J. Differential Equations 252 (2012) 6306–6323

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Algebraic L^2 decay for weak solutions of a viscous Boussinesq system in exterior domains

Pigong Han [1]

Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

ARTICLE INFO

Article history:
Received 4 October 2011
Revised 8 February 2012
Available online 9 March 2012

Keywords:
Boussinesq system
Weak solution
Decay rate
Exterior domain

ABSTRACT

The algebraic decay rates for the total kinetic energy of weak solutions of the n-dimensional viscous Boussinesq system in exterior domains are established by means of the spectral decomposition method of fractional powers of the Laplacian and Stokes operators. The results reveal that L^2-decay of the velocity field with zero divergence becomes slower than that of the temperature.

1. Introduction and main results

The problem of the heat transfer inside viscous incompressible flows is considered in n-dimensional exterior domains. By an exterior domain we mean a connected open set Ω whose complement is the closure of the union of a finite number of bounded domains with smooth boundaries. Accordingly with the Boussinesq approximation, we neglect the variations of the density in the continuity equation and the local heat source due to the viscous dissipation, and consider the variations of the temperature by putting an additional vertical buoyancy force term in the equation of the fluid motion. That is,

$$\begin{cases} \partial_t\theta - k\Delta\theta + (u\cdot\nabla)\theta = 0 & \text{in } \Omega\times(0,\infty), \\ \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \beta\theta e_n & \text{in } \Omega\times(0,\infty), \\ \nabla\cdot u = 0 & \text{in } \Omega\times(0,\infty), \\ \theta(x,t) = u(x,t) = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = a,\ \theta(x,0) = b & \text{in } \Omega, \end{cases} \tag{1.1}$$

E-mail address: pghan@amss.ac.cn.

[1] Supported by NSF of China under grant No. 11071239.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2012.03.002

where $n \geqslant 3$ and Ω is an exterior domain in $\mathbb{R}^n$; the velocity $u = u(x,t)$ is an n-component vector field with zero divergence, the scalar function $\theta = \theta(x,t)$ denotes the density or the temperature and $p = p(x,t)$ the pressure of the fluid; while $u(x,0) = a(x)$ and $\theta(x,0) = b(x)$ are given initial velocity vector with zero divergence in the sense of distribution, and density or the temperature field respectively. Moreover, $e_n = (0,0,\dots,0,1)$, and $\beta \in \mathbb{R}^1$ is a physical constant. $\nu > 0$ and $k > 0$ are the viscous and the thermal diffusion coefficient. By rescaling the unknowns, without loss of generality, we take $\nu = k = \beta = 1$.

Definition. (u,θ) is called a weak solution of (1.1) if $u \in L^\infty_{\mathrm{loc}}(0,\infty; L^2_\sigma(\Omega)) \cap L^2_{\mathrm{loc}}(0,\infty; H^1_0(\Omega))$ and $\theta \in L^\infty_{\mathrm{loc}}(0,\infty; L^2(\Omega)) \cap L^2_{\mathrm{loc}}(0,\infty; H^1_0(\Omega))$ satisfy

$$\begin{aligned} &-\int_0^\infty\int_\Omega u\partial_t\phi\,dx\,dt + \int_0^\infty\int_\Omega \nabla u\cdot\nabla\phi\,dx\,dt + \int_0^\infty\int_\Omega u\cdot\nabla u\cdot\phi\,dx\,dt \\ &\quad = \int_\Omega a\phi(0)\,dx + \int_0^\infty\int_\Omega \theta e_n\cdot\phi\,dx\,dt \quad \text{for all } \phi\in C_0^\infty\big([0,\infty); C^\infty_{0,\sigma}(\Omega)\big), \end{aligned}$$

and

$$\begin{aligned} &-\int_0^\infty\int_\Omega \theta\partial_t\psi\,dx\,dt + \int_0^\infty\int_\Omega \nabla\theta\cdot\nabla\psi\,dx\,dt + \int_0^\infty\int_\Omega (u\cdot\nabla)\theta\psi\,dx\,dt \\ &\quad = \int_\Omega b\psi(0)\,dx \quad \text{for all } \psi\in C_0^\infty\big([0,\infty); C^\infty_0(\Omega)\big), \end{aligned}$$

where $(a,b) \in L^2_\sigma(\Omega) \times L^2(\Omega)$.

The Boussinesq system is widely used to model the dynamics of the ocean or the atmosphere, see e.g. [23]. It arises from the density dependent incompressible Navier–Stokes equations by using the so-called Boussinesq approximation, which consists in neglecting the density dependence in all the terms but the one involving the gravity. This system has lately received significant attention in mathematical fluid dynamics due to its connection to three-dimensional incompressible flows.

Note that when the initial density b is identically zero (or constant), then the system (1.1) reduces to the classical incompressible Navier–Stokes equations. The existence of global weak solutions in the energy space for 3D Navier–Stokes equations was established by Leray [20] and Hopf [17] respectively for an arbitrary L^2-initial velocity. The uniqueness and the regularity of 3D Leray–Hopf's weak solutions are still open questions, which are only known in space dimension two. Meanwhile it is also well known that smooth solutions are global for three and higher dimensions when the data are small in some critical spaces, see for instance [19] for more detailed discussions.

This L^2 decay problem was first raised by Leray [20] in the case of the Cauchy problem in R^3 and then was affirmatively solved by Kato [18] for the Cauchy problem in R^n with $n = 3,4$. Many interesting and important results on the decay properties have been achieved for Navier–Stokes flows, and the readers are referred to [2–9,11,13,15,16,18,24–26] and the references therein.

In this paper we are interested in the L^2 asymptotic behavior of weak solutions of the exterior problem (1.1). However, it is in general not easy to deduce the expected L^2 decay property for problem (1.1) in unbounded domains. The goal of this paper is to study in which way the variations of the temperature affect the asymptotic behavior of the velocity field. Few works are devoted to the study of the large time behavior of solutions to (1.1). By using Fourier transform, Brandolese and Schonbek [12] recently considered the decay properties of weak and strong solutions of system (1.1) in the three-dimensional whole space, however the methods employed by them (Fourier splitting method for example) seem not applicable to the present case. Our main results read as follows.

Theorem 1.1. *Let $a \in L^2_\sigma(\Omega)$ and $b \in L^1(\Omega) \cap L^2(\Omega)$, $n \geqslant 3$. Then problem* (1.1) *admits a weak solution (u,θ) satisfying for any $t > 0$*

$$\|\theta(t)\|_{L^2(\Omega)} \leqslant C(1+t)^{-\frac{n}{4}}, \tag{1.2}$$

and

$$\|u(t)\|_{L^2(\Omega)} \leqslant \begin{cases} C(1+t)^{\frac{1}{4}} & \text{if } n=3, \\ C\log_e(1+t) & \text{if } n=4, \\ C & \text{if } n \geqslant 5. \end{cases} \tag{1.3}$$

Moreover, if there is a small number $\eta > 0$ such that

$$\|b\|_{L^1(\Omega)} \leqslant \eta \quad \text{and} \quad \|e^{t\Delta}b\|_{L^1(\Omega)} \leqslant C(1+t)^{-\frac{1}{2}} \quad \forall t \in (0,\infty) \tag{1.4}$$

hold, then the weak solution (u,θ) satisfies for any $t > 0$

$$\|\theta(t)\|_{L^2(\Omega)} \leqslant C(1+t)^{-\frac{n+2}{4}} \quad \text{and} \quad \|u(t)\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to +\infty. \tag{1.5}$$

Further if $a \in L^{\frac{n}{n-1}}(\Omega)$ holds, then for any $t > 0$,

$$\|u(t)\|_{L^2(\Omega)} \leqslant C_\epsilon(1+t)^{\epsilon-\frac{n-2}{4}} \tag{1.6}$$

with any small $\epsilon > 0$.

Remark. (1) The above estimate (1.2) for the temperature looks optimal, since the decay agrees with that of the heat kernel. On the other hand, the optimality of the estimate (1.3) for the velocity field is not so clear. Theorem 1.1 shows that the estimates (1.2), (1.3) of weak solutions of (1.1) can be improved to (1.5) and (1.6) if the initial fields satisfy additionally suitable assumptions.

(2) The assumption (1.4) is technical, however, from which, we cannot conclude the regularity on the weak solution of (1.1). In addition, it is not difficult to verify that if $\Omega = \mathbb{R}^n_+$, $b, x_n b \in L^1(\mathbb{R}^n_+)$, then the solution $e^{t\Delta}b$ of the linear parabolic equation satisfies the estimate $\|e^{t\Delta}b\|_{L^1(\mathbb{R}^n_+)} \leqslant C(1+t)^{-\frac{1}{2}}\|x_n b\|_{L^1(\mathbb{R}^n_+)}$ for any $t > 0$.

(3) It is not clear whether the decay estimate (1.6) remains true for $\epsilon = 0$, and the main difficulty arises from the effect of the boundary $\partial\Omega$.

To conclude this introduction, we explain some notations used in what follows: Let $C^\infty_{0,\sigma}(\Omega)$ denote the set of all C^∞ real vector-valued functions $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ with compact support in Ω, such that $\nabla \cdot \phi = 0$ in Ω. $L^q_\sigma(\Omega)$ $(1 < q < \infty)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to $\|\cdot\|_{L^q(\Omega)}$, where $L^q(\Omega)$ represents the usual Lebesgue space of real-valued functions. In addition, the norm of $L^\infty(\Omega)$ is denoted by $\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega}|u(x)|$. By symbol C, we denote a generic constant whose value may change from line to line.

2. Decay rates for the weak solutions of problem (1.1)

Let $A = -P\Delta : D(A) \to L^2_\sigma(\Omega)$ be the Stokes operator, where $P : L^2(\Omega) \to L^2_\sigma(\Omega)$ is the Helmholtz projection operator. Then A is positive self-adjoint operator with dense domain $D(A) \subseteq L^2_\sigma(\Omega)$, and there exists a uniquely determined resolution $\{E_\lambda \mid \lambda \geqslant 0\}$ of identity in $L^2_\sigma(\Omega)$ such that the positive self-adjoint operator A^α $(0 < \alpha \leqslant 1)$ is defined as follows (see [27]):

$$A^\alpha = \int_0^\infty \lambda^\alpha \, dE_\lambda \quad \text{with domain } D\big(A^\alpha\big) = \left\{ v \in L^2_\sigma(\Omega) \;\Big|\; \int_0^\infty \lambda^{2\alpha} \, d\|E_\lambda v\|^2_{L^2(\Omega)} < \infty \right\}.$$

Similarly, we can define the fractional powers of the positive self-adjoint operator $-\Delta$ with dense domain $D(-\Delta) = W^{2,2}(\Omega) \cap H^1_0(\Omega) \subseteq L^2(\Omega)$. There exists a uniquely determined resolution $\{F_\lambda | \lambda \geqslant 0\}$ of identity in $L^2(\Omega)$ such that for any $0 < \alpha \leqslant 1$

$$(-\Delta)^\alpha = \int_0^\infty \lambda^\alpha \, dF_\lambda \quad \text{with domain } D\big((-\Delta)^\alpha\big) = \left\{ v \in L^2(\Omega) \;\Big|\; \int_0^\infty \lambda^{2\alpha} \, d\|F_\lambda v\|^2_{L^2(\Omega)} < \infty \right\}.$$

Lemma 2.1. *(See [10].) For any $h \in L^q_\sigma(\Omega)$. Then for any $t > 0$*

$$\big\|e^{-tA} h\big\|_{L^r(\Omega)} \leqslant C_{q,r} t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})} \|h\|_{L^q(\Omega)}, \quad \forall 1 < q \leqslant r < \infty \quad \text{or} \quad 1 \leqslant q < r \leqslant \infty.$$

Lemma 2.2. *(See [10].) Let $0 < \epsilon < \frac{1}{4}$ and $\rho_1 + \rho_2 = 1 + 2\epsilon$ with $\rho_1, \rho_2 \geqslant 0$. Then there is a constant $C = C(\epsilon, \rho_1, \rho_2, n, \Omega)$ such that*

$$\big\|E_\lambda P(u \cdot \nabla)v\big\|_{L^2(\Omega)} \leqslant C\lambda^{\frac{n}{4}-\epsilon} \big\|A^{\frac{\rho_1}{2}} u\big\|_{L^2(\Omega)} \|A^{\frac{\rho_2}{2}} v\|_{L^2(\Omega)}$$

for all $\lambda \geqslant 0$, $u \in D(A^{\frac{\rho_1}{2}})$ and $v \in D(A^{\frac{\rho_2}{2}})$.

Lemma 2.3. *It holds true for all $\lambda > 0$*

$$\big\|F_\lambda (u \cdot \nabla)v\big\|_{L^2(\Omega)} \leqslant C\lambda^{\frac{n+2}{4}} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

for any $u \in H^1_{0,\sigma}(\Omega)$ and any scalar function $v \in H^1_0(\Omega)$.

Proof. Note that

$$\big\|F_\lambda (u \cdot \nabla)v\big\|_{L^2(\Omega)} = \sup_{w \in L^2(\Omega)} \big|\big(F_\lambda (u \cdot \nabla)v, w\big)\big|. \tag{2.1}$$

Since $u \in H^1_{0,\sigma}(\Omega)$, we have for any scalar function $v \in H^1_0(\Omega)$

$$\begin{aligned}
\big|\big(F_\lambda (u \cdot \nabla)v, w\big)\big| &= \big|(v, u \cdot \nabla F_\lambda w)\big| \\
&\leqslant C\|uv\|_{L^1(\Omega)} \|\nabla F_\lambda w\|_{L^\infty(\Omega)} \\
&\leqslant C\|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \|\nabla F_\lambda w\|^{\frac{1}{2}}_{L^{2n}(\Omega)} \big\|\nabla^2 F_\lambda w\big\|^{\frac{1}{2}}_{L^{2n}(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\left\|(-\Delta)^{\frac{1}{2}}F_\lambda w\right\|_{L^{2n}(\Omega)}^{\frac{1}{2}}\left\|(-\Delta)F_\lambda w\right\|_{L^{2n}(\Omega)}^{\frac{1}{2}} \\
&\leqslant C\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\left\|(-\Delta)^{\frac{1}{2}+\frac{n-1}{4}}F_\lambda w\right\|_{L^2(\Omega)}^{\frac{1}{2}}\left\|(-\Delta)^{1+\frac{n-1}{4}}F_\lambda w\right\|_{L^2(\Omega)}^{\frac{1}{2}} \\
&\leqslant C\lambda^{\frac{n+2}{4}}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}, \qquad (2.2)
\end{aligned}
$$

where we have used the fact: $\|(-\Delta)^\alpha F_\lambda w\|_{L^2(\Omega)} \leqslant C\lambda^{-\alpha}\|w\|_{L^2(\Omega)}$ for $\lambda, \alpha > 0$ (see [22]); and the Sobolev inequality (see [10]): Let $g \in D((-\Delta)^\beta)$, $1 < r < \infty$, $0 < \beta < 1$ and if $0 < \frac{1}{q} = \frac{1}{r} - \frac{2\beta}{n} < 1$, then $g \in L^q(\Omega)$ and the estimate holds $\|g\|_{L^q(\Omega)} \leqslant C\|(-\Delta)^\beta g\|_{L^r(\Omega)}$; and the Gagliardo–Nirenberg inequality:

$$
\|f\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^{2n}(\Omega)}^{\frac{1}{2}}\|\nabla f\|_{L^{2n}(\Omega)}^{\frac{1}{2}}. \qquad (2.3)
$$

Indeed the Gagliardo–Nirenberg inequality still remains valid for functions f on the exterior domain Ω. To see this, set $\Omega_{\delta,R} = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > \delta\} \cap \{x \in \mathbb{R}^n; |x| < R\}$, where $R > 0$ is sufficiently large and $\delta > 0$ sufficiently small. Take $\varphi_{\delta,R} \in C_0^\infty(\Omega_{\frac{\delta}{2},2R})$, $\varphi_{\delta,R} \equiv 1$ on $\overline{\Omega_{\delta,R}}$, $0 \leqslant \varphi_{\delta,R} \leqslant 1$ on $\mathbb{R}^n$. Extending $f\varphi_{\delta,R}$ to be 0 from $\Omega_{\frac{\delta}{2},2R}$ to $\mathbb{R}^n$. Use the Gagliardo–Nirenberg inequality on the whole space $\mathbb{R}^n$: Let $n < s < \infty$. It holds for any $g \in W^{1,s}(\mathbb{R}^n)$

$$
\|g\|_{L^\infty(\mathbb{R}^n)} \leqslant C\|g\|_{L^s(\mathbb{R}^n)}^{1-\frac{n}{s}}\|\nabla g\|_{L^s(\mathbb{R}^n)}^{\frac{n}{s}}.
$$

We have for any sufficiently large number $R > 0$ and small number $\delta > 0$

$$
\begin{aligned}
\|f\|_{L^\infty(\Omega_{\delta,R})} &\leqslant \|f\varphi_{\delta,R}\|_{L^\infty(\Omega_{\frac{\delta}{2},2R})} \\
&\leqslant \|f\varphi_{\delta,R}\|_{L^\infty(\mathbb{R}^n)} \\
&\leqslant C\|f\varphi_{\delta,R}\|_{L^{2n}(\mathbb{R}^n)}^{\frac{1}{2}}\left\|\nabla(f\varphi_{\delta,R})\right\|_{L^{2n}(\mathbb{R}^n)}^{\frac{1}{2}} \\
&\leqslant C\|f\|_{L^{2n}(\Omega)}^{\frac{1}{2}}\left(\|\varphi_{\delta,R}\nabla f\|_{L^{2n}(\mathbb{R}^n)} + \|f\nabla\varphi_{\delta,R}\|_{L^{2n}(\mathbb{R}^n)}\right)^{\frac{1}{2}} \\
&\leqslant C\|f\|_{L^{2n}(\Omega)}^{\frac{1}{2}}\left(\|\nabla f\|_{L^{2n}(\Omega)} + R^{-1}\|f\|_{L^{2n}(\Omega)}\right)^{\frac{1}{2}}, \qquad (2.4)
\end{aligned}
$$

where the constant $C = C(n) > 0$ is independent of δ, R.

Letting $(\delta, R) \to (0, \infty)$ in (2.4), we get the desired inequality (2.3). □

Proof of Theorem 1.1. Let $a \in L^2_\sigma(\Omega)$ and $b \in L^1(\Omega) \cap L^2(\Omega)$, $n \geqslant 3$, and consider the successive approximation for $0 \leqslant t < \infty$:

$$
\begin{cases}
u_0(t) = e^{-tA}a, \quad \theta_0(t) = e^{t\Delta}b, \\
\theta_{j+1}(t) = \theta_0(t) - \displaystyle\int_0^t e^{(t-s)\Delta}u_j(s)\cdot\nabla\theta_{j+1}(s)\,ds, \\
u_{j+1}(t) = u_0(t) - \displaystyle\int_0^t e^{-(t-s)A}P(u_j(s)\cdot\nabla u_{j+1}(s) - \theta_j e_n)\,ds
\end{cases}
\qquad (2.5)
$$

for $j = 0, 1, 2, \ldots$.

Problem (2.5) admits a unique strong solution (θ_{j+1}, u_{j+1}) (see [21] for example). It is not difficult to verify that for $j = 0, 1, 2, \ldots$ and $t > 0$

$$\begin{cases} \partial_t \theta_{j+1} - \Delta\theta_{j+1} + (u_j \cdot \nabla)\theta_{j+1} = 0 & \text{in } \Omega \times (0, \infty), \\ \partial_t u_{j+1} + Au_{j+1} + P(u_j \cdot \nabla)u_{j+1} = P\theta_j e_n & \text{in } \Omega \times (0, \infty), \\ \nabla \cdot u_{j+1} = 0 & \text{in } \Omega \times (0, \infty), \\ \theta_{j+1}(x,t) = u_{j+1}(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u_{j+1}(x,0) = a, \quad \theta_{j+1}(x,0) = b & \text{in } \Omega. \end{cases} \tag{2.6}$$

Moreover, if $b \in L^q(\Omega)$ with $1 \leqslant q < \infty$, it holds for each $j = 0, 1, 2, \ldots$ and $t > 0$

$$\left\|\theta_{j+1}(t)\right\|_{L^q(\Omega)} \leqslant \|b\|_{L^1(\Omega)}(c_0 + c_1 t)^{-\frac{n}{2}(1-\frac{1}{q})}, \tag{2.7}$$

where

$$c_0 = \max\left\{\left(\frac{\|b\|_{L^1(\Omega)}}{\|b\|_{L^q(\Omega)}}\right)^{\frac{2q}{n(q-1)}}, \left(\frac{\|b\|_{L^1(\Omega)}}{\|b\|_{L^2(\Omega)}}\right)^{\frac{4}{n}}\right\} \quad \text{and} \quad c_1 = \max\left\{\frac{8S_n}{nq}, \frac{4S_n}{n}\right\};$$

S_n is the Sobolev best constant, which is defined by (see [1])

$$S_n = \inf\left\{\frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^{\frac{2n}{n-2}}(\Omega)}} \;\middle|\; v \in C_0^\infty(\Omega)\right\}.$$

The proof of (2.7) is similar to that of Lemma 3.2 in [12], which is proved to be true on the three-dimensional whole space. Here for the readers' convenience, we give a sketch of the proof of (2.7). Let $2 \leqslant q < \infty$. Multiplying the first equation in (2.6) by $q|\theta_{j+1}(t)|^{q-2}\theta_{j+1}(t)$ and integrating by parts we get for all $t > 0$ and $j = 0, 1, 2, \ldots$

$$\frac{d}{dt}\left\|\theta_{j+1}(t)\right\|^q_{L^q(\Omega)} + \frac{4(q-1)}{q}\left\|\nabla\left(|\theta_{j+1}|^{\frac{q}{2}}\right)(t)\right\|^2_{L^2(\Omega)} = 0.$$

The interpolation inequality and the Sobolev embedding theorem yield for each $j = 0, 1, 2, \ldots$ and $t > 0$

$$\begin{aligned} \left\|\theta_{j+1}(t)\right\|_{L^q(\Omega)} &\leqslant \left\|\theta_{j+1}(t)\right\|^{\frac{2}{2+n(q-1)}}_{L^1(\Omega)} \left\|\theta_{j+1}(t)\right\|^{\frac{n(q-1)}{2+n(q-1)}}_{L^{\frac{nq}{n-2}}(\Omega)} \\ &= \left\|\theta_{j+1}(t)\right\|^{\frac{2}{2+n(q-1)}}_{L^1(\Omega)} \left\||\theta_{j+1}|^{\frac{q}{2}}(t)\right\|^{\frac{2n(q-1)}{q(2+n(q-1))}}_{L^{\frac{2n}{n-2}}(\Omega)} \\ &\leqslant S_n^{-\frac{n(q-1)}{q(2+n(q-1))}} \|b\|^{\frac{2}{2+n(q-1)}}_{L^1(\Omega)} \left\|\nabla|\theta_{j+1}|^{\frac{q}{2}}(t)\right\|^{\frac{2n(q-1)}{q(2+n(q-1))}}_{L^2(\Omega)}, \end{aligned}$$

which implies that

$$\left\|\nabla\left(|\theta_{j+1}|^{\frac{q}{2}}\right)(t)\right\|^2_{L^2(\Omega)} \geqslant S_n\|b\|^{-\frac{2q}{n(q-1)}}_{L^1(\Omega)} \left\|\theta_{j+1}(t)\right\|^{\frac{q(2+n(q-1))}{n(q-1)}}_{L^q(\Omega)},$$

where we have used a basic estimate (see [14]): $\|\theta_{j+1}(t)\|_{L^q(\Omega)} \leqslant \|b\|_{L^q(\Omega)}$ for all $1 \leqslant q \leqslant \infty$.

Therefore for each $j = 0, 1, 2, \dots$ and $t > 0$,

$$\frac{d}{dt}\left\|\theta_{j+1}(t)\right\|_{L^q(\Omega)}^q \leqslant -4S_n\left(1-\frac{1}{q}\right)\|b\|_{L^1(\Omega)}^{-\frac{2q}{n(q-1)}}\left\|\theta_{j+1}(t)\right\|_{L^q(\Omega)}^{\frac{q(2+n(q-1))}{n(q-1)}},$$

and then

$$\begin{aligned}\left\|\theta_{j+1}(t)\right\|_{L^q(\Omega)}^q &\leqslant \left(\|b\|_{L^q(\Omega)}^{-\frac{2q}{n(q-1)}} + \frac{8S_n}{nq}\|b\|_{L^1(\Omega)}^{-\frac{2q}{n(q-1)}}t\right)^{-\frac{n(q-1)}{2}} \\ &\leqslant \|b\|_{L^1(\Omega)}^q\left(\left(\frac{\|b\|_{L^1(\Omega)}}{\|b\|_{L^q(\Omega)}}\right)^{\frac{2q}{n(q-1)}} + \frac{8S_nt}{nq}\right)^{-\frac{n(q-1)}{2}},\end{aligned}$$

which implies that (2.7) holds for $2 \leqslant q < \infty$. Furthermore the interpolation inequality yields that for $1 \leqslant q < 2$,

$$\left\|\theta_{j+1}(t)\right\|_{L^q(\Omega)} \leqslant \left\|\theta_{j+1}(t)\right\|_{L^1(\Omega)}^{\frac{2}{q}-1}\left\|\theta_{j+1}(t)\right\|_{L^2(\Omega)}^{2-\frac{2}{q}} \leqslant \|b\|_{L^1(\Omega)}\left(\left(\frac{\|b\|_{L^1(\Omega)}}{\|b\|_{L^2(\Omega)}}\right)^{\frac{4}{n}} + \frac{4S_nt}{n}\right)^{-\frac{n}{2}(1-\frac{1}{q})}.$$

From the above arguments on $2 \leqslant q < \infty$ and $1 \leqslant q < 2$, we conclude that (2.7) holds true for $1 \leqslant q < \infty$.

Now we continue the proof of Theorem 1.1. $\theta_{j+1} = \theta_{j+1}(x,t)$ and $u_{j+1} = u_{j+1}(x,t)$ satisfy for all $t > 0$ and $j = 0, 1, 2, \dots$

$$\frac{d}{dt}\left\|u_{j+1}(t)\right\|_{L^2(\Omega)}^2 + 2\left\|\nabla u_{j+1}(t)\right\|_{L^2(\Omega)}^2 = 2\int_\Omega \theta_j(x,t)e_n\cdot u_{j+1}(x,t)\,dx \tag{2.8}$$

and

$$\frac{d}{dt}\left\|\theta_{j+1}(t)\right\|_{L^2(\Omega)}^2 + 2\left\|\nabla\theta_{j+1}(t)\right\|_{L^2(\Omega)}^2 = 0. \tag{2.9}$$

Observe that for any $\rho, t > 0$ and $j = 0, 1, 2, \dots$

$$\begin{aligned}\left\|\nabla u_{j+1}(t)\right\|_{L^2(\Omega)}^2 &= \left\|A^{\frac{1}{2}}u_{j+1}(t)\right\|_{L^2(\Omega)}^2 \\ &= \int_0^\infty \lambda\, d\left\|E_\lambda u_{j+1}(t)\right\|_{L^2(\Omega)}^2 \\ &\geqslant \rho\int_\rho^\infty d\left\|E_\lambda u_{j+1}(t)\right\|_{L^2(\Omega)}^2 \\ &= \rho\left(\left\|u_{j+1}(t)\right\|_{L^2(\Omega)}^2 - \left\|E_\rho u_{j+1}(t)\right\|_{L^2(\Omega)}^2\right).\end{aligned} \tag{2.10}$$

Inserting (2.10) into (2.8), we get for all $\rho, t > 0$ and $j = 0, 1, 2, \ldots$

$$\begin{aligned}&\frac{d}{dt}\|u_{j+1}(t)\|^2_{L^2(\Omega)} + 2\rho\|u_{j+1}(t)\|^2_{L^2(\Omega)}\\&\quad\leqslant 2\rho\|E_\rho u_{j+1}(t)\|^2_{L^2(\Omega)} + 2\|u_{j+1}(t)\|_{L^2(\Omega)}\|\theta_j(t)\|_{L^2(\Omega)}.\end{aligned}\tag{2.11}$$

Similar to the proof of (2.11), one has for all $\rho, t > 0$ and $j = 0, 1, 2, \ldots$

$$\frac{d}{dt}\|\theta_{j+1}(t)\|^2_{L^2(\Omega)} + 2\rho\|\theta_{j+1}(t)\|^2_{L^2(\Omega)} \leqslant 2\rho\|F_\rho\theta_{j+1}(t)\|^2_{L^2(\Omega)}.\tag{2.12}$$

Let $0 < \delta < \frac{1}{4}$. Then for any $t > 0$ and $u, v \in H^1_{0,\sigma}(\Omega)$

$$\begin{aligned}&\int_0^t \|A^{\frac{1+2\delta}{4}}u(s)\|_{L^2(\Omega)}\|A^{\frac{1+2\delta}{4}}v(s)\|_{L^2(\Omega)}\,ds\\&\quad\leqslant \int_0^t \left(\|A^{\frac12}u(s)\|_{L^2(\Omega)}\|A^{\frac12}v(s)\|_{L^2(\Omega)}\right)^{\frac{1+2\delta}{2}}\left(\|u(s)\|_{L^2(\Omega)}\|v(s)\|_{L^2(\Omega)}\right)^{\frac{1-2\delta}{2}}ds\\&\quad\leqslant \left(\int_0^t\|\nabla u(s)\|^2_{L^2(\Omega)}\,ds\int_0^t\|\nabla v(s)\|^2_{L^2(\Omega)}\,ds\right)^{\frac{1+2\delta}{4}}\\&\qquad\times\left(\int_0^t\|u(s)\|^2_{L^2(\Omega)}\,ds\int_0^t\|v(s)\|^2_{L^2(\Omega)}\,ds\right)^{\frac{1-2\delta}{4}},\end{aligned}\tag{2.13}$$

where we have used the interpolation inequality for fractional powers (see [27]): Let $0 \leqslant \alpha \leqslant 1$

$$\|A^{\frac{\alpha}{2}}u\|_{L^2(\Omega)} \leqslant C\|A^{\frac12}u\|^{\alpha}_{L^2(\Omega)}\|u\|^{1-\alpha}_{L^2(\Omega)} \leqslant C\|\nabla u\|^{\alpha}_{L^2(\Omega)}\|u\|^{1-\alpha}_{L^2(\Omega)}, \quad \forall u \in H^1_{0,\sigma}(\Omega).$$

Using (2.5), (2.13) and Lemmata 2.1, 2.2, one has for any $\rho > 0$ and $t > 0$

$$\begin{aligned}\|E_\rho u_{j+1}(t)\|_{L^2(\Omega)} &\leqslant \|E_\rho e^{-tA}a\|_{L^2(\Omega)}\\&\quad+\left\|E_\rho\int_0^t\left(\int_0^\rho+\int_\rho^\infty\right)e^{-(t-s)\lambda}\,d\left[E_\lambda P\left(\left(u_j(s)\cdot\nabla\right)u_{j+1}(s)\right)\right]ds\right\|_{L^2(\Omega)}\\&\quad+\int_0^t\|E_\rho e^{-(t-s)A}P\left(e_n\theta_j(s)\right)\|_{L^2(\Omega)}\,ds\\&\leqslant \|e^{-tA}a\|_{L^2(\Omega)}+\int_0^t e^{-(t-s)\rho}\|E_\rho P\left(u_j(s)\cdot\nabla\right)u_{j+1}(s)\|_{L^2(\Omega)}\,ds\end{aligned}$$

$$+\int_0^t (t-s)\left\{\int_0^\rho e^{-(t-s)\lambda}\left\|E_\lambda P\big(u_j(s)\cdot\nabla\big)u_{j+1}(s)\right\|_{L^2(\Omega)}\,d\lambda\right\}ds$$

$$+C\int_0^{\frac{t}{2}}(t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}\left\|\theta_j(s)\right\|_{L^r(\Omega)}ds+C\int_{\frac{t}{2}}^t\left\|\theta_j(s)\right\|_{L^2(\Omega)}ds$$

$$\leqslant\left\|e^{-tA}a\right\|_{L^2(\Omega)}+C\rho^{\frac{n}{4}-\delta}\int_0^t\left\|A^{\frac{1+2\delta}{4}}u_j(s)\right\|_{L^2(\Omega)}\left\|A^{\frac{1+2\delta}{4}}u_{j+1}(s)\right\|_{L^2(\Omega)}ds$$

$$+C\int_0^{\frac{t}{2}}(t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}\left\|\theta_j(s)\right\|_{L^r(\Omega)}ds+C\int_{\frac{t}{2}}^t\left\|\theta_j(s)\right\|_{L^2(\Omega)}ds$$

$$\leqslant\left\|e^{-tA}a\right\|_{L^2(\Omega)}+C\int_0^{\frac{t}{2}}(t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}\left\|\theta_j(s)\right\|_{L^r(\Omega)}ds+C\int_{\frac{t}{2}}^t\left\|\theta_j(s)\right\|_{L^2(\Omega)}ds$$

$$+C\rho^{\frac{n}{4}-\delta}\left(\int_0^t\left\|\nabla u_j(s)\right\|^2_{L^2(\Omega)}ds\int_0^t\left\|\nabla u_{j+1}(s)\right\|^2_{L^2(\Omega)}ds\right)^{\frac{1+2\delta}{4}}$$

$$\times\left(\int_0^t\left\|u_j(s)\right\|^2_{L^2(\Omega)}ds\int_0^t\left\|u_{j+1}(s)\right\|^2_{L^2(\Omega)}ds\right)^{\frac{1-2\delta}{4}}\quad\text{with any } 1<r\leqslant 2. \tag{2.14}$$

From (2.11) and (2.14), we obtain for any $\rho, t>0$ and $j=0,1,\dots$

$$\frac{d}{dt}\left\|u_{j+1}(t)\right\|^2_{L^2(\Omega)}+2\rho\left\|u_{j+1}(t)\right\|^2_{L^2(\Omega)}$$

$$\leqslant C\rho\left\{\left\|e^{-tA}a\right\|_{L^2(\Omega)}+\rho^{\frac{n}{4}-\delta}\left(\int_0^t\left\|\nabla u_j(s)\right\|^2_{L^2(\Omega)}ds\int_0^t\left\|\nabla u_{j+1}(s)\right\|^2_{L^2(\Omega)}ds\right)^{\frac{1+2\delta}{4}}\right.$$

$$\times\left(\int_0^t\left\|u_j(s)\right\|^2_{L^2(\Omega)}ds\int_0^t\left\|u_{j+1}(s)\right\|^2_{L^2(\Omega)}ds\right)^{\frac{1-2\delta}{4}}$$

$$\left.+C\int_0^{\frac{t}{2}}(t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}\left\|\theta_j(s)\right\|^{\frac{2}{r}-1}_{L^1(\Omega)}\left\|\theta_j(s)\right\|^{2(1-\frac{1}{r})}_{L^2(\Omega)}ds+\int_{\frac{t}{2}}^t\left\|\theta_j(s)\right\|_{L^2(\Omega)}ds\right\}^2$$

$$+2\left\|u_{j+1}(t)\right\|_{L^2(\Omega)}\left\|\theta_j(t)\right\|_{L^2(\Omega)}\quad\text{with any } 0<\delta<\frac{1}{4},\ 1<r\leqslant 2. \tag{2.15}$$

Using Lemma 2.3, we get for any $\rho, t > 0$ and $j = 0, 1, \ldots$

$$\begin{aligned}\left\|F_\rho \theta_{j+1}(t)\right\|_{L^2(\Omega)} \leqslant{}& \left\|\theta_0(t)\right\|_{L^2(\Omega)} + \int_0^t e^{-(t-s)\rho} \left\|F_\rho\big(u_j(s)\cdot\nabla\big)\theta_{j+1}(s)\right\|_{L^2(\Omega)} ds \\ &+ \int_0^t (t-s)\left\{\int_0^\rho e^{-(t-s)\lambda}\left\|F_\lambda\big(u_j(s)\cdot\nabla\big)\theta_{j+1}(s)\right\|_{L^2(\Omega)} d\lambda\right\} ds \\ \leqslant{}& \left\|\theta_0(t)\right\|_{L^2(\Omega)} + C\rho^{\frac{n+2}{4}} \int_0^t \left\|u_j(s)\right\|_{L^2(\Omega)}\left\|\theta_{j+1}(s)\right\|_{L^2(\Omega)} ds. \end{aligned} \tag{2.16}$$

From (2.12) and (2.16), one has for any $\rho, t > 0$ and $j = 0, 1, \ldots$

$$\begin{aligned}&\frac{d}{dt}\left\|\theta_{j+1}(t)\right\|^2_{L^2(\Omega)} + \rho\left\|\theta_{j+1}(t)\right\|^2_{L^2(\Omega)} \\ &\quad\leqslant C\rho\left\{\left\|\theta_0(t)\right\|_{L^2(\Omega)} + \rho^{\frac{n+2}{4}}\int_0^t \left\|u_j(s)\right\|_{L^2(\Omega)}\left\|\theta_{j+1}(s)\right\|_{L^2(\Omega)} ds\right\}^2. \end{aligned} \tag{2.17}$$

It follows from (2.7) that for any $t > 0$

$$\sup_{j\geqslant 0}\left\|\theta_{j+1}(t)\right\|_{L^2(\Omega)} \leqslant C\|b\|_{L^1(\Omega)}(1+t)^{-\frac{n}{4}}.$$

In addition, since $b \in L^1(\Omega)$, we infer for any $t > 0$

$$\left\|\theta_0(t)\right\|_{L^2(\Omega)} = \left\|e^{t\Delta}b\right\|_{L^2(\Omega)} \leqslant C\|b\|_{L^1(\Omega)}(1+t)^{-\frac{n}{4}}.$$

Whence it holds true for all $t > 0$

$$\sup_{j\geqslant 0}\left\|\theta_j(t)\right\|_{L^2(\Omega)} \leqslant C\|b\|_{L^1(\Omega)}(1+t)^{-\frac{n}{4}}. \tag{2.18}$$

It follows from (2.8) that for any $t > 0$ and $j = 0, 1, \ldots$

$$\frac{d}{dt}\left\|u_{j+1}(t)\right\|^2_{L^2(\Omega)} + 2\left\|\nabla u_{j+1}(t)\right\|^2_{L^2(\Omega)} \leqslant 2\left\|u_{j+1}(t)\right\|_{L^2(\Omega)}\left\|\theta_j(t)\right\|_{L^2(\Omega)},$$

which implies that

$$\frac{d}{dt}\left\|u_{j+1}(t)\right\|_{L^2(\Omega)} \leqslant \left\|\theta_j(t)\right\|_{L^2(\Omega)}, \tag{2.19}$$

and

$$\left\|u_{j+1}(t)\right\|^2_{L^2(\Omega)} + 2\int_0^t \left\|\nabla u_{j+1}(s)\right\|^2_{L^2(\Omega)} ds \leqslant \|a\|^2_{L^2(\Omega)} + 2\int_0^t \left\|u_{j+1}(s)\right\|_{L^2(\Omega)}\left\|\theta_j(s)\right\|_{L^2(\Omega)} ds. \tag{2.20}$$

Note that

$$\|u_0(t)\|_{L^2(\Omega)} = \|e^{-tA}a\|_{L^2(\Omega)} \leqslant A_0\|a\|_{L^2(\Omega)} \quad \text{with } A_0 > 1. \tag{2.21}$$

From (2.18), (2.19) and (2.21), we derive for any $t > 0$

$$\sup_{j\geqslant 0}\|u_j(t)\|_{L^2(\Omega)} \leqslant \begin{cases} C(1+t)^{\frac{1}{4}} & \text{if } n=3, \\ C\log_e(1+t) & \text{if } n=4, \\ C & \text{if } n\geqslant 5. \end{cases} \tag{2.22}$$

Assume that (1.4) holds, that is $\|e^{t\Delta}b\|_{L^1(\mathbb{R}^n_+)} \leqslant C(1+t)^{-\frac{1}{2}}$ with $t > 0$, and $\|b\|_{L^1(\Omega)} \leqslant \eta$ for some small number $\eta > 0$. We first show that for $n = 3$ and any $t > 0$

$$\sup_{j\geqslant 0}\|u_j(t)\|_{L^2(\Omega)} \leqslant A\|a\|_{L^2(\Omega)} + M(1+t)^{\frac{1}{8}}, \tag{2.23}$$

where $M > 0$ is some constant independent of j to be determined.

From (2.21), we suppose that there exists a $j \geqslant 0$ such that it holds for any $i \in [0, j]$ and $t > 0$

$$\|u_i(t)\|_{L^2(\Omega)} \leqslant A\|a\|_{L^2(\Omega)} + M(1+t)^{\frac{1}{8}} \quad \text{with } n=3. \tag{2.24}$$

By the assumption (1.4) and Lemma 2.1, we get for $n \geqslant 3$ and any $t > 0$

$$\|\theta_0(t)\|_{L^2(\Omega)} = \|e^{t\Delta}b\|_{L^2(\Omega)} \leqslant Ct^{-\frac{n}{2}(1-\frac{1}{2})}\|e^{\frac{t}{2}\Delta}b\|_{L^1(\Omega)} \leqslant C(1+t)^{-\frac{n+2}{4}}. \tag{2.25}$$

Setting $\rho = k(t+1)^{-1}$ with some large positive integer k, and multiplying both sides of (2.17) by $(t+1)^k$, together with (2.18), (2.24), (2.25), we conclude for $n = 3$, any $i \in [0, j]$ with given j, and $t > 0$

$$\begin{aligned} &\frac{d}{dt}\left((t+1)^k\|\theta_{i+1}(t)\|^2_{L^2(\Omega)}\right) \\ &\leqslant C(t+1)^{k-1}\left(\|\theta_0(t)\|_{L^2(\Omega)} + C(t+1)^{-\frac{5}{4}}\int_0^t \|u_i(s)\|_{L^2(\Omega)}\|\theta_{i+1}(s)\|_{L^2(\Omega)}\,ds\right)^2 \\ &\leqslant C(t+1)^{k-1}\left((1+t)^{-\frac{5}{4}} + \|b\|_{L^1(\Omega)}(t+1)^{-\frac{5}{4}}\int_0^t \left(\|a\|_{L^2(\Omega)} + M(1+s)^{\frac{1}{8}}\right)(1+s)^{-\frac{3}{4}}\,ds\right)^2 \\ &\leqslant C(t+1)^{k-1}\left(1 + \|b\|_{L^1(\Omega)}\left(\|a\|_{L^2(\Omega)} + M\right)\right)(1+t)^{-\frac{7}{4}}. \end{aligned} \tag{2.26}$$

By taking $k > 0$ suitably large in (2.26), we infer for $n = 3$, any $i \in [0, j]$ with given j, and $t > 0$

$$\begin{aligned} \|\theta_{i+1}(t)\|_{L^2(\Omega)} &\leqslant (t+1)^{-\frac{k}{2}}\|b\|_{L^2(\Omega)} + \left(1 + \|b\|_{L^1(\Omega)}\left(\|a\|_{L^2(\Omega)} + M\right)\right)(1+t)^{-\frac{7}{8}} \\ &\leqslant C\left(\|b\|_{L^2(\Omega)} + 1 + \|b\|_{L^1(\Omega)}\left(\|a\|_{L^2(\Omega)} + M\right)\right)(1+t)^{-\frac{7}{8}}. \end{aligned} \tag{2.27}$$

It follows from (2.19) and (2.27) that for $n=3$, any $i \in [0,j]$ with given j, and $t>0$

$$\begin{aligned} \|u_{i+2}(t)\|_{L^2(\Omega)} &\leqslant A\|a\|_{L^2(\Omega)} + C\big(\|b\|_{L^2(\Omega)} + 1 + \|b\|_{L^1(\Omega)}\big(\|a\|_{L^2(\Omega)} + M\big)\big)(1+t)^{\frac{1}{8}} \\ &\leqslant A\|a\|_{L^2(\Omega)} + M(1+t)^{\frac{1}{8}}, \end{aligned} \tag{2.28}$$

by taking $M>0$ is suitably large and $\delta>0$ suitably small with $\|b\|_{L^1(\Omega)} \leqslant \delta$.

Therefore (2.23) is verified to be true by (2.21), (2.24) and (2.28). It follows from (2.22) and (2.23) that for any $t>0$

$$\sup_{j\geqslant 0}\|u_j(t)\|_{L^2(\Omega)} \leqslant \begin{cases} C(1+t)^{\frac{1}{8}}\log_e(1+t) & \text{if } n=3,4, \\ C & \text{if } n\geqslant 5. \end{cases} \tag{2.29}$$

Setting $\rho = k(t+1)^{-1}$ with some large positive integer k, multiplying both sides of (2.17) by $(t+1)^k$, together with (2.18), (2.25), (2.29), we conclude for $n=3,4$, $t>0$ and any $j=0,1,2,\dots$

$$\begin{aligned} &\frac{d}{dt}\big((t+1)^k\|\theta_{j+1}(t)\|^2_{L^2(\Omega)}\big) \\ &\quad\leqslant C(t+1)^{k-1}\left((1+t)^{-\frac{n+2}{4}} + (t+1)^{-\frac{n+2}{4}}\int_0^t (1+s)^{-\frac{n}{4}+\frac{1}{8}}\log_e(1+s)\,ds\right)^2 \\ &\quad\leqslant C(t+1)^{k-1}\big((1+t)^{-\frac{n+2}{4}} + (t+1)^{-\frac{n+2}{4}-\frac{n}{4}+\frac{9}{8}}\log_e(1+t)\big)^2, \end{aligned}$$

which implies that for $n=3,4$ and any $t>0$

$$\sup_{j\geqslant 0}\|\theta_{j+1}(t)\|_{L^2(\Omega)} \leqslant C(t+1)^{-\frac{n}{2}+\frac{5}{8}}\log_e(1+t). \tag{2.30}$$

Setting $\rho = k(t+1)^{-1}$ with some large positive integer k, and multiplying both sides of (2.17) by $(t+1)^k$, together with (2.25), (2.29), (2.30), we conclude for $n=3,4$, $t>0$ and any $j=0,1,2,\dots$

$$\begin{aligned} &\frac{d}{dt}\big((t+1)^k\|\theta_{j+1}(t)\|^2_{L^2(\Omega)}\big) \\ &\quad\leqslant C(t+1)^{k-1}\left((1+t)^{-\frac{n+2}{4}} + (t+1)^{-\frac{n+2}{4}}\int_0^t (1+s)^{-\frac{n}{2}+\frac{5}{8}+\frac{1}{8}}\big(\log_e(1+s)\big)^2\,ds\right)^2 \\ &\quad\leqslant C(t+1)^{k-1}\big((1+t)^{-\frac{n+2}{4}} + (1+t)^{-\frac{n+2}{4}}L(t)\big)^2, \end{aligned} \tag{2.31}$$

where

$$L(t) = \begin{cases} (1+t)^{\frac{1}{4}}(\log_e(1+t))^2 & \text{if } n=3, \\ 1 & \text{if } n=4. \end{cases}$$

Whence, from (2.31) we derive that for any $t>0$ and $j=0,1,\dots$

$$\sup_{j\geqslant 0}\|\theta_{j+1}(t)\|_{L^2(\Omega)} \leqslant \begin{cases} C(1+t)^{-1}(\log_e(1+t))^2 & \text{if } n=3, \\ C(1+t)^{-\frac{3}{2}} & \text{if } n=4. \end{cases} \tag{2.32}$$

Setting $\rho = k(t+1)^{-1}$ with some large positive integer k, and multiplying both sides of (2.17) by $(t+1)^k$, together with (2.25), (2.29), (2.32), we conclude for $n=3$, $t>0$ and any $j=0,1,2,\ldots$

$$\begin{aligned}
&\frac{d}{dt}\big((t+1)^k\big\|\theta_{j+1}(t)\big\|^2_{L^2(\Omega)}\big)\\
&\quad\leqslant C(t+1)^{k-1}\left((1+t)^{-\frac{5}{4}}+(1+t)^{-\frac{5}{4}}\int\limits_0^t(1+s)^{-1+\frac{1}{8}}\big(\log_e(1+s)\big)^3\,ds\right)^2\\
&\quad\leqslant C(t+1)^{k-1}\big((1+t)^{-\frac{5}{4}}+(1+t)^{-\frac{5}{4}+\frac{1}{8}}\big(\log_e(1+t)\big)^3\big)^2,
\end{aligned}$$

which implies that for $n=3$ and any $t>0$

$$\sup_{j\geqslant 0}\big\|\theta_{j+1}(t)\big\|_{L^2(\Omega)}\leqslant C(1+t)^{-\frac{5}{4}+\frac{1}{8}}\big(\log_e(1+t)\big)^3. \tag{2.33}$$

Like the proof of (2.33), we get for $n=3$, $t>0$ and any $j=0,1,2,\ldots$

$$\begin{aligned}
&\frac{d}{dt}\big((t+1)^k\big\|\theta_{j+1}(t)\big\|^2_{L^2(\Omega)}\big)\\
&\quad\leqslant C(t+1)^{k-1}\left((1+t)^{-\frac{5}{4}}+(1+t)^{-\frac{5}{4}}\int\limits_0^t(1+s)^{-\frac{5}{4}+\frac{1}{8}+\frac{1}{8}}\big(\log_e(1+s)\big)^4\,ds\right)^2\\
&\quad\leqslant C(t+1)^{k-1}\big((1+t)^{-\frac{5}{4}}+(1+t)^{-\frac{5}{4}}\big(\log_e(1+t)\big)^5\big)^2,
\end{aligned}$$

from which, it holds for $n=3$ and $t>0$

$$\sup_{j\geqslant 0}\big\|\theta_{j+1}(t)\big\|_{L^2(\Omega)}\leqslant C(1+t)^{-\frac{5}{4}}\big(\log_e(1+t)\big)^5. \tag{2.34}$$

Repeating the proof of (2.34) yields for $n=3$, $t>0$ and any $j=0,1,2,\ldots$

$$\begin{aligned}
&\frac{d}{dt}\big((t+1)^k\big\|\theta_{j+1}(t)\big\|^2_{L^2(\Omega)}\big)\\
&\quad\leqslant C(t+1)^{k-1}\left((1+t)^{-\frac{5}{4}}+(1+t)^{-\frac{5}{4}}\int\limits_0^t(1+s)^{-\frac{5}{4}+\frac{1}{8}}\big(\log_e(1+s)\big)^6\,ds\right)^2\\
&\quad\leqslant C(t+1)^{k-1-\frac{5}{2}},
\end{aligned}$$

which implies that for $n=3$ and $t>0$

$$\sup_{j\geqslant 0}\big\|\theta_{j+1}(t)\big\|_{L^2(\Omega)}\leqslant C(1+t)^{-\frac{5}{4}}. \tag{2.35}$$

Combining (2.25), (2.32) and (2.35), we conclude that for $n\geqslant 3$ and any $t>0$

$$\sup_{j\geqslant 0}\big\|\theta_j(t)\big\|_{L^2(\Omega)}\leqslant C(1+t)^{-\frac{n+2}{4}}. \tag{2.36}$$

It follows from (2.19), (2.20), (2.21) and (2.36) that for any $t > 0$

$$\sup_{j\geqslant 0}\|u_j(t)\|_{L^2(\Omega)} \leqslant C \quad \text{and} \quad \sup_{j\geqslant 0}\int_0^t \|\nabla u_j(s)\|^2_{L^2(\Omega)}\,ds \leqslant C. \tag{2.37}$$

Inserting (2.36) and (2.37) into (2.15) with $\frac{n+2}{n} < r < 2$, taking $\rho = k(t+1)^{-1}$ with some large positive integer k in (2.15), and integrating from 0 to t, we derive for any $t > 0$

$$\begin{aligned}
&\sup_{j\geqslant 0}\|u_{j+1}(t)\|^2_{L^2(\Omega)} \\
&\quad\leqslant C(1+t)^{-k}\|a\|^2_{L^2(\Omega)} + C(1+t)^{-k}\int_0^t (1+s)^{k-\frac{n+2}{4}}\,ds \\
&\qquad + C(1+t)^{-k}\int_0^t (1+s)^{k-1}\Big(\|e^{-sA}a\|^2_{L^2(\Omega)} + (1+s)^{-\frac{n(2-r)}{2r}} + (1+s)^{-\frac{n-2}{2}}\Big)\,ds \\
&\quad\to 0 \quad \text{as } t\to\infty.
\end{aligned} \tag{2.38}$$

Here we used the fact: $\|e^{-tA}a\|_{L^2(\Omega)} \to 0$ as $t\to\infty$ for any $a\in L^2_\sigma(\Omega)$.

In the following arguments, we further assume $a \in L^{\frac{n}{n-1}}(\Omega)$. Using (2.5), (2.36) and (2.37), we conclude that for $j = 0, 1, \ldots$ and $t > 0$

$$\begin{aligned}
\|\theta_{j+1}(t)\|_{L^1(\Omega)} &\leqslant \|e^{t\Delta}b\|_{L^1(\Omega)} + C\int_0^t \|e^{(t-s)\Delta}(u_j\cdot\nabla\theta_{j+1})(s)\|_{L^1(\Omega)}\,ds \\
&\leqslant C(1+t)^{-\frac12} + C\int_0^t (t-s)^{-\frac12}\|u_j(s)\|_{L^2(\Omega)}\|\theta_{j+1}(s)\|_{L^2(\Omega)}\,ds \\
&\leqslant C(1+t)^{-\frac12} + C\left(\int_0^{\frac t2} + \int_{\frac t2}^t\right)(t-s)^{-\frac12}(1+s)^{-\frac{n+2}{4}}\,ds \\
&\leqslant C(1+t)^{-\frac12}.
\end{aligned} \tag{2.39}$$

Using (2.36) and (2.39), one has for any $t > 0$ and $j = 0, 1, \ldots$

$$\begin{aligned}
&\int_0^{\frac t2} (t-s)^{-\frac n2(\frac1r-\frac12)}\|\theta_j(s)\|^{\frac2r-1}_{L^1(\Omega)}\|\theta_j(s)\|^{2(1-\frac1r)}_{L^2(\Omega)}\,ds + \int_{\frac t2}^t \|\theta_j(s)\|_{L^2(\Omega)}\,ds \\
&\quad\leqslant Ct^{-\frac n2(\frac1r-\frac12)}\int_0^{\frac t2}(1+s)^{-\frac12(\frac2r-1)-\frac{n+1}{2}(1-\frac1r)}\,ds + C\int_{\frac t2}^t (1+s)^{-\frac{n+2}{4}}\,ds \\
&\quad\leqslant C(1+t)^{-\frac12(\frac n2-1)} \quad \text{by taking } 1 < r < \frac{n}{n-1}.
\end{aligned} \tag{2.40}$$

Note that by Lemma 2.1, it holds true for $a \in L^2_\sigma(\Omega) \cap L^{\frac{n}{n-1}}(\Omega)$ and $t > 0$

$$\|u_0(t)\|_{L^2(\Omega)} = \|e^{-tA}a\|_{L^2(\Omega)} \leqslant C(1+t)^{-\frac{1}{2}(\frac{n}{2}-1)}\|a\|_{L^{\frac{n}{n-1}}(\Omega)}. \tag{2.41}$$

Inserting (2.40) into (2.15) with $1 < r < \frac{n}{n-1}$, using (2.36), (2.37) and (2.41), we deduce for any $t > 0$ and $j = 0, 1, \ldots$

$$\begin{aligned}
&\frac{d}{dt}\|u_{j+1}(t)\|^2_{L^2(\Omega)} + \rho\|u_{j+1}(t)\|^2_{L^2(\Omega)} \\
&\quad\leqslant C\rho\left((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)}\left(1+\|a\|_{L^{\frac{n}{n-1}}(\Omega)}\right) + \rho^{\frac{n}{4}-\delta}\left(\int_0^t \|u_j(s)\|^2_{L^2(\Omega)}\,ds \int_0^t \|u_{j+1}(s)\|_{L^2(\Omega)}\,ds\right)^{\frac{1-2\delta}{4}}\right)^2 \\
&\qquad + C\|u_{j+1}(t)\|_{L^2(\Omega)}(1+t)^{-\frac{n+2}{4}} \quad \text{with any } \delta \in \left(0, \frac{1}{4}\right).
\end{aligned} \tag{2.42}$$

Setting $\rho = k(t+1)^{-1}$ with large positive integer k, and multiplying both sides of (2.42) by $(t+1)^k$, using (2.37), we obtain for any $t > 0$ and $j = 0, 1, \ldots$

$$\begin{aligned}
&\frac{d}{dt}\left((1+t)^k\|u_{j+1}(t)\|^2_{L^2(\Omega)}\right) \\
&\quad\leqslant C(1+t)^{k-1}\left((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)} + (1+t)^{-\frac{n}{4}+\delta+\frac{1-2\delta}{2}}\right)^2 + C(1+t)^{k-1-\frac{1}{2}(\frac{n}{2}-1)} \\
&\quad\leqslant C(1+t)^{k-1}\left((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)} + (1+t)^{-\frac{1}{4}(\frac{n}{2}-1)}\right)^2.
\end{aligned} \tag{2.43}$$

It follows from (2.41) and (2.43) that for any $t > 0$

$$\sup_{j\geqslant 0}\|u_j(t)\|_{L^2(\Omega)} \leqslant C(1+t)^{-\frac{1}{4}(\frac{n}{2}-1)}. \tag{2.44}$$

Inserting (2.44) into (2.42), setting $\rho = k(t+1)^{-1}$ with large positive integer k, and multiplying both sides of (2.42) by $(t+1)^k$. Then for any $t > 0$ and $j = 0, 1, \ldots$

$$\begin{aligned}
&\frac{d}{dt}\left((1+t)^k\|u_{j+1}(t)\|^2_{L^2(\Omega)}\right) \\
&\quad\leqslant C(1+t)^{k-1}\left((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)} + (1+t)^{-\frac{n}{4}+\delta}\left(\int_0^t (1+s)^{-\frac{1}{2}(\frac{n}{2}-1)}\,ds\right)^{\frac{1-2\delta}{2}}\right)^2 \\
&\qquad + C(1+t)^{k-1-\frac{1}{4}(\frac{n}{2}-1)-\frac{1}{2}(\frac{n}{2}-1)} \quad \text{with any } \delta \in \left(0, \frac{1}{4}\right).
\end{aligned} \tag{2.45}$$

It follows from (2.45) that for any $t > 0$

$$\begin{aligned}
&\sup_{j\geqslant 0}\|u_{j+1}(t)\|_{L^2(\Omega)} \\
&\quad\leqslant C\left((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)} + (1+t)^{-\frac{3}{8}(\frac{n}{2}-1)} + (1+t)^{-\frac{n}{4}+\delta}\left(\int_0^t (1+s)^{-\frac{1}{2}(\frac{n}{2}-1)}\,ds\right)^{\frac{1-2\delta}{2}}\right)
\end{aligned}$$

$$\leqslant C\big((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)}+(1+t)^{-\frac{3}{8}(\frac{n}{2}-1)}\big)+C(1+t)^{-\frac{n}{4}+\delta}\begin{cases}1 & \text{if } n\geqslant 7,\\ (\log_e(1+t))^{\frac{1}{2}-\delta} & \text{if } n=6,\\ (1+t)^{(\frac{3}{2}-\frac{n}{4})(\frac{1}{2}-\delta)} & \text{if } 3\leqslant n\leqslant 5,\end{cases}$$
$$\leqslant C(1+t)^{-\frac{3}{8}(\frac{n}{2}-1)}. \tag{2.46}$$

Setting $\rho=k(t+1)^{-1}$ with some large positive integer k, and multiplying both sides of (2.42) by $(t+1)^k$. From (2.41) and (2.46), we conclude for any $t>0$ and $j=0,1,\dots$

$$\begin{aligned}&\frac{d}{dt}\big((1+t)^k\|u_{j+1}(t)\|^2_{L^2(\Omega)}\big)\\ &\quad\leqslant C(1+t)^{k-1}\Bigg((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)}+(1+t)^{-\frac{n}{4}+\delta}\Bigg(\int_0^t(1+s)^{-\frac{3}{4}(\frac{n}{2}-1)}\,ds\Bigg)^{\frac{1}{2}-\delta}\\ &\qquad+(1+t)^{-\frac{3}{16}(\frac{n}{2}-1)-\frac{1}{4}(\frac{n}{2}-1)}\Bigg)^2.\end{aligned}\tag{2.47}$$

Taking $k>0$ suitably large in (2.47). Then for any $t>0$

$$\begin{aligned}\sup_{j\geqslant 0}\|u_{j+1}(t)\|_{L^2(\Omega)}&\leqslant C\big((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)}+(1+t)^{-\frac{7}{16}(\frac{n}{2}-1)}\big)\\ &\qquad+C(1+t)^{-\frac{n}{4}+\delta}\begin{cases}1 & \text{if } n\geqslant 5,\\ (1+t)^{(\frac{7}{4}-\frac{3n}{8})(\frac{1}{2}-\delta)} & \text{if } n=3,4,\end{cases}\\ &\leqslant C(1+t)^{-\frac{7}{16}(\frac{n}{2}-1)}.\end{aligned}\tag{2.48}$$

Inserting (2.41), (2.48) into (2.42), setting $\rho=k(t+1)^{-1}$ with large positive integer k, and multiplying both sides of (2.42) by $(t+1)^k$. Then for any $t>0$ and $j=0,1,\dots$

$$\begin{aligned}&\frac{d}{dt}\big((1+t)^k\|u_{j+1}(t)\|^2_{L^2(\Omega)}\big)\\ &\quad\leqslant C(1+t)^{k-1}\Bigg((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)}+(1+t)^{-\frac{n}{4}+\delta}\Bigg(\int_0^t(1+s)^{-\frac{7}{8}(\frac{n}{2}-1)}\,ds\Bigg)^{\frac{1}{2}-\delta}\Bigg)^2\\ &\qquad+C(1+t)^{k-1-\frac{7}{16}(\frac{n}{2}-1)-\frac{1}{2}(\frac{n}{2}-1)},\end{aligned}$$

from which, it holds true for any $t>0$

$$\begin{aligned}\sup_{j\geqslant 0}\|u_{j+1}(t)\|_{L^2(\Omega)}&\leqslant C\big((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)}+(1+t)^{-\frac{15}{32}(\frac{n}{2}-1)}\big)\\ &\qquad+C(1+t)^{-\frac{n}{4}+\delta}\begin{cases}1 & \text{if } n\geqslant 5,\\ (1+t)^{(\frac{15}{8}-\frac{7n}{16})(\frac{1}{2}-\delta)} & \text{if } n=3,4,\end{cases}\\ &\leqslant C(1+t)^{-\frac{15}{32}(\frac{n}{2}-1)}.\end{aligned}$$

Repeating the above proof steps, we find a sequence $\{\alpha_m\}$ with the properties: for $m = 1, 2, \ldots$

$$\alpha_{m+1} = \frac{1}{2}\left(-\frac{1}{2}\left(\frac{n}{2} - 1\right) + \alpha_m\right), \qquad \alpha_1 = -\frac{1}{4}\left(\frac{n}{2} - 1\right).$$

Moreover, for any $t > 0$

$$\sup_{j \geqslant 0} \left\| u_{j+1}(t) \right\|_{L^2(\Omega)} \leqslant C_m\left((1+t)^{-\frac{1}{2}(\frac{n}{2}-1)} + (1+t)^{\alpha_m}\right) \quad \text{with } m = 1, 2, \ldots. \tag{2.49}$$

After a direct calculation, the sequence $\{\alpha_m\}$ can be rewritten as follows:

$$\alpha_m = -\frac{1}{2}\left(\frac{n}{2} - 1\right) + \frac{1}{2^{m+1}}\left(\frac{n}{2} - 1\right), \quad m = 1, 2, \ldots,$$

with the properties:

$$\lim_{m \to \infty} \alpha_m = -\frac{1}{2}\left(\frac{n}{2} - 1\right) \quad \text{and} \quad \alpha_m > -\frac{1}{2}\left(\frac{n}{2} - 1\right), \quad m = 1, 2, \ldots.$$

For any small $\epsilon > 0$, there exists a large number $m_0 = m_0(\epsilon) > 0$ such that $\alpha_{m_0} < -\frac{1}{2}(\frac{n}{2} - 1) + \epsilon$. Whence from (2.49), one has for any $t > 0$

$$\sup_{j \geqslant 0} \left\| u_{j+1}(t) \right\|_{L^2(\Omega)} \leqslant C_{m_0}(1+t)^{-\frac{1}{2}(\frac{n}{2}-1)+\epsilon}. \tag{2.50}$$

In addition, it follows from (2.9) and (2.37) that any $t > 0$ and $j = 0, 1, \ldots$

$$\int_0^\infty \left(\left\| \nabla u_{j+1}(t) \right\|^2_{L^2(\Omega)} + \left\| \nabla \theta_{j+1}(t) \right\|^2_{L^2(\Omega)} \right) dt \leqslant C. \tag{2.51}$$

From (2.18), (2.22), (2.36), (2.38), (2.50) and (2.51), and after a standard weak converging argument, we can find two functions $u \in L^\infty(0, \infty; L^2_\sigma(\Omega)) \cap L^2(0, \infty; H^1_0(\Omega))$, $\theta \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1_0(\Omega))$, and (u, θ) is a weak solution of (1.1) satisfying the estimates (1.2), (1.3), (1.5), (1.6). $\square$

Acknowledgment

The author is very grateful to Professor Maria E. Schonbek for a number of stimulating and helpful discussions and valuable suggestions on the Boussinesq system.

References

[1] R.A. Adams, Sobolev Space, Academic Press, 1975.
[2] H. Bae, Temporal decays in L^1 and L^∞ for the Stokes flow, J. Differential Equations 222 (2006) 1–20.
[3] H. Bae, Temporal and spatial decays for the Stokes flow, J. Math. Fluid Mech. 10 (2008) 503–530.
[4] H. Bae, H. Choe, Decay rate for the incompressible flows in half spaces, Math. Z. 238 (2001) 799–816.
[5] H. Bae, B. Jin, Asymptotic behavior for the Navier–Stokes equations in 2D exterior domains, J. Funct. Anal. 240 (2006) 508–529.
[6] H. Bae, B. Jin, Temporal and spatial decay rates of Navier–Stokes solutions in exterior domains, Bull. Korean Math. Soc. 44 (2007) 547–567.

[7] H. Bae, B. Jin, Upper and lower bounds of temporal and spatial decays for the Navier–Stokes equations, J. Differential Equations 209 (2005) 365–391.
[8] H. Bae, B. Jin, Temporal and spatial decays for the Navier–Stokes equations, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005) 461–477.
[9] H. Bae, J. Roh, Weighted estimates for the incompressible fluid in exterior domains, J. Math. Anal. Appl. 355 (2009) 846–854.
[10] W. Borchers, T. Miyakawa, Algebraic L^2 decay for Navier–Stokes flows in exterior domains, Acta Math. 165 (1990) 189–227.
[11] L. Brandolese, Space–time decay of Navier–Stokes flows invariant under rotations, Math. Ann. 329 (2004) 685–706.
[12] L. Brandolese, M.E. Schonbek, Large time decay and growth for solutions of a viscous Boussinesq system, Trans. Amer. Math. Soc., in press, http://www.ams.org/cgi-bin/mstrack/accepted_papers?jrnl=tran.
[13] L. Brandolese, F. Vigneron, New asymptotic profiles of nonstationary solutions of the Navier–Stokes system, J. Math. Pures Appl. 88 (2007) 64–86.
[14] A. Córdoba, D. Córdoba, A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (2004) 511–528.
[15] C. He, T. Miyakawa, Nonstationary Navier–Stokes flows in a two-dimensional exterior domain with rotational symmetries, Indiana Univ. Math. J. 55 (2006) 1483–1555.
[16] C. He, T. Miyakawa, On L^1-summability and asymptotic profiles for smooth solutions to Navier–Stokes equations in a 3D exterior domain, Math. Z. 245 (2003) 387–417.
[17] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachr. 4 (1951) 213–231.
[18] T. Kato, Strong L^p-solutions of the Navier–Stokes equation in R^m, with applications to weak solutions, Math. Z. 187 (1984) 471–480.
[19] P.-G. Lemarié, Recent Developments in the Navier–Stokes Problem, CRC Press, 2002.
[20] J. Leray, Sur le mouvement d'un liquide visqueux remplissant l'espace, Acta Math. 63 (1934) 193–248.
[21] T. Miyakawa, H. Sohr, On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier–Stokes equations in exterior domains, Math. Z. 199 (1988) 455–478.
[22] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci., vol. 44, Springer-Verlag, New York, 1983.
[23] J. Pedloski, Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.
[24] M.E. Schonbek, L^2 decay for weak solutions of the Navier–Stokes equations, Arch. Ration. Mech. Anal. 88 (1985) 209–222.
[25] M.E. Schonbek, Lower bounds of rates of decay for solutions to the Navier–Stokes equations, J. Amer. Math. Soc. 4 (1991) 423–449.
[26] M.E. Schonbek, Asymptotic behavior of solutions to the three-dimensional Navier–Stokes equations, Indiana Univ. Math. J. 41 (1992) 809–823.
[27] H. Sohr, The Navier–Stokes Equations, Birkhäuser, Basel, 2001.