau}^n\right\|_V^2\,ds\right) \leqslant C\left(E\left|X_0^n\right|_H^2, T, \sum_{i=1}^n \sigma_i^2, \int_{|x|\leqslant 1} x^2\lambda(dx)\right). \tag{3.7}$$

Lemma 3.1. *Suppose $(X_t^{(1)})$ and $(X_t^{(2)})$ be two solutions of Eq.* (3.2) *on interval $[0,T]$. Set $\Delta_t = X_t^{(1)} - X_t^{(2)}$ and $C_{n,B}$ be a constant such that $\langle B_n(x,y),x\rangle_H \leqslant C_{n,B}|x|_H^2|y|_H$, $x,y\in H_n$. Then*

$$|\Delta_t|_H \leqslant |\Delta_0|_H \exp\left\{2C_{n,B}\int_0^t \left|X_s^{(2)}\right|_H ds\right\}.$$

The proof is similar to the case of white noise as in [5].

Theorem 3.2. *For every $\mathcal{F}_0$-measurable $X_0^n : \Omega \to H_n$, there exists a unique càdlàg adapted solution $(X_t^n)_{t\geqslant 0}$ of Eq.* (3.2) *on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$. If the initial conditions x^m converge to x in H_n, the corresponding solutions converge P-a.s. uniformly in time on bounded intervals. And $(X_t^n)_{t\geqslant 0}$ has Feller property.*

Proof. *Step* 1 (*Existence for bounded initial value* X_0^n). Assume that $|X_0^n|_H \leqslant C$ for some positive constant C. For any $m > C$, let $B_n^m(\cdot) : H_n \to H_n$ be a Lipschitz continuous function such that $B_n^m(x) = B_n(x, x)$ for every $|x|_{H_n} \leqslant m$.

Consider the equation

$$dY_t^{(m)} = \left[-\nu A_n Y_t^{(m)} - B_n^m\left(Y_t^{(m)}, Y_t^{(m)}\right)\right] dt + \sum_{i=1}^{n} \sigma_i e_i\, dL_i(t).$$

It has globally Lipschitz coefficients, so there exists a unique *càdlàg* adapted solution $(Y_t^{(m)})_{t\geqslant 0}$, this is a classical result which can be obtained by contraction principle.

Defined $\tau_m = \inf\{t \geqslant 0\colon\ |Y_t^{(m)}|_H \geqslant m\} \wedge T$. Up to τ_m, the solution $Y_t^{(m)}$ is also a solution of the original equation (3.2). Since $|X_0^n|_H < C$, we have

$$E\Big(\sup_{t\in[0,T]} \big|Y_{t\wedge\tau_m}^{(m)}\big|_H^2\Big) \leqslant C\left(E\big|X_0^n\big|_H^2, T, \sum_{i=1}^{n} \sigma_i^2, \int\limits_{|x|\leqslant 1} x^2 \lambda(dx)\right).$$

In particular,

$$E\Big(1_{\{\tau_m<T\}} \sup_{t\in[0,T]} \big|Y_{t\wedge\tau_m}^{(m)}\big|_H^2\Big) \leqslant C\left(E\big|X_0^n\big|_H^2, T, \sum_{i=1}^{n} \sigma_i^2, \int\limits_{|x|\leqslant 1} x^2 \lambda(dx)\right),$$

which implies

$$P(\tau_m < T) \leqslant \frac{1}{m^2} C\left(E\big|X_0^n\big|_H^2, T, \sum_{i=1}^{n} \sigma_i^2, \int\limits_{|x|\leqslant 1} x^2 \lambda(dx)\right).$$

If $N > m,\ \tau_N \geqslant \tau_m,\ P(Y_t^{(N)} = Y_t^{(m)}, t \in [0, \tau_m]) = 1$. Let $\tau_\infty = \sup_{m>C} \tau_m$, we can define a process $Y_t^{(\infty)}$ for $t \in [0, \tau_\infty)$ uniquely, which equal to $Y_t^{(m)}$ on $[0, \tau_m]$ for every m. Hence $Y_t^{(\infty)}$ is a solution on $[0, \tau_\infty)$. Since for any m,

$$P(\tau_\infty < T) \leqslant P(\tau_m < T) \leqslant \frac{C}{m^2},$$

$P(\tau_\infty < T) = 0$. Thus $Y_t^{(\infty)}$ is a solution for $t \in [0, T - \epsilon]$ for any small $\epsilon > 0$, which shows that there exists a global solution.

Step 2 (*Existence for general initial value* X_0^n). For general case, let $\Omega_m = \{|X_0^n|_H^2 \leqslant m\}$. Define $X_0^{(m)}$ as X_0^n on Ω_m. Let $(Y_t^{(m)})_{t\leqslant 0}$ be the unique solution of equation (3.2) with initial condition $X_0^{(m)}$. If $N > m$, then

$$P\left(\Omega_m \cap \left(Y_t^{(N)} = Y_t^{(m)} \text{ for every } t \geqslant 0\right)\right) = P(\Omega_m).$$

We may then uniquely define a process $Y_t^{(\infty)}$ on $\Omega' = \bigcup_m \Omega_m$ as $Y_t^{(\infty)} = Y_t^{(m)}$ on Ω_m, it is clear that $Y_t^{(\infty)}$ solves Eq. (3.2) on Ω'. But $P(\Omega') = 1$, hence we have proved the existence of a global solution. Below we will let $(Y_t^x)_{t\geqslant 0}$ be the unique solution with initial condition $x \in H_n$.

Step 3. Uniqueness and continuous of the solution are follows from Lemma 3.1, which also implies his implies Markov property and the Feller property. □

3.3. Solution to the martingale problem

Let $\Omega = D_{D(A)'}[0,\infty)$, denote $\mathcal{T}$ the Skorokhod topology of Ω, and $\mathcal{T}_t$ the Skorokhod topology of $D_{D(A)'}[0,t]$, let $\mathcal{F} = \sigma(\mathcal{T})$, $\mathcal{F}_t = \sigma(\mathcal{T}_t)$.

Definition 3.1. Given a probability measure μ_0 on H, a probability measure P on $(\Omega, \mathcal{F})$ is called a solution of the martingale problem associated to Eq. (3.1) with initial law μ_0, if

(1) for every $T > 0$,

$$P\left(\sup_{t\in[0,T]} |\xi_t|_H^2 + \int_0^T \|\xi_t\|_V^2\, ds < \infty\right) = 1. \tag{3.8}$$

(2) for every $\varphi \in \mathscr{D}^\infty$ the process M_t^φ defined P-a.s. on $(\Omega, \mathcal{F})$ as

$$M_t^\varphi(\xi) := \langle \xi_t - \xi_0, \varphi\rangle_H + \int_0^t \nu\langle \xi_s, A\varphi\rangle_H\, ds - \int_0^t \big\langle B(\xi_s,\varphi), \xi_s\big\rangle_H\, ds. \tag{3.9}$$

$(M_t^\varphi, \mathcal{F}_t, P)$ is a Lévy process. Further more, $\{M^{e_i}\}_{i\in\mathbb{N}}$ are independent Lévy processes defined on the complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$, and the characteristic function is

$$E^P \exp\left\{iu\left(\frac{M_{t_2}^{e_i} - M_{t_1}^{e_i}}{\sigma_i}\right)\right\} = \exp\left\{(t_2 - t_1)\int_{|x|\leqslant 1} \big(e^{iux} - 1 - iux\big)\lambda(dx)\right\} \tag{3.10}$$

for $t_2 \geqslant t_1 \geqslant 0$.

(3) $\mu_0 = \pi_0 P$.

Theorem 3.3. *Assume $\sigma^2 = \sum_{i=1}^\infty \sigma_i^2 < \infty$. Let μ be a probability measure on H such that $m_2 = \int_H |x|_H^2 \mu(dx) < \infty$. Then there exists at least one solution to the martingale problem* (3.1) *with initial condition μ.*

Proof. Let $(W, (\mathcal{W}_t)_{t\geqslant 0}, Q, (L_i(t))_{t\geqslant 0, i\in N})$ be a stochastic basis supporting. $u_0 : W \to H$ is $\mathcal{W}_0$-measurable random variable with law μ. Let $X_0^n = \pi_n u_0$.

For every n, there exists a unique *càdlàg* adapted solution $(X_t^n)_{t\geqslant 0}$ of Eq. (3.2) in H_n, with initial condition X_0^n. $(X_t^n)_{t\geqslant 0}$ is a *càdlàg* adapted process in H_n. Since $H_n \subset D(A)'$. It defines a probability measure P_n on $D_{D(A)'}[0,\infty)$.

Step 1 (*Tightness*). Let $D = D(A)$. From Lemma 2.4, we have to prove that X^n is D-weakly tight and for every $\epsilon > 0$, $t > 0$,

$$\lim_{N\to\infty} \lim_{n\to\infty} \sup P\big(r_N^2\big(X^n(s)\big) > \epsilon \text{ for some } s \in [0,t]\big) = 0.$$

First (D-weakly tightness).

For every $\varphi \in D(A)$, we prove that $\{\langle X^n(\cdot), \varphi\rangle_{D(A)', D(A)}, t \in [0, T]\}$ is tight for every $T > 0$.

For τ_n, δ_n satisfies (a), (b), since

$$\begin{aligned}&\big\langle X^n(\tau_n+\delta_n) - X^n(\tau_n), \varphi\big\rangle_{D(A)', D(A)} \\ &\quad = \sum_{i=1}^n \big\langle \sigma_i h_i\big[L_i(\tau_n+\delta_n) - L_i(\tau_n)\big], \varphi\big\rangle_H - \int_{\tau_n}^{\tau_n+\delta_n} \big[\big\langle \nu A_n X^n(s) + B_n\big(X^n(s), X^n(s)\big), \varphi\big\rangle_H\big]\,ds,\end{aligned}$$

we have

$$\begin{aligned}&Q\left(\left|\big\langle X^n(\tau_n+\delta_n) - X^n(\tau_n), \varphi\big\rangle_{D(A)', D(A)}\right| \geqslant \epsilon\right) \\ &\quad \leqslant Q\left[\int_{\tau_n}^{\tau_n+\delta_n} \left|\big\langle \nu A_n X^n(s), \varphi\big\rangle_H\right| ds \geqslant \frac{\epsilon}{3}\right] + Q\left[\int_{\tau_n}^{\tau_n+\delta_n} \left|\big\langle B_n\big(X^n(s), X^n(s)\big), \varphi\big\rangle_H\right| ds \geqslant \frac{\epsilon}{3}\right] \\ &\qquad + Q\left[\left|\left\langle \sum_{i=1}^n \sigma_i e_i\big[L_i(\tau_n+\delta_n) - L_i(\tau_n)\big], \varphi\right\rangle_H\right| \geqslant \frac{\epsilon}{3}\right] \\ &\quad = I_1 + I_2 + I_3.\end{aligned}$$

Since $\lim_{n\to\infty} \delta_n = 0$, $\sum_{i=1}^\infty \sigma_i^2 < \infty$,

$$\begin{aligned}I_1 &\leqslant Q\left(\int_{\tau_n}^{\tau_n+\delta_n} \left|X^n(s)\right|_H^2 + |A\varphi|_H^2\,ds \geqslant \frac{2\epsilon}{3\nu}\right) \\ &\leqslant Q\left(\Big(\sup_{0\leqslant s\leqslant T} \left|X^n(s)\right|_H^2 + |A\varphi|_H^2\Big)\delta_n \geqslant \frac{2\epsilon}{3\nu}\right) \to 0, \quad \text{as } n \to \infty, \\ I_3 &\leqslant Q\left(\sum_{i=1}^\infty \sigma_i^2\big[L_i(\tau_n+\delta_n) - L_i(\tau_n)\big]^2 |\varphi|_H^2 \geqslant \left(\frac{\epsilon}{3}\right)^2\right) \to 0, \quad as\ n \to \infty.\end{aligned}$$

Since for every $\xi \in V$, $\int_0^t |\langle B(u_s, \xi), u_s\rangle|\,ds \leqslant C(\xi)\int_0^t |u_s|_H^{1/2} \|u_s\|_V^{3/2}\,ds$,

$$\begin{aligned}I_2 &\leqslant Q\left(C(\varphi)\sup_{s\in[0,T]}\left|X^n_{(s)}\right|_H^{1/2} \int_{\tau_n}^{\tau_n+\delta_n} \left\|X^n(s)\right\|_V^{3/2}\,ds \geqslant \frac{\epsilon}{3}\right) \\ &\leqslant Q\left(C(\varphi)\sup_{s\in[0,T]}\left|X^n_{(s)}\right|_H^{1/2} \left(\int_{\tau_n}^{\tau_n+\delta_n} \left\|X^n(s)\right\|_V^2\,ds\right)^{3/4} \delta_n^{1/4} \geqslant \frac{\epsilon}{3}\right) \\ &\leqslant Q\left(C(\varphi)\left(\sup_{s\in[0,T]}\left|X^n_{(s)}\right|_H^2 + \int_{\tau_n}^{\tau_n+\delta_n} \left\|X^n(s)\right\|_V^2\,ds\right)\delta_n^{1/4} \geqslant \frac{\epsilon}{3}\right) \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

so $\langle X^n(.), \varphi\rangle_{D(A)', D(A)}$ satisfies (A).

Since

$$Q\left(\langle X_0^n,\varphi\rangle^2_{D(A)',D(A)}\right)\leqslant|\varphi|_H^2 Q\left(|X_0^n|_H^2\right)\leqslant m_2|\varphi|_H^2$$

and

$$J\left(\langle X^n(\cdot),\varphi\rangle_{D(A)',D(A)}\right)\leqslant\max_{i\in N}\left\{|\sigma_i|\right\}\cdot|\varphi|_H,$$

$J(\langle X^n,\varphi\rangle_{D(A)',D(A)})$ and $\langle X_0^n,\varphi\rangle^2_{D(A)',D(A)}$ are tight on line. Thus $\langle X^n(\cdot),\varphi\rangle_{D(A)',D(A)}$ is tight in $D_R[0,T]$ by Lemma 2.5.

Second. Let $h_k=\lambda_i e_i$, which forms a complete orthonormal system of $D(A)'$. Since

$$E^Q\Big[\sup_{s\in[0,t]} r_N^2\big(X^n(s)\big)\Big]=E^Q\Big[\sup_{s\in[0,t]}\Big(\sum_{k\geqslant N+1}\langle X^n(s),h_k\rangle^2_{D(A)'}\Big)\Big]\leqslant\frac{E^Q[\sup_{s\in[0,t]}|X^n(s)|_H^2]}{\lambda_{N+1}^2}\leqslant\frac{C_t}{\lambda_{N+1}^2},$$

(2.4) is proved by Lemma 2.4.

Denote $P_n=Q\circ(X^n)^{-1}$, since $\{X^n\}$ is tight in $D_{D(A)'}[0,\infty)$, there exist n_k such that $P_{n_k}\xrightarrow{w}P$ in $D_{D(A)'}[0,\infty)$.

Step 2 (*P is a martingale solution*).

We need to check the properties (1)–(3) in Definition 3.1.

(1) *of Definition* 3.1. Given $R,N>0$,

$$f_{N,R}^T(\xi)=\sup_{t\in[0,T]}\sum_{n=1}^N\langle\xi(t),e_n\rangle_H^2\wedge R=\sup_{t\in[0,T]}\sum_{n=1}^N\lambda_n^2\langle\xi(t),h_n\rangle^2_{D(A)'}\wedge R.$$

By [5], we know that, if X is process with sample paths in $D_E[0,\infty)$, E is a metric space, then the complement in $[0,\infty)$ of $D(X)\equiv\{t\in[0,\infty)\colon P\{X(t)=X(t-)\}=1\}$ is at most countable. Let $Z=\{t\in[0,\infty)\colon P\{\xi(t)=\xi(t-)\}=1\}$. For every $T\in Z$, denote $\Omega_T^0=\{\xi\in D_{D(A)'}[0,\infty)\colon \xi(T)=\xi(T-)\}$, then $P(\Omega_T^0)=1$. By Lemmas 2.1, 2.3 and Theorem 3.1,

$$\lim_{k\to\infty}P_{n_k}f_{N,R}^T=Pf_{N,R}^T\leqslant C\left(m_2,T,\sum_{i=1}^n\sigma_i^2,\int_{|x|\leqslant1}x^2\lambda(dx)\right),$$

so we have (1).

(2) *of Definition* 3.1. We just prove that $\{M^{e_i}\}_{i\in N}$ are independent Lévy processes.

(i) Obviously, $M^{e_i}(0)=0$, P-a.e.

(ii) We prove (3.10).

Choose $t_1,t_2\in Z$, let $\Omega_{t_1,t_2}=\{\xi\in D_{D(A)'}[0,\infty)\colon \xi(t_i)=\xi(t_i-),\ i=1,2\}$, then $P(\Omega_{t_1,t_2})=1$. Let

$$F_m^{e_i}(\xi)=\exp\left\{iu\frac{M_{t_2}^{e_i}(\pi_m\xi)-M_{t_1}^{e_i}(\pi_m\xi)}{\sigma_i}\right\}.$$

Suppose $\lim_{n\to\infty}\xi_n=\xi$, $\xi_n\in D_{D(A)'}[0,\infty)$, $\xi\in\Omega_{t_1,t_2}$, it is easy to check that $\lim_{n\to\infty}F_m^{e_i}(\xi_n)=F_m^{e_i}(\xi)$. By Lemma 2.3,

$$\lim_{k\to\infty}E^{P_{n_k}}F_m^{e_i}(\xi)=E^PF_m^{e_i}(\xi).$$

Since

$$\begin{aligned}
&E^{P_{n_k}}\left[\left|\int_{t_1}^{t_2}\big\langle B(\pi_m\xi_s,\pi_m\xi_s),e_i\big\rangle ds-\int_{t_1}^{t_2}\big\langle B(\xi_s,\xi_s),e_i\big\rangle ds\right|\right]\\
&\quad\leqslant E^{P_{n_k}}\left[\int_{t_1}^{t_2}\left|\big\langle B(\pi_m\xi_s-\xi_s,\pi_m\xi_s),e_i\big\rangle ds\right|\right]+E^{P_{n_k}}\left[\int_{t_1}^{t_2}\left|\big\langle B(\xi_s,\pi_m\xi_s-\xi_s),e_i\big\rangle ds\right|\right]\\
&\quad\leqslant C\|e_i\|_{A^\beta}E^{P_{n_k}}\left[\int_{t_1}^{t_2}|\pi_m\xi_s-\xi_s|_H|\xi_s|_H\,ds\right]\\
&\quad\leqslant C\|e_i\|_{A^\beta}\left(E^{P_{n_k}}\left[\int_{t_1}^{t_2}|\pi_m\xi_s-\xi_s|_H^2\,ds\right]\right)^{1/2}\left(E^{P_{n_k}}\left[\int_{t_1}^{t_2}|\xi_s|_H^2\,ds\right]\right)^{1/2}\\
&\quad\leqslant\frac{C\|e_i\|_{A^\beta}}{\lambda_{m+1}^{1/2}}E^{P_{n_k}}\left[\int_{t_1}^{t_2}|\xi_s|_V^2\,ds\right]\\
&\quad\leqslant\frac{\widetilde{C}\|e_i\|_{A^\beta}}{\lambda_{m+1}^{1/2}},
\end{aligned}$$

we have for every $\varepsilon>0$, there exists M, for $m>M$,

$$\begin{aligned}
&\left|E^{P_{n_k}}\left[e^{iu(\frac{M_{t_2}^{e_i}(\pi m(\xi))-M_{t_1}^{e_i}(\pi m(\xi))}{\sigma_i})}\right]-E^{P_{n_k}}\left[e^{iu(\frac{M_{t_2}^{e_i}(\xi)-M_{t_1}^{e_i}(\xi)}{\sigma_i})}\right]\right|\\
&\quad\leqslant E^{P_{n_k}}\left|1-e^{iu\frac{\int_{t_1}^{t_2}\langle B(\pi_m\xi_s,\pi_m\xi_s),e_i\rangle ds-\int_{t_1}^{t_2}\langle B(\xi_s,\xi_s),e_i\rangle ds}{\sigma_i}}\right|\\
&\quad\leqslant\varepsilon,
\end{aligned}$$

then we get $\lim_{(k,m)\to\infty}E^{P_{n_k}}e^{iu(\frac{M_{t_2}^{e_i}(\pi m(\xi))-M_{t_1}^{e_i}(\pi m(\xi))}{\sigma_i})}=\exp\{(t_2-t_1)\int_{|x|\leqslant1}(e^{iux}-1-iux)\lambda(dx)\}$.

Note that

$$\lim_{m\to\infty}E^PF_m^{e_i}(\xi)=E^Pe^{iu(\frac{M_{t_2}^{e_i}(\xi)-M_{t_1}^{e_i}(\xi)}{\sigma_i})},$$

(2) is proved. By the same argument, it can be proved that $\{M^{e_i}\}_{1\in N}$ are independent.

(3) *of Definition* 3.1. Since $\lim_{k\to\infty}P_{n_k}(\varphi)=P(\varphi)$, $\varphi\in C_b(D_{D(A)'}[0,\infty))$. $\varphi(\xi)=e^{i\langle u,\xi(0)\rangle}\in C_b(D_{D(A)'}[0,\infty))$, for every $u\in D(A)$, $\lim_{k\to\infty}P_{n_k}\varphi(\xi)=P\varphi(\xi)$, hence $\Pi_0P=\mu$. $\square$

Remark 3.1. In fact, P_n is tight in $L^2(0,T;H)$ also. Let K be a separable Hilbert space, $W^{\alpha,p}(0,T;K)$ be the space (cf. [5]) of all measurable functions $f:[0,T]\to K$ such that

$$\|f\|^p_{W^{\alpha,p}(0,T;K)} := \int_0^T \big|f(t)\big|_K^p dt + \int_0^T\int_0^T \frac{|f(t)-f(r)|_K^p}{|t-r|^{1+\alpha p}}\,dt\,dr < \infty.$$

Note that for $p=2$, $0<\alpha<\frac{1}{2}$,

$$\begin{aligned}
E^Q\left\|\sum_{i=1}^n \sigma_i e_i L_i(\cdot)\right\|^2_{W^{\alpha,2}(0,T;H)} &= E^Q\int_0^T\left|\sum_{i=1}^n \sigma_i e_i L_i(t)\right|_H^2 dt + E^Q\int_0^T\int_0^T \frac{|\int_r^t \sum_{i=1}^n \sigma_i e_i\,dL_i(s)|_H^2}{|t-r|^{1+2\alpha}}\,dt\,dr \\
&= \int_0^T E^Q\left|\sum_{i=1}^n \sigma_i e_i L_i(t)\right|_H^2 dt + 2\int_0^T\int_r^T \frac{E^Q|\int_r^t \sum_{i=1}^n \sigma_i e_i\,dL_i(s)|_H^2}{|t-r|^{1+2\alpha}}\,dt\,dr \\
&\leqslant \sigma^2 T^2 \int_{|x|\leqslant 1} x^2\lambda(dx) + \sigma^2\int_{|x|\leqslant 1} x^2\lambda(dx)\int_0^T\int_0^T \frac{1}{|t-r|^{2\alpha}}\,dt\,dr \\
&< \infty.
\end{aligned}$$

Let $J_t^n = -\int_0^t[\nu A_n X_s^n + B_n(X_s^n, X_s^n)]\,ds$, we have (cf. [5])

$$\|J_\cdot^n\|^2_{W^{1,2}(0,T;D(A^{-\gamma}))} \leqslant C_\nu \int_0^T \|X_s^n\|_V^2\,ds + C\sup_{s\in[0,T]}\big|X_s^n\big|_H^2\int_0^T \|X_s^n\|_V^2\,ds,$$

for $\gamma\in(3/2,2)$, therefore

$$E^{P_n}\big[\|\xi\|_{W^{\alpha,2}([0,T];D(A^{-\gamma}))}\big] \leqslant C\left(\nu, m_2, T, \int_{|x|\leqslant 1} x^2\lambda(dx), \sigma^2\right)$$

for every $\alpha\in(0,1/2)$. From Theorem 4.6 in [5], the family of measures $\{P_n\}$ is tight in $L^2(0,T;H)$.

4. General case

For the multiplicative noise, in addition to the hypotheses of Section 1, we assume that there exists $0<p<2$ such that for every $u,u_1,u_2\in H$ and $x\in K$.

$$\int_{|x|\leqslant 1} \big|F(u_1,x)-F(u_2,x)\big|_H^2\lambda(dx) \leqslant C_1\big\{|u_1-u_2|_H^2 \wedge |u_1-u_2|_H^p\big\}, \tag{4.1}$$

$$\int_{|x|\leqslant 1} \big|F(u,x)\big|_H^2\lambda(dx) \leqslant C_2\big(1+|u|_H^2\big). \tag{4.2}$$

Let $\Omega_\alpha = D_{V'_\alpha}[0,\infty)$, $\mathcal{T}_\alpha$ the Skorokhod topology of Ω_α, $\mathcal{T}_t^\alpha$ the Skorokhod topology of $D_{V'_\alpha}[0,t]$, and $\mathcal{F}_t^\alpha$ the σ-algebra generated by $\mathcal{T}_t^\alpha$. And we assume $\nu=1$.

Definition 4.1. Given a probability measure μ_0 on H, a probability measure P on $(\Omega_\alpha, \mathcal{F}_\alpha)$ is called a solution of the martingale problem associated to Eq. (1.6) with initial law μ_0, if

(1) for every $T > 0$,

$$P\left(\sup_{t\in[0,T]} |\xi_t|_H^2 + \int_0^T \|\xi_s\|_V^2\, ds < \infty\right) = 1.$$

(2) for every $\varphi \in V_\alpha$ the process $\langle \xi_t, \varphi\rangle_{V'_\alpha, V_\alpha}$ is a semi-martingale, $(M_t^\varphi, \mathcal{F}_t^\alpha, P)$ is a square integrable càdlàg martingale. Here M_t^φ is defined P-a.s. on $(\Omega_\alpha, \mathcal{F}_\alpha)$ as

$$M_t^\varphi(\xi) = \langle \xi_t - \xi_0, \varphi\rangle_H + \int_0^t \langle \xi_s, A\varphi\rangle_H\, ds - \int_0^t \big\langle B(\xi_s, \varphi), \xi_s\big\rangle_H\, ds - \int_0^t \big\langle \varphi, f(s)\big\rangle_{V,V'}\, ds,$$

satisfies

$$M_t^\varphi(\xi) = \sum_{0\leqslant s\leqslant t} \triangle\langle \xi_s, \varphi\rangle_H - \int_0^t \int_{|x|\leqslant 1} \big\langle F(\xi_s, x), \varphi\big\rangle_H \lambda(dx)\, ds.$$

(3) $\mu_0 = \pi_0 P$.

In the following of the paper, we denote $(\Omega, \mathcal{F}, \mathcal{F}_t)$ as $(\Omega_\alpha, \mathcal{F}_\alpha, \mathcal{F}_t^\alpha)$.

Remark 4.1. $\sum_{0\leqslant s\leqslant t} \triangle\langle \xi_s, \varphi\rangle_H - \int_0^t \int_{|x|\leqslant 1} \langle F(\xi_s, x), \varphi\rangle_H \lambda(dx)\, ds$ may have no sensible in path, in Definition 4.1, but it has meaning in mean. This can be seen from following. Let $\epsilon_n \downarrow 0$, define a continuous function g_n on R,

$$g_n(x) = \begin{cases} x, & |x| \geqslant 2\epsilon_n, \\ 0, & |x| \leqslant \epsilon_n. \end{cases}$$

Let

$$G_n(\xi)(t) = \sum_{0\leqslant s\leqslant t} g_n\big(\triangle\langle \xi_s, \varphi\rangle_H\big) - \int_0^t \int_{|x|\leqslant 1} g_n\big(\big\langle F(\xi_s, x), \varphi\big\rangle_H\big) \lambda(dx)\, ds.$$

$(G_n, \mathcal{F}_t, P)$ is a square integrable càdlàg martingale, and there exists G, $\lim_{n\to\infty} G_n = G$ in $(\Omega, P) \times L^2([0,T]; R)$, so G is a square integrable càdlàg martingale, and we denote $G(\xi)(t) = \sum_{0\leqslant s\leqslant t} \triangle\langle \xi_s, \varphi\rangle_H - \int_0^t \int_{|x|\leqslant 1} \langle F(\xi_s, x), \varphi\rangle_H \lambda(dx)\, ds$.

Theorem 4.1. *Assume* (4.1)–(4.2). *For any probability measure μ on H with $m_2 = \int_H |x|_H^2 \mu(dx) < \infty$, there exists a martingale solution of Eq.* (1.6).

The proof of this theorem is based on a classical Galerkin approximation scheme.

Let $u_n(t)$ be the *càdlàg* adapted solution of the following equation (4.3) in H_n,

$$\begin{cases} du_n(t) + A_n u_n(t)\,dt + B_n\big(u_n(t), u_n(t)\big)\,dt = f_n(t)\,dt + \displaystyle\int_{|x|\leqslant 1} F_n\big(u_n(t-), x\big)\widetilde{N}_p(dt, dx), \\ u_0^n \in H_n, \end{cases} \tag{4.3}$$

where $f_n(t) = \pi_n f(t)$, $G_n(u) = \pi_n G(u)$, $F_n(u, x) = \pi_n F(u, x)$.

Lemma 4.1. *Assume $E|u_n(0)|_H^2 < \infty$, then for every $T > 0$,*

$$E \sup_{0\leqslant t\leqslant T} \big|u_n(t)\big|_H^2 + E\int_0^T \big\|u_n(s)\big\|_V^2\,ds \leqslant C\left(T, \int_0^T \big\|f_n(s)\big\|_{V'}^2\,ds, E\big|u_n(0)\big|_H^2\right) \tag{4.4}$$

for some positive $C(\cdot)$.

Proof. Apply Itô's formula on (4.3), and note that $\langle x, B_n(x,x)\rangle_H = 0$,

$$\begin{aligned} \big|u_n(t)\big|_H^2 = {} & \big|u_n(0)\big|_H^2 - 2\int_0^t \big\langle u_n(s), A_n u_n(s)\big\rangle_H\,ds - 2\int_0^t \big\langle u_n(s), B_n\big(u_n(s), u_n(s)\big)\big\rangle_H\,ds \\ & + 2\int_0^t \big\langle u_n(s), f_n(s)\big\rangle_{V,V'}\,ds \\ & + \int_0^{t+}\int_{|x|\leqslant 1} \big|u_n(s-) + F_n\big(u_n(s-), x\big)\big|_H^2 - \big|u_n(s-)\big|_H^2 \widetilde{N}(ds, dx) \\ & + \int_0^t \int_{|x|\leqslant 1} \big|u_n(s) + F_n\big(u_n(s), x\big)\big|_H^2 - \big|u_n(s)\big|_H^2 - 2\big\langle u_n(s), F_n\big(u_n(s), x\big)\big\rangle_H \lambda(dx)\,ds, \end{aligned}$$

$$\begin{aligned} \big|u_n(t)\big|_H^2 + 2\int_0^t \big\|u_n(s)\big\|_V^2\,ds = {} & \big|u_n(0)\big|_H^2 + 2\int_0^t \big\langle u_n(s), f_n(s)\big\rangle_{V,V'}\,ds \\ & + \int_0^{t+}\int_{|x|\leqslant 1} \big|u_n(s-) + F_n\big(u_n(s-), x\big)\big|_H^2 - \big|u_n(s-)\big|_H^2 \widetilde{N}(ds, dx) \\ & + \int_0^t \int_{|x|\leqslant 1} \big|F_n\big(u_n(s), x\big)\big|_H^2 \lambda(dx)\,ds. \end{aligned}$$

Since $2\int_0^t \langle u_n(s), f_n(s)\rangle_{V,V'}\,ds \leqslant \int_0^t \|u_n(s)\|_V^2\,ds + \int_0^t |f_n(s)|_{V'}^2\,ds$, we have

$$\big|u_n(t)\big|_H^2 + \int_0^t \big\|u_n(s)\big\|_V^2\,ds \leqslant \big|u_n(0)\big|_H^2 + \int_0^t \big|f_n(s)\big|_{V'}^2\,ds$$

$$+\int_0^{t+}\int_{|x|\leqslant 1}\left|F_n\left(u_n(s-),x\right)\right|_H^2+2\left\langle u_n(s-),F_n\left(u_n(s-),x\right)\right\rangle_H\widetilde{N}(ds,dx)$$

$$+\int_0^{t}\int_{|x|\leqslant 1}\left|F_n\left(u_n(s),x\right)\right|_H^2\lambda(dx)\,ds.$$

By Burkholder–Davis–Gundy inequality,

$$\begin{aligned}
&E\sup_{t\in[0,T]}\left|\int_0^{t+}\int_{|x|\leqslant 1}\left|F_n\left(u_n(s-),x\right)\right|_H^2+2\left\langle u_n(s-),F_n\left(u_n(s-),x\right)\right\rangle_H\widetilde{N}(ds,dx)\right|\\
&\quad\leqslant E\sup_{t\in[0,T]}\left|\int_0^{t+}\int_{|x|\leqslant 1}\left|F_n\left(u_n(s-),x\right)\right|_H^2\widetilde{N}(ds,dx)\right|\\
&\qquad+E\sup_{t\in[0,T]}\left|\int_0^{t+}\int_{|x|\leqslant 1}2\left\langle u_n(s-),F_n\left(u_n(s-),x\right)\right\rangle_H\widetilde{N}(ds,dx)\right|\\
&\quad\leqslant E\left[\int_0^{T+}\int_{|x|\leqslant 1}4\left\langle u_n(s-),F_n\left(u_n(s-),x\right)\right\rangle_H^2N_p(ds,dx)\right]^{1/2}\\
&\qquad+E\left[\int_0^{T+}\int_{|x|\leqslant 1}\left|F_n\left(u_n(s-),x\right)\right|_H^4N_p(ds,dx)\right]^{1/2}\\
&\quad\leqslant 2E\left\{\sup_{t\in[0,T]}\left|u_n(s)\right|_H\cdot\left[\int_0^{T+}\int_{|x|\leqslant 1}\left|F_n\left(u_n(s-),x\right)\right|_H^2N_p(ds,dx)\right]^{1/2}\right\}\\
&\qquad+E\left[\int_0^{T+}\int_{|x|\leqslant 1}\left|F_n\left(u_n(s-),x\right)\right|_H^2N_p(ds,dx)\right]\\
&\quad\leqslant\frac{1}{4}E\sup_{t\in[0,T]}\left|u_n(s)\right|_H^2+5E\left[\int_0^{T+}\int_{|x|\leqslant 1}\left|F_n\left(u_n(s-),x\right)\right|_H^2N_p(ds,dx)\right].
\end{aligned}$$

So

$$\frac{3}{4}E\sup_{t\in[0,T]}\left|u_n(t)\right|_H^2\leqslant\left|u_n(0)\right|_H^2+\int_0^T\left|f(s)\right|_{V'}^2\,ds+6E\int_0^T\int_{|x|\leqslant 1}\left|F_n\left(u_n(s),x\right)\right|_H^2\lambda(dx)\,ds$$

$$\leqslant |u_n(0)|_H^2 + \int_0^T |f(s)|_{V'}^2\, ds + 6CE\int_0^T \left(1 + |u_n(s)|_H^2\right) ds$$

$$\leqslant |u_n(0)|_H^2 + \int_0^T |f(s)|_{V'}^2\, ds + 6CT + 6C\int_0^T E\Big(\sup_{s'\in[0,s]} |u_n(s)|_H^2\Big)\, ds.$$

By Gronwall lemma,

$$E\sup_{s\in[0,T]} |u_n(s)|_H^2 \leqslant C\left(T, |u_n(0)|_H^2, \int_0^T |f_n(s)|_{V'}^2\, ds\right).$$

Since

$$E\int_0^T \|u_n(s)\|_V^2\, ds \leqslant |u_n(0)|_H^2 + \int_0^T \|f(s)\|_{V'}^2\, ds + E\int_0^T \int_{|x|\leqslant 1} |F_n(u_n(s), x)|_H^2 \lambda(dx)\, ds$$

$$\leqslant |u_n(0)|_H^2 + \int_0^T \|f(s)\|_{V'}^2\, ds + E\int_0^T C\left(1 + |u_n(s)|_H^2\right) ds$$

$$\leqslant |u_n(0)|_H^2 + \int_0^T \|f(s)\|_{V'}^2\, ds + CT + CTE\sup_{s\in[0,T]} |u_n(s)|_H^2.$$

(4.4) is proved. □

Corollary 4.1. *Let $\tau \geqslant 0$ be a stopping time and $(u_n(t))_{t\geqslant 0}$ a càdlàg adapted process that P-a.s. satisfies Eq. (4.3) for $t \in [0, \tau(\omega)]$. Assume $E|u_n(0)|_{H_n}^2 < \infty$. Then, for every $T > 0$,*

$$E\Big(\sup_{t\in[0,T]} |u_n(t\wedge\tau)|_{H_n}^2\Big) \leqslant C\left(T, \int_0^T \|f_n(s)\|_{V'}^2\, ds, E|u_n(0)|_{H_n}^2\right).$$

Lemma 4.2. *For every $\mathcal{F}_0$-measurable $u_n(0) : \Omega \to H_n$, there exists a unique càdlàg adapted solution $(u_n(t))_{t\geqslant 0}$ of Eq. (4.3) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$.*

Proof. *Step* 1 (*Existence for bounded initial condition $u_n(0)$*). Assume that $|u_n(0)|_{H_n} \leqslant C$ for some constant $C > 0$. For any $m > C$, let $B_n^m(\cdot) : H_n \to H_n$ be a Lipschitz continuous function such that $B_n^m(x) = B_n(x, x)$ for every $|x|_{H_n} \leqslant m$. Consider the equation

$$du_n(t) + A_n u_n(t)\, dt + B_n^m(u_n(t), u_n(t))\, dt = f_n(t)\, dt + \int_{|x|\leqslant 1} F_n(u_n(t-), x)\widetilde{N}_p(dt, dx)$$

with initial condition $u_n(0)$. It has globally Lipschitz coefficients, so there exists a unique *càdlàg* adapted solution $(u_n^{(m)}(t))_{t\geqslant 0}$ by the standard classical argument. Let τ_m be defined as

$$\tau_m = \inf\{t \geqslant 0\colon \left|u_n^{(m)}\right|_{H_n} \geqslant m\} \wedge T,$$

the solution $(u_n^{(m)}(t))_{t\in[0,\tau_m]}$ is also a solution of Eq. (4.3). (Just note that for $u_n^{(m)}(\tau_m)$ is decided by $t \in [0, \tau_m)$.) Therefore, by Corollary 4.1, we have

$$E\Big(\sup_{t\in[0,T]}\left|u_n^{(m)}(t\wedge\tau_m)\right|_{H_n}^2\Big) \leqslant C\left(T, E\left|u_n(0)\right|_{H_n}^2, \int_0^T \left\|f(s)\right\|_{V'}^2\,ds\right).$$

In particular

$$E\big(1_{\tau_m<T}\left|u_n^{(m)}(\tau_m)\right|_{H_n}^2\big) \leqslant C\left(T, E\left|u_n(0)\right|_{H_n}^2, \int_0^T \left\|f(s)\right\|_{V'}^2\,ds\right),$$

which implies

$$P(\tau_m < T) \leqslant \frac{1}{m^2} C\left(T, E\left|u_n(0)\right|_{H_n}^2, \int_0^T \left\|f(s)\right\|_{V'}^2\,ds\right).$$

Since $|u_n^{(m)}(T\wedge\tau_m)|_{H_n}^2 \geqslant m^2$ on $\{\tau_m < T\}$, $\tau_M \geqslant \tau_m$ as $M > m$ and

$$P\big(u_n^{(M)}(t) = u_n^{(m)}(t),\ t \in [0, \tau_m)\big) = 1,$$

therefore, if $\tau_\infty := sup_{m>C}\{\tau_m\}$, we may uniquely define a process $u_n^{(\infty)}(t)$ for $t \in [0, \tau_\infty)$, which equal to $u_n^{(m)}(t)$ on $[0, \tau_m)$ for every m. Hence $u_n^{(\infty)}(t)$ is a solution on $[0, \tau_\infty)$. Since

$$P(\tau_\infty < T) \leqslant P(\tau_m < T) \leqslant \frac{C}{m^2},$$

for every m, $P(\tau_\infty < T) = 0$. Thus $u_n^{(\infty)}$ is a solution for $t \in [0, T-\epsilon]$ for every small $\epsilon > 0$. Since T is arbitrary, we have proved global existence.

Step 2 (*Existence for general initial condition* $u_n(0)$). Let $\Omega_m \in \mathcal{F}$ be defined as $\Omega_m = \{|u_n(0)|_{H_n}^2 \leqslant m\}$. Define $u_n^{(m)}(0)$ as $u_n(0)$ on Ω_m, 0 otherwise. Let $(u_n^{(m)}(t))_{t\geqslant 0}$ be the unique solution of Eq. (4.3) with initial condition $u_n^{(m)}(0)$. If $M > m$, then

$$P\big(\Omega_m \cap \big(u_n^{(M)}(t) = u_n^{(m)}(t) \text{ for every } t \geqslant 0\big)\big) = P(\Omega_m).$$

We may then uniquely define a process u_n on $\Omega' = \bigcup_m \Omega_m$ as $u_n(t) = u_n^{(m)}(t)$ on Ω_m, it is clear that u_n solves Eq. (2.4) on Ω'. Since $P(\Omega') = 1$, we have get a global solution. $\square$

5. Proof of Theorem 4.1

Proof. Let $(W, \mathcal{W}, (\mathcal{W}_t)_{t\geqslant 0}, Q, p)$ be a stochastic basis supporting, a $\mathcal{W}_0$-measurable random variable $u_0 : W \to H$ with law μ, p is a stationary Poisson point process defined as in Section 1. Let $X_0^n = \pi_n u_0$.

For every n, there exists a unique *càdlàg* adapted solution $(X^n(t))_{t\geqslant 0}$ of Eq. (4.3) in H_n, with initial condition X_0^n. Since $H_n \subset H$, $(X^n(t))_{t\geqslant 0}$ can be viewed as *càdlàg* adapted process in V'_α, so it defines a probability measure P_n on $D_{V'_\alpha}[0,\infty)$.

Step 1 (*Tightness in* $\Omega \cap L^2([0,T]; H)$). From Lemma 2.4, choose $\alpha \geqslant 1$, let $D = V_\alpha$, we prove X^n is D-weakly tight and for every $\epsilon > 0$ and $t > 0$,

$$\lim_{N\to\infty}\lim_{n\to\infty}\sup P\big(r_N^2\big(X^n(s)\big) > \epsilon \text{ for some } s \in [0,t]\big) = 0.$$

First, D-weakly tightness: by using Lemma 2.5, we only need to prove that for every $\varphi \in D$, $\{\langle X^n(.), \varphi\rangle_{D',D}\}$ is tight in $D_R[0,T]$.

For every $T > 0$, by Lemma 4.1

$$E^Q\Big[\sup_{t\in[0,T]}\big(\langle X^n(t),\varphi\rangle_{D',D}^2\big)\Big] \leqslant |\varphi|_H^2 E\Big[\sup_{t\in[0,T]}\big|X^n(t)\big|_H^2\Big] \leqslant |\varphi|_H^2 C\left(T, \int_0^T \big\|f(s)\big\|_{V'}^2\,ds, m_2\right), \quad (5.1)$$

so $\{\langle X^n(.), \varphi\rangle_{D',D}\}$ is tight on the line for each $t \in [0,T]$.

For any τ_n, δ_n satisfy (a), (b), we have

$$\begin{aligned}
&\big|\big\langle X^n(\tau_n+\delta_n) - X^n(\tau_n), \varphi\big\rangle_{D',D}\big| \\
&\leqslant \int_{\tau_n}^{\tau_n+\delta_n}\big|\big\langle\varphi, B_n\big(X^n(s), X^n(s)\big)\big\rangle_H\big|ds + \int_{\tau_n}^{\tau_n+\delta_n}\big|\big\langle\varphi, AX^n(s)\big\rangle_H\big|ds + \int_{\tau_n}^{\tau_n+\delta_n}\big|\big\langle\varphi, f_n(s)\big\rangle_{V,V'}\big|ds \\
&\quad + \left|\int_0^T\int_{|x|\leqslant 1} 1_{(\tau_n,\tau_n+\delta_n]}\big\langle F_n\big(X^n(s-),x\big),\varphi\big\rangle_H \widetilde{N}_p(ds,dx)\right| \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Since $\lim_{n\to\infty}\delta_n = 0$ and $|\langle B(u,v),w\rangle| \leqslant C|u|_H^{1/4}\|u\|_V^{3/4}\|v\|_V|w|_H^{1/4}\|w\|_V^{3/4}$,

$$E^Q[I_2] \leqslant \|\varphi\|_V\left[E\int_0^T\big\|X^n(s)\big\|_V^2\,ds\right]^{1/2}|\delta_n|^{1/2} \to 0, \quad \textit{as } n\to\infty,$$

$$\begin{aligned}
E^Q[I_1] &\leqslant C_\varphi E^Q\left[\int_{\tau_n}^{\tau_n+\delta_n}\big|X^n(s)\big|_H^{1/2}\big\|X^n(s)\big\|_V^{3/2}\,ds\right] \\
&\leqslant C_\varphi\Big[E^Q\sup_{s\in[0,T]}\big|X^n(s)\big|_H^2\Big]^{1/4}\cdot\left[E^Q\int_0^T\big\|X^n(s)\big\|_V^2\,ds\right]^{3/4}\sigma_n^{1/4} \\
&\to 0, \quad \text{as } n\to\infty.
\end{aligned}$$

It's easy to see that $E^Q[I_3] \to 0$, as $n \to \infty$. Since

$$E^Q\left[\left|\int_0^T \int_{|x|\leqslant 1} 1_{(\tau_n,\tau_n+\delta_n]}\big\langle F_n\big(X^n(s-),x\big),\varphi\big\rangle_H \widetilde{N}_p(ds,dx)\right|^2\right]$$
$$= E^Q\left[\int_0^T \int_{|x|\leqslant 1} 1_{(\tau_n,\tau_n+\delta_n]}\big\langle F_n\big(X^n(s-),x\big),\varphi\big\rangle_H^2 \lambda(dx)\,ds\right]$$
$$\leqslant E^Q\left[\int_0^T 1_{(\tau_n,\tau_n+\delta_n]}|\varphi|_H^2 C\big(\big|X^n(s)\big|_H^2+1\big)\,ds\right]$$
$$\leqslant C|\varphi|_H^2 E^Q\left[\sup_{s\in[0,T]}\big|X^n(s)\big|_H^2+1\right]\cdot\delta_n,$$

we get $I_4 \to 0$, in probability. Thus $\{\langle X^n(t),\varphi\rangle_{D',D}, t\in[0,T]\}$ satisfy (A), and by Lemma 2.5, $\{X^n(\cdot)\}$ is D-weakly tight. Finally, since for $h_k = \lambda_k^{\alpha/2} e_k$,

$$E^Q\Big[\Big(\sup_{s\in[0,t]} r_N^2\big(X^n(s)\big)\Big)\Big] = E^Q\left[\sup_{s\in[0,t]}\left(\sum_{k\geqslant N+1}\big\langle X^n(s),h_k\big\rangle_{D'}^2\right)\right]$$
$$\leqslant \frac{E^Q[\sup_{s\in[0,t]}|X^n(s)|_H^2]}{\lambda_{N+1}^{\alpha}}$$
$$\leqslant \frac{C_t}{\lambda_{N+1}^{\alpha}},$$

$\{X^n\}$ is tight in $D_{D'}[0,\infty)$.

Next, I will prove that $\{X^n\}$ is tight in $L^2([0,T];H)$.

Since

$$X_n(t) = X_n(0) - \int_0^t A_n X_n(s)\,ds - \int_0^t B_n\big(X_n(s),X_n(s)\big)\,ds$$
$$+ \int_0^t f_n(s)\,ds + \int_0^{t+} F_n\big(X_n(s-),x\big)\widetilde{N}_p(ds,dx)$$
$$= I_5(t) + I_6(t) + I_7(t) + I_8(t) + I_9(t).$$

For $0<\theta<1/2$,

$$E^Q\big[\big\|I_9(\cdot)\big\|_{W^{\theta,2}([0,T];H)}^2\big] = E^Q\left[\int_0^T \big|I_9(t)\big|_H^2\,dt\right] + E^Q\left[\int_o^T\int_0^T \frac{|I_9(t)-I_9(s)|_H^2}{|t-s|^{1+2\theta}}\,dt\,ds\right]$$
$$= \int_0^T E^Q\big[\big|I_9(t)\big|_H^2\big]\,dt + \int_o^T\int_0^T \frac{E^Q[|I_9(t)-I_9(s)|_H^2]}{|t-s|^{1+2\theta}}\,dt\,ds$$

$$
\begin{aligned}
&= \int_0^T E^Q\left[\int_0^t \int_{|x|\leqslant 1} \left|F_n\big(X_n(s),x\big)\right|_H^2 \lambda(dx)\,ds\right] dt \\
&\quad + \int_0^T \int_0^T \frac{E^Q[\int_s^t \int_{|x|\leqslant 1} |F_n(X_n(z),x)|_H^2 \lambda(dx)\,dz]}{|t-s|^{1+2\theta}}\,dt\,ds \\
&\leqslant C(T,\theta)\left\{\int_0^T E^Q\left[\left|X_n(s)\right|_H^2\right] ds + \int_0^T \int_0^T \frac{\int_s^t E^Q[|X_n(z)|_H^2]\,dz}{|t-s|^{1+2\theta}}\,dt\,ds\right\} \\
&\leqslant C\left(T,\theta,\int \left\| f(s)\right\|_{V'}^2\,ds, m_2\right).
\end{aligned}
$$

Since $E^Q[|I_5(t)|_H^2] \leqslant m_2$ and $E^Q[\|I_8(.)\|^2_{W^{\theta,2}([0,T];V')}] \leqslant C_4$, by Remark 3.1 and Lemma 4.1, for $\gamma \in (3/2,2)$,

$$
E^Q\left[\left\|I_6(\cdot)+I_7(\cdot)\right\|_{W^{1,2}([0,T];D(A^{-\gamma}))}\right] \leqslant C\left(m^2, T, \int_0^T \left\| f(s)\right\|_{V'}^2\,ds\right).
$$

Therefore

$$
E^{P_n}\left[\left\|\xi(\cdot)\right\|_{W^{\theta,2}([0,T];D(A^{-\gamma}))}\right] \leqslant C\left(m^2, T, \int_0^T \left\| f(s)\right\|_{V'}^2\,ds\right).
$$

By [5], Theorem 4.6, P_n is tight in $L^2([0,T];H)$. Hence there exists a probability measure P on $\Omega \cap L^2_{loc}([0,\infty);H)$, which is the weak limit of a sub-sequence $\{P_{n_k}\}$.

Step 2 (*Prove P is a martingale solution*). (1) and (3) can be proved by the same argument as in the proof of Theorem 3.3.

For checking (2), let $g(\cdot)$ be a continuous function from R to R,

$$
g(x) = \begin{cases} x, & |x| \geqslant 1, \\ 0, & |x| \leqslant 1/2. \end{cases}
$$

with $|g(x)| \leqslant |x|$ and $|g'(x)| \leqslant C$.

For any $\varphi \in D$, choose $t \in Z$ (using in Theorem 3.3), let

$$
Y_n^m(\xi)(t) = \sum_{s\in[0,t]} \frac{1}{m} g\big(m\cdot \langle \triangle\xi(s),\varphi\rangle_{D',D}\big) - \int_0^t \int_{|x|\leqslant 1} \frac{1}{m} g\big(m\cdot \langle F_n\big(\xi(s),x\big),\varphi\rangle\big)\lambda(dx)\,ds,
$$

and

$$
Y^m(\xi)(t) = \sum_{s\in[0,t]} \frac{1}{m} g\big(m\cdot \langle \triangle\xi(s),\varphi\rangle_{D',D}\big) - \int_0^t \int_{|x|\leqslant 1} \frac{1}{m} g\big(m\cdot \langle F\big(\xi(s),x\big),\varphi\rangle\big)\lambda(dx)\,ds.
$$

First, $(X^m, \mathcal{F}_s, P)$ is a square integrable martingale.

Set $\Omega_t^0 = \{\xi \in \Omega\colon\ \xi(t) = \xi(t-)\}$. If $\xi \in \Omega_t^0$ and $\xi_n \to \xi$ in $\Omega \cap L^2_{loc}([0,\infty); H)$, by Lemma 2.2,

$$\lim_{n\to\infty} \sum_{s\in[0,t]} \frac{1}{m} g\big(m \cdot \langle \triangle \xi_n(s), \varphi\rangle_{D',D}\big) = \frac{1}{m} g\big(m \cdot \langle \triangle \xi(s), \varphi\rangle_{D',D}\big).$$

Since

$$\begin{aligned}\int_0^t \int_{|x|\leqslant 1} \left|\frac{1}{m} g\big(m \cdot \big\langle F_n\big(\xi_n(s), x\big), \varphi\big\rangle\big)\right| \lambda(dx)\, ds &\leqslant 2m \int_0^t \int_{|x|\leqslant 1} \big|\big\langle F_n\big(\xi_n(s), x\big), \varphi\big\rangle\big|_H^2 \lambda(dx)\, ds \\ &\leqslant 2mC|\varphi|_H^2 \left[\int_0^t \big|\xi_n(s)\big|_H^2\, ds + t\right],\end{aligned}$$

$Y_n^m(t)$ is sensible P_n-a.s.

Set

$$\begin{aligned} A_n^s &= \left\{x\colon\ |x| \leqslant 1 \text{ and } \big|\big\langle F_n\big(\xi_n(s), x\big), \varphi\big\rangle\big| \geqslant \frac{1}{2m}\right\}, \\ A^s &= \left\{x\colon\ |x| \leqslant 1 \text{ and } \big|\big\langle F\big(\xi(s), x\big), \varphi\big\rangle\big| \geqslant \frac{1}{2m}\right\},\end{aligned}$$

we have

$$\begin{aligned}\lambda\big(A_n^s\big) = \int_{|x|\leqslant 1} I_{A_n^s}(x)\lambda(dx) &\leqslant 4m^2 \int_{|x|\leqslant 1} \big|\big\langle F_n\big(\xi_n(s), x\big), \varphi\big\rangle\big|^2 \lambda(dx) \\ &\leqslant Cm^2 |\varphi|_H^2 \big(\big|\xi_n(s)\big|_H^2 + 1\big),\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \int_{|x|\leqslant 1} \frac{1}{m}\big|g\big(m \cdot \big\langle F_n\big(\xi_n(s), x\big), \varphi\big\rangle\big) - g\big(m \cdot \big\langle F\big(\xi(s), x\big), \varphi\big\rangle\big)\big|\lambda(dx)\, ds \\ &\quad = \int_0^t \int_{A^s\cup A_n^s} \frac{1}{m}\big|g\big(m \cdot \big\langle F_n\big(\xi_n(s), x\big), \varphi\big\rangle\big) - g\big(m \cdot \big\langle F\big(\xi(s), x\big), \varphi\big\rangle\big)\big|\lambda(dx)\, ds \\ &\quad \leqslant C \int_0^t \int_{A^s\cup A_n^s} \big|\big\langle F_n\big(\xi_n(s), x\big), \varphi\big\rangle - \big\langle F\big(\xi(s), x\big), \varphi\big\rangle\big|\lambda(dx)\, ds \\ &\quad \leqslant C\left[\int_0^t \int_{|x|\leqslant 1} \big|\big\langle F_n\big(\xi_n(s), x\big), \varphi\big\rangle - \big\langle F\big(\xi(s), x\big), \varphi\big\rangle\big|^2 \lambda(dx)\, ds\right]^{1/2} \left[\int_0^t \lambda\Big(A^s \cup A_n^s\Big)\, ds\right]^{1/2}\end{aligned}$$

$$\leqslant Cm|\varphi|_H \cdot \left[\int_0^t \int_{|x|\leqslant 1} \left|\left\langle F\left(\xi_n(s), x\right) - F\left(\xi(s), x\right), \pi_n \varphi \right\rangle\right|^2 + \left|\left\langle F\left(\xi(s), x\right), \pi_n^c \varphi \right\rangle\right|^2 \lambda(dx)\, ds \right]^{1/2}$$

$$\cdot \left[\int_0^t \left|\xi_n(s)\right|_H^2 ds + \int_0^t \left|\xi(s)\right|_H^2 ds + t \right]^{1/2}$$

$$\leqslant Cm|\varphi|_H \cdot \left[\int_0^t \int_{|x|\leqslant 1} \left|F\left(\xi_n(s), x\right) - F\left(\xi(s), x\right)\right|_H^2 \lambda(dx)\, ds \cdot |\varphi|_H^2 \right.$$

$$\left. + \left|\pi_n^c \varphi\right|_H^2 \cdot \int_0^t \int_{|x|\leqslant 1} \left|F\left(\xi(s), x\right)\right|_H^2 \lambda(dx)\, ds \right]^{1/2} \cdot \left[\int_0^t \left|\xi_n(s)\right|_H^2 ds + \int_0^t \left|\xi(s)\right|_H^2 ds + t \right]^{1/2}$$

$$\leqslant Cm|\varphi|_H \left[\int_0^t \left|\xi_n(s) - \xi(s)\right|_H^2 ds \cdot |\varphi|_H^2 + \left|\pi_n^c \varphi\right|_H^2 \cdot \int_0^t \left[\left|\xi(s)\right|_H^2 + 1\right] ds \right]^{1/2}$$

$$\cdot \left[\int_0^t \left|\xi_n(s)\right|_H^2 ds + \int_0^t \left|\xi(s)\right|_H^2 ds + t \right]^{1/2}.$$

Thus

$$\lim_{n\to\infty} \int_0^t \int_{|x|\leqslant 1} \frac{1}{m} g\left(m \cdot \left\langle F_n\left(\xi_n(s), x\right), \varphi \right\rangle\right) \lambda(dx)\, ds = \int_0^t \int_{|x|\leqslant 1} \frac{1}{m} g\left(m \cdot \left\langle F\left(\xi(s), x\right), \varphi \right\rangle\right) \lambda(dx)\, ds.$$

Since $(Y_n^m(\xi)(t), P_n)$ is a square integrable *càdlàg* martingale, and for $t_1 < t_2 \in Z$,

$$\begin{aligned}
E^{P_n}\left|Y_n^m(\xi)(t_2) - Y_n^m(\xi)(t_1)\right|^2 &= E^Q \left| \int_{t_1}^{t_2} \int_{|x|\leqslant 1} \frac{1}{m} g\left(m \cdot \left\langle F_n\left(X_n(s-), x\right), \varphi \right\rangle\right) \widetilde{N}_p(dx, ds) \right|^2 \\
&\leqslant E^Q \int_{t_1}^{t_2} \int_{|x|\leqslant 1} \left[\frac{1}{m} g\left(m \cdot \left\langle F_n\left(X_n(s), x\right), \varphi \right\rangle\right) \right]^2 \lambda(dx)\, ds \\
&\leqslant |\varphi|_H^2 E^Q \int_{t_1}^{t_2} \int_{|x|\leqslant 1} \left|F_n\left(X_n(s), x\right)\right|_H^2 \lambda(dx)\, ds \\
&\leqslant C|\varphi|_H^2 E^Q \int_{t_1}^{t_2} \left(\left|X_n(s)\right|_H^2 + 1\right) ds \\
&\leqslant C\left(t_2 - t_1, \int_{t_1}^{t_2} \left\|f(s)\right\|_{V'}^2 ds, E\left|X(0)\right|_H^2 \right) |\varphi|_H^2.
\end{aligned}$$

Let $t_1=0$ and using Lemma 2.5,

$$\begin{aligned}
E^P\big|Y^m(\xi)(t_2)\big|^{1+\varepsilon} &= \lim_{n\to\infty} E^{P_n}\big|Y_n^m(\xi)(t_2)\big|^{1+\varepsilon} \\
&\leqslant \sup_n\big[E^{P_n}\big|Y_n^m(\xi)(t_2)\big|^2\big]^{(1+\varepsilon)/2} \\
&\leqslant \left\{C\left(t_2, \int_0^{t_2}\|f(s)\|_{V'}^2\,ds, E\big|u(0)\big|_H^2\right)|\varphi|_H^2\right\}^{(1+\varepsilon)/2} \\
&< \infty,
\end{aligned}$$

for every $0<\varepsilon<1$. So $E^P|Y^m(\xi)(t_2)|^2<\infty$. Since for every $\mathcal{F}_{t_1}$-measurable bounded random variable Z,

$$E^P\big\{\big[Y^m(t_2)-Y^m(t_1)\big]\cdot Z\big\} = \lim_{n\to\infty} E^{P_n}\big\{\big[Y_n^m(t_2)-Y_n^m(t_1)\big]\cdot Z\big\}=0,$$

$(Y^m,\mathcal{F}_s,P)$ is a square integrable càdlàg martingale.

Second, we prove that (Y^m,P) is a Cauchy sequence in $L^2(\Omega\times[0,T];R)$, and denote $\lim_{m\to\infty}Y^m=Y^\varphi$.

By Skorokhod embedding theorem, there exists a stochastic basis $(\Omega',\mathcal{F}',\{\mathcal{F}'_t\}_{t\geqslant 0},P')$ and, on this basis, $\Omega\cap L^2_{loc}([0,\infty);H)$-valued random variables X', X'_k, $k\geqslant 1$, such that X'_k has the same law of P_{n_k} on $\Omega\cap L^2_{loc}([0,\infty);H)$, and $X'_k\to X'$ in $\Omega\cap L^2_{loc}([0,\infty);H)$ P'-a.s.

Let

$$G_{m_1,m_2,n}(\xi)(t)=Y_n^{m_2}(\xi)(t)-Y_n^{m_1}(\xi)(t),\ G_{m_1,m_2}(\xi)(t)=Y^{m_2}(\xi)(t)-Y^{m_1}(\xi)(t).$$

Then

$$\begin{aligned}
E^P\int_0^T\big|G_{m_1,m_2}(\xi)(t)\big|^2\,dt &= \int_0^T E^{P'}\big[\big|G_{m_1,m_2}\big(Y'\big)(t)\big|^2\big]\,dt \\
&\leqslant \int_0^T \liminf_{n\to\infty} E^{P'}\big[\big|G_{m_1,m_2,n}\big(Y'_n\big)(t)\big|^2\big]\,dt
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
&E^{P'}\big[G_{m_1,m_2,n}\big(Y'_n\big)(t)\big]^2 \\
&\quad= E^Q\left[\int_0^{t+}\int_{|x|\leqslant 1}\frac{1}{m_1}g\big(m_1\cdot\big\langle F_n\big(X_n(s-),x\big),\varphi\big\rangle\big)-\frac{1}{m_2}g\big(m_2\cdot\big\langle F_n\big(X_n(s-),x\big),\varphi\big\rangle\big)\widetilde{N}_p(dx,ds)\right]^2 \\
&\quad= E^Q\left[\int_0^{t}\int_{|x|\leqslant 1}\left|\frac{1}{m_1}g\big(m_1\cdot\big\langle F_n\big(X_n(s),x\big),\varphi\big\rangle\big)-\frac{1}{m_2}g\big(m_2\cdot\big\langle F_n\big(X_n(s),x\big),\varphi\big\rangle\big)\right|^2\lambda(dx)\,ds\right]
\end{aligned}$$

$$= E^{P'}\left[\int_0^t \int_{|x|\leqslant 1} \left|\frac{1}{m_1} g\big(m_1\cdot\big\langle F_n\big(X_n'(s),x\big),\varphi\big\rangle\big) - \frac{1}{m_2} g\big(m_2\cdot\big\langle F_n\big(X_n'(s),x\big),\varphi\big\rangle\big)\right|^2 \lambda(dx)\,ds\right].$$

Note that

$$\begin{aligned}
&E^{P'}\left[\int_0^t \int_{|x|\leqslant 1} \left|\frac{1}{m} g\big(m\cdot\big\langle F\big(X'(s),x\big),\varphi\big\rangle\big) - \frac{1}{m} g\big(m\cdot\big\langle F_n\big(X_n'(s),x\big),\varphi\big\rangle\big)\right|^2 \lambda(dx)\,ds\right]\\
&\quad\leqslant C_\varphi E^{P'}\left[\int_0^t \big|X'(s)-X_n'(s)\big|^p\right] ds + \big|\pi_n^c\varphi\big|^2 E^{P'}\left[\int_0^t \int_{|x|\leqslant 1} \big|F\big(X'(s),x\big)\big|^2 \lambda(dx)\,ds\right],
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t \big|X'(s)-X_n'(s)\big|_H^2\,ds = 0, \quad P'\text{-a.s.},\\
&E^{P'}\int_0^t \big|X'(s)-X_n'(s)\big|_H^2\,ds \leqslant C\left(m^2, t, \int \|f(s)\|_{V'}^2\,ds\right),
\end{aligned}$$

by Lemma 2.5,

$$\begin{aligned}
(5.2) &= E^{P'}\left[\int_0^t \int_{|x|\leqslant 1} \left|\frac{1}{m_1} g\big(m_1\cdot\big\langle F\big(X'(s),x\big),\varphi\big\rangle\big) - \frac{1}{m_2} g\big(m_2\cdot\big\langle F\big(X'(s),x\big),\varphi\big\rangle\big)\right|^2 \lambda(dx)\,ds\right]\\
&\leqslant C_\varphi \int_0^T E^{P'}\left[\int_0^t \int_{|x|\leqslant 1} \big|F\big(X'(s),x\big)\big|_H^2 \wedge \left(\frac{1}{m_2^2}\vee\frac{1}{m_1^2}\right)\lambda(dx)\,ds\right] dt\\
&\to 0 \quad \text{as } (m_1,m_2)\to\infty.
\end{aligned}$$

Third, we prove that $E^P e^{iu[M_t^{e_i} - Y^{e_i}(t)]} = 1$.

By using the same method as above, we have for fixed n, (Y_n^m, P_n) is Cauchy sequence in $L^2(\Omega\times[0,T];R)$ and denote ${}^mY = \lim_{n\to\infty} Y_n^m$. For any fixed $n > i$, let

$$M_m^n(\xi)(t) = \langle \xi_t - \xi_0, e_i\rangle_H + \int_0^t \langle \xi_s, Ae_i\rangle_H\,ds - \int_0^t \big\langle B(\pi_m\xi_s, e_i), \pi_m\xi_s\big\rangle_H\,ds - \int_0^t \big\langle e_i, f_n(s)\big\rangle_{V,V'}\,ds,$$

$$M^n(\xi)(t) = \langle \xi_t - \xi_0, e_i\rangle_H + \int_0^t \langle \xi_s, Ae_i\rangle_H\,ds - \int_0^t \big\langle B(\xi_s, e_i), \xi_s\big\rangle_H\,ds - \int_0^t \big\langle e_i, f_n(s)\big\rangle_{V,V'}\,ds,$$

$$M_m(\xi)(t) = \langle \xi_t - \xi_0, e_i\rangle_H + \int_0^t \langle \xi_s, Ae_i\rangle_H\,ds - \int_0^t \big\langle B(\pi_m\xi_s, e_i), \pi_m\xi_s\big\rangle_H\,ds - \int_0^t \big\langle e_i, f(s)\big\rangle_{V,V'}\,ds,$$

then

$$
\begin{aligned}
&\left|E^P e^{iu[M_t^{e_i}-Y^{e_i}(t)]}-1\right| \\
&\quad=\lim_{m\to\infty}\left|E^P e^{iu[M_m(t)-Y^{e_i}(t)]}-1\right| \\
&\quad=\lim_{m\to\infty}\lim_{k\to\infty}\left|E^P e^{iu[M_m(t)-Y^k(t)]}-1\right| \\
&\quad=\lim_{m\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\left|E^{P_n} e^{iu[M_m^n(t)-Y_n^k(t)]}-1\right| \\
&\quad=\lim_{m\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\left|E^{P_n} e^{iu[M_m^n(t)-Y_n^k(t)]}-E^{P_n} e^{iu[M^n(t)-{}^nY(t)]}\right| \\
&\quad\leqslant\lim_{m\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}E^{P_n}\left|e^{iu[\int_0^t\langle B(\xi_s,e_i),\xi_s\rangle_H-\langle B(\pi_m\xi_s,e_i),\pi_m\xi_s\rangle_H\,ds]+iu[{}^nY(t)-Y_n^k(t)]}-1\right| \\
&\quad\leqslant\lim_{m\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\left[E^{P_n}\left|e^{iu[\int_0^t\langle B(\xi_s,e_i),\xi_s\rangle_H-\langle B(\pi_m\xi_s,e_i),\pi_m\xi_s\rangle_H\,ds]}-1\right|+E^{P_n}\left|e^{iu[{}^nY(t)-Y_n^k(t)]}-1\right|\right].
\end{aligned}
$$

We only need to prove the second part equals 0, the first part is similar to prove as Theorem 3.3. Since

$$
\begin{aligned}
&\lim_{n\to\infty}E^{P'}\left[\int_0^t\int_{|x|\leqslant 1}\left|\frac{1}{m_1}g\big(m_1\cdot\big\langle F_n\big(X_n'(s),x\big),\varphi\big\rangle\big)-\frac{1}{m_2}g\big(m_2\cdot\big\langle F_n\big(X_n'(s),x\big),\varphi\big\rangle\big)\right|^2\lambda(dx)\,ds\right] \\
&\quad=E^{P'}\left[\int_0^t\int_{|x|\leqslant 1}\left|\frac{1}{m_1}g\big(m_1\cdot\big\langle F\big(X'(s),x\big),\varphi\big\rangle\big)-\frac{1}{m_2}g\big(m_2\cdot\big\langle F\big(X'(s),x\big),\varphi\big\rangle\big)\right|^2\lambda(dx)\,ds\right],
\end{aligned}
$$

we have

$$
\lim_{k\to\infty}\lim_{n\to\infty}E^{P_n}\left|{}^nY(t)-Y_n^k(t)\right|^2\leqslant\lim_{k\to\infty}E^P\int_0^t\int_{|x|\leqslant 1}\left|F\big(\xi(s),x\big)\right|^2\wedge\frac{1}{k^2}\lambda(dx)\,ds=0,
$$

thus $\lim_{m\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}E^{P_n}|e^{iu[{}^nY(t)-Y_n^k(t)]}-1|=0$. $\square$

6. Markov selection

We start by giving a few definitions and notations. Let $\mathcal{V}\subset\mathcal{H}\subset\mathcal{V}'$ be a Gelfand triple of separable Hilbert spaces with continuous injections. Set $\Omega=D([0,\infty);\mathcal{V}')$. Denote by $\mathcal{B}$ the Borel σ-field of Ω with Skorokhod topology and by $Pr(\Omega)$ the set of all probability measures on $(\Omega,\mathcal{B})$. Define the canonical process $\xi:\Omega\to\mathcal{V}'$ as $\xi_t(\omega)=\omega(t)$.

6.1. Preliminaries on the state space

For each $t\geqslant 0$, let $\Omega_t=D([0,t];\mathcal{V}')$ (resp. $\Omega^t=D([t,\infty);\mathcal{V}')$). Denote by $\mathcal{B}_t$ (resp. $\mathcal{B}^t$) be the Borel σ-field of Ω_t (resp. Ω^t) with Skorokhod topology respectively. Define for each given $t>0$, the map $\Phi_t:\Omega\to\Omega^t$ as

$$
\Phi_t(\omega)(s)=\omega(s-t),\quad s\geqslant t.
$$

Denote $\mathscr{C}_T$ the σ-field generated by simple cylindrical subsets of Ω_T. Recall that

$$\mathscr{C}_T = \sigma\big(\xi_t^{-1}\big(\mathcal{B}\big(\mathcal{V}'\big)\big) \,\big|\, t \in [0,T]\big).$$

Lemma 6.1. *(See [1].) If $\mathcal{V}'$ is a separable metric space, then*

$$\mathscr{C}_T = \mathcal{B}_T.$$

Similarly, we can define $\mathscr{C} = \sigma(\xi_t^{-1}(\mathcal{B}(\mathcal{V}')) \mid t \in [0,\infty))$ (resp. $\mathscr{C}^T = \sigma(\xi_t^{-1}(\mathcal{B}(\mathcal{V}')) \mid t \in [T,\infty))$), and we have

$$\mathscr{C} = \mathcal{B} \quad \big(\text{resp. } \mathscr{C}^T = \mathcal{B}^T\big).$$

Lemma 6.2. *The set $L^\infty_{loc}([0,\infty);\mathcal{H}) \cap \Omega$ is a Borel set in Ω. Moreover,*

$$L^\infty_{loc}\big([0,\infty);\mathcal{H}\big) \cap \Omega = D\big([0,\infty);\mathcal{H}_\sigma\big) \cap \Omega$$

where $\mathcal{H}_\sigma$ denotes the space $\mathcal{H}$ endowed with the weak topology. Finally, the set $L^2_{loc}([0,\infty);\mathcal{V}) \cap \Omega$ is Borel in Ω as well.

Lemma 6.3. *Let $P \in Pr(\Omega)$ be such that*

$$P\big(D\big([0,\infty);\mathcal{H}_\sigma\big) \cap \Omega\big) = 1.$$

Then, for any given $t \geqslant 0$, the mapping $\omega \to \omega(t)$ has a P-modification on $\mathcal{B}_t$ which is $\mathcal{B}_t$-measurable with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$, where $\mathcal{B}(\mathcal{H})$ is the Borel σ-field of $\mathcal{H}$.

For proving Lemma 6.2 we need the following lemma.

Lemma 6.4. *Let X and Y be two Banach spaces, such that $X \subset Y$ with a continuous injection. If a function $\phi(t)$ belongs to $L^\infty([0,T+\epsilon],X)$, $\epsilon > 0$ and is weakly right continuous and has weakly left limits with values in Y for $t \in [0,T+\epsilon]$, then ϕ is weakly right continuous and has weakly left limits with values in X, when $t \in [0,T]$.*

Proof. If we replace Y by the closure of X in Y, we may suppose that X is dense in Y. Hence the dense continuous imbedding of X into Y gives by duality a dense continuous imbedding of Y' (dual of Y), into X' (dual of X):

$$Y' \subset X'.$$

By assumption, for each $\eta \in Y'$,

$$\big\langle\phi(t),\eta\big\rangle_{Y,Y'} \to \big\langle\phi(t_0),\eta\big\rangle_{Y,Y'}, \quad \text{as } t \downarrow t_0,\ \forall t_0 \in [0,T+\epsilon), \tag{6.1}$$

and $\exists\ y_{t_0} \in Y$ such that

$$\big\langle\phi(s),\eta\big\rangle_{Y,Y'} \to \langle Y_{t_0},\eta\rangle_{Y,Y'} \quad \text{as } s \uparrow t_0,\ \forall t_0 \in (0,T+\epsilon]. \tag{6.2}$$

We next to prove that (6.1) and (6.2) are true for $\eta \in X'$, $\phi(t), y_t \in X$, $t \in (0,T]$.

We first prove that for each $t \in (0, T]$, $\phi(t)$, $y_t \in X$ and

$$\left\|\phi(t)\right\|_X \leqslant 2\|\phi\|_{L^\infty([0,T+\epsilon],X)}, \tag{6.3}$$

$$\|y_t\|_X \leqslant 2\|\phi\|_{L^\infty([0,T+\epsilon],X)}. \tag{6.4}$$

For every $t_0 \in [0, T]$, define

$$\widetilde{\Phi}_{t_0}(t) = \begin{cases} 0, & 0 \leqslant t < t_0, \ or\ t > T + \epsilon, \\ \phi(t), & t_0 \leqslant t \leqslant T + \epsilon. \end{cases}$$

We can find a sequence of smooth functions ϕ_m from $[0, T + \epsilon]$ into X, such that

$$\left\|\phi_m(t)\right\|_X \leqslant \|\widetilde{\Phi}_{t_0}\|_{L^\infty([0,T+\epsilon];X)} \leqslant \|\phi\|_{L^\infty([0,T+\epsilon];X)}$$

and

$$\left\langle\phi_m(t), \eta\right\rangle_{Y,Y'} \to \frac{\lim_{s\uparrow t}\langle\widetilde{\Phi}_{t_0}(s), \eta\rangle_{Y,Y'} + \lim s \downarrow t\langle\widetilde{\Phi}_{t_0}(s), \eta\rangle_{Y,Y'}}{2}, \quad \text{for } m \to \infty, \ \forall\eta \in Y'.$$

Since

$$\left\langle\phi_m(t_0), \eta\right\rangle_{Y,Y'} \to \left\langle\frac{\phi_{t_0}(s), \eta}{2}\right\rangle_{Y,Y'}, \quad \text{for } m \to \infty, \ \forall\eta \in Y',$$

and

$$\left|\left\langle\phi_m(t), \eta\right\rangle_{Y,Y'}\right| \leqslant \|\phi\|_{L^\infty([0,T+\epsilon];X)}\|\eta\|_{X'}, \quad \forall m, \ \forall t \in [0, T + \epsilon], \ \forall\eta \in Y',$$

we have the limit

$$\left|\left\langle\frac{\phi_m(t_0)}{2}, \eta\right\rangle_{Y,Y'}\right| \leqslant \|\phi\|_{L^\infty([0,T+\epsilon];X)}\|\eta\|_{X'}, \quad \forall\eta \in Y'.$$

This inequality shows that $\phi(t) \in X$, for every $t \in [0, T + \epsilon]$ and that (6.3) holds. For any $t \in (0, T]$, since

$$\left|\left\langle\frac{y_t}{2}, \eta\right\rangle_{Y,Y'}\right| = \lim_{s\uparrow t}\left|\left\langle\frac{\phi(s)}{2}, \eta\right\rangle_{Y,Y'}\right| \leqslant \|\phi\|_{L^\infty([0,T+\epsilon];X)}\|\eta\|_{X'}, \quad \forall\eta \in Y',$$

$y_t \in X$, for $t \in (0, T + \epsilon]$, and thus (6.4) holds.

Finally let us prove (6.1), (6.2) hold for η in X'. Since Y' is dense in X', there exists, for each $\delta > 0$, some $\eta_\delta \in Y'$ such that

$$\|\eta - \eta_\delta\|_{X'} \leqslant \delta,$$

then for $t > t_0$,

$$\left\langle\phi(t), \phi(t_0), \eta\right\rangle_{X,X'} = \left\langle\phi(t) - \phi(t_0), \eta - \eta_\delta\right\rangle_{X,X'} + \left\langle\phi(t) - \phi(t_0), \eta_\delta\right\rangle_{X,X'},$$

we have

$$\langle \phi(t), \phi(t_0), \eta \rangle_{X,X'} \leqslant 4\delta \|\phi\|_{L^\infty([0,T+\epsilon];X)} + \left|\langle \phi(t) - \phi(t_0), \eta_\delta \rangle_{X,X'}\right|.$$

Since $\eta_\delta \in Y'$, the right continuity assumption implies that

$$\langle \phi(t) - \phi(t_0), \eta_\delta \rangle_{X,X'} \to 0, \quad \text{as } t \downarrow t_0$$

and hence

$$\overline{\lim_{t\downarrow t_0}} \left|\langle \phi(t) - \phi(t_0), \eta_\delta \rangle_{X,X'}\right| \leqslant 4\delta \|\phi\|_{L^\infty([0,T+\epsilon];X)}.$$

Since $\delta > 0$ is arbitrarily small, (6.1) is proved. (6.2) can be proved similarly. □

Proof of Lemma 6.2. Firstly, we prove the equality. By the resonance theorem and the covering theorem, $D([0,\infty);\mathcal{H}_\sigma) \cap \Omega$ is in $L^\infty_{loc}([0,\infty);\mathcal{H}) \cap \Omega$. The other inclusion follows from Lemma 6.4.

Secondly, we prove measurability. Notice that the map

$$f^n_t(\omega) = \left|\omega(t)\right|_{\mathcal{H}_n}, \quad \Omega \to [0,\infty)$$

is continuous for each n. Let $\{h_i\}_{i\in N}$ be a complete orthonormal system of $\mathcal{H}$, $\mathcal{H}_n = span\{h_i, i = 1,\dots,n\}$. For each $t \geqslant 0$, by Lemma 6.1

$$\left\{\omega \in \Omega:\ \left|\omega(t)\right|_{\mathcal{H}} \leqslant R\right\} = \bigcap_{n\geqslant 1} \left\{f^n(\omega)(t) \leqslant R\right\} \in \mathcal{B}. \tag{6.5}$$

Let $D \subset [0,+\infty)$ be a continuous dense set. It's easy to prove that

$$L^\infty_{loc}\big([0,\infty);\mathcal{H}\big) \cap \Omega = \bigcap_{T=1}^{\infty} \bigcup_{R=1}^{\infty} \bigcap_{t\in D\cap[0,T]\cup\{T\}} \left\{\omega \in \Omega:\ \left|\omega(t)\right|_{\mathcal{H}} \leqslant R\right\}.$$

By (6.5), $L^\infty_{loc}([0,\infty);\mathcal{H}) \cap \Omega$ is a Borel subset in $\mathcal{B}$.

Similarly, for each n, $T \geqslant 0$,

$$g^n_T(\omega) = \int_0^T \left|\omega(s)\right|^2_{\mathcal{V}_n} ds$$

is continuous in $\mathcal{V}$, and

$$\left\{\omega \in \Omega \,\middle|\, \int_0^T \left|\omega(s)\right|^2_{\mathcal{V}} ds \leqslant R\right\} = \bigcup_{n\geqslant 1} \left\{\omega \in \Omega \,\middle|\, g^n_T(\omega) \leqslant R\right\} \in \mathcal{B},$$

hence,

$$L^2_{loc}\big([0,\infty);\mathcal{V}\big) \cap \Omega = \bigcap_{T=1}^{\infty} \bigcup_{R=1}^{\infty} \left\{\omega \in \Omega \,\middle|\, \int_0^T \left|\omega(s)\right|^2_{\mathcal{V}} ds \leqslant R\right\},$$

which implies that $L^2_{loc}([0,\infty);\mathcal{V})$ is a Borel subset in $\mathcal{B}$. □

6.1.1. Preliminaries on disintegration and reconstruction of probabilities

Given $P \in Pr(\Omega)$ and $t > 0$, we will denote by $\omega \to P|^{\omega}_{\mathcal{B}_t} : \Omega \to Pr(\Omega^t)$ a regular conditional probability distribution (RCPD) of P on $\mathcal{B}_t$. Since Ω is a Polish space and every σ-field $\mathcal{B}_t$ is finitely generated, such a function exists and is unique, up to P-null sets. In particular,

$$P|^{\omega}_{\mathcal{B}_t}\big[\xi_t = \omega(t)\big] = 1$$

for all $\omega \in \Omega$, and, if $A \in \mathcal{B}_t$ and $B \in \mathcal{B}^t$,

$$P(A \cap B) = \int\limits_A P|^{\omega}_{\mathcal{B}_t}(B)P(d\omega).$$

As conditional probabilities correspond to disintegration with respect to a σ-field, we define below the reconstruction, which is a sort of inverse procedure to disintegration.

Definition 6.1. (See [7].) Consider a probabilities $P \in Pr(\Omega)$, a time instant $t > 0$ and a $\mathcal{B}_t$-measurable map $Q : \Omega \to Pr(\Omega^t)$ such that

$$Q_\omega\big[\xi_t = \omega(t)\big] = 1, \quad \text{for all } \omega \in \Omega.$$

Then denote by $P \otimes_t Q$ the unique probability measure on Ω such that

1. $P \otimes_t Q$ and P agree on $\mathcal{B}_t$.
2. $(Q_\omega)_{\omega\in\Omega}$ is a regular conditional probability distribution of $P \otimes_t Q$ on $\mathcal{B}_t$.

The existence of the $P \otimes_t Q$ can be proved from the following two lemmas, which are similar with the case of $C([0,\infty); \mathcal{V}')$, as in Lemma 6.1.1 and Theorem 6.1.2 in [15]. For the readers convenient, we give the proof in the following.

Lemma 6.5. *Fixed $s \geqslant 0$, suppose that P is a probability measure on $(\Omega, \mathcal{B})$. If $\eta \in \Omega_s$ and $P(\xi(s) = \eta(s)) = 1$, then there is a unique probability measure $\delta_\eta \otimes_s P$ on $(\Omega, \mathcal{B})$ such that $\delta_\eta \otimes_s P(\xi(t) = \eta(t),\ 0 \leqslant t \leqslant s) = 1$ and $\delta_\eta \otimes_s P(A) = P(A)$ for all $A \in \mathcal{B}^s$.*

Proof. The uniqueness is obvious. Let δ_η be the Dirac measure on Ω_s at η, i.e. $\delta_\eta(\{\alpha \in \Omega_s\colon\ \alpha(t) = \eta(t),\ 0 \leqslant t \leqslant s\}) = 1$ and $\Phi : \Omega \to \Omega^s$ be the map defined by $\Phi(\Omega)(t) = \omega(t)$, $t \geqslant s$. By Lemma 6.1, Φ is measurable on $(\Omega, \mathcal{B}^s)$, and therefore $P \circ \Phi^{-1}$ is well defined. Define $\widetilde{P} = \delta_\eta \times (P \circ \Phi^{-1})$ on $\widetilde{X} \equiv \Omega_s \times \Omega^s$. Set $X = \{(\alpha, \beta) \in \widetilde{X}\colon\ \alpha(s) = \beta(s)\}$, X is a Borel subset of $\widetilde{X}$ (denote $f(\alpha,\beta) = \alpha(s) - \beta(s)$: $\Omega_s \times \Omega^s \to \mathcal{V}'$, it's easy to see that X is a Borel subset of $\widetilde{X}$). Note that $\widetilde{P}(X) \geqslant \delta_\eta(\{\alpha \in \Omega_s\colon\ \alpha(s) = \eta(s)\})P \circ \Phi^{-1}(\{\beta \in \Omega^s\colon\ \beta(s) = \eta(s)\}) = 1$. Thus $\widetilde{P}$ can be restricted to X. Define $\psi : X \to \Omega$ as

$$\psi\big((\alpha,\beta)\big)(t) = \begin{cases} \alpha(t), & 0 \leqslant t < s, \\ \beta(t), & t \geqslant s. \end{cases}$$

It is a continuous map from X to Ω, and the restriction of $\widetilde{P}$ to X determines, via ψ, a probability measure on $(\Omega, \mathcal{B})$. This is the desired measure $\delta_\eta \otimes_s P$. $\square$

Remark 6.1. Note that if $x, y \in \Omega$, $d(x, y) = 0$ if and only if $x(t) = y(t)$ for every $t \geqslant 0$; if $x, y \in \Omega_s$, $d(x, y) = 0$, if and only if $x(t) = y(t)$ for every $0 \leqslant t \leqslant s$.

Lemma 6.6. *Fixed $t \geqslant 0$, suppose that $\omega \to Q_\omega$ is a mapping of Ω into probability measures on $(\Omega, \mathcal{B})$ satisfies*

(i) $\omega \to Q_\omega(N)$ *is* $\mathcal{B}_t$*-measurable for all* $N \in \mathcal{B}$,
(ii) $Q_\omega(\xi_t(\cdot) = \omega(t)) = 1$ *for all* $\omega \in \Omega$.

Given a probability measure P *on* $(\Omega, \mathcal{B})$, *there exists a unique probability measure* $P \otimes_t Q_\cdot$ *on* $(\Omega, \mathcal{B})$ *such that* $P \otimes_t Q_\cdot$ *equals* P *on* $(\Omega, \mathcal{B}_t)$ *and* $\{\delta_\omega \otimes_t Q_\omega\}$ *is a RCPD of* $P \otimes_t Q_\cdot|\mathcal{B}_t$.

Proof. The uniqueness is obvious. We prove the existence of $P \otimes_t Q_\cdot$.

Let $N = \{\xi \in \Omega\colon\ \xi(s_1) \in \Lambda_1, \ldots, \xi(s_n) \in \Lambda_n\}$, where $n \geqslant 1$, $0 \leqslant s_1 < \cdots < s_n$, and $\Lambda_1, \ldots, \Lambda_n \in \mathcal{B}(\mathcal{V}')$, then

$$\begin{aligned}\delta_\omega \otimes_t Q_\omega(N) = {} & \mathcal{X}_{[0,s_1)}(t) Q_\omega(N)\\ & + \sum_{k=1}^{n-1} \mathcal{X}_{[s_k,s_{k+1})}(t) \mathcal{X}_{\Lambda_1}\big(\xi(s_1)\big) \ldots \mathcal{X}_{\Lambda_k}\big(\xi(s_k)\big) \times Q_\omega(\xi_{s_{k+1}} \in \Gamma_{k+1}, \ldots, \xi_{s_n} \in \Lambda_n)\\ & + \mathcal{X}_{[s_n,\infty)}(t) \mathcal{X}_{\Lambda_1}\big(\xi(s_1)\big) \ldots \mathcal{X}_{\Lambda_n}\big(\xi(s_n)\big).\end{aligned}$$

It is clear that $\omega \to \delta_\omega \otimes_t Q_\omega(N)$ is $\mathcal{B}_t$-measurable. By Lemma 6.1 and the monotone class theorem, the map $\omega \to \delta_\omega \otimes_t Q_\omega(N)$ is $\mathcal{B}_t$-measurable for all $N \in \mathcal{B}$.

Set

$$G(N) = E^P\big[\delta_\cdot \otimes_t Q_\cdot(N)\big], \quad N \in \mathcal{B}.$$

It is easy to prove that G has the desired properties of $P \otimes_t Q_\cdot$. $\square$

6.2. The Markov property and existence of Markov selections

We first extended some concepts and Theorems in [7] to the space $D([0,\infty); \mathcal{H}_\sigma)$. Since the proving is almost the same as in [7], we only state them without proving.

Given a family $(P_x)_{x \in \mathcal{H}}$ of probability measures, the Markov property can be stated as

$$P_x|^\omega_{\mathcal{B}_t} = \Phi_t P_{\omega(t)}, \quad \text{for } P_x\text{-a.s. } \omega \in \Omega$$

for each $x \in \mathcal{H}$ and $t \geqslant 0$.

Definition 6.2 *(Almost sure Markov property).* Let $x \to P_x$ be a measurable map from $\mathcal{H}$ to $Pr(\Omega)$ such that

$$P_x\big[D\big([0,\infty); \mathcal{H}_\sigma\big) \cap \Omega\big] = 1 \quad \text{for all } x \in \mathcal{H}.$$

The family $(P_x)_{x \in \mathcal{H}}$ has the almost sure Markov property if for each $x \in \mathcal{H}$ there is a set $T \subset (0,\infty)$ with null Lebesgue measure, such that

$$P_x|^\omega_{\mathcal{B}_t} = \Phi_t P_{\omega(t)}, \quad \text{for } t \notin T,\ \omega \in \Omega,\ P_x\text{-a.s.}$$

Denote by $Comp(Pr(\Omega))$ the family of all compact subsets of $Pr(\Omega)$.

Definition 6.3 *(Almost sure pre-Markov family).* Consider a measurable map $\mathcal{C} : \mathcal{H} \to Comp(Pr(\Omega))$ such that $P_x[D([0,\infty); \mathcal{H}_\sigma) \cap \Omega] = 1$ for all $x \in \mathcal{H}$ and $P \in \mathcal{C}(x)$.

The family $(\mathcal{C}(x))_{x \in \mathcal{H}}$ is almost surely pre-Markov if for each $x \in \mathcal{H}$ and $P \in \mathcal{C}(x)$, there is a set $T \subset (0,\infty)$ with null Lebesgue measure, such that for all $t \notin T$, the following properties hold:

1. (Disintegration) there exists $N \in \mathcal{B}_t$ with $P(N) = 0$ such that for all $\omega \notin N$,

$$\omega \in \mathcal{H} \quad \text{and} \quad P|_{\mathcal{B}_t}^{\omega} \in \Phi_t \mathcal{C}\big(\omega(t)\big);$$

2. (Reconstruction) for each $\mathcal{B}_t$-measurable map $\omega \to Q_\omega : \Omega \to Pr(\Omega^t)$ such that there is $N \in \mathcal{B}_t$ with $P(N) = 0$ and for all $\omega \notin N$,

$$\omega(t) \in \mathcal{H} \quad \text{and} \quad Q_\omega \in \Phi_t \mathcal{C}\big(\omega(t)\big);$$

then $P \otimes_t Q \in \mathcal{C}(x)$.

Remark 6.2. If every $\mathcal{C}(x)$ is a singleton, the a.s. pre-Markov family is indeed an a.s. Markov family of probability measures, as stated in Definition 6.2.

For each $f \in C_b(\mathcal{V}', R)$, $\lambda > 0$, $P \in Pr(\Omega)$, $x \in \mathcal{H}$, let

$$J_{\lambda,f}(P) = E^P\left[\int_0^\infty e^{-\lambda t} f\big(\xi(t)\big)\,dt\right],$$

$$R_\lambda^+ f(x) = \sup_{P \in \mathcal{C}(x)} J_{\lambda,f}(P),$$

$$\mathcal{C}_{\lambda,f}(x) = \big\{P \in \mathcal{C}(x) \bigm| J_{\lambda,f}(P) = R_\lambda^+ f(x)\big\}.$$

Lemma 6.7. *(See [7].) Let $(\mathcal{C}(x))_{x\in\mathcal{H}}$ be an a.s. pre-Markov family with non-empty convex values, $\lambda > 0$ and $f \in C_b(\mathcal{V}', R)$. Then $R_\lambda^+ f(x)$ is well defined and $(\mathcal{C}_{\lambda,f}(x))_{x\in\mathcal{H}}$ is again an a.s. pre-Markov family with non-empty convex values.*

Theorem 6.1. *(See [7].) Let $(\mathcal{C}(x))_{x\in\mathcal{H}}$ be an a.s. pre-Markov family with non-empty convex values. Then there is a measurable map $x \to P_x$ on $\mathcal{H}$ with values in $Pr(\Omega)$ such that $P_x \in \mathcal{C}(x)$ for all $x \in \mathcal{H}$ and $(P_x)_{x\in\mathcal{H}}$ has the a.s. Markov property.*

If in Definition 6.3 each set T of exceptional times is empty, call it a pre-Markov family.

Theorem 6.2. *(See [7].) Let $(\mathcal{C}(x))_{x\in\mathcal{H}}$ be a pre-Markov family with non-empty convex values. Then there is a measurable map $x \to P_x$ from $\mathcal{H}$ to $Pr(\Omega)$ such that $P_x \in \mathcal{C}(x)$ for all $x \in \mathcal{H}$ and $(P_x)_{x\in\mathcal{H}}$ has the Markov property.*

7. Markov selection for the Navier–Stokes equations

The Markov selection for the 3D Navier–Stokes equations with Wiener process has been considered in [5,7,8] recently. Using their argument, we extended the results to the Lévy Noise.

Let $\mathcal{T} = [0,1]^3$ be the 3D torus and let $\mathscr{D}^\infty$ be the space of infinitely differentiable divergence-free periodic vector fields on R^3 with zero mean, H be the closure of $\mathscr{D}^\infty$ in the norm of $L^2(\mathcal{T}, R^3)$, and $D(A) = \{u \in H \mid \Delta u \in H\}$. Let $A : D(A) \to H$ be the stokes operator

$$Au = -\Delta u, \quad u \in D(A).$$

It is a positive linear self-adjoint operator on H and we can define the powers A^α, $\alpha \in R$, with domain $D(A^\alpha)$. By proper identifications of dual spaces, $V \subset H \subset V' \subset D(A)'$. The bi-linear operator $B : V \times V \to V'$ is defined as

$$B(u,v)=\mathbb{P}_{\text{div}}(u\cdot\Delta)v$$

where $\mathbb{P}_{div}$ is the projection onto divergence-free vector fields.

Let $(e_i)_{i\in N}$ be a complete orthonormal system of eigenvectors and denote $\sigma^2=\sum_{i=1}^{\infty}\sigma_i^2$, $\sigma_i\in R$.

We shall consider the 3D Navier–Stokes equations in its abstract form

$$du+\big(\nu Au+B(u,u)\big)\,dt=\sum_i^{\infty}\sigma_i e_i\,dL_i(t) \tag{7.1}$$

where $\{L_i(t)=\int_0^{t+}\int_{|x|\leqslant 1}x\widetilde{N}_i(ds,dx)\}_{i\in N}$ are independent Lévy processes on a complete probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},P)$, $\widetilde{N}_i(dt,dx)=N_i(dt,dx)-dt\lambda(dx)$, $\{N_i(dt,dx)\}_{i\in N}$ are independent Poisson random measure with the characteristic measure $\lambda(dx)$ satisfying: $\int_{|x|\leqslant 1}x^2\lambda(dx)<\infty$.

7.1. The solutions to the martingale problem

In view of the results of previous, we consider the particular case where $\mathcal{V}=V$, $\mathcal{H}=H$ and $\mathcal{V}'=D(A)'$. We set

$$\Omega_{NS}=D\big([0,\infty);D(A)'\big).$$

This space will play the role of state space for the solutions to (7.1). By Lemma 6.1, we denote by $\mathcal{B}_{NS}$ the σ-field of Borel sets of Ω_{NS}, and, for each $t\geqslant 0$, by $\mathcal{B}_t^{NS}=\sigma(\omega|_{[0,t]}\colon\ \omega\in\Omega_{NS})$ and $\mathcal{B}_{NS}^t=\sigma(\omega|_{[t,\infty)}\colon\ \omega\in\Omega_{NS})$ the σ-fields of past and future, with respect to time t, events. For Markov selection, we define the solutions to the martingale problem (7.1) as follows

Definition 7.1. Given $x\in H$, a probability P_x on $(\Omega_{NS},\mathcal{B}_{NS})$ is a solution starting at x to the martingale problem associated to the Navier–Stokes equation (7.1) if

[MP1] $P_x[L^{\infty}_{loc}([0,\infty);H)\cap L^2_{loc}([0,\infty);V)]=1$.

[MP2] For each $\{e_i\}_{i\in N}$ the process $M_t^{e_i}$, defined P_x-a.s. on $(\Omega_{NS},\mathcal{B}_{NS})$ as

$$M_t^{e_i}=\langle\xi_t-\xi_0,e_i\rangle+\nu\int_0^t\langle\xi_s,Ae_i\rangle\,ds-\int_0^t\big\langle B(\xi_s,e_i),\xi_s\big\rangle ds$$

is square integrable. Furthermore, $\{M_t^{e_i}\}_{i\in N}$ are independent Lévy process on $(\Omega_{NS},\mathcal{B}_{NS},(\mathcal{B}_t^{NS})_{t\geqslant 0},P_x)$, and the characteristic function is

$$E^{P_x}\big[e^{iu(\frac{M_{t_2}^{e_i}-M_{t_1}^{e_i}}{\sigma_i})}\big]=\exp^{\{(t_2-t_1)\int_{|x|\leqslant 1}(e^{iux}-1-iux)\lambda(dx)\}}$$

for $t_2\geqslant t_1\geqslant 0$.

[MP3] δ_x is the marginal of P_x at time $t=0$.

[MP4] There exists a constant $C>0$ and a Lebesgue null set $T_{P_x}\subset(0,\infty)$ such that for all $0\leqslant s\notin T_{P_x}$ and all $t\geqslant s$,

$$E^{P_x}\left(\sup_{r\in[s,t]}\big|\xi(r)\big|_H^2+\int_s^t\big\|\xi(r)\big\|_V^2\,dr\;\Big|\;\mathcal{B}_s^{NS}\right)\leqslant C\left[\big|\xi(s)\big|_H^2+\sigma^2\int_{|x|\leqslant 1}x^2\lambda(dx)(t-s)\right],\quad P_x\text{-a.s.}$$

Theorem 7.1. *Assume $\sigma^2 < \infty$. For each $x \in H$, there exists at least one martingale solution $P_x \in Pr(\Omega_{NS})$ to Eq.* (7.1) *in the sense of Definition* 7.1; *and there exists a Markov selection $(P_x)_{x\in H}$.*

To prove Theorem 7.1, we need the following three lemmas about regular probabilities. The continuous version refer to [8] and the proof is similar to [8]. The symbol used in the following three lemmas refer to Section 6.

Lemma 7.1. *(See [8].) Let P be a probability measure on $(\Omega, \mathcal{B})$, and for $s \geqslant 0$, denote $Q_x^s := P(. \mid \mathcal{B}_s)(x)$ be an RCPD of P with respect to $\mathcal{B}_s$. Then for any $\zeta \in L^1(\Omega, \mathcal{B}, P)$ and $t \geqslant s$ there exists a P-null set $\Lambda_{t,\zeta} \in \mathcal{B}_s$, satisfying for all $x \in \Lambda_{t,\zeta}^c$,*

$$E^P(\zeta \mid \mathcal{B}_t) = E^{Q_x^s}(\zeta \mid \mathcal{B}_t) = E^{Q_x^s}\big(\zeta \bigm| \mathcal{B}_s^t\big), \quad Q_x^s\text{-a.s.}$$

Lemma 7.2. *(See [8].) Let $D := \{(t,s)\colon 0 \leqslant s \leqslant t < \infty\}$, let $\xi, \eta : D \to R_+$ be two measurable processes on $(\Omega, \mathcal{B})$. Given $P \in Pr(\Omega)$ and $r \geqslant 0$, suppose that*

(i) *for each $s \geqslant 0$, the map $t \to \xi(t,s)$ is a.s. increasing, and $t \to \eta(t,s)$ is a.s. right continuous, $\eta(t,s)$ is $\mathcal{B}_s$-measurable for any $t \geqslant s$;*
(ii) *for each $(t,s) \in D$,*

$$\xi(t,s), \eta(t,s) \in L^1(\Omega, \mathcal{B}, P)$$

and

$$\xi(t,.), \eta(t,.) \in L^1\big(0,t; L^1(\Omega, \mathcal{B}, P)\big);$$

(iii) *for any $x \in \Omega$, and $t \geqslant s \geqslant r$,*

$$\xi(t,s,\Phi_r x) = \xi(t-r, s-r, x)$$

and

$$\eta(t,s,\Phi_r x) = \eta(t-r, s-r, x).$$

Then the following three statements are equivalent:

(1) *There is a Lebesgue null set $T_r \subset (r,\infty)$ such that for any $r \leqslant s \notin T_r$ and $t \geqslant s$,*

$$E^P\big(\xi(t,s) \bigm| \mathcal{B}_s\big) \leqslant \eta(t,s), \quad P\text{-a.s.}$$

(2) *For some P-null set $N \in \mathcal{B}_r$ and each $x \in N^c$, there is a Lebesgue null set $T_{r,x} \subset (r,\infty)$ such that for any $r \leqslant s \notin T_{r,x}$ and any $t \geqslant s$,*

$$E^{Q_x^r}\big(\xi(t,s) \bigm| \mathcal{B}_s^r\big) \leqslant \eta(t,s), \quad Q_x^r\text{-a.s.}$$

(3) *For some P-null set $N \in \mathcal{B}_r$ and each $x \in N^c$, there is a Lebesgue null set $T_{r,x} \subset (0,\infty)$ such that for any $0 \leqslant s \notin T_{r,x}$ and any $t \geqslant s$,*

$$E^{Q_x^r \circ \Phi_r}\big(\xi(t,s) \bigm| \mathcal{B}_s\big) \leqslant \eta(t,s), \quad Q_x^r \circ \Phi_r\text{-a.s.}$$

Moreover $T_r = \emptyset \Leftrightarrow T_{r,x} = \emptyset$.

Lemma 7.3. *(See [8].) Let $(M(t))_{t\geqslant 0}$ and $(K(t))_{t\geqslant 0}$ be $\mathcal{B}_t$-adapted real valued process on $(\Omega, \mathcal{B})$ which satisfy for $x \in \Omega$, $t \geqslant r \geqslant 0$,*

$$M(t, \Phi_r x) = M(t-r, x), \qquad K(t, \Phi_r x) = K(t-r, x). \tag{7.2}$$

Given $P \in Pr(\Omega)$ and $r \geqslant 0$, assume that for each $t \geqslant 0$, $E^P(K(t)) < \infty$. Then the following statements are equivalent:

(1) *$(M_t, \mathcal{B}_t, P)_{t\geqslant r}$ is a càdlàg martingale with square variation process $(K(t))_{t\geqslant r}$.*
(2) *There exists a P-null set $N \in \mathcal{B}_r$ such that for all $x \notin N$, $(M_t, \mathcal{B}_t, Q_x^r)_{t\geqslant s}$ is a càdlàg martingale with square variation process $(K(t))_{t\geqslant r}$ and $E^P[E^{Q_\cdot^r}[K(t)]] < \infty$.*
(3) *There exists a P-null set $N \in \mathcal{B}_r$ such that for all $x \notin N$, $(M_t, \mathcal{B}_t, Q_x^r \circ \Phi_r)_{t\geqslant 0}$ is a càdlàg martingale with square variation process $(K(t))_{t\geqslant 0}$.*

Proof. (1) $\Rightarrow$ (2) First, we prove that if $r \leqslant t_1 \leqslant t_2$, then there is a P-null set $N_{t_1,t_2} \in \mathcal{B}_r$, such that for all $x \notin N_{t_1,t_2}$,

$$E^{Q_x^r}[M_{t_2} \mid \mathcal{B}_{t_1}] = M_{t_1}, \quad Q_x^r\text{-a.s.} \tag{7.3}$$

Indeed, let $A \in \mathcal{B}_{t_1}$, then for each $B \in \mathcal{B}_r$ we have that $A \cap B \in \mathcal{B}_{t_1}$ and

$$\begin{aligned} E^P\big[I_B \cdot E^{Q_\cdot^r}[M_{t_2} I_A]\big] &= E^P[M_{t_2} I_{A\cap B}] \\ &= E^P[M_{t_1} I_{A\cap B}] \\ &= E^P\big[I_B \cdot E^{Q_\cdot^r}[M_{t_1} I_A]\big] \end{aligned}$$

so that $E^{Q_x^r}[M_{t_2} I_A] = E^{Q_x^r}[M_{t_1} I_A]$ out of a P-null set in $\mathcal{B}_r$. Since $\mathcal{B}_r$ is countably generated, the P-null set can be chosen independently of A.

Next, let D be a dense set in $[r, \infty)$, then by the previous argument we can find a P-null set $N \in \mathcal{B}_r$ such that (7.3) is true for $x \notin N$ and $t_1, t_2 \in D$. By Lemma 1.29 in [15], (7.3) is true for all $t \geqslant r$.

One can proceed similarly to prove that M_t is Q_x^r-square integrable with quadratic variation $(K_t)_{t\geqslant r}$, since M_t^2 is a sub-martingale and $M_t^2 - K_t$ is a martingale. Finally, $E^P[E^{Q_\cdot^r}[K_t]] = E^P[K_t]$.

(2) $\Rightarrow$ (1) Since $x \to Q_\cdot^r[K_t]$ is P-integrable, M_t is P-square integrable and it is easy to see that M_t is a martingale with quadratic variation K_t.

(2) $\Leftrightarrow$ (3) is direct from (7.2). Indeed, for any $A \in \mathcal{B}_{s-r}$,

$$\begin{aligned} E^{Q_x^r \circ \Phi_r}(I_A \cdot M_{s-r}) &= E^{Q_x^r}\big(I_{\Phi_r A} \cdot M\big(s-r, \Phi_r^{-1}(.)\big)\big) \\ &= E^{Q_x^r}\big(I_{\Phi_r A} \cdot M(s)\big) \\ &= E^{Q_x^r}\big(I_{\Phi_r A} \cdot M(t)\big) \\ &= E^{Q_x^r \circ \Phi_r}(I_A \cdot M_{t-r}), \end{aligned}$$

this completes the proof. $\square$

As a consequence, we have the following BDG's inequality.

Corollary 7.1. *Let $(M_t, \mathcal{B}_t, P)_{t\geqslant r}$ be a càdlàg square integrable martingale with $M_r = 0$, then P-a.s.*

$$E^P\Big(\sup_{s\in[r,t]} |M_s| \,\Big|\, \mathcal{B}_r\Big) \leqslant C E^P\big([M_t]^{1/2} \,\big|\, \mathcal{B}_r\big).$$

Proof of Theorem 7.1. The proof will be developed in the following lemmas.

Lemma 7.4. *Assume $\sigma^2 < \infty$. For each $x \in H$, there exists at least one martingale solution $P_x \in Pr(\Omega_{NS})$ to Eq.* (7.1) *in the sense of Definition* 7.1.

Proof. Refer to Section 3, we only prove [MP4]. From Theorem 3.3 we have P_n-a.e.

$$\left|\xi(t)\right|_H^2 + 2\nu \int_s^t \left\|\xi(r)\right\|_V^2 dr = \left|\xi(s)\right|_H^2 + 2\int_s^t \left\langle \xi(r), \sum_{i=1}^n \sigma_i e_i \, dL_i^n(r) \right\rangle_H + \sum_{i=1}^n \int_s^t \int_{|x|\leqslant 1} \sigma_i^2 x^2 \, dN_i^n(dr, dx).$$

Let

$$y(t,s,\xi) = \sup_{s'\in[s,t]} \left|\xi\left(s'\right)\right|_H^2 + 2\nu \int_s^t \left\|\xi\left(s'\right)\right\|_V^2 ds',$$

$$y_n(t,s,\xi) = \sup_{s'\in[s,t]} \left|\xi\left(s'\right)\right|_{H_n}^2 + 2\nu \int_s^t \left\|\xi\left(s'\right)\right\|_{V_n}^2 ds'.$$

Then

$$\begin{aligned}
E^{P_n}\left[y(t,r,\xi) \mid \mathcal{B}_r^{NS}\right] &\leqslant \left|\xi(r)\right|_H^2 + 2E^{P_n}\left[\sup_{s'\in[r,t]} \int_r^{s'} \left\langle \xi\left(t'\right), \sum_{i=1}^n \sigma_i e_i \, dL_i^n\left(t'\right)\right\rangle \;\Big|\; \mathcal{B}_r^{NS}\right] \\
&\quad + E^{P_n}\left[\sum_{i=1}^n \int_r^t \int_{|x|\leqslant 1} \sigma_i^2 x^2 \, dN_i^n(dr, dx)\right] \\
\text{by Corollary 7.1} &\leqslant \left|\xi(r)\right|_H^2 + C_1 E^{P_n}\left\{\left[\int_r^t \left\langle \xi\left(t'\right), \sum_{i=1}^n \sigma_i e_i x\right\rangle^2 dN_i^n\left(dt', dx\right)\right]^{1/2} \;\Big|\; \mathcal{B}_r^{NS}\right\} \\
&\quad + (t-r)\sigma^2 \int_{|x|\leqslant 1} x^2 \lambda(dx) \\
&\leqslant \left|\xi(r)\right|_H^2 + C_1 E^{P_n}\left\{\sup_{s'\in[r,t]} \left|\xi\left(s'\right)\right|_H \left[\int_r^t \sum_{i=1}^n \sigma_i^2 x^2 \, dN_i^n\left(dt', dx\right)\right]^{1/2} \;\Big|\; \mathcal{B}_r^{NS}\right\} \\
&\quad + (t-r)\sigma^2 \int_{|x|\leqslant 1} x^2 \lambda(dx) \\
&\leqslant \left|\xi(r)\right|_H^2 + 1/2 E^{P_n}\left[\sup_{s'\in[r,t]} \left|\xi\left(s'\right)\right|_H^2 \;\Big|\; \mathcal{B}_r^{NS}\right] + C_2 (t-r)\sigma^2 \int_{|x|\leqslant 1} x^2 \lambda(dx).
\end{aligned}$$

So $E^{P_n}[y(t,r,\xi) | \mathcal{B}_r^{NS}] \leqslant C(|\xi(r)|_H^2 + (t-r)\sigma^2 \int_{|x|\leqslant 1} x^2 \lambda(dx))$.

By Theorem 3.3, Remark 3.1 and Skorokhod embedding theorem, there exists a stochastic basis $(\Omega', \mathcal{F}', P')$ and $\Omega_{NS} \cap L^2_{loc}([0,\infty); H)$-valued random variables $\widetilde{x}$, $\widetilde{x}_n$ $n \geqslant 1$ such that $\widetilde{x}_n$, $\widetilde{x}$ have the law of P_n, P on $\Omega_{NS} \cap L^2_{loc}([0,\infty); H)$ respectively, and $\widetilde{x}_n \to \widetilde{x}$ in $\Omega_{NS} \cap L^2_{loc}([0,\infty); H)$, P'-a.s. (by choosing a sub-sequence if necessary). So, for any $T > 0$, we have $\lim_{n\to\infty} \int_0^T E^{P'}[|\widetilde{x}_n(s) - \widetilde{x}(s)|_H^2]\,ds = 0$. Thus there exists a Lebesgue null set $T \subset (0,\infty)$ such that for all $s \notin T$, $\lim_{n\to\infty} E^{P'}[|\widetilde{x}_n(s) - \widetilde{x}(s)|_H^2] = 0$.

For any $r \notin T$ and $t \geqslant r$, we want to prove P-a.s.

$$E^P\big[y(t,r,\xi) \bigm| \mathcal{B}_r^{NS}\big] \leqslant C\bigg(\big|\xi(r)\big|_H^2 + (t-r)\sigma^2 \int\limits_{|x|\leqslant 1} x^2\lambda(dx)\bigg)$$

which is equivalent to prove that for any $\mathcal{B}_r^{NS}$-measurable and bounded continuous function g on Ω_{NS},

$$E^P\big[y(t,r,\xi)g(\xi)\big] \leqslant C E^P\bigg[\bigg(\big|\xi(r)\big|_H^2 + (t-r)\sigma^2 \int\limits_{|x|\leqslant 1} x^2\lambda(dx)\bigg)g(\xi)\bigg].$$

By Fatou's lemma

$$\begin{aligned}
E^P\big[y_n(t,r,\xi)g(\xi)\big] &= E^{P'}\big[y_n(t,r,\widetilde{x})g(\widetilde{x})\big] \\
&\leqslant \liminf_{m\to\infty} E^{P'}\big[y_n(t,r,\widetilde{x}_m)g(\widetilde{x}_m)\big] \\
&= \liminf_{m\to\infty} E^{P_m}\big[y_n(t,r,\xi)g(\xi)\big] \\
&\leqslant C \liminf_{m\to\infty} E^{P_m}\bigg[\bigg(\big|\xi(r)\big|_H^2 + (t-r)\sigma^2 \int\limits_{|x|\leqslant 1} x^2\lambda(dx)\bigg)g(\xi)\bigg] \\
&= C \liminf_{m\to\infty} E^{P'}\bigg[\bigg(\big|\widetilde{x}_m(r)\big|_H^2 + (t-r)\sigma^2 \int\limits_{|x|\leqslant 1} x^2\lambda(dx)\bigg)g(\widetilde{x}_m)\bigg] \\
&= C E^{P'}\bigg[\bigg(\big|\widetilde{x}(r)\big|_H^2 + (t-r)\sigma^2 \int\limits_{|x|\leqslant 1} x^2\lambda(dx)\bigg)g(\widetilde{x})\bigg] \\
&= C E^{P}\bigg[\bigg(\big|\xi(r)\big|_H^2 + (t-r)\sigma^2 \int\limits_{|x|\leqslant 1} x^2\lambda(dx)\bigg)g(\xi)\bigg]
\end{aligned}$$

which means that [MP4] in Definition 7.1 holds for P by taking limitation in n. $\square$

Define for each $x \in H$ the subset of $Pr(\Omega_{NS})$ as

$$\mathcal{C}_{NS}(x) = \big\{P \in Pr(\Omega_{NS}) \bigm| P \text{ solves the martingale problem (7.1) starting at } \delta_x\big\}. \tag{7.4}$$

Lemma 7.5. *Given $x \in H$, the set $\mathcal{C}_{NS}(x)$ is non-empty, convex and compact, and satisfies*

(1) *for every $P \in \mathcal{C}_{NS}(x)$, $P[D([0,\infty); H_\sigma)] = 1$,*
(2) *the map $\mathcal{C}_{NS} : H \to Comp(Pr(\Omega_{NS}))$ is Borel measurable.*

Proof. Due to Lemma 7.4, the set $\mathcal{C}_{NS}(x)$ is non-empty. And, from [MP1] of Definition 7.1 and Lemma 6.2, it follows that $P[D([0,\infty); H_\sigma)] = 1$ for every $P \in \mathcal{C}_{NS}(x)$. And it is easy to check each $\mathcal{C}_{NS}(x)$ is convex.

Refer to [7], compactness and measurability follow from the following claim:

For each sequence $\{x_n\}_{n\in N} \subset H$ and $P_n \in \mathcal{C}_{NS}(x_n)$, if $x_n \to x$ in H, then there exists $n_k \uparrow \infty$ and $P \in \mathcal{C}_{NS}(x)$, such that $P_{n_k} \to P$ with respect to weak convergence in $Pr(\Omega_{NS})$.

In order to prove the claim, let $x_n \to x$ in H and $P_n \in \mathcal{C}_{NS}(x_n)$, we first show that $(P_n)_{n\in N}$ is tight on $\Omega_{NS} \cap L^2_{loc}([0,\infty); H)$.

By [MP4] of Definition 7.1, we have that for all $T > 0$,

$$E^{P_n}\left[\sup_{t\in[0,T]} |\xi_t|_H^2 + \int_0^T \|\xi_s\|_V^2\, ds\right] \leqslant C\left(\sigma^2, T, |x_n|_H^2, \int_{|x|\leqslant 1} x^2 \lambda(dx)\right).$$

Next, let $(\Sigma, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$ be a filtered probability space, $\{u(t)\}_{t\geqslant 0}$ be a process on Σ whose law is P_n and such that $\{u(t)\}_{t\geqslant 0}$ is a weak martingale solution to (3.1). In particular,

$$u(t) = x_n - \int_0^t \big(Au(s) + B\big(u(s), u(s)\big)\big)\, ds + \sum_{i=0}^{\infty} \sigma_i L_i(t) e_i, \quad P\text{-a.s.}$$

in $D(A)'$.

Using the same method in Theorem 3.3 and Remark 3.1, we have that $(P_n)_{n\in N}$ is tight on $\Omega_{NS} \cap L^2_{loc}([0,\infty); H)$, denote $P = \lim_{k\to\infty} P_{n_k}$ and we also have P satisfy [MP1], [MP2], [MP3]. By using the method as in Lemma 7.4, [MP4] follows. $\square$

Lemma 7.6. *The disintegration property of Definition* 6.3 *holds for the family* $(\mathcal{C}_{NS}(x))_{x\in H}$.

Proof. Fix $x_0 \in H$ and $P \in \mathcal{C}_{NS}(x_0)$. Let $Q_x^r := P(\cdot \mid \mathcal{B}_r^{NS})(x)$ be an RCPD of P with respect to $\mathcal{B}_r$. We want to show that there is a P-null set $N \in \mathcal{B}_r$ such that for all $x \notin N$,

$$Q_x^r \circ \Phi_r \in \mathcal{C}_{NS}\big(x(r)\big).$$

That is, we need to check $Q_x^r \circ \Phi_r$ satisfies [MP1]–[MP4].

[MP1] Set

$$A_t = \big\{\xi \in \Omega_{NS}\colon \xi|_{[0,t]} \in L^\infty(0,t; H) \cap L^2(0,t; V)\big\} \in \mathcal{B}_t^{NS},$$
$$A^t = \big\{\xi \in \Omega_{NS}\colon \xi|_{[t,\infty)} \in L^\infty_{loc}\big([t,\infty); H\big) \cap L^2_{loc}\big([t,\infty); V\big)\big\} \in \mathcal{B}^t_{NS}.$$

Notice that $P(A_t \cap A^t) = 0$ by property [MP1]. Hence,

$$1 = P\big[A_t \cap A^t\big] = \int_{A_t} Q_x^r\big[A^t\big] P(dx),$$

and thus there is a P-null set $N_1 \in \mathcal{B}_t^{NS}$ such that $Q_x^r[S^t] = 1$ for all $x \notin N_1$.

[MP2] By (3) of Lemma 7.3 there exists a P-null set $N_2 \in \mathcal{B}_r^{NS}$ such that for all $x \notin N_2$, $Q_x^r \circ \Phi_r$ satisfies [MP2].

[MP3] We choose ξ and η in Lemma 7.2 as follows

$$\xi(t,s) := \sup_{s'\in[s,t]} \left|x(s')\right|_H^2 + \int_s^t \left\|x(s')\right\|_V^2 ds', \quad \eta(t,s) = C\left[\left|x(s)\right|_H^2 + \sigma^2 \int_{|x|\leqslant 1} x^2\lambda(dx)(t-s)\right].$$

It's clear that for each $s \in [0,t]$, $\eta(t,s)$ is $\mathcal{B}_s^{NS}$-measurable, $t \to \eta(t,s)$ is continuous, $t \to \xi(t,s)$ is increasing, and (iii) in Lemma 7.2 holds. The integrability conditions on ξ and η in Lemma 7.2 follow from [MP4], i.e.

$$E^P\big(\xi(t,0)\big) \leqslant C\left[|x_0|_H^2 + \sigma^2 \int_{|x|\leqslant 1} x^2\lambda(dx)t\right].$$

Thus, by (2) of Lemma 7.2, there exists a P-null set $N_3 \in \mathcal{B}_r^{NS}$ such that for all $x \notin N_3$, $Q_x^r \circ \Phi_r$ satisfies [MP4].

Finally, letting $N := N_1 \cup N_2 \cup N_3$, we obtain the desired result. $\square$

Lemma 7.7. *The reconstruction property of Definition* 6.3 *holds for the family* $(\mathcal{C}_{NS}(x))_{x\in H}$.

Proof. Fix $x_0 \in H$, and $P \in \mathcal{C}_{NS}(x_0)$, let $Q_x^r \in Pr(\Omega^r)$ satisfying the assumptions in Definition 6.3. Our aim is to show $P \otimes_r Q^r \in \mathcal{C}_{NS}(x_0)$.

[MP1] $P \otimes_r Q^r[A_t \cap A^t] = \int_{A_t} Q_x^r[A^t]P(dx) = P(A_t) = 1$, since $Q_x^r[A^t] = 1$ holds due to [MP1] for $Q_x^r \in \Phi_t\mathcal{C}_{NS}(x(t))$.

[MP3] Since P agrees with $P \otimes_r Q^r$ on $\mathcal{B}_r^{NS}$, $P \otimes_r Q^r(y : y(0) = x_0) = 1$.

[MP2] and [MP4] can be obtained directly from Lemmas 7.2 and 7.3 and the fact that P agrees with $P \otimes_r Q^r$ on $\mathcal{B}_r^{NS}$. $\square$

So Theorem 7.1 is proved. $\square$

Acknowledgments

We are grateful to Professor Michael Röckner for his valuable discussions when we visited Bielefeld University and Professor Yong Liu for his valuable discussions and the anonymous' referee suggestions.

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