J. Differential Equations 252 (2012) 4842–4861

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

# Traveling wave solutions in partially degenerate cooperative reaction–diffusion systems

Bingtuan Li [1]

*Department of Mathematics, University of Louisville, Louisville, KY 40292, United States*

ARTICLE INFO

*Article history:*
Received 2 May 2011
Revised 17 October 2011
Available online 4 February 2012

*MSC:*
92D40
92D25

*Keywords:*
Reaction–diffusion system
Integral system
Linear determinacy
Spreading speed
Traveling wave solution

ABSTRACT

We study the existence of traveling wave solutions for partially degenerate cooperative reaction–diffusion systems that can have three or more equilibria. We show via integral systems that there exist traveling wave solutions in a partially degenerate reaction–diffusion system with speeds above two well-defined extended real numbers. We prove that the two numbers are the same and may be characterized as the spreading speed as well as the slowest speed of a class of traveling wave solutions provided that the linear determinacy conditions are satisfied. We demonstrate our theoretical results by examining a partially degenerate Lotka–Volterra competition model with advection terms.



## 1. Introduction

Mathematical modelling has long been central to the development of general invasion theory (e.g., [14–17,19]). Systems in the forms of reaction–diffusion equations and integro-difference equations are commonly used to describe biological invasion processes. Studies on existence of traveling waves in such systems have received considerable attention, and many noteworthy findings have come out of this field. Weinberger, Lewis, and Li [23,5,7,24] established spreading speeds and traveling wave solutions for cooperative recursions which include cooperative reaction–diffusion systems and cooperative integro-difference systems as special models. They showed that in a cooperative system with more than two equilibria, different components can spread at different speeds, but if certain linear determinacy conditions are satisfied then all the components spread at the same spreading speed which can be computed through linearization. They also showed that the slowest spreading speed can

*E-mail address:* bing.li@louisville.edu.

[1] This research was partially supported by NSF grant DMS 0616445.

0022-0396/$ – see front matter 
doi:10.1016/j.jde.2012.01.018

be characterized as the slowest speed of a class of traveling wave solutions. The proof of the existence of traveling wave solutions given in [7] depends critically on compactness of the recursion operator. For a reaction–diffusion system, compactness is ensured by positivity of all diffusion coefficients. However, there are many biological reaction–diffusion models where at least one diffusion coefficient is zero; see for example Lewis and Schmitz [6], Hadeler and Lewis [3], and Capasso and Maddalena [1].

Liang et al. [11] introduced the Kuratowski measure of noncompactness to weaken the compactness assumption for (periodic) reaction–diffusion systems, and established the existence of traveling wave solutions if the associated solution maps are $\alpha$-contractions. However it is often difficult to prove that the solution maps of a system with high nonlinearity are $\alpha$-contractions. Fang and Zhao [2] employed the iteration method involving lower and upper solutions to establish the existence of traveling wave solution for a partially degenerate cooperative reaction–diffusion system and provided the conditions that ensure the existence of traveling wave solutions. The iteration method has proven to be useful in establishing traveling wave solutions for population models; see Weinberger [21], Wu and Zou [25], and Volkov and Lui [20] for the development of the method and its applications in different contexts.

In this paper we provide new results on the existence of traveling wave solutions for partially degenerate cooperative reaction–diffusion systems. We first show that a traveling wave solution of a partially degenerate cooperative reaction–diffusion system with an appropriate speed is a fixed point of a compact integral operator. We prove that a traveling wave solution for a partially degenerate reaction–diffusion system can be obtained by taking a limit of a sequence of functions that are the fixed points for integral operators. We show via integral systems that there exist traveling wave solutions in a partially degenerate reaction–diffusion system with speeds above two well-defined extended real numbers. We prove that the two numbers are the same and may be characterized as the spreading speed as well as the slowest speed of a class of traveling wave solutions provided that the linear determinacy conditions given in Weinberger et al. [23] are satisfied. The linear determinacy conditions simplify the conditions given in [2]. The hypothesis that there are only two constant equilibria in reaction–diffusion systems made in both [2] and [11] is dropped in the present paper. As shown in Weinberger et al. [23] and Li et al. [7], the spatial dynamics of a system with three or more equilibria can be very different from those of a system with only two equilibria.

This paper is organized as follows. In Section 2, we present the hypotheses for cooperative reaction–diffusion systems and summarize the results on spreading speeds obtained in [23,7,24]. Section 3 shows that a traveling wave solution of a partially generate reaction–diffusion system is equivalent to a fixed point of a compact integral operator. In Section 4, we define two extended real numbers, and relate them to the speeds of traveling wave solutions. Section 5 is devoted to exploring how the linear determinacy conditions can be used to determine the slowest speed of traveling wave solutions. We demonstrate our theoretical results by examining a partially degenerate Lotka–Volterra competition model in Section 6. Some concluding remarks are provided in Section 7.

## 2. Hypotheses and spreading speeds

We study the existence of traveling wave solutions for the reaction–diffusion system

$$\frac{\partial \mathbf{u}}{\partial t} = D\frac{\partial^2 \mathbf{u}}{\partial x^2} - E\frac{\partial \mathbf{u}}{\partial x} + \mathbf{f}\big(\mathbf{u}(t,x)\big), \tag{2.1}$$

where the vector-valued function $\mathbf{u}(t,x) = (u_1(t,x), u_2(t,x), \ldots, u_k(t,x))$ represents densities of the populations of $k$ species or classes at the point $x$ and the time $t$, $D = \operatorname{diag}(d_1, \ldots, d_k)$ and $E = \operatorname{diag}(e_1, \ldots, e_k)$ are constant diagonal matrices, $D$ has nonnegative but not necessarily positive diagonal entries, and $\mathbf{f}(\mathbf{u}) = (f_1, f_2, \ldots, f_k)$ is independent of $x$ and $t$.

We introduce some notation. We shall use boldface Roman symbols like $\mathbf{u}(x)$ to denote $k$-vector-valued functions of $x$, and boldface Greek letters like $\boldsymbol{\alpha}$ to stand for $k$-vectors, which may be thought of as constant vector-valued functions. We define $\mathbf{u}(x) \geqslant \mathbf{v}(x)$ to mean that $u_i(x) \geqslant v_i(x)$ for all $i$ and $x$, and $\mathbf{u}(x) \gg \mathbf{v}(x)$ to mean that $u_i(x) > v_i(x)$ for all $i$ and $x$. We also define $\max\{\mathbf{u}(x), \mathbf{v}(x)\}$

($\min\{\mathbf{u}(x), \mathbf{v}(x)\}$) to mean the vector-valued function whose $i$th component at $x$ is $\max\{u_i(x), v_i(x)\}$ ($\min\{u_i(x), v_i(x)\}$). We use $|\cdot|$ to denote the Euclidean norm. We use the notation $\mathbf{0}$ for the constant vector all of whose components are 0. We shall also use the notation

$$\mathcal{C}_{\boldsymbol{\alpha}} := \big\{\mathbf{u}\colon\ \mathbf{u}(x) \text{ is continuous, and } \mathbf{0} \leqslant \mathbf{u}(x) \leqslant \boldsymbol{\alpha} \text{ for all } x\big\}.$$

We shall make the following hypotheses about the system (2.1).

**Hypotheses 2.1.**

i. *There is a proper subset $\Sigma_0$ of $\{1, \dots, k\}$ such that $d_i = 0$ for $i \in \Sigma_0$ and $d_i > 0$ for $i \notin \Sigma_0$.*
ii. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, *there is a constant $\boldsymbol{\beta} \gg \mathbf{0}$ such that $\mathbf{f}(\boldsymbol{\beta}) = \mathbf{0}$ which is minimal in the sense that there are no constant $\boldsymbol{\nu}$ other than $\boldsymbol{\beta}$ such that $\mathbf{f}(\boldsymbol{\nu}) = \mathbf{0}$ and $\mathbf{0} \ll \boldsymbol{\nu} \leqslant \boldsymbol{\beta}$, and the equation $\mathbf{f}(\boldsymbol{\alpha}) = \mathbf{0}$ has a finite number of constant roots.*
iii. *The system is cooperative*; *i.e., $f_i(\boldsymbol{\alpha})$ is nondecreasing in all components of $\boldsymbol{\alpha}$ with the possible exception of the $i$th one.*
iv. $\mathbf{f}(\boldsymbol{\alpha})$ *is uniformly Lipschitz continuous in $\boldsymbol{\alpha}$ so that there is $\rho > 0$ such that for any $\boldsymbol{\alpha}_i \geqslant \mathbf{0}$, $i = 1, 2$, $|\mathbf{f}(\boldsymbol{\alpha}_1) - \mathbf{f}(\boldsymbol{\alpha}_2)| \leqslant \rho|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2|$.*
v. $\mathbf{f}$ *has the Jacobian $\mathbf{f}'(\mathbf{0})$ at $\mathbf{0}$ with the property that $\mathbf{f}'(\mathbf{0})$ has a positive eigenvalue whose eigenvector has positive components.*

Hypothesis 2.1.i assumes that there is at least one zero diffusion coefficient in (2.1). Hypotheses 2.1.ii–v are essentially the same as those given in Theorem 4.1 in Li et al. [7].

We first recall the framework developed in Weinberger et al. [23] in establishing spreading speeds for (2.1). Let $Q$ denote the time one solution map of (2.1). A result of Szarski (Theorem 65.1 of [18]) shows that $Q$ is order-preserving in $\mathcal{C}_{\boldsymbol{\beta}}$ in the sense that for $\mathbf{u}, \mathbf{v} \in \mathcal{C}_{\boldsymbol{\beta}}$ if $\mathbf{u} \geqslant \mathbf{v}$ then $Q[\mathbf{u}] \geqslant Q[\mathbf{v}]$. Define the sequence $\mathbf{a}_n(c; x)$ by the recursion

$$\mathbf{a}_{n+1}(c; x) = \max\big\{\boldsymbol{\phi}(x), \big[Q\left(\mathbf{a}_n(c; \cdot)\right)\big](x + c)\big\} \tag{2.2}$$

where $\mathbf{a}_0(c; x) = \boldsymbol{\phi}(x)$, and $\boldsymbol{\phi}(x)$ is any nonincreasing continuous function with $\boldsymbol{\phi}(x) = \mathbf{0}$ for $x \geqslant 0$ and $0 \ll \boldsymbol{\phi}(-\infty) \ll \boldsymbol{\beta}$. By definition $\mathbf{a}_0 \leqslant \mathbf{a}_1$, and an induction argument shows that for all $n$, $\mathbf{a}_n \leqslant \mathbf{a}_{n+1} \leqslant \boldsymbol{\beta}$, and $\mathbf{a}_n(c; x)$ is nonincreasing in $c$ and $x$. Thus the sequence $\mathbf{a}_n$ increases to a limit function $\mathbf{a}(c; x)$ that is again nondecreasing in $c$ and $x$ and bounded by $\boldsymbol{\beta}$. The results from Lui [13] show that $\mathbf{a}(c; -\infty) = \boldsymbol{\beta}$, and that the constant vector $\mathbf{a}(c; \infty)$ is a fixed point of $Q$, which is nondecreasing in $c$ and independent of the choice of $\boldsymbol{\phi}$. Define

$$c^* := \sup\big\{c;\ \mathbf{a}(c; \infty) = \boldsymbol{\beta}\big\}, \tag{2.3}$$

and

$$c^*_+ := \sup\big\{c;\ \mathbf{a}(c; \infty) \neq \mathbf{0}\big\}. \tag{2.4}$$

Clearly $c^*_+ \geqslant c^*$. It was shown in [23] that $c^*$ is the slowest spreading speed and $c^*_+$ is an upper bound for all the spreading speeds for (2.1). In the case that there are only two equilibria $\mathbf{0}$ and $\boldsymbol{\beta}$, $c^*_+ = c^*$ so that all the components spread at the same speed $c^*$. The fastest spreading speed $c^*_f$ was defined in Li et al. [7]. Theorem 4.2 in [23] provides the linear determinacy conditions under which $c^*_+ = c^*_f = c^* = \bar{c}$ where $\bar{c}$ is the spreading speed of the linearized system. In [23], the reflection invariance (i.e., $e_i = 0$ for all $i$) was assumed, but it was not used in Theorem 4.2. Consequently Theorem 4.2 in [23] still works for the reaction–advection–diffusion system (2.1).

The existence of traveling wave solutions of (2.1) was studied in Li et al. [7]. The authors showed that $c^*$ can be characterized as the slowest speed of a class of traveling wave solutions when all the

$d_i$ are positive. The assumption $d_i > 0$ for all $i$ implies that the time one solution map $Q$ is compact, which is used in the proof of the existence of traveling wave solutions in [7]. Consequently, the results on traveling wave solutions given in [7] do not apply to (2.1) when at least one of the $d_i$ is zero.

## 3. Integral systems

We use $\mathbf{w}(x-ct)$ to denote a nonincreasing traveling wave solution of (2.1) with speed $c$ connecting two different constant equilibria $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ with $\boldsymbol{\nu}_1 \geqslant \boldsymbol{\nu}_2$. It satisfies

$$-c\mathbf{w}' = D\mathbf{w}'' - E\mathbf{w}' + \mathbf{f}(\mathbf{w}) \tag{3.1}$$

and

$$\mathbf{w}(-\infty) = \boldsymbol{\nu}_1, \qquad \mathbf{w}(\infty) = \boldsymbol{\nu}_2.$$

An important observation from (3.1) is that

$$\lim_{x\to\infty} \mathbf{w}'(x) = \lim_{x\to-\infty} \mathbf{w}'(x) = \mathbf{0}.$$

This can be easily shown by using the so-called fluctuation lemma that can be found in [4].

Choose $\kappa > \rho$ where $\rho$ is given in Hypothesis 2.1.iv. Define

$$\mathbf{H}(\mathbf{u}) = \big(\mathbf{f}(\mathbf{u}) + \kappa\mathbf{u}\big)/\kappa. \tag{3.2}$$

Clearly $f(\boldsymbol{\alpha}) = 0$ if and only if $H(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$. It follows from Hypothesis 2.1.iv that

$$\mathbf{H}(\mathbf{u}) - \mathbf{H}(\mathbf{v}) = (1/\kappa)\big[\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v}) + \kappa(\mathbf{u}-\mathbf{v})\big] \geqslant \frac{\kappa-\rho}{\kappa}(\mathbf{u}-\mathbf{v}) \geqslant \mathbf{0} \tag{3.3}$$

for $0 \leqslant \mathbf{v} \leqslant \mathbf{u} \leqslant \boldsymbol{\beta}$.

For $i \in \Sigma_0$, if $c - e_i > 0$, define

$$(\mathbf{m}_c)_i(x) = \begin{cases} 0 & \text{when } x > 0, \\ \frac{\kappa}{c-e_i} e^{\frac{\kappa}{c-e_i}x} & \text{when } x \leqslant 0, \end{cases}$$

and if $c - e_i < 0$, define

$$(\mathbf{m}_c)_i(x) = \begin{cases} \frac{\kappa}{e_i-c} e^{\frac{\kappa}{c-e_i}x} & \text{when } x \geqslant 0, \\ 0 & \text{when } x < 0. \end{cases}$$

For $i \notin \Sigma_0$, define

$$(\mathbf{m}_c)_i(x) = \frac{\kappa}{d_i(\lambda_{i1} - \lambda_{i2})} \begin{cases} e^{-\lambda_{i1}x} & \text{when } x \geqslant 0, \\ e^{-\lambda_{i2}x} & \text{when } x < 0, \end{cases} \tag{3.4}$$

where

$$\lambda_{i1} = \frac{(c-e_i) + \sqrt{(c-e_i)^2 + 4\kappa d_i}}{2d_i} > 0, \qquad \lambda_{i2} = \frac{(c-e_i) - \sqrt{(c-e_i)^2 + 4\kappa d_i}}{2d_i} < 0. \tag{3.5}$$

$\lambda_{i1}$ and $\lambda_{i2}$ are the two solutions of the equation

$$d_i z^2 - (c - e_i)z - \kappa = 0.$$

Wu and Zou [25] used $(\mathbf{m}_c)_i$ defined above and studied traveling wave solutions for delayed reaction–diffusion systems with $d_i > 0$ and $e_i = 0$ for all $i$. Fang and Zhao [2] introduced the functions similar to $(\mathbf{m}_c)_i$ for $d_i = 0$ and investigated the existence of traveling wave solutions for (2.1) with $e_i = 0$ for all $i$. The authors found lower and upper solutions via differential-integral inequalities, and used the iteration method to establish the existence of traveling wave solutions.

One can further verify that each $m_c^i(x)$ defined above has the properties that $m_c^i(x) \geqslant 0$, $m_c^i(x)$ is bounded, and $\int_{-\infty}^{+\infty} m_c^i(x)\,dx = 1$, so that $m_c^i$ represents a probability density function. Let

$$\mathbf{m}_c(x) = \operatorname{diag}\big((\mathbf{m}_c)_1(x), \ldots, (\mathbf{m}_c)_k(x)\big).$$

We have that

$$\int_{-\infty}^{\infty} \mathbf{m}_c(x)\,dx = \mathbf{I}.$$

Define

$$T_c[\mathbf{u}](x) = \int_{-\infty}^{\infty} \mathbf{m}_c(x-y)\mathbf{H}(\mathbf{u})(y)\,dy. \tag{3.6}$$

We have the following important result.

**Theorem 3.1.** *Assume that $d_i \geqslant 0$ for all $i$ and that Hypotheses* 2.1.ii–v *are satisfied. Let $c \neq e_i$ for all $i$ with $d_i = 0$. Then $\mathbf{w}(x - ct)$ is a nonincreasing traveling wave solution of* (2.1) *connecting two different constant equilibria $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ if and only if $\mathbf{w}$ is a continuous nonincreasing function satisfying*

$$\mathbf{w}(x) = T_c[\mathbf{w}](x) \tag{3.7}$$

*and connecting $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$.*

**Proof.** Assume that $\mathbf{w}(x-ct)$ is a nonincreasing traveling wave solution of (2.1) connecting $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$. $\mathbf{w}$ satisfies the wave equation (3.1). If $d_i = 0$, the $i$th equation in (3.1) is given by

$$(c - e_i)w_i' - \kappa w_i = -\big(f_i(\mathbf{w}) + \kappa w_i\big). \tag{3.8}$$

We first consider the case of $c - e_i > 0$. We view the right-hand side of Eq. (3.8) as a nonhomogeneous term and solve the differential equation to obtain

$$w_i(x) = w_i(x_0)e^{\frac{\kappa}{c-e_i}(x-x_0)} + \frac{\kappa}{(c-e_i)} \int_x^{x_0} e^{\frac{\kappa}{c-e_i}(x-y)} H_i(\mathbf{w})(y)\,dy \tag{3.9}$$

where $x_0$ is any real number. Using the fact that $\int_x^{\infty} e^{-\frac{\kappa}{c-e_i}y} H_i(\mathbf{w})(y)\,dy$ is convergent, $w_i(x)$ is bounded and $\lim_{x_0\to\infty} e^{\frac{-\kappa}{c-e_i}x_0} = 0$, we take the limit $x_0 \to \infty$ in (3.9) to obtain

$$w_i(x) = \frac{\kappa}{c - e_i} \int_x^\infty e^{(c-e_i)(x-y)} H_i(\mathbf{w})(y)\,dy$$

which is equivalent to

$$w_i(x) = \int_{-\infty}^{\infty} (\mathbf{m}_c)_i(x-y) H_i(\mathbf{w})(y)\,dy.$$

Similarly we can show that this equation holds in the case of $c - e_i < 0$.

If $d_i > 0$, the $i$th equation in (3.1) is given by

$$d_i w_i'' + (c - e_i) w_i' - \kappa w_i = -\big(f_i(\mathbf{w}) + \kappa w_i\big). \tag{3.10}$$

Again we view the right-hand side of (3.10) as a nonhomogeneous term. Then $w_i$ is given by

$$\begin{aligned} w_i(x) = {} & -\frac{\lambda_{i2} w_i(x_0) + w_i'(x_0)}{\lambda_{i1} - \lambda_{i2}} e^{\lambda_{i1}(x_0 - x)} + \frac{\lambda_{i1} w_i(x_0) + w_i'(x_0)}{\lambda_{i1} - \lambda_{i2}} e^{\lambda_{i2}(x_0 - x)} \\ & + \frac{\kappa}{d_i(\lambda_{i1} - \lambda_{i2})} \int_{x_0}^{x} e^{-\lambda_{i1}(x-y)} H_i(\mathbf{w})(y)\,dy \\ & + \frac{\kappa}{d_i(\lambda_{i1} - \lambda_{i2})} \int_{x}^{x_0} e^{-\lambda_{i2}(x-y)} H_i(\mathbf{w})(y)\,dy \end{aligned} \tag{3.11}$$

where $x_0$ is any real number.

We multiply Eq. (3.10) by the factor $e^{\lambda_{i1} x}$ and then use integration by parts to obtain that

$$\begin{aligned} & \big[d_i w_i'(x_0) + (c - e_i - d_i\lambda_{i1}) w_i(x_0)\big] e^{\lambda_{i1} x_0} - \big[d_i w_i'(-x_0) + (c - e_i - d_i\lambda_{i1}) w(-x_0)\big] e^{-\lambda_{i1} x_0} \\ & \quad + \int_{-x_0}^{x_0} \big[d_i \lambda_1^2 - (c - e_i)\lambda_1 - \kappa\big] w_i(y) e^{\lambda_{i1} y}\,dy + \kappa \int_{-x_0}^{x_0} e^{\lambda_{i1} y} H_i(\mathbf{w})(y)\,dy = 0. \end{aligned} \tag{3.12}$$

Since $c - e_i - d_i\lambda_{i1} = d_i\lambda_{i2}$ and $d\lambda_{i1}^2 - (c - e_i)\lambda_{i1} - \kappa = 0$, it follows from (3.12) that

$$\begin{aligned} & \big[d_i w_i'(x_0) + d_i\lambda_{i2} w_i(x_0)\big] e^{\lambda_{i1} x_0} - \big[d_i w_i'(-x_0) + d_i\lambda_{i2} w_i(-x_0)\big] e^{-\lambda_{i1} x_0} \\ & \quad + \kappa \int_{-x_0}^{x_0} e^{\lambda_{i1} y} H_i(\mathbf{w})(y)\,dy = 0. \end{aligned} \tag{3.13}$$

Since $\lambda_{i1} > 0$ the second term in (3.13) approaches zero as $x_0 \to \infty$. Note that both $w_i(x_0)$ and $w_i'(x_0)$ are bounded. We therefore have that

$$\lim_{x_0 \to \infty} \left\{ \big[d_i w_i'(x_0) + d_i\lambda_{2i} w_i(x_0)\big] e^{\lambda_{i1} x_0} + \kappa \int_{-x_0}^{x_0} e^{\lambda_{i1} y} H_i(\mathbf{w})(y)\,dy \right\} = 0. \tag{3.14}$$

We rewrite (3.11) as

$$\begin{aligned} w_i(x) &= \frac{\lambda_{i1} w_i(x_0) + w_i'(x_0)}{\lambda_{i1} - \lambda_{i2}} e^{\lambda_{i2}(x_0 - x)} \\ &\quad - \frac{1}{(\lambda_{i1} - \lambda_{i2})} \left\{ \left[ w_i'(x_0) + \lambda_{2i} w_i(x_0) \right] e^{\lambda_{i1} x_0} + \frac{\kappa}{d_i} \int_{-x_0}^{x_0} e^{\lambda_{i1} y} H_i(\mathbf{w})(y)\, dy \right\} e^{-\lambda_{i1} x} \\ &\quad + \frac{k}{d_i(\lambda_{i1} - \lambda_{i2})} \left[ \int_{-x_0}^{x} e^{-\lambda_{i1}(x-y)} H_i(\mathbf{w})(y)\, dy + \int_{x}^{x_0} e^{-\lambda_{i2}(x-y)} H_i(\mathbf{w})(y)\, dy \right]. \end{aligned} \tag{3.15}$$

Note that the first term on the right-hand side of (3.15) approaches 0 as $x_0 \to \infty$. Letting $x_0 \to \infty$ in (3.15) and using (3.14), we obtain that

$$\begin{aligned} w_i(x) &= \frac{\kappa}{d_i(\lambda_{i1} - \lambda_{i2})} \left[ \int_{x}^{\infty} e^{-\lambda_{i2}(x-y)} H_i(\mathbf{w})(y)\, dy + \int_{-\infty}^{x} e^{-\lambda_{i1}(x-y)} H_i(\mathbf{w})(y)\, dy \right] \\ &= \int_{-\infty}^{\infty} (\mathbf{m}_c)_i (x-y) H_i(\mathbf{w})(y)\, dy. \end{aligned}$$

We have shown that $\mathbf{w}$ satisfies (3.7).

We now show that if a nonincreasing continuous function $\mathbf{w}$ satisfies (3.7) and connects two different equilibria $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ then it is a traveling wave solution of (2.1). The definition of $T_c$ and continuity of $\mathbf{w}$ show that $\mathbf{w}$ is differentiable. Direct calculations show that for $d_i = 0$ and $c - e_i > 0$

$$\big(w_i(x)\big)' = \frac{d}{dx} \int_{x}^{\infty} \big(\kappa/(c-e_i)\big) e^{(\kappa/(c-e_i))(x-y)} H_i\big(\mathbf{w}(y)\big)\, dy = \big(\kappa/(c-e_i)\big)\big[w_i(x) - H_i\big(\mathbf{w}(x)\big)\big],$$

so that

$$-c(w_i)' = -e_i(w_i)' + f_i(\mathbf{w}). \tag{3.16}$$

Similarly one can show that (3.16) holds for $d_i = 0$ and $c - e_i < 0$.

If $d_i > 0$

$$\begin{aligned} \big(w_i(x)\big)' &= \frac{\kappa}{d_i(\lambda_{i1} - \lambda_{i2})} \frac{d}{dx} \left( \int_{x}^{\infty} e^{-\lambda_{i2}(x-y)} H_i(\mathbf{w})(y)\, dy + \int_{-\infty}^{x} e^{-\lambda_{i1}(x-y)} H_i(\mathbf{w})(y)\, dy \right) \\ &= \frac{\kappa}{d_i(\lambda_{i1} - \lambda_{i2})} \left( -\lambda_{i2} \int_{x}^{\infty} e^{-\lambda_{i2}(x-y)} H_i(\mathbf{w})(y)\, dy - \lambda_{i1} \int_{-\infty}^{x} e^{-\lambda_{i1}(x-y)} H_i(\mathbf{w})(y)\, dy \right). \end{aligned} \tag{3.17}$$

From this we find that

$$\big(w_i(x)\big)'' = \frac{\kappa}{d_i(\lambda_{i1} - \lambda_{i2})}\Bigg[(\lambda_{i2} - \lambda_{i1})H_i(\mathbf{w})(y) + \lambda_{i2}^2 \int_x^\infty e^{-\lambda_{i2}(x-y)} H_i(\mathbf{w})(y)\,dy$$
$$+ \lambda_{i1}^2 \int_{-\infty}^x e^{-\lambda_{i1}(x-y)} H_i(\mathbf{w})(y)\,dy\Bigg].$$

Using this, (3.2), (3.17), and (3.5), we obtain that for any $i$ with $d_i > 0$,

$$-c(w_i)' = d_i(w_i)'' - e_i(w_i)' + f_i(\mathbf{w}).$$

It follows from this and (3.16) that $\mathbf{w}$ satisfies (3.1) so that $\mathbf{w}$ is a traveling wave solution of (2.1). The proof of the theorem is complete. $\square$

Theorem 3.1 shows that $\mathbf{w}(x - ct)$ with $c \neq e_i$ for $i \in \Sigma_0$ is a traveling wave solution of (2.1) if and only if it is a fixed point of $T_c$.

## 4. Existence of traveling wave solutions

Define

$$D^{(\ell)} = D + (1/\ell)\mathbf{I}$$

with $\ell \geqslant 1$ and $\mathbf{I}$ the identity matrix. Clearly, as $\ell \to \infty$, $D^{(\ell)}$ approaches $D$. $D^{(\ell)}$ is a diagonal matrix with positive diagonal entries. Consequently, the solution map operators for

$$\frac{\partial \mathbf{u}}{\partial t} = D^{(\ell)} \frac{\partial^2 \mathbf{u}}{\partial x^2} - E \frac{\partial \mathbf{u}}{\partial x} + \mathbf{f}\big(\mathbf{u}(t,x)\big) \tag{4.1}$$

are compact, and the results on the existence of traveling wave solutions given in [7] apply to (4.1).

**Lemma 4.1.** *Assume that $\mathbf{w}^{(\ell)}(x - ct)$ is a nonincreasing traveling wave solution of* (4.1) *with speed $c \neq e_i$ for $i \in \Sigma_0$. Then the family $\mathbf{w}^{(\ell)}$ is an equicontinuous family of functions.*

**Proof.** $D^{(\ell)}$ can be written as $D^{(\ell)} = \mathrm{diag}(d_1^{(\ell)}, d_2^{(\ell)}, \ldots, d_k^{(\ell)})$ where $d_i^{(\ell)} = d_i + 1/\ell$. Theorem 3.1 shows that

$$\mathbf{w}^{(\ell)}(x) = \int_{-\infty}^{\infty} \mathbf{m}_c^{(\ell)}(x-y)\mathbf{H}\big(\mathbf{w}^{(\ell)}\big)(y)\,dy \tag{4.2}$$

where

$$\big(\mathbf{m}_c^{(\ell)}\big)_i(x) = \frac{\kappa}{d_i^{(\ell)}(\lambda_{i1}^{(\ell)} - \lambda_{i2}^{(\ell)})} \begin{cases} e^{-\lambda_{i1}^{(\ell)} x} & \text{when } x \geqslant 0, \\ e^{-\lambda_{i2}^{(\ell)} x} & \text{when } x < 0, \end{cases}$$

with

$$\lambda_{i1}^{(\ell)} = \frac{(c - e_i) + \sqrt{(c-e_i)^2 + 4\kappa d_i^{(\ell)}}}{2d_i^{(\ell)}} > 0, \qquad \lambda_{i2}^{(\ell)} = \frac{(c - e_i) - \sqrt{(c-e_i)^2 + 4\kappa d_i^{(\ell)}}}{2d_i^{(\ell)}} < 0.$$

Define

$$\Delta_i^{(\ell)}(M) := \int_{-\infty}^{-M} \left(\mathbf{m}_c^{(\ell)}\right)_i(x)\,dx + \int_{M}^{\infty} \left(\mathbf{m}_c^{(\ell)}\right)_i(x)\,dx.$$

Direct calculations show that

$$\Delta_i^{(\ell)}(M) = \frac{d_i^{(\ell)}}{\sqrt{(c-e_i)^2 + 4\kappa d_i^{(\ell)}}}\left[\lambda_{i1}^{(\ell)} e^{\lambda_{i2}^{(\ell)} M} - \lambda_{i2}^{(\ell)} e^{-\lambda_{i1}^{(\ell)} M}\right]. \tag{4.3}$$

Let

$$\lambda_{i+}^{(\ell)} = \frac{\sqrt{(c-e_i)^2 + 4\kappa d_i^{(\ell)}} + |c-e_i|}{2d_i^{(\ell)}}$$

and

$$\lambda_{i-}^{(\ell)} = -\frac{\sqrt{(c-e_i)^2 + 4\kappa d_i^{(\ell)}} - |c-e_i|}{2d_i^{(\ell)}}.$$

Then $\lambda_{i+}^{(\ell)} > 0$, $\lambda_{i-}^{(\ell)} < 0$, and furthermore

$$\left|\lambda_{i-}^{(\ell)}\right| \leqslant \lambda_{i1}^{(\ell)}, \qquad \left|\lambda_{i-}^{(\ell)}\right| \leqslant \left|\lambda_{i2}^{(\ell)}\right|,$$

and

$$\lambda_{i+}^{(\ell)} \geqslant \lambda_{i1}^{(\ell)}, \qquad \lambda_{i+}^{(\ell)} \geqslant \left|\lambda_{i2}^{(\ell)}\right|.$$

It follows from this and (4.3) that

$$\begin{aligned}
\Delta_i^{(\ell)}(M) &\leqslant 2\frac{d_i^{(\ell)}}{\sqrt{(c-e_i)^2 + 4\kappa d_i^{(\ell)}}}\lambda_{i+}^{(\ell)} e^{\lambda_{i-}^{(\ell)} M} \\
&= \frac{\sqrt{(c-e_i)^2 + 4\kappa d_i^{(\ell)}} + |c-e_i|}{\sqrt{(c-e_i)^2 + 4\kappa d_i^{(\ell)}}} e^{\lambda_{i-}^{(\ell)} M}.
\end{aligned} \tag{4.4}$$

Let $d_{\min} = \inf\{d_i^{(\ell)},\ i \notin \Sigma_0\}$. Then $d_{\min} > 0$. It follows from (4.4) that for $i \notin \Sigma_0$,

$$\Delta_i^{(\ell)}(M) \leqslant \frac{\sqrt{(c-e)^2 + 4\kappa d_i^{(\ell)}} + |c-e|}{\sqrt{(c-e_i)^2 + 4\kappa d_{\min}}} e^{\lambda_{i-}^{(\ell)} M}. \tag{4.5}$$

For $i \in \Sigma_0$, (4.4) shows that

$$\Delta_i^{(\ell)}(M) \leqslant \frac{\sqrt{(c-e_i)^2 + 4\kappa d_i^{(\ell)}} + |c-e_i|}{|c-e_i|} e^{\lambda_{i-}^{(\ell)} M}. \tag{4.6}$$

Note that

$$\lim_{d_i^{(\ell)} \to 0} \lambda_{i-}^{(\ell)} = -\kappa/|c-e_i| < 0.$$

It follows from this, (4.5), (4.6), the convergence of $d_i^{(\ell)}$ to $d_i$, the continuity of $\lambda_-^{(\ell)}$ in $d_i^{(\ell)}$, and the assumption that $c \neq e_i$ for $i \in \Sigma_0$ that there exist $\delta_1 > 0, \delta_2 > 0$ independent on $\ell \geqslant 1$ and $i$ such that

$$\Delta_i^{(\ell)}(M) \leqslant \delta_1 e^{-\delta_2 M}.$$

We therefore have that for any positive $\epsilon > 0$ there exists $M_\epsilon > 0$ independent on $\ell \geqslant 1$ such that for all $\ell \geqslant 1$ and $i$

$$\int_{-\infty}^{-M_\epsilon} \left(\mathbf{m}_c^{(\ell)}\right)_i(x)\,dx + \int_{M_\epsilon}^{\infty} \left(\mathbf{m}_c^{(\ell)}\right)_i(x)\,dx < \epsilon.$$

An argument similar to what is given on page 331 in Li et al. [8] shows that $\mathbf{w}^{(\ell)}$ forms an equicontinuous family of functions. The proof is complete. □

As for (2.1), one can define a function sequence $\mathbf{a}_n^{(\ell)}(c;x)$ by (2.2) with $Q$ replaced by $Q^{(\ell)}$ where $Q^{(\ell)}$ is the time one solution map of (4.1). Let $\mathbf{a}^{(\ell)}$ denote the limit of $\mathbf{a}_n^{(\ell)}(c;x)$ as $n \to \infty$. Define

$$c(\ell)^* := \sup\left\{c;\ \mathbf{a}^{(\ell)}(c;\infty) = \boldsymbol{\beta}\right\},$$

and

$$c(\ell)_+^* := \sup\left\{c;\ \mathbf{a}^{(\ell)}(c;\infty) \neq \mathbf{0}\right\}.$$

Let

$$\tilde{c}^* = \liminf_{\ell\to\infty} c(\ell)^* \tag{4.7}$$

and

$$\tilde{c}_+^* = \liminf_{\ell\to\infty} c(\ell)_+^*. \tag{4.8}$$

Clearly, both $\tilde{c}^*$ and $\tilde{c}_+^*$ are well-defined extended real numbers with $\tilde{c}_+^* \geqslant \tilde{c}^*$. They can be related to the speeds of traveling wave solutions.

We first show that the slowest speed of a class of traveling wave solutions in (2.1) connecting $\boldsymbol{\beta}$ with an equilibrium other than $\boldsymbol{\beta}$ cannot be bigger than $\tilde{c}^*$ and smaller than $c^*$.

**Theorem 4.1.** *Assume that Hypotheses* 2.1 *are satisfied. Then the following statements are true for the system* (2.1):

i. *for* $c \geqslant \tilde{c}^*$ *and* $c \neq e_i$ *for all* $i \in \Sigma_0$, *there is a nonincreasing traveling wave solution* $\mathbf{w}(x - ct)$ *with* $\mathbf{w}(-\infty) = \boldsymbol{\beta}$ *and* $\mathbf{w}(\infty)$ *an equilibrium other than* $\boldsymbol{\beta}$; *and*

ii. *if there is a nonincreasing traveling wave* $\mathbf{w}(x - ct)$ *with* $\mathbf{w}(-\infty) = \boldsymbol{\beta}$ *and* $\mathbf{w}(\infty)$ *an equilibrium other than* $\boldsymbol{\beta}$, *then* $c \geqslant c^*$.

**Proof.** The proof of the statement ii is similar to the second part of the proof of Theorem 3.1 in [7] and is omitted here.

The definition of $\tilde{c}^*$ shows that there exists a subsequence of $\{\ell\}$ still denoted by $\{\ell\}$ such that

$$\lim_{\ell \to \infty} c(\ell)^*_+ = \tilde{c}^*.$$

Then for $c > \tilde{c}^*$, there exists $N_c > 0$ such that $c > c(\ell)^*_+$ for $\ell \geqslant N_c$. It follows from Theorem 4.1 of [7] that for $c > \tilde{c}^*$ and $\ell \geqslant N_c$ the system (4.1) has a nonincreasing traveling wave solution $\mathbf{w}^{(\ell)}(x - ct)$ with $\mathbf{w}^{(\ell)}(-\infty) = \boldsymbol{\beta}$ and $\mathbf{w}^{(\ell)}(-\infty)$ an equilibrium other than $\boldsymbol{\beta}$. Since $\boldsymbol{\beta}$ is the only equilibrium in the interior of $\mathcal{C}_{\boldsymbol{\beta}}$, we can choose $\eta > 0$ so small that there is no constant equilibrium other than $\boldsymbol{\beta}$ in the set $\{\mathbf{w} \in \mathcal{C}_{\boldsymbol{\beta}}\colon\ |\boldsymbol{\beta} - \mathbf{w}| \leqslant \eta\}$.

Since the continuous function $|\boldsymbol{\beta} - \mathbf{w}^{(\ell)}(x)|$ increases from 0 to a positive number as $x$ increases from $-\infty$ to $\infty$, the intermediate value theorem states that there is a real number at which $|\boldsymbol{\beta} - \mathbf{w}^{(\ell)}| = \eta$. We can assume that the real number is 0 by translating if necessary. We therefore have that

$$\left|\boldsymbol{\beta} - \mathbf{w}^{(\ell)}(0)\right| = \eta.$$

Lemma 4.1 shows that $\mathbf{w}^{(\ell)}$ is an equicontinuous family of functions. Then Ascoli's theorem implies that $\mathbf{w}^{(\ell)}$ has a subsequence $\mathbf{w}^{(\ell_j)}$ such that $\mathbf{w}^{(\ell_j)}(x)$ converges to $\mathbf{w}(x)$ uniformly on every bounded interval. Clearly

$$\left|\boldsymbol{\beta} - \mathbf{w}(0)\right| = \eta. \tag{4.9}$$

We now show that

$$\lim_{\ell \to \infty} \int_{-\infty}^{\infty} \left|\mathbf{m}_c^{(\ell)}(x) - \mathbf{m}_c(x)\right| dx = 0. \tag{4.10}$$

For $i \in \Sigma_0$ and $c - e_i > 0$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left|\left(\mathbf{m}_c^{(\ell)}\right)_i(x) - (\mathbf{m}_c)_i(x)\right| dx \\
&\quad = \int_{0}^{\infty} \left|\left(\mathbf{m}_c^{(\ell)}\right)_i(x) - (\mathbf{m}_c)_i(x)\right| dx + \int_{-\infty}^{0} \left|\left(\mathbf{m}_c^{(\ell)}\right)_i(x) - (\mathbf{m}_c)_i(x)\right| dx \\
&\quad \leqslant \frac{\kappa}{d_i^{(\ell)}(\lambda_{i1}^{(\ell)} - \lambda_{i2}^{(\ell)})} \int_{0}^{\infty} e^{-\lambda_{i1}^{(\ell)} x} dx + \frac{\kappa}{d_i^{(\ell)}(\lambda_{i1}^{(\ell)} - \lambda_{i2}^{(\ell)})} \left| \int_{-\infty}^{0} \left(e^{-\lambda_{i2}^{(\ell)} x} - e^{\kappa/(c - e_i) x}\right) dx \right|
\end{aligned}$$

$$+\left|\frac{\kappa}{d_i^{(\ell)}(\lambda_{i1}^{(\ell)}-\lambda_{i2}^{(\ell)})}-\kappa/(c-e_i)\right|\int\limits_{-\infty}^{0} e^{\kappa/(c-e_i)x}\,dx$$

$$=\frac{\kappa}{d_i^{(\ell)}(\lambda_{i1}^{(\ell)}-\lambda_{i2}^{(\ell)})}\frac{1}{\lambda_{i1}^{(\ell)}}+\frac{\kappa}{d_i^{(\ell)}(\lambda_{i1}^{(\ell)}-\lambda_{i2}^{(\ell)})}\left|\frac{1}{-\lambda_{i2}^{(\ell)}}-(c-e_i)/\kappa\right|$$

$$+(c-e_i)/\kappa\left|\frac{\kappa}{d_i^{(\ell)}(\lambda_{i1}^{(\ell)}-\lambda_{i2}^{(\ell)})}-\kappa/(c-e_i)\right|. \tag{4.11}$$

Here we have used the simple fact that two exponential functions $e^{ax}$ and $e^{bx}$ with $a$ and $b$ positive and different coincide only at $x=0$ so that $\int_{-\infty}^{0}|e^{ax}-e^{bx}|\,dx=|\int_{-\infty}^{0}(e^{ax}-e^{bx})\,dx|$.

Simple calculations show that as $d_i^{(\ell)}\to 0$,

$$\frac{\kappa}{d_i^{(\ell)}(\lambda_{i1}^{(\ell)}-\lambda_{i2}^{(\ell)})}\to\kappa/(c-e_i),\qquad \frac{1}{\lambda_{i1}^{(\ell)}}\to 0,\qquad \frac{1}{-\lambda_{i2}^{(\ell)}}\to(c-e_i)/\kappa.$$

It follows from this and (4.11) that for $i\in\Sigma_0$ and $c-e_i>0$

$$\lim_{\ell\to\infty}\int\limits_{-\infty}^{\infty}\left|\left(\mathbf{m}_c^{(\ell)}\right)_i(x)-(\mathbf{m}_c)_i(x)\right|dx=0.$$

Similarly we can show that this is true for $i\in\Sigma_0$ and $c-e_i<0$ and for $i\notin\Sigma_0$, i.e., $d_i>0$. We shall omit the proofs for these two cases. We conclude that (4.10) holds.

Observe that $\mathbf{w}^{(\ell_j)}$ satisfies

$$\mathbf{w}^{(\ell_j)}(x)=\int\limits_{-\infty}^{\infty}\mathbf{m}^{(\ell_j)}(x-y)\mathbf{H}\big(\mathbf{w}^{(\ell_j)}\big)(y)\,dy \tag{4.12}$$

which can be written as

$$\mathbf{w}^{(\ell_j)}(x)=\int\limits_{-\infty}^{\infty}\mathbf{m}_c(x-y)\mathbf{H}\big(\mathbf{w}^{(\ell_j)}\big)(y)\,dy+\int\limits_{-\infty}^{\infty}\big(\mathbf{m}_c^{(\ell_j)}(y)-\mathbf{m}_c(y)\big)\mathbf{H}\big(\mathbf{w}^{(\ell_j)}\big)(x-y)\,dy. \tag{4.13}$$

Since $\int_{-\infty}^{\infty}\mathbf{m}_c(x-y)\mathbf{H}(\mathbf{u})(y)\,dy$ is continuous in $\mathbf{u}$ and $\mathbf{w}^{(\ell_j)}(x)$ converges to $\mathbf{w}(x)$ uniformly on every bounded interval as $j\to\infty$, $\int_{-\infty}^{\infty}\mathbf{m}_c(x-y)\mathbf{H}(\mathbf{w}^{(\ell_j)})(y)\,dy$ converges to $\int_{-\infty}^{\infty}\mathbf{m}(x-y)\mathbf{H}(\mathbf{w})(y)\,dy$. On the other hand since $\mathbf{H}(\mathbf{w}^{(\ell_j)})(x-y)$ is bounded, (4.10) shows that the second term on the right-hand side of (4.13) converges to zero. We then take limits in (4.13) to obtain

$$\mathbf{w}(x)=\int\limits_{-\infty}^{\infty}\mathbf{m}_c(x-y)\mathbf{H}(\mathbf{w})(y)\,dy \tag{4.14}$$

so that $\mathbf{w}$ is a traveling wave solution of (2.1). $\mathbf{w}^{(\ell_j)}(x)$ are nonincreasing functions so is $\mathbf{w}(x)$. The condition (4.9) and the definition of $\eta$ shows that $\mathbf{w}(-\infty)=\boldsymbol{\beta}$ and $\mathbf{w}(x)$ is not a constant function. By taking $x\to\infty$ in (4.14), we see that $\mathbf{w}(\infty)$ is a constant equilibrium other than $\boldsymbol{\beta}$.

We now show that the existence of a traveling wave solution with speed $\tilde{c}^*$ in the case that $\tilde{c}^* \neq e_i$ for all $i \in \Sigma_0$. In this case, there is a number $r > 0$ such that $|\tilde{c}^* - e_i| > r$ for all $i \in \Sigma_0$. We choose a sequence of numbers $c_n$ such that $\tilde{c}^* + r/2 > c_n > \tilde{c}^*$ and $c_n \to \tilde{c}^*$ as $n \to \infty$. Then there exists a sequence of nonincreasing traveling wave solutions for (2.1) which satisfy

$$\mathbf{w}_{c_n}(x) = \int_{-\infty}^{\infty} \mathbf{m}_{c_n}(x-y)\mathbf{H}\big(\mathbf{w}_{c_n}(y)\big)\,dy, \tag{4.15}$$

with $\mathbf{w}_{c_n}(-\infty) = \boldsymbol{\beta}$ and $\mathbf{w}_{c_n}(-\infty) \neq \boldsymbol{\beta}$. Eq. (4.15) shows that for $\delta > 0$

$$\begin{aligned}\big|\mathbf{w}_{c_n}(x+\delta) - \mathbf{w}_{c_n}(x)\big| &\leqslant \int_{-\infty}^{\infty} \big|\mathbf{m}_{c_n}(x+\delta-y) - \mathbf{m}_{c_n}(x-y)\big|\mathbf{H}\big(\mathbf{w}_{c_n}(y)\big)\,dy \\ &\leqslant |\boldsymbol{\beta}| \int_{-\infty}^{\infty} \big|\mathbf{m}_{c_n}(x+\delta) - \mathbf{m}_{c_n}(x)\big|\,dx. \end{aligned} \tag{4.16}$$

In view of (4.16), the choice of $c_n$, and the definition of $\mathbf{m}_c(x)$, we have that for any $\epsilon > 0$ there exists $\delta_0 > 0$ such that $|\mathbf{w}_{c_n}(x+\delta) - \mathbf{w}_{c_n}(x)| < \epsilon$ whenever $\delta < \delta_0$. This shows that the sequence $\mathbf{w}_{c_n}(x)$ forms an equicontinuous family of functions. We choose a small positive number $\eta$ such that the system (2.1) does not have a constant equilibrium other than $\boldsymbol{\beta}$ in the set $\{\mathbf{u} \in \mathcal{C}_{\boldsymbol{\beta}}\colon\ |\boldsymbol{\beta} - \mathbf{u}| \leqslant \eta\}$. Without loss of generality, we may assume that

$$\big|\boldsymbol{\beta} - \mathbf{w}_{c_n}(0)\big| = \eta.$$

Since $\mathbf{w}_{c_n}(x)$ is an equicontinuous family of nonincreasing functions, there exists a subsequence of $\mathbf{w}_{c_n}(x)$ still denoted by $\mathbf{w}_{c_n}(x)$ such that $\mathbf{w}_{c_n}(x)$ converges to a nondecreasing continuous function $\mathbf{w}(x)$ uniformly on every bounded interval. We can then take limits in (4.15) to see that

$$\mathbf{w}(x) = \int_{-\infty}^{\infty} \mathbf{m}_{\tilde{c}^*}(x-y)H\big(\mathbf{w}(y)\big)\,dy,$$

and $|\boldsymbol{\beta} - \mathbf{w}(0)| = \eta$. It follows that $\mathbf{w}$ is a traveling wave solution of (2.1) with speed $\tilde{c}^*$, $\mathbf{w}(-\infty) = \boldsymbol{\beta}$, and $\mathbf{w}(\infty)$ an equilibrium of (2.1) other than $\boldsymbol{\beta}$. This completes the proof of the theorem. $\square$

We next show that the slowest speed of a class of traveling wave solutions connecting $\mathbf{0}$ with an equilibrium other than $\mathbf{0}$ cannot be bigger than $\tilde{c}_+^*$, and the slowest speed of a subclass of traveling wave solutions connecting $\mathbf{0}$ with $\boldsymbol{\beta}$ cannot be smaller than $c_+^*$.

**Theorem 4.2.** *Assume that Hypotheses* 2.1 *are satisfied. Then the following statements are true for the system* (2.1):

i. *for $c \geqslant \tilde{c}_+^*$ and $c \neq e_i$ for $i \in \Sigma_0$, there is a nonincreasing traveling wave solution $\mathbf{w}(x-ct)$ with $\mathbf{w}(\infty) = \mathbf{0}$ and $\mathbf{w}(-\infty)$ an equilibrium other than $\mathbf{0}$*; *and*
ii. *if there is a nonincreasing traveling wave $\mathbf{w}(x-ct)$ with $\mathbf{w}(\infty) = \mathbf{0}$ and $\mathbf{w}(-\infty) = \boldsymbol{\beta}$, then $c \geqslant c_+^*$.*

**Proof.** We can modify the proof of Theorem 3.1 in [7] by replacing

$$\big|\boldsymbol{\beta} - \mathbf{a}\big(c;\kappa;\ell(\kappa)\big)\big| = \eta$$

by

$$\left|\mathbf{a}\big(c;\kappa;\ell(\kappa)\big)\right| = \eta$$

and by assuming that there is no constant equilibrium other than $\mathbf{0}$ in the set $\{\mathbf{w} \in \mathcal{C}_{\boldsymbol{\beta}}\colon\ |\mathbf{w}| \leqslant \eta\}$. One can then use the arguments similar to what in the proofs of Theorem 3.1, Theorem 4.1 and Theorem 4.2 in [7] and in Theorem 4.1 to show that the statement i holds. We shall omit the details here.

The proof of the statement ii is similar to the second part of the proof of Theorem 3.1 in [7] and is omitted. The proof is complete. □

Theorem 4.1 and Theorem 4.2 show that in general there exist traveling wave solutions with speeds above certain numbers in a partially degenerate cooperative reaction–diffusion system.

## 5. Linear determinacy

An interesting question is how $\tilde{c}^*$ is related to $c^*$, and how $\tilde{c}^*_+$ is related to $c^*_+$. In this section, we shall show that when the linear determinacy conditions given in Weinberger et al. [23] are satisfied by (2.1), $\tilde{c}^* = \tilde{c}^*_+ = c^* = c^*_+$ and they are all equal to the unique spreading speed of (2.1) for which a formula can be found.

We need the following hypotheses.

**Hypotheses 5.1.**

i. *The matrix* $\mathbf{f}'(\mathbf{0})$ *is in Frobenius normal form, so that the same is true of*

$$C_\mu = \mu^2 D + \mu E + \mathbf{f}'(\mathbf{0}).$$

*There is a positive entry to the left of each of the irreducible diagonal blocks other than the first (uppermost) one. The blocks are ordered starting at the uppermost block.*

ii. *Let* $\gamma_\sigma(\mu)$ *be the principal eigenvalue of the* $\sigma$*th irreducible diagonal block of* $C_\mu$ *such that*
   a. $\gamma_1(0) > 1$; *and*
   b. $\gamma_1(0) > \gamma_\sigma(0)$ *for all* $\sigma > 0$.

iii. *Let* $\boldsymbol{\xi}(\mu)$ *be the eigenvector of* $C_\mu$ *which corresponds to* $\lambda_1(\mu)$. *The infimum*

$$\bar{c} := \inf_{\mu>0}(1/\mu)\gamma_1(\mu) \tag{5.1}$$

*is attained at an extended positive value* $\bar{\mu}$ *of* $\mu$. *Either*

(a) $\bar{\mu}$ *is finite*

$$\gamma_1(\bar{\mu}) > \gamma_\sigma(\bar{\mu}), \tag{5.2}$$

*and*

$$\mathbf{f}\big(\min\big\{\tau\boldsymbol{\xi}(\bar{\mu}), \boldsymbol{\beta}\big\}\big) - \mathbf{f}'(0)\tau\boldsymbol{\xi}(\bar{\mu}) \leqslant \mathbf{0} \tag{5.3}$$

*for all positive* $\tau$;
*or*

(b) *there is a sequence* $\mu_\nu \nearrow \bar{\mu}$ *such that for each* $\nu$ *the inequalities* (5.2) *and* (5.3) *with* $\bar{\mu}$ *replaced by* $\mu_\nu$ *are valid.*

These hypotheses are a proper subset of a variant of Hypotheses 4.1 given in Weinberger et al. [23]. As shown in [23], Hypotheses 5.1 provide the linear determinacy conditions, which together with Hypotheses 2.1 guarantee that $c^*$ and $c_+^*$ are the same and equal to the spreading speed of the linearized system of (2.1). Here we have dropped the hypothesis in [23] that $e_i = 0$ for all $i$, i.e., the time one solution operator $Q$ is reflection invariant. In [23] the reflection invariance was assumed, but it was not used in the proof of Theorem 4.2. Consequently Theorem 4.2 in [23] is still valid without the reflection invariance assumption.

**Lemma 5.1.** *Assume that Hypotheses* 2.1 *and Hypotheses* 5.1 *are satisfied. Then*

$$c^* = c_+^* = \tilde{c}^* = \tilde{c}_+^* = \bar{c}$$

*where $\bar{c}$ is given by* (5.1)*, and $\bar{c}$ represents the unique spreading speed of* (2.1)*.*

**Proof.** It follows immediately from Theorem 4.2 in [23] that

$$c^* = c_+^* = \bar{c}$$

and $\bar{c}$ represents the unique spreading speed of (2.1).

For system (4.1)

$$C_\mu^{(\ell)} = \mu^2 D^{(\ell)} + \mu E + \mathbf{f}'(\mathbf{0})$$

which can be written as

$$C_\mu^{(\ell)} = C_\mu + \left(\mu^2/\ell\right)\mathbf{I}.$$

Let $\gamma_1^{(\ell)}(\mu)$ be the principal eigenvalue of $C_\mu^{(\ell)}$. Clearly

$$\gamma_1^{(\ell)}(\mu) = \gamma_1(\mu) + \mu^2/\ell.$$

It is easily seen that the principal eigenvector $\boldsymbol{\xi}(\mu)$ of the matrix $C_\mu$ is also the principal eigenvector of the matrix $C_\mu^{(\ell)}$. We apply Theorem 4.2 in [23] to (4.1) and find that

$$c^*(\ell) = c^*(\ell)_+ = \inf_{\mu>0}(1/\mu)\left(\gamma_1(\mu) + \mu^2/\ell\right).$$

As $\ell \to \infty$, $(1/\mu)(\gamma_1(\mu) + \mu^2/\ell)$ decreases to $(1/\mu)\gamma_1(\mu)$ uniformly on every bounded interval in the form $[a, b]$ with $0 < a < b$. It follows that

$$\tilde{c}_+^* = \tilde{c}^* = \liminf_{\ell\to\infty} \inf_{\mu>0}(1/\mu)\left(\gamma_1(\mu) + \mu^2/\ell\right) = \inf_{\mu>0}(1/\mu)\gamma_1(\mu) = \bar{c}.$$

The proof is complete. □

By using Theorem 4.1 and Lemma 5.1, we obtain the following results.

**Theorem 5.1.** *Assume that Hypotheses* 2.1 *and Hypotheses* 5.1 *are satisfied. Then the following statements are true for the system* (2.1)*:*

i. *for* $c \geqslant \bar{c}$ *and* $c \neq e_i$ *for* $i \in \Sigma_0$, *there is a nonincreasing traveling wave solution* $\mathbf{w}(x - ct)$ *with* $\mathbf{w}(-\infty) = \boldsymbol{\beta}$ *and* $\mathbf{w}(\infty)$ *an equilibrium other than* $\boldsymbol{\beta}$; *and*
ii. *if there is a nonincreasing traveling wave* $\mathbf{w}(x - ct)$ *with* $\mathbf{w}(-\infty) = \boldsymbol{\beta}$ *and* $\mathbf{w}(\infty)$ *an equilibrium other than* $\boldsymbol{\beta}$, *then* $c \geqslant \bar{c}$.

This theorem characterizes $\bar{c}$ as the slowest speed of a class of traveling wave solutions under appropriate assumptions.

The following theorem, obtained from Theorem 4.2 and Lemma 5.1, shows that the slowest speed of a class of traveling wave solutions with $\mathbf{0}$ at $\infty$ cannot be bigger than $\bar{c}$, and the slowest speed of a subclass of traveling wave solutions connecting $\mathbf{0}$ with $\boldsymbol{\beta}$ cannot be smaller than $\bar{c}$.

**Theorem 5.2.** *Assume that Hypotheses* 2.1 *and Hypotheses* 5.1 *are satisfied. Then the following statements are true for the system* (2.1):

i. *for* $c \geqslant \bar{c}$ *and* $c \neq e_i$ *for* $i \in \Sigma_0$, *there is a nonincreasing traveling wave solution* $\mathbf{w}(x - ct)$ *with* $\mathbf{w}(\infty) = \mathbf{0}$ *and* $\mathbf{w}(-\infty)$ *an equilibrium other than* $\mathbf{0}$; *and*
ii. *if there is a nonincreasing traveling wave* $\mathbf{w}(x - ct)$ *with* $\mathbf{w}(\infty) = \mathbf{0}$ *and* $\mathbf{w}(-\infty) = \boldsymbol{\beta}$, *then* $c \geqslant \bar{c}$.

Hypotheses 5.1 represent a simplification over the hypotheses made in [2] where (5.3) is required to hold with $\bar{\mu}$ replaced by $\mu$ for all $0 < \mu \leqslant \bar{\mu}$ if $\bar{\mu}$ is finite. Note that we have dropped the hypotheses made in [2] that the system (2.1) has only two equilibria, that $C_0$ is irreducible, and that $e_i = 0$ for all $i$.

## 6. Applications to a Lotka–Volterra competition model

We consider the Lotka–Volterra two-species competition model system

$$\begin{aligned}
\frac{\partial p}{\partial t} &= d_1 \frac{\partial^2 p}{\partial x^2} - e_1 \frac{\partial p}{\partial x} + r_1 p(1 - p - a_1 q), \\
\frac{\partial q}{\partial t} &= -e_2 \frac{\partial q}{\partial x} + r_2 q(1 - q - a_2 p),
\end{aligned} \tag{6.1}$$

where $p(t,x)$ and $q(t,x)$ are densities of two competing species, $e_1$ and $e_2$ are real numbers, and other parameters are positive numbers. Note that the diffusion coefficient of the species $q$ is 0. This system has, in general, four constant equilibria: The unpopulated state $(0,0)$; the $p$ species mono-culture state $(1,0)$; the $q$ species mono-culture state $(0,1)$; and the coexistence state $(p^*, q^*)$ where

$$p^* = \frac{1 - a_1}{1 - a_1 a_2}, \qquad q^* = \frac{1 - a_2}{1 - a_1 a_2}.$$

The last state is in the first quadrant if and only if $(1 - a_1)(1 - a_2) > 0$, and is otherwise irrelevant. The stability of the equilibria can be easily determined through the standard linearization analysis.

We assume that

$$a_1 < 1$$

so that the mono-culture equilibrium $(0,1)$ is invadable. As is well known, the change of variables $u = p$, $v = 1 - q$ converts the system (6.1) into the cooperative system

$$\begin{aligned}
\frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} - e_1 \frac{\partial u}{\partial x} + r_1 u(1 - a_1 - u + a_1 v), \\
\frac{\partial v}{\partial t} &= -e_2 \frac{\partial v}{\partial x} + r_2 (1 - v)(a_2 u - v).
\end{aligned} \tag{6.2}$$

For this system, $\boldsymbol{\beta} = ((1-a_1)/(1-a_1a_2), a_2(1-a_1)/(1-a_1a_2))$ if $a_2 \leqslant 1$, and $\boldsymbol{\beta} = (1,1)$ if $a_2 > 1$. Note that if $a_2 \leqslant 1$, then $\mathbf{0}$ and $\boldsymbol{\beta}$ are the only equilibria in $\mathcal{C}_{\boldsymbol{\beta}}$, and if $a_2 > 1$ then there is an extra equilibrium $(0,1)$ in $\mathcal{C}_{\boldsymbol{\beta}}$. It is easily seen that the model (6.2) satisfies Hypotheses 2.1.

Theorem 4.1 and Theorem 4.2 show that (6.2) always has nonincreasing traveling wave solutions with speeds above certain numbers connecting $\boldsymbol{\beta}$ (or $\mathbf{0}$) with an equilibrium other than $\boldsymbol{\beta}$ (or $\mathbf{0}$). Observe that a nonincreasing traveling wave solution of (6.2) connecting $(0,0)$ with $(0,1)$ has a zero component, and a nonincreasing traveling wave solution of (6.2) connecting $(1,0)$ with $\boldsymbol{\beta} = (1,1)$ has a constant component with value 1. Such a traveling wave solution is equivalent to a traveling wave solution for a scalar equation. We shall show, by using Theorem 5.2, the existence of nonincreasing traveling wave solutions in (6.2) that connect $\mathbf{0}$ with $\boldsymbol{\beta}$.

The matrix $C_\mu$ for (6.2) is given by

$$C_\mu = \begin{pmatrix} d_1\mu^2 + e_1\mu + r_1(1-a_1) & 0 \\ r_2a_2 & e_2\mu - r_2 \end{pmatrix}.$$

Clearly

$$\gamma_1(\mu) = d_1\mu^2 + e_1\mu + r_1(1-a_1), \qquad \gamma_2(\mu) = e_2\mu - r_2.$$

It is easily seen that $\bar{c}$ defined by (5.1) is given by

$$\bar{c} = \inf_{\mu>0} \gamma_1(\mu)/\mu = e_1 + 2\sqrt{d_1(1-a_1)} \tag{6.3}$$

and the infimum is attained at $\bar{\mu} = \sqrt{(r_1(1-a_1))/d_1}$.

One can follow the proof of Theorem 3.1 in Lewis et al. [5] to show that the linear determinacy hypotheses, i.e., Hypotheses 5.1, are satisfied by (6.2) if

$$e_1 + 2\sqrt{d_1(1-a_1)} \geqslant e_2 + r_2 \max\{a_1a_2 - 1, 0\}\sqrt{d_1/\big(r_1(1-a_1)\big)}. \tag{6.4}$$

Lemma 5.1 shows that under this condition

$$\tilde{c}^* = \tilde{c}^*_+ = c^* = c^*_+ = \bar{c}.$$

Observe that the condition (6.4) is equivalent to

$$\bar{c} \geqslant e_2 + r_2 \max\{a_1a_2 - 1, 0\}\sqrt{d_1/\big(r_1(1-a_1)\big)},$$

so that $\bar{c} \geqslant e_2$. It is possible that $\bar{c} = e_2$ when $a_1a_2 \leqslant 1$.

**Theorem 6.1.** *Assume that* (6.4) *holds and* $a_1 < 1$. *Let* $\bar{c}$ *be given by* (6.3). *Then the following statements hold for the system* (6.2).

i. *If* $\bar{c} > e_2$, *or if* $\bar{c} = e_2$ *and* $a_2 \leqslant 1$, *then for* $c \geqslant \bar{c}$ *the system* (6.2) *has a nonincreasing traveling wave solution with speed* $c$ *connecting* $\mathbf{0}$ *with* $\boldsymbol{\beta}$;
ii. *If* $\bar{c} = e_2$ *and* $a_2 > 1$, *then* (6.2) *has no classical nonincreasing traveling wave solution with speed* $\bar{c} = e_2$ *connecting* $\mathbf{0}$ *with* $\boldsymbol{\beta}$; *and*
iii. (6.2) *has no nonincreasing traveling wave solution with speed* $c$ *connecting* $\mathbf{0}$ *with* $\boldsymbol{\beta}$ *if* $c < \bar{c}$.

**Proof.** If $\bar{c} > e_2$, Theorem 5.2 shows that for $c \geqslant \bar{c}$, the system (6.2) has a nonincreasing traveling wave solution $(u(x-ct), v(x-ct))$ which connect $\mathbf{0}$ with $\boldsymbol{\beta}$ or with $(0,1)$. If it connects $\mathbf{0}$ with $(0,1)$, then $u \equiv 0$ so that $v(x-ct)$ connects 0 with 1 and is a nonincreasing function satisfying

$$(c-e_2)v' = r_2 v(1-v). \tag{6.5}$$

Since $c > e_2$, (6.5) implies that $v$ is a nondecreasing function, a contradiction. Therefore the traveling wave must connect $\mathbf{0}$ with $\boldsymbol{\beta}$. The second equation of (6.2) shows that a traveling wave solution $(u(x-e_2t), v(x-e_2t))$ of (6.2) with speed $e_2$ satisfies that for $-\infty < z < \infty$

$$\big(1-v(z)\big)\big(a_2u(z)-v(z)\big) = 0. \tag{6.6}$$

This equation shows that $v(z)$ is either $a_2u(z)$ or 1 for any real number $z$. Assume that $\bar{c} = e_2$ and $a_2 \leqslant 1$. We substitute $v = a_2u$ into the second equation of (6.2) to obtain

$$\frac{\partial u}{\partial t} = d_1\frac{\partial^2 u}{\partial x^2} - e_1\frac{\partial u}{\partial x} + r_1u\big(1-a_1-(1-a_1a_2)u\big). \tag{6.7}$$

This is the Fisher equation with an advection term, which has the minimal traveling wave speed $\bar{c} = e_2$. Eq. (6.7) has a nonincreasing traveling wave solution $u(x-e_2t)$ connecting 0 with $(1-a_1)/(1-a_1a_2)$. Then $(u(x-e_2t), v(x-e_2t))$ with $v(x-e_2t) = a_2u(x-e_2t)$ is a nonincreasing traveling wave solution of (6.2) connecting $\mathbf{0}$ with $\boldsymbol{\beta} = ((1-a_1)/(1-a_1a_2), a_2(1-a_1)/(1-a_1a_2))$. This completes the proof of the statement i.

We next consider the case of $\bar{c} = e_2$ and $a_2 > 1$. In this case $\boldsymbol{\beta} = (1,1)$. If $(u(x-e_2t), v(x-e_2t))$ is a classical nonincreasing traveling wave solution of (6.2) connecting $\mathbf{0}$ with $\boldsymbol{\beta} = (1,1)$, we derive a contradiction as follows. Since $u(z)$ and $v(z)$ are continuous and both decrease from 1 to 0 as $z$ increases from $-\infty$ to $\infty$, (6.6) and the assumption $a_2 > 1$ imply

$$v(z) = \min\big\{1, a_2u(z)\big\}. \tag{6.8}$$

Substituting this to the second equation of (6.2), we obtain that

$$d_1u'' + (e_2-e_1)u' + r_1u\big(1-a_1-u+a_1\min\{1, a_2u\}\big) = 0. \tag{6.9}$$

Assume that $a_2u(z_0) = 1$ for some real number $z_0$. (6.9) shows that

$$d_1u''(z_0) + (e_2-e_1)u'(z_0) + r_1u(z_0)\big(1-u(z_0)\big) = 0.$$

This shows that $u''(z_0) < 0$ if $u'(z_0) = 0$. It follows that if $u'(z_0) = 0$ then $u'(z) > 0$ for $z < z_0$ and $z$ sufficiently close to $z_0$. This is impossible as $u(z)$ is a nonincreasing function. We therefore have that $u'(z_0) < 0$. It follows from this and (6.8) that $v(z)$ is not differentiable at $z_0$, which contradicts that $(u(x-e_2t), v(x-e_2t))$ is a classical nonincreasing traveling solution. This completes the proof of the statement ii.

The statement iii follows from Theorem 5.2.ii. The proof is complete. □

The conditions given in [2] require that (5.3) holds with $\bar{u}$ replaced by all $0 \leqslant \mu \leqslant \bar{\mu}$ and $e_i = 0$ for all $i$, which leads to

$$\sqrt{d_1r_1(1-a_1)} \geqslant r_2\max\{a_1a_2-1, 0\}\sqrt{d_1/\big(r_1(1-a_1)\big)}$$

for (6.2). This condition is much stronger than (6.4) in the case of $e_1 = e_2 = 0$.

## 7. Discussion

We studied the existence of traveling wave solutions for a large class of partially degenerate cooperative reaction–diffusion systems. We showed that a traveling wave solution of a partially degenerate cooperative reaction–diffusion system with an appropriate speed is a fixed point of a compact integral operator. We proved that a partially degenerate cooperative reaction–diffusion system has traveling wave solutions with speeds above two extended real numbers. We also demonstrated that the two numbers are the same and may be characterized as the spreading speed as well as the slowest speed of a class of traveling wave solutions provided that the linear determinacy conditions given in Weinberger et al. [23] are satisfied.

The framework developed in this paper might be used to establish the existence of traveling wave solutions for other different kinds of spatial-temporal systems. There have been extensive studies regarding traveling solutions in delayed reaction–diffusion systems with positive diffusion coefficients; see for example Wu and Zou [25], Li et al. [10], Liang and Zhao [12], and Li and Zhang [9]. To establish existence of traveling wave solutions for a partially degenerate delayed cooperative reaction–diffusion systems, one might first show that a traveling wave solution of such a system is equivalent to a fixed point of a compact integral operator and then show that a traveling wave solution can be obtained by taking a limit of a sequence of functions that are fixed points of related integral systems. A similar approach might be used to show existence of traveling wave solutions in cooperative integral-differential systems.

The present paper only treated reaction–diffusion systems for a one-dimension habitat. However, it is known (see, e.g., [22,13,23]) how to use the one-dimensional results to determine the spreading speeds and traveling waves in higher-dimensional habitats by looking at one direction at a time. One chooses each unit direction vector $\boldsymbol{\xi}$, and uses the framework developed in this paper to study the existence of traveling waves in the direction $\boldsymbol{\xi}$ of a homogeneous habitat, which are functions of the single variable $\boldsymbol{\xi} \cdot \mathbf{x}$.

## Acknowledgments

I am grateful to Professor Hans F. Weinberger for his constructive comments which have considerably improved the paper. I would also like to thank the anonymous referee for his/her helpful comments and suggestions.

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